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Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2015**

1 hour 45 minutes

* 2 3 9 0 7 4 1 5 6 8 *

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF9)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

This document consists of **3** printed pages and **1** blank page.

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CAMBRIDGE International Examinations

1 Solve the inequality $|2x - 5| > 3|2x + 1|$. [4]

2 Using the substitution $u = 3^x$, solve the equation $3^x + 3^{2x} = 3^{3x}$ giving your answer correct to 3 significant figures. [5]

3 The angles θ and ϕ lie between 0° and 180°, and are such that

$$\tan(\theta - \phi) = 3 \quad \text{and} \quad \tan\theta + \tan\phi = 1.$$

Find the possible values of θ and ϕ. [6]

4 The equation $x^3 - x^2 - 6 = 0$ has one real root, denoted by α.

(i) Find by calculation the pair of consecutive integers between which α lies. [2]

(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

5 The equation of a curve is $y = e^{-2x}\tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where a and b are constants. [5]

(ii) Explain why the gradient of the curve is never negative. [1]

(iii) Find the value of x for which the gradient is least. [1]

6 The polynomial $8x^3 + ax^2 + bx - 1$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(2x + 1)$ the remainder is 1.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, factorise $p(x)$ completely. [3]

7 The points A, B and C have position vectors, relative to the origin O, given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

The plane m is perpendicular to AB and contains the point C.

(i) Find a vector equation for the line passing through A and B. [2]

(ii) Obtain the equation of the plane m, giving your answer in the form $ax + by + cz = d$. [2]

(iii) The line through A and B intersects the plane m at the point N. Find the position vector of N and show that $CN = \sqrt{(13)}$. [5]

8 The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = (x + 2)\sin^2 2\theta,$$

and it is given that $x = 0$ when $\theta = 0$. Solve the differential equation and calculate the value of x when $\theta = \frac{1}{4}\pi$, giving your answer correct to 3 significant figures. [9]

9 The complex number $3 - \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

(i) On an Argand diagram with origin O, show the points A, B and C representing the complex numbers u, u^* and $u^* - u$ respectively. What type of quadrilateral is $OABC$? [4]

(ii) Showing your working and without using a calculator, express $\dfrac{u^*}{u}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\dfrac{u^*}{u}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right).$$ [3]

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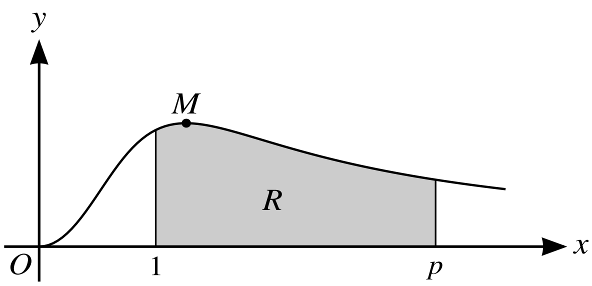

The diagram shows the curve $y = \dfrac{x^2}{1 + x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. [6]

1 Solve the inequality $|2x - 5| > 3|2x + 1|$. [4]

$$(2x-5)(2x-5) > 9(2x+1)(2x+1)$$

$$4x^2 - 20x + 25 > 9(4x^2 + 4x + 1)$$

$$4x^2 - 20x + 25 > 36x^2 + 36x + 9$$

$$32x^2 + 56x - 16 < 0$$

$$x = \frac{1}{4}, \quad x = -2$$

$$-2 < x < \frac{1}{4}$$

test $x = -1$

$$|2(-1)-5| > 3(2(-1)+1)$$

$$7 > 3$$

✓

2 Using the substitution $u = 3^x$, solve the equation $3^x + 3^{2x} = 3^{3x}$ giving your answer correct to 3 significant figures. [5]

$$3^x + (3^x)^2 = (3^x)^3$$

$$u + u^2 = u^3$$

$$1 + u = u^2$$

$$u^2 - u - 1 = 0$$

$u = 1.618$ $\qquad$ $u = -0.618$ ✗ rejected

$$3^x = 1.618$$

$$x = \frac{\ln 1.618}{\ln 3}$$

$$x = 0.438$$

3 The angles θ and ϕ lie between 0° and 180°, and are such that

$$\tan(\theta - \phi) = 3 \quad \text{and} \quad \tan\theta + \tan\phi = 1.$$

Find the possible values of θ and ϕ. [6]

3

$$\frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = 3$$

$$\tan\theta = 1 - \tan\phi$$

$$\frac{1 - \tan\phi - \tan\phi}{1 + (1 - \tan\phi)\tan\phi} = 3$$

$$\frac{1 - 2\tan\phi}{1 + \tan\phi - \tan^2\phi} = 3$$

$$1 - 2\tan\phi = 3\tan\phi - 3\tan^2\phi + 3$$

$$3\tan^2\phi - 5\tan\phi - 2 = 0$$

$$\tan\phi = \frac{5 \pm \sqrt{25 - 4(3)(-2)}}{2(3)}$$

$$\tan\phi = \frac{5 \pm 7}{6}$$

$\tan\phi = 2$ $\qquad$ $\tan\phi = -\frac{1}{3}$

$\phi = 63.4°$, $161.6°$

$$\theta = \tan^{-1}(3) + \phi$$

$$= 135$$

$$\theta = 53.1$$

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(ii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \sqrt{\left(x_n + \frac{6}{x_n}\right)}$$

converges, then it converges to α. [2]

(iii) Use this iterative formula to determine α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

4) 2 and 3

as when $x = 2$, value was < 0

" " $= 3$, " " > 0

2

ii) $x^3 = x^2 + 6$

$x^2 = x + \frac{6}{x}$

$x = \sqrt{\left(x + \frac{6}{x}\right)}$

iii) let $x_1 = 2$

$x_2 = 2.21796$

$x_3 = 2.21882$

$x_4 = 2.21877$

$x_5 = 2.21878$

$x = 2.219$

5 The equation of a curve is $y = e^{-2x}\tan x$, for $0 \leqslant x < \frac{1}{2}\pi$.

(i) Obtain an expression for $\frac{dy}{dx}$ and show that it can be written in the form $e^{-2x}(a + b\tan x)^2$, where a and b are constants. [5] 5

(ii) Explain why the gradient of the curve is never negative. [1] 1

(iii) Find the value of x for which the gradient is least. [1] 1

i) $e^{-2x}(\sec^2 x) + (-2e^{-2x})\tan x$

$\frac{dy}{dx} = e^{-2x}\sec^2 x - 2e^{-2x}\tan x$

$e^{-2x}(\sec^2 x - 2\tan x)$ $\tan x - 1$

4 $e^{-2x}(\tan^2 x + 1 - 2\tan x)$

↓

5 $e^{-2x}(\tan x - 1)^2$

$e^{-2x}(-1 + \tan x)^2$

ii) $(-1+\tan x)$ is squared which means its always +ve and e^{-2x} is also always +ve. so $\frac{dy}{dx}$ is always +ve

iii) $e^{-2x}(-1+\tan x)^2 = 0$

$-1 + \tan x =$

$x = \tan^{-1}(1)$

$= \frac{1}{4}\pi$

6 The polynomial $8x^3 + ax^2 + bx - 1$, where a and b are constants, is denoted by $p(x)$. It is given that $(x + 1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(2x + 1)$ the remainder is 1.

(i) Find the values of a and b. ✓ [5]

(ii) When a and b have these values, factorise $p(x)$ completely. ✗ [3]

i)

$x+1=0$
$x=-1$

$-8+a-b-1=0$

$a-b=9$

$a=9+b$

$2x+1=0$
$x=-\frac{1}{2}$

$8(-\frac{1}{2})^3 + a(-\frac{1}{2})^2 + b(-\frac{1}{2}) - 1 = 1$

$-1 + \frac{1}{4}a - \frac{1}{2}b - 1 = 1$

$\frac{1}{4}a - \frac{1}{2}b = 3$

$a - 2b = 12$

$9+b-2b=12$

$9-b=12$

$\underline{b=-3}$

$\therefore a = 9-3 = \underline{6}$

ii) $(x+1)(cx^2+dx+e)$

$8x^2+6x^2-3x-1$

$cx^3 + +dx^2 + ex + cx^2 + dx + e$

$e=-1$

$c=8$

$d+c = 6$

$d = 6-8$

$= -2$

$ac=-8$

$8x^2-2x-1$

$8x^2-4x+2x-1$

$4x(2x-1)+1(2x-1)$

$(4x+1)(2x-1)$

$\therefore p(x) = (x+1)(4x+1)(2x-1)$

7 The points A, B and C have position vectors, relative to the origin O, given by

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The plane m is perpendicular to AB and contains the point C.

(i) Find a vector equation for the line passing through A and B. [2] 2

(ii) Obtain the equation of the plane m, giving your answer in the form $ax + by + cz = d$. [2] 0

(iii) The line through A and B intersects the plane m at the point N. Find the position vector of N and show that $CN = \sqrt{(13)}$. [5] 0

i) $i + 2j + \lambda(2i - 2j + k)$

8 The variables x and θ satisfy the differential equation

$$\frac{dx}{d\theta} = (x + 2)\sin^2 2\theta,$$

and it is given that $x = 0$ when $\theta = 0$. Solve the differential equation and calculate the value of x when $\theta = \frac{1}{4}\pi$, giving your answer correct to 3 significant figures. [9] 9

$\int \frac{1}{x+2}\,dx = \int \sin^2 2\theta \, d\theta$

$\cos 4\theta = 1 - 2\sin^2 2\theta$

$\sin^2 2\theta = \frac{1}{2} - \frac{\cos 4\theta}{2}$

$\ln|x+2| = \int \frac{1}{2} - \frac{\cos 4\theta}{2}\, d\theta$

$\ln|x+2| = \frac{1}{2}\theta - \frac{1}{2}\int \cos 4\theta \, d\theta$

$\ln|x+2| = \frac{1}{2}\theta - \frac{1}{2}\left(\frac{1}{4}\sin 4\theta\right)$

$\ln(x+2) = \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta + C$

$\ln 2 = 0 - 0 + C$

$C = \ln 2$

$\ln(x+2) = \frac{1}{8}\pi - \frac{1}{8}\sin\left(4 \times \frac{1}{4}\pi\right) + \ln 2$

$\ln(x+2) = \frac{1}{8}\pi + \ln 2$

$\ln(x+2) - \ln 2 = \frac{1}{8}\pi$

$\ln \frac{1}{2}x + 1 = \frac{1}{8}\pi$

$\frac{1}{2}x + 1 = e^{\frac{1}{8}\pi}$

$\frac{1}{2}x = e^{\frac{1}{8}\pi} - 1$

$x = \underline{\underline{0.962}}$

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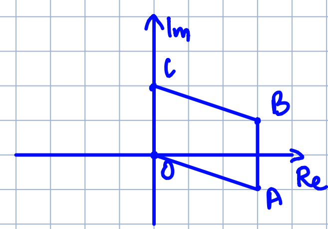
(iii) By considering the argument of $\frac{u^*}{u}$, prove that

$$\tan^{-1}\left(\tfrac{3}{4}\right) = 2\tan^{-1}\left(\tfrac{1}{3}\right).$$ [3]

i) $u = 3 - i$

$u^* = 3 + i$

$u^* - u = 3+i - 3-i$
$= 0 + 2i$



$\therefore$ parallelogram

ii) $\frac{3+i}{3-i} \times \frac{(3+i)}{(3+i)}$

$\frac{9+6i-1}{9-i^2}$

$\frac{8+6i}{9+1} = \frac{8}{10} + \frac{6}{10}i$

$\frac{4}{5} + \frac{3}{5}i$

iii) $\tan^{-1}\left(\frac{3/\cancel{5}}{4/\cancel{5}}\right) = \arg\left(\frac{u^*}{u}\right)$

$\arg\left(\frac{u^*}{u}\right) = \arg u^* - \arg u$

$\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(-\frac{1}{3}\right)$

$\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$\tan^{-1}\left(\frac{3}{4}\right) = 2\tan^{-1}\left(\frac{1}{3}\right)$

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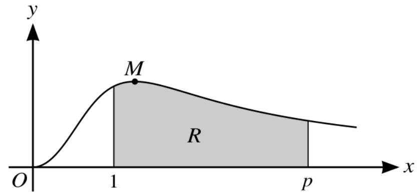

The diagram shows the curve $y = \dfrac{x^2}{1+x^3}$ for $x \geqslant 0$, and its maximum point M. The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = p$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Calculate the value of p for which the area of R is equal to 1. Give your answer correct to 3 significant figures. [6]

i) $\dfrac{dy}{dx} = 0 = \dfrac{(1+x^3)(2x) - (3x^2)(x^2)}{(1+x^3)^2}$

$= 2x + 2x^4 - 3x^4 = 0$

$2x - x^4 = 0$

$x(2-x^3) = 0$

$x = 0$ ✗ $\qquad x^3 = 2$

$x = \sqrt[3]{2}$

ii) $\dfrac{1}{3}\displaystyle\int \dfrac{3x^2}{1+x^3}\,dx$

$\left[\dfrac{1}{3}\ln(1+x^3)\right]_1^p = 1$

$\dfrac{1}{3}\ln(1+p^3) - \dfrac{1}{3}\ln(2) = 1$

$\dfrac{1}{3}\ln\left(\dfrac{1+p^3}{2}\right) = 1$

$\ln\dfrac{1+p^3}{2} = 3$

$\dfrac{1+p^3}{2} = e^3$

$p = \sqrt[3]{2e^3 - 1}$

$= 3.40$

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