Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

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CENTRE NUMBER

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MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2017**

1 hour 45 minutes

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF9)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, candidate number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.

This document consists of **19** printed pages and **1** blank page.

1 Find the quotient and remainder when x^4 is divided by $x^2 + 2x - 1$. [3]

6

$$
\begin{array}{r|l}
 & x^2 - 2x + 5 \\
x^2+2x-1 & x^4 + 0x^3 + 0x^2 + 0x + 0 \\
 & - \quad x^4 + 2x^3 - x^2 \\
 & 0 - 2x^3 + x^2 + 0x \\
 & - \quad -2x^3 - 4x^2 + 2x \\
 & 0 + 5x^2 - 2x + 0x \\
 & - \quad 5x^2 + 10x - 5 \\
 & 0 - 12x + 5
\end{array}
$$

$x=2 : \frac{16}{7}$

quotient $= x^2 - 2x + 5$ ✓

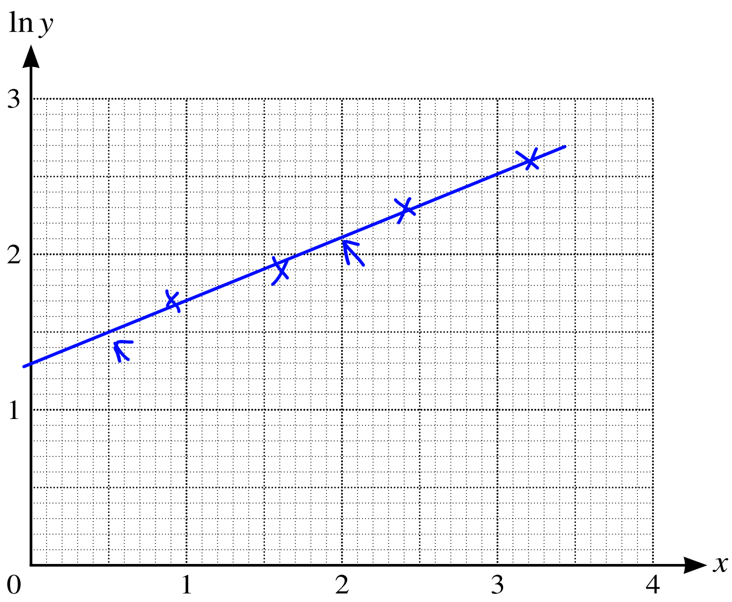
remainder $= -12x + 5$ ✓

$x^2 - 2x + 5 + \frac{-12x+5}{x^2+2x-1}$

2 Two variable quantities x and y are believed to satisfy an equation of the form $y = C(a^x)$, where C and a are constants. An experiment produced four pairs of values of x and y. The table below gives the corresponding values of x and $\ln y$.

x	0.9	1.6	2.4	3.2
$\ln y$	1.7	1.9	2.3	2.6

By plotting $\ln y$ against x for these four pairs of values and drawing a suitable straight line, estimate the values of C and a. Give your answers correct to 2 significant figures. [5]



1

$y = Ca^x$ $\qquad$ $1.5 = Ca^{0.5}$ ④

① $2.1 = Ca^2$

$C = \dfrac{2.1}{a^2}$ ③

sub 3 into 4

$1.5 = \dfrac{2.1a^{0.5}}{a^2}$

$1.5 = 2.1a^{-1.5}$

$\ln a^{-1.5} = \ln \dfrac{1.5}{2.1}$

$-1.5\ln a = \ln \dfrac{5}{7}$

$a = \exp\left(\dfrac{\ln(5/7)}{-1.5}\right)$

$a = 1.25146$

$a \approx 1.3$

$C = \dfrac{2.1}{1.25146^2}$

$C = 1.340855$

$C \approx 1.3$

3 The equation $x^3 = 3x + 7$ has one real root, denoted by α.

(i) Show by calculation that α lies between 2 and 3. [2]

Let $f(x) = x^3 - 3x - 7$

$f(2) = 2^3 - 3(2) - 7$
$= -5$

$f(3) = 3^3 - 3(3) - 7$
$= 11$

change of sign between $x =$ and $x = 3$
so root is between 2 and 3

Two iterative formulae, A and B, derived from this equation are as follows:

$$x_{n+1} = (3x_n + 7)^{\frac{1}{3}}, \qquad (A)$$

$$x_{n+1} = \frac{x_n^3 - 7}{3}. \qquad (B)$$

Each formula is used with initial value $x_1 = 2.5$.

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [4]

A: $x_1 = 2.5$

$x_2 = (3(2.5)+7)^{\frac{1}{3}} = 2.4385$

$x_3 = 2.4281$

$x_4 = 2.4263$

$x_5 = 2.4261$

$x_6 = 2.4260$

$x_7 = 2.4260$

$\therefore x = 2.43$

B: $x_1 = 2.5$

$x_2 = \frac{(2.5)^3 - 7}{3} = 2.8750$

$x_3 = 5.5879$

$x_4 = 55.8264$

B fails to converge

4 **(i)** Prove the identity $\tan(45° + x) + \tan(45° - x) \equiv 2\sec 2x$. [4]

$$\frac{\tan 45 + \tan x}{1 - \tan 45 \tan x} + \frac{\tan 45 - \tan x}{1 + \tan 45 \tan x}$$

$$\frac{1 + \tan x}{1 - \tan x} + \frac{1 - \tan x}{1 + \tan x}$$

$$= \frac{(1 + \tan x)(1 + \tan x) + (1 - \tan x)(1 - \tan x)}{(1 - \tan x)(1 + \tan x)}$$

$$= \frac{1 + 2\tan x + \tan^2 x + (1 + \tan^2 x - 2\tan x)}{(1 - \tan x)(1 + \tan x)}$$

$$= \frac{2\tan^2 x + 2}{1 - \tan^2 x}$$

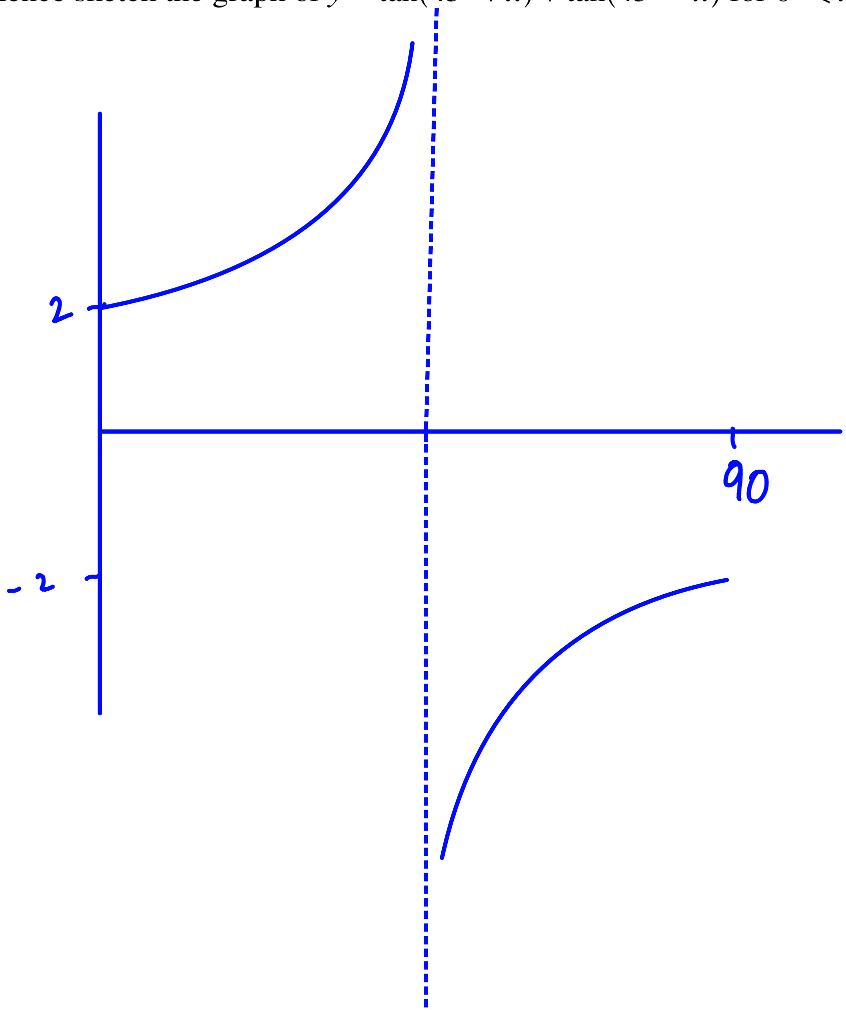
$$= \frac{2\frac{\sin^2 x}{\cos^2 x} + 2}{1 - \frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{\frac{2\sin^2 x + 2\cos^2 x}{\cancel{\cos^2 x}}}{\frac{\cos^2 x - \sin^2 x}{\cancel{\cos^2 x}}}$$

$$= \frac{2(\sin^2 x + \cos^2 x)}{\cos^2 x - \sin^2 x}$$

$$= \frac{2(1)}{\cos 2x} = 2\sec 2x$$

(ii) Hence sketch the graph of $y = \tan(45° + x) + \tan(45° - x)$ for $0° \leqslant x \leqslant 90°$. [3]



5 The equation of a curve is $2x^4 + xy^3 + y^4 = 10$.

(3)

(i) Show that $\frac{dy}{dx} = -\frac{8x^3 + y^3}{3xy^2 + 4y^3}$. [4]

$8x^3 + (1)y^3 + (x)(3y^2)\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$

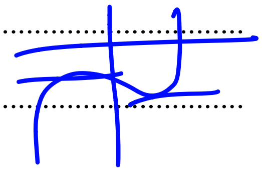
$8x^3 + y^3 + 3xy^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$

$8x^3 + y^3 = -(3xy^2 + 4y^3)\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{8x^2 + y^3}{3xy^2 + 4y^3}$

(ii) Hence show that there are two points on the curve at which the tangent is parallel to the x-axis and find the coordinates of these points. [4]

4



$$\frac{dy}{dx} = \frac{-8x^3 + y^3}{3xy^2 + 4y^3} = 0$$

$$-8x^2 + y^3 = 0 \rightarrow y = \sqrt[3]{8x^2} = \sqrt[3]{8} \times \sqrt[3]{x^2}$$

$$y = 2\sqrt[3]{x^2}$$

$$8x^2 = y^3$$

$$x = \sqrt{\frac{y^3}{8}} = \frac{\sqrt{y^2 \times y}}{\sqrt{8}} = \frac{y\sqrt{y}}{\sqrt{8}}$$

$$2x^4 + xy^3 + y^4 = 10$$

$$y = 2 \times \sqrt[3]{x^2} = 2(x^2)^{\frac{1}{3}}$$

$$y^4 = 16x^{2 \times \frac{4}{3}} = 16x^{\frac{8}{3}}$$

$$2x^4 + x \times 8x^2 + 16x^{\frac{8}{3}} = 10$$

$$2x^4 + 8x^3 + 16x^{\frac{8}{3}} = 10$$

6 The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = 4\cos^2 y \tan x,$$

for $0 \leqslant x < \frac{1}{2}\pi$, and $x = 0$ when $y = \frac{1}{4}\pi$. Solve this differential equation and find the value of x when $y = \frac{1}{3}\pi$. [8]

$$\int \frac{1}{\cos^2 y}\, dy = 4\int \tan x \, dx$$

$$\int \sec^2 y \, dy = 4\int \frac{\sin x}{\cos x}\, dx$$

$$\tan y = -4\ln\cos x + c$$

$$\tan\frac{1}{4}\pi = -4\underbrace{\ln 1}_{0} + c$$

$$1 = c$$

$$\tan\frac{1}{3}\pi = -4\ln\cos x + 1$$

$$1-\sqrt{3} = 4\ln\cos x$$

$$\ln\cos x = -0.183$$
$$\cos x = 0.83276$$
$$x = \cos^{-1}(0.83276)$$
$$= 0.58672$$
$$\approx 0.587$$

7 (a) The complex number u is given by $u = 8 - 15i$. Showing all necessary working, find the two square roots of u. Give answers in the form $a + ib$, where the numbers a and b are real and exact. [5]

(2)

$\sqrt{8-15i} = x+iy$

$(x+iy)^2 = 8-15i$ ✓

$(x+iy)(x+iy) = 8-15i$

$x^2 + 2ixy + i^2y^2 = 8-15i$

$x^2 + 2ixy - y^2 = 8-15i$

$x^2 - y^2 = 8$ ✓

$x^2 = 8+y^2$

$x = \sqrt{8+y^2}$

$2xy = -15$ ✓

$x = \dfrac{-15}{2y}$

$\left(\dfrac{-15}{2y}\right)^2 = 8+y^2$

$\dfrac{225}{4y^2} = 8+y^2$

$225 = 32y^2 + 4y^4$

$4y^4 + 32y^2 - 225$

$y_{1,2} = \dfrac{9}{2}$ (A), $\dfrac{-25}{2}$ (B)

$x = \dfrac{-15}{2\left(\frac{9}{2}\right)}$ (A) or $\dfrac{-15}{2\left(\frac{-25}{2}\right)}$ (B)

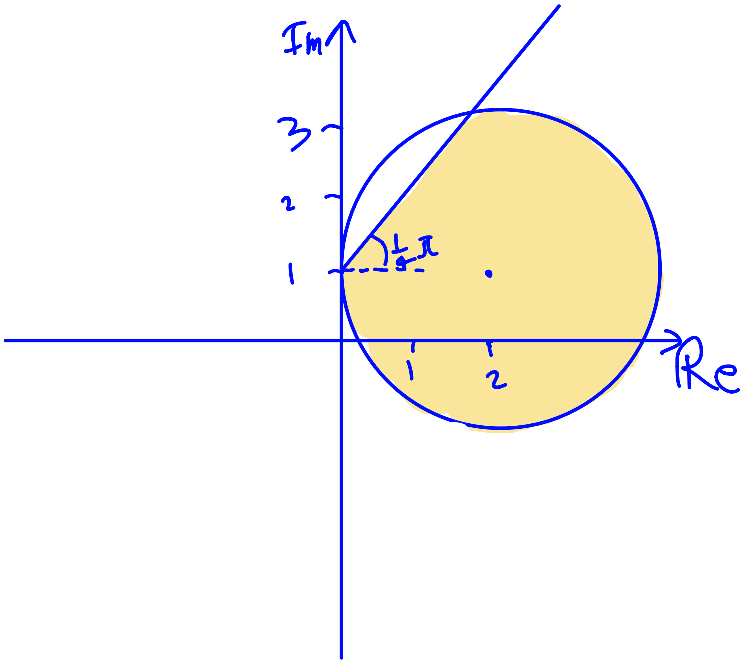
$= \dfrac{-15}{9}$ or $\dfrac{15}{25}$

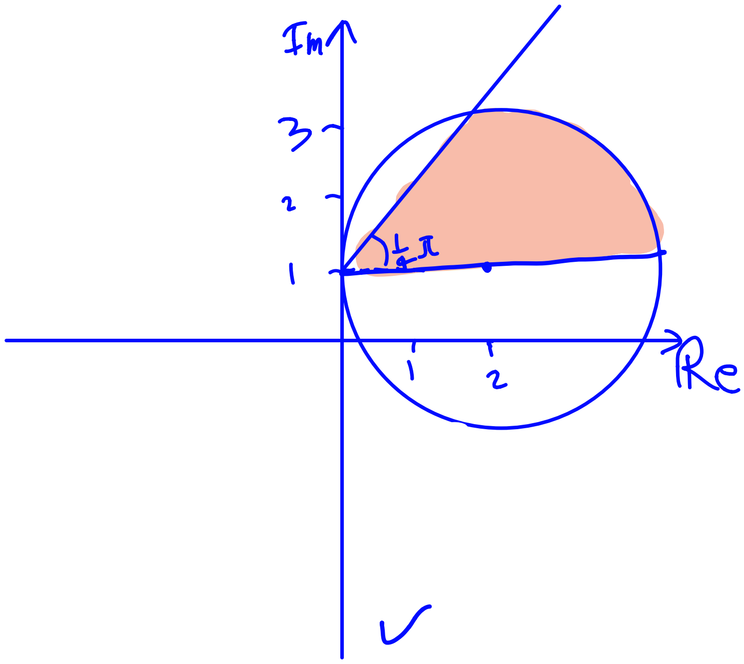
$= -\dfrac{5}{3}$ or $\dfrac{3}{5}$

$-\dfrac{5}{3} + \dfrac{9}{2}i$ or $\dfrac{3}{5} - \dfrac{25}{2}i$

(b) On an Argand diagram, shade the region whose points represent complex numbers satisfying both the inequalities $|z - 2 - \mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z - \mathrm{i}) \leqslant \frac{1}{4}\pi$. [4]

$z-(2+i) < 2$ $z-(0+i)$





8 Let $\mathrm{f}(x) = \dfrac{4x^2 + 9x - 8}{(x+2)(2x-1)}$.

(i) Express $\mathrm{f}(x)$ in the form $A + \dfrac{B}{x+2} + \dfrac{C}{2x-1}$. [4]

2

$A(x+2)(2x-1) + B(2x-1) + C(x+2)$

$A(2x^2+3x-2) + 2Bx - B + Cx + 2C$

$4x^2+9x-8 = 2Ax^2 + 3Ax - 2A + 2Bx - B + Cx + 2C$

$4 = 2A$

$A = 2$

$9 = 3(2) + 2B + C$

$9 = 6 + 2B + C$

$3 = 2B + C$

$C = 3 - 2B$

$-8 = -2(2) - B + 2C$

$-8 = -4 - B + 2C$

$-4 = -B + 2C$

$B = 2C + 4$

$C = 3 - 2(2C+4)$

$C = 3 - 4C - 8$

$5C = -5$

$C = -1$

$\therefore B = 2(-1) + 4$

$B = 2$

(ii) Hence show 4 that $\int_1^4 \mathrm{f}(x)\,\mathrm{d}x = 6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$. [5]

4

$$\int_1^4 2 + \frac{2}{x+2} - \frac{1}{2x-1} \, dx$$

$$2x + 2\ln(x+2) - \frac{1}{2}\ln(2x-1)$$

sub: 4, $8 + 2\ln 6 - \frac{1}{2}\ln 7$

sub: 1, $1 + 2\ln 3 - \frac{1}{2}\ln 1$

$= 2 + 2\ln 3$

$\therefore \quad 8 + 2\ln 6 - \frac{1}{2}\ln 7 - 2 - 2\ln 3$

$6 + 2\ln 2 - \frac{1}{2}\ln 7$

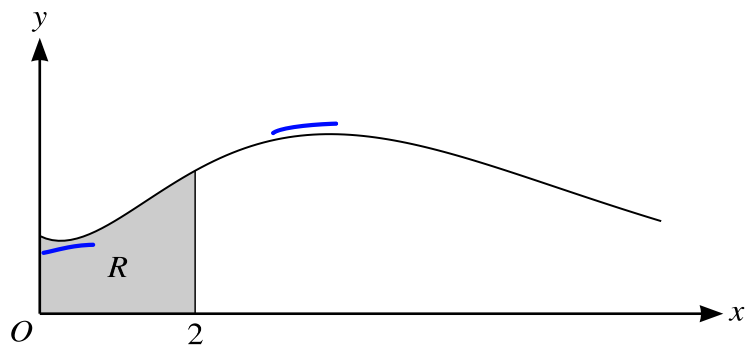
$6 + \ln 4 - \frac{1}{2}\ln 7$

$6 + \ln 16^{\frac{1}{2}} - \frac{1}{2}\ln 7$

$6 + \frac{1}{2}\ln 16 - \frac{1}{2}\ln 7$

$6 + \frac{1}{2}\ln\left(\frac{16}{7}\right)$

9



The diagram shows the curve $y = (1 + x^2)e^{-\frac{1}{2}x}$ for $x \geqslant 0$. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 2$.

(i) Find the exact values of the x-coordinates of the stationary points of the curve. [4]

$$\frac{dy}{dx} = (1+x^2)\left(-\frac{1}{2}\right)\left(e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x}(2x) = 0$$

$$= -\frac{1}{2}e^{-\frac{1}{2}x} - \frac{1}{2}x^2 e^{-\frac{1}{2}x} + 2xe^{-\frac{1}{2}x} = 0$$

$$2x\cancel{e^{-\frac{1}{2}x}} = \frac{1}{2}\cancel{e^{-\frac{1}{2}x}}(1+x^2)$$

$$4x = 1 + x^2$$

$$x^2 - 4x + 1 = 0$$

$$x = 2 + \sqrt{3}$$

and

$$x = 2 - \sqrt{3}$$

(ii) Show that the exact value of the area of R is $18 - \dfrac{42}{e}$. [5]

$$\int (1+x^2)\, e^{-\frac{1}{2}x}\, dx$$

$u \quad v'$

$u = 1+x^2 \qquad u' = 2x$

$v = -2e^{-\frac{1}{2}x} \qquad v' = e^{-\frac{1}{2}x}$

$(1+x^2)(-2e^{-\frac{1}{2}x}) - \int (-2e^{-\frac{1}{2}x})(2x)\, dx$

$-2e^{-\frac{1}{2}x} - 2x^2 e^{-\frac{1}{2}x} + 4\int x e^{-\frac{1}{2}x}\, du$

$u = x \qquad u' = 1$

$v = -2e^{-\frac{1}{2}x} \quad v' = e^{-\frac{1}{2}x}$

$-2e^{-\frac{1}{2}x} - 2x^2 e^{-\frac{1}{2}x} + 4\left[(x)(-2e^{-\frac{1}{2}x}) - \int -2e^{-\frac{1}{2}x}\, dx\right]$

$-2e^{-\frac{1}{2}x} - 2x^2 e^{-\frac{1}{2}x} + 4\left[-2xe^{-\frac{1}{2}x} + 2(-2e^{-\frac{1}{2}x})\right]$

$-2e^{-\frac{1}{2}x} - 2x^2 e^{-\frac{1}{2}x} + 4\left(-2xe^{-\frac{1}{2}x} - 4e^{-\frac{1}{2}x}\right)$

$-2e^{-\frac{1}{2}x} - 2x^2 e^{-\frac{1}{2}x} - 8xe^{-\frac{1}{2}x} - 16e^{-\frac{1}{2}x}$

$-18e^{-\frac{1}{2}x} - 8xe^{-\frac{1}{2}x} - 2x^2 e^{-\frac{1}{2}x}$

$$-2e^{-\frac{1}{2}x}(9 + 4x + x^2)\Big|_0^2$$

sub2: $-2e^{-1}(9+8+4) = \dfrac{-2}{e}(21) = \dfrac{-42}{e}$

sub0: $-2(1)(9) = -18$

$\dfrac{-42}{e} - (-18) \Rightarrow 18 - \dfrac{42}{e}$

10 The equations of two lines l and m are $\mathbf{r} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$ respectively.

l m

(i) Show that the lines do not intersect. [3]

$l: \begin{pmatrix} 3-\lambda \\ -1+\lambda \\ -2+4\lambda \end{pmatrix}$ $m: \begin{pmatrix} 4+2\mu \\ 4+\mu \\ -3-2\mu \end{pmatrix}$

① $3-\lambda = 4+2\mu$ ② $-1+\lambda = 4+\mu$

$2\mu + \lambda = -1$ $\lambda = 5+\mu$

$\lambda = -1-2\mu$

equate ① and ②, $-1-2\mu = 5+\mu$

$3\mu = -6$

$\mu = -2$

$\therefore \lambda = -1-2(-2)$

$\lambda = 3$

$-2+4(3) = -3-2(-2)$

$10 \neq 1$

$\therefore$ lines don't intersect

(ii) Calculate the acute angle between the directions of the lines. [3]

$\begin{array}{c} x \\ y \\ z \end{array} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$\cos\theta = \dfrac{-2+1-8}{\sqrt{(-1)^2+1^2+4^2} \cdot \sqrt{2^2+1^2+2^2}}$

$= \dfrac{-9}{\sqrt{18} \cdot \sqrt{9}} = \dfrac{-9}{3\sqrt{18}}$ $\dfrac{\pi}{4}$

$\cos^{-1}\left(\dfrac{-9}{3\sqrt{18}}\right) = \pi - \cos^{-1}\left(\dfrac{9}{3\sqrt{18}}\right)$

$=$

(iii) Find the equation of the plane which passes through the point $(3, -2, -1)$ and which is parallel to both l and m. Give your answer in the form $ax + by + cz = d$. [5]

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