A Note on Tikhonov Regularization of Linear Ill-Posed Problems

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1 Linear Ill-Posed Problems

In this note I describe Tikhonov regularization for finding a stable approximate solution to a linear ill-posed problem represented in the form of an operator equation

$$Au = f \, . \tag{1}$$

where, instead of the exact data f, noisy data f_δ is available with

$$\|f - f_\delta\|_F \leq \delta \, . \tag{2}$$

Here the operator A is a linear compact injective operator between Hilbert spaces U and F. The solution u and data f belong to U and F, respectively. The inner products in U and F are denoted by $(\cdot, \cdot)_U$ and $(\cdot, \cdot)_F$.

The problem (1) is *ill-posed* in the sense that the inverse operator A^{-1} of A exists but it is not continuous. Hence, although the problem (1) has a unique solution, solving it directly will not give a *right* solution. Indeed, the linear operator A is so badly conditioned that any numerical attempt to directly solve (1) may fail.

2 Tikhonov Regularization

In order to find a solution in stable manner, Tikhonov proposed to solve

$$u_\alpha = \arg\min_{w \in U} \; J_\alpha(w) = \|Aw - f_\delta\|_F^2 + \alpha\|w\|_U^2 \ , \tag{3}$$

where the regularization parameter α is found such that

$$\|Au_\alpha - f_\delta\|_F = \delta \ . \tag{4}$$

It can be easily shown that for every positive parameter α there exists a unique $u_\alpha \in U$ for which the functional J_α attains its minimal. Furthermore, it can be shown that there exists a positive value of α for which the condition (4) is satisfied.

The computation of the approximate solution u_α consists in solving the Euler equation corresponding to the functional J_α. This equation has the form

$$\big(A^*A + \alpha I\big)u_\alpha = A^* f_\delta \ , \tag{5}$$

where A^* is the adjoint operator of A and I is the identity operator.

The regularization parameter α satisfying the condition (4) and thus the associated solution u_α is typically determined by the Morozov's discrepancy principle. This is done as follows

- Choose α_0, $0 < d < 1$
- Set $j = 0$ and solve (5) for u_0 corresponding to α_0
- while $\|Au_{\alpha_j} - f_\delta\| > \delta$
  - $j = j + 1$
  - $\alpha_j = d\alpha_{j-1}$
  - Compute u_{α_j} from (5)
- end

- $\alpha_{\max} = \alpha_{j-1}$, $\alpha_{\min} = \alpha_j$
- While not $(\|Au_\alpha - f_\delta\| = \delta)$
  - $\alpha = (\alpha_{\max} + \alpha_{\min})/2$
  - Compute u_α from (5)
  - If $\|Au_\alpha - f_\delta\| > \delta$ then $\alpha_{\max} = \alpha$ else $\alpha_{\min} = \alpha$
- end

3 A Simple Example

Let us look at the problem of solving the Fredholm integral equation of the first kind

$$\int_a^b K(x,y)u(x)dx = f(y), \quad c \leq y \leq d, \tag{6}$$

where $u(x)$ is the unknown function in a Hilbert space U, and $f(y)$ is a known function in a space F. Let us assume that the kernel $K(x,y)$ is continuous with respect to the variable y and that it has a continuous partial derivative $\partial K/\partial y$.

It is known that the problem (6) is ill-posed. For example, if we replace the equation (6) with the system of linear algebraic equations

$$\sum_{j=1}^{n} K_{ij}\Delta x_j u_j = f(x_i), \quad i = 1, \ldots, n \tag{7}$$

by approximating the integral with a sum by means of Simpson's rule, then the resulting system is very badly conditioned. Consequently, attempt to find a solution u by inverting the matrix A^n with $A^n_{ij} = K_{ij}\Delta x_j$ will fail. In this event, Tikhonov regularization of (7) will lead to a stable approximate solution.

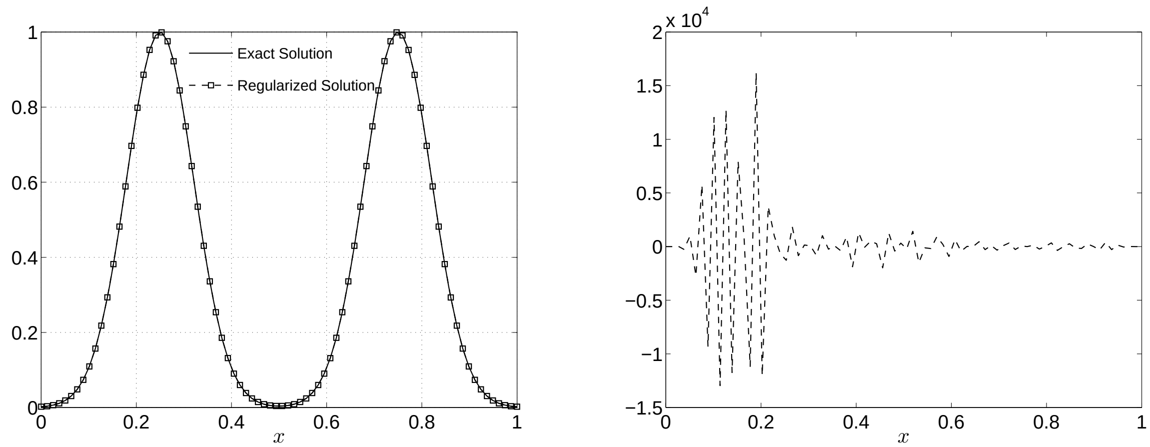
To illustrate the point of our discussion, we consider a particular problem in which $a = 0$, $b = 1$,

$c = -2$, $d = 2$, $K(x, y) = \frac{1}{1+100(x-y)^2}$, the data f is chosen such that

$$u = e^{-100(x-0.25)^2} + e^{-100(x-0.75)^2}$$

is the solution of the problem (6). Below we present numerical results.

We first show in Figure 1(a) the exact solution and the regularized solution obtained by using Tikhonov regularization. We see that the regularized solution is almost distinguishable from the exact one. We next show in Figure 1(b) the result obtained by inverting the matrix A^n. The saw-toothed broken line has nothing in common with the exact solution.



(a) (b)

Figure 1: Numerical results for the simple example: (a) comparison of the exact solution and the regularized solution and (b) this result is obtained by inverting the matrix A^n.