APPLYING SATISFIABILITY TO THE ANALYSIS OF CRYPTOGRAPHY

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WHY USE SAT FOR CRYPTOGRAPHY?

- Cryptographic algorithms are critical
  - Central to commerce, private communication, etc.
  - We want them to be correct
- Cryptography is hard
  - Rare expertise
  - Top experts have designed algorithms now known vulnerable
  - Design $\rightarrow$ implementation can introduce other problems
- Key primitives used in cryptography amenable to SAT
  - Can be given denotational semantics in propositional logic
  - Many interesting problems surprisingly tractable

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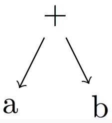
- Many primitives naturally made of bit vector operations
  - Block ciphers
  - Stream ciphers
  - Hash functions
  - Pseudo-random number generators (PRNGs)
- Public-key algorithms representable, but trickier
  - Lots of number theory, including multiplication
  - SMT can alleviate some of this (but tends to be slower on bit-vectory things)
- We'll focus on primitives with type $\{0,1\}^n \rightarrow \{0,1\}^m$

- A function from the ZUC stream cipher

```
uint8_t c = a + b;
if (c & 0x80) {
  c = (c & 0x7F) + 1;
}
return c;
```



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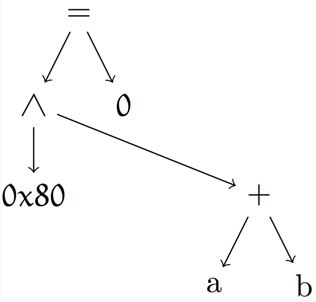

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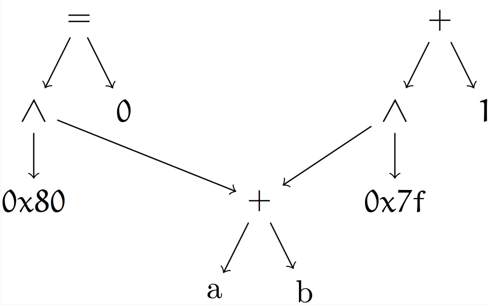

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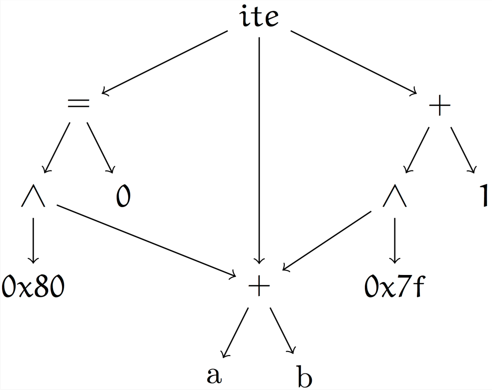

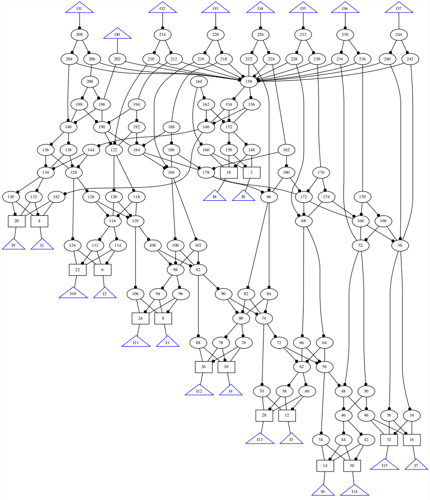
O1 O2 O3 O4 O5 O6 O7
208 O0 214 220 226 232 238 244
204 206 202 210 212 216 218 222 224 228 230 234 236 240 242
200 164 158
198 196 194 162 154 156
140 190 192 188 146 152
136 138 144 122 184 186 160 150 148 182
134 128 104 178 18 2 180 176
130 132 142 126 120 118 I8 I0 86 172 174 170
20 4 116 110 68 168 166
I9 I1 124 112 114 108 100 102 52 38
22 6 98 92
I10 I2 106 94 96 90 82 84
24 8 80 74
I11 I3 88 78 76 72 66 64
26 10 62 56
I12 I4 70 58 60 48 50
28 12 46 40 36 34
I13 I5 54 44 42 32 16
14 30 I15 I7
I6 I14

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- Transalg (Russian Academy of Sciences) [4]
- Cryptol and the Software Analysis Workbench (SAW) (Galois)
  - Cryptol for describing algorithms concisely
  - SAW for proving properties about them
- TODO: others

- Each with a concise formula summarizing the SAT problem

$$\forall x.\ P \text{ or } \exists y.\ Q$$

- Cryptol specifications available online
- Analyzed using SAW + ABC
- Cryptol or SAW file names mentioned in slides (e.g. [file.saw])

https://github.com/galoisinc/sat2015-crypto

file.saw $\longrightarrow$ examples/file.saw

- Randomness is critical to cryptography
  - Keys must be unpredictable, or they're vulnerable
- Typical structure:

  seed $\longrightarrow$ pseudo-random function $\longrightarrow$ pseudo-random value

- Seed often comes from "true" random source
- But it's critical not to lose entropy from this source!

- Bug in Android cryptographic PRNG discarded entropy
- Intended: 160 bits of entropy, from system PRNG to SHA-1

0	160	224 (255)
seed	counter	0x80000000

- Implemented: 64 bits of entropy survive

0	64	96	160 (255)
counter	0x80000000	seed (rest)	$0 \ldots 00$

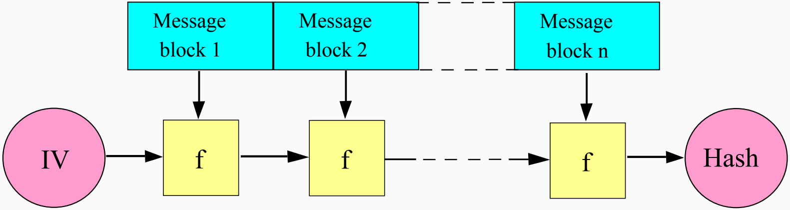
- Used to steal around $6k of BitCoin
- Fixed in Android 4.4 (KitKat)
- Similar weaknesses existed in Debian, FreeBSD at times

INJECTIVITY OF PRNG SEEDING (CONT.)

$$\forall x, y.\ x \neq y \Rightarrow f(x) \neq f(y)$$

- Here, x and y come from system PRNG, f is code between system PRNG and SHA-1
- Easily provable with SAT solver [android-prng.saw]
  - Symbolic execution: Java code $\rightarrow$ AIG
  - Annotation on system RNG variables, input to SHA-1
  - ABC can find collision (0.017s), prove fixed version (0.010s)
- Dörre and Klebanov used a different approach to prove fixed code [1]
  - Information flow annotations on methods using a contract verification tool (KeY) and 95 manual proof steps

HASH FUNCTIONS



- Key property: hard to find two messages that have the same hash (a collision)
- Often built using Merkle–Damgård construction, iterating compression function
- Compression function f usually n iterations of simpler function g

$$\exists x, y.\ x \neq y \land f(x) = f(y)$$

- Discovering a collision
- Black box search in $O(2^{n/2})$

$$\exists x.\ f(x) = a \text{ (for some known value } a\text{)}$$

- Discovering message given hash value (inversion)
- Harder than finding a collision ($O(2^n)$)
- We want to know that it's hard to solve these problems

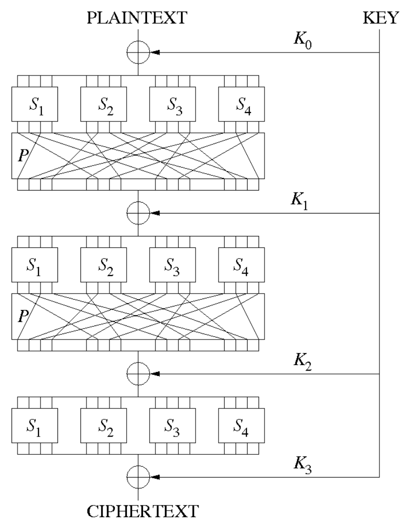
- Mironov and Zhang analyzed collisions in MD4, MD5, SHA-0
  - Direct translation of MD4 $\rightarrow 2^{22}$ solutions in < 1h
  - Collision on full MD5 in around 100h
    - Reduced rounds much easier (25 in 38s with ABC) [MD5.saw]
    - Used differential path derived manually (more on this later)
  - Estimated around 3 million (2006-era) CPU hours for SHA-0
- No known collisions on SHA-1 from this, but it may be a matter of time
  - Algorithm (not SAT-based) to find collisions in 2^{63} operations

- Many candidate algorithms for the SHA-3 standard
- No preimages discoverable by SAT on full algorithms
- Homsirikamol et al. found preimages for fewer rounds
  - Direct translation, with no manual cryptanalysis

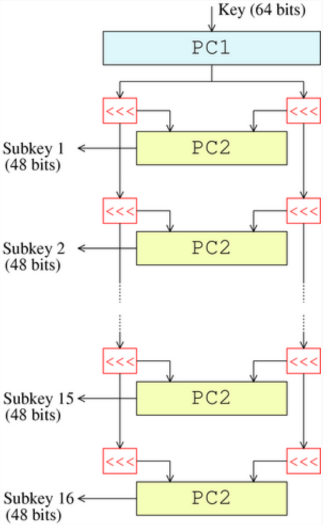
Algorithm	Rounds	Security margin	Code
SHA-1	21	74% (21/80)	SHA-0-1.saw (36s)
SHA-256	16	75% (16/64)	SHA265.saw (1.3s)
Keccak-256	2	92% (2/24)	N/A
BLAKE-256	1	93% (1/14)	Blake256.saw (17s)
Groestl-256	0.5	95% (0.5/10)	N/A
JH-256	2	96% (2/42)	N/A
Skein-512-256	1	99% (1/72)	N/A

BLOCK CIPHERS

- Given a key, a block cipher is a pseudo-random permutation
  - Therefore, invertable (for a fixed key)
- Often built as a substitution-permutation network
  - Will return to S-boxes



KEY EXPANSION



- Symmetric encryption often involves a single shared key
  - Block ciphers typically need one key per round
  - Stream ciphers need one "key" per message block
- So we need to expand the initial key in an unpredictable way
- Should preserve size of key space

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INJECTIVITY OF KEY EXPANSION

$$\forall x, y.\ x \neq y \Rightarrow f(x) \neq f(y)$$

- Provable injectivity of SIMON, Salsa20 key expansion
  - Around 10-20s with ABC [simon.saw] [Salsa20.saw]
- Provable injectivity of AES key expansion
  - A little under a minute with ABC [AES.saw]
- ZUC, a stream cipher for GSM, had a vulnerability
  - Key expansion not injective in ZUC 1.4 (0.5s) [zuc.saw]
  - Provably fixed in ZUC 1.5 (0.6s) [zuc.saw]
  - Originally shown with custom search procedure taking 3m
- Not injective for DES!
  - Known to have weak keys
  - Can show quickly with ABC (0.08s) [DES.saw]

$$\exists m.\ E(m, k) = c \text{ for known } k \text{ and } c$$

- Decrypting using encryption code
- This actually works!
  - Relatively efficient for DES (0.2s), 3DES (0.8s) [DES.saw] [3DES.saw]
  - More modern ciphers slower:
    - AES (1.5m) [AES.saw]
    - SIMON (3.6m) [simon.saw]
    - Speck (20s) [speck.saw]
- Can also run the encryption directly:
  - $\exists c.\ E(m, k) = c$ for known m and k
  - Usually takes $1/5$ to $1/2$ the time of decryption
  - Not really useful, but illustrates the flexibility of SAT

BLOCK CIPHER CONSISTENCY

$$\forall m, k.\ D(E(m, k), k) = m$$

- For a block cipher with encryption function E, decryption D
- Feasible to show for many ciphers
  - DES (4s) [DES.saw]
  - 3DES (8s) [3DES.saw]
  - SIMON (128-256) (6.2m) [simon.saw]
  - Speck (2.7m) [speck.saw]
- Hard to show for AES (at least with encodings and solvers we've tried) [AES.saw]

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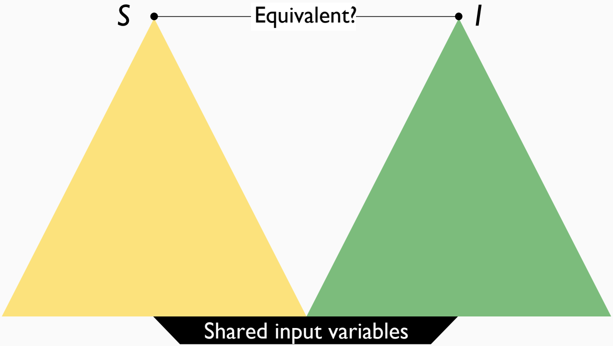
$$\forall x.\ f(x) = g(x)$$

- Two functions give equivalent output for all inputs
- AIGs give distinct benefits over direct CNF creation
    - Intuitive construction
    - Sharing subterms reduces overall expression size
    - SAT sweeping helps identify candidate equivalences
        - Especially effective for cryptography
    - Use best available SAT solver for final phase
- Works on many cryptographic primitives, including AES (~6.4m) [AES-eq.saw]

EQUIVALENCE CHECKING ILLUSTRATED



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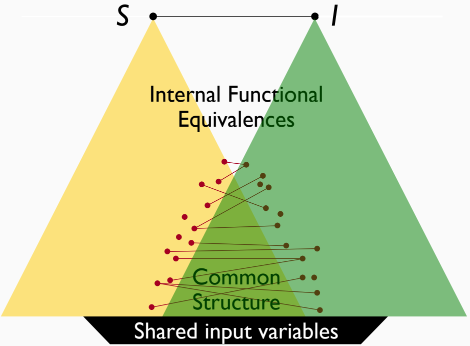

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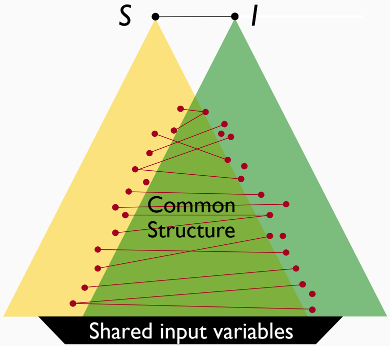
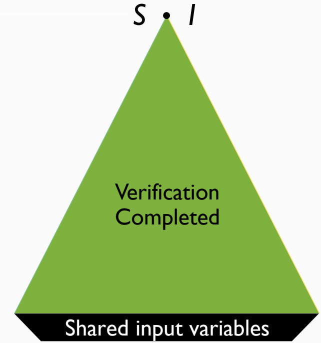
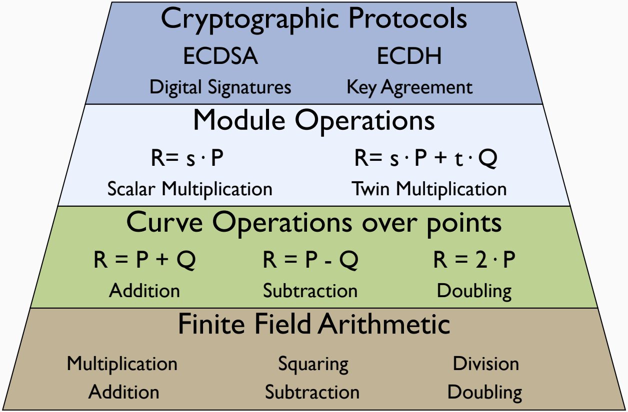



S
I
Verification
Completed
Shared input variables

COMPOSITIONAL EQUIVALENCE CHECKING: MOTIVATION



COMPOSITIONAL EQUIVALENCE CHECKING

- Key tool: uninterpreted functions
- Symbolic execution turns imperative code into functional code
  - So procedure calls can be uninterpreted functions
  - If we know all inputs and outputs
- Used for checking equivalence between Cryptol and Java ECDSA
  - Takes around 5 minutes to run
  - Takes ~1500 lines of script (mostly I/O mapping, a few rewrite rules)
  - ABC for leaves, rewriting + Z3 for higher layers

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- Attack on symmetric block ciphers
- Known plaintext attack: assumes attacker has some set of (m, c) pairs
- Basic idea: can we approximate the encryption function by a linear function?
    - Where *linear* here means made up entirely of XOR operations
- Any time the encryption function agrees with a linear function too often, this can ease cryptanalysis
- Can use #SAT to count how often it behaves linearly

- Like linear cryptanalysis, known plaintext attack on symmetric block ciphers
- Analysis of how differences in input affect differences in output
- Disproportional effects can be exploitable
- A differential characteristic is a set of differences as they traverse a path through the algorithm
- Can use #SAT to calculuate the distributions of differential characteristics

- SIMON is a lightweight block cipher published recently
- Kölbl *et al.* used SAT for linear and differential cryptanlysis of SIMON [3] [simon-diff.saw]
- Using #SAT to calculate differential characteristic distributions
- Not direct analysis of code, but of manually-derived simplification
- Serves as an additonal tool for cryptanalysis, not push-putton
- Suggests slightly different parameters possibly preferable to published numbers

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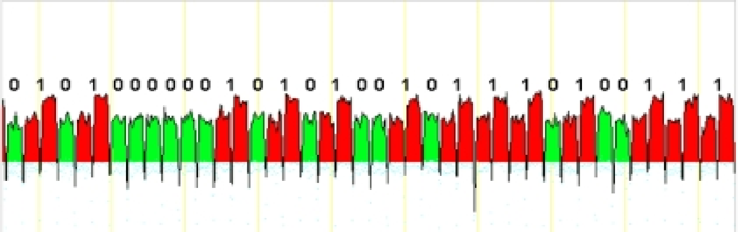
SIDE CHANNEL ANALYSIS



- Traditional side channel analysis
  - Observe specific part of last round of block cipher
  - Combine with pre-calculated formulas to recover key
- Caveat: only works if you can observe the right signals

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USING SAT FOR SIDE CHANNEL ANANLYSIS

- Example: hardware implementation
  - Encode entire algorithm (as implemented) as a boolean circuit
  - Observe any internal values possible
  - Constrain internal variables accordingly
  - Try to solve for key (or other aspects of state)
- More information than just inputs or outputs
- Can use any internal variable in the algorithm
- Hamming weights plus a handful of (m, c) pairs enough to recover AES key [@mohamed2012improved-sc]

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CREATING CODE: OPTIMAL S-BOXES

- S-boxes intended to unpredictably substitute bits
  - Ideally indistinguishable from a random function
  - But in practice representable as a short program using only linear operations
- Generating optimal S-boxes:
  - Or any function on a small domain
  - $\exists p.\ [\![p]\!](x_0) = y_0 \wedge \cdots \wedge [\![p]\!](x_n) = y_n$
- Fuhs *et al.* found a 23-instruction AES S-box program [2]
  - Less than a minute with MiniSat
  - Proving unsatisfiability of 22 instructions took 106 hours (CryptoMiniSat)

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- General synthesis:
  - $\exists p.\ \forall x.\ [\![p]\!](x) = f(x)$
- Generating efficient implementations:
  - $\min p.\ \forall x.\ [\![p]\!](x) = f(x)$
- Generally, a hard problem
  - But QBF solvers are getting powerful
  - Many papers in SMT community about this problem recently

OTHER EXAMPLES

- Cryptographic protocol analysis
- Diffusion analysis
- Differential fault analysis
- Constructing new encryption schemes
- What's next?

CURRENTLY DIFFICULT PROBLEMS

- Some problems are difficult with current solvers, but potentially tractable
  - AES equivalence in CNF (tractable using SAT sweeping!)
  - AES consistency
  - Good benchmarks for solver improvement!
- Some problems that are intrinsically difficult (we hope)
  - Multiplication difficulty seems related to hardness of factoring
  - Finding a hash collisions had better be hard
  - Finding k given m, c should be hard, too
- Synthesis is currently hard
  - But provers are getting better at this (QBF and $\exists\forall$-SMT)
- Very similar problems can be different in difficulty
  - Reversing AES vs. finding k from m, c

TOOLS

- Cryptol is a DSL for cryptographic code
  - Allows expression of algorithms at a high-level of abstraction
  - Built-in connection to SMT solvers
  - BSD-licensed
    - `http://cryptol.net`
- The Software Analysis Workbench (SAW) allows analysis of implementations
  - Symbolic execution of Cryptol, C, Java
  - Bindings to SMT and SAT solvers, including ABC
  - Freely available (with source) for non-commercial use
    - `http://saw.galois.com`
- All the examples from this talk are available
  - `https://github.com/galoisinc/sat2015-crypto`

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SUMMING IT UP

- Cryptography and SAT go very nicely together
  - Easily representable as propositional formulas
  - AIGs are super handy! (not just for hardware)
- More use of SAT during algorithm development will make our crypto stronger
  - And it's happening: see SHA-3, Trivium, SIMON, etc.
- Nice source of hard benchmark problems for solvers

https://github.com/galoisinc/sat2015-crypto

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[1] Felix Dörre and Vladimir Klebanov. **Pseudo-random number generator verification - a case study.** In *7th Working Conference on Verified Software: Theories, Tools, and Experiments*, 2015.

[2] Carsten Fuhs and Peter Schneider-Kamp. **Optimizing the AES S-Box using SAT.** In Geoff Sutcliffe, Stephan Schulz, and Eugenia Ternovska, editors, *IWIL 2010. The 8th International Workshop on the Implementation of Logics*, volume 2 of *EasyChair Proceedings in Computing*, pages 64–70. EasyChair, 2010.

[3] Stefan Kölbl, Gregor Leander, and Tyge Tiessen. **Observations on the SIMON block cipher family.** In Rosario Gennaro and Matthew Robshaw, editors, *Advances in Cryptology – CRYPTO 2015*, volume 9215 of *Lecture Notes in Computer Science*, pages 161–185. Springer Berlin Heidelberg, 2015.

[4] Ilya Otpuschennikov, Alexander Semenov, and Stepan Kochemazov. **Transalg: a tool for translating procedural descriptions of discrete functions to SAT (tool paper), 2014.**

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