Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

CANDIDATE NAME		
CENTRE NUMBER		CANDIDATE NUMBER

7084725728

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **May/June 2018**

1 hour 45 minutes

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF9)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, candidate number and name in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions in the space provided. If additional space is required, you should use the lined page at the end of this booklet. The question number(s) must be clearly shown.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.

This document consists of **18** printed pages and **2** blank pages.

CAMBRIDGE International Examinations

1 Showing all necessary working, solve the equation $\ln(x^4 - 4) = 4\ln x - \ln 4$, giving your answer correct to 2 decimal places. [4]

$$\ln(x^4 - 4) = 4\ln x - \ln 4$$

$$\ln(x^4 - 4) = \ln\left(\frac{x^4}{4}\right)$$

$$x^4 - 4 = \frac{x^4}{4}$$

$$4x^4 - 16 = x^4$$

$$3x^4 = 16$$

$$x^4 = \frac{16}{3}$$

$$x = \sqrt[4]{\frac{16}{3}} = 1.52$$

2 **(i)** Given that $\sin(x - 60°) = 3\cos(x - 45°)$, find the exact value of $\tan x$. [4]

$$\sin x \cos 60 - \sin 60 \cos x = 3(\cos x \cos 45 + \sin x \sin 45)$$

$$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = 3\left(\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x\right)$$

$$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = \frac{3\sqrt{2}}{2}\cos x + \frac{3\sqrt{2}}{2}\sin x$$

$\div \cos x$

$$\frac{1}{2}\tan x - \frac{\sqrt{3}}{2} = \frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}\tan x$$

$$\tan x\left(\frac{1}{2} - \frac{3\sqrt{2}}{2}\right) = \frac{\sqrt{3} + 3\sqrt{2}}{2}$$

$$\tan x = \frac{\frac{\sqrt{3} + 3\sqrt{2}}{\cancel{2}}}{\frac{1 - 3\sqrt{2}}{\cancel{2}}}$$

$$\tan x = \frac{\sqrt{3} + 3\sqrt{2}}{1 - 3\sqrt{2}}$$

(ii) Hence solve the equation $\sin(x - 60°) = 3\cos(x - 45°)$, for $0° < x < 360°$. [2]

$$\tan x = \frac{\sqrt{3} + 3\sqrt{2}}{1 - 3\sqrt{2}}$$

$$x = \tan^{-1}\left(\frac{\sqrt{3} + 3\sqrt{2}}{1 - 3\sqrt{2}}\right)$$

$$= 118.48998$$

$$= 11\cancel{0}.5° \quad \text{or} \quad 298.5°$$

3 A curve has equation $y = \dfrac{e^{3x}}{\tan\frac{1}{2}x}$. Find the x-coordinates of the stationary points of the curve in the interval $0 < x < \pi$. Give your answers correct to 3 decimal places. [6]

5

$$y = \frac{e^{3x}}{\tan\left(\frac{1}{2}x\right)}$$

$$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{\tan\left(\frac{1}{2}x\right)(3e^{3x}) - (e^{3x})\left(\frac{1}{2}\sec^2\left(\frac{1}{2}x\right)\right)}{\tan\left(\frac{1}{2}x\right)^2} = 0$$

$$3\cancel{e^{3x}}\tan\left(\frac{1}{2}x\right) = \frac{1}{2}\cancel{e^{3x}}\sec^2\left(\frac{1}{2}x\right)$$

$$6\frac{\sin\left(\frac{1}{2}x\right)}{\cancel{\cos\left(\frac{1}{2}x\right)}} = \frac{1}{\cos^{\cancel{2}}\left(\frac{1}{2}x\right)}$$

$$6\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right) = 1$$

$$3\left(2\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)\right) = 1$$

$$\sin x = \frac{1}{3}$$

sin +v

2	1

$x = \sin\left(\frac{1}{3}\right)$ or $x = \pi - \sin^{-1}\left(\frac{1}{3}\right)$

$= 0.3398369$ or $x = 2.80176$

$\therefore x = 0.340$ or $x = 2.802$

4 The polynomial $x^4 + 2x^3 + ax + b$, where a and b are constants, is divisible by $x^2 - x + 1$. Find the values of a and b. [5]

$$x^2 + 3x + 2$$

$$x^2 - x + 1 \,\big|\, x^4 + 2x^3 + 0x^2 + ax + b$$

$$- \quad x^4 - x^3 + x^2$$

$$0 + 3x^3 - x^2 + ax$$

$$- \quad 3x^3 - 3x^2 + 3x$$

$$0 + 2x^2 + (a-3)x + b$$

$$- \quad 2x^2 - 2x + 2$$

$$0$$

$(a-3)x + 2x = 0$

$ax - 3x + 2x = 0$

$ax = 1x$

$a = 1$

$b - 2 = 0$

$b = 2$

5 Let $I = \int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx$.

(i) Using the substitution $x = \cos^2\theta$, show that $I = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\,d\theta$. [4]

when $x = \frac{1}{4}$, $\cos^2\theta = \frac{1}{4}$

$\theta = \cos^{-1}\left(\sqrt{\frac{1}{4}}\right) = \frac{1}{3}\pi$

when $x = \frac{3}{4}$, $\cos^2\theta = \frac{3}{4}$

$\theta = \cos^{-1}\left(\sqrt{\frac{3}{4}}\right) = \frac{1}{6}\pi$

$x = \cos^2\theta$

$\frac{dx}{d\theta} = -2\cos\theta\sin\theta$, $dx = -2\sin\theta\cos\theta\,d\theta$

$\int_{\frac{1}{3}\pi}^{\frac{1}{6}\pi} \sqrt{\frac{\cos^2\theta}{1-\cos^2\theta}} \times -2\sin\theta\cos\theta\,d\theta$

$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sqrt{\frac{\cos^2\theta}{\sin^2\theta}} \times 2\sin\theta\cos\theta\,d\theta$

$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{\cos\theta}{\cancel{\sin\theta}} \times 2\cancel{\sin\theta}\cos\theta\,d\theta$

$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} 2\cos^2\theta\,d\theta$

(ii) Hence find the exact value of I. [4]

$2\int \cos^2\theta \, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$2\int \frac{1}{2} + \frac{\cos 2\theta}{2} \, d\theta$

$2\left[\left(\frac{1}{2}\theta\right) + \frac{1}{2}\int \cos 2\theta \, d\theta\right]$

$2\left(\frac{1}{2}\theta + \frac{1}{2}\left(\frac{1}{2}\right)\sin 2\theta\right)$

$2\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right.$

let $f(x) = \theta + \frac{1}{2}\sin 2\theta \Big|_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$

$f\left(\frac{1}{3}\pi\right) = \frac{1}{3}\pi + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{3}\pi + \frac{\sqrt{3}}{4}$

$f\left(\frac{1}{6}\pi\right) = \frac{1}{6}\pi + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{6}\pi + \frac{\sqrt{3}}{4}$

$\frac{1}{3}\pi + \cancel{\frac{\sqrt{3}}{4}} - \frac{1}{6}\pi - \cancel{\frac{\sqrt{3}}{4}} = \frac{1}{6}\pi$

6 In a certain chemical reaction the amount, x grams, of a substance is decreasing. The differential equation relating x and t, the time in seconds since the reaction started, is

$$\frac{dx}{dt} = -kx\sqrt{t},$$

where k is a positive constant. It is given that $x = 100$ at the start of the reaction.

(i) Solve the differential equation, obtaining a relation between x, t and k. [5]

$\int \frac{1}{x}\,dx = -k\int \sqrt{t}\,dt$

$\ln x = -k\int t^{\frac{1}{2}}\,dt$

$\ln x = -\frac{kt^{\frac{3}{2}}}{\frac{3}{2}} + C$

$\ln x = \frac{-2kt^{\frac{3}{2}}}{3} + C$

$\ln 100 = \frac{0}{3} + C$

$C = \ln 100$

$\ln x = \ln 100 - \frac{2kt^{\frac{3}{2}}}{3}$

(ii) Given that $t = 25$ when $x = 80$, find the value of t when $x = 40$. [3]

1

$\ln x = \ln 100 - \frac{2}{3}kt^{\frac{3}{2}}$

$\ln 80 = \ln 100 - \frac{250k}{3}$

$\frac{250k}{3} = \ln \frac{100}{80}$

$k = 3\ln \frac{100}{80}$

$k = \frac{3}{250}\ln\left(\frac{100}{80}\right) = 0.0026777$

$\frac{2}{3}kt^{\frac{3}{2}} = \ln \frac{100}{40}$

$kt^{\frac{3}{2}} = \frac{3\ln \frac{100}{40}}{2}$

$k\left(\sqrt{t}\right)^3 = 1.3744$

$\left(\sqrt{t}\right)^3 = 513.2719$

$\sqrt{t} = 8.0066$

$t = 64.1$

7 **(i)** Showing all working and without using a calculator, solve the equation $z^2 + (2\sqrt{6})z + 8 = 0$, giving your answers in the form $x + iy$, where x and y are real and exact. [3]

$z^2 + (2\sqrt{6})z + 8 = 0$

$\dfrac{-2\sqrt{6} \pm \sqrt{(2\sqrt{6})^2 - 4(1)(8)}}{2}$

$\dfrac{-2\sqrt{6} \pm \sqrt{24-32}}{2}$

$\dfrac{-2\sqrt{6} \pm \sqrt{-8i}}{2}$ $\rightarrow$ $\sqrt{-8i} = \sqrt{8 \times -1}$

$= \sqrt{8} \times \sqrt{-1}$

$= \sqrt{4 \times 2} \times \sqrt{-1}$

$= \sqrt{4} \times \sqrt{2} \times \sqrt{-1}$

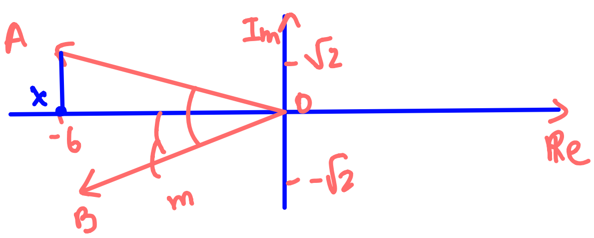
$= 2\sqrt{2}i$

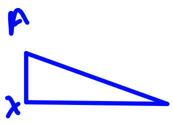
$\dfrac{-\cancel{2}\sqrt{6} \pm \cancel{2}\sqrt{2}i}{\cancel{2}}$

$-6 \pm \sqrt{2}i$

$z = -6 + \sqrt{2}i$ or $z = -6 - \sqrt{2}i$

(ii) Sketch an Argand diagram showing the points representing the roots. [1]





(iii) The points representing the roots are A and B, and O is the origin. Find angle AOB. [3] 0

$\therefore AOB = 2 \times \text{angle } m$

$= 2 \times \tan^{-1}\left(\frac{\sqrt{2}}{6}\right)$

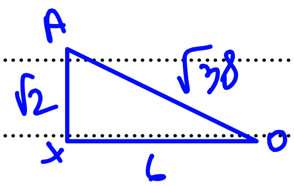
$= 0.46295$

$= 0.463$

(iv) Prove that triangle AOB is equilateral. [1]

$AO = \sqrt{(\sqrt{2})^2 + 6^2} = \sqrt{38}$

$BO = \sqrt{(-\sqrt{2}^2)_+ - (6^2)} = \sqrt{38}$



$AB = 2 \times Ax$

$Ax = \sqrt{\sqrt{38} - 6^2} = \sqrt{2}$

8 The positive constant a is such that $\int_0^a x e^{-\frac{1}{2}x}\,dx = 2$.

(i) Show that a satisfies the equation $a = 2\ln(a+2)$. [5]

$\int_0^a x e^{-\frac{1}{2}x}\,dx$

$u \quad v'$

$u = x \qquad u' = 1$

$v = -2e^{-\frac{1}{2}x} \qquad v' = e^{-\frac{1}{2}x}$

$x(-2e^{-\frac{1}{2}x}) - \int -2e^{-\frac{1}{2}x}(1)\,dx$

$-2xe^{-\frac{1}{2}x} + 2\int e^{-\frac{1}{2}x}\,dx$

$-2xe^{-\frac{1}{2}x} + 2(-2e^{-\frac{1}{2}x})$

$\left[-2e^{-\frac{1}{2}a}(x+2)\right]_0^a = 2$

$\left(-2e^{-\frac{1}{2}a}(a+2)\right) - \left(-2e^0(2)\right) = 2$

$-2e^{-\frac{1}{2}a}(a+2) - (-4) = 2$

$\cancel{-}2e^{-\frac{1}{2}a}(a+2) = \cancel{-}2$

$\ln 2e^{-\frac{1}{2}a} = \ln \frac{2}{a+2}$

$\ln 2 + \left(-\frac{1}{2}a\right) = \ln \frac{2}{a+2}$

$\ln 2 - \ln\left(\frac{2}{a+2}\right) = \frac{1}{2}a$

$\frac{1}{2}a = \ln \frac{\cancel{2}}{} \times \frac{a+2}{\cancel{2}}$

$a = 2\ln(a+2)$

(ii) Verify by calculation that a lies between 3 and 3.5. [2]

$a = 2\ln(a+2)$

let $f(x) = 2\ln(x+2) - x$

$\therefore f(3) = 2\ln(5) - 3 = 0.218$

$f(3.5) = -0.0905$

change of sign $\therefore$ root.

(iii) Use an iteration based on the equation in part **(i)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

let $a_1 = 3.1$

$a_2 = 2\ln(3.1+2) = 3.25848$

$a_3 = 3.31968$

$a_4 = 3.34283$

$a_5 = 3.35151$

$a_6 = 3.35478$

$a_7 = 3.35597$

$a_8 = 3.35642$ $\therefore a = 3.36$

$a_9 = 3.35659$

$a_{10} = 3.35665$

9 Let $f(x) = \dfrac{12x^2 + 4x - 1}{(x-1)(3x+2)}$.

3

(i) Express $f(x)$ in partial fractions. [5]

$$\frac{12x^2+4x-1}{(x-1)(3x+2)} = A + \frac{B}{x-1} + \frac{C}{3x+2}$$

$$12x^2+4x-1 = A(x-1)(3x+2) + B(3x+2) + C(x-1)$$

$$= A(3x^2-x-2) + 3Bx + 2B + Cx - C$$

$$3Ax^2 - Ax - 2A + 3Bx + 2B + Cx - C$$

$12\cancel{x^2} = 3A\cancel{x^2}$

$A = 4$

$4 = -4 + 3B + C$

$8 = 3B + C$

$\therefore 8 - 3B = C$

$-1 = -2(4) + 2B - C$

$-1 = -8 + 2B - C$

$7 = 2B + C$

$\therefore C = 7 - 2B$

$8 - 3B = 7 - 2B$

$1 = B$

$\therefore C = 7 - 2(1)$

$= 5$

$$\therefore 4 + \frac{1}{x-1} + \frac{5}{3x+2}$$

(ii) Hence obtain the expansion of $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^2. [5]

0

$4 + \frac{1}{x-1} + \frac{5}{3x+2}$

$4 + (x-1)^{-1} + 5(3x+2)^{-1}$

$(x-1)^{-1} = (-1+x)^{-1} = -1^{-1}(1-x)^{-1} = -(1-x)^{-1}$

$-\left[1 + (-1)(-x) + \frac{(-1)(-2)(x)^2}{2\times 1}\right]$

$= -(1 + x + x^2)$

$= -1 - x - x^2$

$5 \times 2^{-1}\left(1 + \frac{3}{2}x\right)^{-1} =$

$\frac{5}{2}\left(1 + (-1)(\frac{3}{2}) + \frac{(-1)(-2)(\frac{3}{2}x)^2}{2}\right)$

$\frac{5}{2}\left(1 - \frac{3}{2}x + \frac{9}{4}x^2\right)$

$\frac{5}{2} - \frac{15}{4}x + \frac{45}{8}x^2$

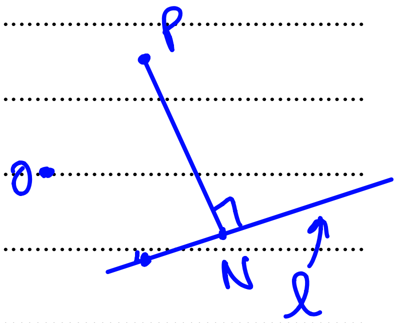
$4 - 1 - x - x^2 - \frac{5}{2} - \frac{15}{4}x + \frac{45}{8}x^2$

$\frac{1}{2} + \frac{11}{4}x + \frac{37}{8}x^2$

10 The point P has position vector $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$.

(i) Find the length of the perpendicular from P to l, giving your answer correct to 3 significant figures. [5]

2



$$\vec{NP} = \vec{OP} - \vec{ON}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4+\mu \\ 2+2\mu \\ 5+3\mu \end{pmatrix}$$

$$= \begin{pmatrix} -1-\mu \\ -4-2\mu \\ -4-3\mu \end{pmatrix}$$

$$\begin{pmatrix} -1-\mu \\ -4-2\mu \\ -4-3\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0$$

$$-1-\mu + (-8-4\mu) + (-12-9\mu) = 0$$

$$-1-\mu-8-4\mu-12-9\mu = 0$$

$$-13-13\mu = 0$$

-21

$$13\mu = 13$$

$$\mu = 1 \quad -\frac{3}{2}$$

sub and check!

$$NP = \begin{pmatrix} -1-1 \\ -4-2 \\ -4-3 \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ -7 \end{pmatrix}$$

$$\therefore |NP| = \sqrt{2^2+6^2+7^2} = 9.43$$

(ii) Find the equation of the plane containing l and P, giving your answer in the form $ax + by + cz = d$. [5]

Additional Page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

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