Cambridge Assessment International Education

Cambridge International AS & A Level

CANDIDATE NAME	

CENTRE NUMBER		CANDIDATE NUMBER	

7688702067

MATHEMATICS **9709/32**

Paper 3 Pure Mathematics 3 **October/November 2020**

1 hour 50 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Blank pages are indicated.

[Turn over

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1 Solve the equation

$$\ln(1 + e^{-3x}) = 2.$$

Give the answer correct to 3 decimal places. [3]

$1 + e^{-3x} = e^2$

$e^{-3x} = e^2 - 1$

$\ln e^{-3x} = \ln e^2 - 1$

$-3x = \ln e^2 - 1$

$x = \dfrac{\ln e^2 - 1}{-3}$

$= -0.618$

2 **(a)** Expand $\sqrt[3]{1+6x}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

$(1+6x)^{\frac{1}{3}}$

$= 1 + (\frac{1}{3})(6x) + \frac{(\frac{1}{3})(\frac{1}{3}-1)(6x)^2}{2!} + \frac{(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)(6x)^3}{3!}$

$= 1 + 2x - 4x^2 + \frac{40}{3}x^3$

(b) State the set of values of x for which the expansion is valid. [1]

$|6x| < 1$

$6|x| < 1$

$|x| < \frac{1}{6}$

3 The variables x and y satisfy the relation $2^y = 3^{1-2x}$.

(a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line. [3]

$\ln 2^y = \ln 3^{1-2x}$

$y \ln 2 = (1-2x) \ln 3$

$y \ln 2 = \ln(3) - 2x \ln 3$

$y = \left(\frac{-2 \ln 3}{\ln 2}\right) x + \frac{\ln 3}{\ln 2}$

$y = m \quad x \quad + \quad c$

$\therefore$ gradient $= \frac{-2\ln 3}{\ln 2}$

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [2]

$3x = \frac{-2x \ln 3}{\ln 2} + \frac{\ln 3}{\ln 2}$

$3x + \frac{2x \ln 3}{\ln 2} = \frac{\ln 3}{\ln 2}$

$3x \ln 2 + 2x \ln 3 = \ln 3$

$x(\ln 8 + \ln 9) = \ln 3$

$x = \underline{\underline{\frac{\ln 3}{\ln 72}}}$

$a = 3$
$b = 72$

4 **(a)** Show that the equation $\tan(\theta + 60°) = 2\cot\theta$ can be written in the form

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0.$$ [3]

$$\tan(\theta+60) = 2 \times \frac{1}{\tan\theta}$$

$$\frac{\tan\theta + \tan 60}{1 - \tan 60 \tan\theta} = \frac{2}{\tan\theta}$$

$$\frac{\sqrt{3} + \tan\theta}{1 - \sqrt{3}\tan\theta} = \frac{2}{\tan\theta}$$

$$\sqrt{3}\tan\theta + \tan^2\theta = 2 - 2\sqrt{3}\tan\theta$$

$$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$$

(b) Hence solve the equation $\tan(\theta + 60°) = 2 \cot \theta$, for $0° < \theta < 180°$. [3]

$\tan^2\theta + 3\sqrt{3}\tan\theta - 2 = 0$

$x = \tan\theta$

$x^2 - 3\sqrt{3}x - 2 = 0$

$x = -0.35996$ or 5.5561

$\tan\theta = -0.35996$ or $\tan\theta = 5.5561$

$\theta_1 = 180 - \tan^{-1}(0.35996)$

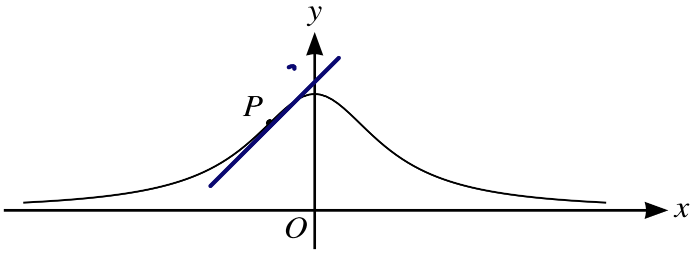
$\theta_1 = 160.2°$

$\theta_2 = \tan^{-1}(5.5561)$

$\theta_2 = 79.797$

$\approx 79.8°$

5



The diagram shows the curve with parametric equations

$$x = \tan\theta, \quad y = \cos^2\theta,$$

for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$. [3]

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dy}{d\theta} = 2\cos\theta(-\sin\theta)$$
$$= -2\cos\theta\sin\theta$$

$$\frac{dx}{d\theta} = \sec^2\theta = \frac{1}{\cos^2\theta}$$

$$\therefore \frac{d\theta}{dx} = \cos^2\theta$$

$$\frac{dy}{dx} = -2\cos\theta\sin\theta \times \cos^2 x$$
$$= -2\sin\theta\cos^3\theta$$

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P. [4]

when $\frac{dy}{dx}$ is max, $\frac{d^2y}{dx^2} = 0$

$\frac{dy}{dx} = \boxed{-2\sin\theta}\boxed{\cos^3\theta}$

$= -2\sin\theta(3\cos^2\theta \times -\sin\theta) + \cos^3\theta(-2\cos\theta)$

$= 6\sin^2\theta\cos^2\theta - 2\cos^4\theta$

$= \cos^2\theta(6\sin^2\theta - 2\cos^2\theta)$

$= \cos^2\theta(6(1-\cos^2\theta) - 2\cos^2\theta)$

$= \cos^2\theta(6 - 8\cos^2\theta) = 0$

$6 - 8\cos^2\theta = 0$

$\cos^2\theta = \frac{3}{4}$

$\cos\theta = \pm\sqrt{0.75}$

$\theta =$

$\theta = \cos^{-1}(\sqrt{0.75})$ or $2\pi - \cos^{-1}\sqrt{0.25}$

$= \frac{\pi}{6}$ $\quad = \frac{11}{6}\pi - 2\pi = -\frac{\pi}{6}$ ✓

6 The complex number u is defined by

$$u = \frac{7 + \mathrm{i}}{1 - \mathrm{i}}.$$

(a) Express u in the form $x + \mathrm{i}y$, where x and y are real. [3]

$u = \frac{7+i\,(1+i)}{(1-i)(1+i)}$

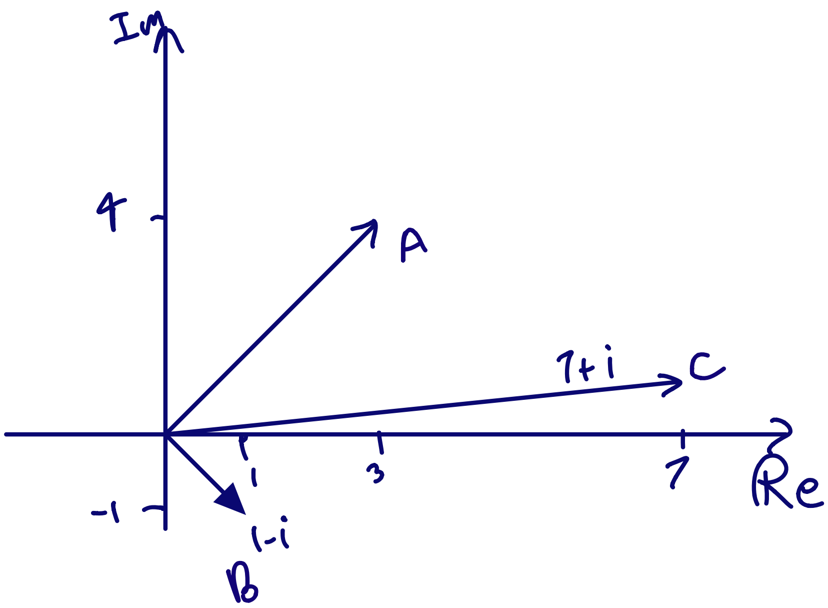
$= \frac{7 + 7i + i + (-1)}{1 - (-1)}$

$= \frac{6+8i}{2}$

$= 3 + 4i$

(b) Show on a sketch of an Argand diagram the points A, B and C representing u, $7 + \mathrm{i}$ and $1 - \mathrm{i}$ respectively. [2]

v w



(c) By considering the arguments of 7 + i and 1 − i, show that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = \tan^{-1}\left(\tfrac{1}{7}\right) + \tfrac{1}{4}\pi.$$ [3]

$\arg\left(\frac{v}{w}\right) = \arg v - \arg w$

$\arg(u) = \arg v - \arg w$

$\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) - \left(-\tan^{-1}\left(\frac{1}{1}\right)\right)$

$\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}(1)$

$\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{7}\right) + \frac{1}{4}\pi$

7 The variables x and t satisfy the differential equation

$$e^{3t}\frac{dx}{dt} = \cos^2 2x,$$

for $t \geqslant 0$. It is given that $x = 0$ when $t = 0$.

(a) Solve the differential equation and obtain an expression for x in terms of t. [7]

$\int \frac{1}{\cos^2 2x}\, dx = \int \frac{1}{e^{3t}}\, dt$

$\int \frac{1}{(\cos 2x)^2}\, dx = \int e^{-3t}\, dt$

$\int \sec^2 2x\, dx = \int e^{-3t}\, dt$

$\int e^{-3t}$

$\frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + c$

when $x = 0$, $t = 0$

$\frac{1}{2}\tan 0 = -\frac{1}{3}e^{0} + c$

$0 = -\frac{1}{3} + c \Rightarrow c = \frac{1}{3}$

$\frac{1}{2}\tan 2x = -\frac{1}{3}e^{-3t} + \frac{1}{3}$

$\tan 2x = \frac{2}{3}(1 - e^{-3t})$

$x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}(1 - e^{-3t})\right)$

(b) State what happens to the value of x when t tends to infinity. [1]

$x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}(1 - e^{-3t})\right)$

as $t \Rightarrow \infty$, $x = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}(1)\right)$, $x = 0.294$

as t tends to ∞ , x becomes 0.294

8 With respect to the origin O, the position vectors of the points A, B, C and D are given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Show that $AB = 2CD$. [3]

$\overrightarrow{AB} = OB - OA$

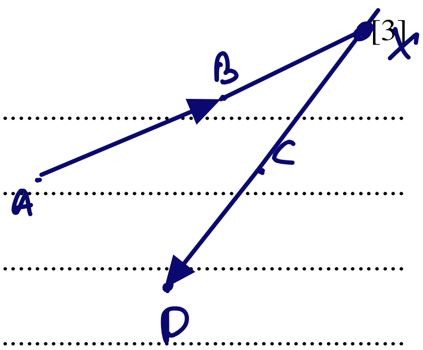
$= \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = \therefore AB = \sqrt{2^2 + 2^2 + 4^2} = 2\sqrt{6}$

same

$CD = OD - OC$

$= \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \therefore \quad 2\overrightarrow{CD} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ so $2CD = \sqrt{4^2 + 2^2 + 2^2} = 2\sqrt{6}$

(b) Find the angle between the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$. [3]



$\lambda \overrightarrow{BA} = \lambda \begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix}$

$\mu CD = \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$\cos\theta = \dfrac{\cancel{-2(2)} + -2(1) + \cancel{4(1)}}{2\sqrt{6} \times \sqrt{6}}$

$\cos\theta = \dfrac{-2}{12}$

$\theta = \pi - \cos^{-1}\left(\dfrac{2}{12}\right)$ or

$= 99.6°$

Check?

(c) Show that the line through A and B does not intersect the line through C and D. [4]

$AB: \; r_1 = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2+2\lambda \\ 1-2\lambda \\ 5-4\lambda \end{pmatrix}$

$CD: \; r_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2\mu \\ 1+\mu \\ 2+\mu \end{pmatrix}$

$X: \; 2+2\lambda = 1+2\mu$ $\qquad$ $Y: \; 1-2\lambda = 1+\mu$

$2\lambda - 2\mu = -1$ $\qquad$ $\mu = -2\lambda$ — 3

3 → sub

$2\lambda - 2(-2\lambda) = -1$ $\qquad$ $\mu = \frac{2}{6} = \frac{1}{3}$

$2\lambda + 4\lambda = -1$

$6\lambda = -1$

$\lambda = -\frac{1}{6}$

$Z: \; r_1: 5 - 4(-\frac{1}{6}) = \frac{17}{3}$ $\qquad$ $\frac{17}{3} \neq \frac{7}{3}$

$r_2: 2 + (\frac{1}{3}) = \frac{7}{3}$

$Z_{AB} \neq Z_{CD}$ $\therefore$ they dont intersect

9 Let $f(x) = \dfrac{7x + 18}{(3x + 2)(x^2 + 4)}$.

(a) Express $f(x)$ in partial fractions. [5]

$$\frac{7x+18}{(3x+2)(x^2+4)} = \frac{A}{3x+2} + \frac{Bx+C}{x^2+4}$$

$7x+18 = A(x^2+4) + (Bx+C)(3x+2)$

$7x+18 = Ax^2+4A + 3Bx^2 + 2Bx + 3Cx + 2C$

① $0 = A + 3B \Rightarrow A = -3B$

② $7 = 2B + 3C \Rightarrow C = \frac{7-2B}{3}$

③ $18 = 4A + 2C \Rightarrow 18 = 4(-3B) + 2\left(\frac{7-2B}{3}\right)$

$18 = -12B + \frac{14-4B}{3}$

$54 = -36B + 14 - 4B$

$40 = -40B$

$B = -1$

$\therefore A = -3(-1) = 3$

$C = \frac{7-2(-1)}{3} = 3$

$A = 3 \quad C = 3$

$B = -1$

$$\frac{3}{3x+2} + \frac{3-x}{x^2+4}$$

(b) Hence find the exact value of $\int_0^2 f(x)\,dx$. [6]

$$\int \frac{3}{3x+2}\,dx + \int \frac{3}{x^2+4}\,dx - \int \frac{x}{x^2+4}\,dx$$

$$3\left(\frac{1}{3}\right)\ln 3x+2 \quad + 3\int \frac{1}{x^2+2^2}\,dx - \int \frac{x}{x^2+4}\,dx$$

$$\ln(3x+2) \quad + 3\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) - \frac{1}{2}\int \frac{2x}{x^2+4}\,dx$$

$$\ln|3x+2| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2} \quad \ln x^2+4$$

$$\left[\ln|3x+2| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln|x^2+4|\right]_0^2$$

$$\left[\ln 8 + \frac{3}{2}\tan^{-1} 1 - \frac{1}{2}\ln 8\right] - \left[\ln 2 + \frac{3}{2}(0) - \frac{1}{2}\ln 4\right]$$

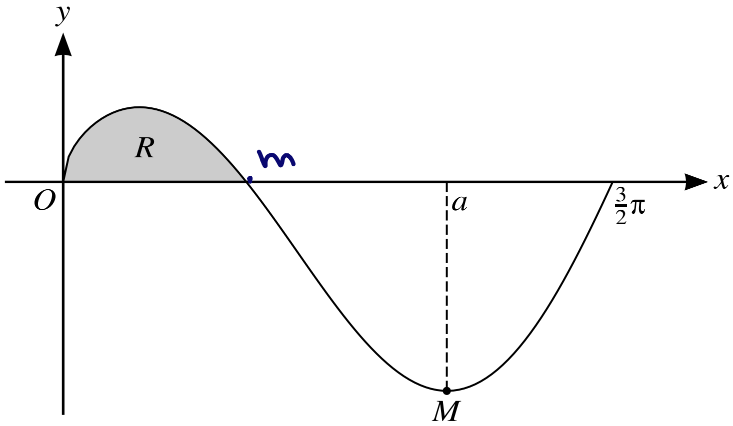
$$\ln 8 + \frac{3\pi}{8} - \frac{1}{2}\ln 8 - \left(\ln 2 - \ln 4^{\frac{1}{2}}\right)$$

$$\frac{1}{2}\ln 8 + \frac{3}{8}\pi$$

$$\frac{1}{2}\ln 2^3 + \frac{3}{8}\pi$$

$$\frac{3}{2}\ln 2 + \frac{3}{8}\pi$$

10



The diagram shows the curve $y = \sqrt{x}\cos x$, for $0 \leqslant x \leqslant \frac{3}{2}\pi$, and its minimum point M, where $x = a$. The shaded region between the curve and the x-axis is denoted by R.

(a) Show that a satisfies the equation $\tan a = \dfrac{1}{2a}$. [3]

$y = x^{\frac{1}{2}} \cos x$

$\frac{dy}{dx} = x^{\frac{1}{2}}(-\sin x) + \cos x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= -\sqrt{x}\sin x + \frac{1}{2\sqrt{x}}\cos x = 0$ at stationary point

$\frac{1}{2\sqrt{x}}\cos x = \sqrt{x}\sin x$ | $\frac{\sin a}{\cos a} = \frac{1}{2\sqrt{a}\times\sqrt{a}}$ | $\tan a = \frac{1}{2a}$

$x = a$

(b) The sequence of values given by the iterative formula $a_{n+1} = \pi + \tan^{-1}\left(\dfrac{1}{2a_n}\right)$, with initial value $x_1 = 3$, converges to a.

Use this formula to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

$x_1 = 3$

$x_2 = \pi + \tan^{-1}\left(\frac{1}{2(3)}\right) = 3.3067$

$x_3 = 3.2917$

$x_4 = 3.2923$

$x_5 = 3.2923$

$x_6 = 3.2923$

$\therefore x = 3.29$

(c) Find the volume of the solid obtained when the region R is rotated completely about the x-axis. Give your answer in terms of π. [6]

$y = \sqrt{x}\cos x$

$y^2 = x\cos^2 x$

$\pi \int y^2 \, dx$

$\pi \int x\cos^2 x \, dx$

finding m

$0 = \sqrt{x}\cos x$

$\cos x = 0$

$x = \frac{\pi}{2}$

LIAT

$u = x \quad u' = 1$

$v = \frac{1}{2}x + \frac{1}{4}\sin 2x \quad v' = \cos^2 x$

$\cos 2x = 2\cos^2 x - 1$

$2\cos^2 x = \cos 2x + 1$

$\cos^2 x = \frac{\cos 2x + 1}{2}$

$\int \frac{\cos 2x}{2} dx + \int \frac{1}{2} dx$

$\frac{1}{2}\left(\frac{1}{2}\right)\sin 2x + \frac{1}{2}x$

$\frac{1}{2}x + \frac{1}{4}\sin 2x$

$\pi \times \left[x\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) - \int \frac{1}{2}x + \frac{1}{4}\sin 2x\right]$

$\pi \left[x\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) - \frac{x^2}{4} + \frac{1}{8}\cos 2x\right]_0^{\frac{\pi}{2}}$

$\pi \left(\frac{\pi}{2}\left(\frac{\pi}{4} - 0\right) - \frac{\pi^2}{16} + \frac{1}{8}(-1)\right)$

$\pi \left(\frac{\pi^2}{8} - \frac{\pi^2}{16} - \frac{1}{8}\right)$

$\pi \left(\frac{2\pi^2 - \pi^2}{16} - \frac{1}{8}\right)$

$\pi \left(\frac{\pi^2}{16} - \frac{1}{8}\right)$

$\pi \left(\frac{\pi^2 - 2}{16}\right)$

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

5 The complex number z is defined by $z = \frac{9\sqrt{3} + 9i}{\sqrt{3} - i}$. Find, showing all your working,

(i) an expression for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, [5]

(ii) the two square roots of z, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

i) $\frac{9}{2} + \frac{9\sqrt{3}}{2}i$

ii) $\sqrt{\left(\frac{9}{2} + \frac{9\sqrt{3}}{2}i\right)} = (x+iy)$

$\frac{9}{2} + \frac{9\sqrt{3}}{2}i = x^2 - y^2 + 2ixy$

$\frac{9}{2} = x^2 - y^2$

$\frac{9\sqrt{3}\cancel{i}}{2} = 2\cancel{i}xy$

$x = \frac{9\sqrt{3}}{4y}$

$\frac{9}{2} = \left(\frac{9\sqrt{3}}{4y}\right)^2 - y^2$

$\frac{9}{2} = \frac{243}{16y^2} - y^2$

$\frac{9}{2} = \frac{243 - 16y^4}{16y^2}$

$144y^2 = 486 - 32y^4$

$32y^4 + 144y^2 - 486 = 0$

$y^2 = \frac{9}{4}$ or $\frac{-27}{4}$

$y = \frac{3}{2}$ or $\cancel{\frac{3\sqrt{3}}{2}i}$

$x^2 = \frac{9}{2} + y^2$

$x_1^2 = \frac{9}{2} + \left(\frac{3}{2}\right)^2$

$x^2 = \frac{9}{2} + \frac{9}{4}$

$x = \frac{3\sqrt{3}}{2}$

$\therefore r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2}$

$r = 3$

$\theta = \tan^{-1}\left(\frac{3/2}{\frac{3\sqrt{3}}{2}}\right) = \frac{1}{6}\pi$

$\frac{3}{2} + \frac{3}{2}i \Rightarrow$ Mod $= \frac{3\sqrt{3}}{2}$, arg $= \frac{1}{4}\pi$

$\frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}i \Rightarrow$ Mod $= \frac{3\sqrt{6}}{2}$, arg $= \frac{1}{4}\pi$

$r_1 = \frac{3\sqrt{6}}{2}$ $\theta = \frac{1}{4}\pi$

$r_2 = \frac{3\sqrt{3}}{2}$ $\theta = \frac{1}{4}\pi$

$re^{i\theta}$

$\frac{3\sqrt{3}}{2}e^{\frac{1}{4}\pi}$