Cambridge International Examinations
Cambridge International Advanced Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2014**

1 hour 45 minutes

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF9)

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READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

This document consists of **3** printed pages and **1** blank page.

CAMBRIDGE International Examinations

1 Use logarithms to solve the equation $e^x = 3^{x-2}$, giving your answer correct to 3 decimal places. [3]

2 **(i)** Use the trapezium rule with 3 intervals to estimate the value of

$$\int_{\frac{1}{6}\pi}^{\frac{2}{3}\pi} \operatorname{cosec} x \, dx,$$

giving your answer correct to 2 decimal places. [3]

(ii) Using a sketch of the graph of $y = \operatorname{cosec} x$, explain whether the trapezium rule gives an overestimate or an underestimate of the true value of the integral in part **(i)**. [2]

3 The polynomial $ax^3 + bx^2 + x + 3$, where a and b are constants, is denoted by $p(x)$. It is given that $(3x + 1)$ is a factor of $p(x)$, and that when $p(x)$ is divided by $(x - 2)$ the remainder is 21. Find the values of a and b. [5]

4 The parametric equations of a curve are

$$x = \frac{1}{\cos^3 t}, \quad y = \tan^3 t,$$

where $0 \leqslant t < \frac{1}{2}\pi$.

(i) Show that $\dfrac{dy}{dx} = \sin t$. [4]

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t is $y = x \sin t - \tan t$. [3]

5 **Throughout this question the use of a calculator is not permitted.**

The complex numbers w and z satisfy the relation

$$w = \frac{z + i}{iz + 2}.$$

(i) Given that $z = 1 + i$, find w, giving your answer in the form $x + iy$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + iy$, where x and y are real. [4]

6 It is given that $\int_1^a \ln(2x)\,dx = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

(ii) Use the iterative formula

$$a_{n+1} = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

7 In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

8 **(i)** By first expanding $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$ [4]

(ii) Show that, after making the substitution $x = \frac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

(iii) Hence solve the equation

$$x^3 - x + \frac{1}{6}\sqrt{3} = 0,$$

giving your answers correct to 3 significant figures. [4]

9 Let $f(x) = \frac{x^2 - 8x + 9}{(1-x)(2-x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

10 The line l has equation $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. The point A has position vector $3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

(i) Show that the length of the perpendicular from A to l is 15. [5]

(ii) The line l lies in the plane with equation $ax + by - 3z + 1 = 0$, where a and b are constants. Find the values of a and b. [5]

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1 Use logarithms to solve the equation $e^x = 3^{x-2}$, giving your answer correct to 3 decimal places. [3]

1)

$x = (x-2)\ln 3$

$x = x\ln 3 - 2\ln 3$

$x - x\ln 3 = -2\ln 3$

$x(1-\ln 3) = -2\ln 3$

$x = \dfrac{-2\ln 3}{1-\ln 3}$

$= 22.281$

5

3 The polynomial $ax^3 + bx^2 + x + 3$, where a and b are constants, is denoted by $p(x)$. It is given that $(3x+1)$ is a factor of $p(x)$, and that when $p(x)$ is divided by $(x-2)$ the remainder is 21. Find the values of a and b. [5]

3)

$(3x+1)(cx^2 + dx + e)$

$cx^2 - 8x + 3$

$e = 3$

$3e + d = 1$

$3(3) + d = 1$

$9 + d = 1$

$d = -8$

$x = -\frac{1}{3}$

$12x^3 - 20x^2 + x + 3$

$(x-2)\overline{)\,ax^3 + bx^2 + x + 3}$

$8a + 4b + 2 + 3 = 21$

$8a + 4b = 16$

$2a + b = 4$ —1

$b = 4 - 2a$

$-\frac{1}{27}a + \frac{1}{9}b = \frac{-8}{3}$

$-\frac{1}{27}a + \frac{1}{9}(4-2a) = \frac{-8}{3}$

$-\frac{1}{27}a + \frac{4}{9} - \frac{2}{9}a = \frac{-8}{3}$

$-\frac{7}{27}a = -\frac{28}{9}$

$a = \underline{12}$

$\therefore b = 4 - 24 = \underline{-20}$

4 The parametric equations of a curve are

$$x = \frac{1}{\cos^3 t}, \quad y = \tan^3 t,$$

$\cos^{-3} t$

where $0 \leqslant t < \frac{1}{2}\pi$.

(i) Show that $\frac{dy}{dx} = \sin t$. [4]

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t is $y = x\sin t - \tan t$. [3]

i) $\frac{dy}{dt} = 3\tan^2 t \sec^2 t = \frac{3\tan^2 t}{\cos^2 t}$

$\frac{dx}{dt} = -3\cos^{-4} t\,(-\sin t) = \frac{3\sin t}{\cos^4 t}$ $\quad \frac{dt}{dx} = \frac{\cos^4 t}{3\sin t}$

$\frac{dy}{dx} = \frac{\cancel{3}\tan^2 t}{\cancel{\cos^2 t}} \times \frac{\cos^{\cancel{4}} t}{\cancel{3}\sin t}$

$= \frac{\tan^2 t \cos^2 t}{\sin t}$

$= \tan^2 t \times \frac{\cos t}{\sin t} \times \cos t$

$= \tan^{\cancel{2}} t \cos t \times \frac{1}{\cancel{\tan t}}$

$= \frac{\sin t}{\cancel{\cos t}} \times \cancel{\cos t}$

$= \sin t$

ii) $y = mx + c$

$c = y - mx$

$c = \tan^3 t - \frac{\sin t}{\cos^3 t}$

$y = x\sin t + \tan^3 t - \frac{\sin t}{\cos^3 t}$

$y = x\sin t + \frac{\sin^3 t - \sin t}{\cos^3 t}$

$y = x\sin t + \frac{\sin(\sin^2 - 1)}{\cos t \times \cos^2 t}$

$= x\sin t + \frac{\sin}{\cos t}\,\frac{-\cos^2}{\cos^2 t}$

$y = x\sin t - \tan t$

5 **Throughout this question the use of a calculator is not permitted.**

The complex numbers w and z satisfy the relation

$$w = \frac{z + \mathrm{i}}{\mathrm{i}z + 2}.$$

(i) Given that $z = 1 + \mathrm{i}$, find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [4]

i) $\dfrac{1+2i}{i(1+i)+2}$

$\dfrac{1+2i}{i^2+i+2} = \dfrac{(1+2i)}{(1+i)}\dfrac{(1-i)}{(1-i)}$

$= \dfrac{1+i+2}{1^2-i^2}$

$= \dfrac{3+i}{2}$

$= \dfrac{3}{2} + \dfrac{1}{2}i$

ii) $x+iy = \dfrac{x+iy+i}{i(x+iy)+2}$ $\quad x<0$

$= \dfrac{x+i(y+1)}{xi-y+2}$ $\quad y>$

$= \dfrac{x+i(y+1)}{(-y+2)+xi}$

$\dfrac{x+i(y+1)\ (-y+2-xi)}{(y+2+xi)(-y+2-xi)}$

$-xy+2x-x^2i$

$w = \dfrac{w+i}{iw+2}$

$iw^2 + w - i$

$\dfrac{-1 \pm \sqrt{1-4(i)(-i)}}{2i}$

$\dfrac{-1 \pm \sqrt{3}\,i}{2i}$

$\pm\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

6 It is given that $\int_1^a \ln(2x)\,\mathrm{d}x = 1$, where $a > 1$.

(i) Show that $a = \frac{1}{2}\exp\left(1 + \frac{\ln 2}{a}\right)$, where $\exp(x)$ denotes e^x. [6]

(ii) Use the iterative formula

$$a_{n+1} = \tfrac{1}{2}\exp\left(1 + \frac{\ln 2}{a_n}\right)$$

to determine the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

i) $\int \ln 2x \, dx$

$u = \ln 2x \qquad u' = \frac{\cancel{2}}{\cancel{2}x} = \frac{1}{x}$

$v = x \qquad v' = 1$

$x \ln 2x - \int \frac{x}{x} dx$

$\left[x \ln 2x - x\right]_1^a = 1$

$a \ln 2a - a \quad - \ln 2 \cancel{+1} = \cancel{1}$

$a \ln 2a = a + \ln 2$

$\ln 2a = 1 + \frac{\ln 2}{a}$

$2a = e^{1 + \frac{\ln 2}{a}}$

$a = \frac{1}{2} e^{\left(1 + \frac{\ln 2}{a}\right)}$

ii)

7 In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation

$$\frac{dR}{dx} = R\left(\frac{1}{x} - 0.57\right),$$

where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

i) $\ln R = \int \frac{1}{x} - 0.57 \, dx$

$\ln R = \ln x - 0.57x + C$

$\ln 16.8 = \ln 0.5 - 0.285 + C$

$\ln(33.6) + 0.285 = C$

$\ln R = \ln x - 0.57x + \ln(33.6) + 0.285$

$R = e^{\ln x - 0.57x + \ln(33.6) + 0.285}$

$R = e^{-0.57x + 0.285} \times x \times 33.6$ ✓

$$R = \frac{33.6x\, e^{0.285}}{e^{0.57x}}$$

8 **(i)** By first expanding $\sin(2\theta + \theta)$, show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta. \qquad [4]$$

(ii) Show that, after making the substitution $x = \dfrac{2\sin\theta}{\sqrt{3}}$, the equation $x^3 - x + \frac{1}{6}\sqrt{3} = 0$ can be written in the form $\sin 3\theta = \frac{3}{4}$. [1]

(iii) Hence solve the equation

$$x^3 - x + \tfrac{1}{6}\sqrt{3} = 0,$$

giving your answers correct to 3 significant figures. [4]

i) $\sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$2\sin\theta\cos^2\theta + (1 - 2\sin^2\theta)\sin\theta$

$2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$

$2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$3\sin\theta - 4\sin^3\theta$

ii) $\dfrac{8\sin^3\theta}{3\sqrt{3}} - \dfrac{2\sin\theta}{\sqrt{3}} + \dfrac{1}{6}\sqrt{3} = 0$

$4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$\sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$

$8\sin^3\theta - 6\sin\theta + \dfrac{3}{2} = 0$

$\cancel{8}^{2}\left(\dfrac{3\sin\theta - \sin 3\theta}{\cancel{4}\,1}\right) - 6\sin\theta + \dfrac{3}{2} = 0$

$\cancel{-}\,2\sin 3\theta = \cancel{-}\dfrac{3}{2}$

$\sin 3\theta = \dfrac{3}{4}$

iii) $3\theta = \sin^{-1}\left(\dfrac{3}{4}\right)$

$\theta = 0.282687$

$x = \dfrac{2\sin}{\sqrt{3}}$

$x = \underline{0.322}$

$3\theta = \pi - \sin^{-1}\left(\dfrac{3}{4}\right)$

$\theta = 0.764510$

$x = \underline{0.799}$

9 Let $f(x) = \dfrac{x^2 - 8x + 9}{(1-x)(2-x)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

i) $\dfrac{A}{1-x} + \dfrac{B}{(2-x)} + \dfrac{C}{(2-x)^2}$

$x^2 - 8x + 9 = A(2-x)^2 + B(2-x)(1-x) + C(1-x)$

$-3 = -C \Rightarrow \underline{C = 3}$

$2 = A \Rightarrow \underline{A = 2}$

$9 = 4A + 2B + C$

$9 = 8 + 2B + 3$

$-2 = 2B$

$\underline{B = -1}$

ii) $2(1-x)^{-1} - (2-x)^{-1} + 3(2-x)^{-2}$

$2\left[1 + x + x^2\right]$

$2 + 2x + 2x^2$

$2^{-1}\left(1 - \frac{x}{2}\right)^{-1}$

$\frac{1}{2}\left[1 + \frac{x}{2} + \frac{1}{4}x^2\right]$

$\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2$

$\frac{3}{4}\left(1 - \frac{x}{2}\right)^{-2}$

$\frac{3}{4}\left[1 + x + \frac{3}{4}x^2\right]$

$\frac{3}{4} + \frac{3}{4}x + \frac{9}{16}x^2$

$\underline{\frac{9}{4} + \frac{5}{2}x + \frac{39}{16}x^2}$

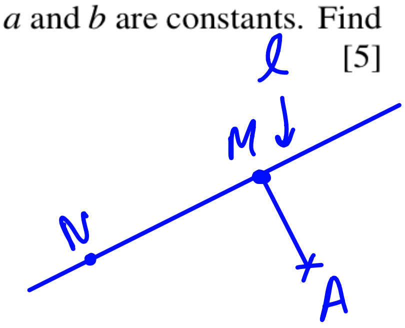
10 The line l has equation $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 9\mathbf{k} + \lambda(-2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. The point A has position vector $3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$.

(i) Show that the length of the perpendicular from A to l is 15. [5]

(ii) The line l lies in the plane with equation $ax + by - 3z + 1 = 0$, where a and b are constants. Find the values of a and b. [5]

i)

$$\begin{pmatrix} 4-2\lambda \\ -9+\lambda \\ 9-2\lambda \end{pmatrix}$$



$AM = OM - OA$

$$\begin{pmatrix} 4-2\lambda \\ -9+\lambda \\ 9-2\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1-2\lambda \\ -17+\lambda \\ 4-2\lambda \end{pmatrix}$$

$AM . NM = 0$

$$\begin{pmatrix} 1-2\lambda \\ -17+\lambda \\ 4-2\lambda \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$$

$-2+4\lambda-17+\lambda-8+4\lambda=0$

$9\lambda-27=0$

$\lambda=3$

$$\therefore AM = \begin{pmatrix} 1-6 \\ -17+3 \\ 4-6 \end{pmatrix} = \begin{pmatrix} -5 \\ -14 \\ -2 \end{pmatrix}$$

$$\sqrt{(-5)^2+(-14)^2+(-2)^2} = 15$$