Analytical Solution with Independent Return

W_t is the wealth just after consumption is deducted from the portfolio. Its evolution follows

$$\begin{aligned} W_{t+1} &= (W_t - \beta_t) \cdot R^f + \beta_t \cdot R_t^r - C_{t+1} \\ &= W_t \cdot R^f + \beta_t(R_t^r - R^f) - C_{t+1} \end{aligned}$$

where

- R^f is the accumulation factor for risk free asset from time t to $t+1$
- R_t^r is the accumulation factor for the risky portfolio from time t to $t+1$
- β_t is the amount of wealth invested in the risky portfolio at time t
- C_{t+1} is the consumption made at time $t+1$

Value function at time 0:

$$V_0(W_0) = \min_{\{C_t\}_{t=1}^{T}, \{\beta_t\}_{t=0}^{T-1}} E[\sum_{t=1}^{T} \delta^t \cdot (C_t^2 - 2\lambda C_t) + \delta^T \cdot (W_T^2 - 2\lambda W_T)]$$

with terminal condition

$$V_T(W_T) = W_T^2 - 2\lambda W_T$$

where

- λ is added for the convenience of calculation

Value function at time t:

$$\begin{aligned} V_t(W_t) &= \min_{\{C_s\}_{s=t}^{T}, \{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^{T} \delta^s \cdot (C_s^2 - 2\lambda C_s) + \delta^T \cdot (W_T^2 - 2\lambda W_T)] \\ &= \min_{\{C_s\}_{s=t}^{T}, \{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^{T} \delta^s \cdot ((C_s - \lambda)^2 - \lambda^2) + \delta^T \cdot ((W_T - \lambda)^2 - \lambda^2)] \end{aligned}$$

The second equality comes from completion of squares.

Define

$$\widetilde{W_t} = W_t - \lambda \qquad c_t = C_t - \lambda$$

Then the evolution of wealth becomes

$$\widetilde{W}_{t+1} = \widetilde{W}_t \cdot R^f + \beta_t(R_t^r - R^f) - c_{t+1} + (R^f - 2) \cdot \lambda$$

After the transformation, the value function at time t and terminal condition becomes

$$\begin{cases} V_t(W_t) = \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^{T} \delta^{s-t} \cdot (c_s^2 - \lambda^2) + \delta^{T-t} \cdot (\widetilde{W}_T^2 - \lambda^2)] \\ \\ V_T(W_T) = \widetilde{W}_T^2 - \lambda^2 \end{cases}$$

Now define

$$J_t = \begin{bmatrix} R_t^r - R^f \\ -1 \end{bmatrix} \qquad Z_t = \begin{bmatrix} \beta_t \\ c_{t+1} \end{bmatrix} \qquad \mathbb{I}^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The evolution of wealth becomes

$$\widetilde{W}_{t+1} = \widetilde{W}_t \cdot R^f + J_t' Z_t + (R^f - 2) \cdot \lambda$$

The value function at time t and terminal condition:

$$\begin{cases} V_t(W_t) = \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t}^{T-1} \delta^{s+1-t} \cdot (Z_t' \mathbb{I}^{22} Z_t - \lambda^2) + \delta^{T-t} \cdot (\widetilde{W}_T^2 - \lambda^2)] \\ \\ V_T(W_T) = \widetilde{W}_T^2 - \lambda^2 \end{cases}$$

The Bellman equation for this problem would be

$$\begin{aligned} V_t(W_t) &= \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t}^{T-1} \delta^{s+1-t} \cdot (Z_t' \mathbb{I}^{22} Z_t - \lambda^2) + \delta^{T-t} \cdot (\widetilde{W}_T^2 - \lambda^2)] \\ &= \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\delta \cdot (Z_t' \mathbb{I}^{22} Z_t - \lambda^2) + \sum_{s=t+1}^{T-1} \delta^{s+1-t} \cdot (Z_s' \mathbb{I}^{22} Z_s - \lambda^2) + \delta^{T-t} \cdot (\widetilde{W}_T^2 - \lambda^2)] \\ &= \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\delta \cdot (Z_t' \mathbb{I}^{22} Z_t - \lambda^2) + E[\sum_{s=t+1}^{T-1} \delta^{s+1-t} \cdot (Z_s' \mathbb{I}^{22} Z_s - \lambda^2) + \delta^{T-t} \cdot (\widetilde{W}_T^2 - \lambda^2)]] \\ &= \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\delta \cdot (Z_t' \mathbb{I}^{22} Z_t - \lambda^2) + \delta \cdot E[V_{t+1}(W_{t+1})]] \end{aligned}$$

The third equality comes from Law of Iterated Expection conditional on the known information at time t