Proposal

Typically, we face the problem of solving

$$V_0(x_0) = \sup_{\mathbf{c}_0} \mathbf{E}[\sum_{t=0}^{T-1} \delta^t u(\mathbf{X}_t, c_t) + U(\mathbf{X}_T)\delta^T]$$

such that $\mathbf{c}_0 = \{c_0, c_1, ...c_{T-1}\}$ and the state evolves according to

$$\mathbf{X}_{t+1} = g(\mathbf{X}_t, \varepsilon_t) \in \mathbf{R}^d$$

[1] exogenous shocks $\varepsilon_t \in \mathbf{R}^d$.

Rewriting in the Bellman equation format, we have

$$V_t(x) = \sup_{\mathbf{c}_t} u(x, c_t) + \mathbf{E}[V_{t+1}(\mathbf{X}_{t+1})|\mathbf{X}_t = \mathbf{x}_t]$$

with the boundary condition

$$V_T(x) = U(x).$$

In order to solve this, the dynamic problem start at $T-1$, one needs to solve the optimization problem at each time and iterate the valuation backwards. In simple dynamic program, problems arise when the dimension d is large. At each time, t, the $\mathbb{E}_t$ is conditioning on $\sigma(\mathbf{X}_t)$, one cannot easily derive it analytically, hence making the optimization cumbersome to solve. Numerical approximation schemes, e.g. Monte-Carlo, can be used together with dynamic problem for solving this type of problems, but its numerical performance and theoretical properties are highly dependent on the structure of the problem considered.

[1] In most cases, the control also affects the evolution of states. A simple change of variable condition allows us to “push-out" the control of the state evolution.

Instead, we consider a single index

$$Y_t = s(\mathbf{X}_t, \beta_t) \in \mathbf{R}$$

and rewrite the Bellman equation into

$$V_t(y) = \sup_{\mathbf{c}_t, \boldsymbol{\beta}_t} u(y, c_t) + \mathbb{E}[V_{t+1}(Y_{t+1})|Y_t = y] \text{ for } \boldsymbol{\beta}_t = \{\beta_t, \beta_{t+1}, ..., \beta_T\} \tag{1}$$

with the boundary condition

$$V_T(y) = U(y).$$

such that

$$Y_{t+1} = s(g(\mathbf{X}_t, \varepsilon_t), \beta_{t+1}).$$

We have now effectively reduced the d-dimensional problem into one, and know the problem becomes to finding the distribution $F_t(y; \beta_t)$ of Y_t in order to evaluate the integral. Our idea is to use the empirical distribution $F_t^n(y; \beta_t)$.

Suppose given this empirical distribution, the optimal controls in equation (1) can be solved exactly. We would like to analysis the convergence behaviour of the resulting $V_0(x_0)^n$ to the true $V_0(x_0)$.

Problem to Solve

- Conditions for the validity of equation (1)
- Given F_t^n, can we evaluate $\mathbb{E}_t$.

Consider the typical high dimensional portfolio selection problem in terms of random process, we can get

$$V_0(x_0) = \sup_{\mathbf{c}_0, \beta_0} \mathbf{E}[u(C_0(W_0, c_0)) + \sum_{t=1}^{T-1} u(C_t(W_t(\beta_{t-1}, \mathbf{r}_{t-1}), c_t))\delta^t + u(W_T(\beta_{T-1}, \mathbf{r}_{T-1}))\delta^T]$$

where

- $\mathbf{r}_t$ is a vector of returns of different assets at time t.
- β_t is a vector of weight of different assets in this portfolio.
- W_t is the wealth at time t, a function of return $\mathbf{r}_{t-1}$ and weight β_{t-1} which are observed or determined from the last period.
- c_t is the control variable (or a vector of the control variables) which in turn determines consumption C_t.
- u is the utility function depending on consumption or the terminal wealth.

$$C_t = W_t \times \frac{\exp}{}$$