Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **May/June 2016**

1 hour 45 minutes

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF9)

4146704902

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

This document consists of **3** printed pages and **1** blank page.

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1 **(i)** Solve the equation $2|x - 1| = 3|x|$. [3]

(ii) Hence solve the equation $2|5^x - 1| = 3|5^x|$, giving your answer correct to 3 significant figures. [2]

2 Find the exact value of $\int_0^{\frac{1}{2}} x\mathrm{e}^{-2x}\,\mathrm{d}x$. [5]

3 By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0° < \theta < 180°$. [5]

4 The variables x and y satisfy the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 - 2x^2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms. [6]

5 The curve with equation $y = \sin x \cos 2x$ has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

6 **(i)** By sketching a suitable pair of graphs, show that the equation

$$5\mathrm{e}^{-x} = \sqrt{x}$$

has one root. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{2}\ln\left(\frac{25}{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

7 The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - 2xy}{x^2 - y^2}$. [4]

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

8 Let $f(x) = \dfrac{4x^2 + 12}{(x+1)(x-3)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

9 With respect to the origin O, the points A, B, C, D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \overrightarrow{OC} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OD} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

(i) Find the equation of the plane containing A, B and C, giving your answer in the form $ax + by + cz = d$. [6]

(ii) The line through D parallel to OA meets the plane with equation $x + 2y - z = 7$ at the point P. Find the position vector of P and show that the length of DP is $2\sqrt{(14)}$. [5]

10 (a) Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})\mathrm{i}$. Give your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

(b) (i) On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2\mathrm{i}| = 1$ and $\arg(z - 1) = \frac{3}{4}\pi$. [4]

(ii) Calculate the least value of $|w - z|$ for points on these loci. [2]

1 **(i)** Solve the equation $2|x - 1| = 3|x|$. [3]

(ii) Hence solve the equation $2|5^x - 1| = 3|5^x|$, giving your answer correct to 3 significant figures. [2]

i) $4(x-1)(x-1) = 9x^2$

$4(x^2-2x+1) = 9x^2$

$$4x^2 - 8x + 4 = 9x^2$$

$5x^2 + 8x - 4$

$\dfrac{-8 \pm \sqrt{64 - 4(-20)}}{10}$

$\dfrac{-8 \pm \sqrt{144}}{10}$

$\dfrac{-8 \pm 12}{10} \Rightarrow$ $x = \dfrac{2}{5}$ $x = -2$

ii) $5^x = \dfrac{2}{5}$ and $5^x = -2$

$x \ln 5 = \ln \dfrac{2}{5}$

$x = -0.569$

2 Find the exact value of $\int_0^{\frac{1}{2}} xe^{-2x}\,dx$. [5]

u v'

$u = x$ $u' = 1$

$v = -\frac{1}{2}e^{-2x}$ $v' = e^{-2x}$

$$-\frac{xe^{-2x}}{2} - \int -\frac{1}{2}e^{-2x}\,dx$$

$$-\frac{xe^{-2x}}{2} + \frac{1}{2}\int e^{-2x}\,dx$$

$$-\frac{xe^{-2x}}{2} + \frac{1}{2}\left(-\frac{1}{2}e^{-2x}\right)$$

$$-\frac{xe^{-2x}}{2} - \frac{1}{4}e^{-2x}$$

$$= \left[\frac{1}{4}e^{-2x}\left(-2x - 1\right)\right]_0^{\frac{1}{2}}$$

sub $\frac{1}{4}$: $\frac{1}{4}e^{-1}(-1-1) \Rightarrow \frac{1}{4e}(-2) = \frac{-2}{4e} = \frac{-1}{2e}$

0: $\frac{1}{4}(1)(-1) = -\frac{1}{4}$

0.066

$$\frac{-1}{2e} + \frac{1}{4}$$

$$\frac{1}{4}\left(1 - \frac{2}{e}\right)$$

3 By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0° < \theta < 180°$. [5]

$$\frac{1}{\sin} = 3\sin\theta + \frac{\cos\theta}{\sin\theta}$$

$$\frac{1}{\cancel{\sin}} = \frac{3\sin^2\theta + \cos\theta}{\cancel{\sin\theta}}$$

$$1 = 3(1-\cos^2\theta) + \cos\theta$$

$$1 = 3 - 3\cos^2\theta + \cos\theta$$

$$3\cos^2\theta - \cos\theta - 2 = 0$$

$$\frac{1 \pm \sqrt{1-4(-6)}}{6}$$

$$\frac{1 \pm 5}{6}$$

$\cos\theta = 1$, $\cos\theta = \frac{-4}{6} = -\frac{2}{3}$

$\theta = \cos^{-1}(1) = 0$

X

$\theta = 180 - \cos\left(\frac{2}{3}\right)$

$= \underline{\underline{131.8°}}$

4 The variables x and y satisfy the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 - 2x^2),$$

and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms. [6]

$$\int \frac{1}{y}\,dy = \int \frac{1}{x} - \frac{2x^2}{x}\,dx$$

$$\ln y = \ln x - 2\int x\,dx$$

$$\ln y = \ln(x) - \frac{\cancel{2}x^2}{\cancel{2}} + c$$

$$\ln y = \ln(x) - x^2 + c$$

$$\ln 2 = 0 - 1 + c$$

$$\ln(2) + 1 = c$$

-2.306

$$\ln y = \ln(x) - x^2 + \ln(2) + 1$$

$$y = e^{\ln(2x) + 1 - x^2}$$

$$y = \frac{e^{\ln(2x)} \times e^1}{e^{x^2}}$$

$$y = \frac{2xe}{e^{x^2}}$$

$$y = 2xe^{1-x^2}$$

5 The curve with equation $y = \sin x \cos 2x$ has one stationary point in the interval $0 < x < \frac{1}{2}\pi$. Find the x-coordinate of this point, giving your answer correct to 3 significant figures. [6]

6

$y = \sin x \cos 2x$

$y' = \sin x(-2\sin 2x) + \cos 2x(\cos x)$

$= -2\sin x \sin 2x + \cos 2x \cos x$

$\cos 2x \cos x = 2\sin 2x \sin x$

$(2\cos^2 x - 1)\cos x = 2(2\sin x \cos x)\sin x$

$2\cos^3 x - \cos x = 4\sin^2 x \cos x$

$\cancel{\cos x}(2\cos^2 x - 1) = 4\sin^2 x \cancel{\cos x}$

$2\cos^2 x - 1 = 4(1 - \cos^2 x)$

$2\cos^2 x - 1 = 4 - 4\cos^2 x$

$6\cos^2 x = 5$

$\cos^2 x = \frac{5}{6}$

$x = \cos^{-1}\left(\sqrt{\frac{5}{6}}\right)$ or $x = \cos^{-1}\left(-\sqrt{\frac{5}{6}}\right)$

$x = \underline{0.421}$ ✓

$\underbrace{x = 2.72 \qquad x = 3.56}_{\text{out of range}}$

6 **(i)** By sketching a suitable pair of graphs, show that the equation

$$5e^{-x} = \sqrt{x}$$

has one root. [2]

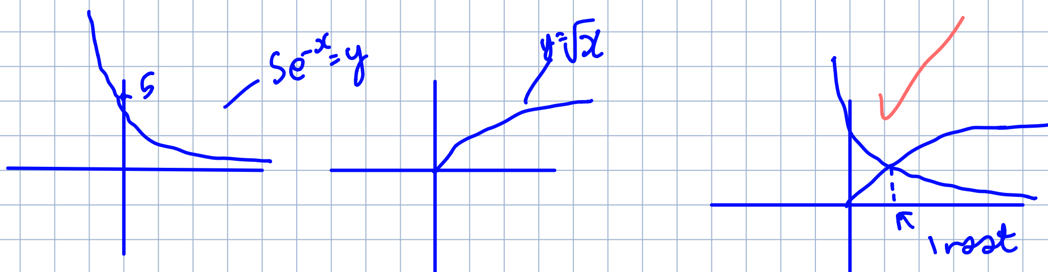
(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \tfrac{1}{2}\ln\left(\frac{25}{x_n}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

i)



ii)

$$x = \ln\left(\frac{25}{x}\right)^{\frac{1}{2}}$$

$$x = \ln\frac{5}{x^{\frac{1}{2}}}$$

$$e^x = \frac{5}{\sqrt{x}}$$

$$e^x = \frac{\frac{5}{e^x}}{\frac{\sqrt{x}}{e^x}}$$

$$e^x = \frac{5e^{-x}}{\sqrt{x}\,e^{-x}}$$

$$(e^x)\sqrt{x}\,e^{-x} = 5e^{-x}$$

$$\frac{\cancel{e^x}\sqrt{x}}{\cancel{e^x}} = 5e^{-x}$$

iii)

$x_1 = 1$

$x_2 = \frac{1}{2}\ln\left(\frac{25}{1}\right) = 1.6094$

$x_3 = 1.3714$

$x_4 = 1.4515$

$x_5 = 1.4231$

$x_6 = 1.4330$

$x_7 = 1.4296$

$x_8 = 1.4308$

$\therefore x = 1.43$

7 The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $\frac{dy}{dx} = \frac{x^2 - 2xy}{x^2 - y^2}$. [4]

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

i) $3x^2 - \left(3x^2\frac{dy}{dx} + y(6x)\right) + 3y^2\frac{dy}{dx} = 0$

$3x^2 - 3x^2\frac{dy}{dx} - 6xy + 3y^2\frac{dy}{dx} = 0$

$\cancel{3}(x^2 - 2xy) = \cancel{3}(x^2 - y^2)\frac{dy}{dx}$

$\underline{\underline{\frac{x^2 - 2xy}{x^2 - y^2}}} = \frac{dy}{dx}$

ii) $x^2 - 2xy = 0$

$x(x - 2y) = 0$

$x = 0$

$0^3 - 0 + y^3 = 3$

$y^3 = 3$

$y = \sqrt[3]{3}$

$\underline{\underline{(0, \sqrt[3]{3})}}$

$x = 2y$

$4y^2$

$(2y)^3 - 3(2y)^2 y + y^3 = 3$

$8y^3 - 12y^3 + y^3 = 3$

$-3y^3 = 3$

$y^3 = -1$

$y = \sqrt[3]{-1} = -1$

$x = 2(-1) = -2$

$\underline{\underline{(-2, -1)}}$

8 Let $f(x) = \dfrac{4x^2 + 12}{(x+1)(x-3)^2}$. $= \dfrac{A}{x+1} + \dfrac{B}{x-3} + \dfrac{C}{(x-3)^2}$

(i) Express $f(x)$ in partial fractions. [5] 5

(ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5] 3

i) $4x^2 + 12 = A(x-3)(x-3) + B(x-3)(x+1) + C(x+1)$

$= A(x^2 - 6x + 9) + B(x^2 - 2x - 3) + Cx + C$

$= Ax^2 - 6Ax + 9A + Bx^2 - 2Bx - 3B + Cx + C$

$4 = A + B$	$0 = -6A - 2B + C$	$12 = 9A - 3B + C$
$B = 4 - A$	$0 = -6A - 2(4-A) + C$	$C = 12 + 3B - 9A$
		$C = 12 + 3(4-A) - 9A$
		$C = 12 + 12 - 3A - 9A$
		$C = 24 - 12A$

$-6A - 8 + 2A + 24 - 12A = 0$

$-16A + 16 = 0$

$16A = 16$

$A = 1$

$\dfrac{1}{(x+1)} + \dfrac{3}{(x-3)} + \dfrac{12}{(x-3)^2}$

$B = 4 - 2 = 3$

$C = 24 - 12(2) = 12$

ii) $(x+1)^{-1} + 3(-3+x)^{-1} + 12(-3+x)^{-2}$

$(1+x)^{-1} \Rightarrow 1 - x + \dfrac{(-1)(-2)(x)^2}{2!} \Rightarrow 1 - x + x^2$

$3(-3+x)^{-1} \to 3 \times -3^{-1}\left(1 - \dfrac{x}{3}\right)^{-1} \Rightarrow -1\left(1 - \dfrac{x}{3}\right)^{-1} = -1\left(1 + \dfrac{x}{3} + \dfrac{(-1)(-2)\left(-\frac{x}{3}\right)^2}{2!}\right)$

$= -1 - \dfrac{x}{3} - \dfrac{1}{9}x^2$

$12 \times 3^{-2}\left(1 - \dfrac{x}{3}\right)^{-2} \Rightarrow \dfrac{4}{3}\left(1 + \dfrac{2}{3}x\right) = \dfrac{4}{3} + \dfrac{8}{9}x$

$\cancel{1} - x + x^2 \quad \cancel{1} - \dfrac{1}{3}x - \dfrac{1}{9}x^2 + \dfrac{4}{3} + \dfrac{8}{9}x$

$\dfrac{4}{3} - \dfrac{4}{9}x + \dfrac{8}{9}x^2$

10 **(a)** Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})i$. Give your answers in the form $x + iy$, where x and y are real and exact. [5]

3

(b) **(i)** On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2i| = 1$ and $\arg(z - 1) = \frac{3}{4}\pi$. [4]

$w - (1+2i)$

(ii) Calculate the least value of $|w - z|$ for points on these loci. [2]

$z - (1 + 0i)$

a) $\sqrt{7 - 6\sqrt{2}i} = x + iy$

$(x+iy)(x+iy) = 7 - 6\sqrt{2}i$

$x^2 + 2ixy - y^2 = 7 - 6\sqrt{2}i$

$7 = x^2 - y^2$ | $2xy = -6\sqrt{2}$

$y^2 = x^2 - 7$ | $4x^2y^2 = 72$

$4x^2(x^2 - 7) = 72$

$4x^4 - 28x^2 - 72 = 0$

$x^4 - 7x^2 - 18$

$\dfrac{7 \pm \sqrt{49 - 4(1)(-18)}}{2 \times 1} = \dfrac{7 \pm 11}{2}$

this is x^2 not x!

$x = 9$ or -2

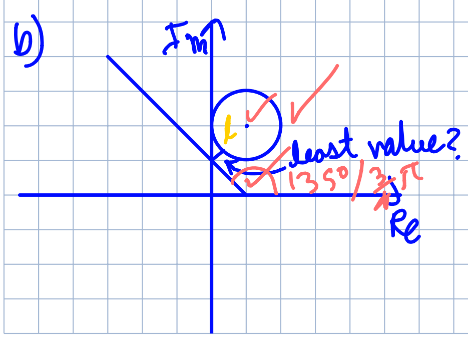
$y^2 = (9^2) - 7$ or $y^2 = (-2)^2 - 7$

$y^2 = 74$ or $y^2 = 9$

$y = \pm\sqrt{74}$ or $y = \pm 3$

$9 \pm \sqrt{74}i$ or $-2 \pm 3i$

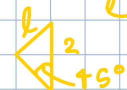
b)



c) $\sin 45 = \dfrac{l}{2}$

SOHCAH TOA

$l = 2\sin 45 = \sqrt{2}$



least value $= \sqrt{2} - 1 \approx 0.414$

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