

FASP User Guide

FASP Developer Team

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Chapter 1

Introduction

1.1 General description

The Fast Auxiliary Space Preconditioning (FASP) package provides C source files¹ to build a library of iterative solvers and preconditioners for the solution of large-scale linear systems of equations. The components of the FASP basic library include several ready-to-use, modern, and efficient iterative solvers used in applications ranging from simple examples of discretized scalar partial differential equations (PDEs) to numerical simulations of complex, multicomponent physical systems via the Auxiliary Space Preconditioning framework [19].

The main components of the FASP basic library are:

- Basic linear iterative methods;
- Standard Krylov subspace methods;
- Geometric and Algebraic Multigrid (G/AMG) methods;
- Incomplete factorization methods.

The FASP distribution also includes several examples for solving simple benchmark problems. The basic (kernel) FASP distribution is open-source and is licensed under GNU Lesser General Public License or LGPL. Other distributions may have different licensing (contact the developer team for details on this).

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¹The code is in the C99 (ISO/IEC 9899:1999) compatible.

1.2 Roadmap: from basics to complex applications

A distinct feature of the FASP software project is that it is an open ended project. It contains a basic kernel of sources and is maintained by a team of developers with the expertise to build efficient solvers for a wide range of complex numerical models.

As typical for an open-source software, the further development of FASP project will also be based on the involvement of the community. While we have our own plans for expanding FASP's capabilities, we also count on the users' input in providing requests for, as well as, contributions to, the expansion of FASP in different application areas. Our team is ready to provide (or help with) the design and the implementation of efficient solvers based on the FASP kernel to best meet the goals and the requirements of our users.

The FASP software has been successfully used to build efficient solvers for several discretized PDEs and systems of PDEs: general scalar elliptic equations; linear elasticity; Brinkman equation; bi-harmonic equation; Stokes and Navier-Stokes equations; $H(\text{curl})/H(\text{div})$ systems; Maxwell's system. The resulting solvers have been applied in simulations from fluid dynamics, underground water simulation, fluid-structure interactions, Oldroyd-B and Johnson-Segelman models, black-oil model, and magnetohydrodynamics (MHD).

Several of these benchmark problems are included as examples in the open-source distribution, others are under development or have more restrictive licensing.

1.3 How to use this guide

This user's guide describes how to use the existing solvers in FASP via a couple of simple tutorial problems. The user's guide is a self-contained document but does *not* provide any details about the algorithms or their implementation. Along with this guide, we provide a reference manual² for technical details on the implementation which includes references. We recommend that the users read these references to better understanding of the code. Furthermore, since FASP is under heavy development, please use this guide with caution because the code might have been changed before this document is updated.

1.4 How to obtain FASP

There are several ways to download the FASP source files. We recommend users download the most updated version from the FASP page on SourceForge.

²Available online at <http://fasp.sourceforge.net>. It is also available in “[fasp solver/doc/doc.zip](#)”.

Downloading from SourceForge

The most updated version of FASP can be downloaded directly from

<http://fasp.sf.net/download/faspsolver.zip>

Downloading from BitBucket

FASP is also hosted on *BitBucket.org*³ using Mercurial (Hg)⁴. A Hg client for GNU Linux, Mac OS X, or Windows can be downloaded from

<http://mercurial.selenic.com/downloads/>

There are also many other third-party clients which provides Hg services, for example: EasyMercurial⁵ (cross platform) and SourceTree⁶ (for Mac OS X only).

As a DVCS (Distributed Version Control System) source-control software, Hg is relatively new. But compared with other tools like Git, Hg is considered *friendlier* with a lower learning curve. This is despite the fact that Hg uses two distinct sets of commands and two distinct vocabularies for operations depending upon whether the repository is local or remote. Documentation for Hg is substantially better, including a book⁷. They've also had the advantage of trying the documentation on a fairly savvy group of developers (Mozilla) who gave them lots of feedback that helped polish the rough edges.

Linux or Mac OS X

First, you need to obtain a free copy of FASP kernel functions from our public Hg repository. If you are downloading FASP for the first time, you can clone the repository to your local machine:

```
"Download FASP kernel subroutines via HTTPS"
$ hg clone https://faspusers@bitbucket.org/fasp/faspsolver
```

If you have any problems when clone this repository, please send us an email to faspdev@gmail.com.

After a long pause⁸, you should have obtained “*faspsolver*” in your current directory successfully. If you have already cloned the repository before, you can just pull a new version and update your local version with it: Go to your local “*faspsolver*” directory and then

³Official website: <https://bitbucket.org/>

⁴Official website: <http://mercurial.selenic.com/>

⁵Official website: <http://easyhg.org>

⁶Official website: <http://www.sourcetreeapp.com>

⁷The hgbook, <http://hgbook.red-bean.com/>

⁸In fact, a very long pause. This is because the initial clone with copy all the history data which is about 400MB in total. Depending on the speed of your network, it could take 15 minutes to one hour.

```
"Pull a new version from BitBucket"
$ hg pull

"Update you local version to the new version"
$ hg update
```

Windows OS

If you are using Windows, you may want to install TortoiseHg⁹. After installing it, the TortoiseHg menu has been merged into the right-click menu of Windows Explore. You could download FASP copy from BitBucket.org. Choose “TortoiseHg” --> “Clone” in the pop-up menu, the source address is

<https://faspusers@bitbucket.org/fasp/faspsovler>

Then press “Clone” and you will obtain “faspsolver” in the directory you set.

1.5 Building and installing the FASP library and examples

FASP has been tested using the compilers and built-in libraries of several Linux distributions (Cent OS, Debian, Fedora, RedHat, Ubuntu) Mac OS X 10.6 and later (Leopard, Snow Leopard, Lion, Mavericks, Yosemite, El Capitan), and Windows (XP, Win 7) with several compilers, including gcc, g++, clang, icc, VC++. FASP also easily links to applications written in Fortran and this has been tested with gfortran, g95, ifort Fortran compilers.

FASP on Linux or Mac OS X

To build the FASP library for these operating systems: (1) go to the “faspsolver” directory; (2) modify the “FASP.mk.example” file to match your system and save it as “FASP.mk”; (3) then execute:

```
$ make config
$ make install
```

Note that, if you do not have “FASP.mk” present in the current directory, default settings will be used for building and installation FASP.

In order to be sure that everything went all right, you can go to the “[faspsolver/test](#)” directory and try to run a test problem as follows:

```
$ ./test.ex
```

⁹Official website: <http://tortoisehg.bitbucket.org/>

If you need help with the available options, type

```
$ make help
```

and you will get the following screen

```

||   Fast Auxiliary Space Preconditioners (FASP)   ||
||-----||

Quick start:
To build FASP, copy "FASP.mk.example" to "FASP.mk" and put user-defined setting
there and then type "make config; make install".

More options:
-----
$ make config           # Configure the building environment
$ make config CC=gcc    # Configure with a specific C compiler
$ make config debug=yes # Configure with compiler debug options ON
$ make config debug=all # Configure with FASP internal debug options ON
-----

$ make           # Compile the library (after "make config")
$ make install   # install FASP libraries and related files
$ make uninstall # Remove installed files by "make install"
$ make headers   # Generate function declarations automatically
$ make docs      # Generate the FASP documentation with Doxygen
$ make clean     # Remove obj files but retain configuration options
$ make distclean # Clean and completely removes the build directory
$ make version   # Show version information
$ make help      # Show this screen
-----

```

To uninstall FASP and clean up the working directory, you can run

```
$ make uninstall
$ make distclean
```

Windows 7

We provide a Visual Studio 2008 (VS08) distribution and a VS10 distribution of FASP for Windows users. For example, you can just open “[faspsolver/vs08/faspsolver-vs08.sln](#)” if you are using VS08 as your default developing environment. Then a single-click at the “Build Solution” on the menu or “F7” key will give you all the FASP libraries and the test programs in “[faspsolver/test/](#)”. The way for building in VS10 is similar.

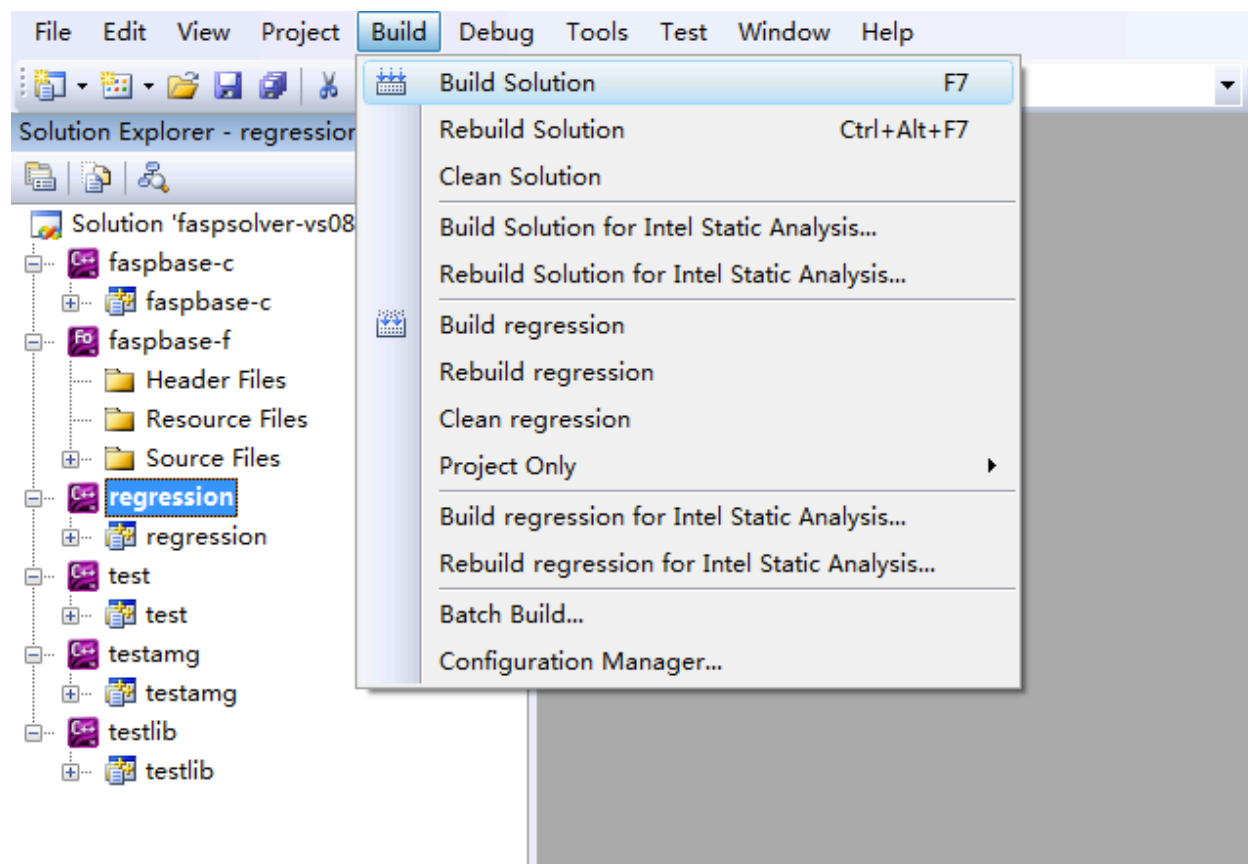


Figure 1.1: Build FASP using Visual Studio 2008.

You need a C/C++ compiler and the Visual Studio to build FASP. For example, the build can be accomplished using either Microsoft Visual C++ or Intel C compiler.

If you are using other versions of Visual Studio (like VS05 or VS12), we advise NOT to convert the "VS08" files to your VS version because the FASP source files might be **removed** by the Visual Studio automatically. In such case, we recommend that you create your own version to build all the libraries and test programs.

If you need to build a VS FASP yourself, you need to create 5 projects:

1. "faspbase-c" contains all the ".c" and ".inl" files in the directory `"/base/src/"`. You should add `"/base/include"` in Additional Directories. This project contains the core subroutines of fapsolver.
2. "faspbase-f" contains all the ".f" files in `"/base/extra/sparsekit"`.

3. “testlib” contains all the “.c” files in “./test/src/”. You should add “./test/include” in Additional Directories.
4. “test” is an executing program for test purpose in FASP. The source file is “./test/main/test.c”.
5. “regression” is another executing program, which contains several methods to test the problems. The source file is “./test/main/regression.c”.

NOTE: If you are using Visual C++, all the C files should be compiled as C++ code (by using the /TP compiling option).

After a successful build on VS, you will have two static libraries named “faspbase-c-vs08.lib” and “faspbase-f-vs08.lib”. You can use the “lib” command to wrap them together as one single file (e.g. FASP.lib) for better portability. For example:

```
C:\FASP> lib /ltcg /out:FASP.lib faspbase-c-vs08.lib faspbase-f-vs08.lib
```

Using a TCL based GUI for installation

For users who like more a GUI based installation, we provide a simple TCL Graphical User Interface (GUI) for building the FASP library. On a machine running Linux or Mac OS X with Tcl/Tk installed, you may invoke the GUI by typing

```
$ wish FASP_install.tcl
```

If all is OK, you should see on your screen the FASP window as shown in Figure 1.2. The rest of the building process is more or less straightforward: After choosing appropriate parameters, click “Config” first, followed by clicking “Install”.

External libraries

There are a few *optional* external libraries that you might want to use, including memory allocation routines, direct solvers, ILU methods, discretization packages, etc. FASP has interfaces to several of them, for example, FASP can be linked to use UMFPack, SuperLU, MUMPS, SparseKit, dlmalloc.

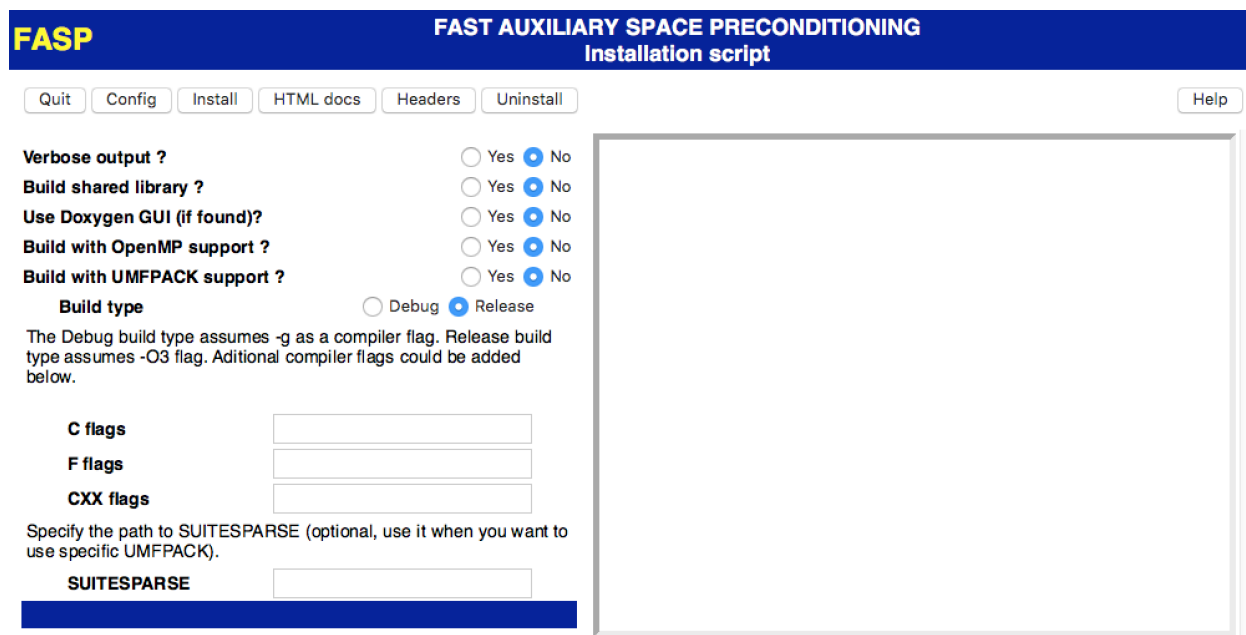


Figure 1.2: Install FASP using the TCL GUI on Mac OS X El Capitan.

Chapter 2

A brief tutorial

In this chapter, we discuss several simple examples included with this FASP distribution and demonstrating how to use the FASP package for solving linear systems. We read the matrices from disk files (the files are also included in the FASP distribution). All the examples in this section can be found inside “[faspsolver/tutorial/](#)”.

After you successfully build FASP (see §1.5), just go to the “[faspsolver/tutorial/](#)” directory and the tutorial examples should be ready to run.

In this section we mainly discuss the C version of these examples; the FASP distribution also includes F90 versions of some of the examples.

In the description below, we display a typical output from runs of each of the examples. Note that the actual output depends on the solver parameters, and, on your computer it may be different than what you see here.

2.1 Example 1: An AMG solver for the Poisson equation

The first example is a standard one: We read a symmetric positive definite matrix A and right-hand side b from harddisk and then we solve $Ax = b$ using the classical AMG method [2, 13, 14]; see §3.6. In this example the matrix A included with the FASP distribution corresponds to a discretization with continuous piecewise linear finite elements of the Poisson equation

$$-\Delta u = f$$

(with the Dirichlet boundary conditions) on a triangulation of a bounded domain Ω .

```
1  /*! \file poisson-amg.c
2  *
3  *  \brief The first test example for FASP: using AMG to solve
4  *         the discrete Poisson equation from P1 finite element.
```

```

5  *          C version.
6  *
7  *  \note   AMG example for FASP: C version
8  *
9  *  Solving the Poisson equation (P1 FEM) with AMG
10 *
11
12 #include "fasp.h"
13 #include "fasp_funcs.h"
14
15 /**
16  * \fn int main (int argc, const char * argv[])
17  *
18  * \brief This is the main function for the first example.
19  *
20  * \author Chensong Zhang
21  * \date   12/21/2011
22  *
23  * Modified by Chensong Zhang on 09/22/2012
24  */
25 int main (int argc, const char * argv[])
26 {
27     input_param    inparam; // parameters from input files
28     AMG_param      amgparam; // parameters for AMG
29
30     printf("\n=====");
31     printf("\n||   FASP: AMG example -- C version   ||");
32     printf("\n=====\\n\\n");
33
34     // Step 0. Set parameters: We can use ini/amg.dat
35     fasp_param_set(argc, argv, &inparam);
36     fasp_param_init(&inparam, NULL, &amgparam, NULL, NULL);
37
38     // Set local parameters using the input values
39     const int print_level = inparam.print_level;
40
41     // Step 1. Get stiffness matrix and right-hand side
42     // Read A and b -- P1 FE discretization for Poisson. The location
43     // of the data files is given in "ini/amg.dat".
44     dCSRmat A;
45     dvector b, x;
46     char filename1[512], *datafile1;
47     char filename2[512], *datafile2;
48
49     // Read the stiffness matrix from matFE.dat
50     strncpy(filename1, inparam.workdir, 128);
51     datafile1="csrmat_FE.dat"; strcat(filename1, datafile1);
52
53     // Read the RHS from rhsFE.dat
54     strncpy(filename2, inparam.workdir, 128);
55     datafile2="rhs_FE.dat"; strcat(filename2, datafile2);
56
57     fasp_dcsrvec2_read(filename1, filename2, &A, &b);

```

```

58
59 // Step 2. Print problem size and AMG parameters
60 if (print_level>PRINT_NONE) {
61     printf("A: m = %d, n = %d, nnz = %d\n", A.row, A.col, A.nnz);
62     printf("b: n = %d\n", b.row);
63     fasp_param_amg_print(&amgparam);
64 }
65
66 // Step 3. Solve the system with AMG as an iterative solver
67 // Set the initial guess to be zero and then solve it
68 // with AMG method as an iterative procedure
69 fasp_dvec_alloc(A.row, &x);
70 fasp_dvec_set(A.row, &x, 0.0);
71
72 fasp_solver_amg(&A, &b, &x, &amgparam);
73
74 // Step 4. Clean up memory
75 fasp_dcsr_free(&A);
76 fasp_dvec_free(&b);
77 fasp_dvec_free(&x);
78
79 return FASP_SUCCESS;
80 }
81
82 /*-----*/
83 /*--      End of File      --*/
84 /*-----*/

```

Since this is the first example, we will explain it in some detail:

- Line 1 tells the Doxygen documentation system¹ that the filename is “poisson-amg.c”. Line 3–5 tells the Doxygen what is the purpose of this file (function).
- Line 12–13 includes the main FASP header file “fasp.h” and FASP function declarations header “fasp_funcs.h”. These two headers shall be included in all files that requires FASP subroutines. Please also be noted that the function declarations in “fasp_funcs.h” are automatically generated from the source files by an `awk` script and we do not recommend modifying this file, since your changes may be lost.
- Line 35–36 sets solver parameters using the default parameters or from the command line options; see more discussions in §2.5. In the “tutorial/ini/amg.dat” file, we can set the location of the data files, type of solvers, maximal number of iteration numbers, convergence tolerance, and many other parameters for iterative solvers.

¹Doxygen <http://www.doxygen.org> is a useful tool for generating documentation from annotated sources. We use it in FASP development.

- Line 44 defines a sparse matrix A in the compressed sparse row (CSR) format. Line 45 defines two vectors: the right-hand side b and the numerical solution x . We refer to §3.1 for definitions of vectors and general sparse matrices.
- Line 57 reads the matrix and the right-hand side from two disk files. Line 46–55 defines the filenames of them.
- Line 60–64 prints basic information of coefficient matrix, right-hand side, and solver parameters.
- Line 69–70 allocates memory for the solution vector x and set its initial value to be all zero.
- Line 72 solves $Ax = b$ using the AMG method. Type of the AMG method and other parameters have been given in “amgparam” at Line 36; see §3.6.
- Line 75–77 frees up memory allocated for A , b , and x .

To run this example, type:

```
$ ./poisson-amg-c.ex
```

A sample output is listed as follows:

```
|| FASP: AMG example — C version ||
```

```
fasp_dcsrvec2_read: reading file ../data/csrmat_FE.dat...
```

```
fasp_dcsrvec2_read: reading file ../data/rhs_FE.dat...
```

```
A: m = 3969, n = 3969, nnz = 27281
```

```
b: n = 3969
```

```
Parameters in AMG_param
```

AMG print level:	2
AMG max num of iter:	1
AMG type:	1
AMG tolerance:	1.00e-06
AMG max levels:	20
AMG cycle type:	1
AMG coarse solver type:	0
AMG scaling of coarse correction:	0
AMG smoother type:	2
AMG smoother order:	1
AMG num of presmoothing:	1
AMG num of postsmoothing:	1
AMG coarsening type:	1
AMG interpolation type:	1
AMG dof on coarsest grid:	500

```

AMG strong threshold:      0.3000
AMG truncation threshold:  0.2000
AMG max row sum:          0.9000
AMG aggressive levels:     0
AMG aggressive path:       1

```

Calling classical AMG ...

Level	Num of rows	Num of nonzeros	Avg. NNZ / row
0	3969	27281	6.87
1	1985	28523	14.37
2	541	7951	14.70
3	141	1803	12.79

Grid complexity = 1.672 | Operator complexity = 2.403

Classical AMG setup costs 0.0068 seconds.

It Num	r / b	r	Conv. Factor
0	1.000000e+00	7.514358e+00	---
1	9.851978e-03	7.403129e-02	0.0099
2	3.507451e-04	2.635624e-03	0.0356
3	1.764023e-05	1.325550e-04	0.0503
4	8.820794e-07	6.628261e-06	0.0500

Number of iterations = 4 with relative residual 8.820794e-07.
 AMG solve costs 0.0017 seconds.
 AMG totally costs 0.0091 seconds.

We also provide a Fortran 90 example, which does the same thing as this C code except it gives less output, in “[tutorial/main/poisson-amg.f90](#)”. Users who would like to call FASP solver from a Fortran based application can see how to do this example.

```

1  !> \file poisson-amg.f90
2  !> \brief The first test example for FASP: using AMG to solve
3  !>       the discrete Poisson equation from P1 finite element.
4  !>       F90 version.
5  !>
6  !> \note  AMG example for FASP: F90 version
7  !>
8  !> Solving the Poisson equation (P1 FEM) with AMG
9  !>
10 !> \author Chensong Zhang
11 !> \date   12/21/2011
12
13 program test
14
15     implicit none
16

```



```

17 double precision, dimension(:), allocatable :: u, b, a
18 integer, dimension(:), allocatable :: ia, ja
19
20 integer :: iufile, n, nnz, i, prt_lvl, maxit
21 double precision :: tol
22
23 print*, ""
24 write(*,"(A)") "=====
25 write(*,"(A)") "|| FASP: AMG example -- F90 version ||"
26 write(*,"(A)") "=====
27 print*, ""
28
29 ! Step 0: user defined variables
30 prt_lvl = 3
31 maxit = 100
32 tol = 1.0d-6
33 iufile = 1
34
35 ! Step 1: read A and b
36
37 !==> Read data A from file
38 open(unit=iufile,file='../data/csrmat_FE.dat')
39
40 read(iufile,*) n
41 allocate(ia(1:n+1))
42 read(iufile,*) (ia(i),i=1,n+1)
43
44 nnz=ia(n+1)-ia(1)
45 allocate(ja(1:nnz),a(1:nnz))
46 read(iufile,*) (ja(i),i=1,nnz)
47 read(iufile,*) (a(i),i=1,nnz)
48
49 close(iufile)
50
51 !==> Read data b from file
52 open(unit=iufile,file='../data/rhs_FE.dat')
53
54 read(iufile,*) n
55 allocate(b(1:n))
56 read(iufile,*) (b(i),i=1,n)
57
58 close(iufile)
59
60 !==> Shift the index to start from 0 (for C routines)
61 forall (i=1:n+1) ia(i)=ia(i)-1
62 forall (i=1:nnz) ja(i)=ja(i)-1
63
64 ! Step 2: Solve the system
65
66 !==> Initial guess
67 allocate(u(1:n))
68 u=0.0d0
69 call fasp_fwrapper_amg(n,nnz,ia,ja,a,b,u,tol,maxit,prt_lvl)

```

```

70
71     ! Step 3: Clean up memory
72     deallocate(ia,ja,a)
73     deallocate(b,u)
74
75 end program test
76
77 /*-----*/
78 /*--          End of File          --*/
79 /*-----*/

```

2.2 Example 2: Conjugate gradient without preconditioning

In the second example, we modify the previous example slightly and solve the Poisson equation using iterative methods (here by default we use the Conjugate Gradient method without preconditioning).

```

1  /*! \file poisson-its.c
2  *  \brief The second test example for FASP: using ITS to solve
3  *         the discrete Poisson equation from P1 finite element.
4  *
5  *  \note  ITS example for FASP: C version
6  *
7  *  Solving the Poisson equation (P1 FEM) with iterative methods
8  */
9
10 #include "fasp.h"
11 #include "fasp_functs.h"
12
13 /**
14  * \fn int main (int argc, const char * argv[])
15  *
16  * \brief This is the main function for the second example.
17  *
18  * \author Feiteng Huang
19  * \date   04/13/2012
20  *
21  * Modified by Chensong Zhang on 09/22/2012
22  */
23 int main (int argc, const char * argv[])
24 {
25     input_param      inparam; // parameters from input files
26     itsolver_param   itparam; // parameters for itsolver
27
28     printf("\n=====");
29     printf("\n||    FASP: ITS example -- C version    ||");
30     printf("\n=====\\n\\n");
31
32     // Step 0. Set parameters: We can use ini/its.dat
33     fasp_param_set(argc, argv, &inparam);
34     fasp_param_init(&inparam, &itparam, NULL, NULL, NULL);

```

```

35
36 // Set local parameters
37 const int print_level = inparam.print_level;
38
39 // Step 1. Get stiffness matrix and right-hand side
40 // Read A and b -- P1 FE discretization for Poisson. The location
41 // of the data files is given in "ini/its.dat".
42 dCSRmat A;
43 dvector b, x;
44 char filename1[512], *datafile1;
45 char filename2[512], *datafile2;
46
47 // Read the stiffness matrix from matFE.dat
48 strncpy(filename1, inparam.workdir, 128);
49 datafile1="csrmat_FE.dat"; strcat(filename1, datafile1);
50
51 // Read the RHS from rhsFE.dat
52 strncpy(filename2, inparam.workdir, 128);
53 datafile2="rhs_FE.dat"; strcat(filename2, datafile2);
54
55 fasp_dcsrvec2_read(filename1, filename2, &A, &b);
56
57 // Step 2. Print problem size and ITS parameters
58 if (print_level > PRINT_NONE) {
59     printf("A: m = %d, n = %d, nnz = %d\n", A.row, A.col, A.nnz);
60     printf("b: n = %d\n", b.row);
61     fasp_param_solver_print(&itparam);
62 }
63
64 // Step 3. Solve the system with ITS as an iterative solver
65 // Set the initial guess to be zero and then solve it using standard
66 // iterative methods, without applying any preconditioners
67 fasp_dvec_alloc(A.row, &x);
68 fasp_dvec_set(A.row, &x, 0.0);
69
70 fasp_solver_dcsr_itsolver(&A, &b, &x, NULL, &itparam);
71
72 // Step 4. Clean up memory
73 fasp_dcsr_free(&A);
74 fasp_dvec_free(&b);
75 fasp_dvec_free(&x);
76
77 return FASP_SUCCESS;
78 }
79
80 /*-----*/
81 /*--          End of File          --*/
82 /*-----*/

```

This example is very similar to the first example and we briefly explain the differences:

- Line 67–68 allocates memory for the solution vector x and set its initial value to be all zero.

- Line 70 solves $Ax = b$ using the general interface for Krylov subspace methods. Type the iterative method and other parameters have been specified in “itparam”; see §3.5 for details.

To run this example, we can simply type:

```
$ ./poisson-its-c.ex
```

A sample output is as follows:

```
|| FASP: ITS example — C version ||
```

```
fasp_dcsrvec2_read: reading file ../data/csrmat_FE.dat...
```

```
fasp_dcsrvec2_read: reading file ../data/rhs_FE.dat...
```

```
A: m = 3969, n = 3969, nnz = 27281
```

```
b: n = 3969
```

```
Parameters in itsolver_param
```

```
Solver print level:      2
Solver type:             1
Solver precondition type: 2
Solver max num of iter:  500
Solver tolerance:        1.00e-06
Solver stopping type:    1
```

```
Calling PCG solver (CSR) ...
```

It Num	r / b	r	Conv. Factor
0	1.000000e+00	7.514358e+00	---
1	5.078029e-01	3.815813e+00	0.5078
2	3.728856e-01	2.801996e+00	0.7343
3	3.359470e-01	2.524426e+00	0.9009
4	2.590574e-01	1.946650e+00	0.7711
5	2.380797e-01	1.789016e+00	0.9190
6	1.992579e-01	1.497295e+00	0.8369
7	1.847971e-01	1.388631e+00	0.9274
8	1.619777e-01	1.217158e+00	0.8765
9	1.513446e-01	1.137257e+00	0.9344
10	1.364935e-01	1.025661e+00	0.9019
11	1.283425e-01	9.644117e-01	0.9403
12	1.179652e-01	8.864327e-01	0.9191
13	1.115146e-01	8.379605e-01	0.9453
14	1.038726e-01	7.805360e-01	0.9315
15	9.863412e-02	7.411721e-01	0.9496
16	9.277360e-02	6.971341e-01	0.9406
17	8.842679e-02	6.644706e-01	0.9531
18	8.378399e-02	6.295829e-01	0.9475

19		8.011023e-02		6.019770e-01		0.9562
20		7.633221e-02		5.735875e-01		0.9528
21		7.317756e-02		5.498824e-01		0.9587
22		7.003292e-02		5.262524e-01		0.9570
23		6.728610e-02		5.056119e-01		0.9608
24		6.461736e-02		4.855580e-01		0.9603
25		6.219614e-02		4.673640e-01		0.9625
26		5.989276e-02		4.500557e-01		0.9630
27		5.773520e-02		4.338429e-01		0.9640
28		5.571758e-02		4.186818e-01		0.9651
29		5.377630e-02		4.040944e-01		0.9652
30		5.198586e-02		3.906404e-01		0.9667
31		5.022413e-02		3.774021e-01		0.9661
32		4.861699e-02		3.653255e-01		0.9680
33		4.700598e-02		3.532197e-01		0.9669
34		4.554874e-02		3.422696e-01		0.9690
35		4.406559e-02		3.311246e-01		0.9674
36		4.273253e-02		3.211075e-01		0.9697
37		4.135901e-02		3.107864e-01		0.9679
38		4.013076e-02		3.015569e-01		0.9703
39		3.885861e-02		2.919975e-01		0.9683
40		3.776252e-02		2.837611e-01		0.9718
41		3.678565e-02		2.764205e-01		0.9741
42		3.648645e-02		2.741722e-01		0.9919
43		3.725368e-02		2.799375e-01		1.0210
44		3.922957e-02		2.947850e-01		1.0530
45		4.003513e-02		3.008383e-01		1.0205
46		3.683219e-02		2.767703e-01		0.9200
47		3.161285e-02		2.375503e-01		0.8583
48		2.944107e-02		2.212307e-01		0.9313
49		2.961834e-02		2.225628e-01		1.0060
50		2.774118e-02		2.084571e-01		0.9366

2.3 Example 3: Conjugate gradient with preconditioning

This example is a bit more involved and is a modification of the previous one. In this example, we wish to demonstrate how to use the FASP library and run a preconditioned conjugate gradient (PCG) method.

```

1  /*! \file poisson-pcg.c
2  *   \brief The third test example for FASP: using PCG to solve
3  *           the discrete Poisson equation from P1 finite element.
4  *           C version.
5  *
6  *   \note   PCG example for FASP: C version
7  *
8  *   Solving the Poisson equation (P1 FEM) with PCG methods
9  */
10
11 #include "fasp.h"

```

```

12 #include "fasp_funcs.h"
13
14 /**
15  * \fn int main (int argc, const char * argv[])
16  *
17  * \brief This is the main function for the third example.
18  *
19  * \author Feiteng Huang
20  * \date 05/17/2012
21  *
22  * Modified by Chensong Zhang on 09/22/2012
23  */
24 int main (int argc, const char * argv[])
25 {
26     input_param      inparam; // parameters from input files
27     itsolver_param    itparam; // parameters for itsolver
28     AMG_param         amgparam; // parameters for AMG
29     ILU_param         iluparam; // parameters for ILU
30
31     printf("\n=====");
32     printf("\n|| FASP: PCG example -- C version ||");
33     printf("\n=====\\n\\n");
34
35     // Step 0. Set parameters: We can use ini/pcg.dat
36     fasp_param_set(argc, argv, &inparam);
37     fasp_param_init(&inparam, &itparam, &amgparam, &iluparam, NULL);
38
39     // Set local parameters
40     const SHORT print_level = itparam.print_level;
41     const SHORT pc_type      = itparam.precond_type;
42     const SHORT stop_type    = itparam.stop_type;
43     const INT   maxit        = itparam.maxit;
44     const REAL  tol          = itparam.tol;
45
46     // Step 1. Get stiffness matrix and right-hand side
47     // Read A and b -- P1 FE discretization for Poisson. The location
48     // of the data files is given in "ini/pcg.dat".
49     dCSRmat A;
50     dvector b, x;
51     char filename1[512], *datafile1;
52     char filename2[512], *datafile2;
53
54     // Read the stiffness matrix from matFE.dat
55     strncpy(filename1, inparam.workdir, 128);
56     datafile1="csrmat_FE.dat"; strcat(filename1, datafile1);
57
58     // Read the RHS from rhsFE.dat
59     strncpy(filename2, inparam.workdir, 128);
60     datafile2="rhs_FE.dat"; strcat(filename2, datafile2);
61
62     fasp_dcsrvec2_read(filename1, filename2, &A, &b);
63
64     // Step 2. Print problem size and PCG parameters

```

```

65     if (print_level>PRINT_NONE) {
66         printf("A: m = %d, n = %d, nnz = %d\n", A.row, A.col, A.nnz);
67         printf("b: n = %d\n", b.row);
68         fasp_param_solver_print(&itparam);
69     }
70
71     // Setp 3. Setup preconditioner
72     // Preconditioner type is determined by pc_type
73     precondition *pc = fasp_precond_setup(pc_type, &amgparam, &iluparam, &A);
74
75     // Step 4. Solve the system with PCG as an iterative solver
76     // Set the initial guess to be zero and then solve it using PCG solver
77     // Note that we call PCG interface directly. There is another way which
78     // calls the abstract iterative method interface; see poisson-its.c for
79     // more details.
80     fasp_dvec_alloc(A.row, &x);
81     fasp_dvec_set(A.row, &x, 0.0);
82
83     fasp_solver_dcsr_pcg(&A, &b, &x, pc, tol, maxit, stop_type, print_level);
84
85     // Step 5. Clean up memory
86     if (pc_type!=PREC_NULL) fasp_mem_free(pc->data);
87     fasp_dcsr_free(&A);
88     fasp_dvec_free(&b);
89     fasp_dvec_free(&x);
90
91     return FASP_SUCCESS;
92 }
93
94 /*-----*/
95 /*--          End of File          --*/
96 /*-----*/

```

This example is very similar to the first example, and the details are as follows.

- Line 36–37 sets default parameters. In this example, we need parameters for iterative methods, AMG preconditioner, and ILU preconditioner.
- Line 73 sets up the desired preconditioner and prepare it for the preconditioned iterative methods.
- Line 83 calls PCG to solve $Ax = b$. One can also call the general iterative method interface as in the previous example.
- Line 86 cleans up auxiliary data associated with the preconditioner in use if necessary.

To run this example, we can simply type:

```
$ ./poisson-pcg-c.ex
```

A sample output is given as follows (note that the actual output depends on the solver parameters and might be different than what you see here):

```

|| FASP: PCG example — C version ||

fasp_dcsrvec2_read: reading file ../data/csrmat_FE.dat...
fasp_dcsrvec2_read: reading file ../data/rhs_FE.dat...
A: m = 3969, n = 3969, nnz = 27281
b: n = 3969

Parameters in itsolver_param
-----
Solver print level:          2
Solver type:                 1
Solver precondition type:    2
Solver max num of iter:     500
Solver tolerance:            1.00e-06
Solver stopping type:        1
-----

Level  Num of rows  Num of nonzeros  Avg. NNZ / row
-----
0       3969        27281          6.87
1       1985        28523          14.37
2        541        7951          14.70
3        141        1803          12.79
-----

Grid complexity = 1.672 | Operator complexity = 2.403

Classical AMG setup costs 0.0042 seconds.

It Num | ||r||/||b|| | ||r|| | Conv. Factor
-----
0 | 1.000000e+00 | 7.514358e+00 | ---
1 | 1.156153e-02 | 8.687750e-02 | 0.0116
2 | 3.127181e-04 | 2.349876e-03 | 0.0270
3 | 4.813471e-06 | 3.617014e-05 | 0.0154
4 | 5.312526e-08 | 3.992022e-07 | 0.0110
Number of iterations = 4 with relative residual 5.312526e-08.

```

We also provide a Fortran 90 example, which does the same thing as this C code except it gives less output, in “[tutorial/main/poisson-pcg.f90](#)”. Users who would like to call FASP solver from a Fortran based application can see how to do this example.

```

1  !> \file poisson-pcg.f90
2  !> \brief The third test example for FASP: using PCG to solve
3  !>       the discrete Poisson equation from P1 finite element.
4  !>       F90 version.

```



```

5  !>
6  !> \note   PCG example for FASP: F90 version
7  !>
8  !> Solving the Poisson equation (P1 FEM) with PCG method.
9  !>
10 !> \author Chensong Zhang
11 !> \date   12/21/2011
12
13 program test
14
15     implicit none
16
17     double precision, dimension(:), allocatable :: u,b
18     double precision, dimension(:), allocatable :: a
19     integer,          dimension(:), allocatable :: ia,ja
20
21     integer          :: iufile, n, nnz, i, prt_lvl, maxit
22     double precision :: tol
23
24     print*, ""
25     write(*,"(A)") "=====
26     write(*,"(A)") "||   FASP: PCG example -- F90 version   ||"
27     write(*,"(A)") "=====
28     print*, ""
29
30     ! Step 0: user defined variables
31     prt_lvl = 3
32     maxit = 500
33     tol = 1.0d-6
34     iufile = 1
35
36     ! Step 1: read A and b
37
38     !==> Read data A from file
39     open(unit=iufile,file='../data/csrmat_FE.dat')
40
41     read(iufile,*) n
42     allocate(ia(1:n+1))
43     read(iufile,*) (ia(i),i=1,n+1)
44
45     nnz=ia(n+1)-ia(1)
46     allocate(ja(1:nnz),a(1:nnz))
47     read(iufile,*) (ja(i),i=1,nnz)
48     read(iufile,*) (a(i),i=1,nnz)
49
50     close(iufile)
51
52     !==> Read data b from file
53     open(unit=iufile,file='../data/rhs_FE.dat')
54
55     read(iufile,*) n
56     allocate(b(1:n))
57     read(iufile,*) (b(i),i=1,n)

```

```

58
59  close(iufile)
60
61  !==> Shift the index to start from 0 (for C routines)
62  forall (i=1:n+1) ia(i)=ia(i)-1
63  forall (i=1:nnz) ja(i)=ja(i)-1
64
65  ! Step 2: Solve the system
66
67  !==> Initial guess
68  allocate(u(1:n))
69  u=0.0d0
70  call fasp_fwapper_krylov_amg(n,nnz,ia,ja,a,b,u,tol,maxit,prt_lvl);
71
72  ! Step 3: Clean up memory
73  deallocate(ia,ja,a)
74  deallocate(b,u)
75
76  end program test
77
78  /*-----*/
79  /*--          End of File          --*/
80  /*-----*/

```

2.4 Example 4: An GMG solver for the Poisson equation

The geometric multigrid method (GMG) is one of the most efficient solving techniques for discrete algebraic systems arising from many types of partial differential equations [1, 16]. GMG utilizes a hierarchy of grids or discretizations and reduces the error at a number of frequencies simultaneously. Because of its plausible linear complexity—i.e., the low computational cost of solving a linear system with N unknowns is $O(N)$ —the GMG method is one of the most popular Poisson solvers. Although the GMG’s applicability is limited as it requires explicit information on the hierarchy of the discrete system, when it can be applied, GMG is far more efficient than its algebraic version, the algebraic multigrid (AMG) method.

We now give a simple example on calling the geometric multigrid for solving the Poisson’s equation in 2D (discretized by the standard five-point finite difference stencil). Consider the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (0,1)^2 \subset \mathbb{R}^2$. The main reason why we choose this simplest possible setting is to emphasize that, even for a simple problem, the new heterogeneous architectures present challenges for numerical implementation. Another reason is to allow us to use explicit stencils and to avoid the bottleneck of sparse matrix-vector production. The standard central finite difference method is applied to discretize the Poisson’s equation. In other words, the Laplace operator is discretized by

the classical second-order central difference scheme. After discretization, we end up with a system of linear equations:

$$\mathbf{A}\vec{u} = \vec{f}.$$

We use the five-point central difference scheme in 2D. Consider a uniform square mesh of $\Omega = [0, 1]^2$ with size $h = \frac{1}{n}$ and in which $x_i = ih$, $y_j = jh$ ($i, j = 0, 1, \dots, n$). Let $u_{i,j}$ be the numerical approximation of $u(x_i, y_j)$. The five-point central difference scheme for solving the Poisson's equation in 2D can be written as follows:

$$-u_{i-1,j} - u_{i,j-1} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1} = h^2 f(x_i, y_j) \quad i, j = 1, 2, \dots, n-1.$$

The sample code for this solver can be found in “[tutorial/main/poisson-gmg.c](#)” and a piece of the source code is listed as follows:

```

1  /*! \file poisson-gmg.c
2  *
3  * \brief The fourth test example for FASP: using GMG to solve
4  *       the discrete Poisson equation from five-point finite
5  *       difference stencil. C version.
6  *
7  * \note  GMG example for FASP: C version
8  *
9  * Solving the Poisson equation (FDM) with GMG
10 /*
11
12 #include <time.h>
13 #include <math.h>
14
15 #include "fasp.h"
16 #include "fasp_functs.h"
17
18 const REAL pi = 3.14159265;
19
20 /**
21  * \fn static REAL f2d(INT i, INT j, INT nx, INT ny)
22  *
23  * \brief Setting f in Poisson equation, where
24  *       f = sin(pi x)*sin(pi y)
25  *
26  * \param i      i-th position in x direction
27  * \param j      j-th position in y direction
28  * \param nx     Number of grids in x direction
29  * \param ny     Number of grids in y direction
30  *
31  * \author Ziteng Wang
32  * \date 06/07/2013
33  */
34 static REAL f2d (INT i,
35                  INT j,
36                  INT nx,

```

```

37         INT ny)
38     {
39         return sin(pi *(((REAL) j)/((REAL) ny)))
40             *sin(pi *(((REAL) i)/((REAL) nx)));
41     }
42
43 /**
44  * \fn static REAL L2NormError2d(REAL *u, INT nx, INT ny)
45  *
46  * \brief Computing Discretization Error, where exact solution
47  *       u = sin(pi x)*sin(pi y)/(2*pi*pi)
48  *
49  * \param u      Vector of DOFs
50  * \param nx      Number of grids in x direction
51  * \param ny      Number of grids in y direction
52  *
53  * \author Ziteng Wang
54  * \date 06/07/2013
55  */
56 static REAL L2NormError2d (REAL *u,
57                             INT nx,
58                             INT ny)
59 {
60     const REAL h = 1.0/nx;
61     REAL l2norm = 0.0, uexact;
62
63     INT i, j;
64     for ( i = 1; i < ny; i++ ) {
65         for ( j = 1; j < nx; j++ ) {
66             uexact = sin(pi*i*h)*sin(pi*j*h)/(pi*pi*2.0);
67             l2norm += pow((u[i*(nx+1)+j] - uexact), 2);
68         }
69     }
70
71     return sqrt(l2norm*h*h);
72 }
73
74 /**
75  * \brief An example of GMG method using Full Multigrid cycle
76  *
77  * \author Chensong Zhang
78  * \date 10/12/2015
79  *
80  * \note Number of grids of nx = ny should be equal to 2^maxlevel.
81  */
82 int main (int argc, const char *argv[])
83 {
84     const REAL rtol = 1.0e-6;
85     const INT prtlvl = PRINT_MORE;
86
87     INT i, j, k, nx, maxlevel;
88     REAL *u, *b, h, error0;
89

```

```

90 // Step 0. Set number of levels for GMG
91 printf("Enter the desired number of levels:  ");
92 scanf("%d", &maxlevel);
93
94 // Step 1. Compute right-hand side b and set approximate solution u
95 nx = (int) pow(2.0, maxlevel);
96 h = 1.0/((REAL) nx);
97
98 u = (REAL *)malloc((nx+1)*(nx+1)*sizeof(REAL));
99 fasp_array_set((nx+1)*(nx+1), u, 0.0);
100
101 b = (REAL *)malloc((nx+1)*(nx+1)*sizeof(REAL));
102 for (i = 0; i <= nx; i++) {
103     for (j = 0; j <= nx; j++) {
104         b[j*(nx+1)+i] = h*h*f2d(i, j, nx, nx);
105     }
106 }
107
108 // Step 2. Solve the Poisson system in 2D with full Multigrid cycle
109 fasp_poisson_fgmg_2D(u, b, nx, nx, maxlevel, rtol, prtlvl);
110
111 // Step 3. Compute error in L2 norm
112 error0 = L2NormError2d(u, nx, nx);
113
114 printf("L2 error ||u-u'|| = %e\n", error0);
115
116 // Step 4. Clean up memory
117 free(u);
118 free(b);
119
120 return FASP_SUCCESS;
121 }
122
123 /*-----*/
124 /*--          End of File          --*/
125 /*-----*/

```

2.5 How to change parameters for solvers/preconditioners

In the previous examples, we have seen how to use the default parameters in FASP. In this section we discuss changing such parameters by reading them from a disk file or from the command line. An example of parameter initialization file is found in the FASP tutorial directory and is named “**tutorial/ini/amg.dat**”.

```
$ ./poisson-amg-c.ex -ini ini/amg.dat
```

We take “tutorial/ini/amg.dat” as an example:

```

1  %-----%
2  % input parameters                                     %
3  % lines starting with % are comments                  %
4  % must have spaces around the equal sign "="          %
5  %-----%
6
7  workdir = ../data/    % work directory, no more than 128 characters
8  print_level = 3       % How much information to print out
9
10 %-----%
11 % parameters for multilevel iteration                  %
12 %-----%
13
14 AMG_type              = C      % C classic AMG
15                               % SA smoothed aggregation
16                               % UA unsmoothed aggregation
17 AMG_cycle_type        = V      % V V-cycle | W W-cycle
18                               % A AMLI-cycle | NA Nonlinear AMLI-cycleA
19 AMG_tol               = 1e-8   % tolerance for AMG
20 AMG_maxit             = 100    % number of AMG iterations
21 AMG_levels            = 20     % max number of levels
22 AMG_coarse_dof        = 500    % max number of coarse degrees of freedom
23 AMG_coarse_scaling    = OFF    % switch of scaling of the coarse grid correction
24 AMG_amli_degree       = 2      % degree of the polynomial used by AMLI cycle
25 AMG_nl_amli_krylov_type = 6    % Krylov method in NLAMLI cycle: 6 FGMRES | 7 GCG
26
27 %-----%
28 % parameters for AMG smoothing                         %
29 %-----%
30
31 AMG_smoother          = GS     % GS | JACOBI | SGS
32                               % SOR | SSOR | GSOR | SGSOR | POLY
33 AMG_ILU_levels        = 0      % number of levels using ILU smoother
34 AMG_Schwarz_levels    = 0      % number of levels using Schwarz smoother
35 AMG_relaxation        = 1.1    % relaxation parameter for SOR smoother
36 AMG_polynomial_degree = 3      % degree of the polynomial smoother
37 AMG_presmooth_iter    = 2      % number of presmoothing sweeps
38 AMG_postsmooth_iter   = 2      % number of postsmoothing sweeps
39
40 %-----%
41 % parameters for classical AMG SETUP                  %
42 %-----%
43
44 AMG_coarsening_type   = 1      % 1 Modified RS
45                               % 3 Compatible Relaxation
46                               % 4 Aggressive
47 AMG_interpolation_type = 1      % 1 Direct | 2 Standard | 3 Energy-min
48 AMG_strong_threshold  = 0.6    % Strong threshold
49 AMG_truncation_threshold = 0.4  % Truncation threshold
50 AMG_max_row_sum       = 0.9    % Max row sum
51
52 %-----%
53 % parameters for aggregation-type AMG SETUP           %

```



```

AMG max levels:          20
AMG cycle type:          1
AMG coarse solver type:  0
AMG scaling of coarse correction: 0
AMG smoother type:       2
AMG smoother order:      1
AMG num of presmoothing:  2
AMG num of postsmoothing: 2
Aggregation type:        1
Aggregation number of pairs: 2
Aggregation quality bound: 8.00

```

Calling SA AMG ...

Level	Num of rows	Num of nonzeros	Avg. NNZ / row
0	3969	27281	6.87
1	541	6531	12.07
2	41	421	10.27

Grid complexity = 1.147 | Operator complexity = 1.255

Smoothed aggregation setup costs 0.0028 seconds.

It	Num	r / b	r	Conv. Factor
0	1.000000e+00	7.514358e+00	---	
1	4.345463e-02	3.265336e-01	0.0435	
2	8.041967e-03	6.043022e-02	0.1851	
3	3.808810e-03	2.862076e-02	0.4736	
4	1.838990e-03	1.381883e-02	0.4828	
5	8.675952e-04	6.519421e-03	0.4718	
6	4.089274e-04	3.072827e-03	0.4713	
7	1.939823e-04	1.457653e-03	0.4744	
8	9.276723e-05	6.970862e-04	0.4782	
9	4.471799e-05	3.360270e-04	0.4820	
10	2.171249e-05	1.631554e-04	0.4855	
11	1.060934e-05	7.972239e-05	0.4886	
12	5.212246e-06	3.916668e-05	0.4913	
13	2.572464e-06	1.933042e-05	0.4935	
14	1.274466e-06	9.576797e-06	0.4954	
15	6.333891e-07	4.759512e-06	0.4970	
16	3.155926e-07	2.371476e-06	0.4983	
17	1.575755e-07	1.184079e-06	0.4993	
18	7.881043e-08	5.922098e-07	0.5001	
19	3.947044e-08	2.965950e-07	0.5008	
20	1.978978e-08	1.487075e-07	0.5014	
21	9.931176e-09	7.462641e-08	0.5018	

Number of iterations = 21 with relative residual 9.931176e-09.

AMG solve costs 0.0053 seconds.

AMG totally costs 0.0083 seconds.

You can compare this with the sample results in §2.1.

The input parameters allowed in FASP are not limited to the ones listed in this example. A list of possible iterative methods and preconditioners can be found in “base/include/fasp_const.h”; see §4.2. For more parameters and their ranges, we refer to the FASP Reference Manual.

Using “-ini [FILE]” is just one example of allowed command line option. To find out more what command line options are acceptable, you can type in a terminal window:

```
$ ./poisson-amg-c.ex -help
```

which will give you something like

```

=====
||   FASP: AMG example — C version   ||
=====

FASP command line options:
=====
-ini           [CharValue] : Ini file name
-print         [IntValue]  : Print level
-output        [IntValue]  : Output to screen or a log file
-solver        [IntValue]  : Solver type
-precond       [IntValue]  : Preconditioner type
-maxit         [IntValue]  : Max number of iterations
-tol           [RealValue] : Tolerance for iterative solvers
-amgmaxit      [IntValue]  : Max number of AMG iterations
-amgtol        [RealValue] : Tolerance for AMG methods
-amgtype       [IntValue]  : AMG type
-amgcycle      [IntValue]  : AMG cycle type
-amgcoarsening [IntValue]  : AMG coarsening type
-amginterpolation [IntValue] : AMG interpolation type
-amgsmoother   [IntValue]  : AMG smoother type
-amgsthreshold [RealValue] : AMG strong threshold
-amgscoupled   [RealValue] : AMG strong coupled threshold
-help          : Brief help messages

```

For example, in order to change the AMG type to the smoothed aggregation (SA) used by the preconditioner for PCG, you can also use the command line options:

```
./poisson-amg-c.ex -amgtype 2 -amgmaxit 100
```

Here we only changed two parameters from the default setting without changing anything else. So it might not give the same output as in the previous example.

Chapter 3

Data structures and basic usage

In this chapter, we discuss the basic data structures and the important building blocks which are useful for constructing auxiliary space preconditioners for systems of PDEs in Chapter 4. In particular, we will discuss vectors, sparse matrices, iterative methods, and multigrid methods.

3.1 Vectors and sparse matrices

The data structures most often used for implementing iterative methods are sparse matrices and vectors. In this section, we first discuss the data structures for vectors and matrices in FASP; and then we discuss BLAS operations for sparse matrices. The definitions can be found in “[base/include/fasp.h](#)”.

Vectors

The data structure for vectors is very simple. It only contains the length of the vector and an array which contains the entries of this vector.

```
330
331 /**
332  * \struct dvector
333  * \brief Vector with n entries of REAL type
334  */
335 typedef struct dvector{
336
337     /*! number of rows
338     INT row;
339
340     /*! actual vector entries
341     REAL *val;
```

Sparse matrices

On the other hand, sparse matrices for PDE applications are very complicated. It depends on the particular applications, discretization methods, as well as solution algorithms. In FASP, there are several types of sparse matrices, COO, CSR, CSRL, BSR, and CSR Block, etc. The presentation closely follows ideas from Pissanetzky [12].

In this section, we use the following sparse matrix as an example to explain different formats for sparse matrices:

Example 3.1.1 Consider the following 4×5 matrix with 12 non-zero entries

$$\begin{pmatrix} 1 & 1.5 & 0 & 0 & 12 \\ 0 & 1 & 6 & 7 & 1 \\ 3 & 0 & 6 & 0 & 0 \\ 1 & 0 & 2 & 0 & 5 \end{pmatrix}$$

(i) COO format

The coordinate (COO) format or IJ format is the simplest sparse matrix format.

```

192  /**
193  * \struct dCOOmat
194  * \brief Sparse matrix of REAL type in COO (or IJ) format
195  *
196  *
197  * Coordinate Format (I,J,A)
198  *
199  * \note The starting index of A is 0.
200  * \note Change I to rowind, J to colind. To avoid with complex.h confliction on I.
201  */
202  typedef struct dCOOmat{
203
204      //! row number of matrix A, m
205      INT row;
206
207      //! column of matrix A, n
208      INT col;
209
210      //! number of nonzero entries
211      INT nnz;
212
213      //! integer array of row indices, the size is nnz
214      INT *rowind;
215
216      //! integer array of column indices, the size is nnz
217      INT *colind;
218
219      //! nonzero entries of A
220      REAL *val;

```

So it clear that the sparse matrix in Example 3.1.1 in COO format is stored as:

```
row = 4
col = 5
nnz = 12

I J  val
-----
0 0  1.0
0 1  1.5
0 4 12.0
1 1  1.0
1 2  6.0
1 3  7.0
1 4  1.0
.....
```

Although the COO format is easy to understand or use, it wastes storage space and has little advantages in sparse BLAS operations.

NOTE: In FASP, the indices always start from 0, instead of from 1. This is often the source of problems related to vectors and matrices.

(ii) CSR format

The most commonly used data structure for sparse matrices nowadays is probably the so-called *compressed sparse row* (CSR) format, according to Saad [15]. The compressed row storage format of a matrix $A \in \mathbb{R}^{n \times m}$ (n rows and m columns) consists of three arrays, as follows:

1. An integer array of row pointers of size $n+1$;
2. An integer array of column indexes of size nnz ;
3. An array of actual matrix entries.

In FASP, we define:

```
132
133 /**
134  * \struct dCSRmat
135  * \brief Sparse matrix of REAL type in CSR format
136  *
137  * CSR Format (IA,JA,A) in REAL
138  *
139  * \note The starting index of A is 0.
140  */
141 typedef struct dCSRmat{
142
```

```

143  //! row number of matrix A, m
144  INT row;
145
146  //! column of matrix A, n
147  INT col;
148
149  //! number of nonzero entries
150  INT nnz;
151
152  //! integer array of row pointers, the size is m+1
153  INT *IA;
154
155  //! integer array of column indexes, the size is nnz
156  INT *JA;
157
158  //! nonzero entries of A
159  REAL *val;

```

The matrix (only nonzero elements) is stored in the array *val* row after row, in a way that *i*-th row begins at $val(IA(i))$ and ends at $val(IA(i + 1) - 1)$. In the same way, $JA(IA(i)$ to $JA(IA(i + 1) - 1)$ will contain the column indexes of the non-zeros in row *i*. Thus *IA* is of size $n + 1$ (number of rows in *val* plus one), *JA* and *val* are of size equal to the number of non-zeroes. The total number of non-zeroes is equal to $IA(n + 1) - 1$.

NOTE: When the sparse matrix *A* is a boolean (i.e. all entries are either 0 or 1), the actual non-zeroes are not stored because it is understood that, if it is nonzero, it could only be 1 and there is no need to store it.

The matrix in Example 3.1.1 in CSR format is represented in the following way:

- *IA* is of size 5 and

$$IA = \parallel 0 \parallel 3 \parallel 7 \parallel 9 \parallel 12 \parallel$$

- *JA* is of size $IA(5) - 1 = 12$

$$JA = \parallel 0 \mid 1 \mid 4 \parallel 1 \mid 3 \mid 2 \mid 4 \parallel 0 \mid 2 \parallel 2 \mid 4 \mid 0 \parallel$$

- *val* is of the same size as *JA* and

$$val = \parallel 1. \mid 1.5 \mid 12. \parallel 1. \mid 7. \mid 6. \mid 1. \parallel 3. \mid 6. \parallel 2. \mid 5. \mid 1. \parallel$$

Here we use double vertical bars to separate rows and single vertical bars to separate values.

NOTE: The indices in JA and entries of val does NOT have to be ordered as seen in this example. Sometimes they are sorted in ascending order in each row. More often, the diagonal entries are stored in the first position in each row and the rest are sorted in ascending order.

Below is a “non-numeric” example.

Example 3.1.2 Consider the following sparse matrix:

$$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}$$

For this matrix, we have that the number of non-zeros $nnz = 10$. Furthermore, the three arrays of in the CSR format are:

$$IA = \parallel 0 \parallel 2 \parallel 5 \parallel 7 \parallel,$$

$$JA = \parallel 0 \mid 2 \parallel 1 \mid 2 \mid 3 \parallel 1 \mid 3 \parallel 0 \mid 1 \mid 2 \parallel,$$

and

$$val = \parallel a_{11} \mid a_{13} \parallel a_{22} \mid a_{23} \mid a_{24} \parallel a_{32} \mid a_{34} \parallel a_{41} \mid a_{42} \mid a_{43} \parallel.$$

NOTE: The CSR format presents challenges to sparse matrix-vector product mainly because of the high cache missing rate due to indirect memory access and irregular access pattern. In order to reduce the cache missing rate, we introduce an improved data format, CSRL.

(iii) CSRL format

CSRL matrix format [10] groups rows with same number of nonzeros together and improves cache hitting rate.

```

253
254 /*!
255  * \struct dCSRLmat
256  * \brief Sparse matrix of REAL type in CSRL format
257  */
258 typedef struct dCSRLmat{
259
260     //! number of rows
261     INT row;
262
263     //! number of cols
264     INT col;
265
266     //! number of nonzero entries

```

```

267     INT nnz;
268
269     //! number of different values in i-th row, i=0:nrows-1
270     INT dif;
271
272     //! nz_diff[i]: the i-th different value in 'nzrow'
273     INT *nz_diff;
274
275     //! row index of the matrix (length-grouped): rows with same nnz are together
276     INT *index;
277
278     //! j in {start[i],...,start[i+1]-1} means nz_diff[i] nnz in index[j]-row
279     INT *start;
280
281     //! column indices of all the nonzeros
282     INT *ja;
283
284     //! values of all the nonzero entries
285     REAL *val;

```

3.2 Block sparse matrices

For PDE applications, we often need to solve systems of partial differential equations. Many iterative methods and preconditioners could take advantages of the structure of PDE systems and improve efficiency. So we often need to use semi-structured (block) sparse data structures to store the coefficient matrix arising from PDE systems.

Depending on different applications and different solving algorithms, we can use two types of block matrices: dBSRmat (or BSR Block Compressed Sparse Row) and block_dCSRmat (CSR Block or Block of CSR matrices).

For more details as well as other specialized block matrices, readers are referred to the header file “`base/include/fasp_block.h`”.

As an example, we consider the following matrix, which have been used in §3.1 for the CSR format. We add structure to this matrix and divide it as a 2×2 block matrix:

Example 3.2.1

$$\left(\begin{array}{cc|cc} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ \hline 0 & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{array} \right)$$

(i) BSR format

This format is a standard data structure for storing block sparse matrices which has been used by the Intel MKL library.

```

33
34 /**
35  * \struct dBSRmat
36  * \brief Block sparse row storage matrix of REAL type
37  *
38  * \note This data structure is adapted from the Intel MKL library. Refer to:
39  * http://software.intel.com/sites/products/documentation/hpc/mkl/lin/index.htm
40  *
41  * \note Some of the following entries are capitalized to stress that they are
42  *       for blocks!
43  */
44 typedef struct dBSRmat {
45
46     //!< number of rows of sub-blocks in matrix A, M
47     INT ROW;
48
49     //!< number of cols of sub-blocks in matrix A, N
50     INT COL;
51
52     //!< number of nonzero sub-blocks in matrix A, NNZ
53     INT NNZ;
54
55     //!< dimension of each sub-block
56     INT nb; // NOTE: for the moment, allow nb*nb full block
57
58     //!< storage manner for each sub-block
59     INT storage_manner; // 0: row-major order, 1: column-major order
60
61     //!< A real array that contains the elements of the non-zero blocks of
62     //!< a sparse matrix. The elements are stored block-by-block in row major
63     //!< order. A non-zero block is the block that contains at least one non-zero
64     //!< element. All elements of non-zero blocks are stored, even if some of
65     //!< them is equal to zero. Within each nonzero block elements are stored
66     //!< in row-major order and the size is (NNZ*nb*nb).
67     REAL *val;
68
69     //!< integer array of row pointers, the size is ROW+1
70     INT *IA;
71
72     //!< Element i of the integer array columns is the number of the column in the
73     //!< block matrix that contains the i-th non-zero block. The size is NNZ.
74     INT *JA;

```

For the matrix in Example 3.2.1, we have that the number of block rows $ROW = 2$, the number of block columns $COL = 2$, and the number of block nonzeros $NNZ = 4$. The block size is $nb = 2$. We can choose different storage manners for storing the small blocks. Suppose that we set it to be

0, i.e. row-major format. Then the three arrays of in the BSR format are:

$$IA = \begin{bmatrix} 0 & 8 & 16 \end{bmatrix},$$

$$JA = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix},$$

and

$$val = \begin{bmatrix} a_{11} & 0 & 0 & a_{22} & a_{13} & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{41} & a_{42} & 0 & a_{34} & a_{43} & 0 \end{bmatrix}.$$

We immediately notice that this format might be not be the best choice for this particular matrix due to all the blocks are nonzero blocks, i.e., contain nonzero entries. However, for PDE applications, this does not usually happen.

(ii) CSR Block format

This format is simple and is derived from the dCSRmat data structure. The following definition explains itself.

```

77
78 /**
79  * \struct block_dCSRmat
80  * \brief Block REAL CSR matrix format
81  *
82  * \note The starting index of A is 0.
83  */
84 typedef struct block_dCSRmat {
85
86     //! row number of blocks in A, m
87     INT brow;
88
89     //! column number of blocks A, n
90     INT bcol;
91
92     //! blocks of dCSRmat, point to blocks[brow][bcol]
93     dCSRmat **blocks;

```

3.3 I/O subroutines for sparse matrices

In FASP, we provided several functions for reading, writing, and printing different formats of sparse matrices in plain text or binary formats. These functions can be found in “[base/src/io.c](#)” and we list the available functions as follows:

```

834 /*----- In file: io.c -----*/

```

```

835
836 void fasp_dcsrvec1_read (const char *filename,
837                         dCSRmat *A,
838                         dvector *b);
839
840 void fasp_dcsrvec2_read (const char *filemat,
841                         const char *filerhs,
842                         dCSRmat *A,
843                         dvector *b );
844
845 void fasp_dcsr_read (const char *filename,
846                    dCSRmat *A);
847
848 void fasp_dcoo_read (const char *filename,
849                    dCSRmat *A);
850
851 void fasp_dcoo1_read (const char *filename,
852                     dCOOmat *A);
853
854 void fasp_dcoo_shift_read (const char *filename,
855                          dCSRmat *A);
856
857 void fasp_dmtx_read (const char *filename,
858                    dCSRmat *A);
859
860 void fasp_dmtxsym_read (const char *filename,
861                      dCSRmat *A);
862
863 void fasp_dstr_read (const char *filename,
864                   dSTRmat *A);
865
866 void fasp_dbsr_read (const char *filename,
867                   dBSRmat *A);
868
869 void fasp_dvecind_read (const char *filename,
870                      dvector *b);
871
872 void fasp_dvec_read (const char *filename,
873                   dvector *b);
874
875 void fasp_ivecind_read (const char *filename,
876                      ivector *b);
877
878 void fasp_ivec_read (const char *filename,
879                    ivector *b);
880
881 void fasp_dcsrvec1_write (const char *filename,
882                        dCSRmat *A,
883                        dvector *b);
884
885 void fasp_dcsrvec2_write (const char *filemat,
886                        const char *filerhs,
887                        dCSRmat *A,

```

```

888         dvector *b );
889
890 void fasp_dcoo_write (const char *filename,
891                     dCSRmat *A);
892
893 void fasp_dstr_write (const char *filename,
894                     dSTRmat *A);
895
896 void fasp_dbsr_write (const char *filename,
897                     dBSRmat *A);
898
899 void fasp_dvec_write (const char *filename,
900                     dvector *vec);
901
902 void fasp_dvecind_write (const char *filename,
903                       dvector *vec);
904
905 void fasp_ivec_write (const char *filename,
906                     ivector *vec);
907
908 void fasp_dvec_print (INT n,
909                     dvector *u);
910
911 void fasp_ivec_print (INT n,
912                     ivector *u);
913
914 void fasp_dcsr_print (dCSRmat *A);
915
916 void fasp_dcoo_print (dCOOmat *A);
917
918 void fasp_dbsr_print (dBSRmat *A);
919
920 void fasp_dbsr_write_coo (const char *filename,
921                         const dBSRmat *A);
922
923 void fasp_dcsr_write_coo (const char *filename,
924                         const dCSRmat *A);
925
926 void fasp_dstr_print (dSTRmat *A);
927
928 void fasp_matrix_read (const char *filename,
929                      void *A);
930
931 void fasp_matrix_read_bin (const char *filename,
932                          void *A);
933
934 void fasp_matrix_write (const char *filename,
935                      void *A,
936                      INT flag);
937
938 void fasp_vector_read (const char *filerhs,
939                      void *b);
940

```

```

941 void fasp_vector_write (const char *filerhs,
942                         void *b,
943                         INT flag);
944
945 void fasp_hb_read (const char *input_file,
946                  dCSRmat *A,
947                  dvector *b);

```

NOTE: The above function declarations are taken from “[base/include/fasp_funcs.h](#)”. This header file is automatically generated based on the source codes. Users are discouraged from changing it by hand; their changes may be lost.

3.4 Sparse matrix-vector multiplication

The matrix-vector multiplication: $y = Ax$ can be performed in the following simple way:

```

1  /**
2   * \fn void fasp_blas_dcsr_m xv (dCSRmat *A, REAL *x, REAL *y)
3   *
4   * \brief Matrix-vector multiplication y = A*x
5   *
6   * \param A   Pointer to dCSRmat matrix A
7   * \param x   Pointer to array x
8   * \param y   Pointer to array y
9   *
10  * \author Chensong Zhang
11  * \date 07/01/2009
12  */
13 void fasp_blas_dcsr_m xv (dCSRmat *A,
14                          REAL *x,
15                          REAL *y)
16 {
17     const INT m = A->row;
18     const INT *ia = A->IA, *ja = A->JA;
19     const REAL *aj = A->val;
20
21     INT i, k, beg, end;
22     register REAL tmp;
23
24     for ( i=0; i<m; ++i ) {
25         tmp = 0.0;
26         beg = ia[i]; end = ia[i+1];
27         for ( k=beg; k<end; ++k ) tmp += aj[k]*x[ja[k]];
28         y[i] = tmp;
29     }
30 }

```

This is only a simple example for sparse matrix-vector multiplication (SpMV) kernel. Since we need many types of sparse matrices, there are various of versions of SpMV for different data structures. See the Reference Manual for more details.

3.5 Iterative methods

In FASP, there are a couple of standard preconditioned iterative methods [15] implemented, including preconditioned CG, BiCGstab, GMRES, Variable Restarting GMRES, Flexible GMRES, etc. In this section, we use the CSR matrix format as example to introduce how to call these iterative methods. To learn more details, we refer to the Reference Manual.

We first notice the abstract interface for the iterative methods. The following code segment is taken from “`base/src/itsolver_csr.c`”:

```

20 /**
21  * \fn INT fasp_solver_dcsr_itsolver (dCSRmat *A, dvector *b, dvector *x,
22  *                                  precondition *pc, itsolver_param *itparam)
23  *
24  * \brief Solve Ax=b by preconditioned Krylov methods for CSR matrices
25  *
26  * \param A      Pointer to the coeff matrix in dCSRmat format
27  * \param b      Pointer to the right hand side in dvector format
28  * \param x      Pointer to the approx solution in dvector format
29  * \param pc     Pointer to the preconditioning action
30  * \param itparam Pointer to parameters for iterative solvers
31  *
32  * \return      Iteration number if converges; ERROR otherwise.
33  *
34  * \author Chensong Zhang
35  * \date 09/25/2009
36  *
37  * \note This is an abstract interface for iterative methods.
38  */
39 INT fasp_solver_dcsr_itsolver (dCSRmat *A,
40                               dvector *b,
41                               dvector *x,
42                               precondition *pc,
43                               itsolver_param *itparam)

```

The names of the input arguments explain themselves mostly and they are explained in the Reference Manual in detail.

We briefly discuss how to call this function; and, once you understand PCG, you can easily call other iterative methods.

```

463 // ILU setup for whole matrix
464 ILU_data LU;
465 if ( (status = fasp_ilu_dcsr_setup(A,&LU,iluparam)) < 0 ) goto FINISHED;

```

```

466
467 // check iludata
468 if ( (status = fasp_mem_iludata_check(&LU)) < 0 ) goto FINISHED;
469
470 // set preconditioner
471 precondition pc;
472 pc.data = &LU;
473 pc.fct = fasp_precond_ilu;
474
475 // call iterative solver
476 status = fasp_solver_dcsr_itsolver(A,b,x,&pc,itparam);

```

Now we explain this code segment a little bit:

- Line 463–465 performs the setup phase for ILU method. The particular type of ILU method is determined by “iluparam”; see §2.5. Line 7 performs a simple memory check for ILU.
- Line 471–473 defines the preconditioner data structure “pc”, which contains two parts: one is the actual preconditioning action “pc.fct”, the other is the auxiliary data which is needed to perform the preconditioning action “pc.data”.
- Line 476 calls iterative methods. “A” is the matrix in dCSRmat format; “b” and “x” are the right-hand side and the solution vectors, respectively. Similar to ILU setup, the type of iterative methods is determined by “itparam”.

Apparently, we should now explain the data structure “itparam”.

```

/**
 * \struct itsolver_param
 * \brief Parameters passed to iterative solvers
 */
typedef struct {

    SHORT itsolver_type; /**< solver type: see message.h */
    SHORT precondition_type; /**< preconditioner type: see message.h */
    SHORT stop_type; /**< stopping criteria type */
    INT maxit; /**< max number of iterations */
    REAL tol; /**< convergence tolerance */
    INT restart; /**< number of steps for restarting: for GMRES etc */
    SHORT print_level; /**< print level: 0--10 */

} itsolver_param; /**< Parameters for iterative solvers */

```

Possible “itsolver_type” includes:

```

/**
 * \brief Definition of solver types for iterative methods
 */
#define SOLVER_DEFAULT 0 /**< Use default solver in FASP */

```

```

//-----
#define SOLVER_CG          1  /**< Conjugate Gradient */
#define SOLVER_BiCGstab    2  /**< Bi-Conjugate Gradient Stabilized */
#define SOLVER_MinRes      3  /**< Minimal Residual */
#define SOLVER_GMRES       4  /**< Generalized Minimal Residual */
#define SOLVER_VGMRES      5  /**< Variable Restarting GMRES */
#define SOLVER_VFGMRES     6  /**< Variable Restarting Flexible GMRES */
#define SOLVER_GCG         7  /**< Generalized Conjugate Gradient */
#define SOLVER_GCR         8  /**< Generalized Conjugate Residual */
//-----
#define SOLVER_SCG         11 /**< Conjugate Gradient with safety net */
#define SOLVER_SBiCGstab   12 /**< BiCGstab with safety net */
#define SOLVER_SMinRes     13 /**< MinRes with safety net */
#define SOLVER_SGMRES      14 /**< GMRES with safety net */
#define SOLVER_SVGMRES     15 /**< Variable-restart GMRES with safety net */
#define SOLVER_SVFGMRES    16 /**< Variable-restart FGMRES with safety net */
#define SOLVER_SGCG        17 /**< GCG with safety net */
//-----
#define SOLVER_AMG         21 /**< AMG as an iterative solver */
#define SOLVER_FMG         22 /**< Full AMG as an solver */

```

3.6 Algebraic multigrid

The classical algebraic multigrid method [14] is an important component in many of our auxiliary space preconditioners. Because of its user-friendly and scalability, AMG becomes increasingly popular in scientific and engineering computing, especially when GMG is difficult or not possible to be applied. Various of new AMG techniques [17, 18, 4, 7, 5, 9, 6, 20, 3, 11, 8] have emerged in recent years.

The following code segment is part of “`base/src/amg.c`” and it is a good example which shows how to call different AMG methods (classical AMG, smoothed aggregation, un-smoothed aggregation) and different multilevel iterative methods (V-cycle, W-cycle, AMLI-cycle, Nonlinear AMLI-cycle, etc).

```

42     const SHORT    max_levels = param->max_levels;
43     const SHORT    prtlvl    = param->print_level;
44     const SHORT    amg_type   = param->AMG_type;
45     const SHORT    cycle_type = param->cycle_type;
46     const INT      nnz = A->nnz, m = A->row, n = A->col;
47
48     // local variables
49     SHORT          status;
50     AMG_data *      mgl = fasp_amg_data_create(max_levels);
51     REAL           AMG_start, AMG_end;
52
53     #if DEBUG_MODE > 0
54         printf("### DEBUG: %s ..... [Start]\n", __FUNCTION__);
55     #endif

```

```

56
57     if ( prtlvl > PRINT_NONE ) fasp_gettime(&AMG_start);
58
59     // check matrix data
60     if ( m != n ) {
61         printf("### ERROR: A is not a square matrix!\n");
62         fasp_chkerr(ERROR_MAT_SIZE, __FUNCTION__);
63     }
64
65     if ( nnz <= 0 ) {
66         printf("### ERROR: A has no nonzero entries!\n");
67         fasp_chkerr(ERROR_MAT_SIZE, __FUNCTION__);
68     }
69
70     // Step 0: initialize mgl[0] with A, b and x
71     mgl[0].A = fasp_dcsr_create(m, n, nnz);
72     fasp_dcsr_cp(A, &mgl[0].A);
73
74     mgl[0].b = fasp_dvec_create(n);
75     fasp_dvec_cp(b, &mgl[0].b);
76
77     mgl[0].x = fasp_dvec_create(n);
78     fasp_dvec_cp(x, &mgl[0].x);
79
80     // Step 1: AMG setup phase
81     switch (amg_type) {
82
83         case SA_AMG: // Smoothed Aggregation AMG setup
84             if ( prtlvl > PRINT_NONE ) printf("\nCalling SA AMG ...\n");
85             status = fasp_amg_setup_sa(mgl, param); break;
86
87         case UA_AMG: // Unsmoothed Aggregation AMG setup
88             if ( prtlvl > PRINT_NONE ) printf("\nCalling UA AMG ...\n");
89             status = fasp_amg_setup_ua(mgl, param); break;
90
91         default: // Classical AMG setup
92             if ( prtlvl > PRINT_NONE ) printf("\nCalling classical AMG ...\n");
93             status = fasp_amg_setup_rs(mgl, param);
94
95     }
96
97     // Step 2: AMG solve phase
98     if ( status == FASP_SUCCESS ) { // call a multilevel cycle
99
100         switch (cycle_type) {
101
102             case AMLI_CYCLE: // AMLI-cycle
103                 fasp_amg_solve_amli(mgl, param); break;
104
105             case NL_AMLI_CYCLE: // Nonlinear AMLI-cycle
106                 fasp_amg_solve_nl_amli(mgl, param); break;
107
108             default: // V,W-cycles (determined by param)

```



```

109         fasp_amg_solve(mgl, param); break;
110
111     }
112
113     fasp_dvec_cp(&mgl[0].x, x);
114
115 }
116
117 else { // call a backup solver
118
119     if ( prtlvl > PRINT_MIN ) {
120         printf("### WARNING: AMG setup failed!\n");
121         printf("### WARNING: Use a backup solver instead.\n");
122     }
123     fasp_solver_dcsr_spgmres (A, b, x, NULL, param->tol, param->maxit,
124                             20, 1, prtlvl);
125
126 }

```

The code above is very simple and we only wish to point out that:

- Line 42–45 reads some of the parameters from “AMG_param”, which can be defined from a input file; see §2.5.
- Line 50–78 initializes the “AMG_data” with a copy of the coefficient matrix, the right-hand side, and the initial solution (it will store the final solution eventually).
- Line 81–95 calls three different AMG setup methods, determined by “amg_type”.
- Line 98–115 calls three different multilevel iterative methods, determined by “cycle_type”.

Parameters for AMG

There are a couple of controlling parameters for algebraic multigrid methods in FASP. Basically, there are four types of parameters for AMG—They control multilevel iterations, smoothing, classical AMG setup, and aggregation AMG setup. The following is a sample from “test/ini/input.dat” and a brief explanation of each parameter is given.

```

55 %-----%
56 % parameters for multilevel iteration      %
57 %-----%
58
59 AMG_type          = C      % C classic AMG
60                      % SA smoothed aggregation
61                      % UA unsmoothed aggregation
62 AMG_cycle_type    = V      % V V-cycle | W W-cycle
63                      % A AMLI-cycle | NA Nonlinear AMLI-cycleA
64 AMG_tol           = 1e-6   % tolerance for AMG
65 AMG_maxit         = 1      % number of AMG iterations

```

```

66 AMG_levels           = 20      % max number of levels
67 AMG_coarse_dof       = 500     % max number of coarse degrees of freedom
68 AMG_coarse_solver    = 0      % coarsest solver: 0 iterative |
69                        % 31 SuperLU | 32 UMFPack | 33 MUMPS
70 AMG_coarse_scaling   = OFF     % switch of scaling of the coarse grid correction
71 AMG_amli_degree      = 2      % degree of the polynomial used by AMLI cycle
72 AMG_nl_amli_krylov_type = 6    % Krylov method in NLAMLI cycle: 6 FGMRES | 7 GCG
73
74 %-----%
75 % parameters for AMG smoothing %
76 %-----%
77
78 AMG_smoother         = GS      % GS | JACOBI | SGS SOR | SSOR |
79                        % GSOR | SGSOR | POLY | L1DIAG | CG
80 AMG_smooth_order     = CF      % NO: natural order | CF: CF order
81 AMG_ILU_levels       = 0      % number of levels using ILU smoother
82 AMG_Schwarz_levels   = 0      % number of levels using Schwarz smoother
83 AMG_relaxation       = 1.0    % relaxation parameter for SOR smoother
84 AMG_polynomial_degree = 3      % degree of the polynomial smoother
85 AMG_presmooth_iter   = 1      % number of presmoothing sweeps
86 AMG_postsmooth_iter  = 1      % number of postsmoothing sweeps
87
88 %-----%
89 % parameters for classical AMG SETUP %
90 %-----%
91
92 AMG_coarsening_type  = 1      % 1 Modified RS
93                        % 3 Compatible Relaxation
94                        % 4 Aggressive
95 AMG_interpolation_type = 1    % 1 Direct | 2 Standard | 3 Energy-min
96 AMG_strong_threshold = 0.3    % Strong threshold
97 AMG_truncation_threshold = 0.1 % Truncation threshold
98 AMG_max_row_sum      = 0.9    % Max row sum
99
100 %-----%
101 % parameters for aggregation-type AMG SETUP %
102 %-----%
103
104 AMG_aggregation_type = 2      % 1 Matching | 2 VMB
105 AMG_pair_number      = 2      % Number of pairs in matching
106 AMG_strong_coupled   = 0.08   % Strong coupled threshold
107 AMG_max_aggregation  = 20     % Max size of aggregations
108 AMG_tentative_smooth = 0.67   % Smoothing factor for tentative prolongation
109 AMG_smooth_filter    = OFF    % Switch for filtered matrix for smoothing
110 AMG_quality_bound    = 8.0    % quality of aggregation: 8.0 sysmm | 10.0 unsymm

```

NOTE: Here we can not discuss the details of these parameters as a full discussion requires more understand of the underlying algorithms which we have completely omitted. So to learn more about, we refer to the Reference Manual.

Chapter 4

More advanced features

In this chapter, we discuss a few more advanced features of FASP. We will discuss parallel versions of FASP and its build-in features for debugging purposes. These features will be helpful for people who would like to develop on the top of FASP. For users who only wish to call a few standard solvers, they can skip this chapter.

4.1 An OpenMP example

OpenMP¹ (Open Multiprocessing) is an API that supports multi-platform shared memory multiprocessing programming in C, C++, and Fortran, on most processor architectures and operating systems. It consists of a set of compiler directives, library routines, and environment variables that influence run-time behavior. Some preliminary OpenMP support has been included since the very beginning of FASP. We consistently improves and expands OpenMP support as multiprocessor architectures become the dominant desktop computing environment.

NOTE: By default, OpenMP is disabled in FASP. In order to turn it on, you need to modify FASP.mk slightly as follows.

To enable OpenMP support in FASP, you can simply use the config option

```
$ make config openmp=yes
```

If you use OpenMP very often and do not want type in this extra command-line option, you need to uncomment one line in “FASP.mk” and set “openmp” to be “yes”.

```
37 #  
38 # You may use multithread version after you enable OpenMP support. To
```

¹Official website: <http://openmp.org/>

```

39 # setup the environment, you need
40 # >> export OMP_NUM_THREADS=4 (for bash)
41 # >> setenv OMP_NUM_THREADS 4 (for tcsh)
42 # If you want to compile with OpenMP support, uncomment the next line:
43 #
44 # openmp=yes
45 #

```

After you build FASP with “openmp=yes”, OpenMP is turned on and the number of threads is determined by the environment variable OMP_NUM_THREADS. For example, to use 8 threads in `sh/bash`, you need to set:

```
$ export OMP_NUM_THREADS=8
```

4.2 Predefined constants

FASP has many predefined constants used in the source files. Using these macros makes the source codes more readable. These constants are defined in “`base/include/fasp_const.h`” and a printout of this file is below:

```

1  /*! \file fasp_const.h
2  *  \brief Definition of all kinds of messages, including error messages,
3  *      solver types, etc.
4  *
5  *  \note This is internal use only. Do NOT change.
6  *
7  *
8  *  Created by Chensong Zhang on 03/20/2010.
9  *  Modified by Chensong Zhang on 12/06/2011.
10 *  Modified by Chensong Zhang on 12/25/2011.
11 *  Modified by Chensong Zhang on 04/22/2012.
12 *  Modified by Ludmil Zikatanov on 02/15/2013: CG -> SMOOTHER.CG.
13 *  Modified by Chensong Zhang on 02/16/2013: GS -> SMOOTHER.GS, etc.
14 *  Modified by Chensong Zhang on 04/09/2013: Add safe Krylov methods.
15 *  Modified by Chensong Zhang on 09/22/2013: Clean up Doxygen.
16 *  Modified by Chensong Zhang on 09/17/2013: Filename changed from message.h.
17 *
18 *
19 */
20
21 #ifndef __FASP_MESSAGES__          /*-- allow multiple inclusions --*/
22 #define __FASP_MESSAGES__
23
24 /**
25  *  \brief Definition of return status and error messages
26  */
27 #define FASP_SUCCESS              0  /**< return from function successfully */
28 //

```

```

29 #define ERROR_OPEN_FILE      -10 /**< fail to open a file */
30 #define ERROR_WRONG_FILE     -11 /**< input contains wrong format */
31 #define ERROR_INPUT_PAR      -13 /**< wrong input argument */
32 #define ERROR_REGRESS        -14 /**< regression test fail */
33 #define ERROR_MAT_SIZE       -15 /**< wrong problem size */
34 #define ERROR_NUM_BLOCKS     -18 /**< wrong number of blocks */
35 #define ERROR_MISC           -19 /**< other error */
36 //
37 #define ERROR_ALLOC_MEM      -20 /**< fail to allocate memory */
38 #define ERROR_DATA_STRUCTURE -21 /**< problem with data structures */
39 #define ERROR_DATA_ZERODIAG  -22 /**< matrix has zero diagonal entries */
40 #define ERROR_DUMMY_VAR      -23 /**< unexpected input data */
41 //
42 #define ERROR_AMG_INTERP_TYPE -30 /**< unknown interpolation type */
43 #define ERROR_AMG_SMOOTH_TYPE -31 /**< unknown smoother type */
44 #define ERROR_AMG_COARSE_TYPE -32 /**< unknown coarsening type */
45 #define ERROR_AMG_COARSEING   -33 /**< coarsening step failed to complete */
46 //
47 #define ERROR_SOLVER_TYPE     -40 /**< unknown solver type */
48 #define ERROR_SOLVER_PRECTYPE -41 /**< unknown precondition type */
49 #define ERROR_SOLVER_STAG     -42 /**< solver stagnates */
50 #define ERROR_SOLVER_SOLSTAG  -43 /**< solver's solution is too small */
51 #define ERROR_SOLVER_TOLSMALL -44 /**< solver's tolerance is too small */
52 #define ERROR_SOLVER_ILUSETUP -45 /**< ILU setup error */
53 #define ERROR_SOLVER_MISC     -46 /**< misc solver error during run time */
54 #define ERROR_SOLVER_MAXIT    -48 /**< maximal iteration number exceeded */
55 #define ERROR_SOLVER_EXIT     -49 /**< solver does not quit successfully */
56 //
57 #define ERROR_QUAD_TYPE       -60 /**< unknown quadrature type */
58 #define ERROR_QUAD_DIM        -61 /**< unsupported quadrature dim */
59 //
60 #define ERROR_LIC_TYPE        -80 /**< wrong license type */
61 //
62 #define ERROR_UNKNOWN         -99 /**< an unknown error type */
63
64 /**
65  * \brief Definition of logic type
66  */
67 #define TRUE                  1 /**< logic TRUE */
68 #define FALSE                 0 /**< logic FALSE */
69
70 /**
71  * \brief Definition of switch
72  */
73 #define ON                    1 /**< turn on certain parameter */
74 #define OFF                   0 /**< turn off certain parameter */
75
76 /**
77  * \brief Print level for all subroutines — not including DEBUG output
78  */
79 #define PRINT_NONE            0 /**< silent: no printout at all */
80 #define PRINT_MIN             1 /**< quiet: print error, important warnings */
81 #define PRINT_SOME            2 /**< some: print less important warnings */

```

```

82 #define PRINT_MORE          4 /**< more:   print some useful debug info */
83 #define PRINT_MOST          8 /**< most:   maximal printouts, no files */
84 #define PRINT_ALL           10 /**< all:    all printouts, including files */
85
86 /**
87  * \brief Definition of matrix format
88  */
89 #define MAT_FREE             0 /**< matrix-free format: only mxv action */
90 #define MAT_CSR              1 /**< compressed sparse row */
91 #define MAT_BSR              2 /**< block-wise compressed sparse row */
92 #define MAT_STR              3 /**< structured sparse matrix */
93 #define MAT_bCSR             4 /**< block matrix of CSR */
94 #define MAT_bBSR            5 /**< block matrix of BSR for bordered systems */
95 #define MAT_CSRL             6 /**< modified CSR to reduce cache missing */
96 #define MAT_SymCSR           7 /**< symmetric CSR format */
97
98 /**
99  * \brief Definition of solver types for iterative methods
100  */
101 #define SOLVER_DEFAULT       0 /**< Use default solver in FASP */
102 //-----
103 #define SOLVER_CG             1 /**< Conjugate Gradient */
104 #define SOLVER_BiCGstab       2 /**< Bi-Conjugate Gradient Stabilized */
105 #define SOLVER_MinRes         3 /**< Minimal Residual */
106 #define SOLVER_GMRES          4 /**< Generalized Minimal Residual */
107 #define SOLVER_VGMRES         5 /**< Variable Restarting GMRES */
108 #define SOLVER_VFGMRES        6 /**< Variable Restarting Flexible GMRES */
109 #define SOLVER_GCG            7 /**< Generalized Conjugate Gradient */
110 #define SOLVER_GCR            8 /**< Generalized Conjugate Residual */
111 //-----
112 #define SOLVER_SCG            11 /**< Conjugate Gradient with safety net */
113 #define SOLVER_SBiCGstab      12 /**< BiCGstab with safety net */
114 #define SOLVER_SMinRes        13 /**< MinRes with safety net */
115 #define SOLVER_SGMRES         14 /**< GMRes with safety net */
116 #define SOLVER_SVGMRES        15 /**< Variable-restart GMRES with safety net */
117 #define SOLVER_SVFGMRES       16 /**< Variable-restart FGMRES with safety net */
118 #define SOLVER_SGCG           17 /**< GCG with safety net */
119 //-----
120 #define SOLVER_AMG            21 /**< AMG as an iterative solver */
121 #define SOLVER_FMG            22 /**< Full AMG as an solver */
122 //-----
123 #define SOLVER_SUPERLU        31 /**< SuperLU Direct Solver */
124 #define SOLVER_UMFPACK         32 /**< UMFPack Direct Solver */
125 #define SOLVER_MUMPS           33 /**< MUMPS   Direct Solver */
126
127 /**
128  * \brief Definition of iterative solver stopping criteria types
129  */
130 #define STOP_REL_RES          1 /**< relative residual ||r||/||b|| */
131 #define STOP_REL_PRECRES       2 /**< relative B-residual ||r||_B/||b||_B */
132 #define STOP_MOD_REL_RES       3 /**< modified relative residual ||r||/||x|| */
133
134 /**

```

```

135 * \brief Definition of preconditioner type for iterative methods
136 */
137 #define PREC_NULL          0 /**< with no precondition */
138 #define PREC_DIAG          1 /**< with diagonal precondition */
139 #define PREC_AMG           2 /**< with AMG precondition */
140 #define PREC_FMG           3 /**< with full AMG precondition */
141 #define PREC_ILU           4 /**< with ILU precondition */
142 #define PREC_SCHWARZ       5 /**< with Schwarz preconditioner */
143
144 /**
145  * \brief Type of ILU methods
146  */
147 #define ILUk                1 /**< ILUk */
148 #define ILUt                2 /**< ILUt */
149 #define ILUtp               3 /**< ILUtp */
150
151 /**
152  * \brief Type of Schwarz smoother
153  */
154 #define SCHWARZ_FORWARD     1 /**< Forward ordering */
155 #define SCHWARZ_BACKWARD    2 /**< Backward ordering */
156 #define SCHWARZ_SYMMETRIC   3 /**< Symmetric smoother */
157
158 /**
159  * \brief Definition of AMG types
160  */
161 #define CLASSIC_AMG         1 /**< classic AMG */
162 #define SA_AMG              2 /**< smoothed aggregation AMG */
163 #define UA_AMG              3 /**< unsmoothed aggregation AMG */
164
165 /**
166  * \brief Definition of aggregation types
167  */
168 #define PAIRWISE            1 /**< pairwise aggregation */
169 #define VMB                 2 /**< VMB aggregation */
170
171 /**
172  * \brief Definition of cycle types
173  */
174 #define V_CYCLE             1 /**< V-cycle */
175 #define W_CYCLE             2 /**< W-cycle */
176 #define AMLI_CYCLE          3 /**< AMLI-cycle */
177 #define NL_AMLI_CYCLE       4 /**< Nonlinear AMLI-cycle */
178
179 /**
180  * \brief Definition of standard smoother types
181  */
182 #define SMOOTHER_JACOBI     1 /**< Jacobi smoother */
183 #define SMOOTHER_GS         2 /**< Gauss-Seidel smoother */
184 #define SMOOTHER_SGS        3 /**< Symmetric Gauss-Seidel smoother */
185 #define SMOOTHER_CG         4 /**< CG as a smoother */
186 #define SMOOTHER_SOR        5 /**< SOR smoother */
187 #define SMOOTHER_SSOR       6 /**< SSOR smoother */

```



```

188 #define SMOOTHER_GSOR          7  /**< GS + SOR smoother */
189 #define SMOOTHER_SGSOR        8  /**< SGS + SSOR smoother */
190 #define SMOOTHER_POLY          9  /**< Polynomial smoother */
191 #define SMOOTHER_L1DIAG       10  /**< L1 norm diagonal scaling smoother */
192
193 /**
194  * \brief Definition of coarsening types
195  */
196 #define COARSE_RS              1  /**< Classical */
197 #define COARSE_RSP            2  /**< Classical, with positive offdiags */
198 #define COARSE_CR             3  /**< Compatible relaxation */
199 #define COARSE_AC             4  /**< Aggressive coarsening */
200 #define COARSE_MIS            5  /**< Aggressive coarsening based on MIS */
201
202 /**
203  * \brief Definition of interpolation types
204  */
205 #define INTERP_DIR             1  /**< Direct interpolation */
206 #define INTERP_STD            2  /**< Standard interpolation */
207 #define INTERP_ENG            3  /**< energy minimization interpolation */
208
209 /**
210  * \brief Type of vertices (DOFs) for coarsening
211  */
212 #define GOPT                  -5  /**< Cannot fit in aggregates */
213 #define UNPT                  -1  /**< Undetermined points */
214 #define FGPT                   0  /**< Fine grid points */
215 #define CGPT                   1  /**< Coarse grid points */
216 #define ISPT                   2  /**< Isolated points */
217
218 /**
219  * \brief Definition of smoothing order
220  */
221 #define NO_ORDER              0  /**< Natural order smoothing */
222 #define CF_ORDER              1  /**< C/F order smoothing */
223
224 /**
225  * \brief Type of ordering for smoothers
226  */
227 #define USERDEFINED           0  /**< User defined order */
228 #define CPFIRST               1  /**< C-points first order */
229 #define FPFIRST              -1  /**< F-points first order */
230 #define ASCEND                12  /**< Ascending order */
231 #define DESCEND               21  /**< Descending order */
232
233 /**
234  * \brief Some global constants
235  */
236 #define BIGREAL                1e+20 /**< A large real number */
237 #define SMALLREAL              1e-20 /**< A small real number */
238 #define SMALLREAL2             1e-40 /**< An extremely small real number */
239 #define MAX_REFINE_LVL         20  /**< Maximal refinement level */
240 #define MAX_AMG_LVL            20  /**< Maximal AMG coarsening level */

```

```

241 #define MIN_CDOF          20  /**< Minimal number of coarsest variables */
242 #define MIN_CRATE         0.9 /**< Minimal coarsening ratio */
243 #define MAX_CRATE         20.0 /**< Maximal coarsening ratio */
244 #define MAX_RESTART       20  /**< Maximal restarting number */
245 #define MAX_STAG          20  /**< Maximal number of stagnation times */
246 #define STAG_RATIO        1e-4 /**< Stagnation tolerance = tol*STAGRATIO */
247 #define OPENMP_HOLDS      2000 /**< Smallest size for OpenMP version */
248
249 #endif                      /* end if for __FASP_MESSAGES__ */
250
251 /*-----*/
252 /*--      End of File      --*/
253 /*-----*/

```

4.3 Debugging and how to enable it

NOTE: The default FSP build is a RELEASE version (/O3 or equivalent compiler options are enabled) and such version is compiled with optimization and no warnings are displayed during the build. How to build the FASP library with debugging enabled is described below.

There is a built-in debug feature which is intended to help developers and users to locate malfunctions and bugs in FASP (and hopefully fix them). In order to turn this feature on, you need to add the **debug** option during the config stage by

```
$ make config debug=all
```

When this debug feature is turned on, there will be a lot more information printed when you run FASP. If you just want to enable the debugging and warnings during the compile stage, you can do so by using

```
$ make config debug=yes
```


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