Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **May/June 2014**

1 hour 45 minutes

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF9)

5374638582

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

This document consists of **3** printed pages and **1** blank page.

1 **(i)** Simplify $\sin 2\alpha \sec \alpha$. [2]

(ii) Given that $3\cos 2\beta + 7\cos\beta = 0$, find the exact value of $\cos\beta$. [3]

2 Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1 + 3\tan x)}}{\cos^2 x}\,dx.$$ [5]

3 The parametric equations of a curve are

$$x = \ln(2t + 3), \qquad y = \frac{3t + 2}{2t + 3}.$$

Find the gradient of the curve at the point where it crosses the y-axis. [6]

4 The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6ye^{3x}}{2 + e^{3x}}.$$

Given that $y = 36$ when $x = 0$, find an expression for y in terms of x. [6]

5 The complex number z is defined by $z = \dfrac{9\sqrt{3} + 9i}{\sqrt{3} - i}$. Find, showing all your working,

(i) an expression for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, [5]

(ii) the two square roots of z, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [3]

6 It is given that $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$.

(i) Show that $16x^3 - 24x^2 - 15x - 2 = 0$. [3]

(ii) By first using the factor theorem, factorise $16x^3 - 24x^2 - 15x - 2$ completely. [4]

(iii) Hence solve the equation $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$. [1]

7 The straight line l has equation $\mathbf{r} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$. The plane p passes through the point $(4, -1, 2)$ and is perpendicular to l.

(i) Find the equation of p, giving your answer in the form $ax + by + cz = d$. [2]

(ii) Find the perpendicular distance from the origin to p. [3]

(iii) A second plane q is parallel to p and the perpendicular distance between p and q is 14 units. Find the possible equations of q. [3]

8 **(i)** By sketching each of the graphs $y = \operatorname{cosec} x$ and $y = x(\pi - x)$ for $0 < x < \pi$, show that the equation

$$\operatorname{cosec} x = x(\pi - x)$$

has exactly two real roots in the interval $0 < x < \pi$. [3]

(ii) Show that the equation $\operatorname{cosec} x = x(\pi - x)$ can be written in the form $x = \dfrac{1 + x^2 \sin x}{\pi \sin x}$. [2]

(iii) The two real roots of the equation $\operatorname{cosec} x = x(\pi - x)$ in the interval $0 < x < \pi$ are denoted by α and β, where $\alpha < \beta$.

(a) Use the iterative formula

$$x_{n+1} = \frac{1 + x_n^2 \sin x_n}{\pi \sin x_n}$$

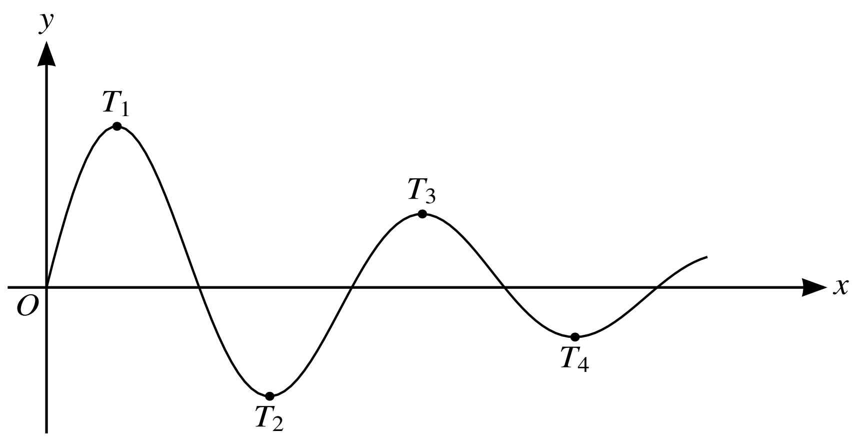
to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(b) Deduce the value of β correct to 2 decimal places. [1]

9 **(i)** Express $\dfrac{4 + 12x + x^2}{(3 - x)(1 + 2x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{4 + 12x + x^2}{(3 - x)(1 + 2x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

10



The diagram shows the curve $y = 10e^{-\frac{1}{2}x} \sin 4x$ for $x \geqslant 0$. The stationary points are labelled T_1, T_2, T_3, … as shown.

(i) Find the x-coordinates of T_1 and T_2, giving each x-coordinate correct to 3 decimal places. [6]

(ii) It is given that the x-coordinate of T_n is greater than 25. Find the least possible value of n. [4]

1 **(i)** Simplify $\sin 2\alpha \sec \alpha$. [2]

(ii) Given that $3\cos 2\beta + 7\cos\beta = 0$, find the exact value of $\cos\beta$. [3]

i) $2\sin\alpha \cancel{\cos\alpha} \times \frac{1}{\cancel{\cos\alpha}} = 2 \quad \alpha$

ii) $6\cos^2\beta - 3 + 7\cos\beta$

$$\frac{-7 \pm \sqrt{49-4(6)(-3)}}{12}$$

$$\frac{-7 \pm \sqrt{121}}{12}$$

$\cos\beta = \frac{1}{3}$ $\qquad \cos\beta = -\frac{3}{2}$

$\beta = 70.5°$

or $\beta = 289.5°$

2 Use the substitution $u = 1 + 3\tan x$ to find the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sqrt{(1 + 3\tan x)}}{\cos^2 x}\, dx.$$

[5]

$$\int_1^4 \frac{u^{\frac{1}{2}}}{\cancel{\cos^2 x}} \times \frac{\cancel{\cos^2 x}}{3}\, dx$$

$\frac{du}{dx} = \frac{3}{\cos^2 x}$

$dx = \frac{\cos^2 x\, du}{3}$

when $x = 0$, $u = 1$

$x = \frac{1}{4}\pi$ $u = 4$

$$\frac{1}{3}\int_1^4 u^{\frac{1}{2}}\, dx$$

$$\left[\frac{1}{3} \times \frac{2u^{\frac{3}{2}}}{3}\right]_1^4$$

$$\left[\frac{2u^{\frac{3}{2}}}{9}\right]_1^4$$

$$\frac{16}{9} - \frac{2}{9} = \frac{14}{9}$$

3 The parametric equations of a curve are

$$x = \ln(2t + 3), \quad y = \frac{3t + 2}{2t + 3}.$$

Find the gradient of the curve at the point where it crosses the y-axis. [6]

6

$$\frac{dx}{dt} = \frac{2}{2t+3}$$

$$\frac{dy}{dt} = \frac{(2t+3)(3) - (2)(3t+2)}{(2t+3)^2} = \frac{6t+9-6t-4}{(2t+3)^2} = \frac{5}{(2t+3)^2}$$

$$3t = -2$$
$$t = \frac{-2}{3}$$

$$x = \ln\left(2 \times \frac{-2}{3} + 3\right)$$
$$=$$

$$\frac{dy}{dt} = \frac{dy}{dt} \times \frac{dt}{dt}$$

$$= \frac{5}{2t+3} \times \frac{2t+3}{2} = \frac{10t+15}{4t+6}$$

$$\frac{10\left(-\frac{2}{3}\right) + 15}{4\left(-\frac{2}{3}\right) + 6} = \frac{5}{2}$$

4 The variables x and y are related by the differential equation

$$\frac{dy}{dx} = \frac{6ye^{3x}}{2 + e^{3x}}.$$

Given that $y = 36$ when $x = 0$, find an expression for y in terms of x. [6]

2

$$\int \frac{1}{6y}\,dy = \int \frac{e^{3x}}{2+e^{3x}}\,dx$$

$$\frac{1}{6}\ln 6^2$$

$$\frac{1}{6}\ln y = \frac{1}{3}\ln(2 + e^{3x}) + c$$

$$\frac{1}{6}\ln 36 = \frac{1}{3}\ln(2) + c \quad 3$$

$$\frac{1}{3}\ln 6 = \frac{1}{3}\ln 2 + c \quad 3$$

$$c = \frac{1}{3}\ln 6 - \frac{1}{3}\ln 2 \Rightarrow \frac{1}{3}\ln 3 \quad 3$$

$$\frac{1}{6}\ln y = \frac{1}{3}\ln(2 + e^{3x}) + \frac{1}{3}\ln 3$$

$$\frac{1}{6}\ln y = \cancel{\frac{1}{3}\ln(6 + 3e^{3x})}$$

$$\ln y = 2\ln(6 + 3e^{3x})$$

$$\frac{1}{6}\ln y = \frac{1}{3}\ln(4 + 2e^{3x})$$

$$\ln y = 2\ln 4 + 2e^{3x}$$

$$\ln = (4 + 2e^{3x})^2$$

$y = e^{2\ln(6+3e^{3x})}$ or $y = e^{\ln(6+3e^{3x})^2}$

$y = (e^2)^{\ln(6+3e^{3x})}$ $\quad y = (6+3e^{3x})^2$

5 The complex number z is defined by $z = \dfrac{9\sqrt{3} + 9\mathrm{i}}{\sqrt{3} - \mathrm{i}}$. Find, showing all your working,

(i) an expression for z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, [5]

(ii) the two square roots of z, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. ~~[3]~~

i) $\dfrac{(9\sqrt{3}+9i)(\sqrt{3}+i)}{(\sqrt{3})^2 - i^2} = \dfrac{27 + 18\sqrt{3}i - 9}{3+1} = \dfrac{18 + 18\sqrt{3}i}{4}$

$= \dfrac{18}{4} + \dfrac{18}{4}\sqrt{3}i$

$r = \sqrt{\left(\frac{18}{4}\right)^2 + \left(\frac{18\sqrt{3}}{4}\right)^2}$

$= 9$

$\theta = \tan^{-1}\left(\dfrac{\left(\frac{18\sqrt{3}}{4}\right)}{\left(\frac{18}{4}\right)}\right) = \frac{1}{3}\pi$

$x+iy \quad x+iy$

$9e^{i\frac{1}{3}\pi}$

ii) $\dfrac{18}{4} + \dfrac{18}{4}\sqrt{3}i = x^2 - y^2 + 2ixy$

$2xy = \dfrac{18\sqrt{3}}{4}$

$\dfrac{18}{4} = x^2 - y^2$

$4x^2y^2 = \dfrac{972}{16}$

$x^2 = \dfrac{18}{4} + y^2$

$4y^2\left(\frac{18}{4} + y^2\right) = \dfrac{972}{16}$

$18y^2 \quad 4y^4 = \dfrac{972}{16}$

$288y^2 \quad 64y^4 = 972$

$64y^2 + 288y^2 - 972 =$

$y^2 = \dfrac{9}{4}, \; \cancel{-\dfrac{27}{4}}$

$y = \pm\dfrac{3}{2}$

$$x = \sqrt{\frac{18}{4} + \frac{9}{4}} = \pm\frac{3\sqrt{3}}{2}$$

$$\pm\frac{3\sqrt{3}}{2} + \frac{3i}{2}$$

$$\sqrt{(1.5)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = 3e^{\frac{1}{6}\pi}$$

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(ii) By first using the factor theorem, factorise $16x^3-24x^2-15x-2$ completely. [4]

(iii) Hence solve the equation $2\ln(4x-5)+\ln(x+1)=3\ln 3$. [1]

i) $\ln[(4x-5)(4x-5)(x+1)] = \ln 27$

$\ln[(16x^2-40x+25)(x+1)] = \ln 27$

$\ln(16x^3+16x^2-40x^2-40x+25x+25) = \ln 27$

$\cancel{\ln}(16x^3-24x^2-15x+25) = \cancel{\ln}\,27$

$16x^3-24x^2-15x-2=0$

ii) $f(x) = 16x^3-24x^2-15x-2$

$f(1) = -25$

$f(2) = 0$

$x-2=0$

$(x-2)(ax^2+bx+c)$

$ax^3+bx^2+cx-2ax^2-2bx-2c$

$\underline{a=16}$ $\quad cx-2bx=-15x$ $\quad 2c=2$

$\underline{c=1}$

$x-2bx=-15x$

$-2b=-16$

$b=8$

$(x-2)(16x^2+8x+1)$ $\quad ac=-1$

$16x^2+$

$(4x+1)$

$16x^2+8x+1$

$(4x+1)^2=0$

$\underline{(x-2)(4x+1)^2}$

iii) $(x-2)(4x+1)^2 = 0$

$x-2 = 0$ | $(4x+1) = 0$

$x = 2$ | $x = -\frac{1}{4}$

8

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(a) Use the iterative formula

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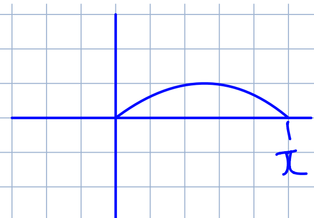
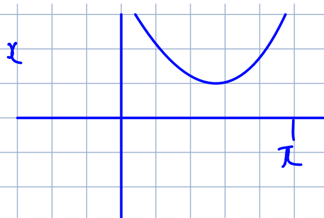
to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(b) Deduce the value of β correct to 2 decimal places. [1]

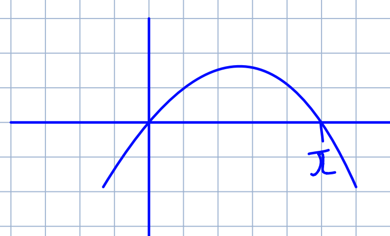
i) $\frac{1}{\sin x}$

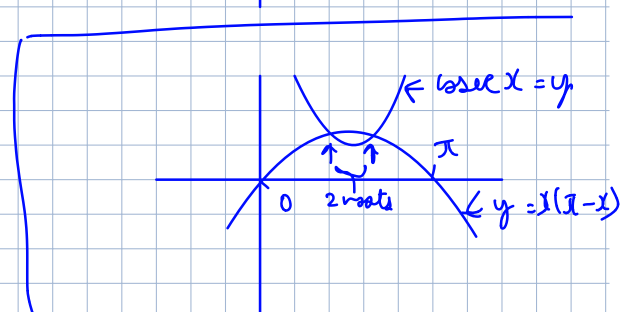


cosec x



$-x^2 + \pi x$





ii) $\frac{1}{\sin x} = \pi x - x^2$

$1 = \pi x \sin x - x^2 \sin x$

$1 + x^2 \sin x = \pi x \sin x$

$x = \dfrac{1 + x^2 \sin x}{\pi \sin x}$

iii) let $x_1 = 1$

$$x_2 = \frac{1 + 1^2 \sin 1}{\pi \sin 1} = 0.6966$$

$x_3 = 0.6506$

$x_4 = 0.6603$

$x_5 = 0.6577$

$x_6 = 0.6584$

$x_7 = 0.6582$

$x_8 = 0.6583$

$x_9 = 0.6583$

$= \underline{0.66}$

iv) $x^2 - \pi x = \dfrac{1}{\sin x}$

$$x = \sqrt{\frac{1}{\sin x} + \pi x}$$

9 **(i)** Express $\dfrac{4+12x+x^2}{(3-x)(1+2x)^2}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{4+12x+x^2}{(3-x)(1+2x)^2}$ in ascending powers of x, up to and including the term in x^2. [5]

i) $4+12x+x^2 = \dfrac{A}{3-x} + \dfrac{B}{1+2x} + \dfrac{C}{(1+2x)^2}$

$4+12x+x^2 = A(1+2x)^2 + B(3-x)(1+2x) + C(3-x)$

sub $x=3$ sub $x=-\frac{1}{2}$

$49 = 49A$ $-\frac{7}{4} = \frac{7}{2}C = -\frac{1}{2}$

$A = \underline{1}$

$12 = 4A + 5B - C$

$12 = 4 + 5B + \frac{1}{2}$

$2 \to \frac{32}{25}$

$B = \frac{3}{2}$

$\dfrac{1}{3-x} + \dfrac{3}{2(1+2x)} - \dfrac{1}{2(1+2x)^2}$

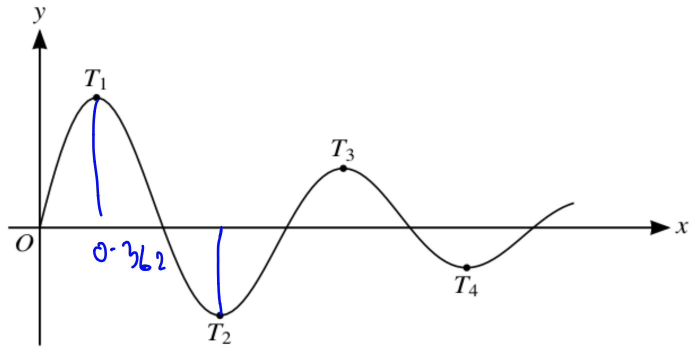
ii) $(3-x)^{-1} \Rightarrow 3\left(1-\frac{x}{3}\right)^{-1} \Rightarrow 3^{-1}\left(1+\frac{x}{3}+\frac{1}{9}x^2\right) = \cancel{3+x}+\frac{1}{3}x^2$ $\frac{1}{3}$

$\frac{3}{2}(1+2x)^{-1} \Rightarrow \frac{3}{2}(1-2x+4x^2) = \frac{3}{2} - 3x + 6x^2$

$\frac{1}{2}(1+2x)^{-2} \Rightarrow \frac{1}{2}(1-4x+12x^2) = \frac{1}{2} - 2x + 6x^2$

$4 + \frac{1}{3}x^2$

10



The diagram shows the curve $y = 10e^{-\frac{1}{2}x}\sin 4x$ for $x \geqslant 0$. The stationary points are labelled T_1, T_2, T_3, … as shown.

(i) Find the x-coordinates of T_1 and T_2, giving each x-coordinate correct to 3 decimal places. [6]

5

(ii) It is given that the x-coordinate of T_n is greater than 25. Find the least possible value of n. ~~[4]~~

i) $\frac{dy}{dx} = -5e^{-\frac{1}{2}x}\sin 4x + (4\cos 4x)(10e^{-\frac{1}{2}x}) = 0$

$= 40\cancel{e^{-\frac{1}{2}x}}\cos 4x = 5\cancel{e^{-\frac{1}{2}x}}\sin 4x$

$5e^{-\frac{1}{2}x}(8\cos 4x - \sin 4x) = 0$

$8\cos 4x = \sin 4x$

$8 = \tan 4x$

$4x = \tan^{-1} 8 \rightarrow 4x = 4.58803$

$x = 1.147$

$x = \frac{1}{4}\tan^{-1} 8$

$T_1 = 0.362 \qquad T_2 = 1.147$

ii) $25 - 0.362 = 24.638$

$\frac{24.638}{0.281} =$

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