Proposal

Typically, we face the problem of solving

$$V_0(x_0) = \sup_{\mathbf{c}_0} \mathbf{E}[\sum_{t=0}^{T-1} \delta^t u(\mathbf{X}_t, c_t) + U(\mathbf{X}_T)\delta^T]$$

such that $\mathbf{c}_0 = \{c_0, c_1, ...c_{T-1}\}$ and the state evolves according to

$$\mathbf{X}_{t+1} = g(\mathbf{X}_t, \varepsilon_t) \in \mathbf{R}^d$$

[1] exogenous shocks $\varepsilon_t \in \mathbf{R}^d$.

Rewriting in the Bellman equation format, we have

$$V_t(x) = \sup_{\mathbf{c}_t} u(x, c_t) + \mathbf{E}[V_{t+1}(\mathbf{X}_{t+1}) | \mathbf{X}_t = \mathbf{x}_t]$$

with the boundary condition

$$V_T(x) = U(x).$$

In order to solve this, the dynamic problem start at $T-1$, one needs to solve the optimization problem at each time and iterate the valuation backwards. In simple dynamic program, problems arise when the dimension d is large. At each time, t, the $\mathbb{E}_t$ is conditioning on $\sigma(\mathbf{X}_t)$, one cannot easily derive it analytically, hence making the optimization cumbersome to solve. Numerical approximation schemes, e.g. Monte-Carlo, can be used together with dynamic problem for solving this type of problems, but its numerical performance and theoretical properties are highly dependent on the structure of the problem considered.

[1]In most cases, the control also affects the evolution of states. A simple change of variable condition allows us to “push-out" the control of the state evolution.

Instead, we consider a single index

$$Y_t = s(\mathbf{X}_t, \beta_t) \in \mathbf{R}$$

and rewrite the Bellman equation into

$$V_t(y) = \sup_{\mathbf{c}_t, \beta_t} u(y, c_t) + \mathbb{E}[V_{t+1}(Y_{t+1})|Y_t = y] \text{ for } \beta_t = \{\beta_t, \beta_{t+1}, ..., \beta_T\} \tag{1}$$

with the boundary condition

$$V_T(y) = U(y).$$

such that

$$Y_{t+1} = s(g(\mathbf{X}_t, \varepsilon_t), \beta_{t+1}).$$

We have now effectively reduced the d-dimensional problem into one, and know the problem becomes to finding the distribution $F_t(y; \beta_t)$ of Y_t in order to evaluate the integral. Our idea is to use the empirical distribution $F_t^n(y; \beta_t)$.

Suppose given this empirical distribution, the optimal controls in equation (1) can be solved exactly. We would like to analysis the convergence behaviour of the resulting $V_0(x_0)^n$ to the true $V_0(x_0)$.

Problem to Solve

- Conditions for the validity of equation (1)
- Given F_t^n, can we evaluate $\mathbb{E}_t$.

Example 1 (Portfolio Choice and Optimal Consumption Decision)**.** *Consider a portfolio choice when a large number of assets are considered with a constraint that the sum of all the weights is equal to one and a weight could be allowed to be negative. In combination with the portfolio choice, we solve optimal consumption and investment decision under stochastic dynamical programming.*

Suppose that there are q number of assets considered for the construction of a target portfolio and their returns and the associated covariance matrix are denoted as $\mathbf{R} = (R_1, R_2, ..., R_q)'$. Let the return of the target portfolio be $R_p = \mathbf{w}'\mathbf{R}$ and hence R_p does not belong to the vector $\mathbf{R}$. Let the return of the target portfolio be μ_p. Under this setup, we can consider the following regression model:

$$\begin{aligned}\mu_p &= \mathbb{E}[w_1 R_{1t} + w_2 R_{2t} + ... + w_q R_{qt}] \\ &= \mathbb{E}[(1 - \sum_{i=2}^{q} w_i) R_{1t} + w_2 R_{2t} ... + w_q R_{qt}]\end{aligned}$$

$$R_{1t} = \mu_p + \sum_{i=2}^{q} w_i X_{it} + \epsilon_t \tag{2}$$

where $X_{it} = R_{1t} - R_{it}$. Under our general setup, $Y_t = \hat{\mathbf{w}}'\mathbf{R}_t$ where $\hat{\mathbf{w}}$ is a OLS estimator for $\mathbf{w}$ in (2).

$$V_0(x_0) = \sup_{\mathbf{c}_0, \beta_0} \mathbf{E}[u(C_0(W_0, c_0)) + \sum_{j=1}^{T-1} u(C_j(W_j(\beta_{j-1}, \mathbf{R}_{j-1}, \omega_{j-1}, r_f), c_j))\delta^j + u(W_T(\beta_{T-1}, \mathbf{R}_{T-1}, \omega_{T-1}, r_f))\delta^T]$$

contional on the boundary condition

$$V_T(x_T) = \sup_{\beta_{T-1}} \mathbf{E}[u(W_T(\beta_{T-1}, \mathbf{r}_{T-1}, \omega_{T-1}, r_f))]$$

where

- $\mathbf{R}_t$ is a vector of returns of different risky assets at time t.
- ω_t is a vector of weight of different assets in risky portfolio.
- β_t is the weight allocated to the risky portfolio comparing to risk free asset
- W_t is the wealth at time t, a function of return $\mathbf{r}_{t-1}$ and weight β_{t-1} which are observed

or determined from the last period.

- c_t is the a vector of 2 control variables which in turn determines consumption C_t.
- u is the utility function depending on consumption or the terminal wealth.

The state of W_t evolves according to

$$Y_t = \hat{\mathbf{w}}'\mathbf{R}_t$$

$$\begin{aligned} W_{t+1} &= (W_t - C_t)(\beta_t \cdot Y_t + (1-\beta_t) \cdot r_f + 1) \\ C_t &= W_t \cdot \frac{\exp\{c_{1,t} + c_{2,t}W_t\}}{1 + \exp\{c_{1,t} + c_{2,t}W_t\}} \end{aligned}$$

where we assume the comsumption is propotional to wealth and the propotion is calculated by applying inverse logit function.

The utility function is assumed to be $u(x) = \log x$.

If we write it into Bellman equation

$$V_t(x_t) = \sup_{\mathbf{c}_t, \beta_t} \mathbf{E}[\sum_{j=t}^{T-1} u(C_j(W_j(\beta_{j-1}, \mathbf{r}_{j-1}, \omega_{j-1}, r_f), c_j))\delta^{j-t} + u(W_T(\beta_{T-1}, \mathbf{r}_{T-1}, \omega_{T-1}, r_f))\delta^{T-t}]$$

This example is a typical high-dimensional optimization problem which cannot be sloved under the current literature, especially when the number of assets to be chosen is quite big. Therefore, we take a novel two steps approach where we first compute the weights using lasso and then using dynamic programming to calculate the appropriate control variable c_t.