

# Proposal

Typically, we face the problem of solving

$$V_0(x_0) = \sup_{\mathbf{c}_0} \mathbf{E}[\sum_{t=0}^{T-1} \delta^t u(\mathbf{X}_t, c_t) + U(\mathbf{X}_T) \delta^T]$$

such that  $\mathbf{c}_0 = \{c_0, c_1, \dots, c_{T-1}\}$  and the state evolves according to

$$\mathbf{X}_{t+1} = g(\mathbf{X}_t, \varepsilon_t) \in \mathbf{R}^d$$

<sup>1</sup> exogenous shocks  $\varepsilon_t \in \mathbf{R}^d$ .

Rewriting in the Bellman equation format, we have

$$V_t(x) = \sup_{\mathbf{c}_t} u(x, c_t) + \mathbf{E}[V_{t+1}(\mathbf{X}_{t+1}) | \mathbf{X}_t = \mathbf{x}_t]$$

with the boundary condition

$$V_T(x) = U(x).$$

In order to solve this, the dynamic problem start at  $T - 1$ , one needs to solve the optimization problem at each time and iterate the valuation backwards. In simple dynamic program, problems arise when the dimension  $d$  is large. At each time,  $t$ , the  $\mathbb{E}_t$  is conditioning on  $\sigma(\mathbf{X}_t)$ , one cannot easily derive it analytically, hence making the optimization cumbersome to solve. Numerical approximation schemes, e.g. Monte-Carlo, can be used together with dynamic problem for solving this type of problems, but its numerical performance and theoretical properties are highly dependent on the structure of the problem considered.

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<sup>1</sup>In most cases, the control also affects the evolution of states. A simple change of variable condition allows us to “push-out” the control of the state evolution.

Instead, we consider a single index

$$Y_t = s(\mathbf{X}_t, \beta_t) \in \mathbf{R}$$

and rewrite the Bellman equation into

$$V_t(y) = \sup_{\mathbf{c}_t, \beta_t} u(y, c_t) + \mathbb{E}[V_{t+1}(Y_{t+1}) | Y_t = y] \text{ for } \beta_t = \{\beta_t, \beta_{t+1}, \dots, \beta_T\} \quad (1)$$

with the boundary condition

$$V_T(y) = U(y).$$

such that

$$Y_{t+1} = s(g(\mathbf{X}_t, \varepsilon_t), \beta_{t+1}).$$

We have now effectively reduced the d-dimensional problem into one, and know the problem becomes to finding the distribution  $F_t(y; \beta_t)$  of  $Y_t$  in order to evaluate the integral. Our idea is to use the empirical distribution  $F_t^n(y; \beta_t)$ .

Suppose given this empirical distribution, the optimal controls in equation (1) can be solved exactly. We would like to analysis the convergence behaviour of the resulting  $V_0(x_0)^n$  to the true  $V_0(x_0)$ .

### **Problem to Solve**

- Conditions for the validity of equation (1)
- Given  $F_t^n$ , can we evaluate  $\mathbb{E}_t$ .

Consider the typical high dimensional portfolio selection problem in terms of random process, we can get

$$V_0(x_0) = \sup_{\mathbf{c}_0, \beta_0} \mathbf{E}[u(C_0(W_0, c_0)) + \sum_{j=1}^{T-1} u(C_j(W_j(\beta_{j-1}, \mathbf{r}_{j-1}), c_j))\delta^j + u(W_T(\beta_{T-1}, \mathbf{r}_{T-1}))\delta^T]$$

contional on the boundary condition

$$V_T(x_T) = \sup_{\beta_{T-1}} \mathbf{E}[u(W_T(\beta_{T-1}, \mathbf{r}_{T-1}))]$$

where

- $\mathbf{r}_t$  is a vector of returns of different assets at time  $t$ .
- $\beta_t$  is a vector of weight of different assets in this portfolio.
- $W_t$  is the wealth at time  $t$ , a function of return  $\mathbf{r}_{t-1}$  and weight  $\beta_{t-1}$  which are observed or determined from the last period.
- $c_t$  is the a vector of 2 control variables which in turn determines consumption  $C_t$ .
- $u$  is the utility function depending on consumption or the terminal wealth.

The state of  $W_t$  evolves according to

$$W_{t+1} = (W_t - C_t)(\beta_t \cdot \mathbf{r}_t + 1)$$

$$C_t = W_t \cdot \frac{\exp\{c_{1,t} + c_{2,t}W_t\}}{1 + \exp\{c_{1,t} + c_{2,t}W_t\}}$$

where we assume the consumption is propotional to wealth and the propotion is calculated by applying inverse logit function.

The utility function is assumed to be  $u(x) = \frac{x^{1-\lambda}}{1-\lambda}$ .

If we write it into Bellman equation

$$V_t(x_t) = \sup_{\mathbf{c}_t, \beta_t} \mathbf{E}[\sum_{j=t}^{T-1} u(C_j(W_j(\beta_{j-1}, \mathbf{r}_{j-1}), c_j))\delta^{j-t} + u(W_T(\beta_{T-1}, \mathbf{r}_{T-1}))\delta^{T-t}]$$

This example is a typical high-dimensional optimization problem which cannot be sloved under the current literature, especially when the number of assets to be chosen is quite big. Therefore, we take a novel two steps approach where we first compute the weights using lasso and then using dynamic programming to calculate the appropriate control variable  $c_t$ .