Cambridge Assessment International Education

32/75

Cambridge International AS & A Level

CANDIDATE NAME	

CENTRE NUMBER		CANDIDATE NUMBER	

1862605169

MATHEMATICS **9709/33**

Paper 3 Pure Mathematics 3 **May/June 2020**

1 hour 50 minutes

You must answer on the question paper.

You will need: List of formulae (MF19)

INSTRUCTIONS

- Answer **all** questions.
- Use a black or dark blue pen. You may use an HB pencil for any diagrams or graphs.
- Write your name, centre number and candidate number in the boxes at the top of the page.
- Write your answer to each question in the space provided.
- Do **not** use an erasable pen or correction fluid.
- Do **not** write on any bar codes.
- If additional space is needed, you should use the lined page at the end of this booklet; the question number or numbers must be clearly shown.
- You should use a calculator where appropriate.
- You must show all necessary working clearly; no marks will be given for unsupported answers from a calculator.
- Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

INFORMATION

- The total mark for this paper is 75.
- The number of marks for each question or part question is shown in brackets [ ].

This document has **20** pages. Blank pages are indicated.

1 Solve the inequality $|2x - 1| > 3|x + 2|$. [4]

1 ✗

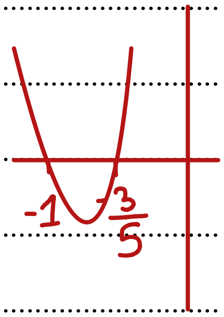
$$(2x-1)^2 > [3(x+2)]^2$$

$$4x^2 - 4x + 1 > 9(x^2 + 4x + 4)$$

$$4x^2 - 4x + 1 > 9x^2 + 4x + 4$$

$$5x^2 + 8x + 3 < 0$$

$$x_{1,2} = -\frac{3}{5}, -1$$



$$x < -1$$

$$x > -\frac{3}{5}$$

checking $x < -1$

$$|2(-2)-1| > 3|-1+2|$$

$$5 > 3$$

✓

checking $x > -\frac{3}{5}$ ✗

$$|2(0)-1| > 3|0+2|$$

$$1 \ngtr 6$$

$\therefore$ $x < -1$

2 Find the exact value of $\int_0^1 (2-x)e^{-2x}\,dx$. [5]

LIAIE

3

$\int_0^1 (2-x)\,e^{-2x}$

$u = 2-x \qquad u' = -1$

$v = -\frac{1}{2}e^{-2x} \qquad v' = e^{-2x}$

$(2-x)\left(-\frac{1}{2}e^{-2x}\right) - \int \left(-\frac{1}{2}e^{-2x}\right)(-1)\,dx$

$\frac{x}{2}e^{-2x} - e^{-2x} - \frac{1}{2}\int e^{-2x}\,dx$

$\frac{x}{2}e^{-2x} - e^{-2x} - \frac{1}{2}\left(-\frac{1}{2}\right)e^{-2x}$

$\frac{x}{2}e^{-2x} - e^{-2x} + \frac{1}{4}e^{-2x}$

$\left[\frac{e^{-2x}}{2\cancel{4}}\left(2x - \frac{3}{\cancel{4}}\right)\right]_0^1$

$\left(\frac{e^{-2}}{4\cancel{4}} \times -1\cancel{4}\right) - \left(\frac{1}{4\cancel{2}} \times -\frac{3}{\cancel{4}}\right)$

$\cancel{\frac{e^{-2}}{8} + \frac{3}{8}} \qquad -\frac{e^{-2}}{4} + \frac{3}{4}$

$\cancel{\frac{1}{8}\left(e^{-2} + 3\right)} \qquad \frac{1}{4}\left(3 - e^{2}\right)$

3 **(a)** Show that the equation

$$\ln(1 + e^{-x}) + 2x = 0$$

*

can be expressed as a quadratic equation in e^x. ? [2]

$\ln(1 + e^{-x}) = -2x$

$1 + e^{-x} = e^{-2x}$

$\frac{1}{e^{2x}} = 1 + e^{-x}$

$e^{2x} + e^{2x}(e^{-x}) = 1$

$e^{2x} + e^{x} - 1 = 0$ ✓

(b) Hence solve the equation $\ln(1 + e^{-x}) + 2x = 0$, giving your answer correct to 3 decimal places. [4]

$\frac{-b \pm \sqrt{b^2 - 4ac}}{ac}$ $\quad x = e^x$ $\quad$ 2

$x^2 + x - 1$

$a = 1, \ b = 1, \ c = -1$

$\frac{-1 \pm \sqrt{1 - 4(1)(-1)}}{2}$

$\frac{-1 \pm \sqrt{1 + 4}}{2}$

$\frac{-1 + \sqrt{5}}{2}, \ \frac{-1 - \sqrt{5}}{2}$

$\cancel{e^x = -1.618}, \ e^x = 0.618$

$x = \ln 0.618 = -0.481$

4 The equation of a curve is $y = x \tan^{-1}\left(\frac{1}{2}x\right)$.

22

(a) Find $\frac{dy}{dx}$. [3]

$\frac{dy}{dx}$ 2

1

$$\frac{dy}{dx} = \tan^{-1}\left(\tfrac{1}{2}x\right)(1) + x\left(\frac{1}{1+\left(\frac{x}{2}\right)^2}\right)$$

$$= \tan^{-1}\left(\tfrac{1}{2}x\right) + \left(\frac{x}{\frac{4+x^2}{4}}\right)$$

$$= \tan^{-1}\left(\tfrac{1}{2}x\right) + \left(x \times \frac{4}{4+x^2}\right)$$

$1 + \frac{x^2}{4}$

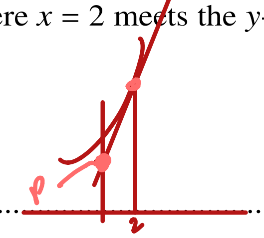
$$= \tan^{-1}\left(\tfrac{1}{2}x\right) + \frac{4x}{x^2+4} \times \frac{1}{2}$$

$$\tan^{-1}\left(\tfrac{1}{2}x\right) + \frac{4x}{2x^2+8}$$

$$+ \frac{2x}{2(x^2+4)} = \frac{2x}{x^2+4}$$

(b) The tangent to the curve at the point where $x = 2$ meets the y-axis at the point with coordinates $(0, p)$.

Find p. [3] 0



$$m = \tan^{-1}(1) + \frac{4}{6} = 1.452$$

$y = 1.452x + c$

$y = c = p$

$p = 1.452(0) + p$

$0 = 0$

5 By first expressing the equation

$$\tan\theta\tan(\theta + 45^\circ) = 2\cot 2\theta$$

as a quadratic equation in $\tan\theta$, solve the equation for $0^\circ < \theta < 90^\circ$. [6]

$$\tan\theta\left(\frac{\tan\theta + 1}{1 - \tan\theta}\right) = 2 \times \frac{1}{\tan 2\theta}$$

$$\frac{\tan^2\theta + \tan\theta}{1 - \tan\theta} = \cancel{2} \times \left(\frac{1 - \tan^2\theta}{\cancel{2}\tan\theta}\right)$$

$$\frac{\tan^2\theta + \tan\theta}{1 - \tan\theta} = \frac{1 - \tan^2\theta}{\tan\theta}$$

$$\tan^3\theta + \tan^2\theta = (1 - \tan^2\theta)(1 - \tan\theta)$$

$$\cancel{\tan^3\theta} + \tan^2\theta = 1 - \tan^2\theta - \tan\theta + \cancel{\tan^3\theta}$$

$$2\tan^2\theta + \tan\theta - 1 = 0$$

$$\frac{-1 \pm \sqrt{1 - 4(-1)(2)}}{4}$$

$$\frac{-1 \pm \sqrt{9}}{4}$$

$$\frac{-1 \pm 3}{4} = \frac{1}{2}, -1$$

$\theta_1 = \tan^{-1}\left(\frac{1}{2}\right) = \underline{26.6}$

$\theta_2 = \tan^{-1}(1) = 45$, $180 - 45 = 135$ ✗

$\theta = 26.6$

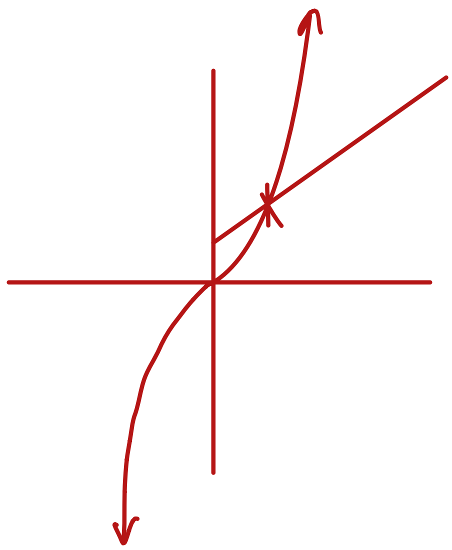
c	1	2	3
y	1	32	243

$y = 2 + x$

x	1	2	3
y	3	4	5

6 ? **(a)** By sketching a suitable pair of graphs, show that the equation $x^5 = 2 + x$ has exactly one real root. [2]

$y = x^5$



$x^5 - x - 2 = 0$

Intersect at 1 point only, between 1 and 2

(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part **(a)**. [2]

0

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$x_1 = \frac{4(1.5)^5 + 2}{5(1.5)^4 - 1} = 1.33162$

$x_2 = 1.27352$

$x_3 = 1.26724$

$x_4 = 1.26717$

$x_5 = 1.26717$

$\therefore x = 1.267$

7 Let $f(x) = \dfrac{2}{(2x-1)(2x+1)}$.

(a) Express $f(x)$ in partial fractions. [2]

$\dfrac{A}{2x-1} + \dfrac{B}{2x+1}$

$2 = A(2x+1) + B(2x-1)$

$2 = 2Ax + A + 2Bx - B$

$\therefore 2 = A - B$

$0 = 2A + 2B$

$A = 1$

$B = -1$ $\quad \dfrac{1}{2x-1} - \dfrac{1}{2x+1}$

(b) Using your answer to part **(a)**, show that

$$(f(x))^2 = \frac{1}{(2x-1)^2} - \frac{1}{2x-1} + \frac{1}{2x+1} + \frac{1}{(2x+1)^2}.$$ [2]

$\left[(2x-1)^{-1} - (2x+1)^{-1}\right] \times \left[(2x-1)^{-1} - (2x+1)^{-1}\right]$

$(2x-1)^{-2} - 2(2x-1)^{-1}(2x+1)^{-1} + (2x+1)^{-2}$

$\dfrac{1}{(2x-1)^2} - 2\left(\dfrac{1}{2x-1}\right)\left(\dfrac{1}{2x+1}\right) + \dfrac{1}{(2x+1)^2}$

$\dfrac{1}{(2x-1)^2} - \dfrac{2}{(2x-1)(2x+1)} + \dfrac{1}{(2x+1)^2}$

$\to f(x)$

$\dfrac{1}{(2x-1)^2} - \dfrac{1}{2x-1} + \dfrac{1}{2x+1} + \dfrac{1}{(2x+1)^2}$

(c) Hence show that $\int_1^2 (\mathrm{f}(x))^2\,\mathrm{d}x = \frac{2}{5} + \frac{1}{2}\ln\left(\frac{5}{9}\right)$. [5]

1

$\int \frac{1}{(2x-1)^2} - \int \frac{1}{2x-1} + \int \frac{1}{2x+1} + \int \frac{1}{(2x+1)^2}$

$u = (2x-1)$

$\int \frac{1}{u^2}$

$\int u^{-2}$

$\int$

$\int (2x-1)^{-2} - \frac{1}{2}\ln 2x-1 + \frac{1}{2}\ln 2x+1 + \int (2x+1)^{-2}$

$-\frac{1}{2}(2x-1)^{-1} - \frac{1}{2}\ln\left[(2x-1)(2x+1)\right] + -\frac{1}{2}(2x+1)^{-1}$

$\left[-\frac{1}{2(2x-1)} - \frac{1}{2}\ln(4x^2-1) - \frac{1}{2(2x+1)}\right]_1^2$

$\left(-\frac{1}{6} - \frac{1}{2}\ln 15 - \frac{1}{10}\right) - \left(-\frac{1}{2} + \frac{1}{2}\ln 3 - \frac{1}{6}\right)$

$-\frac{4}{15} - \frac{1}{2}\ln 15 - \left(\frac{1}{2}\ln 3 - \frac{2}{3}\right)$

$-\frac{4}{15} - \frac{1}{2}\ln 15 - \frac{1}{2}\ln 3 + \frac{2}{3}$

$\frac{2}{5} - \frac{1}{2}\ln\frac{15}{3}$

2

X

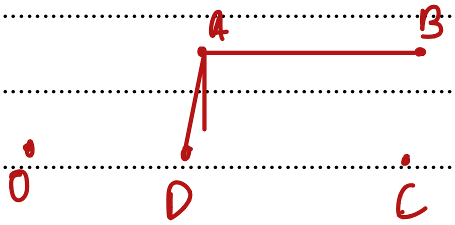
8 Relative to the origin O, the points A, B and D have position vectors given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OD} = 3\mathbf{i} + 2\mathbf{k}.$$

A fourth point C is such that $ABCD$ is a parallelogram.

(a) Find the position vector of C and verify that the parallelogram is not a rhombus. [5]

$$\overrightarrow{AB} = \begin{pmatrix}2\\5\\3\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix}$$



$$\overrightarrow{BD} = \begin{pmatrix}3\\0\\2\end{pmatrix} - \begin{pmatrix}2\\5\\3\end{pmatrix} = \begin{pmatrix}1\\-5\\-1\end{pmatrix}$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \cancel{\overrightarrow{AB}} + \overrightarrow{BD} - \cancel{\overrightarrow{AB}}$$

$$= \begin{pmatrix}1\\2\\1\end{pmatrix} + \begin{pmatrix}1\\-5\\-1\end{pmatrix} = \begin{pmatrix}2\\-3\\0\end{pmatrix}$$

Rhombus, all 4 sides are equal, Parallel, 2 pairs of sides are equal

$\therefore$ AB = CD and AC = BD

$$\sqrt{1^2+3^2+2^2} = \sqrt{1^2+3^2+2^2} \checkmark \qquad \sqrt{1^2+5^2+1^2} = \sqrt{1^2+5^2+1^2} \checkmark$$

$$CD = \begin{pmatrix}3\\0\\2\end{pmatrix} - \begin{pmatrix}2\\-3\\0\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix}$$

$$AC = \begin{pmatrix}2\\-3\\0\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\-5\\-1\end{pmatrix}$$

(b) Find angle *BAD*, giving your answer in degrees. [3]

AB , AD

$AB = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$AD = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$$\theta = \frac{2 - 6 + 2}{\sqrt{1^2+3^2+2^2} \times \sqrt{2^2+2^2+1^2}}$$

$$= \frac{-2}{\sqrt{14} \times 3}$$

$\cos^{-1}\left(\frac{2}{\sqrt{14} \times 3}\right) = 79.736$, $\theta = 100.3°$

SOH CAH TOA

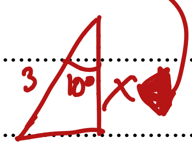
(c) Find the area of the parallelogram correct to 3 significant figures. [2]

Area $= x \times AB$

$= 2.9544 \times \sqrt{14}$

$= 11.0543$

$= 11.1$



$3 \cos 10 = x$

$x = 2.9544$

9 **(a)** The complex numbers u and w are such that

$$u - w = 2\mathrm{i} \quad \text{and} \quad uw = 6.$$

Find u and w, giving your answers in the form $x + \mathrm{i}y$, where x and y are real and exact. [5]

$u \times w = 6$ 0

$u = \frac{6}{w}$

$\frac{6}{w} - w = 2i$

$\frac{6 - w^2}{w} = 2i$

$6 - w^2 = 2i\,w$

$w(2i + w) = 6$

$2i + w = 6$

$w = 6 - 2i$

$u = \frac{6}{6-2i}$

$= \frac{6 \times (6+2i)}{(6-2i)(6+2i)}$

$= \frac{36+12i}{36 - 4(-1)}$

$= \frac{36+12i}{40}$

$u = \frac{9}{10} + \frac{3i}{10}$

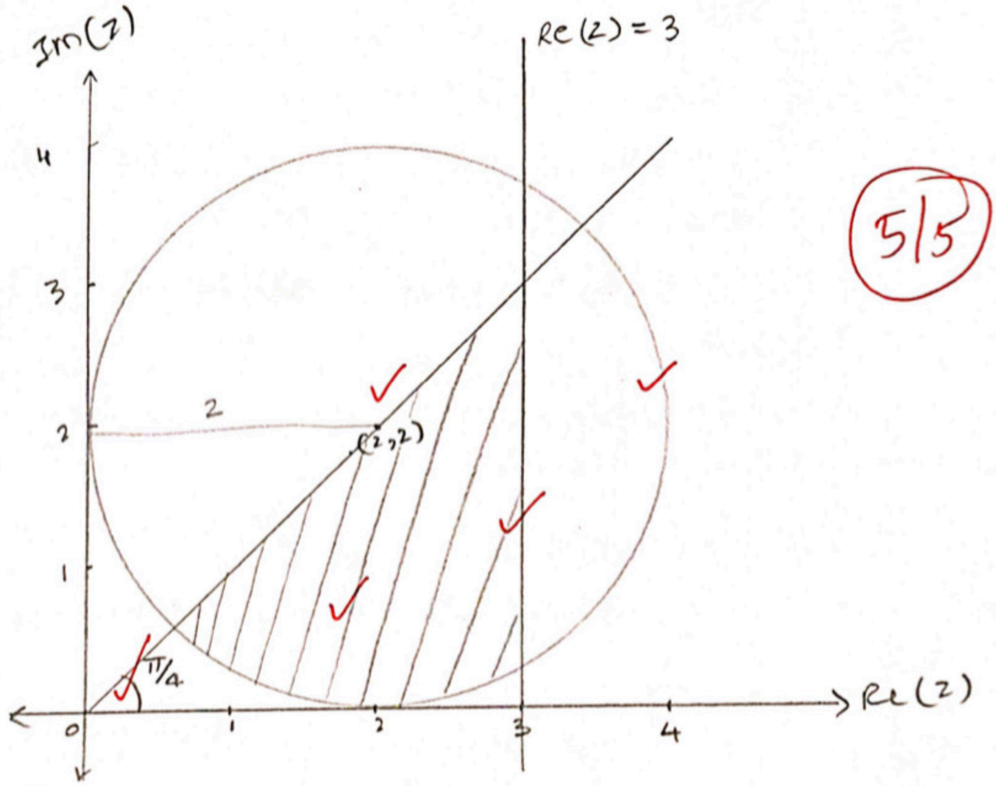
(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers z satisfying the inequalities

??

$$|z - 2 - 2\mathrm{i}| \leqslant 2, \quad 0 \leqslant \arg z \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad \operatorname{Re} z \leqslant 3. \qquad [5]$$

2

$|z-(2+2i)| \leq 2$

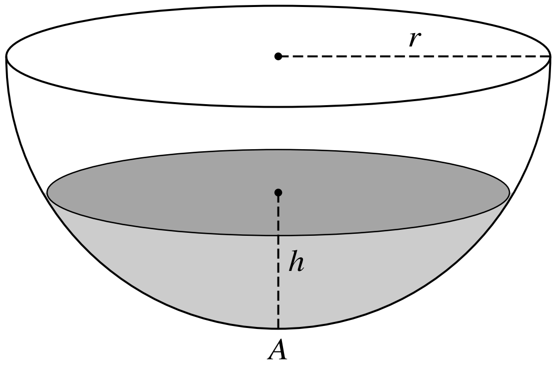


5/5

10

$\frac{dV}{dt} = k\sqrt{h}$

$V = \frac{1}{3}\pi(3rh^2 - h)$



A tank containing water is in the form of a hemisphere. The axis is vertical, the lowest point is A and the radius is r, as shown in the diagram. The depth of water at time t is h. At time $t = 0$ the tank is full and the depth of the water is r. At this instant a tap at A is opened and water begins to flow out at a rate proportional to $\sqrt{h}$. The tank becomes empty at time $t = 14$.

The volume of water in the tank is V when the depth is h. It is given that $V = \frac{1}{3}\pi(3rh^2 - h^3)$.

(a) Show that h and t satisfy a differential equation of the form

$$\frac{dh}{dt} = -\frac{B}{2rh^{\frac{1}{2}} - h^{\frac{3}{2}}},$$

where B is a positive constant. 0 [4]

??

$3rh^2 - h^3 = \frac{3V}{\pi}$

0

(b) Solve the differential equation and obtain an expression for t in terms of h and r. [8]

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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