UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

MATHEMATICS **9709/33**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2013**

1 hour 45 minutes

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF9)

6391576228

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.
Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.

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1 Given that $2\ln(x+4) - \ln x = \ln(x+a)$, express x in terms of a. [4]

2 Use the substitution $u = 3x + 1$ to find $\displaystyle\int \frac{3x}{3x+1}\,dx$. [4]

3 The polynomial $f(x)$ is defined by

$$f(x) = x^3 + ax^2 - ax + 14,$$

where a is a constant. It is given that $(x+2)$ is a factor of $f(x)$.

(i) Find the value of a. [2]

(ii) Show that, when a has this value, the equation $f(x) = 0$ has only one real root. [3]

4 A curve has equation $3e^{2x}y + e^{x}y^3 = 14$. Find the gradient of the curve at the point $(0, 2)$. [5]

5 It is given that $\displaystyle\int_0^p 4xe^{-\frac{1}{2}x}\,dx = 9$, where p is a positive constant.

(i) Show that $p = 2\ln\left(\dfrac{8p+16}{7}\right)$. [5]

(ii) Use an iterative process based on the equation in part **(i)** to find the value of p correct to 3 significant figures. Use a starting value of 3.5 and give the result of each iteration to 5 significant figures. [3]

6 Two planes have equations $3x - y + 2z = 9$ and $x + y - 4z = -1$.

(i) Find the acute angle between the planes. [3]

(ii) Find a vector equation of the line of intersection of the planes. [6]

7 **(i)** Given that $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec}2\theta$, show that $2\sin\theta + 4\cos\theta = 3$. [3]

(ii) Express $2\sin\theta + 4\cos\theta$ in the form $R\sin(\theta+\alpha)$ where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(iii) Hence solve the equation $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec}2\theta$ for $0° < \theta < 360°$. [4]

8 **(i)** Express $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in partial fractions. [5]

(ii) Hence expand $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [5]

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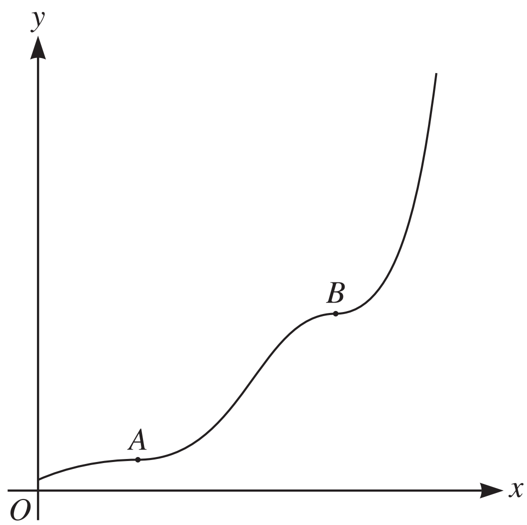
9 **(a)** Without using a calculator, use the formula for the solution of a quadratic equation to solve

$$(2 - \mathrm{i})z^2 + 2z + 2 + \mathrm{i} = 0.$$

Give your answers in the form $a + b\mathrm{i}$. [5]

(b) The complex number w is defined by $w = 2\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$. In an Argand diagram, the points A, B and C represent the complex numbers w, w^3 and w^* respectively (where w^* denotes the complex conjugate of w). Draw the Argand diagram showing the points A, B and C, and calculate the area of triangle ABC. [5]

10



A particular solution of the differential equation

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 4(y^3 + 1)\cos^2 x$$

is such that $y = 2$ when $x = 0$. The diagram shows a sketch of the graph of this solution for $0 \leqslant x \leqslant 2\pi$; the graph has stationary points at A and B. Find the y-coordinates of A and B, giving each coordinate correct to 1 decimal place. [10]

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1 Given that $2\ln(x+4) - \ln x = \ln(x+a)$, express x in terms of a. [4]

$$\ln(x+4)^2 - \ln x = \ln(x+a)$$

$$\frac{x^2+8x+16}{x} = x+a$$

$$\cancel{x^2}+8x+16 = \cancel{x^2}+ax$$

$$x(8-a)+16 = 0$$

$$x = \frac{16}{8-a}$$

2 Use the substitution $u = 3x+1$ to find $\int \frac{3x}{3x+1}\,dx$. [4]

2)

$3x = u-1$

$u = 3x+1$

$\frac{du}{3} = dx$

$$\frac{1}{3}\int \frac{u-1}{u}\,du$$

$$\frac{1}{3}\int 1 - \frac{1}{u}\,du$$

$$\frac{1}{3}u - \frac{1}{3}\ln u$$

$$\frac{1}{3}(u - \ln u)$$

$$\frac{1}{3}(3x+1 - \ln(3x+1))$$

3 The polynomial $f(x)$ is defined by

$$f(x) = x^3 + ax^2 - ax + 14,$$

where a is a constant. It is given that $(x + 2)$ is a factor of $f(x)$.

(i) Find the value of a. [2]

(ii) Show that, when a has this value, the equation $f(x) = 0$ has only one real root. [3]

i) $x = -2$

$-8 + 4a - 2a + 14 = 0$

$6 + 2a = 0$

$a = -3$

ii) $x^3 - 3x^2 + 3a + 14$

$(x+2)(cx^2 + dx + e)$

$c = 1$

$2e = 14$

$e = 7$

$\cancel{ex} + 2d\cancel{x} = 3$

$7 + 2d = 3$

$2d = -4$

$d = -2$

$\underbrace{\substack{(x+2) = 0 \\ x = -2}}_{\text{real root}}$

$x^2 - 2x + 7$

$b^2 - 4ac$

$4 - 4(1)(7)$

$-24 < 0$

determinant is < 0 $\therefore$ 2 complex roots

4 A curve has equation $3e^{2x}y + e^x y^3 = 14$. Find the gradient of the curve at the point $(0, 2)$. [5]

$6e^{2x}y + 3e^{2x}\frac{dy}{dx} + e^x y^3 + 3e^x y^2 \frac{dy}{dx} = 0$

$6e^{2x}y + e^x y^3 = \left(-3e^x y^2 - 3e^{2x}\right)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{6e^{2x}y + e^x y^3}{-3e^x y^2 - 3e^{2x}}$

$\frac{6(2) + (1)(8)}{-3(4) - 3(1)} = \frac{12 + 8}{-15} = \frac{20}{-15} = -\frac{4}{3}$

5 It is given that $\int_0^p 4xe^{-\frac{1}{2}x}\,dx = 9$, where p is a positive constant.

(i) Show that $p = 2\ln\left(\frac{8p+16}{7}\right)$. [5]

(ii) Use an iterative process based on the equation in part **(i)** to find the value of p correct to 3 significant figures. Use a starting value of 3.5 and give the result of each iteration to 5 significant figures. [3]

i) $\int 4xe^{-\frac{1}{2}x}\,dx$

u v'

$u = 4x$ $\quad u' = 4$

$v = -2e^{-\frac{1}{2}x}$ $\quad v' = e^{-\frac{1}{2}x}$

$-8xe^{-\frac{1}{2}x} - \int -8e^{-\frac{1}{2}x}\,dx$

$-8xe^{-\frac{1}{2}x} + 8\int e^{-\frac{1}{2}x}\,dx$

$\left[-8xe^{-\frac{1}{2}x} - 16e^{-\frac{1}{2}x}\right]_0^p$

$\left(-8pe^{-\frac{1}{2}p} - 16e^{-\frac{1}{2}p}\right) - (0 - 16) = 9$

$-8pe^{-\frac{1}{2}p} - 16e^{\frac{1}{2}p} + 16 - 9 = 0$

$-e^{-\frac{1}{2}p}(8p+16) + 7 = 0$

$e^{\frac{1}{2}p}(8p+16) = 7$

$e^{-\frac{1}{2}p} = \frac{7}{8p+16}$

$-\frac{1}{2}p = \ln\left(\frac{7}{8p+16}\right)$

$p = -2\ln\left(\frac{7}{8p+16}\right)$

$p = 2\ln\left(\frac{8p+16}{7}\right)$

7 **(i)** Given that $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec}2\theta$, show that $2\sin\theta + 4\cos\theta = 3$. [3]

(ii) Express $2\sin\theta + 4\cos\theta$ in the form $R\sin(\theta + \alpha)$ where $R > 0$ and $0° < \alpha < 90°$, giving the value of α correct to 2 decimal places. [3]

(iii) Hence solve the equation $\sec\theta + 2\operatorname{cosec}\theta = 3\operatorname{cosec}2\theta$ for $0° < \theta < 360°$. [4]

i)
$$\frac{1}{\cos\theta} + \frac{2}{\sin\theta} = \frac{3}{\sin 2\theta}$$

$$\frac{\sin\theta + 2\cos\theta}{\sin\theta\cos\theta} = \frac{3}{2\cos\theta\sin\theta}$$

$$(\sin\theta + 2\cos\theta)(2\cancel{\cos\theta\sin\theta}) = 3\cancel{\sin}\theta\cancel{\cos}\theta$$

$$2\sin\theta + 4\cos\theta = 3$$

ii)
$$R = \sqrt{4^2+2^2} = \sqrt{16+4} = \sqrt{20}$$

$$\alpha = \tan^{-1}\left(\frac{4}{2}\right) = 63.43$$

$$\sqrt{20}\sin(\theta + 63.43°)$$

iii)
$$\sqrt{20}\sin(\theta + 63.43349) = 3$$

$$\theta + 63.4349 = \sin^{-1}\left(\frac{3}{\sqrt{20}}\right) \quad \text{or} \quad 180 - \sin^{-1}\left(\frac{3}{\sqrt{20}}\right)$$

$$\theta = -21.30\cancel{44}85 \qquad \text{or} \qquad \underline{\underline{74.4°}}$$

$$\downarrow +360$$

$$\theta = 338.6955$$

$$= \underline{\underline{338.7°}}$$

8 **(i)** Express $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in partial fractions. $= \dfrac{A}{1+x} + \dfrac{B}{(1+x)^2} + \dfrac{C}{2-3x}$ [5]

(ii) Hence expand $\dfrac{7x^2+8}{(1+x)^2(2-3x)}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [5]

i) $7x^2+8 = A(1+x)(2-3x) + B(2-3x) + C(1+x)^2$

let $x=-1$

$7+8 = B(2-3(-1))$

$15 = 5B$

$B = 3$

$2-3x=0$
$3x=2$
$x=\frac{2}{3}$

let $x=\frac{2}{3}$

$\frac{100}{\cancel{9}} = \frac{25}{\cancel{9}}C$

$C = 4$

$7 = -3A + C$

$7 = -3A + 4$

$3 = -3A$

$A = -1$

$-1(1+x)^{-1} + 3(1+x)^{-2} + 4(2-3x)^{-1}$

ii)

$-1(1+x)^{-1}$	$3(1+x)^{-2}$
$-1(1-x+x^2)$	$3(1-2x+3x^2)$
$-1+x-x^2$	$3-6x+9x^2$

$4(2)^{-1}(1-\frac{3}{2}x)^{-1}$

$2(1+\frac{3}{2}+\frac{9}{4}x^2)$

$2+3+\frac{9}{2}x^2$

$(-1+3+2)+(1-6+3)x+(-1+9+\frac{9}{2})x^2$

$4-2x+\frac{25}{2}x^2$

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a) :)

$a = (2-i)$ $\quad c = 2+i$

$b = 2$

$b^2 - 4ac = 4 - 4(2-i)(2+i)$
$= 4 - 4(4+1)$
$= 4 - 20 = -16$

$$\frac{-2 \pm \sqrt{-16}}{4-2i}$$

$$\frac{-2 \pm 4i}{4-2i}$$

$$z = \frac{(-2+4i)(4+2i)}{(4-2i)(4+2i)} \quad \text{or} \quad z = \frac{(-2-4i)(4+2i)}{20}$$

$$= \frac{-8-4i+16i-8}{16-(-4)} \qquad = \frac{\cancel{-8}-4i-16i\cancel{+8}}{20}$$

$$= \frac{-16+12i}{20} \qquad = \frac{-20i}{20}$$

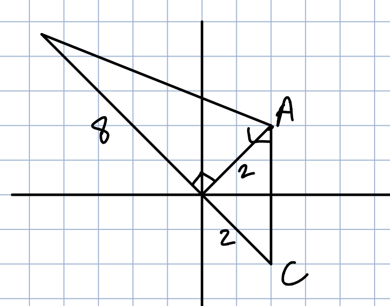
$$-\frac{4}{5} + \frac{3}{5}i \qquad = -i$$

b)

$w = 2e^{\frac{1}{4}\pi i}$

$w^3 = 8e^{\frac{3}{4}\pi i}$

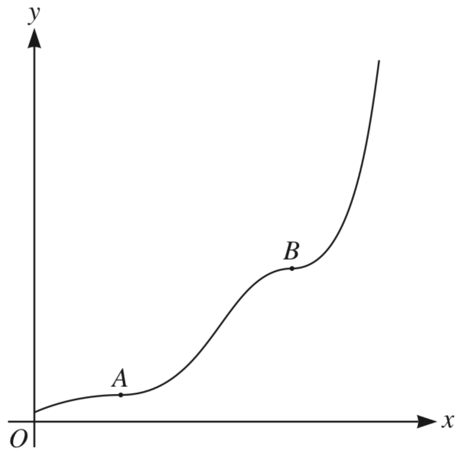
$w^* = 2e^{-\frac{1}{4}\pi i}$



$\text{area} = \left(\frac{1}{2} \times 2 \times 8\right) + \left(\frac{1}{2} \times 2 \times 2\right)$

$= 10$

10



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$$\frac{1}{4}\int \frac{3y^2}{y^3+1}\,dy = \int \cos^2 x \; dx$$

$$\frac{1}{4}\ln(y^3+1) = \frac{1}{2}\int \cos 2x + 1 \; dx$$

$$\frac{1}{4}\ln(y^3+1) = \frac{1}{2}\left[\left(\frac{1}{2}\sin 2x\right) + x\right]$$

$$\frac{1}{4}\ln(y^3+1) = \frac{1}{4}\sin 2x + \frac{1}{2}x + C$$

$$\frac{1}{4}\ln 9 = C$$

$$\frac{1}{4}\ln(y^3+1) = \frac{1}{4}\sin 2x + \frac{1}{2}x + \frac{1}{4}\ln 9$$

$$\ln(y^3+1) = \sin 2x + 2x + \ln 9$$

$$y^3 + 1 = 9e^{\sin 2x + 2x}$$

$$\frac{dy}{dx} = 0 = 4(y^3+1)\cos^2 x$$

$$= 4(y^3+1)(\sin^2 x - 1)$$

$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

$$4\left(9e^{\sin 2x + 2x}\right)(\sin^2 x - 1)$$

$$\cancel{36\sin^2 x\, e^{\sin 2x + 2x}} = \cancel{36\, e^{\sin 2x + 2x}}$$

$$\sin^2 x = 1$$

$$\sin x = 1$$

$$x = \frac{1}{2}\pi$$

$$y = \sqrt[3]{9e^{\sin(\pi) + \pi} - 1}$$

$$y = 5.9$$