PNAS
www.pnas.org

Supplementary Information for

Comparative analysis of new approach to discrete Ricci curvature on undirected graphs

Fergus O'Hanlon

Fergus O'Hanlon
E-mail: fergus.ohanlon@st-annes.ox.ac.uk

This PDF file includes:

Figs. S1 to S2
Tables S1 to S8

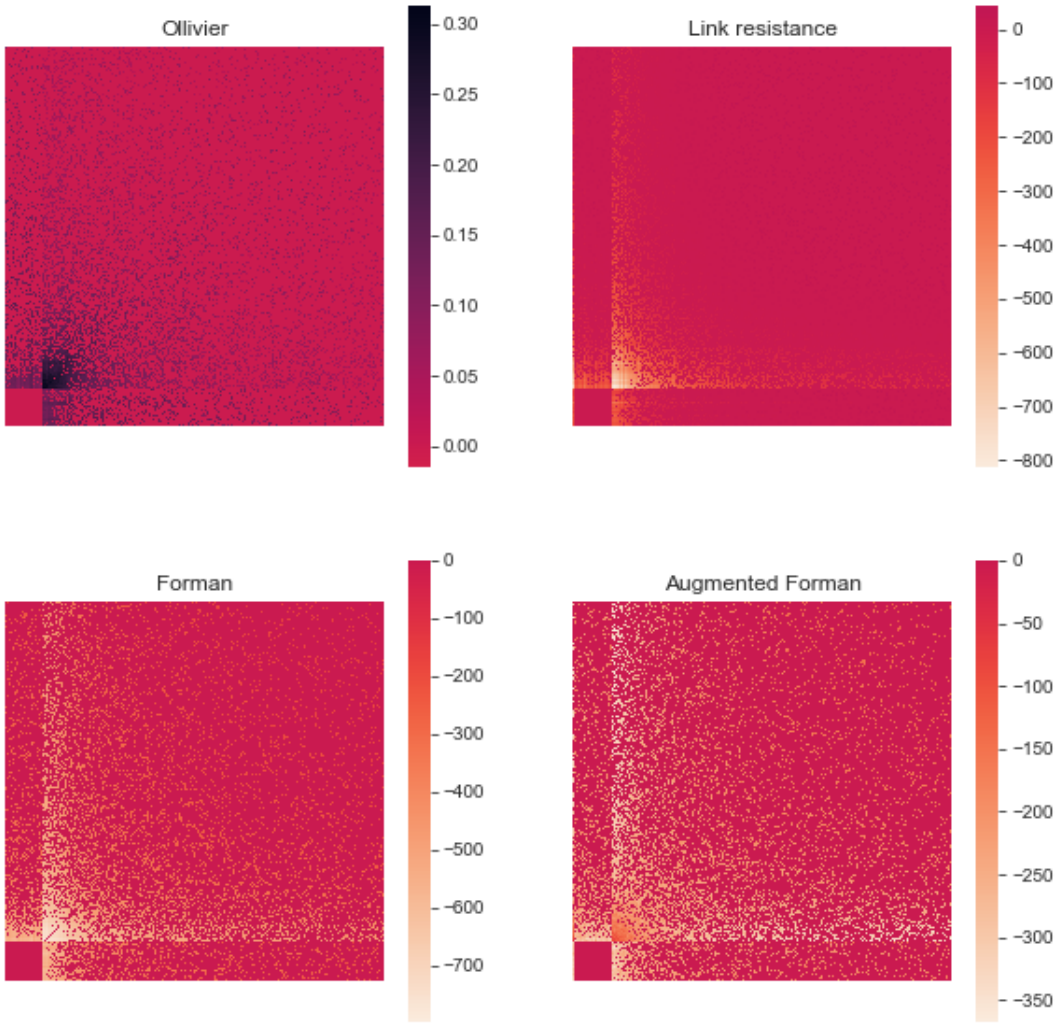

Fig. S1. Various Ricci curvatures for the Barabási-Albert (BA) model with $n = 1000$, $m = 100$ with weighting scheme as described in the main manuscript.

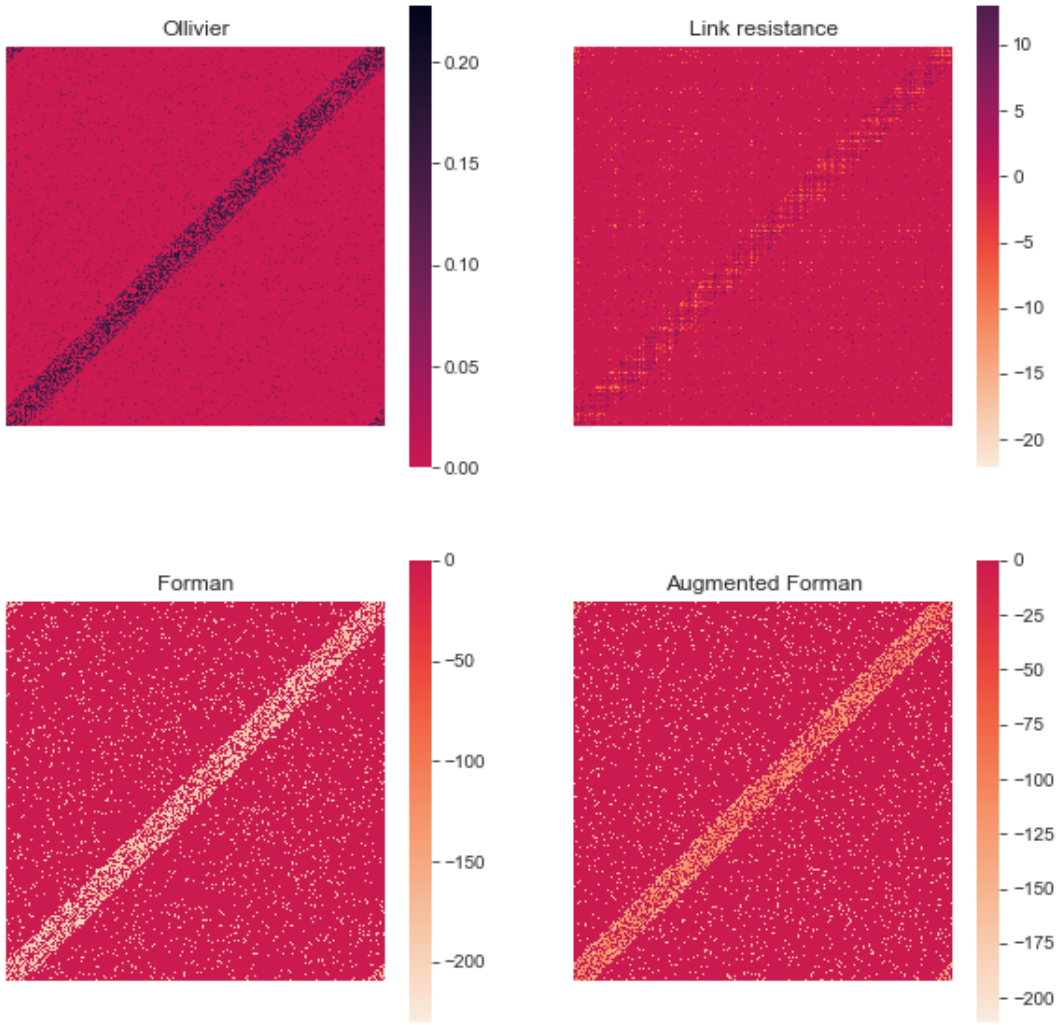

Fig. S2. Various Ricci curvatures for the Watts-Strogatz model with $n = 1000$, $k = 100$, $p = 0.5$ with weighting scheme as described in the main manuscript.

Table S1. Spearman correlations between discrete link Ricci curvatures with other discrete link Ricci curvatures

Network	OR versus FR	OR versus AFR	OR versus RES	FR versus RES	AFR versus RES
Model networks					
ER $n = 1000$, $p = 0.003$	0.89	0.9	0.81	0.91	0.91
ER $n = 1000$, $p = 0.007$	0.39	0.44	0.42	0.92	0.91
ER $n = 1000$, $p = 0.01$	-0.04	0.03	0.05	0.94	0.92
WS $n = 1000$, $k = 2$, $p = 0.5$	0.92	0.92	0.89	0.96	0.96
WS $n = 1000$, $k = 8$, $p = 0.5$	0.18	0.7	0.22	0.95	0.65
WS $n = 1000$, $k = 10$, $p = 0.5$	0.1	0.69	0.14	0.95	0.61
BA $n = 1000$, $m = 2$	0.74	0.74	0.78	0.99	0.99
BA $n = 1000$, $m = 4$	0.33	0.35	0.38	0.98	0.98
BA $n = 1000$, $m = 5$	0.13	0.16	0.17	0.98	0.98
Weighted model networks					
ER $n = 1000$, $p = 0.003$	0.79	0.8	0.79	0.84	0.84
ER $n = 1000$, $p = 0.007$	0.08	0.12	0.34	0.86	0.85
ER $n = 1000$, $p = 0.01$	-0.26	-0.2	0.02	0.89	0.87
WS $n = 1000$, $k = 2$, $p = 0.5$	0.9	0.9	0.82	0.91	0.91
WS $n = 1000$, $k = 8$, $p = 0.5$	0.1	0.62	0.22	0.87	0.63
WS $n = 1000$, $k = 10$, $p = 0.5$	0.04	0.63	0.15	0.89	0.6
BA $n = 1000$, $m = 2$	-0.01	0	0.24	0.93	0.93
BA $n = 1000$, $m = 4$	-0.19	-0.18	0.11	0.93	0.93
BA $n = 1000$, $m = 5$	-0.23	-0.21	0.06	0.93	0.93
Real networks					
Sister cities	0.67	0.76	0.71	0.89	0.86
US Power Grid	0.59	0.76	0.66	0.9	0.84
Euro Road	0.8	0.87	0.8	0.9	0.87
Dolphin	0.07	0.71	0.15	0.86	0.27
Contiguous USA States	0.69	0.91	0.7	0.85	0.63
Zachary karate club	0.69	0.91	0.7	0.85	0.63
Jazz Musicians	0.11	0.9	0.2	0.87	0.3
Zebra	-0.02	0.62	0.67	0.51	0.26
Weighted real networks					
Les Misérables	-0.29	0.77	0.06	0.73	0.05
Windsurfers	-0.82	0.92	-0.41	0.64	-0.49

Table S2. Spearman correlations between discrete node Ricci curvatures with other discrete node Ricci curvatures

Network	OR versus FR	OR versus AFR	OR versus RES	FR versus RES	AFR versus RES
Model networks					
ER $n = 1000$, $p = 0.003$	0.97	0.97	0.8	0.82	0.82
ER $n = 1000$, $p = 0.007$	0.97	0.97	0.92	0.88	0.88
ER $n = 1000$, $p = 0.01$	0.96	0.96	0.96	0.91	0.91
WS $n = 1000$, $k = 2$, $p = 0.5$	0.89	0.89	0.63	0.69	0.69
WS $n = 1000$, $k = 8$, $p = 0.5$	0.79	0.93	0.8	0.89	0.86
WS $n = 1000$, $k = 10$, $p = 0.5$	0.77	0.92	0.79	0.91	0.87
BA $n = 1000$, $m = 2$	0.6	0.6	0.76	0.32	0.32
BA $n = 1000$, $m = 4$	0.6	0.6	0.93	0.56	0.56
BA $n = 1000$, $m = 5$	0.63	0.64	0.95	0.61	0.61
Weighted model networks					
ER $n = 1000$, $p = 0.003$	0.95	0.95	0.84	0.8	0.8
ER $n = 1000$, $p = 0.007$	0.9	0.9	0.96	0.88	0.88
ER $n = 1000$, $p = 0.01$	0.85	0.85	0.96	0.91	0.91
WS $n = 1000$, $k = 2$, $p = 0.5$	0.93	0.94	0.7	0.67	0.67
WS $n = 1000$, $k = 8$, $p = 0.5$	0.74	0.87	0.83	0.89	0.86
WS $n = 1000$, $k = 10$, $p = 0.5$	0.71	0.85	0.81	0.91	0.87
BA $n = 1000$, $m = 2$	0	0	0.68	0.45	0.45
BA $n = 1000$, $m = 4$	0.03	0.04	0.62	0.65	0.65
BA $n = 1000$, $m = 5$	0.09	0.09	0.63	0.7	0.71
Real networks					
Sister cities	0.91	0.95	0.54	0.57	0.52
US Power Grid	0.67	0.82	0.66	0.63	0.61
Euro Road	0.89	0.91	0.62	0.55	0.55
Dolphin	0.04	0.49	0.14	0.86	0.66
Contiguous USA States	0.61	0.89	0.5	0.81	0.63
Zachary karate club	0.24	0.7	-0.18	0.66	0.31
Jazz Musicians	-0.79	0.02	-0.58	0.84	0.44
Zebra	-0.72	0.99	-0.37	0.67	-0.36
Weighted real networks					
Les Misérables	-0.72	0.82	-0.69	0.86	-0.8
Windsurfers	-0.93	0.95	-0.94	0.92	-0.87

Table S3. Spearman correlations between node resistance curvature with other node-based measures

Network	DEG	BC	CC
Model networks			
ER $n = 1000$, $p = 0.003$	-0.94	-0.89	-0.08
ER $n = 1000$, $p = 0.007$	-0.94	-0.93	-0.19
ER $n = 1000$, $p = 0.01$	-0.95	-0.95	-0.20
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.93	-0.76	0
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.96	-0.88	0.12
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.96	-0.89	0.15
BA $n = 1000$, $m = 2$,	-0.92	-0.70	-0.21
BA $n = 1000$, $m = 4$,	-0.94	-0.83	-0.12
BA $n = 1000$, $m = 5$,	-0.94	-0.86	0.02
Weighted model networks			
ER $n = 1000$, $p = 0.003$	-0.88	-0.93	-0.08
ER $n = 1000$, $p = 0.007$	-0.91	-0.92	-0.2
ER $n = 1000$, $p =$0.01	-0.94	-0.78	-0.21
Real networks			
Sister cities	-0.89	-0.87	-0.37
US Power Grid	-0.93	-0.83	-0.23
Euro Road	-0.88	-0.67	-0.24
Dolphin	-0.93	-0.85	-0.33
Contiguous USA States	-0.9	-0.86	0.72
Zachary karate club	-0.95	-0.89	0.56
Jazz Musicians	-0.87	-0.92	0.54
Zebra	-0.79	-0.96	0.96
Weighted real networks			
Les Misérables	-0.91	-0.64	-0.40
Windsurfers	-0.91	-0.46	0.01

Table S4. Spearman correlations between of link resistance curvature with other link-based measures

Network	EBC	DIS
Model networks		
ER $n = 1000$, $p = 0.003$	-0.75	0
ER $n = 1000$, $p = 0.007$	-0.83	-0.02
ER $n = 1000$, $p = 0.01$	-0.81	-0.05
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.49	0
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.54	-0.05
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.47	-0.05
BA $n = 1000$, $m = 2$	-0.77	-0.17
BA $n = 1000$, $m = 4$	-0.83	-0.35
BA $n = 1000$, $m = 5$	-0.84	-0.41
Weighted model networks		
ER $n = 1000$, $p = 0.003$	-0.68	nan
ER $n = 1000$, $p = 0.007$	-0.28	-0.03
ER $n = 1000$, $p = 0.01$	-0.1	-0.05
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.5	nan
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.24	-0.05
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.16	-0.05
BA $n = 1000$, $m = 2$	-0.32	-0.16
BA $n = 1000$, $m = 4$	-0.45	-0.35
BA $n = 1000$, $m = 5$	-0.46	-0.41
Real networks		
Sister cities	-0.56	-0.28
US Power Grid	-0.37	-0.16
Euro Road	-0.44	-0.07
Dolphin	-0.05	-0.11
Contiguous USA States	-0.58	-0.60
Zachary karate club	-0.63	-0.40
Jazz Musicians	-0.39	-0.23
Zebra	-0.70	-0.19
Weighted real networks		
Les Misérables	-0.04	-0.38
Windsurfers	0.41	-0.27

Table S5. Pearson correlations between discrete link Ricci curvatures with other discrete link Ricci curvatures

Network	OR versus FR	OR versus AFR	OR versus RES	FR versus RES	AFR versus RES
Model networks					
ER $n = 1000$, $p = 0.003$	0.79	0.79	0.62	0.9	0.9
ER $n = 1000$, $p = 0.007$	0.42	0.47	0.36	0.9	0.89
ER $n = 1000$, $p = 0.01$	-0.02	0.06	0.04	0.92	0.9
WS $n = 1000$, $k = 2$, $p = 0.5$	0.85	0.85	0.85	1	1
WS $n = 1000$, $k = 8$, $p = 0.5$	0.18	0.76	0.21	0.94	0.63
WS $n = 1000$, $k = 10$, $p = 0.5$	0.11	0.74	0.14	0.95	0.59
BA $n = 1000$, $m = 2$	0.51	0.51	0.35	0.69	0.67
BA $n = 1000$, $m = 4$	0.17	0.2	-0.04	0.75	0.71
BA $n = 1000$, $m = 5$	-0.02	0.02	-0.2	0.77	0.72
Weighted model networks					
ER $n = 1000$, $p = 0.003$	0.7	0.7	0.6	0.84	0.84
ER $n = 1000$, $p = 0.007$	0.12	0.16	0.3	0.84	0.83
ER $n = 1000$, $p = 0.01$	-0.25	-0.19	0.01	0.87	0.85
WS $n = 1000$, $k = 2$, $p = 0.5$	0.85	0.85	0.76	0.94	0.94
WS $n = 1000$, $k = 8$, $p = 0.5$	0.11	0.68	0.22	0.87	0.62
WS $n = 1000$, $k = 10$, $p = 0.5$	0.05	0.67	0.15	0.88	0.59
BA $n = 1000$, $m = 2$	-0.01	0	0.25	0.61	0.6
BA $n = 1000$, $m = 4$	-0.16	-0.15	0.07	0.67	0.63
BA $n = 1000$, $m = 5$	-0.2	-0.19	0	0.71	0.67
Real networks					
Sister cities	0.34	0.47	0.21	0.76	0.7
US Power Grid	0.52	0.69	0.54	0.9	0.81
Euro Road	0.72	0.77	0.67	0.92	0.88
Dolphin	0.1	0.73	0.08	0.79	0.09
Contiguous USA States	0.69	0.89	0.69	0.86	0.69
Zachary karate club	0.71	0.81	0.44	0.87	0.28
Jazz Musicians	0.13	0.86	0.2	0.83	0.32
Zebra	-0.1	0.79	0.51	0.27	0.27
Weighted real networks					
Les Misérables	-0.44	0.45	-0.17	0.86	-0.62
Windsurfers	-0.81	0.36	-0.45	0.79	-0.83

Table S6. Pearson correlations between discrete node Ricci curvatures with other discrete node Ricci curvatures

Network	OR versus FR	OR versus AFR	OR versus RES	FR versus RES	AFR versus RES
Model networks					
ER $n = 1000$, $p = 0.003$	0.94	0.94	0.78	0.78	0.78
ER $n = 1000$, $p = 0.007$	0.96	0.96	0.92	0.87	0.87
ER $n = 1000$, $p = 0.01$	0.95	0.95	0.96	0.9	0.91
WS $n = 1000$, $k = 2$, $p = 0.5$	0.86	0.86	0.76	0.82	0.82
WS $n = 1000$, $k = 8$, $p = 0.5$	0.82	0.93	0.82	0.91	0.87
WS $n = 1000$, $k = 10$, $p = 0.5$	0.8	0.92	0.81	0.93	0.88
BA $n = 1000$, $m = 2$	0.89	0.89	0.98	0.86	0.86
BA $n = 1000$, $m = 4$	0.89	0.89	0.99	0.88	0.89
BA $n = 1000$, $m = 5$	0.9	0.9	0.99	0.9	0.9
Weighted model networks					
ER $n = 1000$, $p = 0.003$	0.91	0.92	0.81	0.73	0.73
ER $n = 1000$, $p = 0.007$	0.88	0.88	0.95	0.85	0.85
ER $n = 1000$, $p = 0.01$	0.84	0.84	0.96	0.9	0.9
WS $n = 1000$, $k = 2$, $p = 0.5$	0.89	0.9	0.78	0.76	0.76
WS $n = 1000$, $k = 8$, $p = 0.5$	0.76	0.88	0.85	0.9	0.87
WS $n = 1000$, $k = 10$, $p = 0.5$	0.73	0.86	0.83	0.92	0.89
BA $n = 1000$, $m = 2$	0.75	0.76	0.94	0.87	0.87
BA $n = 1000$, $m = 4$	0.73	0.74	0.91	0.9	0.9
BA $n = 1000$, $m = 5$	0.73	0.73	0.89	0.91	0.91
Real networks					
Sister cities	0.59	0.72	0.66	0.43	0.45
US Power Grid	0.66	0.81	0.74	0.7	0.77
Euro Road	0.86	0.88	0.76	0.71	0.72
Dolphin	0.02	0.51	0.13	0.81	0.67
Contiguous USA States	0.74	0.92	0.58	0.78	0.65
Zachary karate club	0.53	0.7	0.44	0.97	0.92
Jazz Musicians	-0.57	0.03	-0.39	0.83	0.69
Zebra	-0.89	0.99	-0.14	0.49	-0.18
Weighted real networks					
Les Misérables	-0.58	0.67	-0.37	0.86	-0.69
Windsurfers	-0.95	0.84	-0.95	0.94	-0.89

Table S7. Pearson correlations between node resistance curvature with other node-based measures

Network	DEG	BC	CC
Model networks			
ER $n = 1000$, $p = 0.003$	-0.94	-0.77	-0.01
ER $n = 1000$, $p = 0.007$	-0.93	-0.9	0.02
ER $n = 1000$, $p = 0.01$	-0.95	-0.94	0.02
WS $n = 1000$, $k = 2$, $p = 0.5$	-1	-0.37	nan
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.96	-0.88	0.21
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.97	-0.89	0.21
BA $n = 1000$, $m = 2$	-0.99	-0.92	0.05
BA $n = 1000$, $m = 4$	-0.99	-0.92	0.1
BA $n = 1000$, $m = 5$	-0.99	-0.92	0.12
Weighted model networks			
ER $n = 1000$, $p = 0.003$	-0.88	-0.87	-0.01
ER $n = 1000$, $p = 0.007$	-0.91	-0.9	0.02
ER $n = 1000$, $p = 0.01$	-0.94	-0.76	0.01
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.95	-0.37	nan
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.94	-0.76	0.2
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.95	-0.62	0.21
BA $n = 1000$, $m = 2$	-0.94	-0.95	0.05
BA $n = 1000$, $m = 4$	-0.97	-0.92	0.07
BA $n = 1000$, $m = 5$	-0.98	-0.91	0.08
Real networks			
Sister cities	-0.73	-0.64	-0.01
US Power Grid	-0.91	-0.23	0.06
Euro Road	-0.91	-0.4	-0.04
Dolphin	-0.88	-0.63	-0.24
Contiguous USA States	-0.9	-0.75	0.65
Zachary karate club	-0.98	-0.94	0.5
Jazz Musicians	-0.86	-0.81	0.42
Zebra	-0.57	-0.86	0.89
Weighted real networks			
Les Misérables	-0.77	-0.88	0.05
Windsurfers	-0.96	-0.16	-0.1

Table S8. Pearson correlations between node resistance curvature with other link-based measures

Network	EBC	DIS
Model networks		
ER $n = 1000$, $p = 0.003$	-0.76	nan
ER $n = 1000$, $p = 0.007$	-0.83	-0.02
ER $n = 1000$, $p = 0.01$	-0.8	-0.05
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.26	nan
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.57	-0.06
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.5	-0.06
BA $n = 1000$, $m = 2$	-0.91	-0.65
BA $n = 1000$, $m = 4$	-0.83	-0.79
BA $n = 1000$, $m = 5$	-0.78	-0.81
Weighted model networks		
ER $n = 1000$, $p = 0.003$	-0.67	nan
ER $n = 1000$, $p = 0.007$	-0.3	-0.03
ER $n = 1000$, $p = 0.01$	-0.11	-0.06
WS $n = 1000$, $k = 2$, $p = 0.5$	-0.26	nan
WS $n = 1000$, $k = 8$, $p = 0.5$	-0.24	-0.06
WS $n = 1000$, $k = 10$, $p = 0.5$	-0.15	-0.05
BA $n = 1000$, $m = 2$	-0.62	-0.68
BA $n = 1000$, $m = 4$	-0.54	-0.81
BA $n = 1000$, $m = 5$	-0.48	-0.81
Real networks		
Sister cities	-0.26	-0.5
US Power Grid	-0.12	-0.28
Euro Road	-0.26	-0.11
Dolphin	-0.01	-0.22
Contiguous USA States	-0.56	-0.56
Zachary karate club	-0.38	-0.76
Jazz Musicians	-0.26	-0.26
Zebra	-0.42	-0.13
Weighted real networks		
Les Misérables	-0.11	-0.57
Windsurfers	0.25	-0.26