**Cambridge Assessment International Education**
Cambridge International Advanced Level

Cambridge International A Level

CANDIDATE NAME: Fuzail Hamid

CENTRE NUMBER: 

CANDIDATE NUMBER: 

* 2 9 6 9 0 0 9 5 4 1 *

**MATHEMATICS** **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **May/June 2019**

**1 hour 45 minutes**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF9)

**READ THESE INSTRUCTIONS FIRST**

Write your centre number, candidate number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions in the space provided. If additional space is required, you should use the lined page at the end of this booklet. The question number(s) must be clearly shown.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.

This document consists of **18** printed pages and **2** blank pages.




Cambridge Assessment International Education

1 Use the trapezium rule with 3 intervals to estimate the value of

$$\int_0^3 \left|2^x - 4\right| \mathrm{d}x. \qquad [3]$$

**2** Showing all necessary working, solve the equation $\ln(2x - 3) = 2\ln x - \ln(x - 1)$. Give your answer correct to 2 decimal places. [4]

$\ln(2x-3) = 2\ln\left(\frac{x}{x-1}\right)$

$2x-3 = \left(\frac{x}{x-1}\right)^2$

$2x-3 = \frac{x^2}{(x-1)^2}$

$2x-3 = \frac{x^2}{(x-1)(x-1)}$

$2x-3 = \frac{x^2}{x^2-2x+1}$

$(x^2-2x+1)(2x-3) = x^2$

$2x^3-3x^2-4x^2-6x+2x-3 = x^2$

$2x^3-8x^2-4x-3=0$

**3** Find the gradient of the curve $x^3 + 3xy^2 - y^3 = 1$ at the point with coordinates $(1, 3)$. [4]

×

$$\frac{dy}{dx} = 3x^2 + \left[3y^2 + 6xy\frac{dy}{dx}\right] - 3y^2\frac{dy}{dx} = 1$$

$$3x^2 + 3y^2 + 6xy\frac{dy}{dx} - 3y^2\frac{dy}{dx} - 1 = 0$$

$$3(x^2 + y^2) - 1 = 3(y^2 - 2xy)\frac{dy}{dx}$$

Redo

$$\frac{dy}{dx} = \frac{3x^2 + 3y^2 - 1}{3(y^2 - 2xy)}$$

$$m = \frac{3(1)^2 + 3(9) - 1}{3(9 - 2(3))}$$

$$m = \frac{3 + 27 - 1}{9}$$

$$= \frac{29}{9} =$$

**4** By first expressing the equation $\cot\theta - \cot(\theta + 45^\circ) = 3$ as a quadratic equation in $\tan\theta$, solve the equation for $0^\circ < \theta < 180^\circ$. [6]

$$\frac{1}{\tan\theta} - \frac{1}{\tan(\theta+45^\circ)} = 3$$

$$\frac{1}{\tan\theta} - \frac{1}{\left(\frac{\tan\theta + \tan 45}{1 - \tan\theta\tan 45}\right)} = 3$$

$$\frac{1}{\tan\theta} - \frac{1}{\left(\frac{\tan\theta + 1}{1 - \tan\theta}\right)} = 3$$

$$\frac{1}{\tan\theta} - \frac{1-\tan\theta}{\tan\theta+1} = 3$$

$$\frac{\tan\theta + 1 - (\tan\theta - \tan^2\theta)}{\tan^2\theta + \tan\theta} = 3$$

$$\cancel{\tan\theta} + 1 - \cancel{\tan\theta} + \tan^2\theta = 3\tan^2\theta + 3\tan\theta$$

$$2\tan^2\theta + 3\tan\theta - 1 = 0$$

$\tan\theta = 0.280776$

$\theta = 15.7$

$\tan\theta = -1.780776$

$\theta = 180 - \tan^{-1}(1.780776)$

or $\cancel{360 - \tan^{-1}(1.780776)}$

$\theta = 119.3$

$\to (\sin\theta)^{-2}$

5 **(i)** Differentiate $\dfrac{1}{\sin^2\theta}$ with respect to $\theta$. [2]

$$\frac{d\theta}{dx} = -2(\sin\theta)^{-3}(\cos\theta)$$

$$= -\frac{2\cos\theta}{\sin^3\theta}$$

$$= \frac{-2\cos\theta}{\sin\theta} \times \frac{1}{\sin^2\theta}$$

$$= -2\cot\ \operatorname{cosec}^2\theta$$

**(ii)** The variables $x$ and $\theta$ satisfy the differential equation

$$x\tan\theta\frac{dx}{d\theta} + \operatorname{cosec}^2\theta = 0,$$

IMP

for $0 < \theta < \frac{1}{2}\pi$ and $x > 0$. It is given that $x = 4$ when $\theta = \frac{1}{6}\pi$. Solve the differential equation, obtaining an expression for $x$ in terms of $\theta$. [6]

$$x\tan\theta\, dx = -\operatorname{cosec}^2\theta\, d\theta$$

$$\int x\, dx = -\int \frac{\operatorname{cosec}^2\theta}{\tan\theta}\, d\theta$$

$$\frac{1}{2}x^2 = -\int \frac{1}{\sin^2\theta}\ \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= -\int \frac{\cos\theta}{\sin^3\theta}\, d\theta$$

$t = \sin\theta$

$\frac{dt}{d\theta} = \cos\theta$

$$= -\int \frac{\cancel{\cos\theta}}{t^3} \times \frac{dt}{\cancel{\cos\theta}}$$

$d\theta = \frac{dt}{\cos\theta}$

$$\frac{1}{2}x^2 = -\int t^{-3}$$

$$\frac{1}{2}x^2 = -\frac{t^{-2}}{-2}$$

$$\frac{1}{2}x^2 = \frac{1}{2\sin^2\theta} + c$$

$$\frac{1}{2}(4)^2 = \frac{1}{2 \times \frac{1}{4}} + c$$

$$8 = 2 + c$$

$$c = 6$$

$$\therefore \quad x^2 = 2\left[\frac{1}{2\sin^2\theta} + 6\right]$$

$$x = \sqrt{\frac{\cancel{2}\,1}{\cancel{2}\sin^2\theta} + 12}$$

$$x = \sqrt{\operatorname{cosec}^2\theta + 12}$$

**6** **(i)** By first expanding $\sin(2x + x)$, show that $\sin 3x \equiv 3\sin x - 4\sin^3 x$. [4]

$\sin 2x \cos x + \cos 2x \sin x$

$2\sin x \cos^2 x + [(1-2\sin^2 x)\sin x]$

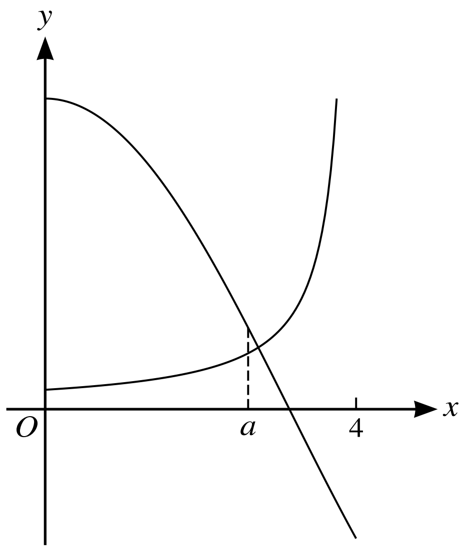
$2\sin x(1-\sin^2 x) + [\sin x - 2\sin^3 x]$

$2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x$

$3\sin x - 4\sin^3 x$

**(ii)** Hence, showing all necessary working, find the exact value of $\int_0^{\frac{1}{3}\pi} \sin^3 x \, dx$. [4]

**7**




The diagram shows the curves $y = 4\cos\frac{1}{2}x$ and $y = \dfrac{1}{4-x}$, for $0 \leqslant x < 4$. When $x = a$, the tangents to the curves are perpendicular.

**(i)** Show that $a = 4 - \sqrt{(2\sin\frac{1}{2}a)}$. [4]

$r_1: \dfrac{dy}{dx} = -2\sin\frac{1}{2}x$ $r_2: \dfrac{dy}{dx} = -(4-x)^{-2}(-1)$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{(4-x)^2}$

$\cancel{/}2\sin\dfrac{a}{2} = \cancel{/}(4-a)^2$

$\sqrt{2\sin\dfrac{a}{2}} = 4-a$

$a = 4 - \sqrt{2\sin\frac{1}{2}a}$

**(ii)** Verify by calculation that $a$ lies between 2 and 3. [2]

$a = 4 - \sqrt{2\sin\left(\frac{a}{2}\right)}$

2      3

LHS RHS      LHS RHS

$2 > 2.70$      $3 < 2.59$

there is a change in sign from 2 ~~to~~ 3

**(iii)** Use an iterative formula based on the equation in part **(i)** to determine $a$ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$a_{n+1} = 4 - \sqrt{2\sin\left(\frac{a_n}{2}\right)}$

$a_1 = 2$

$a_2 = 2.70272$

$a_3 = 2.60285$

$a_4 = 2.61152$

$a_5 = 2.61070$

$a_6 = 2.61077$

$a_7 = 2.61077$

$a_8 = 2.61077$

$\therefore \underline{\underline{a = 2.61}}$

**8** Let $\mathrm{f}(x) = \dfrac{16 - 17x}{(2 + x)(3 - x)^2}$.

**(i)** Express $\mathrm{f}(x)$ in partial fractions. [5]

$$\frac{16-17x}{(2+x)(3-x)^2} = \frac{A}{(2+x)} + \frac{B}{(3-x)} + \frac{C}{(3-x)^2}$$

$$16-17x = A(3-x)(3-x) + B(2+x)(3-x) + C(2+x)$$
$$16-17x = A(9-6x+x^2) + B(6+x-x^2) + 2C + Cx$$
$$16-17x = 9A - 6Ax + Ax^2 + 6B + Bx - Bx^2 + 2C + Cx$$

① $A - B = 0$
② $9A + 6B + 2C = 16$
③ $-6A + B + C = -17$

from ①, $B = A$ — ④

sub 4 into 3, $-6B + B + C = -17$
$C = 5B - 17$ — ⑤

sub 5 and 4 into 2

$$9B + 6B + 2(5B - 17) = 16$$
$$15B + 10B - 34 = 16$$
$$25B = 50$$
$$B = 2$$
$$\therefore A = 2 \text{ and } C = -7$$

$$\frac{2}{2+x} + \frac{2}{3-x} - \frac{7}{(3-x)^2}$$

**(ii)** Hence obtain the expansion of $f(x)$ in ascending powers of $x$, up to and including the term in $x^2$. [5]

$2(2+x)^{-1} + 2(3-x)^{-1} - 7(3-x)^{-2}$

$2 \cdot 2^{-1}\left(1+\frac{x}{2}\right)^{-1} + 2 \cdot 3^{-1}\left(1-\frac{x}{3}\right)^{-1} - 7 \cdot 3^{-2}\left(1-\frac{x}{3}\right)^{-2}$

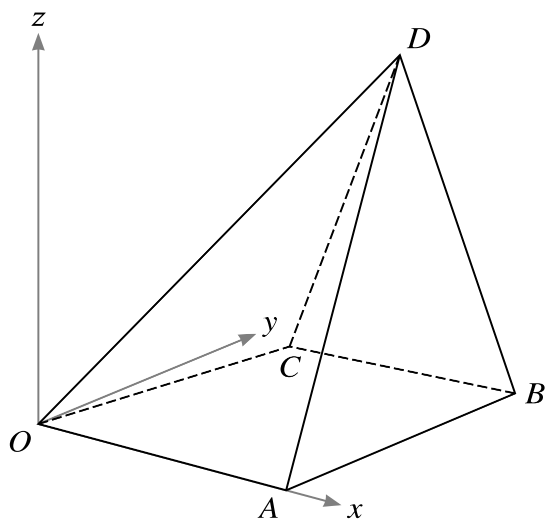
$\left[1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)\left(\frac{x}{2}\right)^2}{2!}\right] + \frac{2}{3}\left[1+(-1)\left(-\frac{x}{3}\right)+\frac{(-1)(-2)\left(-\frac{x}{3}\right)^2}{2!}\right] - \frac{7}{9}\left[1+(-2)\left(-\frac{x}{3}\right)+\frac{(-2)(-3)\left(-\frac{x}{3}\right)^2}{2!}\right]$

$\left(1-\frac{1}{2}x+\frac{1}{4}x^2\right)+\frac{2}{3}\left(1+\frac{1}{3}x+\frac{1}{9}x^2\right)-\frac{7}{9}\left(1+\frac{2}{3}x+\frac{1}{3}x^2\right)$

$1-\frac{1}{2}x+\frac{1}{4}x^2+\frac{2}{3}+\frac{2}{9}x+\frac{2}{27}x^2-\frac{7}{9}-\frac{14}{27}x-\frac{7}{27}x^2$

$\frac{8}{9}-\frac{43}{54}x+\frac{7}{108}x^2$

**9**




The diagram shows a set of rectangular axes $Ox$, $Oy$ and $Oz$, and four points $A$, $B$, $C$ and $D$ with position vectors $\overrightarrow{OA} = 3\mathbf{i}$, $\overrightarrow{OB} = 3\mathbf{i} + 4\mathbf{j}$, $\overrightarrow{OC} = \mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OD} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$.

**(i)** Find the equation of the plane $BCD$, giving your answer in the form $ax + by + cz = d$. [6]

X

**(ii)** Calculate the acute angle between the planes $BCD$ and $OABC$. [4]

$\times$

**10 Throughout this question the use of a calculator is not permitted.**

The complex number $(\sqrt{3}) + \mathrm{i}$ is denoted by $u$.

**(i)** Express $u$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$, giving the exact values of $r$ and $\theta$. Hence or otherwise state the exact values of the modulus and argument of $u^4$. [5]

$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{6}\pi$

$\therefore u = 2e^{\frac{\pi}{6}i}$

$u^4 = \left(2e^{\frac{\pi}{6}i}\right)^4$

$= 2^4 e^{\frac{\pi}{6}i \times 4}$

$= 16 e^{\frac{2}{3}\pi i}$

mod of $u^4 = 16$

arg of $u^4 = \frac{2}{3}\pi$

$\tan^{-1}(3) : 1.249$

$\tan 1.249$



**(ii)** Verify that $u$ is a root of the equation $z^3 - 8z + 8\sqrt{3} = 0$ and state the other complex root of this equation. [3]

$z^3 = (\sqrt{3}+i)^3 = (\sqrt{3}+i)(\sqrt{3}+i)(\sqrt{3}+i)$

$= (3 + 2\sqrt{3}i + (-1))(\sqrt{3}+i)$

$(2 + 2\sqrt{3}i)(\sqrt{3}+i)$

$2\sqrt{3} + 2i + 6i + 2\sqrt{3}(-1)$

$2\sqrt{3} + 8i - 2\sqrt{3}$

$8i$

$8i - 8(\sqrt{3}+i) + 8\sqrt{3} = 0$

$8i - 8\sqrt{3} - 8i + 8\sqrt{3} = 0$

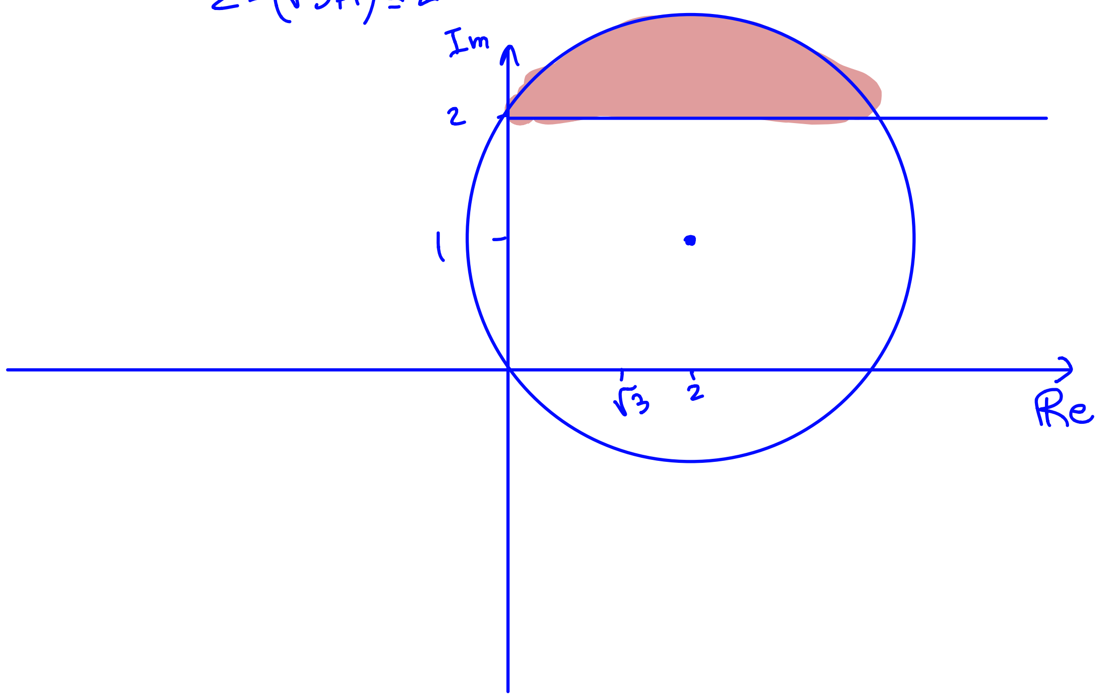
$0 = 0$

shown

other root: $\sqrt{3} - i$

**(iii)** On a sketch of an Argand diagram, shade the region whose points represent complex numbers $z$ satisfying the inequalities $|z - u| \leqslant 2$ and $\operatorname{Im} z \geqslant 2$, where $\operatorname{Im} z$ denotes the imaginary part of $z$. [5]

$z - (\sqrt{3}+i) \leq 2$



## Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

**BLANK PAGE**

**BLANK PAGE**

(b) A buffer solution is to be made using 1.00 mol $dm^{-3}$ ethanoic acid, $CH_3CO_2H$, and 1.00 mol $dm^{-3}$ sodium ethanoate, $CH_3CO_2Na$.
Calculate to the nearest 1 $cm^3$ the volumes of each solution that would be required to make 100 $cm^3$ of a buffer solution with pH 5.50.
Clearly show all steps in your working.
$K_a(CH_3CO_2H) = 1.79 \times 10^{-5}$ mol $dm^{-3}$

1mol$dm^{-3}$ - $CH_3CO_2H \rightleftharpoons CH_3CO_2^- + H^+$

$[H^+] = 10^{-5.5}$
$= 3.162 \times 10^{-6}$ mol $dm^{-3}$

Vol of acid = $x$ , ∴ mol of acid = $cV$ = $1(x) = x$    $c = \frac{n}{V}$

∴ vol of $CH_3COONa = 0.1 - x$ , ∴ mol of salt = $1(0.1 - x) = 0.1 - x$

$K_a(CH_3CO_2H) = \frac{[H^+][CH_3COONa]}{CH_3COOH} = \frac{3.162 \times 10^{-6} \times \frac{0.1-x}{\cancel{0.1}}}{\frac{x}{\cancel{0.1}}} = 1.79 \times 10^{-5}$

$= 3.162 \times 10^{-7} - 3.162 \times 10^{-6}x = 1.79 \times 10^{-5}x$

$2.1062 \times 10^{-5}x = 3.162 \times 10^{-7}$
$x = 1.5 \times 10^{-2} = 15cm^3$

∴ vol of $CH_3COO^-Na^+ = 100 - 15 = 85cm^3$