Analytical Solution with Independent Return

W_t is the wealth just after consumption is deducted from the portfolio. Its evolution follows

$$\begin{aligned} W_{t+1} &= (W_t - \beta_t) \cdot R^f + \beta_t \cdot R_t^r - C_{t+1} \\ &= W_t \cdot R^f + \beta_t (R_t^r - R^f) - C_{t+1} \end{aligned}$$

where

- R^f is the accumulation factor for risk free asset from time t to $t+1$
- R_t^r is the accumulation factor for the risky portfolio from time t to $t+1$
- β_t is the amount of wealth invested in the risky portfolio at time t
- C_{t+1} is the consumption made at time $t+1$

The objective function at time 0:

$$f_0(W_0) = \min_{\{C_t\}_{t=1}^T, \{\beta_t\}_{t=0}^{T-1}} E[\sum_{t=1}^T \delta^t \cdot (C_t^2 - 2\lambda C_t) + \delta^T \cdot (W_T^2 - 2\lambda W_T)]$$

with terminal condition

$$f_T(W_T) = W_T^2 - 2\lambda W_T$$

where

- λ is added for the convenience of calculation

The objective function at time t:

$$\begin{aligned} f_t(W_t) &= \min_{\{C_s\}_{s=t}^T, \{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^T \delta^s \cdot (C_s^2 - 2\lambda C_s) + \delta^T \cdot (W_T^2 - 2\lambda W_T)] \\ &= \min_{\{C_s\}_{s=t}^T, \{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^T \delta^s \cdot ((C_s - \lambda)^2 - \lambda^2) + \delta^T \cdot ((W_T - \lambda)^2 - \lambda^2)] \end{aligned}$$

The second equality comes from completion of squares.

Define

$$\widetilde{W_t} = W_t - \lambda \qquad c_t = C_t - \lambda$$

Then the evolution of wealth becomes

$$\widetilde{W}_{t+1} = \widetilde{W}_t \cdot R^f + \beta_t (R_t^r - R^f) - c_{t+1} + (R^f - 2) \cdot \lambda$$

After the transformation, the objective function at time t and terminal condition becomes

$$\begin{cases} f_t(W_t) = \min_{\{C_s\}_{s=t}^{T},\{\beta_s\}_{s=t}^{T-1}} E[\sum_{s=t+1}^{T} \delta^{s-t} \cdot (c_s^2 - \lambda^2) + \delta^{T-t} \cdot (\widetilde{W}_T^2 - \lambda^2)] \\ \\ f_T(W_T) = \widetilde{W}_T^2 - \lambda^2 \end{cases}$$

Now define

$$J_t = \begin{bmatrix} R_t^r - R^f \\ -1 \end{bmatrix} \qquad Z_t = \begin{bmatrix} \beta_t \\ c_{t+1} \end{bmatrix} \qquad \mathbb{I}^{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The evolution of wealth becomes

$$\widetilde{W}_{t+1} = \widetilde{W}_t \cdot R^f + J_t' Z_t + (R^f - 2) \cdot \lambda$$

The objective function at time t and terminal condition:

$$\begin{cases} f_t(W_t) = \min_{\{Z_s\}_{s=t}^{T}} E[\sum_{s=t}^{T-1} \delta^{s+1-t} \cdot (Z_t' \mathbb{I}^{22} Z_t - \lambda^2) + \delta^{T-t} \cdot (\widetilde{W}_T^2 - \lambda^2)] \\ \\ f_T(W_T) = \widetilde{W}_T^2 - \lambda^2 \end{cases} \tag{1}$$

Then the optimization problem (1) is equivalent to the following problem

$$\begin{cases} V_t(W_t) = \min_{\{Z_s\}_{s=t}^{T}} E[\sum_{s=t}^{T-1} \delta^{s+1-t} Z_t' \mathbb{I}^{22} Z_t + \delta^{T-t} \widetilde{W}_T^2] \\ \\ V_T(W_T) = \widetilde{W}_T^2 \end{cases}$$

We have $f_t(W_t) = V_t(W_t) - \lambda^2 (\sum_{s=t}^{T-1} \delta^{s+1-t} + \delta^{T-t})$

The Bellman equation for this problem would be

$$\begin{aligned} V_t(W_t) &= \min_{\{Z_s\}_{s=t}^{T}} E[\sum_{s=t}^{T-1} \delta^{s+1-t} Z_t' \mathbb{I}^{22} Z_t + \delta^{T-t} \widetilde{W}_T^2] \\ &= \min_{\{Z_s\}_{s=t}^{T}} E[\delta Z_t' \mathbb{I}^{22} Z_t + \sum_{s=t+1}^{T-1} \delta^{s+1-t} Z_s' \mathbb{I}^{22} Z_s + \delta^{T-t} \widetilde{W}_T^2] \\ &= \min_{\{Z_s\}_{s=t}^{T}} E[\delta Z_t' \mathbb{I}^{22} Z_t + E[\sum_{s=t+1}^{T-1} \delta^{s+1-t} Z_s' \mathbb{I}^{22} Z_s + \delta^{T-t} \widetilde{W}_T^2]] \\ &= \min_{\{Z_s\}_{s=t}^{T}} E[\delta Z_t' \mathbb{I}^{22} Z_t + \delta \cdot E[V_{t+1}(W_{t+1})]] \end{aligned}$$

The third equality comes from Law of Iterated Expection conditional on the known information at time t.

At time $T-1$

$$
\begin{aligned}
V_{T-1}(W_{T-1}) &= \min_{C_{T-1},\beta_{T-1}} E[\delta Z'_{T-1}\mathbb{I}^{22}Z_{T-1} + \delta \cdot E[V_T(W_T)]] \\
&= \min_{C_{T-1},\beta_{T-1}} E[\delta Z'_{T-1}\mathbb{I}^{22}Z_{T-1} + \delta \cdot E[\widetilde{W}_T^2]] \qquad (2)
\end{aligned}
$$

Set

$$
\psi_{T-1}(W_{T-1}) = E[\delta Z'_{T-1}\mathbb{I}^{22}Z_{T-1} + \delta \cdot E[\widetilde{W}_T^2]]
$$

We solve the FOC for the above euqation as follows

$$
\begin{aligned}
\frac{\partial \psi_{T-1}(W_{T-1})}{\partial Z_{T-1}} &= 2\delta\mathbb{I}^{22}Z_{T-1} + 2\delta E[\widetilde{W}_T \frac{\partial \widetilde{W}_T}{\partial Z_{T-1}}] \\
&= 2\delta\mathbb{I}^{22}Z_{T-1} + 2\delta E[(\widetilde{W}_{T-1} \cdot R^f + J'_{T-1}Z_{T-1} + (R^f-2)\cdot\lambda)J_{T-1}]
\end{aligned}
$$

$$
Z_{T-1} = -(\mathbb{I}^{22} + E[J_{T-1}J'_{T-1}])^{-1}(\widetilde{W}_{T-1}R^f E[J_{T-1}] + (R^f-2)\lambda E[J_{T-1}]) \qquad (3)
$$

which is the solution at $T-1$, provided $\mathbb{I}^{22} + E[J_{T-1}J'_{T-1}]$ is invertible where

$$
J_{T-1}J'_{T-1} = \begin{bmatrix} R^r_{T-1} - R^f \\ -1 \end{bmatrix} \begin{bmatrix} R^r_{T-1} - R^f & -1 \end{bmatrix} = \begin{bmatrix} (R^r_{T-1} - R^f)^2 & -(R^r_{T-1} - R^f) \\ -(R^r_{T-1} - R^f) & 1 \end{bmatrix}
$$

Substituting (3) into (2) and simplifying it, we obtain

$$
\begin{aligned}
&V_{T-1}(W_{T-1}) \\
&= E[\delta \cdot ((\mathbb{I}^{22} + E[J_{T-1}J'_{T-1}])^{-1}(\widetilde{W}_{T-1}R^f E[J_{T-1}] + (R^f-2)\lambda E[J_{T-1}])'\mathbb{I}^{22}(\mathbb{I}^{22} + E[J_{T-1}J'_{T-1}])^{-1}(\widetilde{W}_{T-1}R^f E \\
&\quad + \delta \cdot E[(\widetilde{W}_{T-1} \cdot R^f + J'_{T-1}Z_{T-1} + (R^f-2)\cdot\lambda)^2 - \lambda^2]]
\end{aligned}
$$