Simulate AR(1) Autoregressive Processes

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Mean and Standard Deviation for AR(1) Autoregressive Process

A first-order autoregressive process can be written as:

- AR1: $X_t = \text{constant} + \text{persistence} \cdot x_{t-1} + \epsilon$
- AR1: $X_t = C + \rho \cdot x_{t-1} + \epsilon$

Assume that ϵ is mean zero

Note that, we know the mean of X:

- $\mu_X = C + \rho \cdot \mu_X + 0$
- $\mu_x = \frac{C}{1-\rho}$

Note that, we also know the standard deviation of X:

- $\text{var}(X) = \rho^2 \cdot \text{var}(X) + \text{var}(\epsilon)$
- $\sigma_x = \sqrt{\frac{\sigma_\epsilon^2}{1-\rho^2}}$

We will let the initial point of the time series follow the stationary distribution of the AR(1) process, then we simulate the time series over 100 periods, in the example below, we use a highly persistent shock process with $\rho = 0.98$, $\sigma_\epsilon = 0.02$, $C = 0.02$. Note that for this process:

- $\mu_x^{\rho=0.98,\sigma_\epsilon=0.02,C=0.02} = \frac{0.02}{1-0.98} = 1$
- $\sigma_x^{\rho=0.98,\sigma_\epsilon=0.02,C=0.02} = \sqrt{\frac{0.02^2}{1-0.98^2}} \approx 0.10$

Simulated one First-Order Autoregressive Time-Series

In the Example below, we simulate an individual for 1000 periods, given $\rho = 0.98$, $\sigma_\epsilon = 0.02$, $C = 0.02$. Given that the process is highly persistent, the individual stays rich or poor for dozens of periods at a time. If each period is a year, look at the results below, and suppose the simulated time series is income, what is the process saying about this person's income rise and fall. Note that we have the same person through all 1000 periods, but if you only look at 50 periods (years), you might this this person during one span is really successful, anothe segment of 50 years, doing really bad, but actually there is nothing changing in the person's type, all that is changing is the person's luck.

First Set Parameters:

```
% Number of Time Periods
it_T = 1000;
% Mean and SD of the Shock Process
fl_constant = 0.02;
fl_normal_sd = 0.02;
% Persistence
fl_persistence = 0.98;
% Bounds on Shocks
fl_shk_bnds = 3;
% Initialize with exo fed point or not, if false initialize at Random Point
% from the stationary distribution
bl_init = true;
fl_init = fl_constant/(1 - fl_persistence);
```

Second, generate a vector of normal shocks:

```
% Generate a normal shock vector (the first draw will be ignored)
it_draws = it_T;
rng(789);
ar_fl_shocks = normrnd(0, fl_normal_sd, 1, it_draws);
disp(ar_fl_shocks(1:20));
```

```
  Columns 1 through 15

   -0.0060   -0.0047    0.0168    0.0118    0.0380    0.0062   -0.0616   -0.0485   -0.0192    0.0023   -0.0197    0.

  Columns 16 through 20

    0.0099   -0.0200   -0.0206   -0.0090   -0.0069
```

Third, replace any values exceeding bounds:

```
% out of bounds indicators
fl_shk_bds_lower = 0 - fl_normal_sd*fl_shk_bnds;
fl_shk_bds_upper = 0 + fl_normal_sd*fl_shk_bnds;
ar_bl_outofbounds = (ar_fl_shocks <= fl_shk_bds_lower | ar_fl_shocks >= fl_shk_bds_upper);
% count out of bounds
disp(strcat('lower:', num2str(fl_shk_bds_lower), ', upper:', num2str(fl_shk_bds_upper)));
```

```
lower:-0.06, upper:0.06
```

```
disp(sum(ar_bl_outofbounds));
```

```
     4
```

```
ar_fl_shocks(ar_fl_shocks <= fl_shk_bds_lower) = fl_shk_bds_lower;
ar_fl_shocks(ar_fl_shocks >= fl_shk_bds_upper) = fl_shk_bds_upper;
```

Fourth, generate the AR(1) time series:

```
% Initialize Output Array
ar_fl_time_series = zeros(size(ar_fl_shocks));
% Loop over time
```

```
for it_t=1:1:length(ar_fl_shocks)
    if (it_t == 1)
        % initialize using the ean of the process
        ar_fl_time_series(1) = fl_constant/(1 - fl_persistence);
        if (bl_init)
            ar_fl_time_series(1) = fl_init;
        end
    else
        fl_ts_t = fl_constant + ar_fl_time_series(it_t-1)*fl_persistence + ar_fl_shocks(it_t);
        ar_fl_time_series(it_t) = fl_ts_t;
    end
end
```

Fifth, show the mean and sd of the process (these are very close to the analytical results):

```
disp(mean(ar_fl_time_series));
```

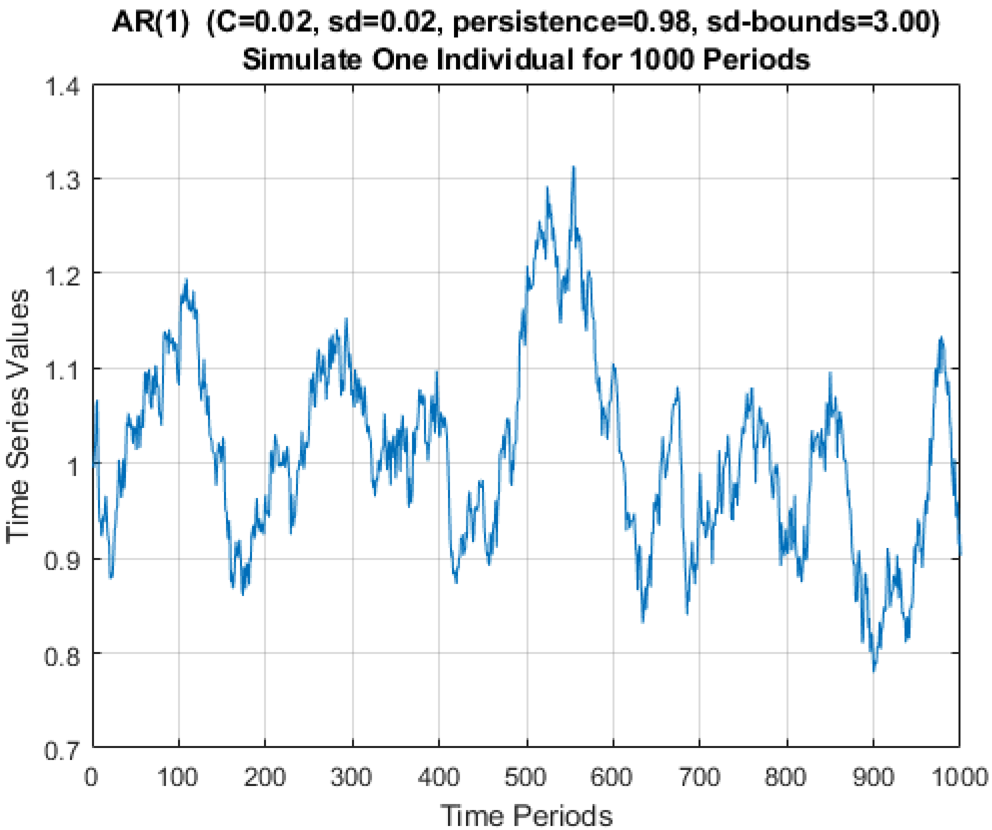
```
1.0104
```

```
disp(std(ar_fl_time_series));
```

```
0.1000
```

Sixth, plot the results:

```
figure();
% x-axis
ar_it_time = 1:1:length(ar_fl_shocks);
% plot
plot(ar_it_time, ar_fl_time_series);
% Generate Title
ar_fl_params_values = [fl_constant, fl_normal_sd, fl_persistence, fl_shk_bnds];
ar_st_parms_names = ["C", "sd", "persistence", "sd-bounds"];
st_rounding = '.2f';
st_title_main = "AR(1) ";
ar_st_params = strcat(ar_st_parms_names, compose(strcat("=%", st_rounding), ar_fl_params_values
st_param_pasted = strjoin(ar_st_params, ', ');
st_title_wth_params = strcat(st_title_main, ' (', st_param_pasted, ')');
title({st_title_wth_params, 'Simulate One Individual for 1000 Periods'});
% X and Y labels
ylabel({'Time Series Values'});
xlabel('Time Periods');
grid on;
```



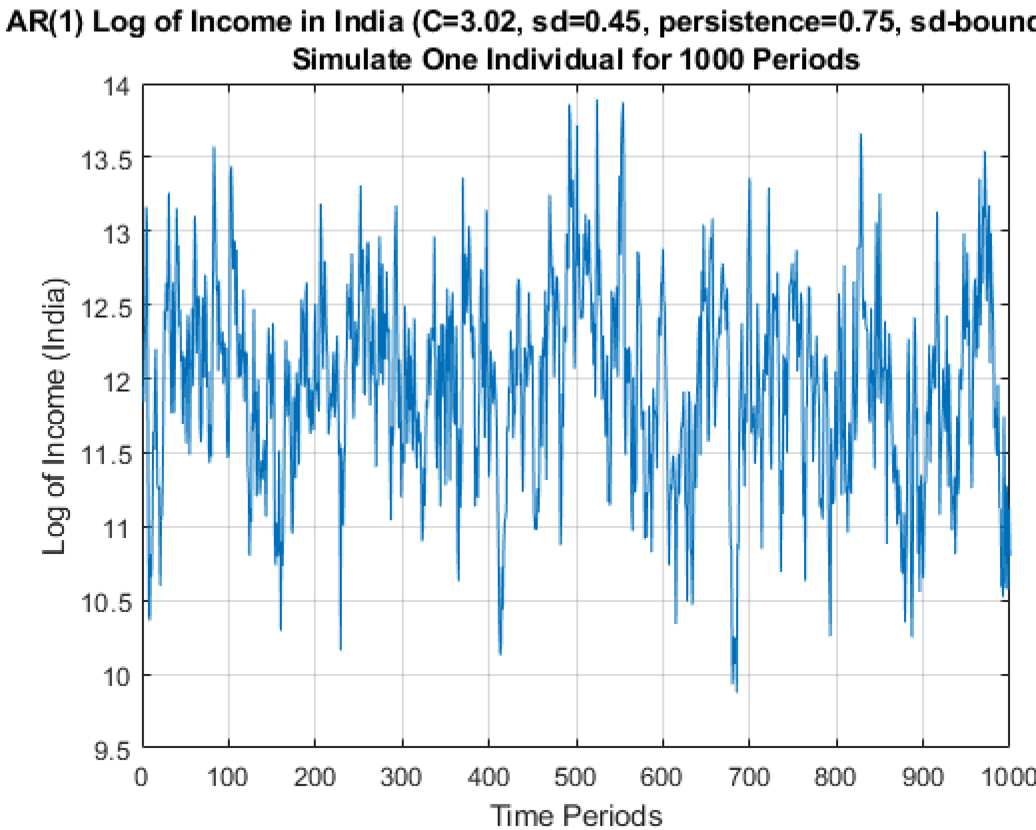
Income Process for India

In the Example below, we simulate an individual for 1000 periods. In the example below, the parameters are from estiamting an AR(1) process for log of income in India, see Udupa and Wang (2020). We can use the ff_simu_stats function from MEconTools to look at the distributional information from this time series, we care about income, so we will exponentiate the log of income we obtained.

Note that for the distributional results, GINI, share of income held by different percentiles of households, is invariant to the choice of the C constant term earlier. Try changing that number, the distributional statistics that we obtain below will be the same. Also note that the exponential of the mean of the log of income is not equal to the mean of income.

```
% Number of Time Periods
it_T = 1000;
% Mean and SD of the Shock Process
fl_constant = 3.024467;
fl_normal_sd = 0.45;
% Persistence
fl_persistence = 0.7468;
% Bounds on Shocks
fl_shk_bnds = 5;
% Initialize with exo fed point or not, if false initialize at Random Point
% from the stationary distribution
bl_init = true;
fl_init = fl_constant/(1 - fl_persistence);
% Generate a normal shock vector (the first draw will be ignored)
```

```
it_draws = it_T;
rng(789);
ar_fl_shocks = normrnd(0, fl_normal_sd, 1, it_draws);
% out of bounds indicators
fl_shk_bds_lower = 0 - fl_normal_sd*fl_shk_bnds;
fl_shk_bds_upper = 0 + fl_normal_sd*fl_shk_bnds;
ar_bl_outofbounds = (ar_fl_shocks <= fl_shk_bds_lower | ar_fl_shocks >= fl_shk_bds_upper);
ar_fl_shocks(ar_fl_shocks <= fl_shk_bds_lower) = fl_shk_bds_lower;
ar_fl_shocks(ar_fl_shocks >= fl_shk_bds_upper) = fl_shk_bds_upper;
% Initialize Output Array
ar_fl_time_series = zeros(size(ar_fl_shocks));
% Loop over time
for it_t=1:1:length(ar_fl_shocks)
    if (it_t == 1)
        % initialize using the ean of the process
        ar_fl_time_series(1) = fl_constant/(1 - fl_persistence);
        if (bl_init)
            ar_fl_time_series(1) = fl_init;
        end
    else
        fl_ts_t = fl_constant + ar_fl_time_series(it_t-1)*fl_persistence + ar_fl_shocks(it_t);
        ar_fl_time_series(it_t) = fl_ts_t;
    end
end
ar_series = (ar_fl_time_series);
fl_mean = mean(ar_series);
fl_std = std(ar_series);
figure();
% x-axis
ar_it_time = 1:1:length(ar_fl_shocks);
% plot
plot(ar_it_time, (ar_fl_time_series));
% Generate Title
ar_fl_params_values = [fl_constant, fl_normal_sd, fl_persistence, fl_shk_bnds];
ar_st_parms_names = ["C", "sd", "persistence", "sd-bounds"];
st_rounding = '.2f';
st_title_main = "AR(1) Log of Income in India";
ar_st_params = strcat(ar_st_parms_names, compose(strcat("=%", st_rounding), ar_fl_params_values
st_param_pasted = strjoin(ar_st_params, ', ');
st_title_wth_params = strcat(st_title_main, ' (', st_param_pasted, ')');
title({st_title_wth_params, 'Simulate One Individual for 1000 Periods'});
% X and Y labels
ylabel({'Log of Income (India)'});
xlabel('Time Periods');
grid on;
```




```
% Set Parameters
mp_cl_mt_xyz_of_s = containers.Map('KeyType','char', 'ValueType','any');
mp_cl_mt_xyz_of_s('log_income') = {(ar_series), zeros(1)};
mp_cl_mt_xyz_of_s('income') = {exp(ar_series), zeros(1)};
mp_cl_mt_xyz_of_s('ar_st_y_name') = ["log_income", "income"];
% Mass
rng(123);
mt_f_of_s = zeros(size(ar_series)) + 1/numel(ar_series);
mt_f_of_s = mt_f_of_s/sum(mt_f_of_s, 'all');
mp_cl_mt_xyz_of_s_out = ff_simu_stats(mt_f_of_s, mp_cl_mt_xyz_of_s);
```

```
xxx tb_outcomes: all stats xxx
    OriginalVariableNames    log_income      income   
    _____________________    __________    __________

    {'mean'          }         11.959       1.946e+05
    {'unweighted_sum'}          11959       1.946e+08
    {'sd'            }        0.67236      1.4024e+05
    {'coefofvar'     }       0.056222         0.72064
    {'gini'          }       0.031635         0.35645
    {'min'           }         9.8742           19423
    {'max'           }         13.893       1.081e+06
    {'pYis0'         }              0               0
    {'pYls0'         }              0               0
    {'pYgr0'         }              1               1
    {'pYisMINY'      }          0.001           0.001
    {'pYisMAXY'      }          0.001           0.001
    {'p1'            }         10.258           28498
    {'p10'           }         11.087           65316
    {'p25'           }         11.518      1.0049e+05
    {'p50'           }         11.983         1.6e+05
```

```
{'p75'              }        12.42     2.4771e+05
{'p90'              }       12.761     3.4829e+05
{'p99'              }       13.479      7.141e+05
{'fl_cov_log_income'}      0.45207          84333
{'fl_cor_log_income'}            1        0.89439
{'fl_cov_income'    }        84333     1.9667e+10
{'fl_cor_income'    }      0.89439              1
{'fracByP1'         }    0.0093239      0.0014385
{'fracByP10'        }     0.089816       0.024687
{'fracByP25'        }       0.2317       0.088239
{'fracByP50'        }      0.47761        0.25477
{'fracByP75'        }      0.73265        0.51224
{'fracByP90'        }      0.89045        0.73747
{'fracByP99'        }      0.98852        0.95275
```