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Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **(P3)** **October/November 2016**

1 hour 45 minutes

* 5 7 1 1 2 0 4 0 2 6 *

Additional Materials: List of Formulae (MF9)

READ THESE INSTRUCTIONS FIRST

An answer booklet is provided inside this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 75.

CAMBRIDGE International Examinations

1 Solve the equation $\dfrac{3^x + 2}{3^x - 2} = 8$, giving your answer correct to 3 decimal places. [3]

2 Expand $(2 - x)(1 + 2x)^{-\frac{3}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

3 Express the equation $\sec\theta = 3\cos\theta + \tan\theta$ as a quadratic equation in $\sin\theta$. Hence solve this equation for $-90° < \theta < 90°$. [5]

4 The equation of a curve is $xy(x - 6y) = 9a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [7]

5 **(i)** Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$. [4]

(ii) Hence show that $\displaystyle\int_0^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \tfrac{1}{2}\ln\tfrac{3}{2}$. [4]

6 **(i)** By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec}\tfrac{1}{2}x = \tfrac{1}{3}x + 1$$

has one root in the interval $0 < x \leqslant \pi$. [2]

(ii) Show by calculation that this root lies between 1.4 and 1.6. [2]

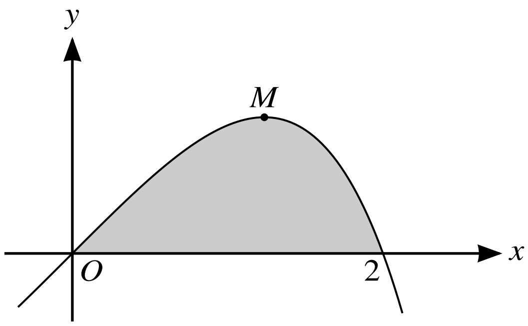
(iii) Show that, if a sequence of values in the interval $0 < x \leqslant \pi$ given by the iterative formula

$$x_{n+1} = 2\sin^{-1}\left(\frac{3}{x_n + 3}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

7



The diagram shows part of the curve $y = (2x - x^2)e^{\frac{1}{2}x}$ and its maximum point M.

(i) Find the exact x-coordinate of M. [4]

(ii) Find the exact value of the area of the shaded region bounded by the curve and the positive x-axis. [5]

8 Two planes have equations $3x + y - z = 2$ and $x - y + 2z = 3$.

(i) Show that the planes are perpendicular. [3]

(ii) Find a vector equation for the line of intersection of the two planes. [6]

9 **Throughout this question the use of a calculator is not permitted.**

(a) Solve the equation $(1 + 2i)w^2 + 4w - (1 - 2i) = 0$, giving your answers in the form $x + iy$, where x and y are real. [5]

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 1 - i| \leqslant 2$ and $-\frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{4}\pi$. [5]

10 A large field of area $4\,\text{km}^2$ is becoming infected with a soil disease. At time t years the area infected is $x\,\text{km}^2$ and the rate of growth of the infected area is given by the differential equation $\frac{dx}{dt} = kx(4 - x)$, where k is a positive constant. It is given that when $t = 0$, $x = 0.4$ and that when $t = 2$, $x = 2$.

(i) Solve the differential equation and show that $k = \frac{1}{4}\ln 3$. [9]

(ii) Find the value of t when 90% of the area of the field is infected. [2]

1 Solve the equation $\dfrac{3^x + 2}{3^x - 2} = 8$, giving your answer correct to 3 decimal places. [3]

let $y = 3^x$

$$\frac{y+2}{y-2} = 8$$

$$y + 2 = 8y - 16$$

$$7y = 18$$

$$y = \frac{18}{7}$$

$$3^x = \frac{18}{7}$$

$$x \ln 3 = \ln \frac{18}{7}$$

$$x = \ln\left(\frac{18}{7}\right) \div \ln 3 = 0.859686$$

$$\approx 0.860$$

2 Expand $(2 - x)(1 + 2x)^{-\frac{3}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

$$(2-x)(1+2x)^{-\frac{3}{2}}$$

$$(2-x)\left[1 + \left(-\frac{3}{2}\right)(2x) + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(2x)^2}{2!}\right]$$

$$(2-x)\left(1 - 3x + \frac{15}{2}x^2\right)$$

$$2 - 6x + 15x^2 - x + 3x^2 - \frac{15}{2}x^3$$

$$2 - 7x + 18x^2$$

3 Express the equation $\sec\theta = 3\cos\theta + \tan\theta$ as a quadratic equation in $\sin\theta$. Hence solve this equation for $-90° < \theta < 90°$. [5]

3

$$\frac{1}{\cos\theta} = 3\cos\theta + \frac{\sin\theta}{\cos\theta}$$

$$\frac{1}{\cancel{\cos\theta}} = \frac{3\cos^2\theta + \sin\theta}{\cancel{\cos\theta}}$$

$$3(1-\sin^2\theta) + \sin\theta = 1$$

$$3 - 3\sin^2\theta + \sin\theta = 1$$

$$\underset{c}{2} - \underset{a}{3}\sin^2\theta + \underset{b}{\sin\theta} = 0$$

$$\frac{-1 \pm \sqrt{1-4(-3)(2)}}{-6}$$

$$\frac{-1 \pm 5}{6}$$

$\sin\theta = \frac{2}{3}$

$\theta = \sin^{-1}\left(\frac{2}{3}\right)$

$= \underline{\underline{41.810}}$

$\sin\theta = -1$

$\theta = 360 - \sin^{-1}(1)$ or $180 + \sin^{-1}(1)$

$= 270$ or 270

$-360 \rightarrow -90$

$\times$

4 The equation of a curve is $xy(x - 6y) = 9a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of ~~this~~ point. [7]

#

$x^2y - 6xy^2 = 9a^3$

$(2x)(y) + (x^2)\left(\frac{dy}{dx}\right) - \left[(6)(y^2) + (6x)\left(2y\frac{dy}{dx}\right)\right] = 0$

$2xy + x^2\frac{dy}{dx} - 6y^2 - 12xy\frac{dy}{dx} = 0$

$2xy - 6y^2 = (12xy - x^2)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2xy - 6y^2}{12xy - x^2}$

$2xy - 6y^2 = 0$

$2y(x - 3y) = 0$

$2y = 0$

$y = 0$

rejected

$x - 3y = 0$

$x = 3y$

$(3y)y(3y - 6y) = 9a^3$

$3y^2(-3y) = 9a^3$

$-9y^3 = 9a^3$

$y^3 = -a^3$

$y = -a$

$x = 3(-a) = -3a$

$(-3a, -a)$

5 **(i)** Prove the identity $\tan 2\theta - \tan\theta \equiv \tan\theta \sec 2\theta$. [4]

(ii) Hence show that $\int_0^{\frac{1}{6}\pi} \tan\theta \sec 2\theta \, d\theta = \frac{1}{2}\ln\frac{3}{2}$. [4]

i) $\frac{\sin 2\theta}{\cos 2\theta} - \frac{\tan\theta}{\cos 2\theta}$

$\frac{\sin 2\theta - \cos 2\theta \tan\theta}{\cos 2\theta}$

$\frac{2\sin\theta\cos\theta - (2\cos^2\theta - 1)\tan\theta}{\cos 2\theta}$

$\frac{2\sin\theta\cos\theta - 2\cos^{\cancel{2}}\theta \frac{\sin\theta}{\cancel{\cos\theta}} + \tan\theta}{\cos 2\theta}$

$\frac{\cancel{2\sin\cos\theta} - \cancel{2\sin\theta\cos\theta} + \tan\theta}{\cos 2\theta}$

$\sec 2\theta \tan\theta$

ii) $\int \tan 2\theta - \tan\theta \, d\theta$

$\int \frac{\sin 2\theta}{\cos 2\theta} - \frac{\sin\theta}{\cos\theta} \, d\theta$

$-\frac{1}{2}\int \frac{-2\sin 2\theta \, d\theta}{\cos 2\theta} - (-)\int \frac{-\sin\theta}{\cos\theta} \, d\theta$

$\left[\frac{-1}{2}\ln\cos 2\theta + \ln\cos\theta\right]_0^{\frac{1}{6}\pi}$

$-\frac{1}{2}\ln\cos\left(\frac{2}{6}\pi\right) + \ln\cos\left(\frac{1}{6}\pi\right)$

$-\frac{1}{2}\ln\left(\frac{1}{2}\right) + \ln\left(\frac{\sqrt{3}}{2}\right)$

$-\frac{1}{2}\ln 1 + \ln 1 \Rightarrow 0$

$\frac{1}{2}\ln 2 + \ln\left(\frac{3}{4}\right)^{\frac{1}{2}}$

$\frac{1}{2}\ln 2 + \frac{1}{2}\ln\frac{3}{4}$

$\frac{1}{2}\ln 2 \times \frac{3}{4}$

$\frac{1}{2}\ln\frac{3}{2}$

6 **(i)** By sketching a suitable pair of graphs, show that the equation

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has one root in the interval $0 < x \leqslant \pi$. [2]

(ii) Show by calculation that this root lies between 1.4 and 1.6. [2]

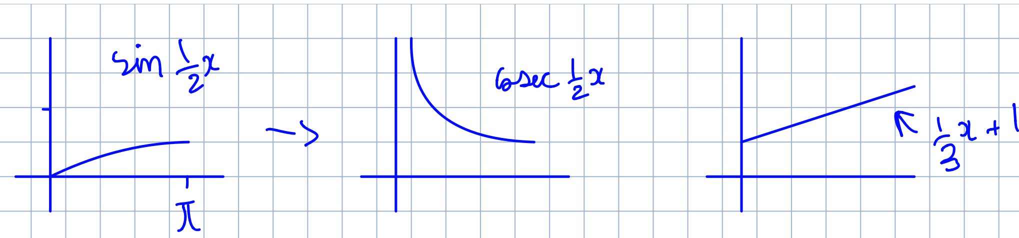
(iii) Show that, if a sequence of values in the interval $0 < x \leqslant \pi$ given by the iterative formula

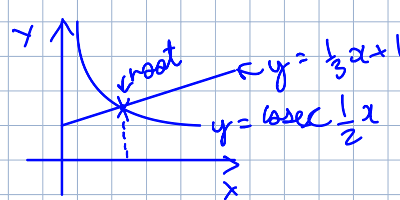
$$x_{n+1} = 2\sin^{-1}\left(\frac{3}{x_n + 3}\right)$$

converges, then it converges to the root of the equation in part **(i)**. [2]

(iv) Use this iterative formula to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

i)





ii)

	LHS		RHS
$x=1.4$	1.552	$>$	1.466
	1.394	$<$	1.533

change of sign, ∴ root is b/w 1.4 and 1.6

iii) $\operatorname{cosec} \frac{1}{2}x = \frac{1}{3}x + 1$

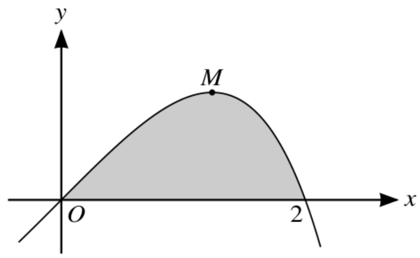
$$\frac{1}{\sin \frac{1}{2}x} = \frac{x+3}{3}$$

$$\sin \tfrac{1}{2}x = \frac{3}{x+3}$$

$$\tfrac{1}{2}x = \sin^{-1}\left(\frac{3}{x+3}\right)$$

$$x = 2\sin^{-1}\left(\frac{3}{x+3}\right)$$

7



The diagram shows part of the curve $y = (2x - x^2)e^{\frac{1}{2}x}$ and its maximum point M.

(i) Find the exact x-coordinate of M. [4] 2

(ii) Find the exact value of the area of the shaded region bounded by the curve and the positive x-axis. 3
[5]

i)
$y = (2x - x^2)e^{\frac{1}{2}x}$

$\frac{dy}{dx} = (2x - x^2)(\frac{1}{2}e^{\frac{1}{2}x}) + (e^{\frac{1}{2}x})(2 - 2x) = 0$

$x e^{\frac{1}{2}x} - \frac{x^2}{2}e^{\frac{1}{2}x} + 2e^{\frac{1}{2}x} - 2xe^{\frac{1}{2}x} = 0$

$e^{\frac{1}{2}x}(x + 2 - 2x - \frac{1}{2}x^2) = 0$

$e^{\frac{1}{2}x}$

$\times$

$-x + 2 - \frac{1}{2}x^2 = 0$

$\frac{1}{2}x^2 + x - 2 = 0$

$\frac{-1 \pm \sqrt{1 - 4(\frac{1}{2})(-2)}}{1}$

$-1 \pm \sqrt{5}$

$-1 + \sqrt{5}$

ii) $\displaystyle\int_0^2 (2x - x^2)\, e^{\frac{1}{2}x}\, dx$

$\uparrow u \qquad \uparrow v'$

$u = 2x - x^2 \qquad u' = 2 - 2x$

$v = 2e^{\frac{1}{2}x} \qquad v' = e^{\frac{1}{2}x}$

$(2x - x^2)(2e^{\frac{1}{2}x}) - \int (2-2x)(2e^{\frac{1}{2}x})\, dx$

$(2x - x^2)(2e^{\frac{1}{2}x}) - 2\int (2-2x)\, e^{\frac{1}{2}x}\, dx$

$u = 2 - 2x \qquad u' = -2$

$v = 2e^{\frac{1}{2}x} \qquad v' = e^{\frac{1}{2}x}$

$4xe^{\frac{1}{2}x} - 2x^2e^{\frac{1}{2}x} - 2\left[2e^{\frac{1}{2}x}(2-2x) - \int -2\,(2e^{\frac{1}{2}x})\, dx\right]$

$4xe^{\frac{1}{2}x} - 2x^2e^{\frac{1}{2}x} - 2\left[2e^{\frac{1}{2}x}(2-2x) + 4\,(2e^{\frac{1}{2}x})\right]$

$4xe^{\frac{1}{2}x} - 2x^2e^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}(2-2x) - 16e^{\frac{1}{2}x}$

$4xe^{\frac{1}{2}x} - 2x^2e^{\frac{1}{2}x} - 8e^{\frac{1}{2}x} + 8xe^{\frac{1}{2}x} - 16e^{\frac{1}{2}x}$

$12xe^{\frac{1}{2}x} - 24e^{\frac{1}{2}x} - 2x^2 e^{\frac{1}{2}x}$

$e^{\frac{1}{2}x}(12x - 24 - 2x^2)$

sub 2: $e^1(24 - 24 - 8) = -8e$

0: $1(0 - 24) = \cancel{-14}\ -24$

$\cancel{14 - 8e} \qquad 24 - 8e$

9 Throughout this question the use of a calculator is not permitted.

(a) Solve the equation $(1 + 2i)w^2 + 4w - (1 - 2i) = 0$, giving your answers in the form $x + iy$, where x and y are real. [5]

0

(b) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequalities $|z - 1 - i| \leqslant 2$ and $-\frac{1}{4}\pi \leqslant \arg z \leqslant \frac{1}{4}\pi$. [5]

4

a)

$$\frac{-4 \pm \sqrt{16 - 4(1+2i)(1-2i)}}{2(1+2i)}$$

$$\frac{-4 \pm \sqrt{16 - 4() + 4)}}{2+4i}$$

$$\frac{-4 \pm \sqrt{16-20}}{2+4i}$$

$$\frac{-4 \pm \sqrt{4(-1)}}{2+4i}$$

$$\frac{-4 \pm 2i}{2+4i}$$

$$\frac{-4 + 2i}{(2+4i)} \frac{(2-4i)}{(2-4i)}$$

$$\frac{\cancel{-8} + 16i + 4i + \cancel{8}}{(4+16)}$$

$$\frac{20i}{20} = i$$

$$\frac{-4 - 2i(2-4i)}{20}$$

$$\frac{-8 + 16i - 4i - 8}{20}$$

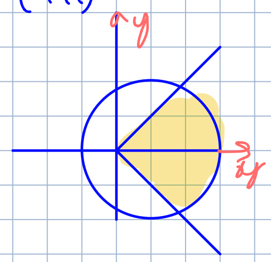
$$\frac{-16 + 14i}{20}$$

$$-\frac{4}{5} + \frac{7}{10}i$$

x

$$\frac{-4}{5} \pm \frac{7}{10}i$$

b) $z - (1+i)$



10 A large field of area 4 km^2 is becoming infected with a soil disease. At time t years the area infected is x km^2 and the rate of growth of the infected area is given by the differential equation $\frac{dx}{dt} = kx(4-x)$, where k is a positive constant. It is given that when $t = 0$, $x = 0.4$ and that when $t = 2$, $x = 2$.

(i) Solve the differential equation and show that $k = \frac{1}{4}\ln 3$. [9]

(ii) Find the value of t when 90% of the area of the field is infected. [2]

i) $\int \frac{1}{x(4-x)}\,dx = k\int dt$

$\frac{1}{x(4-x)} = \frac{A}{x} + \frac{B}{4-x}$

$1 = 4A - Ax + Bx$

$A = \frac{1}{4}$

$-\frac{1}{4} + B = 0$

$B = \frac{1}{4}$

$\int \frac{1}{4x} + \frac{1}{4(4-x)}\,dx = kt$

$\int \frac{1}{4x}\,dx + \int \frac{1}{16-4x}\,dx = kt$

$\frac{1}{4}\int \frac{4}{4x}\,dx - \frac{1}{4}\int \frac{-4}{16-4x}\,dx = kt$

$\frac{1}{4}\ln 4x - \frac{1}{4}\ln(16-4x) + c = kt$

$\frac{1}{4}\ln \frac{4x}{16-4x} + c = kt$

$\frac{1}{4}\ln \frac{1.6}{14.4} = 0$

$\frac{1}{4}\ln\left(\frac{1}{9}\right) + c = 0$

$c = -\frac{1}{4}\ln\frac{1}{9} = \frac{1}{4}\ln 9$

$\frac{1}{4}\ln\frac{8}{16-8} + \frac{1}{4}\ln 9 = 2k$

$\frac{1}{4}\ln 9 = 2k$

$\frac{1}{8}\ln 9 = k$

$\frac{1}{8}\ln 3^2 = k$

$\frac{2}{8}\ln 3 = k$

$\frac{1}{4}\ln 3 = k$

ii) $0.9 \times 4 = 3.6$

$\frac{1}{4}\ln\frac{14.4}{16-14.4} + \frac{1}{4}\ln 9 = \frac{1}{4}\ln(3)t$

$\frac{1}{4}\ln 9 + \frac{1}{4}\ln 9 = \frac{1}{4}\ln(3)t$

$\frac{\frac{1}{4}\ln 81}{\frac{1}{4}\ln 3} = t$

$t = 4$

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