UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

MATHEMATICS **9709/32**

Paper 3 Pure Mathematics 3 **(P3)** **May/June 2013**

1 hour 45 minutes

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF9)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

This document consists of **3** printed pages and **1** blank page.

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UNIVERSITY *of* CAMBRIDGE
International Examinations

[Turn over

1 Solve the equation $|x - 2| = \left|\frac{1}{3}x\right|$. [3]

2 The sequence of values given by the iterative formula

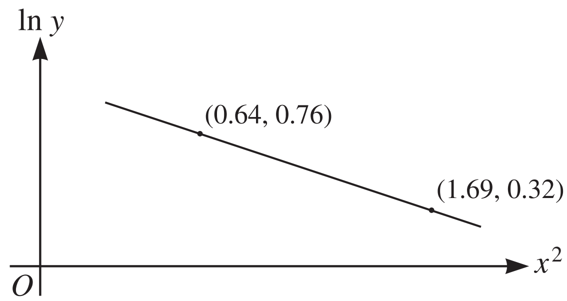
$$x_{n+1} = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)},$$

with initial value $x_1 = 3.5$, converges to α.

(i) Use this formula to calculate α correct to 4 decimal places, showing the result of each iteration to 6 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

3



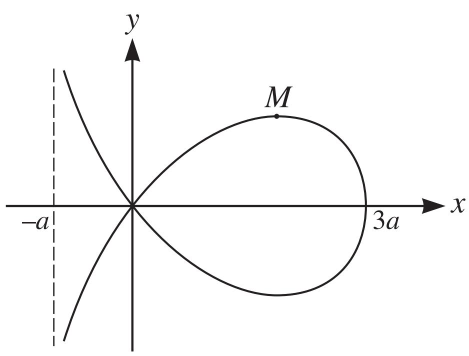
The variables x and y satisfy the equation $y = Ae^{-kx^2}$, where A and k are constants. The graph of $\ln y$ against x^2 is a straight line passing through the points (0.64, 0.76) and (1.69, 0.32), as shown in the diagram. Find the values of A and k correct to 2 decimal places. [5]

4 The polynomial $ax^3 - 20x^2 + x + 3$, where a is a constant, is denoted by $p(x)$. It is given that $(3x + 1)$ is a factor of $p(x)$.

(i) Find the value of a. [3]

(ii) When a has this value, factorise $p(x)$ completely. [3]

5



The diagram shows the curve with equation

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0,$$

where a is a positive constant. The maximum point on the curve is M. Find the x-coordinate of M in terms of a. [6]

6 **(i)** By differentiating $\dfrac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\dfrac{dy}{dx} = \sec x$. [4]

(ii) Using the substitution $x = (\sqrt{3}) \tan \theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3 + x^2)}} \, dx,$$

expressing your answer as a single logarithm. [4]

7 **(i)** By first expanding $\cos(x + 45°)$, express $\cos(x + 45°) - (\sqrt{2}) \sin x$ in the form $R \cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of R correct to 4 significant figures and the value of α correct to 2 decimal places. [5]

(ii) Hence solve the equation

$$\cos(x + 45°) - (\sqrt{2}) \sin x = 2,$$

for $0° < x < 360°$. [4]

8 **(i)** Express $\dfrac{1}{x^2(2x + 1)}$ in the form $\dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{2x + 1}$. [4]

(ii) The variables x and y satisfy the differential equation

$$y = x^2(2x + 1)\frac{dy}{dx},$$

and $y = 1$ when $x = 1$. Solve the differential equation and find the exact value of y when $x = 2$. Give your value of y in a form not involving logarithms. [7]

9 **(a)** The complex number w is such that $\operatorname{Re} w > 0$ and $w + 3w^* = iw^2$, where w^* denotes the complex conjugate of w. Find w, giving your answer in the form $x + iy$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 2i| \leqslant 2$ and $0 \leqslant \arg(z + 2) \leqslant \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for points in this region, giving your answer correct to 2 decimal places. [6]

10 The points A and B have position vectors $2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ respectively. The plane p has equation $x + y = 5$.

(i) Find the position vector of the point of intersection of the line through A and B and the plane p. [4]

(ii) A second plane q has an equation of the form $x + by + cz = d$, where b, c and d are constants. The plane q contains the line AB, and the acute angle between the planes p and q is 60°. Find the equation of q. [7]

1 Solve the equation $|x - 2| = \left|\frac{1}{3}x\right|$. [3]

$(x-2)(x-2) = \frac{1}{9}x^2$

$x^2 - 4x + 4 = \frac{1}{9}x^2$

$\frac{8}{9}x^2 - 4x + 4$

$x = 3$ and $\frac{3}{2}$

2 The sequence of values given by the iterative formula

$$x_{n+1} = \frac{x_n(x_n^3 + 100)}{2(x_n^3 + 25)},$$

with initial value $x_1 = 3.5$, converges to α.

(i) Use this formula to calculate α correct to 4 decimal places, showing the result of each iteration to 6 decimal places. [3]

(ii) State an equation satisfied by α and hence find the exact value of α. [2]

i) $x_1 = 3.5$

$x_2 = \frac{3.5(3.5^3 + 100)}{2(3.5^3 + 25)} = 3.683702$

$x_3 = 3.684031$

$x_4 = 3.684031$

$\therefore x = 3.6840$

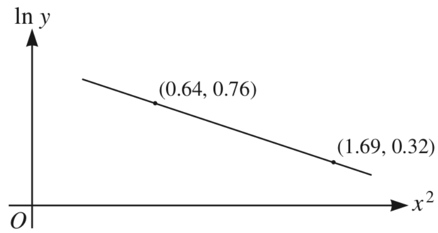
ii) $2x(x^3 + 25) = x^4 + 100x$

$2x^4 + 50x = x^4 + 100x$

$x^{\cancel{4}\,3} = 50\cancel{x}$

$x = \sqrt[3]{50}$

3



The variables x and y satisfy the equation $y = Ae^{-kx^2}$, where A and k are constants. The graph of $\ln y$ against x^2 is a straight line passing through the points (0.64, 0.76) and (1.69, 0.32), as shown in the diagram. Find the values of A and k correct to 2 decimal places. [5]

$y = m x + c$

$\ln y = \ln Ae^{-kx^2}$

$\ln y = \ln A + \ln e^{-kx^2}$

$\ln y = \underbrace{\ln A}_{y\text{ intercept}} \underbrace{- k}_{\text{gradient}} x^2$

$\frac{0.76 - 0.32}{0.64 - 1.69} = -0.419047$

$-k = -0.419047$

$k = 0.42$

$c = y - mx$

$= 0.76 - (-0.419047)(0.64)$

$c = 1.02819$

$1.02819 = \ln A$

$A = 2.796$

$= 2.80$

4 The polynomial $ax^3 - 20x^2 + x + 3$, where a is a constant, is denoted by $p(x)$. It is given that $(3x + 1)$ is a factor of $p(x)$.

(i) Find the value of a. [3]

$12x^3 - 20x^2 + x + 3$

(ii) When a has this value, factorise $p(x)$ completely. [3]

i) $x = -\frac{1}{3}$

$\frac{-a}{27} - \frac{20}{9} - \frac{1}{3} + 3 = 0$

$\frac{-a}{27} = -\frac{4}{9}$

$-a = \frac{-108}{9}$

$a = 12$

ii) $(3x+1)(cx^2 + dx + e)$

$3c = 12$ $\quad e = 3$

$c = 4$

$3e + d = 1$

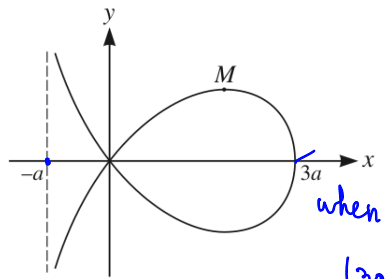
$9 + d = 1$

$d = -8$

$4x^2 - 8x + 3$

$(3x+1)(2x-3)(2x-1)$

5



$y^2 = \dfrac{3a^2 - x^2}{x+a}$

$0^2 = \dfrac{3a^2 - (3a)^2}{4a} = \dfrac{0}{4a} = 0$

when $x = 3a$, $y = 0$

$(3a)^3 - 3a(3a)^2 =$

$27a^3 - 27a^3 = 0$

$x^2 + y^2(x+a) - 3ax^2$

The diagram shows the curve with equation

$$x^3 + xy^2 + ay^2 - 3ax^2 = 0,$$

where a is a positive constant. The maximum point on the curve is M. Find the x-coordinate of M in terms of a. [6]

$3x^2 + (1)(y^2) + (x)(2y)\left(\frac{dy}{dx}\right) + 2ay\frac{dy}{dx} - 6ax = 0$

$3x^2 + y^2 + 2xy\frac{dy}{dx} + 2ay\frac{dy}{dx} - 6ax = 0$

$3x^2 + y^2 - 6ax = (-2xy - 2ay)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3x^2 + y^2 - 6ax}{-2xy - 2ay} = 0$

$3x^2 + y^2 - 6ax = 0$

$3x^2 - 6ax + y^2 = 0$

$y^2 = 6ax - 3x^2$

$x^3 + x(6ax - 3x^2) + a(6ax - 3x^2) - 3ax^2 = 0$

$x^3 + \cancel{6ax^2} - 3x^3 + 6a^2x - \cancel{3ax^2} - \cancel{3ax^2} = 0$

$\cancel{2}x^{\cancel{3}\,2} = \cancel{6}\,3\,a^2\cancel{x}$

$x^2 = 3a^2$

$x = \sqrt{3}\,a$

6 **(i)** By differentiating $\frac{1}{\cos x}$, show that the derivative of $\sec x$ is $\sec x \tan x$. Hence show that if $y = \ln(\sec x + \tan x)$ then $\frac{dy}{dx} = \sec x$. [4]

(ii) Using the substitution $x = (\sqrt{3}) \tan \theta$, find the exact value of

$$\int_1^3 \frac{1}{\sqrt{(3 + x^2)}}\, dx,$$

expressing your answer as a single logarithm. [4]

i) $\frac{d}{dx}\frac{1}{\cos x} = \frac{(0)\cos x + \sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$

$= \tan x \sec x$

$y = \ln(\sec x + \tan x) \qquad \frac{dy}{dx} = \frac{1}{\sec x + \tan x}(\sec x \tan x) + (\sec^2 x)$

$= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$

$= \frac{\sec x \cancel{(\sec x + \tan x)}}{\cancel{\sec x + \tan x}}$

$\frac{dy}{dx} = \sec x$

ii) $x = \sqrt{3}\tan\theta \qquad \theta = \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$

$x^2 = 3\tan^2\theta$

$dx = \sqrt{3}\sec^2\theta\, d\theta \qquad \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{\sqrt{3 + 3\tan^2\theta}} \sqrt{3}\sec^2\theta\, d\theta$

$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\sqrt{3}\sec^2\theta}{\sqrt{3(1+\tan^2\theta)}}\, d\theta \Rightarrow \int \frac{\sqrt{3}\sec^2\theta}{\sqrt{3\sec^2\theta}}\, d\theta \Rightarrow \int \frac{\cancel{\sqrt{3}}\sec\cancel{\theta}}{\cancel{\sqrt{3}}\,\cancel{\sec\theta}}$

$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sec\theta\, d\theta$

$\left[\ln(\sec\theta + \tan\theta)\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \qquad \sec\theta = \sqrt{\tan^2\theta + 1}$

$\ln(2 + \sqrt{3}) - \ln\left(\frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{3}\right)$

$\ln(2 + \sqrt{3}) - \ln(\sqrt{3}) \Rightarrow \ln\left(\frac{2 + \sqrt{3}}{\sqrt{3}}\right)$

7 **(i)** By first expanding $\cos(x + 45°)$, express $\cos(x + 45°) - (\sqrt{2})\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of R correct to 4 significant figures and the value of α correct to 2 decimal places. [5]

(ii) Hence solve the equation

$$\cos(x + 45°) - (\sqrt{2})\sin x = 2,$$

for $0° < x < 360°$. [4]

i) $\cos x \cos 45 - \sin x \sin 45 \quad - \sqrt{2}\sin x$

$\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x \quad -\sqrt{2}\sin x$

$\frac{\sqrt{2}}{2}\cos x - \frac{3\sqrt{2}}{2}\sin x$

$R = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{-3\sqrt{2}}{2}\right)^2}$ $\qquad \theta = \tan^{-1}\left(\frac{\left(\frac{3\sqrt{2}}{2}\right)}{\left(\frac{\sqrt{2}}{2}\right)}\right) = \underline{\underline{71.57}}$

$= 2.236$

$2.236\cos(x + 71.57)$

ii) $x + 71.57 = \cos^{-1}\left(\frac{2}{2.236}\right)$ or $x + 71.57 = 360 - \cos^{-1}\left(\frac{2}{2.236}\right)$

$x + 71.57 = 26.5615$ or $x + 71.57 = 333.438$

$x = -45.0084$ ✗ $\qquad x = \underline{\underline{261.9°}}$

$\downarrow +360$

or $= \underline{\underline{315°}}$

8 **(i)** Express $\dfrac{1}{x^2(2x+1)}$ in the form $\dfrac{A}{x^2}+\dfrac{B}{x}+\dfrac{C}{2x+1}$. [4]

$x^2(2x+1)$

(ii) The variables x and y satisfy the differential equation

$$y = x^2(2x+1)\frac{dy}{dx},$$

and $y = 1$ when $x = 1$. Solve the differential equation and find the exact value of y when $x = 2$. Give your value of y in a form not involving logarithms. [7]

i) $1 = A(2x+1) + B(x)(2x+1) + Cx^2$

sub $x = 0$

$1 = A$

sub $x = -\frac{1}{2}$

$1 = 0 + 0 + \frac{C}{4}$

$C = 4$

$0 = 2A + B$

$0 = 2 + B$

$B = -2$

$\frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x+1}$

ii) $\int \frac{1}{x^2(2x+1)}\,dx = y\,dy$

$\int \frac{1}{x^2} - \frac{2}{x} + \frac{4}{2x+1}\,dx = \int \frac{1}{y}\,dy$

$-\frac{1}{x} - 2\ln x + 2\ln(2x+1) = \ln y + C$

$-1 + 2\ln 3 =$

$2\ln(3) - 1 = C$

$\ln y = -\frac{1}{2} - 2\ln 2 + 2\ln 5 + 1 - 2\ln 3$

$\ln y = \frac{1}{2} + 2\ln 5 - 2\ln 2 - 2\ln 3$

$\ln y = \frac{1}{2} + 2\ln\left(\frac{5}{6}\right) \rightarrow \frac{1}{2} + \ln\frac{25}{36}$

$y = e^{\frac{1}{2}} \times e^{\ln\frac{25}{36}}$

$= \frac{25e^{\frac{1}{2}}}{36}$

9 **(a)** The complex number w is such that $\operatorname{Re} w > 0$ and $w + 3w^* = \mathrm{i}w^2$, where w^* denotes the complex conjugate of w. Find w, giving your answer in the form $x + \mathrm{i}y$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z which satisfy both the inequalities $|z - 2\mathrm{i}| \leqslant 2$ and $0 \leqslant \arg(z + 2) \leqslant \frac{1}{4}\pi$. Calculate the greatest value of $|z|$ for points in this region, giving your answer correct to 2 decimal places. [6]

$2-(0+2i)$ $\quad 2-(-2+0i)$

$(x+iy)(x+iy)$

x^2+2ixy

a) $(x+iy)+3(x-iy) = i(x^2-y^2+2ixy)$

$x+iy+3x-3iy = x^2i - y^2i - 2xy$

$4x - 2iy = x^2i - y^2i - 2xy$

$4\cancel{x} = -2\cancel{x}y$

$y = -2$

$-2y = x^2-(-2)^2$

$-2(-2) = x^2-4$

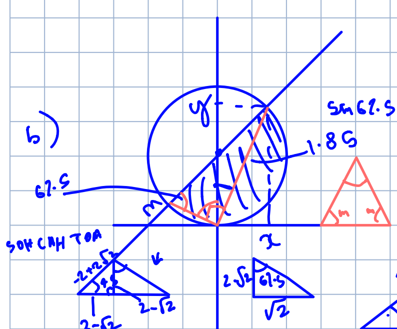
$4 = x^2 - 4$

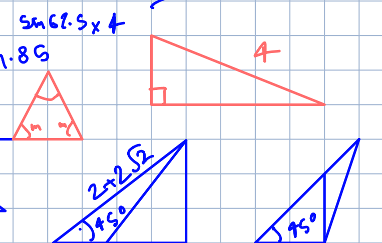
$8 = x^2$

$x = \sqrt{8}$

$w = \sqrt{8} - 2i$

b)

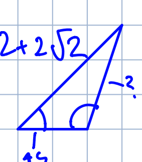




SOH CAH

$\cos 45 = \frac{2}{H}$

$H = \frac{2}{\cos 45}$



$\frac{2}{\sin 45} = \frac{2+2\sqrt{2}}{}$

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