UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2013**

1 hour 45 minutes

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF9)

5238158802

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

This document consists of **4** printed pages.

JC13 11_9709_31/RP

UNIVERSITY *of* CAMBRIDGE International Examinations

[Turn over

1 The equation of a curve is $y = \dfrac{1+x}{1+2x}$ for $x > -\frac{1}{2}$. Show that the gradient of the curve is always negative. [3]

2 Solve the equation $2|3^x - 1| = 3^x$, giving your answers correct to 3 significant figures. [4]

3 Find the exact value of $\displaystyle\int_1^4 \frac{\ln x}{\sqrt{x}}\,dx$. [5]

4 The parametric equations of a curve are

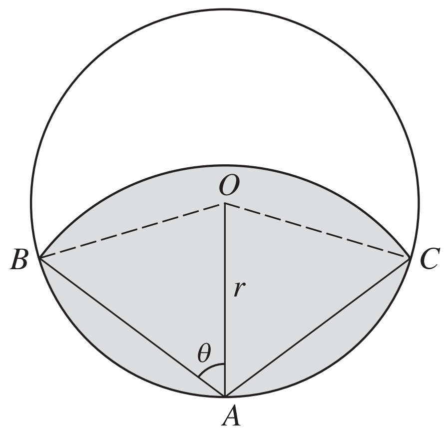
$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t.$$

Show that $\dfrac{dy}{dx} = \tan\left(t - \frac{1}{4}\pi\right)$. [6]

5 **(i)** Prove that $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$. [3]

(ii) Hence show that $\displaystyle\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta\,d\theta = \frac{1}{2}\ln 3$. [4]

6



In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is θ radians. The shaded region is bounded by the circumference of the circle and the arc with centre A joining B and C. The area of the shaded region is equal to half the area of the circle.

(i) Show that $\cos 2\theta = \dfrac{2\sin 2\theta - \pi}{4\theta}$. [5]

(ii) Use the iterative formula

$$\theta_{n+1} = \frac{1}{2}\cos^{-1}\left(\frac{2\sin 2\theta_n - \pi}{4\theta_n}\right),$$

with initial value $\theta_1 = 1$, to determine θ correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

7 Let $\mathrm{f}(x) = \dfrac{2x^2 - 7x - 1}{(x-2)(x^2+3)}$.

(i) Express $\mathrm{f}(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [5]

8 **Throughout this question the use of a calculator is not permitted.**

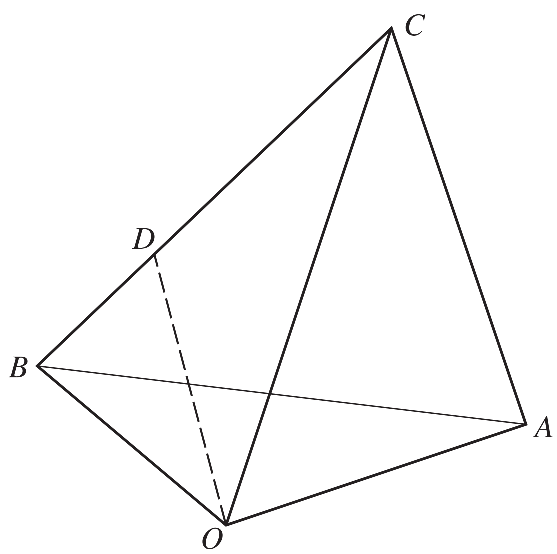
(a) The complex numbers u and v satisfy the equations

$$u + 2v = 2\mathrm{i} \quad \text{and} \quad \mathrm{i}u + v = 3.$$

Solve the equations for u and v, giving both answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + \mathrm{i}| = 1$ and the locus representing complex numbers w satisfying $\arg(w - 2) = \frac{3}{4}\pi$. Find the least value of $|z - w|$ for points on these loci. [5]

9



The diagram shows three points A, B and C whose position vectors with respect to the origin O are given by $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$. The point D lies on BC, between B and C, and is such that $CD = 2DB$.

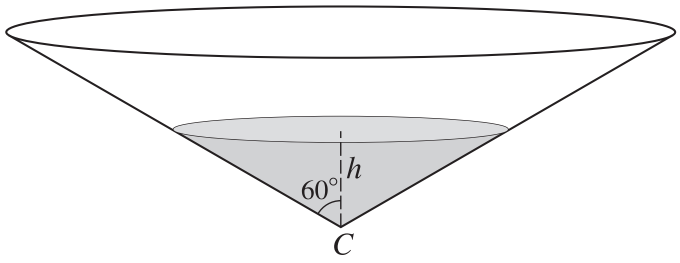
(i) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [6]

(ii) Find the position vector of D. [1]

(iii) Show that the length of the perpendicular from A to OD is $\frac{1}{3}\sqrt{(65)}$. [4]

[Question 10 is printed on the next page.]

10



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is $60°$, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.

(i) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}},$$

where A is a positive constant. [4]

(ii) Solve the differential equation given in part **(i)** and obtain an expression for t in terms of h and H. [6]

(iii) Find the time at which the depth reaches $\frac{1}{2}H$. [1]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]

1 The equation of a curve is $y = \dfrac{1+x}{1+2x}$ for $x > -\frac{1}{2}$. Show that the gradient of the curve is always negative. [3]

$$\frac{dy}{dx} = \frac{(1)(1+2x) - (2)(1+x)}{(1+2x)^2}$$

$$= \frac{1+\cancel{2x} - 2 - \cancel{2x}}{(1+2x)^2}$$

$$\frac{-1}{(1+2x)^2}$$

$-ve \therefore \frac{dy}{dx}$ is always $-ve$

2 Solve the equation $2|3^x - 1| = 3^x$, giving your answers correct to 3 significant figures. [4]

$u = 3^x$

$2(u-1) = u$

$4(u^2 - 2u + 1) = u^2$

$4u^2 - 8u + 4 = u^2$

$3u^2 - 8u + 4$

$u = 2$

$3^x = 2$

$x = \frac{\ln 2}{\ln 3}$

$= 0.631$

$u = \frac{2}{3}$

$3^x = \frac{2}{3}$

$x = \frac{\ln \frac{2}{3}}{\ln 3}$

$= -0.369$

3 Find the exact value of $\int_1^4 \frac{\ln x}{\sqrt{x}}\,dx$. [5]

$\int x^{-\frac{1}{2}} \ln x \, dx$

$u = \ln x \qquad u' = \frac{1}{x} = x^{-1}$

$v = 2x^{\frac{1}{2}} \qquad v' = x^{-\frac{1}{2}} \qquad 2x^{\frac{1}{2}}$

$-\frac{1}{2x^{\frac{3}{2}}}$

$2x^{\frac{1}{2}}\ln x - \int 2x^{\frac{1}{2}} \times x^{-1}\,dx$

$2x^{\frac{1}{2}}\ln x - 2\int x^{-\frac{1}{2}}\,dx$

$2x^{\frac{1}{2}}\ln x - 2(2x^{\frac{1}{2}})$

$2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}}$

$\left[2x^{\frac{1}{2}}(\ln x - 2)\right]_1^4$

$4(\ln 4 - 2) - [2(-2)]$

$4\ln 4 - 8 - (-4)$

$4\ln 4 - 4$

4 The parametric equations of a curve are

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t.$$

Show that $\dfrac{dy}{dx} = \tan\left(t - \frac{1}{4}\pi\right)$. [6]

4) $\dfrac{dx}{dt} = -te^{-t}\cos t - e^{-t}\sin t$

$-e^{t}(\sin t + t\cos t)$

$\dfrac{dy}{dt} = -te^{-t}\sin t + e^{-t}\cos t$

$e^{-t}(\cos t - t\sin t)$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$= \dfrac{\cancel{e^{-t}}(\cos t - t\sin t)}{-\cancel{e^{-t}}(\sin t + t\cos t)}$

$= \dfrac{t\sin t - \cos t}{\sin t + t\cos t}$

5 **(i)** Prove that $\cot\theta + \tan\theta \equiv 2\operatorname{cosec} 2\theta$. [3]

(ii) Hence show that $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2\theta \, d\theta = \frac{1}{2}\ln 3$. [4]

$\operatorname{cosec} 2\theta = \frac{\cot\theta}{2} + \frac{\tan\theta}{2}$

$= \frac{\cos\theta}{2\sin\theta} + \frac{\tan\theta}{2}$

$\frac{\cos t}{\sin t} + \frac{\sin t}{\cos t}$

$\frac{\cos^2 t + \sin^2 t}{\sin t \cos t}$

$\frac{1}{\sin\theta\cos\theta} \times \frac{2}{2} = \frac{2}{2\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\operatorname{cosec} 2\theta$

ii) $\frac{1}{2}\int \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \, d\theta$

$\frac{1}{2}\left(\ln\sin\theta - \ln\cos\theta\right)$

$\frac{1}{2}\ln\frac{\sin\theta}{\cos\theta} = \left[\frac{1}{2}\ln\tan\theta\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$

$\frac{1}{3}\pi : \frac{1}{2}\ln\tan\left(\frac{1}{3}\pi\right) = \frac{1}{2}\ln\sqrt{3}$

$\frac{1}{6}\pi : \frac{1}{2}\ln\tan\frac{1}{6}\pi = \frac{1}{2}\ln\frac{\sqrt{3}}{3}$

$\frac{1}{2}\ln\sqrt{3} \quad -\frac{1}{2}\ln\frac{\sqrt{3}}{3}$

$\frac{1}{2}\ln\left(\frac{\cancel{\sqrt{3}}}{\frac{\cancel{\sqrt{3}}}{3}}\right)$

$\frac{1}{2}\ln 3$ ✓

6

$\frac{1}{2}r^2(2\pi-4\theta)$

$\frac{r^2(2\pi-4\theta)}{2}$

Shaded: ABC

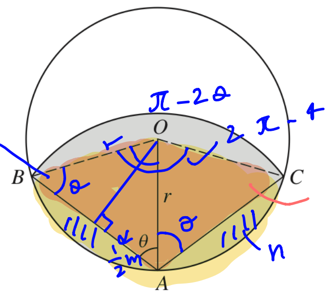
$\cos\theta = \frac{\frac{1}{2}m}{r}$

$\frac{1}{2}m = r\cos\theta$

$m = 2r\cos\theta$

$\frac{1}{2}(2r\cos\theta)^2\, 2\theta$

$4\theta r^2\cos^2\theta$



$r \times m \times \sin\theta$

$r\times(2r\cos\theta)\sin\theta$

$2r^2\cos\theta\sin\theta$

$r^2\sin 2\theta$

In the diagram, A is a point on the circumference of a circle with centre O and radius r. A circular arc with centre A meets the circumference at B and C. The angle OAB is θ radians. The shaded region is bounded by the circumference of the circle and the arc with centre A joining B and C. The area of the shaded region is equal to half the area of the circle.

(i) Show that $\cos 2\theta = \dfrac{2\sin 2\theta - \pi}{4\theta}$. [5]

$\cos 2\theta = 2\cos^2\theta - 1$

(ii) Use the iterative formula

$$\theta_{n+1} = \tfrac{1}{2}\cos^{-1}\left(\frac{2\sin 2\theta_n - \pi}{4\theta_n}\right),$$

with initial value $\theta_1 = 1$, to determine θ correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

Shaded: $4\theta r^2\cos^2\theta + \left(\frac{r^2(2\pi-4\theta)}{2} - r^2\sin 2\theta\right) = \frac{\pi r^2}{2}$

$4\theta r^2\cos^2\theta + \left(\frac{2\pi r^2 - 4r^2\theta - 2r^2\sin 2\theta}{2}\right) = \frac{\pi r^2}{2}$

$8\theta r^2\cos^2\theta + 2\pi r^2 - 4r^2\theta - 2r^2\sin 2\theta = \pi r^2$

$4r^2\theta(2\cos^2\theta - 1) + 2\pi r^2 - 2r^2\sin 2\theta = \pi r^2$

$4\cancel{r^2}\theta(\cos 2\theta) + \pi \cancel{r^2} - 2\cancel{r^2}\sin 2\theta = 0$

$4\theta\cos 2\theta + \pi - 2\sin 2\theta = 0$

$4\theta\cos 2\theta = 2\sin 2\theta - \pi$

$\cos 2\theta = \frac{2\sin 2\theta - \pi}{4\theta}$

1 : 42

7 Let $f(x) = \dfrac{2x^2 - 7x - 1}{(x-2)(x^2+3)}$. $= \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+3}$

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

i) $2x^2 - 7x - 1 = A(x^2+3) + (Bx+C)(x-2)$

let $x=2$

$-7 = 7A$

$A = -1$

$-1 = -3 + -2C$

$2 = -2C$

$C = -1$

$2 = A + B$

$2 = -1 + B$

$B = 3$

$\dfrac{-1}{x-2} + \dfrac{3x-1}{x^2+3}$

ii) $-1(-2+x)^{-1} + (3x-1)(x^2+3)^{-1}$

$-1(-2)^{-1}\left(1 - \frac{1}{2}x\right)^{-1}$

$\frac{1}{2}\left(1 + \frac{1}{2}x + \frac{1}{4}x^2\right)$

$\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2$

$(3x-1)(3+x^2)^{-1}$

$\frac{1}{3}\left(1 + \frac{x^2}{3}\right)^{-1}$

$\frac{1}{3}\left(1 - \frac{1}{3}x^2\right)$

$(3x-1)\left(\frac{1}{3} - \frac{1}{9}x^2\right)$

$x - \cancel{\frac{1}{3}x^3} - \frac{1}{3} + \frac{1}{9}x^2$

$\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + -\frac{1}{3} + x + \frac{1}{9}x^2$

$\frac{1}{6} + \frac{5}{4}x + \frac{17}{72}x^2$

8 Throughout this question the use of a calculator is not permitted.

(a) The complex numbers u and v satisfy the equations

$$u + 2v = 2\mathrm{i} \quad \text{and} \quad \mathrm{i}u + v = 3.$$ $\times 2 \longrightarrow 2iu + 2v = 6$

Solve the equations for u and v, giving both answers in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + \mathrm{i}| = 1$ ($z-(0-i)$) and the locus representing complex numbers w satisfying $\arg(w - 2) = \frac{3}{4}\pi$ ($w-(2+0i)$). Find the least value of $|z - w|$ for points on these loci. [5]

a)

$u + 2v = 2i$

$-\ \ 2iu + 2v = 6$

$u - 2iu = 2i - 6$

$(x+iy) - 2i(x+iy) = 2i - 6$

$x + iy - 2ix + 2y = 2i - 6$

$x + 2y = -6 \qquad y - 2x = 2$

$x = -6 - 2y \qquad 2x = y - 2$

$x = \frac{y-2}{2}$

$-6 - 2y = \frac{y-2}{2}$

$-12 - 4y = y - 2$

$-10 = 5y$

$y = -2$

$\therefore x = -6 - 2(-2)$
$= -2$

$u = -2 - 2i$

$v = 1 + 2i$

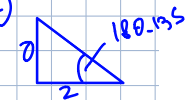
$v = 3 - iu$

$v = 3 - i(-2 - 2i)$

$v = 3 - (-2i + 2)$

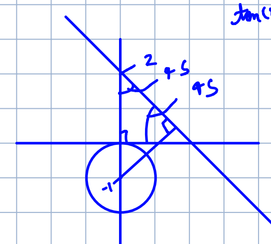
$v = 3 + 2i - 2$

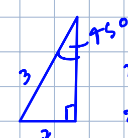
$v = 1 + 2i$



$\tan 135 = \frac{0}{2}$

ii)



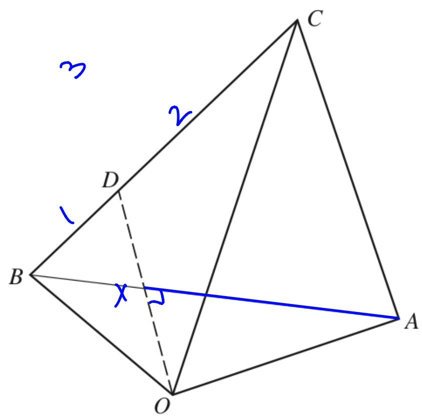


$3\sin 45 = x$

$x = \frac{3\sqrt{2}}{2}$

$\frac{3\sqrt{2}}{2} - 1 = 1.12$

9



The diagram shows three points A, B and C whose position vectors with respect to the origin O are given by $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$. The point D lies on BC, between B and C, and is such that $CD = 2DB$.

~~(i)~~ Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [6]

(ii) Find the position vector of D. [1]

(iii) Show that the length of the perpendicular from A to OD is $\frac{1}{3}\sqrt{(65)}$. [4]

ii) $\vec{OD} = OB + \frac{1}{3}BC$

$BC = OC - OB$

$= \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \times \frac{1}{3} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ ✓

iii) $XA = OA - \overbrace{OX}^{\lambda OD}$

$= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1\lambda \\ 2\lambda \\ 2\lambda \end{pmatrix}$

$= \begin{pmatrix} 2-\lambda \\ -1-2\lambda \\ 2-2\lambda \end{pmatrix}$

$XA \cdot OD = 0$

$\begin{pmatrix} 2-\lambda \\ -1-2\lambda \\ 2-2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$2 - \lambda + (-2 - 4\lambda) + (4 - 2\lambda) = 0$

$2 - \lambda - 2 \quad -4\lambda + 4 - 4\lambda = 0$

$-9\lambda + 4 = 0$

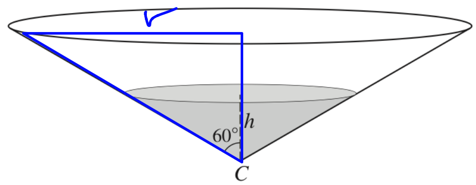
$9\lambda = 4$

$\lambda = \frac{4}{9}$

$XA: \cancel{\begin{pmatrix} 2 - \frac{4}{9} \\ -1-2\left(\frac{4}{9}\right) \\ 2-2\left(\frac{4}{9}\right) \end{pmatrix}}$

$= \begin{pmatrix} 10/7 \\ -15/7 \\ 6/7 \end{pmatrix}$ $\sqrt{\left(\frac{10}{7}\right)^2 + \left(\frac{-15}{7}\right)^2 + \left(\frac{6}{7}\right)^2}$

10



A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°, as shown in the diagram. At time $t = 0$, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$, where h is the depth of water at time t. The tank becomes empty when $t = 60$.

$h \tan 60 = r$

$r = \sqrt{3} h$

$r^2 = 3h^2$

(i) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -Ah^{-\frac{3}{2}},$$

where A is a positive constant. [4]

(ii) Solve the differential equation given in part **(i)** and obtain an expression for t in terms of h and H. [6]

(iii) Find the time at which the depth reaches $\frac{1}{2}H$. [1]

[The volume V of a cone of vertical height h and base radius r is given by $V = \frac{1}{3}\pi r^2 h$.]

i) $V = \frac{1}{3}\pi r^2 h$

$\frac{1}{3}\pi 3h^3$

πh^3

$V = \pi h^3$

$\frac{dV}{dt} \propto -h^{\frac{1}{2}}$

$\frac{dV}{dh} = 3\pi h^2$

$\frac{dV}{dt} = -kh^{\frac{1}{2}}$

$\frac{dh}{dV} = \frac{1}{3\pi h^2}$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$= \frac{-kh^{\frac{1}{2}}}{3\pi h^2}$ $\quad A = \frac{k}{3\pi}$

$\frac{dh}{dt} = -Ah^{-\frac{3}{2}}$

ii) $\frac{dh}{dt} = \frac{-A}{h^{\frac{3}{2}}}$

$\int_H^0 h^{\frac{3}{2}}\, dh = \int_0^{60} -A\, dt$ $\quad 0 = -60A + C$

$\frac{2h^{\frac{5}{2}}}{5} = -At + C$ $\quad \underline{C = 60A}$

$$\frac{2}{5}h^{\frac{5}{2}} = -At + 60A \longrightarrow \frac{2}{5}H^{\frac{5}{2}} = 0 + 60A$$

$$A = \frac{1}{150}H^{\frac{5}{2}}$$

$$At = 60A - \frac{2}{5}h^{\frac{5}{2}}$$

$$t = 60 - \frac{2h^{\frac{5}{2}}}{5A}$$

$$= 60 - \frac{2h^{\frac{5}{2}}}{\frac{5H^{\frac{5}{2}}}{150}}$$

$$t = 60 - \frac{6h^{\frac{5}{2}}}{H^{\frac{5}{2}}}$$

$$t = 60 - \frac{60h^{\frac{5}{2}}}{H^{\frac{5}{2}}}$$

$$= 60\left(1 - \left(\frac{h}{H}\right)^{\frac{5}{2}}\right)$$

iii) $t = 60\left(1 - \left(\frac{0.5H}{H}\right)^{\frac{5}{2}}\right)$

$$t = 60\left(1 - 0.5^{\frac{5}{2}}\right)$$

$$= 49.4s$$