Cambridge International A Level

Cambridge Assessment International Education
Cambridge International Advanced Level

60/75

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CENTRE NUMBER

CANDIDATE NUMBER

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MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2019**

1 hour 45 minutes

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF9)

READ THESE INSTRUCTIONS FIRST

Write your centre number, candidate number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions in the space provided. If additional space is required, you should use the lined page at the end of this booklet. The question number(s) must be clearly shown.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.

This document consists of **19** printed pages and **1** blank page.

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1 Given that $\ln(1 + e^{2y}) = x$, express y in terms of x. [3]

$$1 + e^{2y} = e^{x}$$
$$e^{2y} = e^{x} - 1$$
$$\ln e^{2y} = \ln(e^{x} - 1)$$
$$2y = \ln(e^{x} - 1)$$
$$y = \frac{1}{2}\ln(e^{x} - 1)$$

2 Solve the inequality $|2x - 3| > 4|x + 1|$. [4]

2

$(2x-3)^2 > (4(x+1))^2$

$(2x-3)(2x-3) > 16(x+1)(x+1)$

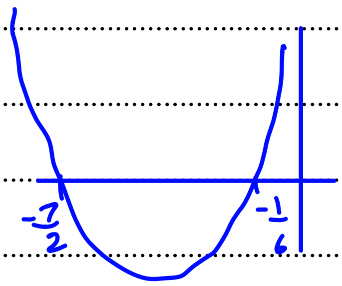
$4x^2 - 12x + 9 > 16(x^2 + 2x + 1)$

$4x^2 - 12x + 9 > 16x^2 + 32x + 16$

$12x^2 + 44x + 7 < 0$

$x_{1,2} = -\frac{1}{6}, \ -\frac{7}{2}$

$-0.1\dot{6}, \ -3.5$



$\therefore \ -\frac{7}{2} < x < -\frac{1}{6}$

test(-2)

$|-4-3| > 4|-2+1|$

$7 > 4$

✓

3 The parametric equations of a curve are

$$x = 2t + \sin 2t, \quad y = \ln(1 - \cos 2t).$$

Show that $\dfrac{dy}{dx} = \operatorname{cosec} 2t$. [5]

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{d}{dx}(1-\cos 2t) = -(-\sin 2t)2 = 2\sin 2t$$

$$> \frac{dy}{dt} = \frac{1}{1-\cos 2t}(2\sin 2t) = \frac{2\sin 2t}{1-\cos 2t}$$

$$> \frac{dx}{dt} = 2 + 2\cos 2t$$

$$\therefore \frac{dt}{dx} = \frac{1}{2(1+\cos 2t)}$$

$$\therefore \frac{dy}{dx} = \frac{2\sin 2t}{1-\cos 2t} \times \frac{1}{2(1+\cos 2t)} = \frac{2\sin 2t}{2(1+\cos 2t)(1-\cos 2t)}$$

$$= \frac{\cancel{2}\sin 2t}{\cancel{2}(1-\cos^2 2t)} = \frac{\cancel{\sin 2t}}{\sin^{\cancel{2}} 2t} = \frac{1}{\sin 2t}$$

$$= \operatorname{cosec} 2t$$

4 The number of insects in a population t weeks after the start of observations is denoted by N. The population is decreasing at a rate proportional to $Ne^{-0.02t}$. The variables N and t are treated as continuous, and it is given that when $t = 0$, $N = 1000$ and $\frac{dN}{dt} = -10$.

(i) Show that N and t satisfy the differential equation

$$\frac{dN}{dt} = -0.01e^{-0.02t}N.$$

[1] 1

$$\frac{dN}{dt} \propto Ne^{-0.02t}$$

$$\frac{dN}{dt} = kNe^{-0.02t}$$

$$-10 = k(1000)e^{0}$$

$$-10 = 1000k$$

$$k = -0.01$$

$$\therefore \frac{dN}{dt} = -0.01e^{-0.02t}N$$

(ii) Solve the differential equation and find the value of t when $N = 800$. [6] 5

$$\int \frac{1}{N}\,dN = -0.01\int e^{-0.02t}\,dt$$

$$\ln N = -0.01\left(\frac{1}{-0.02}e^{-0.02t}\right) + c$$

$$\ln N = \frac{1}{2}e^{-0.02t} + c$$

$$\ln 1000 = \frac{1}{2}e^{0} + c$$

$$c = \ln(1000) - \frac{1}{2}$$

$$c = 6.407755$$

$\ln 800 = \frac{1}{2} e^{-0.02t} + 6.40755$

$e^{-0.02t} = 0.553713$

$-0.02t = \ln(0.553713)$

$t = 29.555$

29.6

$\therefore \cancel{t = 30}$

(iii) State what happens to the value of N as t becomes large. [1]

$\ln N = \frac{1}{2} e^{-0.02t} + 6.40755$

Let $t = 10^9$

$\therefore \ln N = \frac{1}{2} e^{-0.02(10^9)} + 6.40775$

$N = e^{\frac{1}{2} e^{-0.02 \times 10^9} + 6.40775} = 606.527$

$\therefore N$ will approach 606.527

5 The curve with equation $y = e^{-2x}\ln(x-1)$ has a stationary point when $x = p$.

(2)

(i) Show that p satisfies the equation $x = 1 + \exp\left(\dfrac{1}{2(x-1)}\right)$, where $\exp(x)$ denotes e^x. [3]

$y = e^{-2x}\ln(x-1)$

!

$\frac{dy}{dx} = e^{-2x}\left(\frac{1}{x-1}\right) + \ln(x-1)(-2e^{-2x})$

$= \frac{e^{-2x}}{(x-1)} - 2e^{-2x}\ln(x-1)$

$\frac{e^{-2p}}{(p-1)} - 2e^{-2p}\ln(p-1) = 0$

$e^{-2p}\left(\frac{1}{(p-1)} - 2\ln(p-1)\right) = 0$

$\frac{1}{(p-1)} = 2\ln(p-1)$

$\ln(x-1) = \frac{1}{2(x-1)}$

$x-1 = \exp\left(\frac{1}{2(x-1)}\right)$

$x = 1 + \exp\left(\frac{1}{2(x-1)}\right)$

(ii) Verify by calculation that p lies between 2.2 and 2.6. [2]

$y = -x + 1 + \exp\left(\frac{1}{2(x-1)}\right)$

when $x = 2.2$

$-2.2 + 1 + e^{\left(\frac{1}{2(2.2-1)}\right)}$

$y = 0.3168$

when $x = 2.6$

$y = -0.233$

change of sign between 2.2 and 2.6, ∴ root is in between 2.2 and 2.6

(iii) Use an iterative formula based on the equation in part **(i)** to determine p correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x = 1 + e^{\left(\frac{1}{2(x-1)}\right)}$

let $x_1 = 2.3$

$x_2 = 1 + e^{\left(\frac{1}{2(2.3-1)}\right)} = 2.5169$

$x_3 = 2.3904$

$x_4 = 2.4328$

$x_5 = 2.4176$

$x_6 = 2.4229$

$x_7 = 2.4210$

$x_8 = 2.4217$

$x_9 = 2.4215$

$x_{10} = 2.4216$

$\therefore x = 2.42$ (2 dp) $= p$

6 **(i)** By differentiating $\frac{\cos x}{\sin x}$, show that if $y = \cot x$ then $\frac{dy}{dx} = -\operatorname{cosec}^2 x$. [2] 2

$$\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \quad) = 1$$

$$- \frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

(ii) Show that $\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} x \operatorname{cosec}^2 x \, dx = \frac{1}{4}(\pi + \ln 4)$. 3 [6]

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} x \operatorname{cosec}^2 x \, dx$$

$u = x \qquad u' = 1$

BRUH

$v = -\frac{\cos x}{\sin x} \qquad v' = \operatorname{cosec}^2 x$

$$\frac{x - \cos x}{\sin x} - \int -\frac{\cos x}{\sin x}(1)\, dx$$

ok ts → not slow tumts?

$$\text{let } f(x) = \left[\frac{x \cos x}{\sin x} + \ln \cancel{\sin x}\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi}$$

$$f\left(\tfrac{1}{2}\pi\right) = \left(\tfrac{1}{2}\pi \frac{\cos\tfrac{1}{2}\pi}{\sin\tfrac{1}{2}\pi} + \ln\left(\sin\tfrac{1}{2}\pi\right)\right)$$

$$= \tfrac{1}{2}\pi\frac{(0)}{1} + \ln 1 \Rightarrow 0$$

$$f\left(\tfrac{1}{4}\pi\right) = \left(\tfrac{1}{4}\pi \frac{\cos\left(\tfrac{1}{4}\pi\right)}{\sin\left(\tfrac{1}{4}\pi\right)} + \ln\left(\sin\tfrac{1}{4}\pi\right)\right)$$

$$= \tfrac{1}{4}\pi \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} + \ln\frac{\sqrt{2}}{2} = \tfrac{1}{4}\pi + \ln\frac{\sqrt{2}}{2}$$

$$\therefore \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} x\,\mathrm{cosec}^2 x\,dx = 0 - \left(\tfrac{1}{4}\pi + \ln\frac{\sqrt{2}}{2}\right)$$

$$= -\tfrac{1}{4}\pi - \ln 4^{-\frac{1}{4}}$$

$$= -\tfrac{1}{4}\pi + \tfrac{1}{4}\ln 4$$

$$\boxed{\tfrac{1}{4}\left(-\pi + \ln 4\right)}$$

7 Two lines l and m have equations $\mathbf{r} = a\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ respectively, where a is a constant. It is given that the lines intersect.

r_1 r_2

(i) Find the value of a. [4] 3

$$r_1: \begin{pmatrix} a \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \qquad r_2: \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$r_1: \begin{pmatrix} a+\lambda \\ 2-2\lambda \\ 3+3\lambda \end{pmatrix} \qquad r_2: \begin{pmatrix} 2+2\mu \\ 1-\mu \\ 2+\mu \end{pmatrix}$$

① $a + \lambda = 2 + 2\mu \Rightarrow$

② $2 - 2\lambda = 1 - \mu \Rightarrow \mu - 2\lambda = -1$

③ $3 + 3\lambda = 2 + \mu \Rightarrow -\mu + 3\lambda = -1$

②+③

$$\begin{aligned} \mu - 2\lambda &= -1 \\ + \ -\mu + 3\lambda &= -1 \\ \hline 0 + \lambda &= -2 \\ \lambda &= -2 \end{aligned}$$

$\therefore$ from ② $\mu = 2\lambda - 1$
$= 2(-2) - 1$
$= -4 - 1$
$= -5$

$\therefore a = 2 + 2\mu - \lambda$
$= 2 + 2(-5) - (-2)$
$= 2 - 10 + \cancel{4}\,2$

$a = -4$ $\quad -6$

(ii) When a has this value, find the equation of the plane containing l and m. [5]

X

5

8 Let $\mathrm{f}(x) = \dfrac{x^2 + x + 6}{x^2(x+2)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

$$\frac{x^2+x+6}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+2)}$$

$$x^2+x+6 = A(x)(x+2) + B(x+2) + C(x^2)$$
$$= A(x^2+2x) + Bx + 2B + Cx^2$$
$$= Ax^2 + 2Ax + Bx + 2B + Cx^2$$

1 $x^2 = Ax^2 + Cx^2$
2 $x = 2Ax + Bx$
3 $6 = 2B$

from 3, $B = 3$

from 2, $1 = 2A + 3$
$2A = -2$
$A = -1$

from 1 $1 = (-1) + C$
$C = 2$

$$\frac{-1}{x} + \frac{3}{x^2} + \frac{2}{(x+2)}$$

(ii) Hence, showing full working, show that the exact value of $\int_1^4 f(x)\,dx$ is $\frac{9}{4}$. [5]

$$\int_1^4 \frac{-1}{x} + \frac{3}{x^2} + \frac{2}{(x+2)}\,dx$$

$$-\ln x + \int 3x^{-2} + 2\ln|x+2|$$

$$-\ln x + \frac{3x^{-1}}{-1} + 2\ln|x+2|$$

let $f(x) = -\ln x - \frac{3}{x} + 2\ln(x+2)$

$$\therefore f(4) = -\ln 4 - \frac{3}{4} + 2\ln(6)$$

$$f(1) = -\ln 1 - 3 + 2\ln 3$$
$$= -3 + 2\ln 3$$

$$f(4) - f(1) = -\ln 4 - \frac{3}{4} + \ln 6^2 - (-3 + \ln 3^2)$$

$$= \ln\frac{36}{4} - \frac{3}{4} + 3 - \ln 9$$

$$= \cancel{\ln 9 - \ln 9} + 3 - \frac{3}{4}$$

$$= 3 - \frac{3}{4} = \frac{9}{4}$$

9 **(i)** By first expanding $\cos(2x + x)$, show that $\cos 3x \equiv 4\cos^3 x - 3\cos x$. [4]

$\cos 2x \cos x - \sin 2x \sin x$

$(2\cos^2 x - 1)\cos x - 2\sin^2 x \cos x$

$2\cos^3 x - \cos x - 2\sin^2 x \cos x$

$2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x$

$2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x$

(ii) Hence solve the equation $\cos 3x + 3\cos x + 1 = 0$, for $0 \leqslant x \leqslant \pi$. [2]

$4\cos^3 \cancel{3}x - 3\cos x + 3\cos x + 1 = 0$

$4\cos^3 \cancel{3}x = -1$

$3x = \cos^{-1}\left(-\frac{1}{4}\right)$

$3x = \pi - \cos^{-1}\left(\frac{1}{4}\right)$ | $3x = \pi + \cos^{-1}\left(\frac{1}{4}\right)$

$3x = \pi - 1.31811$ | $3x = \pi + 1.31811$

$x = 0.6078$ | $x = 1.4866$

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \cos^3 x \, dx$. [4]

3

$\cos 3x = 4\cos^3 x - 3\cos x$

$4\cos^3 x = \cos 3x + 3\cos x$

$\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \cos^3 x \, dx = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{4}(\cos 3x + 3\cos x)$$

$$\frac{1}{4}\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \cos 3x + 3\cos x \, dx$$

$$\frac{1}{4}\left(\frac{1}{3}\sin 3x + 3\sin x\right)_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$$

let $x = \frac{1}{3}\pi \therefore \frac{1}{4}\left(\frac{1}{3}\sin\pi + 3\sin\frac{1}{3}\pi\right)$

$= \frac{1}{4}\left(0 + \frac{3\sqrt{3}}{2}\right)$

$x = \frac{1}{6}\pi \therefore \frac{1}{4}\left(\frac{1}{3} - \frac{3}{2}\right)$

$\therefore \frac{1}{4}\left(\frac{3\sqrt{3}}{2} + \frac{7}{6}\right)$

10 **(a)** The complex number u is given by $u = -3 - (2\sqrt{10})\mathrm{i}$. Showing all necessary working and without using a calculator, find the square roots of u. Give your answers in the form $a + \mathrm{i}b$, where the numbers a and b are real and exact. [5]

$(x + iy) = (-3 - (2\sqrt{10})i)(-3 - (2\sqrt{10})i)$

$x + iy = 9 + 6\sqrt{10}i + 6\sqrt{10}i + 40i^2$

$= 9 + 12\sqrt{10}i - 40$

$-31 + 12\sqrt{10}i$

$\therefore a = -31$

$b = 12\sqrt{10}$

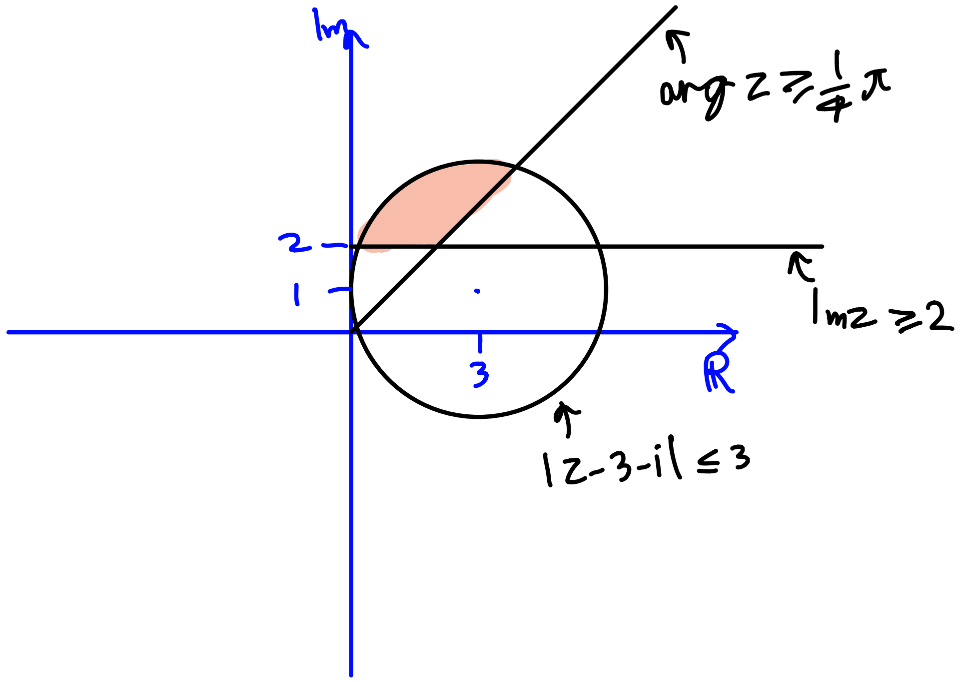
$\therefore -31 + (12\sqrt{10})i$

$\therefore -31 \pm (12\sqrt{10})i$

(b) On a sketch of an Argand diagram shade the region whose points represent complex numbers z satisfying the inequalities $|z - 3 - \mathrm{i}| \leqslant 3$, $\arg z \geqslant \frac{1}{4}\pi$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of the complex number z. [5]

$z-(3+i)$

$t^{-1}(x) = \frac{1}{4}\pi$



Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.