Comparative analysis of new approach to discrete Ricci curvature on undirected graphs

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Ollivier-Ricci and Forman-Ricci curvatures are two well-studied approaches for discretising Ricci curvature for nodes and links of graphs. Recently, a novel approach based on the concept of effective resistance has been introduced, namely resistance curvature. This report introduces various notions of discrete Ricci curvature for undirected, weighted graphs and compares those that are well-established with the new resistance curvatures.

network geometry | discrete Ricci curvature | graph measures

Network science has provided practical methods to model and analyse complex systems across a vast range of disciplines such as computer science (1), biology (2), epidemiology (3), and sociology (4). One of the goals of network analysis is to characterise the underlying structure of both real and model networks. This focus has led to the development of describing the geometric characterisation of graphs. A central area of this topic is curvature which measures the deviation of an object from being flat. This is a fundamental concept to the discipline of differential geometry.

Differential geometry studies the geometry of smooth spaces called smooth manifolds. A significant branch within this discipline is Riemannian geometry which has been integral to the mathematical description of Einstein's theory of general relativity. The notion of *Ricci curvature* originates within this framework. The other form of curvature that is of interest to us is *scalar curvature*. Ricci curvature associates a tensor to each point on a manifold to quantify curvature, whereas scalar curvature associates a single number.

Graphs are discrete objects, and so in order to have an analogous notions of curvature to analyse their geometry, a discrete theory must be formulated. Two notions of generalised Ricci curvature that have been well-studied and applied to graphs for this purpose were introduced by Yann Ollivier (5) and Robin Forman (6). These approaches are fundamentally different; Ollivier's approach is based on optimal transport theory, and Forman's is based on graph Laplacian. Investigations of the relationships between these approaches to Ricci curvatures have been carried out (7, 8). Ricci curvature of graphs has been applied to community detection using Ricci flow (9).

Recently, a new approach to measure Ricci curvature on graphs has been introduced in (10), based on the definition of effective resistance. In this work, we compare and analyse this new notion of Ricci curvature with the current ones on a variety of different networks.

Preliminaries

Graphs and Basic Definitions. Networks are represented by graphs. A weighted graph $\mathcal{G} = (V, E, w)$ consists of a set of N vertices (nodes) V, a set of M edges (links) $E \subseteq V \times V$, and a weight function $w : E \to \mathbb{R}_{>0}$. A link $e_{ij} := (v_i, v_j) \in E$ connects a node v_i with a node v_j and we denote $i \sim j$ if $e_{ij} \in E$. We denote the weight of a link $e_{ij} \in E$ by w_{ij}, and the weight of a node $v_i \in V$ by w_i. We focus on simple, undirected graphs that do not contain self loops throughout this work which means $e_{ij} \in E \iff e_{ji} \in E$, and $e_{ii} \notin E$.

The *Laplacian matrix* L is useful for describing and analysing associated graphs. It is an $N \times N$ matrix, defined as follows

$$(L)_{ij} := \begin{cases} -w_{ij} & \text{if } i \sim j \\ k_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

where $k_i := \sum_{j \sim i} w_{ij}$ is the weighted degree of node v_i. The *combinatorial degree* of a node $v_i \in V$ is $\deg(v_i) := \sum_{i \sim j} 1$ and will also be of interest in our work.

Lastly, we define π_i to be the set of neighbours of a node $v_i \in V$, that is $\pi_i := \{v_j \in V : \exists\, e_{ij} \in E\}$

Discrete Ricci curvatures on undirected, weighted graphs. In the following section we introduce several approaches to measuring discrete Ricci curvature on undirected, weighted graphs, namely Ollivier, Forman, augmented Forman, and effective resistance.

Ollivier. To describe Ollivier's approach to Ricci curvature which based on optimal transport, we introduce the notion of probability measures on graphs, transportation plans, and the 1-Wasserstein distance (11). A probability measure μ on a graph is a function $\mu : V \to [0, 1]$ such that

$$\sum_{v_i \in V} \mu(v_i) = 1$$

Significance Statement

Ricci curvature is a valuable measure in network analysis. It can help shed light on underlying structures of networks, such as the identification of communities . Ricci curvature is rooted in the field of differential geometry, which studies the geometry of smooth spaces. As such, there is no unique way to measure Ricci curvature on graphs due to their discrete structure. Therefore, it is essential to compare the different approaches that are developed for carrying out this discretisation. Our computational comparisons of a newly introduced method with well-established ones provide insights into its behaviour for a variety of networks.

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A transport plan ξ is a map $\xi : V \times V \to [0,1]$ such that $\forall\, (v_i, v_j) \in V \times V$:

$$\sum_{v_i \in V} \xi(v_i, v_j) = \mu(v_j), \qquad \sum_{v_j \in V} \xi(v_i, v_j) = \mu(v_j)$$

$\xi(v_i, v_j)$ is the mass transported from nodes v_i to v_j along the shortest path between them, $d(v_i, v_j)$ at *cost* $d(v_i, v_j)\xi(v_i, v_j)$. The *total cost* of the transport plan is

$$\text{cost}(\xi) := \sum_{v_i \in V} \sum_{v_j \in V} d(v_i, v_j)\xi(v_i, v_j)$$

The set of all transport plans from μ to ν is denoted $\Pi(\mu, \nu)$, and the 1-Wasserstein ditance, also known as the *earth mover's distance* is defined as

$$W_1(\mu, \nu) := \inf_{\xi \in \Pi(\mu,\nu)} \text{cost}(\xi)$$

We lastly note that a transport plan $\xi \in \Pi(\mu, \nu)$ is called *optimal* if $W_1(\mu, \nu) = \text{cost}(\xi)$.

By considering a graph as a metric space with a random walk μ_i associated to each node $v_i \in V$ we can now define the following:

Definition 1 (Ollivier-Ricci curvature of an edge)**.** The *Ollivier-Ricci* curvature (OR) of an edge $e_{ij} \in E$ is defined as (5):

$$\kappa_{ij}^{(OR)} := 1 - \frac{W_1(\mu_i, \mu_j)}{d(v_i, v_j)}$$

where $d(v_i, v_j)$ is the length of the shortest path between nodes v_i and v_j.

The probability measure μ_i for $v_i \in V$ that we focus on is the one proposed by (12):

$$\mu_i^{\alpha,p}(v_j) = \begin{cases} \alpha & \text{if } v_j = v_i \\ \frac{1-\alpha}{C} \cdot \exp(-d(v_j, v_i)^p) & \text{if } v_j \in \pi_i \\ 0 & \text{otherwise} \end{cases} \qquad [1]$$

where $C = \sum_{x_i \in \pi(v_i)} \exp(-d(v_j, v_i)^p)$ is a normalisation factor. The laziness of the random walk is described by the parameter α which is the probability of remaining at node v_i. The power parameter p determines how much the neighbour v_j of v_i is discounted with respect to the distance $d(v_j, v_i)$.

Definition 2 (Ollivier-Ricci curvature of a node)**.** The *Ollivier-Ricci* (OR) curvature of a node $v_i \in V$ can naturally be defined as

$$p_i^{(OR)} = \sum_{j \sim i} \kappa_{ij}^{(OR)} \qquad [2]$$

This is anaolous to the notion of scalar curvature in Riemannian geometry. We note that a degree-normalised version is another natural way to define this notion but we focus on the above approach.

Forman. CW complexes are a broad class of geometric objects and in particular, a locally finite graph is a 1-complex. That is, a graph is a 1-dimensional CW complex that consists of 0-cells (nodes) and 1-cells (links). Forman proposed a general discretisation of Ricci curvature $\mathcal{F}(\alpha^p)$ for CW complexes (6) based on topological arguments. This fomrulation is based on an abstraction of the *Bochner-Weittzenböck formula* (13) and takes the form of eqn.3. We're interested in the case for networks and so we will focus on the form of $\mathcal{F}(\alpha^p)$ for the $p = 1$ case.

Definition 3 (Forman-Ricci curvature for links)**.** The *Forman-Ricci* (FR) curvature of a link $e_{ij} \in E$ is defined as

$$\kappa_{ij}^{(FR)} := w_i \left(2 - \sum_{k \sim i} \sqrt{\frac{w_{ij}}{w_{ik}}} \right) + w_j \left(2 - \sum_{k \sim j} \sqrt{\frac{w_{ij}}{w_{jk}}} \right)$$

When a graph is unweighted i.e $w_{ij} = 1 \;\forall\; e_{ij} \in E$, and $w_i = 1 \;\forall\; v_i \in V$ then the Forman-Ricci curvature for links simplifies to

$$\kappa_{ij}^{(FR)} = 4 - \deg(v_i) - \deg(v_j)$$

Similar to Ollivier-Ricci curvature, there are two natural ways to define Forman-Ricci curvature for nodes and we focus on the non-normalised definition.

Definition 4 (Forman-Ricci curvature for nodes)**.** The *Forman-Ricci* (FR) curvature of a node $v_i \in V$ can naturally be defined as

$$p_i^{(FR)} = \sum_{j \sim i} \kappa_{ij}^{(FR)} \qquad [4]$$

One of the main advantages to Forman's approach is that it is able to take into account the weights of nodes as well as links. Ollivier-Ricci curvature is better suited for community detection as it is more capable of capturing the network's structure since it is more geometric (9). However Forman-Ricci curvature is computationally cheaper to calculate.

augmented Forman. The previously mentioned Forman-Ricci curvature for graphs arose upon considering graphs as a 1-complex. This construction can be augmented (14), by considering graphs as a 2-complex in eqn.3, thus providing the so-called *augmented Forman-Ricci* (AFR) curvature $\kappa_{ij}^{(AFR)}$ of a link $e_{ij} \in E$:

$$\kappa_{ij}^{(AFR)} := w_{ij} \Bigg[\left(\sum_{e_{ij} < f} \frac{w_{ij}}{w(f)} + \sum_{v < e_{ij}} \frac{w(v)}{w_{ij}} \right) - \sum_{\hat{e} \| e_{ij}} \left| \sum_{\hat{e}, e_{ij} < f} \frac{\sqrt{w_{ij} \cdot w(\hat{e})}}{w(f)} - \sum_{v < e_{ij}, v < \hat{e}} \frac{w(v)}{\sqrt{w_{ij} \cdot w(\hat{e})}} \right| \Bigg]$$

where $w(\hat{e})$ denotes the weight of an edge $\hat{e}$, $w(v)$ denotes the weight of a vertex v, $w(f)$ denotes the weight of a face f, $\alpha < \beta$ means that α is a face of β, and $\|$ denotes parallelism, which is when two cells have a common higher dimensional face, known as a parent. The augmented Forman-Ricci curvature of a node is similar to before.

$$\mathcal{F}(\alpha^p) := w(\alpha^p)\left[\left(\sum_{\beta^{p+1}>\alpha^p}\frac{w(\alpha^p)}{w(\beta^{p+1})}+\sum_{\gamma^{p-1}<\alpha^p}\frac{w(\gamma^{p-1})}{w(\alpha^p)}\right)-\sum_{\substack{\alpha_1^p\|\alpha^p\\ \alpha_1^p\neq\alpha^p}}\left|\sum_{\substack{\beta^{p+1}>\alpha_1^p\\ \beta^{p+1}>\alpha^p}}\frac{\sqrt{w(\alpha^p)w(\alpha_1^p)}}{w(\beta^{p+1})}-\sum_{\substack{\gamma^{p-1}<\alpha_1^p\\ \gamma^{p-1}<\alpha^p}}\frac{w(\gamma^{p-1})}{\sqrt{w(\alpha^p)w(\alpha_1^p)}}\right|\right] \quad [3]$$

where $w(e)$ denotes the weight of a cell/simplice, $\alpha < \beta$ means that α is a face of β, and $\alpha_1 \parallel \alpha_2$ means cells α_1 and α_2 are parallel, that is they have a common higher dimensional face, known as a parent.

effective resistance. Recently, a new approach to measure discrete curvature on graphs was introduced based on effective resistance (10). A scalar curvature on nodes $p^{(RES)} : E \to \mathbb{R}$ and a Ricci curvature on links $\kappa^{(RES)} : V \to \mathbb{R}$ were established. Effective resistance emerged from electircal circuit theory, where it measures the resistance exerted by the whole network on a current flowing between any two nodes. The *effective resistance* r_{ij} (15) between a pair of nodes $(v_i, v_j) \in V \times V$ is defined as

$$r_{ij} := \begin{cases} (\mathbf{e}_i - \mathbf{e}_j)^T L^\dagger (\mathbf{e}_i - \mathbf{e}_j) & \text{if } v_i, v_j \text{ in same component} \\ \infty & \text{otherwise} \end{cases}$$

where $\mathbf{e}_i$ is the i^{th} unit vector and $L^\dagger$ is the *Moore-Penrose pseudoinverse* of the laplacian matrix L. This notion in fact defines a metric between nodes of the network. Our focus will be on the term $r_{ij}w_{ij}$, the so-called the *relative resistance* of nodes (v_i, v_j) which refelcts the importance of the link e_{ij} for connectivity of the graph. With this term, we have the following definition:

Definition 5 (resistance curvature)**.** The *resistance curvature* of a node $v_i \in V$ is defined as

$$p_i^{(RES)} := 1 - \frac{1}{2}\sum_{j\sim i} r_{ij}w_{ij}$$

Definition 6 (link resistance curvature)**.** The link resistance curvature of a link $e_{ij} \in E$ is defined as

$$\kappa_{ij}^{(RES)} := \frac{2(p_i^{(RES)} + p_j^{(RES)})}{r_{ij}}$$

We note that (10) also proposes a 'degree-normalised' version of link resistance curvature given by $\kappa_{ij} = 2(p_i/k_i + p_j/k_j)$. These notions of link and node curvatures Effective resistance is computationally efficient and highly susceptible to analysis with their various amounts of theoretical properties (10).

Other edge and node based measures. We introduce some other measures that will be relevant for our work. A few of these are edge-based, namely edge betweeness centrality and dispersion. Others are node-based, namely node betweeness centrality and clustering coefficient.

betweenness of edge. The *betweenness centrality* of an edge $e_{ij} \in E$ is defined as

$$\text{betweenness}_{ij} = \frac{2}{(N-1)(N-2)}\sum_{\substack{k=1\\k\neq i}}^{N}\sum_{\substack{l=1\\l\neq i}}^{k-1}\frac{\sigma_{kl}^{ij}}{\sigma_{kl}}$$

where σ_{kl} is the number of shortest paths connecting a pair of nodes $(v_i, v_j) \in V \times V$, and σ_{kl}^{ij} is the number of such shortest paths that pass throught the edge e_{ij}.

dispersion. The (normalised) *dispersion* between a pair of nodes $(v_i, v_j) \in V \times V$ (4) is defined as

$$\text{dispersion}_{ij} = \frac{1}{\text{embeddedness}_{ij}}\sum_{s,t\in\pi_i\cap\pi_j} d_j(s,t)$$

where embeddedness$_{i,j}$ is the number of mutual links between nodes v_i and v_j, and d_j is a distance function on the nodes of $\pi_i \cap \pi_j$. We focus on the distance function d_j defined such that $d_j(s,t)$ equals 1 when there is no link between s and t and also the only common neighbours of s and t in π_i are v_i and v_j, and $d_j(s,t)$ equals 0 otherwise. The embeddedness term is a normalisation factor for this form of the dispersion measure.

betweenness of node. The *betweenness centrality* of node $v_i \in V$ is defined as

$$\text{betweenness}_i = \frac{2}{(N-1)(N-2)}\sum_{\substack{k=1\\k\neq i}}^{N}\sum_{\substack{l=1\\l\neq i}}^{k-1}\frac{\sigma_{kl}^{i}}{\sigma_{kl}}$$

where σ_{kl} is the number of shortest paths connecting a pair of nodes $(v_k, v_l) \in V \times V$, and σ_{kl}^i is the number of such shortest paths that pass throught the node v_i.

clustering coefficient of a node. The *clustering coefficient* C_i of a node $v_i \in V$ measures the abundance of triangles in the neighbourhood of v_i. A weighted version, $\widetilde{C}_i$ for weighted graphs is defined by (16) as

$$\widetilde{C}_i = \frac{2}{\deg(v_i)(\deg(v_i)-1)}\sum_{k=1}^{N}\sum_{l=1}^{N}(\tilde{w}_{i,k}\tilde{w}_{k,l}\tilde{w}_{l,i})^{1/3}$$

where $\tilde{w}_{ij} = w_{ij}/\max(w_{ij})$. This definition coincides with the unweighted clustering coefficient, that is $\widetilde{C}_i = C_i$ when the weights are binary.

Materials and Methods

The NetworkX package (17) was used extensively to generate graphs for model networks and construct graphs from real world datasets. The Ollivier-Ricci, Forman-Ricci, and augmented Forman-Ricci link curvatures were calculated using the GraphRicciCurvature package (12) in python. This package calculates the degree-normalised definitions of node Ricci-curvatures so we implemented our own function to calculate the node curvatures as in eqns.2,4 as to compare with link resistance curvature in which is not degree-normalised and to compare with the results in (7). In calculating the Olivier-Ricci curvature, we use the probability measure in eqn.1 with laziness parameter $\alpha = 0.5$ and power parameter $p = 2$.

These are the default values in the GraphRicci curvature (12) package.

We implemented our own functions in Python to calculate the resistance and link resistance curvatures for undirected, weighted graphs. SciPy (18) was used to calculate the Spearman and Pearson correlations. The pandas (19, 20) and NumPy (21) packages were also used throughout all of our work.

We also implemented the following normalised weighting scheme from (22) to impose on model networks (Supplementary Tables S1-S8): For a node $v_i \in V$ impose the weight $w_i = \tilde{w}_i := w_i / \max(w_i)$ and for a link $e_{ij} \in E$ set $w_{ij} = \tilde{w}_{ij} := w_{ij} / \max(w_{ij})$ where

$$w_i = \frac{1}{\deg(v_i)} \sum_{j \sim i} \deg(v_i), \qquad w_{ij} = \sqrt{\tilde{w}_i^2 + \tilde{w}_j^2}$$

Dataset descriptions. We considered three undirected model networks, namely, the Erdős-Rényi (ER) model (23), the Watts-Strogatz (WS) model (24), and the Barabási-Albert (BA) model (25). several undirected real networks including two that are weighted. The model networks were generated using the NetworkX package (17).

- **Erdős-Rényi model** generates graphs of n nodes such that the probability that each possible edge exists between any pair of nodes is governed by p.
- **Watts-Strogatz model** generates graphs of n nodes, initially in a ring configuration such that each node is linked to its k nearest neighbours. Each link in the initial configuration is then removed with probability p and replaced by a link with another node chosen at random (uniformly). This is a called a shortcut link and these establish small-world properties for the graph.
- **Barabási-Albert model** generates graphs that initially have of m nodes. New nodes are added to the graph one at a time, and each node has m links that are preferentially attached to the existing nodes with high degree. This is known as preferential attachement.

Real empirical networks were also considered including two that have weighted edges. These datasets were accessed through the konect database (26), imported using the pandas package (19), and associated graphs were generated using NetworkX (17) in Python.

- **Sister cities** (27) represents the network of cities of the world connected by *"sister city"* or *"twin city"* relationships. In this network of 14,274 nodes and 20,573 links, the nodes are cities and a link between two nodes denotes that they are sister or twin cities.
- **US power grid** (28) represents the high-voltage power grid in the Western States of the USA. In this network of 4,941 nodes and 6,594 links, the nodes are transformers, substations, and generators, and the links are high-voltage transmission lines.
- **Euroroads** (29) represents the international E-road network, a road network located mostly in Europe. In this network of 1,174 nodes and 1,417 links, the nodes represent cities and a link between two nodes denotes that they are connected by an E-road.
- **Dolphins** (30) represents a social network of a bottlenose dolphin community living off Doubtful Sound, New Zealand. In this network of 62 nodes and 159 links, the nodes represent bottlenose dolphins and a link between two nodes indicates a frequent association.
- **Contiguous USA** (31) represents the network of shared borders amongst the states of USA and Washington DC. In this network of 49 nodes and 107 links, the nodes represent USA states and a link between two nodes indicates that they share a border.
- **Zachary karate club** (32) represents a social network of friendships of a karate club at a US university in the 1970s. In this network of 34 nodes and 78 links, the nodes represent members of the club and a link between two nodes represents a tie between two members of the club.
- **Jazz musicians** (33) represents a collaboration network between Jazz musicians. In this network of 198 nodes and 2,742 links, the nodes represent a Jazz musician and a link between two nodes denotes that two musicians have played in a band.
- **Zebra** (34) represents a network of interactions amongst a zebra community in Kenya. In this network of 27 nodes and 111 links, a node represents a zebra and a link between two zebras shows that there was an interaction between them during the study.
- **Les Misérables** (35) represents a weighted network of the co-occurances of characters in Victor Hugo's novel *'Les Misérables'*. In this network of 77 nodes and 254 links, a node represents a charachter and a link between two nodes indicates that these two characters appeared in the same chapter of the book. The weigh of each link indiciates how often such a co-appearance occured.
- **Windsurfers** (36) represents a weighted network of interpersonal contacts between windsurfers in southern California during the fall of 1986. In this network of 43 nodes and 336 links, a node represents a windsurfer and an edge between two nodes shoes that there was inerpersonal contact.

Results

Comparison between resistance curvatures and other curvature measures. We compared the link resistance curvature with Ollivier-Ricci, Forman-Ricci, and augmented Forman-Ricci curvature of edges in model networks and in real networks, including weighted networks (Table 1 and SI Table S1). In ER, WS, and BA model networks, we found that the correlation between the link resistance curvature and the other link curvatures mostly decreased as the average degree of the model networks increased. This relationship is very similar to that of the correlation between the Ollivier-Ricci and (augmented) Forman-Ricci curvatures, as observed by (7). We observe that this relationship is much stronger for the Ollivier-Ricci curvature than the (augmented) Forman-Ricci curvatures, particularly the vanilla Forman-Ricci curvature.

For unweighted real networks the correlation between the curvatures were all positive, ranging from moderately low to moderately high. The Forman-Ricci curvature had the highest correlation for all the networks considered and also had the least variance across the networks. The correlations significantly decreased overall for the weighed, real networks. This includes the Forman-Ricci curvaturef, but it was still proportionately much higher than that for the Ollivier-Ricci and augmented Forman-Ricci curvatures.

We compared the resistance curvature with Ollivier-Ricci, Forman-Ricci, and augmented Forman-Ricci curvature of nodes in model networks and in real networks including weighted networks (Table 2 and SI Tables S2,S6). In ER, WS, and BA model networks we found that the correlation between the node resistance curvature and the other node curvatures increased as the degree of the model networks increased. This relationship is the opposite to the correlation between the Ollivier-Ricci and (augmented) Forman-Ricci curvatures as observed by (7). We observe that this relationship is much stronger for the Ollivier-Ricci curvature than the (augmented) Forman-Ricci curvatures, particularly the vanilla Forman-Ricci curvature.

In nearly every network, we observe that the correlation between the link resistance curvature and the Forman-Ricci curvature for links is higher than the resistance curvature and the Forman-Ricci curvature for nodes.

Comparison between node resistance curvature and other node-based measures. We compared resistance curvature with three other node-based measures for graphs, namely weighted

Table 1. Comparisons of link resistance curvature with other discrete link Ricci curvatures

Network	OR	FR	AFR
Model networks			
ER $n = 1000$, $p = 0.003$	0.81	0.91	0.91
ER $n = 1000$, $p = 0.007$	0.42	0.92	0.91
ER $n = 1000$, $p = 0.01$	0.05	0.94	0.92
WS $n = 1000$, $k = 2$, $p = 0.5$	0.89	0.96	0.96
WS $n = 1000$, $k = 8$, $p = 0.5$	0.22	0.95	0.65
WS $n = 1000$, $k = 10$, $p = 0.5$	0.14	0.95	0.61
BA $n = 1000$, $m = 2$	0.78	0.99	0.99
BA $n = 1000$, $m = 4$	0.38	0.98	0.98
BA $n = 1000$, $m = 5$	0.17	0.98	0.98
Real networks			
Sister cities	0.71	0.89	0.86
US Power Grid	0.66	0.90	0.84
Euro Road	0.80	0.90	0.87
Dolphin	0.15	0.86	0.27
Contiguous USA States	0.70	0.85	0.63
Zachary karate club	0.75	0.98	0.63
Jazz Musicians	0.20	0.87	0.30
Zebra	0.67	0.51	0.26
Weighted real networks			
Les Misérables	0.06	0.73	0.05
Windsurfers	−0.41	0.64	−0.49

This table lists the Spearman correlation between the link resistance curvature $\kappa^{(RES)}$ with the Ollivier-Ricci (OR), Forman-Ricci (FR) and augmented Forman-Ricci (AFR) link curvatures respectively. The values are rounded to 2 decimal places. For model networks, the correlation value is mean over a sample of 30 networks, each generated with the specified parameters. The comparisons between the other discrete measures are listed in Supplementary Table S1. Furthermore, the Pearson correlation of all mentioned relationships can be found in Supplementary Table S5.

Table 2. Comparisons of node resistance curvature with other discrete node Ricci curvatures

Network	OR	FR	AFR
Model networks			
ER $n = 1000$, $p = 0.003$	0.80	0.82	0.82
ER $n = 1000$, $p = 0.007$	0.92	0.88	0.88
ER $n = 1000$, $p = 0.01$	0.96	0.91	0.91
WS $n = 1000$, $k = 2$, $p = 0.5$	0.63	0.69	0.69
WS $n = 1000$, $k = 8$, $p = 0.5$	0.80	0.89	0.86
WS $n = 1000$, $k = 10$, $p = 0.5$	0.79	0.91	0.87
BA $n = 1000$, $m = 2$	0.76	0.32	0.32
BA $n = 1000$, $m = 4$	0.93	0.56	0.56
BA $n = 1000$, $m = 5$	0.95	0.61	0.61
Real networks			
Sister cities	0.54	0.57	0.52
US Power Grid	0.66	0.63	0.61
Euro Road	0.62	0.55	0.55
Dolphin	0.14	0.86	0.66
Contiguous USA States	0.50	0.81	0.63
Zachary karate club	−0.18	0.66	0.31
Jazz Musicians	−0.58	0.84	0.44
Zebra	−0.37	0.67	−0.36
Weighted real networks			
Les Misérables	−0.69	0.86	−0.80
Windsurfers	−0.94	0.92	−0.87

This table lists the Spearman correlation between the node resistance curvature $p^{(RES)}$ with the Ollivier-Ricci (OR), Forman-Ricci (FR) and augmented Forman-Ricci (AFR) node curvatures respectively. The values are rounded to 2 decimal places. For model networks, the correlation value is mean over a sample of 30 networks, each generated with the specified parameters. The comparisons between the other discrete measures are listed in Supplementary Table S2. Furthermore, the Pearson correlation of all mentioned relationships can be found in Supplementary Table S6.

degree, betweenness centrality, and clustering coefficient. We found that resistance curvature has a very high negative correlation with weighted degree in nearly all models (Table 3 and SI Tables S3,S6). In particular, for both weighted graphs that we considered, the correlation was very high. This relationship seen across all networks is not surprising as resistance curvature is based on effective resistance, which is related to the weights of the graph. In both model and unweighted, real networks, we found a significant negative correlation between resistance curvature and betweenness centrality. In model networks, this correlation got stronger as the average degree of the network increased. However, this relationship was not as strong or consistent amongst the weighted real networks. In contrast, there was no consistent relationship between node resistance curvature and clustering coefficient across all types of networks.

Comparison between link resistance curvatures and other link-based measures. We compared link resistance curvature with two other link-based measures for graphs, namely edge betweenness centrality and dispersion. In both model and unweighted real networks, we found that link resistance curvature had a significant negative correlation with edge betweenness centrality (Table 4 and SI Tables S4,S8). This correlation was stronger in model networks than in unweighted real networks. In weighted networks, this relationship did not appear to be as strong. We found no consistent relationship between the link resistance and dispersion, but we found that no network had a positive correlation for these measures.

Discussion

We have performed a preliminary analysis comparing the effective resistance-based approach to discrete Ricci curvature with other approaches and other measures. While our computational experiments have been relatively extensive, further exploration of other networks, particularly weighted real networks is justified. Other limitations of our study include the small sample size of 30 for our analysis on random networks.

We also began an initial exploration into the numerical implementation of the resistance flow introduced in (10):

$$\frac{dr_{ij}}{dt} = -2(p_i + p_j) \text{ for all } i \neq j \text{ in the same component}$$

Our initial implementation appears to be working for undirected, unweighted and weighted graphs. This code can be found with the supplemental information. We aim to do a comparison with the Ollivier-Ricci flows (12) and to explore the application of this flow to community detection similar to (9). Not much progress has been made in this study so far but we hope to continue this research and aim to have a submission in the following edition of this journal.

Table 3. Comparisons of node resistance curvature with other node-based measures

Network	DEG	BC	CC
Model networks			
ER $n = 1000$, $p = 0.003$	-0.94	-0.89	-0.08
ER $n = 1000$, $p = 0.007$	-0.94	-0.93	-0.19
ER $n = 1000$, $p = 0.01$	-0.95	-0.95	-0.20
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.93	-0.76	0
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.96	-0.88	0.12
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.96	-0.89	0.15
BA $n = 1000$, $m = 2$,	-0.92	-0.70	-0.21
BA $n = 1000$, $m = 4$,	-0.94	-0.83	-0.12
BA $n = 1000$, $m = 5$,	-0.94	-0.86	0.02
Real networks			
Sister cities	-0.89	-0.87	-0.37
US Power Grid	-0.93	-0.83	-0.23
Euro Road	-0.88	-0.67	-0.24
Dolphin	-0.93	-0.85	-0.33
Contiguous USA States	-0.9	-0.86	0.72
Zachary karate club	-0.95	-0.89	0.56
Jazz Musicians	-0.87	-0.92	0.54
Zebra	-0.79	-0.96	0.96
Weighted real networks			
Les Misérables	-0.91	-0.64	-0.40
Windsurfers	-0.91	-0.46	0.01

This table lists the Spearman correlation between the node resistance curvature $p^{(RES)}$ with weighted degree (DEG), betweeness centrality (BC), and clustering coefficient (CC) respectively. The values are rounded to 2 decimal places. For model networks, the correlation value is mean over a sample of 30 networks, each generated with the specified parameters. The comparisons between the other discrete measures are listed in Supplementary Table S3. Furthermore, the Pearson correlation of all mentioned relationships can be found in Supplementary Table S7.

Table 4. Comparisons of link resistance curvature with other link-based measures

Network	EBC	DIS
Model networks		
ER $n = 1000$, $p = 0.003$	-0.75	0
ER $n = 1000$, $p = 0.007$	-0.83	-0.02
ER $n = 1000$, $p = 0.01$	-0.81	-0.05
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.49	0
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.54	-0.05
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.47	-0.05
BA $n = 1000$, $m = 2$	-0.77	-0.17
BA $n = 1000$, $m = 4$	-0.83	-0.35
BA $n = 1000$, $m = 5$	-0.84	-0.41
Real networks		
Sister cities	-0.56	-0.28
US Power Grid	-0.37	-0.16
Euro Road	-0.44	-0.07
Dolphin	-0.05	-0.11
Contiguous USA States	-0.58	-0.60
Zachary karate club	-0.63	-0.40
Jazz Musicians	-0.39	-0.23
Zebra	-0.70	-0.19
Weighted real networks		
Les Misérables	-0.04	-0.38
Windsurfers	0.41	-0.27

This table lists the Spearman correlation between the link resistance curvature $\kappa^{(RES)}$ with edge betweeness centrality (EBC) and dispersion (DIS), respectively. The values are rounded to 2 decimal places. For model networks, the correlation value is mean over a sample of 30 networks, each generated with the specified parameters. The comparisons between the other discrete measures are listed in Supplementary Table S4. Furthermore, the Pearson correlation of all mentioned relationships can be found in Supplementary Table S8.

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