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Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

7240366882

MATHEMATICS **9709/33**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2016**

1 hour 45 minutes

Additional Materials: List of Formulae (MF9)

READ THESE INSTRUCTIONS FIRST

An answer booklet is provided inside this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.

This document consists of **4** printed pages and **1** insert.

1 It is given that $z = \ln(y + 2) - \ln(y + 1)$. Express y in terms of z. [3]

2 The equation of a curve is $y = \dfrac{\sin x}{1 + \cos x}$, for $-\pi < x < \pi$. Show that the gradient of the curve is positive for all x in the given interval. [4]

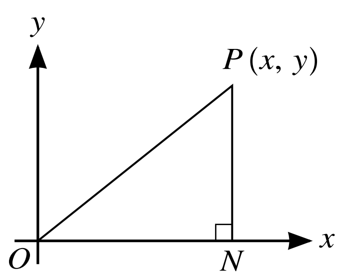
3 Express the equation $\cot 2\theta = 1 + \tan\theta$ as a quadratic equation in $\tan\theta$. Hence solve this equation for $0° < \theta < 180°$. [6]

4 The polynomial $4x^4 + ax^2 + 11x + b$, where a and b are constants, is denoted by $\mathrm{p}(x)$. It is given that $\mathrm{p}(x)$ is divisible by $x^2 - x + 2$.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the real roots of the equation $\mathrm{p}(x) = 0$. [2]

5



The diagram shows a variable point P with coordinates (x, y) and the point N which is the foot of the perpendicular from P to the x-axis. P moves on a curve such that, for all $x \geqslant 0$, the gradient of the curve is equal in value to the area of the triangle OPN, where O is the origin.

(i) State a differential equation satisfied by x and y. [1]

The point with coordinates $(0, 2)$ lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. [5]

(iii) Sketch the curve. [1]

6 Let $I = \displaystyle\int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})}\,dx$.

(i) Using the substitution $u = \sqrt{x}$, show that $I = \displaystyle\int_1^2 \frac{u - 1}{u + 1}\,du$. [3]

(ii) Hence show that $I = 1 + \ln\frac{4}{9}$. [6]

7 **Throughout this question the use of a calculator is not permitted.**

The complex number z is defined by $z = (\sqrt{2}) - (\sqrt{6})\mathrm{i}$. The complex conjugate of z is denoted by z^*.

(i) Find the modulus and argument of z. [2]

(ii) Express each of the following in the form $x + \mathrm{i}y$, where x and y are real and exact:

(a) $z + 2z^*$;

(b) $\dfrac{z^*}{\mathrm{i}z}$.

[4]

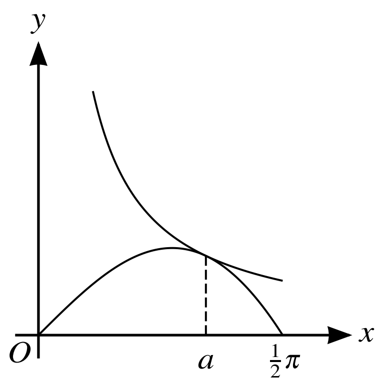
(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z^* and $\mathrm{i}z$ respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$. [3]

8 Let $\mathrm{f}(x) = \dfrac{3x^2 + x + 6}{(x+2)(x^2+4)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [5]

9



The diagram shows the curves $y = x\cos x$ and $y = \dfrac{k}{x}$, where k is a constant, for $0 < x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = a$.

(i) Show that a satisfies the equation $\tan a = \dfrac{2}{a}$. [5]

(ii) Use the iterative formula $a_{n+1} = \tan^{-1}\left(\dfrac{2}{a_n}\right)$ to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iii) Hence find the value of k correct to 2 decimal places. [2]

[Question 10 is printed on the next page.]

10 The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

(i) Find the position vectors of the two points on the line whose distance from the origin is $\sqrt{(10)}$. [5]

(ii) The plane p has equation $ax + y + z = 5$, where a is a constant. The acute angle between the line l and the plane p is equal to $\sin^{-1}\left(\frac{2}{3}\right)$. Find the possible values of a. [5]

12:19

1 It is given that $z = \ln(y+2) - \ln(y+1)$. Express y in terms of z. [3]

$$z = \ln \frac{y+2}{y+1}$$

$$\frac{y+2}{y+1} = e^z$$

$$y+2 = ye^z + e^z$$

$$y - ye^z = e^z - 2$$

$$y(1-e^z) = e^z - 2$$

$$y = \frac{e^z - 2}{1 - e^z}$$

2 The equation of a curve is $y = \dfrac{\sin x}{1+\cos x}$, for $-\pi < x < \pi$. Show that the gradient of the curve is positive for all x in the given interval. [4]

1

$$\frac{dy}{dx} = \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{1+\cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$\frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{\cos x + 1}$$

$$= \frac{2}{(1+\cos x)^2}$$

gradient is positive as square of the denominator will always be positive $\therefore \frac{dy}{dx}$ will always be +ve

3 Express the equation $\cot 2\theta = 1 + \tan\theta$ as a quadratic equation in $\tan\theta$. Hence solve this equation for $0° < \theta < 180°$. [6]

$$\frac{1}{\tan 2\theta} = 1 + \tan\theta$$

$$1 = \tan 2\theta + \tan 2\theta \tan\theta$$

$$1 = \frac{2\tan\theta}{1-\tan^2\theta} + \frac{2\tan\theta}{1-\tan^2\theta}(\tan\theta)$$

$$1 = \frac{2\tan\theta + 2\tan^2\theta}{1-\tan^2\theta}$$

$$2\tan\theta + 2\tan^2\theta = 1 - \tan^2\theta$$

$$3\tan^2\theta + 2\tan\theta - 1 = 0$$

$$\tan\theta = \frac{-2 \pm \sqrt{4-4(3)(-1)}}{6}$$

$$= \frac{-2 \pm 4}{6}$$

$\tan\theta = \frac{1}{3}$ or $\tan\theta = -1$

$\theta = \tan^{-1}\left(\frac{1}{3}\right)$ or $\theta = 180 - \tan^{-1}(1)$ or $360 - \tan^{-1}(1)$

$= 18.4°$ ✓ $\qquad = 135°$ ✓

4 The polynomial $4x^4 + ax^2 + 11x + b$, where a and b are constants, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $x^2 - x + 2$.

(i) Find the values of a and b. [5] 5

(ii) When a and b have these values, find the real roots of the equation $p(x) = 0$. [2] 2

i) $(x^2 - x + 2)(4x^2 + ?x + \quad)$

$-x(?) + 2(?x) = 11x$

1 6

$12x - 1x = 11x$

$$
\begin{array}{r|l}
 & 4x^2 + 4x + (a-4) \\
\hline
x^2 - x + 2 & 4x^4 + 0x^3 + ax^2 + 11x + b \\
- & 4x^4 - 4x^3 + 8x^2 \\
\hline
 & 0 + 4x^3 + ax^2 - 8x^2 + 11x \\
- & 4x^3 - 4x^2 + 8x \\
\hline
 & 0 + ax^2 - 4x^2 + 3x + b \\
- & ax^2 - 4x^2 - ax + 4x + 2a - 8 \\
\hline
 & 0 \quad 3x + ax - 4x + b - 2a + 8 = 0
\end{array}
$$

$ax - x + b - 2a + 8 = 0$

$a(x-2) + b + 8 = 0$

$(x^2 - x + 2)(4x^2 + 4x + ?)$

$-x? + 2(4x) = 11x$

$-x? + 8x = 11x$

$-?x = 3x$

$? = -3$

$(x^2 - x + 2)(4x^2 + 4x - 3)$

$a - 4 = -3$

$a = 1$

$b = 2(-3) = -6$

ii) $4x^2 + 4x - 3$

$\dfrac{-4 \pm \sqrt{16 - 4(4)(-3)}}{8}$

$\dfrac{-4 \pm \sqrt{64}}{8}$

$\dfrac{-4 \pm 8}{8}$

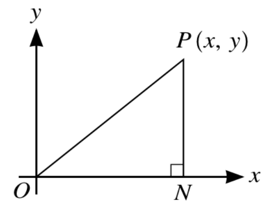
$x = \dfrac{1}{2}$ or $x = \dfrac{-3}{2}$

$(2x - 1) = 0$

$(2x + 3) = 0$

roots $(2x - 1)$ and $(2x + 3)$

5



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The point with coordinates $(0, 2)$ lies on the curve.

(ii) Solve the differential equation to obtain the equation of the curve, expressing y in terms of x. [5]

(iii) Sketch the curve. [1]

i) $\frac{dy}{dx} = \frac{1}{2} x y$

ii) $\int \frac{1}{y}\, dy = \frac{1}{2} \int x\, dx$

$\ln y = \frac{1}{2} \times \frac{x^2}{2} + C$

$\ln y = \frac{x^2}{4} + C$

$\ln 2 = C$

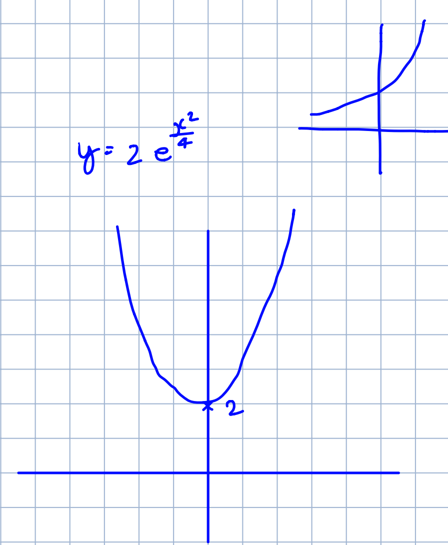
$\therefore \ln y = \frac{x^2}{4} + \ln 2$

$y = e^{\frac{x^2}{4} + \ln 2}$

$y = e^{\frac{x^2}{4}} \times e^{\ln 2}$

$y = 2e^{\frac{x^2}{4}}$

iii) $y = 2e^{\frac{x^2}{4}}$



6 Let $I = \int_1^4 \frac{(\sqrt{x}) - 1}{2(x + \sqrt{x})} \, dx$.

(i) Using the substitution $u = \sqrt{x}$, show that $I = \int_1^2 \frac{u-1}{u+1} \, du$. [3]

(ii) Hence show that $I = 1 + \ln \frac{4}{9}$. [6]

i) $u = x^{\frac{1}{2}}$ $\quad x = u^2$ $\quad$ when $x = 1$ $\quad u = 1$

$\frac{du}{dx} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $\quad x = 4$ $\quad u = 2$

$dx = 2\sqrt{x}\,du \rightarrow 2u\,du$

$$\int_1^2 \frac{u-1}{\cancel{2}(u^2+u)} \cancel{2}u\,du$$

$$\int_1^2 \frac{u^2-u}{u^2+u}\,du \Rightarrow \int_1^2 \frac{\cancel{u}(u-1)}{\cancel{u}(u+1)}\,du \Rightarrow \int_1^2 \frac{u-1}{u+1}\,du$$

ii) $\int_1^2 \frac{u-1}{u+1}\,du \Rightarrow \int_1^2 \frac{u}{u+1} - \frac{1}{u+1}\,du$ $\quad$ onkil!

$A + \frac{B}{u+1}$

$u = Au + A + B$

$A = 1$

$1 + B = 0$

$B = -1$

$1 - \frac{1}{u+1}$

$$\int_1^2 1 - \frac{1}{u+1}\,du - \int \frac{1}{u+1}\,du$$

$u - \ln|u+1| - \ln|u+1|$

$$\left[u - 2\ln|u+1|\right]_1^2$$

$(2 - 2\ln 3) - (1 - 2\ln 2)$

$2 - 2\ln 3 - 1 + 2\ln 2$

$1 + 2\ln 2 - 2\ln 3$

$1 + 2\ln\frac{2}{3}$

$1 + \ln\left(\frac{2}{3}\right)^2$

$1 + \ln\frac{4}{9}$

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The complex number z is defined by $z = (\sqrt{2}) - (\sqrt{6})\mathrm{i}$. The complex conjugate of z is denoted by z^*.

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(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z^* and $\mathrm{i}z$ respectively. Prove that angle AOB is equal to $\frac{1}{6}\pi$. [3]

7i) mod : $\sqrt{(-\sqrt{6})^2+(\sqrt{2})^2}$

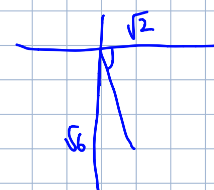
$= \sqrt{6+2}$

$= \sqrt{8}$

arg: $\tan^{-1}\left(\dfrac{\sqrt{6}}{\sqrt{2}}\right)$

$60°$

$\therefore \arg = -60° = -\dfrac{1}{3}\pi$



ii) a) $(\sqrt{2} - \sqrt{6}i) + 2(\sqrt{2} + \sqrt{6}i)$

$\sqrt{2} - \sqrt{6}i + 2\sqrt{2} + 2\sqrt{6}i$

$3\sqrt{2} + \sqrt{6}i$

b) $\dfrac{\sqrt{2}+\sqrt{6}i}{i(\sqrt{2}-\sqrt{6}i)}$

$\dfrac{\sqrt{2}+\sqrt{6}i}{\sqrt{2}i - \sqrt{6}i^2} = \dfrac{\sqrt{2}+\sqrt{6}i}{\sqrt{6}+\sqrt{2}i}$

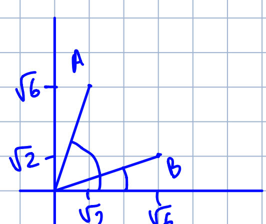
$\dfrac{(\sqrt{2}+\sqrt{6}i)(\sqrt{6}-\sqrt{2}i)}{(\sqrt{6}+\sqrt{2}i)(\sqrt{6}-\sqrt{2}i)}$

$\dfrac{2\sqrt{3} - 2i + 6i + 2\sqrt{3}}{6+2} = \dfrac{4\sqrt{3}+4i}{8}$

$= \dfrac{\sqrt{3}}{\sqrt{2}} + \dfrac{1}{2}i$

iii) A: $\sqrt{2} + \sqrt{6}i$

B: $\sqrt{6} + \sqrt{2}i$



$\angle AOB = \arg A - \arg b$

$= \tan^{-1}\left(\dfrac{\sqrt{6}}{\sqrt{2}}\right) - \tan^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{6}}\right)$

$= \dfrac{1}{3}\pi - \dfrac{1}{6}\pi = \dfrac{1}{6}\pi$

8 Let $f(x) = \dfrac{3x^2 + x + 6}{(x+2)(x^2+4)}$. $= \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

i) $3x^2+x+6 = A(x^2+4) + (Bx+C)(x+2)$

$3x^2+x+6 = Ax^2+4A + Bx^2+2Bx+Cx+2C$

$3 = A+B$

$A = 3-B$

$A = 3-1$

$A = 2$

$1 = 2B+C$

$C = 1-2B$

$C = 1-2(1)$

$C = -1$

$6 = 4A + 2C$

$6 = 4(3-B) + 2C$

$6 = 12-4B+2(1-2B)$

$6 = 12-4B+2-4B$

$6 = 14-8B$

$8B = 8$

$B = 1$

$\dfrac{2}{x+2} + \dfrac{x-1}{x^2+4}$

ii) $2(2+x)^{-1} + (x-1)(4+x^2)^{-1}$

$2\left[2\left(1+\frac{x}{2}\right)\right]^{-1} = 2(2)^{-1}\left(1+\frac{x}{2}\right)^{-1} = \left(1+\frac{x}{2}\right)^{-1} = 1 + (-1)\left(\frac{x}{2}\right) + \dfrac{(-1)(-2)\left(\frac{x}{2}\right)^2}{2}$

$= 1 - \dfrac{x}{2} + \dfrac{1}{4}x^2$

$-1 < \dfrac{x^2}{4} < 1$

$-4 < x^2 < 4$

?? $\sqrt{-4} < x < 2$

$(x-1)\left[4\left(1+\frac{x^2}{4}\right)\right]^{-1} = (x-1)(4)^{-1}\left(1+\dfrac{x^2}{4}\right)^{-1}$

$= (x-1)\left(\frac{1}{4}\right)\left[1+(-1)\left(\frac{x^2}{4}\right)\right]$

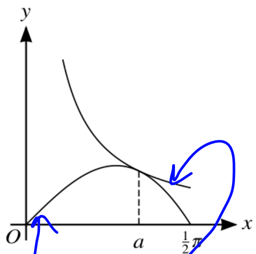
$= (x-1)\left(\frac{1}{4}\right)\left(1 - \frac{x^2}{4}\right) \Rightarrow (x-1)\left(\frac{1}{4} - \frac{x^2}{16}\right)$

$\dfrac{x}{4} - \dfrac{x^3}{16} - \dfrac{1}{4} + \dfrac{x^2}{16}$

$\therefore 1 - \dfrac{x}{2} + \dfrac{1}{4}x^2 - \dfrac{1}{4} + \dfrac{x}{4} + \dfrac{x^2}{16}$

$\dfrac{3}{4} - \dfrac{1}{4}x + \dfrac{5}{16}x^2$

9



The diagram shows the curves $y = x\cos x$ and $y = \dfrac{k}{x}$, where k is a constant, for $0 < x \leqslant \frac{1}{2}\pi$. The curves touch at the point where $x = a$.

kx^{-1}

(i) Show that a satisfies the equation $\tan a = \dfrac{2}{a}$. 2 ~~[5]~~

(ii) Use the iterative formula $a_{n+1} = \tan^{-1}\left(\dfrac{2}{a_n}\right)$ to determine a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

(iii) Hence find the value of k correct to 2 decimal places. [2]

i)

$$a\cos a = \frac{k}{a} \quad x$$

$$\frac{dy_1}{dx} = x(-\sin x) + (1)(\cos x)$$

$$= -x\sin x + \cos x$$

$$\frac{dy_2}{dx} = \frac{-k}{x^2}$$

$$\frac{-k}{x^2} = \cos x - x\sin x$$

$$-k = x^2\cos x - x^3\sin x$$

$$\frac{-k}{\cos x} = x^2 - x^3\tan x$$

$$x\cos x = \frac{k}{x}$$

$$k = x^2\cos x$$

$$\frac{-x^2\cos x}{x^2} = \cos x - x\sin x$$

$$-1 = 1 - x\tan x$$

$$x\tan x = 2$$

$$\tan x = \frac{2}{x}$$

ii) let $a_1 = \frac{1}{4}\pi$

$a_2 = \tan^{-1}\left(\dfrac{2}{(\frac{1}{4}\pi)}\right) = 1.19660$

$a_3 = 1.03163$

$a_4 = 1.09458$

$a_5 = 1.07004$

$a_6 = 1.07953$

$a_7 = 1.07585$

$a_8 = 1.07727$

$a_9 = 1.07672$

$\therefore\ a = 1.077$

iii) $y = 1.077\cos 1.077 = 0.51047$

$k = xy$

$= 1.077 \times 0.51047$

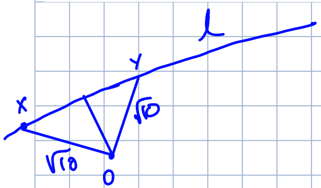
$k = 0.54977$

$k = 0.55$

10 The line l has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

(i) Find the position vectors of the two points on the line whose distance from the origin is $\sqrt{(10)}$. [5]

(ii) The plane p has equation $ax + y + z = 5$, where a is a constant. The acute angle between the line l and the plane p is equal to $\sin^{-1}\left(\frac{2}{3}\right)$. Find the possible values of a. [5]



$$\begin{pmatrix} 1+2\lambda \\ 2-\lambda \\ 1+\lambda \end{pmatrix}$$

$$\sqrt{(1+2\lambda)^2+(2-\lambda)^2+(1+\lambda)^2} = \sqrt{10}$$

$$1+\cancel{4\lambda}+4\lambda^2+4-\cancel{4\lambda}+\lambda^2+1+2\lambda+\lambda^2 = 10$$

$$6\lambda^2+2\lambda+6=10$$

$$6\lambda^2+2\lambda-4=0$$

$$\lambda = \frac{-2 \pm \sqrt{4-4(6)(-4)}}{12}$$

$$\lambda = \frac{-2 \pm \sqrt{100}}{12} = \frac{-2 \pm 10}{12}$$

$$\lambda_1 = \frac{2}{3} \qquad \lambda_2 = -1$$

Point 1: $\begin{pmatrix} 1+2\left(\frac{2}{3}\right) \\ 2-\frac{2}{3} \\ 1+\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 7/3 \\ 4/3 \\ 5/3 \end{pmatrix}$ ✓

Point 2: $\begin{pmatrix} 1+2(-1) \\ 2-(-1) \\ 1+(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$ ✓

ii)