Introduction to Aerial Robotics
Lecture 2

Shaojie Shen
Associate Professor
Dept. of ECE, HKUST

9 February 2021

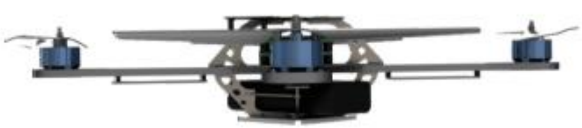
Outline

- Rigid Body Transformations
- Rotational Motions
- Rotation Representations
- Rigid Body Motions
- Rigid Body Velocities
- Quadrotor Dynamics

Rigid Body Transformations

Rigid Body

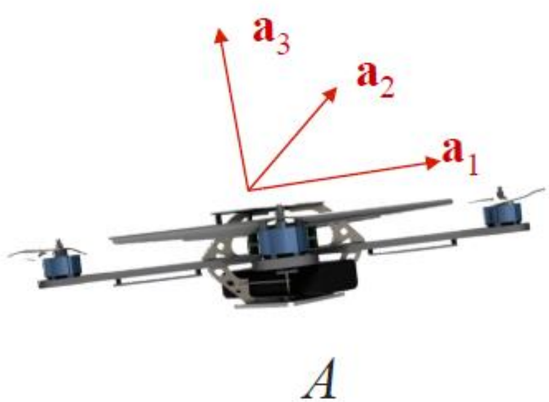
- Two distinct positions and orientations of the same rigid body
  - Let $\mathbf{p}$ and $\mathbf{q}$ be two points on a rigid body
  - $\|\boldsymbol{p}(t)-\boldsymbol{q}(t)\| = \|\boldsymbol{p}(0)-\boldsymbol{q}(0)\| = \text{constant}$

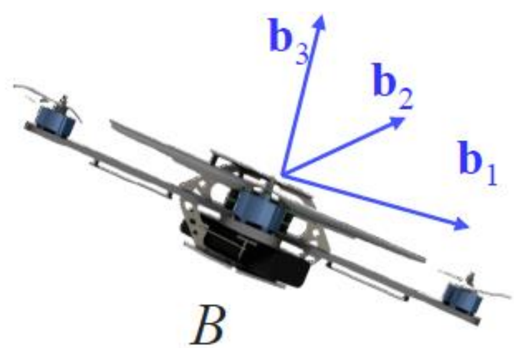


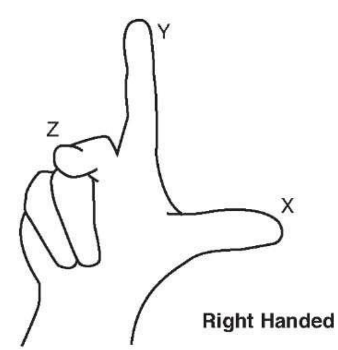
Courtesy Vijay Kumar

Reference Frames

- We associate any position and orientation with a reference frame
  - We use **right-handed** coordinate frames
  - We can find three linearly independent vectors $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_3$ that are basis vectors for reference frame A
  - We can write any vector as a linear combination of basis vectors in either frame $\boldsymbol{v} = \boldsymbol{v}_\mathbf{1}\mathbf{a}_\mathbf{1} + v_2\mathbf{a}_2 + v_3\mathbf{a}_3$







Notation

- Vectors
  - $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{a}$, …
- Matrices
  - $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, …
- Reference frames
  - $A, B, C,$ …
  - $a, b, c,$ …
- Transformations
  - ${}^{A}\mathbf{A}_{B}$, ${}^{A}\mathbf{R}_{B}$ …
  - $\mathbf{A}_{ab}$, $\mathbf{R}_{ab}$ …
  - $g_{ab}(.)$, $h_{ab}(.)$ …

Be Aware of **Potential Confusion!!!**

Rigid Body Displacement

- Object:

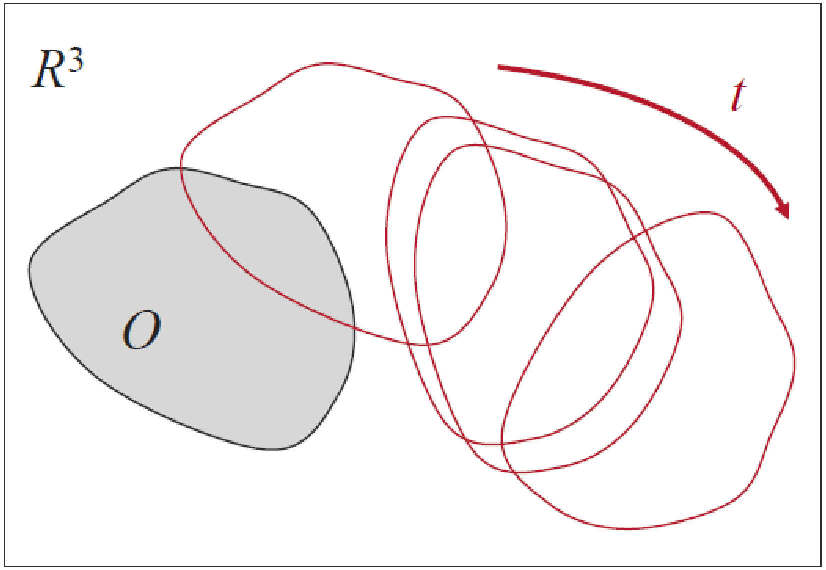
$$O \in \mathbb{R}^3$$

- Rigid body displacement
  - Map

$$g : O \longrightarrow \mathbb{R}^3$$

- Rigid body motion
  - Continuous family of maps

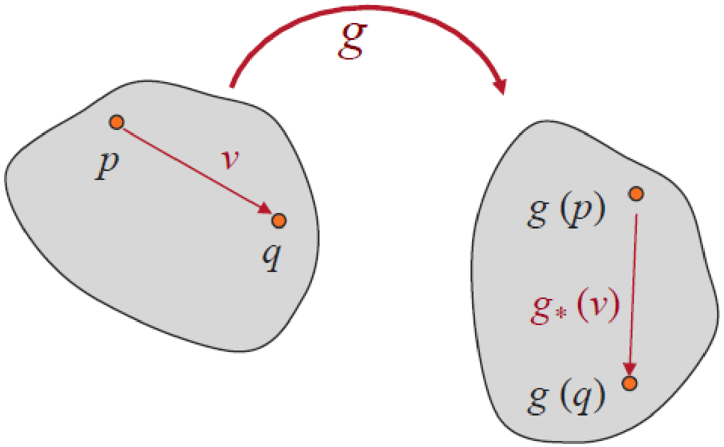
$$g(t) : O \longrightarrow \mathbb{R}^3$$



Rigid Body Displacement

- A displacement of a transformation of points
  - Transformation (g) of points induces an action (g*) on vectors

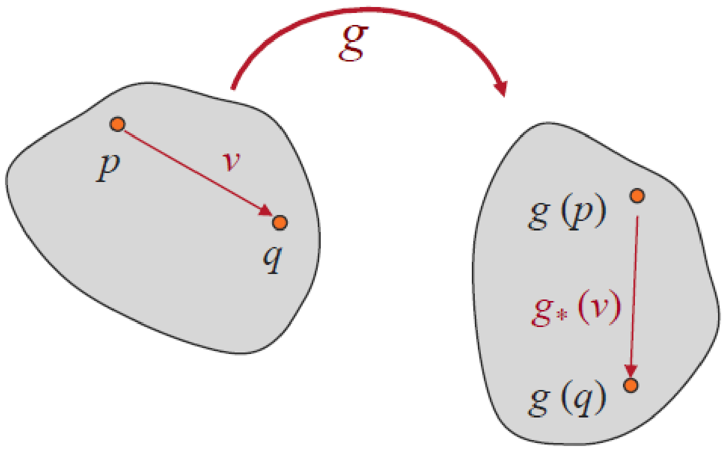
$$g_*(\mathbf{v}) = g(\mathbf{q}) - g(\mathbf{p})$$



Definition of Rigid Body Displacement

- Lengths are preserved

$$\|g(\mathbf{q}) - g(\mathbf{p})\| = \|\mathbf{q} - \mathbf{p}\|$$



- Cross products are preserved

$$g_*(\mathbf{v}) \times g_*(\mathbf{w}) = g_*(\mathbf{v} \times \mathbf{w})$$

Why?

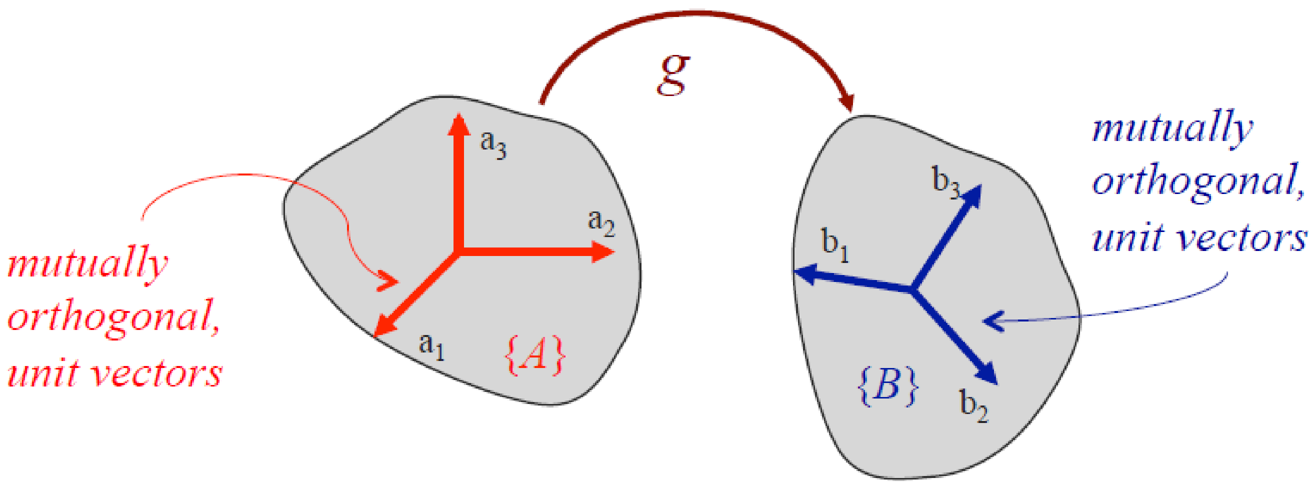
Eliminate internal reflection: $(x, y, z) \rightarrow (x, y, -z)$

Properties of Rigid Body Displacement

- Inner products are also preserved

$$g_*(\mathbf{v}) \cdot g_*(\mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$$

- Orthogonal vectors are mapped to orthogonal vectors



Rigid Body Displacement

- Rigid body displacements are transformations that satisfy two important properties:
  1. Lengths are preserved
  2. Cross products are preserved

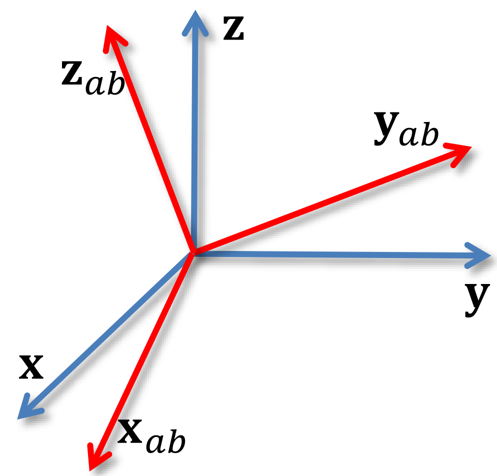
- Rigid body transformations and rigid body displacements are often used interchangeably
  - Transformations generally used to describe relationship between reference frames attached to different rigid bodies.
  - Displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body

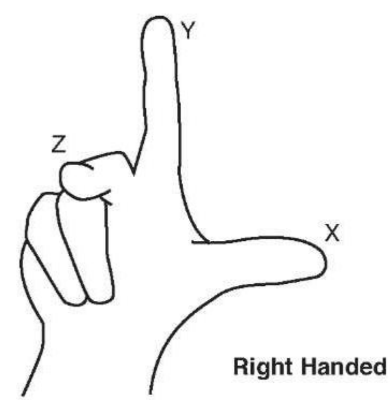
Rotational Motions

THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY 香港科技大學
DEPARTMENT OF ELECTRONIC & COMPUTER ENGINEERING 電子及計算機工程學系

Rotation

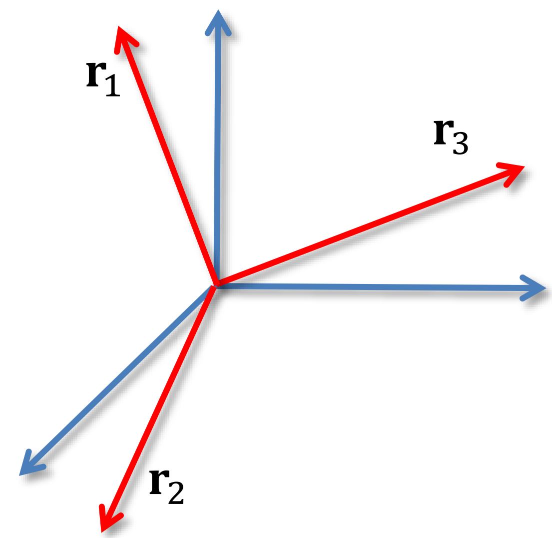
- Coordinate frames are right-handed
- Principle axes of frame *A*:
  - $\mathbf{x} = [1\ 0\ 0]^T$
  - $\mathbf{y} = [0\ 1\ 0]^T$
  - $\mathbf{z} = [0\ 0\ 1]^T$
- Principle axes of frame *B*:
  - $\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab} \subset \mathbb{R}^3$
- The Rotation Matrix:
  - $\mathbf{R}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}]$
  - Coordinates of principle axes of *B* related to *A*





Properties of a Rotation Matrix

- Let $\mathbf{R} = [\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3]$ be a rotation matrix
- Orthogonal:
  - $\mathbf{r}_i{}^T \cdot \mathbf{r}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$
  - $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$
- Special orthogonal:
  - $\det\mathbf{R} = \mathbf{r}_1{}^T \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = \mathbf{r}_1{}^T \cdot \mathbf{r}_1 = 1$
- The set of all rotations forms the Special Orthogonal Group
  - Special orthogonal group
  - 3D rotations: $SO(3)$
  - 2D rotations: $SO(2)$
  - $SO(n) = \{\mathbf{R} \in \mathbb{R}^{n\times n} | \mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}, \det\mathbf{R} = 1\}$



Properties of a Rotation Matrix

- $SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} | \mathbf{R}\cdot\mathbf{R}^T = \mathbf{I}, \det\mathbf{R} = 1\}$
- $SO(3)$ is a group under the operation of matrix multiplication
  1. Closure: If $\mathbf{R}_1, \mathbf{R}_2 \in SO(3)$, then $\mathbf{R}_1\cdot\mathbf{R}_2 \in SO(3)$
  2. Identity: The identity matrix is the identity element
  3. Inverse: If $\mathbf{R} \in SO(3)$, then $\mathbf{R}^{-1} \in SO(3)$
  4. Associativity: $\mathbf{R}_1\cdot(\mathbf{R}_2\cdot\mathbf{R}_3) = (\mathbf{R}_1\cdot\mathbf{R}_2)\cdot\mathbf{R}_3$

$(G,\cdot)$ is a group if:

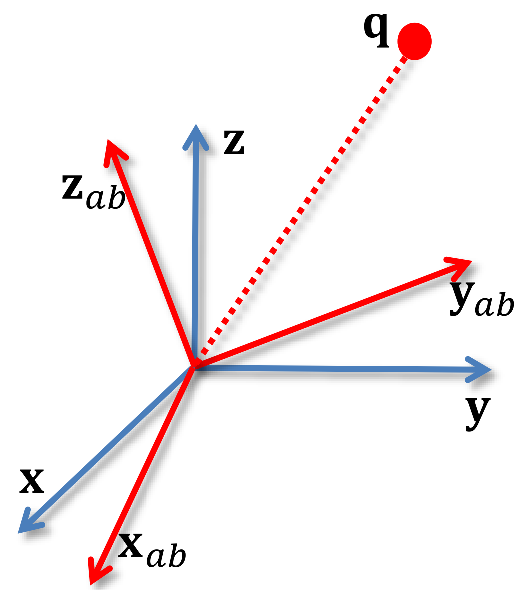
1) $g_1, g_2 \in G \Rightarrow g_1\cdot g_2 \in G$
2) $\exists!\, \mathrm{e} \in G,\ s.t.\ g\cdot e = e\cdot g = g,\ \forall g \in G$
3) $\forall g \in G,\ \exists!\ g^{-1} \in G,\ s.t.\ g\cdot g^{-1} = g^{-1}\cdot g = e$
4) $g_1\cdot(g_2\cdot g_3) = (g_1\cdot g_2)\cdot g_3$

Group examples:

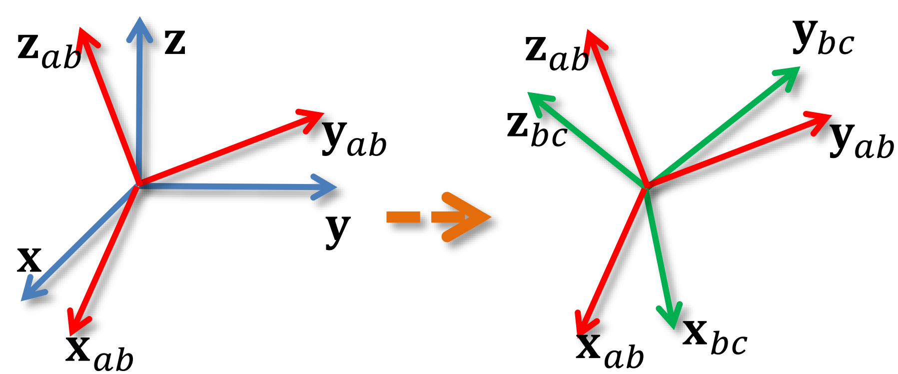
1. The set of all integers with addition operation
2. The set of all real numbers with arithmetic operations

Properties of a Rotation Matrix

- A transformation that rotates the coordinates of a point from frame *B* to frame *A*
  - Let $\mathbf{q}_b = [x_b, y_b, z_b]^T \in \mathbb{R}^3$ be coordinate of point $\mathbf{q}$ in frame *B*
  - Let $\mathbf{q}_a$ be coordinate of point $\mathbf{q}$ in frame *A*
  - $\mathbf{q}_a = x_b \cdot \mathbf{x}_{ab} + y_b \cdot \mathbf{y}_{ab} + z_b \cdot \mathbf{z}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \mathbf{R}_{ab} \cdot \mathbf{q}_b$



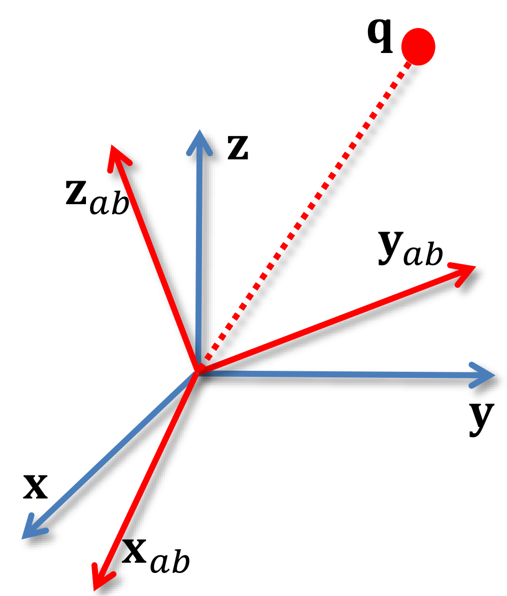
- Composition Rule
  - $\mathbf{R}_{ac} = \mathbf{R}_{ab} \cdot \mathbf{R}_{bc}$



Rotation is Rigid Body Transformation

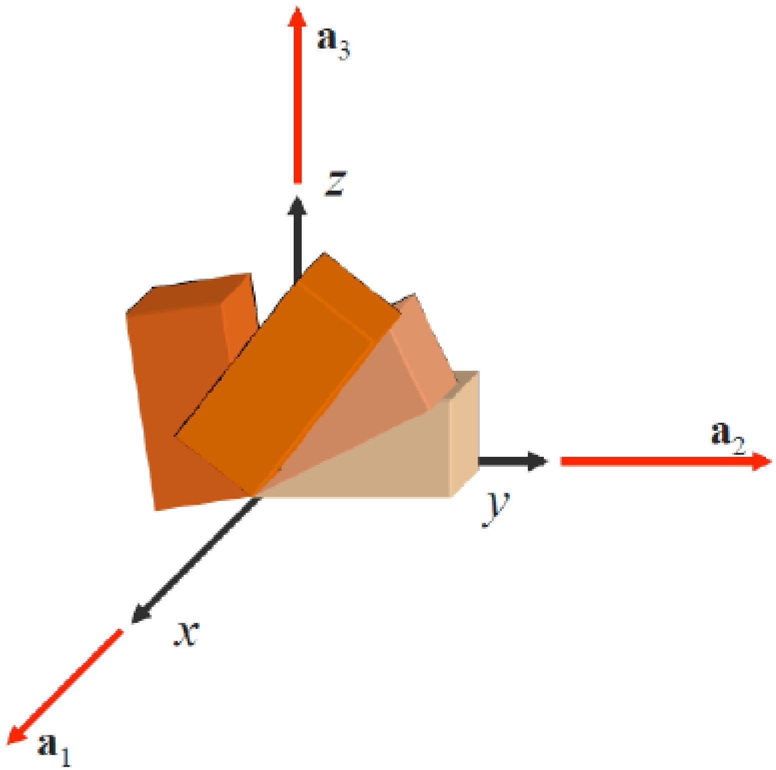
$\mathbf{R}_{ab} = [\mathbf{x}_{ab}, \mathbf{y}_{ab}, \mathbf{z}_{ab}]$ preserves:

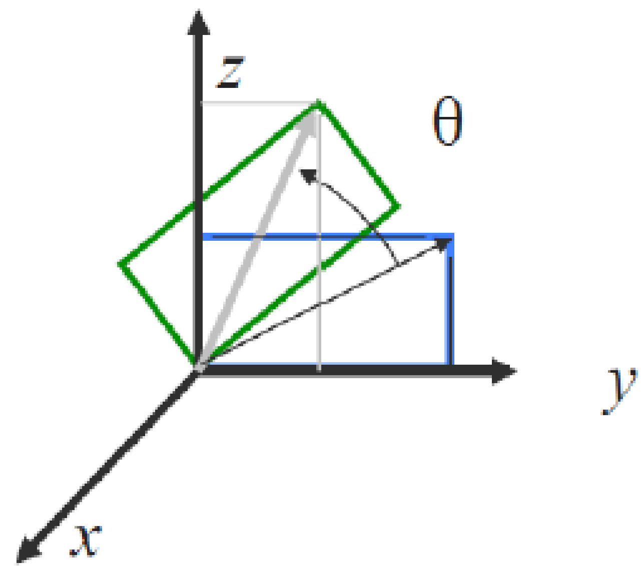
- Length:
  - $\|\mathbf{R}_{ab}(\mathbf{p}_b - \mathbf{q}_b)\| = \|\mathbf{p}_b - \mathbf{q}_b\|$
- Cross product:
  - $\mathbf{R}_{ab}(\mathbf{v} \times \boldsymbol{w}) = (\mathbf{R}_{ab}\mathbf{v}) \times (\mathbf{R}_{ab}\mathbf{w})$
  - Use the fact $\mathbf{R}(\mathbf{v})^{\wedge}\mathbf{R}^{T} = (\mathbf{R}\mathbf{v})^{\wedge}$ to prove, where $(\mathbf{a})^{\wedge} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$ is the skew-symmetric matrix, and $\mathbf{a} \times \mathbf{b} = (\mathbf{a})^{\wedge}\mathbf{b}$



Example - Rotation

$$\mathbf{R}_{\mathrm{x}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix}$$

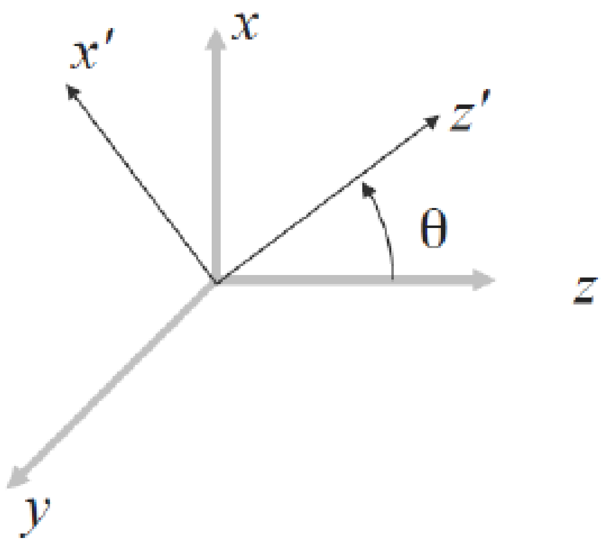


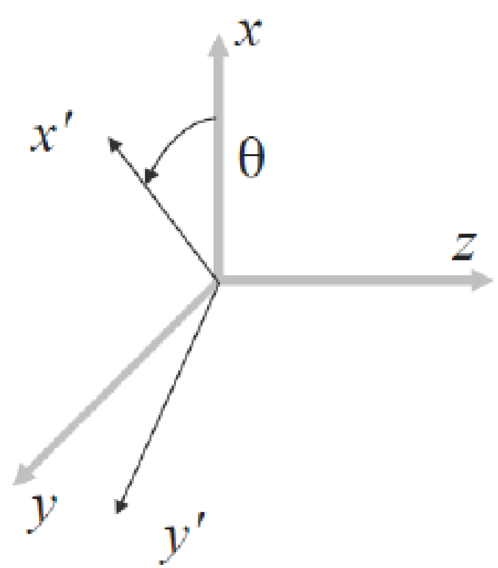


Example - Rotation

$$\mathbf{R}_{\mathrm{y}}(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$$

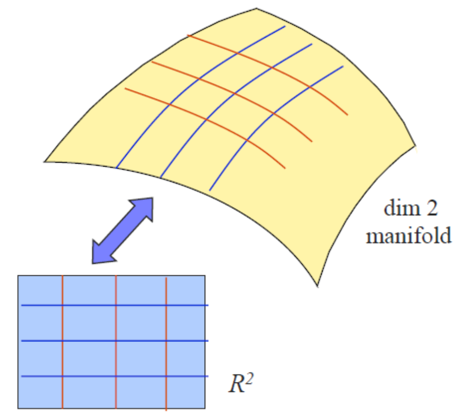
$$\mathbf{R}_{\mathrm{z}}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

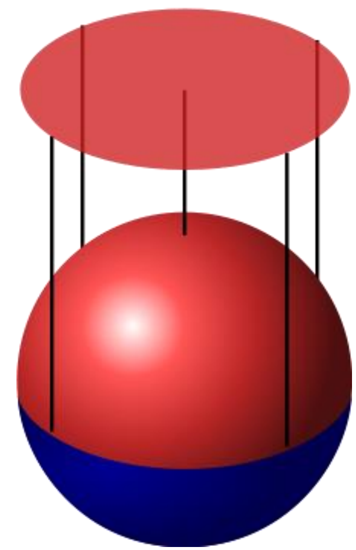


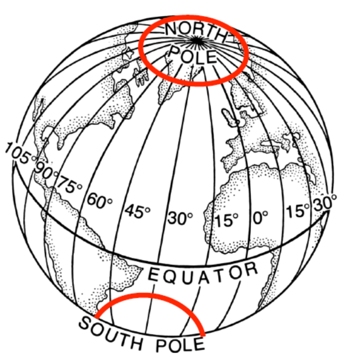


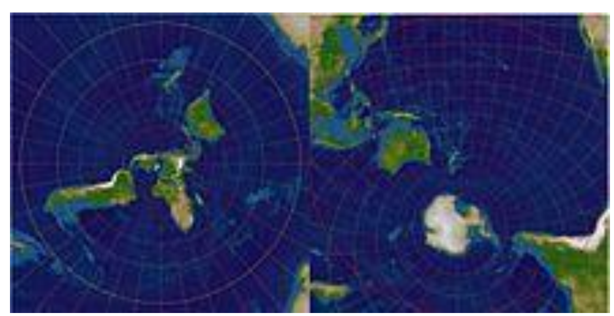
Properties of Rotation

- $SO(3)$ is a continuous group
  - The multiplication operation is a continuous operation
  - The inverse is a continuous function
- $SO(3)$ is a smooth manifold
  - A manifold of dimension n is a set M which is locally resembled to Euclidean space $\mathbb{R}^n$ near each point
  - Example: sphere is a differentiable manifold that is locally resembled to $\mathbb{R}^2$









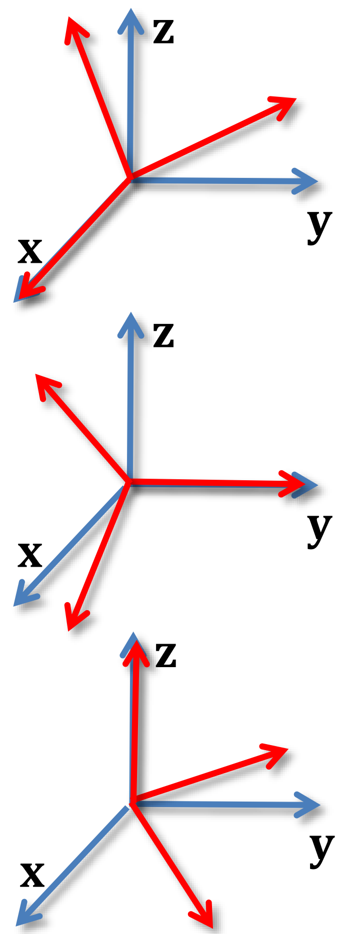
Rotation Representations

Rotation Representations

- Rotation matrices
- Euler angles
- Exponential coordinates
- Quaternions
  - Will be discussed later in the semester
  - Slides are provided as appendix for this lecture

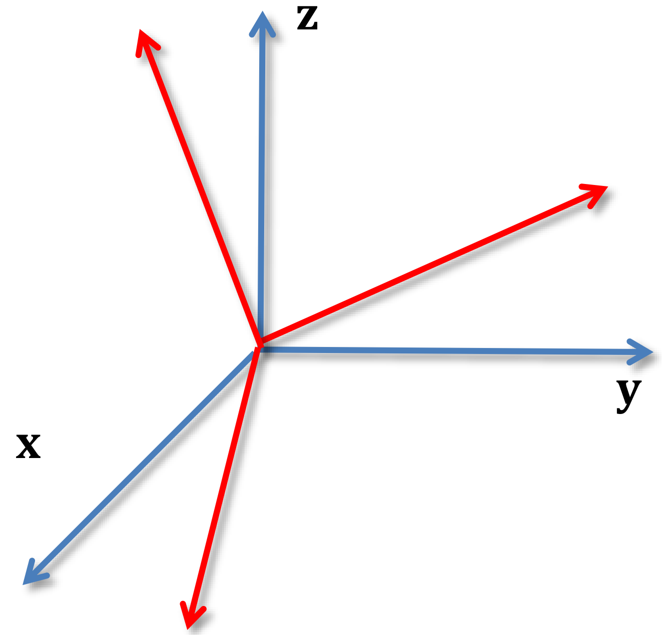
Euler Angles

- Elementary rotations:
  - $\boldsymbol{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$
  - $\boldsymbol{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
  - $\boldsymbol{R}_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Euler Angles

- Any rotation can be described by three successive rotations about linear independent axes
- However, this is an almost 1-1 transform with singularities:
  - $\boldsymbol{R}_z(\psi) \cdot \boldsymbol{R}_x(\phi) \cdot \boldsymbol{R}_y(\theta) \Rightarrow \boldsymbol{R}$
  - $\boldsymbol{R}_z(\psi) \cdot \boldsymbol{R}_x(\phi) \cdot \boldsymbol{R}_y(\theta) \not\Leftarrow \boldsymbol{R}$



Euler Angles

- Different Euler angle conversions:

Proper Euler angles	Tait-Bryan angles
$X_1Z_2X_3 = \begin{bmatrix} c_2 & -c_3s_2 & s_2s_3 \\ c_1s_2 & c_1c_2c_3 - s_1s_3 & -c_3s_1 - c_1c_2s_3 \\ s_1s_2 & c_1s_3 + c_2c_3s_1 & c_1c_3 - c_2s_1s_3 \end{bmatrix}$	$X_1Z_2Y_3 = \begin{bmatrix} c_2c_3 & -s_2 & c_2s_3 \\ s_1s_3 + c_1c_3s_2 & c_1c_2 & c_1s_2s_3 - c_3s_1 \\ c_3s_1s_2 - c_1s_3 & c_2s_1 & c_1c_3 + s_1s_2s_3 \end{bmatrix}$
$X_1Y_2X_3 = \begin{bmatrix} c_2 & s_2s_3 & c_3s_2 \\ s_1s_2 & c_1c_3 - c_2s_1s_3 & -c_1s_3 - c_2c_3s_1 \\ -c_1s_2 & c_3s_1 + c_1c_2s_3 & c_1c_2c_3 - s_1s_3 \end{bmatrix}$	$X_1Y_2Z_3 = \begin{bmatrix} c_2c_3 & -c_2s_3 & s_2 \\ c_1s_3 + c_3s_1s_2 & c_1c_3 - s_1s_2s_3 & -c_2s_1 \\ s_1s_3 - c_1c_3s_2 & c_3s_1 + c_1s_2s_3 & c_1c_2 \end{bmatrix}$
$Y_1X_2Y_3 = \begin{bmatrix} c_1c_3 - c_2s_1s_3 & s_1s_2 & c_1s_3 + c_2c_3s_1 \\ s_2s_3 & c_2 & -c_3s_2 \\ -c_3s_1 - c_1c_2s_3 & c_1s_2 & c_1c_2c_3 - s_1s_3 \end{bmatrix}$	$Y_1X_2Z_3 = \begin{bmatrix} c_1c_3 + s_1s_2s_3 & c_3s_1s_2 - c_1s_3 & c_2s_1 \\ c_2s_3 & c_2c_3 & -s_2 \\ c_1s_2s_3 - c_3s_1 & c_1c_3s_2 + s_1s_3 & c_1c_2 \end{bmatrix}$
$Y_1Z_2Y_3 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_1s_2 & c_3s_1 + c_1c_2s_3 \\ c_3s_2 & c_2 & s_2s_3 \\ -c_1s_3 - c_2c_3s_1 & s_1s_2 & c_1c_3 - c_2s_1s_3 \end{bmatrix}$	$Y_1Z_2X_3 = \begin{bmatrix} c_1c_2 & s_1s_3 - c_1c_3s_2 & c_3s_1 + c_1s_2s_3 \\ s_2 & c_2c_3 & -c_2s_3 \\ -c_2s_1 & c_1s_3 + c_3s_1s_2 & c_1c_3 - s_1s_2s_3 \end{bmatrix}$
$Z_1Y_2Z_3 = \begin{bmatrix} c_1c_2c_3 - s_1s_3 & -c_3s_1 - c_1c_2s_3 & c_1s_2 \\ c_1s_3 + c_2c_3s_1 & c_1c_3 - c_2s_1s_3 & s_1s_2 \\ -c_3s_2 & s_2s_3 & c_2 \end{bmatrix}$	$Z_1Y_2X_3 = \begin{bmatrix} c_1c_2 & c_1s_2s_3 - c_3s_1 & s_1s_3 + c_1c_3s_2 \\ c_2s_1 & c_1c_3 + s_1s_2s_3 & c_3s_1s_2 - c_1s_3 \\ -s_2 & c_2s_3 & c_2c_3 \end{bmatrix}$
$Z_1X_2Z_3 = \begin{bmatrix} c_1c_3 - c_2s_1s_3 & -c_1s_3 - c_2c_3s_1 & s_1s_2 \\ c_3s_1 + c_1c_2s_3 & c_1c_2c_3 - s_1s_3 & -c_1s_2 \\ s_2s_3 & c_3s_2 & c_2 \end{bmatrix}$	$Z_1X_2Y_3 = \begin{bmatrix} c_1c_3 - s_1s_2s_3 & -c_2s_1 & c_1s_3 + c_3s_1s_2 \\ c_3s_1 + c_1s_2s_3 & c_1c_2 & s_1s_3 - c_1c_3s_2 \\ -c_2s_3 & s_2 & c_2c_3 \end{bmatrix}$

Euler Angles

- Example: Z-Y-Z Euler angles:
  - Sequence of three rotations about body-fixed axes
  - $\boldsymbol{R} = \boldsymbol{R}_z(\phi) \cdot \boldsymbol{R}_y(\theta) \cdot \boldsymbol{R}_z(\psi)$
  - $\boldsymbol{R} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi c\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$
- If $\sin\theta \neq 0$:
  - $\theta = \operatorname{acos}(r_{33})$
  - $\psi = \operatorname{atan2}(\frac{r_{32}}{\sin\theta}, -\frac{r_{31}}{\sin\theta})$
  - $\phi = \operatorname{atan2}(\frac{r_{23}}{\sin\theta}, \frac{r_{13}}{\sin\theta})$

Euler Angles

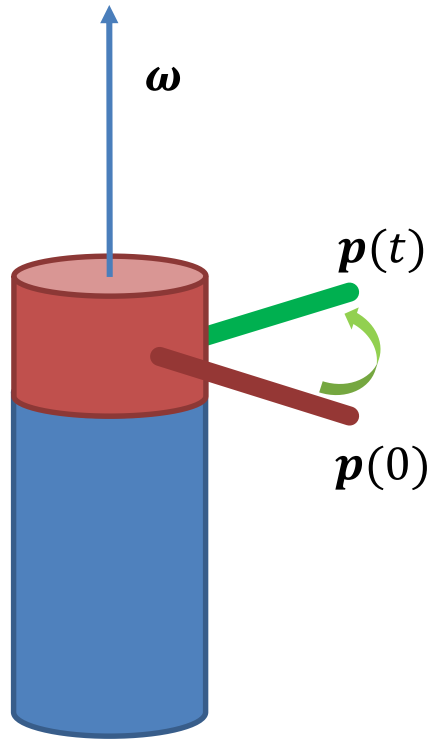
- Example: Z-Y-Z Euler angles:
  - Sequence of three rotations about body-fixed axes
  - $\boldsymbol{R} = \boldsymbol{R}_z(\phi) \cdot \boldsymbol{R}_y(\theta) \cdot \boldsymbol{R}_z(\psi)$
  - $\boldsymbol{R} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & -c\phi c\theta s\psi - s\phi c\psi & c\phi s\theta \\ s\phi c\theta c\psi + c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi c\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$
- If $\sin\theta = 0$:
  - $\boldsymbol{R} = \begin{bmatrix} c\phi c\psi - s\phi s\psi & -c\phi s\psi - s\phi c\psi & 0 \\ c\phi s\psi + s\phi c\psi & -s\phi s\psi + c\phi c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{R}_z(\phi + \psi)$
  - As long as $\phi + \psi$ is preserved, we have infinite set of Euler angles!

Exponential Coordinates

- Scalar differential equation:
  - $\begin{cases} \dot{x}(t) = ax(t) \\ x(0) = x_0 \end{cases} \Longrightarrow x(t) = e^{at}x_0$
- Matrix differential equation:
  - $\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \Longrightarrow \boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0$
  - $e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{1}{2}\boldsymbol{A}^2 + \frac{1}{3!}\boldsymbol{A}^3 + \cdots + \frac{1}{n!}\boldsymbol{A}^n + \cdots$

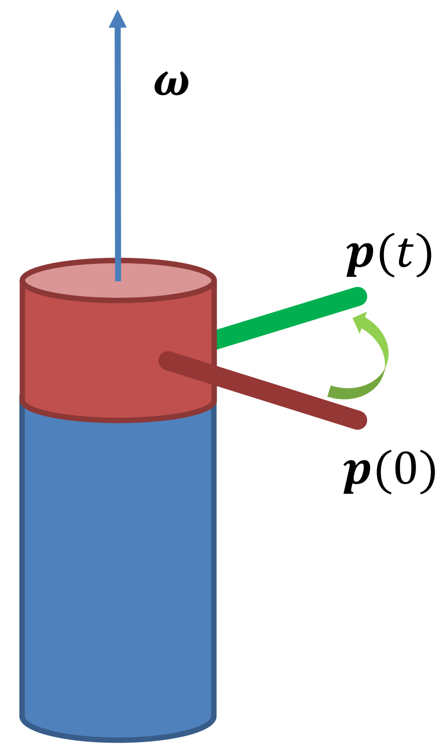
Exponential Coordinates

- Degree-of-freedom of $SO(3)$:
  - $\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$
  - $\boldsymbol{r}_i{}^T \cdot \boldsymbol{r}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \Longrightarrow$ 6 constraints
  - $\boldsymbol{R}$ has only 3 independent parameters
- Consider the motion of a point about a rotating link $\boldsymbol{\omega}$ at constant unit velocity:
  - $\begin{cases} \dot{\boldsymbol{p}}(t) = \boldsymbol{\omega} \times \boldsymbol{p}(t) = \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{p}(t) \\ \boldsymbol{p}(0) = \boldsymbol{p}_0 \end{cases} \Longrightarrow \boldsymbol{p}(t) = e^{\widehat{\boldsymbol{\omega}}t}\boldsymbol{p}_0$



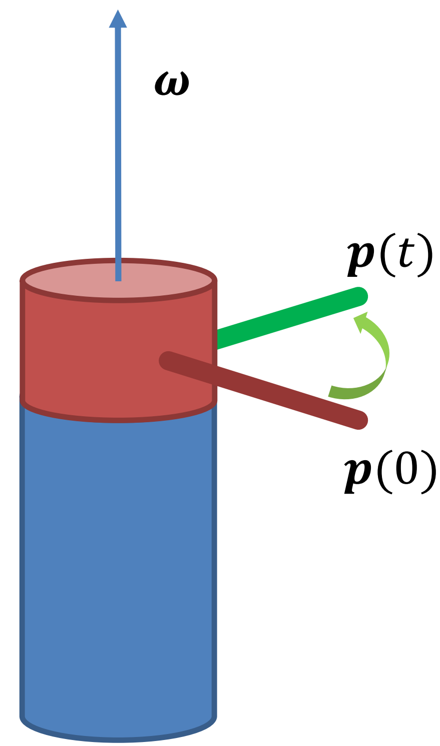
Exponential Coordinates

- Consider the motion of a point about a rotating link $\boldsymbol{\omega}$ at constant unit velocity:
  - $\begin{cases} \dot{\boldsymbol{p}}(t) = \boldsymbol{\omega} \times \boldsymbol{p}(t) = \widehat{\boldsymbol{\omega}} \cdot \boldsymbol{p}(t) \\ \boldsymbol{p}(0) = \boldsymbol{p}_0 \end{cases} \Longrightarrow \boldsymbol{p}(t) = e^{\widehat{\boldsymbol{\omega}}t}\boldsymbol{p}_0$
  - $\widehat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$
- Rotating about ω at unit velocity for θ units:
  - $\boldsymbol{R}(\boldsymbol{\omega}, \theta) = e^{\widehat{\boldsymbol{\omega}}\theta}$



Exponential Coordinates

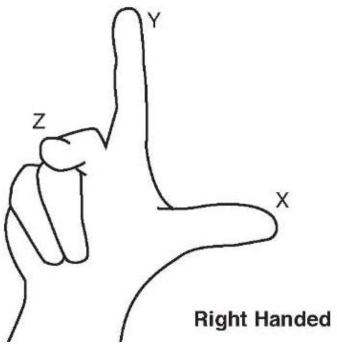
- The vector space of all 3×3 skew-symmetric matrices is denoted as $so(3)$:
  - $so(3) = \{\boldsymbol{S} \in \mathbb{R}^{3\times 3}: \boldsymbol{S}^T = -\boldsymbol{S}\}$
- The exponential map:
  - $\boldsymbol{R}(\boldsymbol{\omega}, \theta) = e^{\widehat{\boldsymbol{\omega}}\theta} = \boldsymbol{I} + \widehat{\boldsymbol{\omega}} \sin\theta + \widehat{\boldsymbol{\omega}}^2(1 - \cos\theta)$
- $e^{\widehat{\boldsymbol{\omega}}\theta} \in SO(3)$
  - $\left[e^{\widehat{\boldsymbol{\omega}}\theta}\right]^{-1} = e^{-\widehat{\boldsymbol{\omega}}\theta} = e^{\widehat{\boldsymbol{\omega}}^T\theta} = \left[e^{\widehat{\boldsymbol{\omega}}\theta}\right]^T$
  - Since $\det e^{\boldsymbol{0}} = 1$, and both determinant and exponential map are continuous functions, we know $\det e^{\widehat{\boldsymbol{\omega}}\theta} = 1$
- The exponential map is onto (many to one)
  - $\theta = 0 \Rightarrow \boldsymbol{\omega}$ can be chosen arbitrary

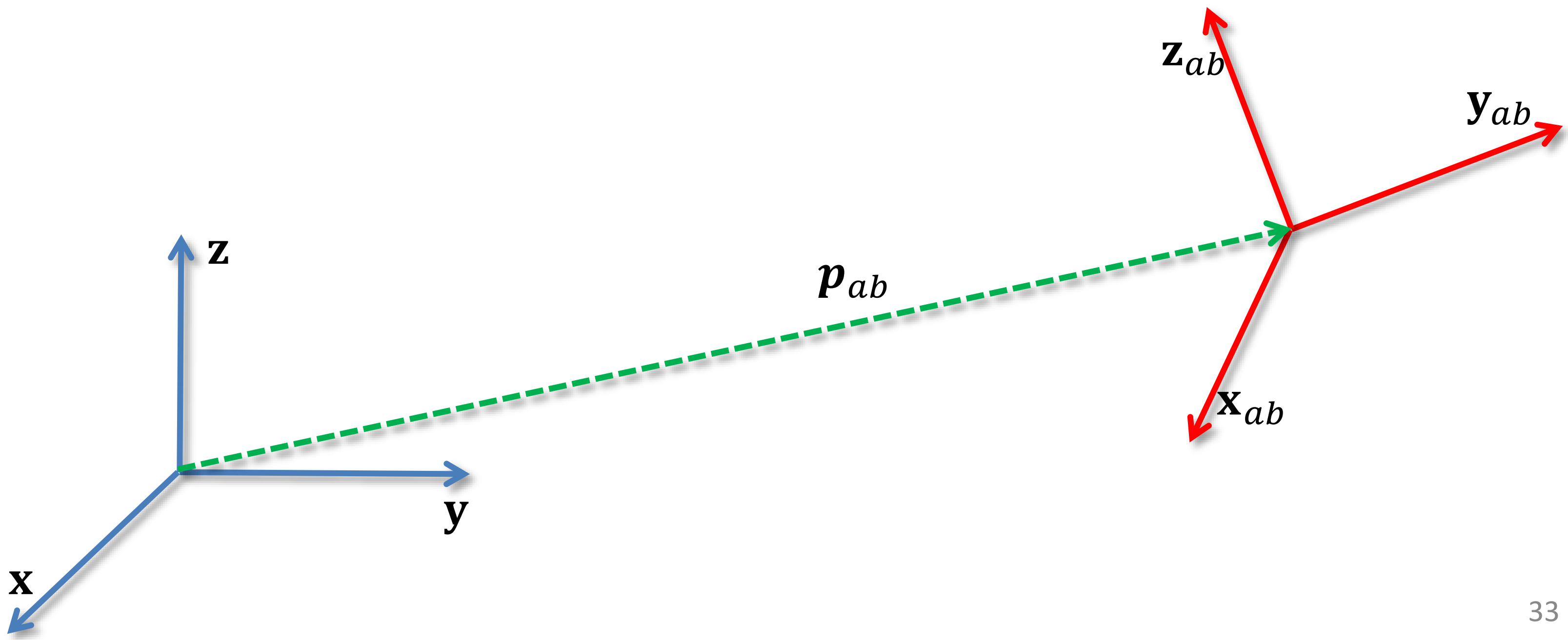


Rigid Body Motions

Rigid Body Motion

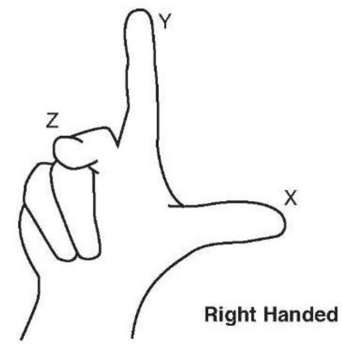
- General rigid body motions that includes both translation and rotation forms the product space of $\mathbb{R}^3$ and $SO(3)$. Denoted as $SE(3)$ – Special Euclidean group.
  - $SE(3) = \{(\boldsymbol{p}, \boldsymbol{R}): \boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$

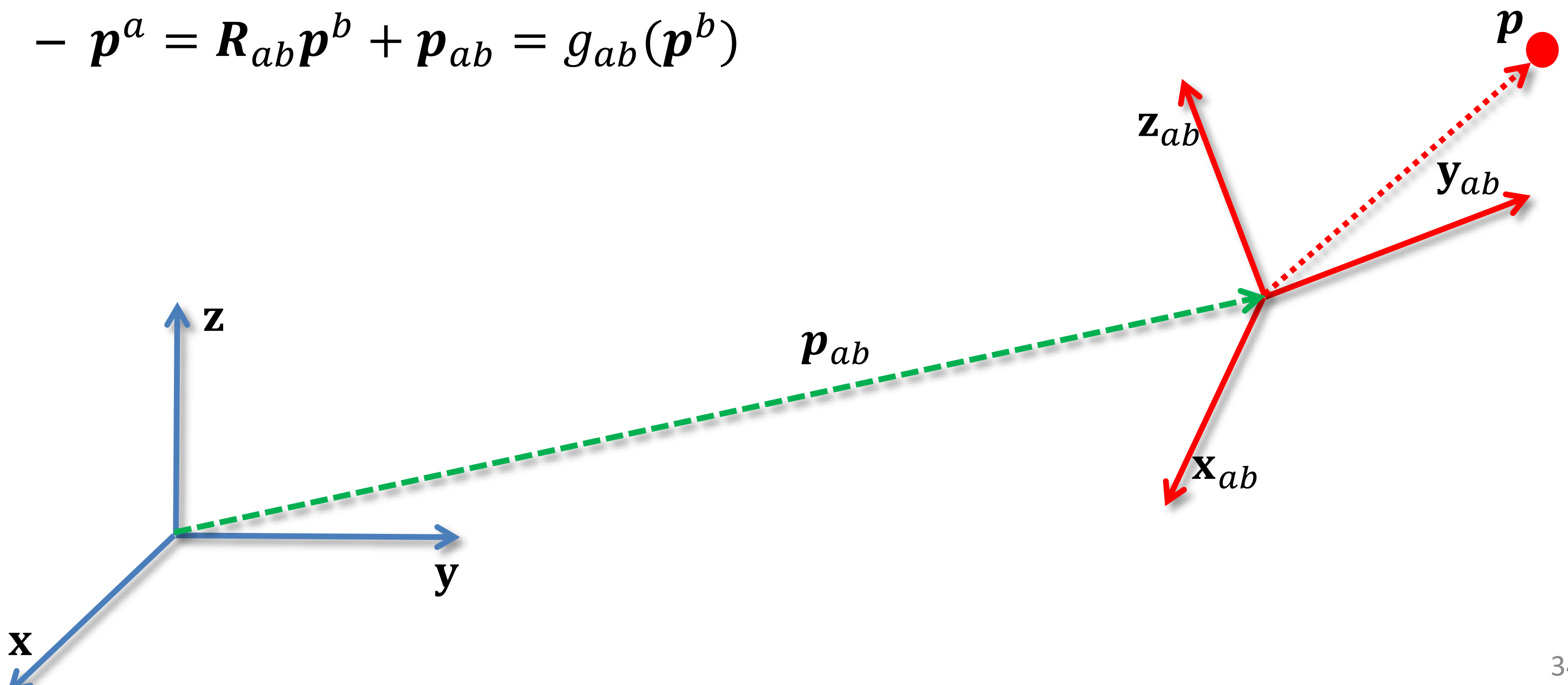




Rigid Body Motion

- Special Euclidean group:
  - $SE(3) = \{(\boldsymbol{p}, \boldsymbol{R}) : \boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$
- Transformation of a point between different coordinate frames:
  - $\boldsymbol{p}^a = \boldsymbol{R}_{ab}\boldsymbol{p}^b + \boldsymbol{p}_{ab} = g_{ab}(\boldsymbol{p}^b)$



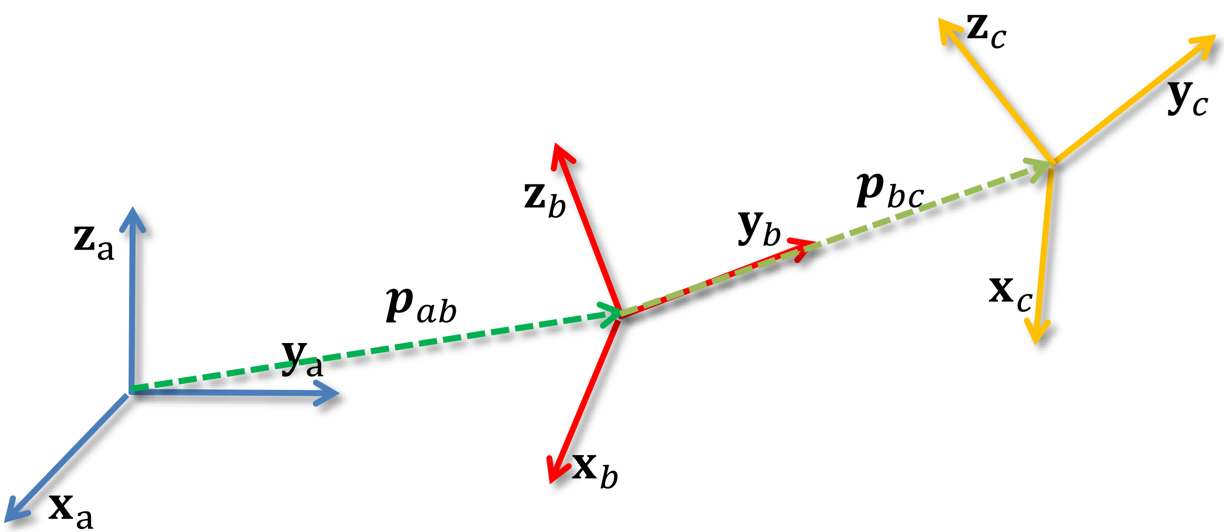


Rigid Body Motion

- Homogeneous coordinates of a point:
  - $\bar{\boldsymbol{p}} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$
- Homogeneous coordinates of a vector:
  - $\bar{\boldsymbol{v}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}$
- Homogeneous representation of rigid body motion:
  - $\bar{\boldsymbol{p}}^a = \begin{bmatrix} \boldsymbol{p}^a \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{ab} & \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}^b \\ 1 \end{bmatrix} = \bar{g}_{ab} \bar{\boldsymbol{p}}^b$

Rigid Body Motion

- Homogeneous representation of rigid body motion:
  - $\bar{g}_{ab} = \begin{bmatrix} \boldsymbol{R}_{ab} & \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix}$
- Composition rule for rigid body motions:
  - $\bar{g}_{ac} = \bar{g}_{ab} \cdot \bar{g}_{bc} = \begin{bmatrix} \boldsymbol{R}_{ab}\boldsymbol{R}_{bc} & \boldsymbol{R}_{ab}\boldsymbol{p}_{bc} + \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix}$
  - Compare with composition of rotational motion: $\boldsymbol{R}_{ac} = \boldsymbol{R}_{ab} \cdot \boldsymbol{R}_{bc}$



Properties of Rigid Body Motion

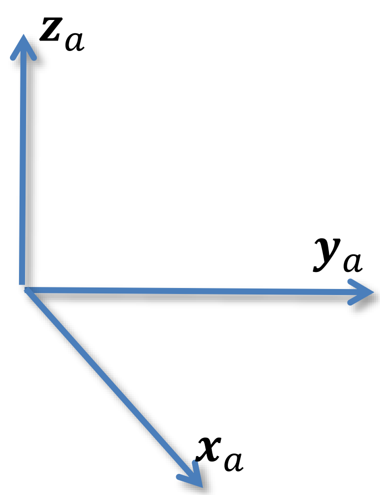
- $SE(3) = \{(\boldsymbol{p}, \boldsymbol{R}) : \boldsymbol{p} \in \mathbb{R}^3, \boldsymbol{R} \in SO(3)\} = \mathbb{R}^3 \times SO(3)$
- $SE(3)$ is a group under the operation of matrix multiplication
  - Closure
  - Identity
  - Inverse
  - Associativity
- $g \in SE(3)$ is a rigid body transformation
  - Lengths are preserved
  - Cross products are preserved

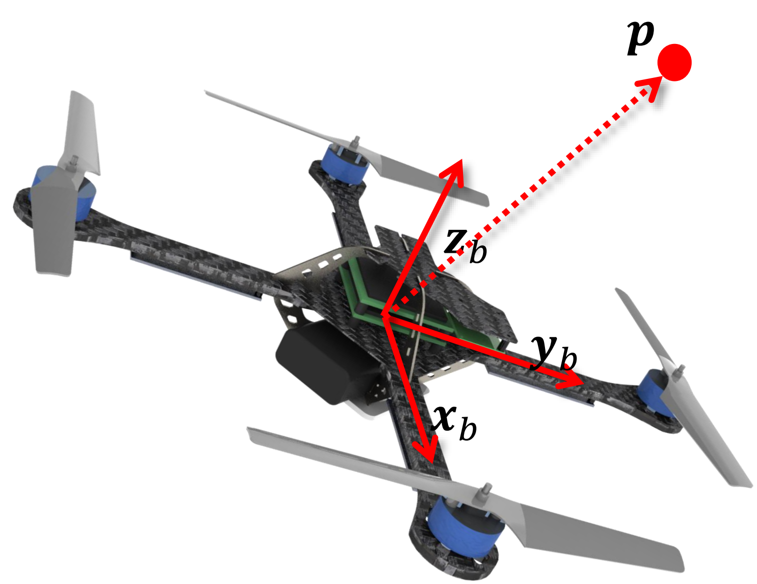
Prove it yourself!

Rigid Body Velocities

Angular Velocity

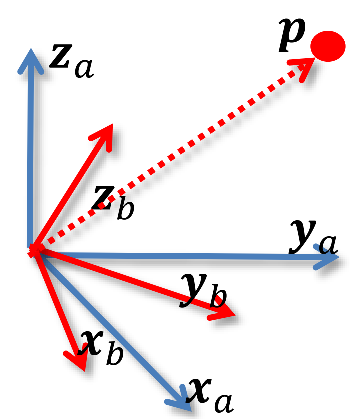
- Coordinate frames:
  - Frame A: spatial frame
  - Frame B: body frame
- A point attached to the body follows a rotational path in spatial frame:
  - $\boldsymbol{p}^a(t) = \boldsymbol{R}_{ab}(t)\boldsymbol{p}^b$





Angular Velocity

- Coordinate frames:
  - Frame A: spatial frame
  - Frame B: body frame
- A point attached to the body follows a rotational path in spatial frame:
  - $\boldsymbol{p}^a(t) = \boldsymbol{R}_{ab}(t)\boldsymbol{p}^b$
- The velocity of the point in spatial frame:
  - $\boldsymbol{v}_p^a(t) = \frac{d}{dt}\boldsymbol{p}^a(t) = \dot{\boldsymbol{R}}_{ab}(t)\boldsymbol{p}^b$
- This can be rewritten as:
  - $\boldsymbol{v}_p^a(t) = \boxed{\dot{\boldsymbol{R}}_{ab}(t)\boldsymbol{R}_{ab}^{-1}(t)}\boldsymbol{R}_{ab}(t)\boldsymbol{p}^b$

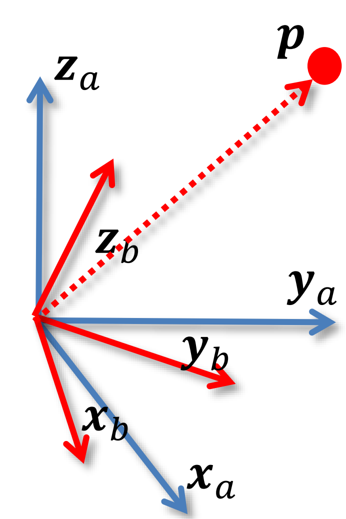


Skew-symmetric matrix.
Why?

Prove $\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I} \implies \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = 0$

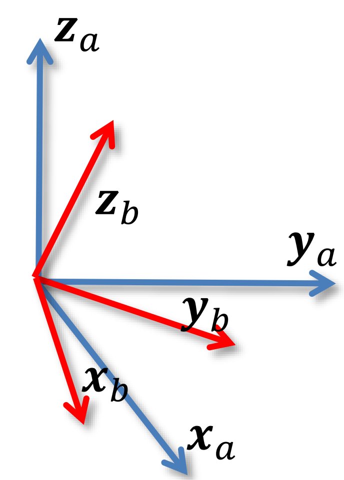
Angular Velocity

- The instantaneous spatial angular velocity $\boldsymbol{\omega}_{ab}^{a}$
  - $\widehat{\boldsymbol{\omega}}_{ab}^{a} = \dot{\boldsymbol{R}}_{ab} \cdot \boldsymbol{R}_{ab}^{-1}$
- The instantaneous body angular velocity $\boldsymbol{\omega}_{ab}^{b}$
  - $\widehat{\boldsymbol{\omega}}_{ab}^{b} = \boldsymbol{R}_{ab}^{-1} \cdot \dot{\boldsymbol{R}}_{ab}$
- Conversion:
  - $\widehat{\boldsymbol{\omega}}_{ab}^{b} = \boldsymbol{R}_{ab}^{-1} \cdot \widehat{\boldsymbol{\omega}}_{ab}^{a} \cdot \boldsymbol{R}_{ab}$
  - $\boldsymbol{\omega}_{ab}^{b} = \boldsymbol{R}_{ab}^{-1} \cdot \boldsymbol{\omega}_{ab}^{a}$
- Velocity induced by rotational motion:
  - $\boldsymbol{v}_{p}^{a} = \widehat{\boldsymbol{\omega}}_{ab}^{a} \cdot \boldsymbol{R}_{ab} \cdot \boldsymbol{p}^{b} = \boldsymbol{\omega}_{ab}^{a} \times \boldsymbol{p}^{a}$
  - $\boldsymbol{v}_{p}^{b} = \boldsymbol{R}_{ab}^{T} \cdot \boldsymbol{v}_{p}^{a} = \boldsymbol{\omega}_{ab}^{b} \times \boldsymbol{p}^{b}$



Angular Velocity

- Numerical Integration
  - $\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\boldsymbol{\omega}}^b \;\Rightarrow \boldsymbol{R}(t+\Delta t) \sim \boldsymbol{R}(t) + \Delta t \cdot \boldsymbol{R}(t)\widehat{\boldsymbol{\omega}}^b$
  - $\dot{\boldsymbol{R}} = \widehat{\boldsymbol{\omega}}^a\boldsymbol{R} \;\Rightarrow \boldsymbol{R}(t+\Delta t) \sim \boldsymbol{R}(t) + \Delta t \cdot \widehat{\boldsymbol{\omega}}^a\boldsymbol{R}(t)$
- Constant speed rotation
  - $\boldsymbol{R}(t) = \boldsymbol{R}_0 \cdot \exp(\widehat{\boldsymbol{\omega}}_0^b \cdot t)$
  - $\boldsymbol{R}(t) = \exp(\widehat{\boldsymbol{\omega}}_0^a \cdot t) \cdot \boldsymbol{R}_0$



Angular Velocity

- Simple example

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}^{\boldsymbol{T}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \dot{\boldsymbol{R}} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta}$$

Angular Velocity

- Simple example

$$\boldsymbol{R} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\widehat{\boldsymbol{\omega}}_{ab}^{b} = \boldsymbol{R}^{\boldsymbol{T}}\dot{\boldsymbol{R}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta} = \widehat{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\dot{\theta}$$

$$\widehat{\boldsymbol{\omega}}_{ab}^{a} = \dot{\boldsymbol{R}}\boldsymbol{R}^{\boldsymbol{T}} = \dot{\theta}\begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta} = \widehat{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\dot{\theta}$$

Angular Velocity

- Two rotations

$$\boldsymbol{R} = \boldsymbol{R}_z(\theta)\,\boldsymbol{R}_x(\varphi)$$

$$\widehat{\boldsymbol{\omega}}^b = \boldsymbol{R}^T\dot{\boldsymbol{R}} = (\boldsymbol{R}_z\,\boldsymbol{R}_x)^T(\dot{\boldsymbol{R}}_z\boldsymbol{R}_x + \boldsymbol{R}_z\dot{\boldsymbol{R}}_x)$$

$$= \boldsymbol{R}_x^T\boldsymbol{R}_z^T\dot{\boldsymbol{R}}_z\boldsymbol{R}_x + \boldsymbol{R}_x^T\dot{\boldsymbol{R}}_x$$

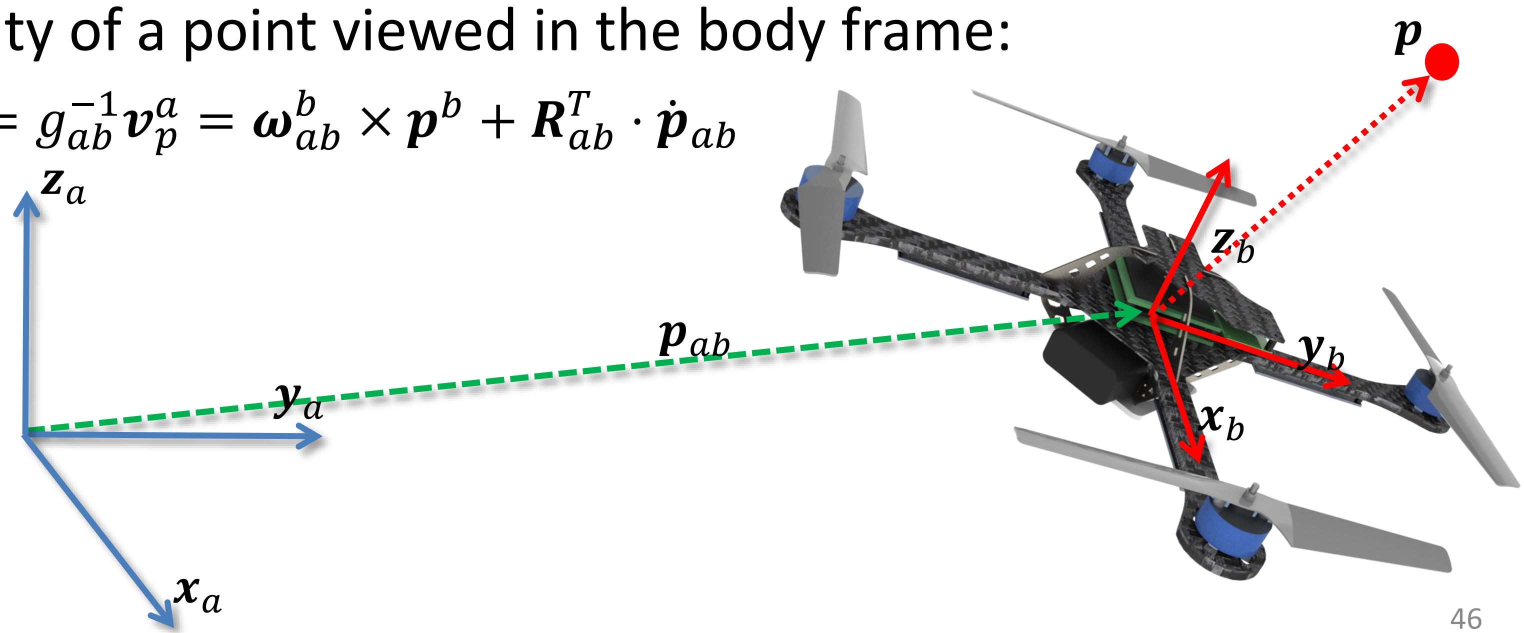
$$\widehat{\boldsymbol{\omega}}^s = \dot{\boldsymbol{R}}\boldsymbol{R}^T = (\dot{\boldsymbol{R}}_z\boldsymbol{R}_x + \boldsymbol{R}_z\dot{\boldsymbol{R}}_x)(\boldsymbol{R}_z\,\boldsymbol{R}_x)^T$$

$$= \dot{\boldsymbol{R}}_z\boldsymbol{R}_z^T + \boldsymbol{R}_z\dot{\boldsymbol{R}}_x\boldsymbol{R}_x^T\boldsymbol{R}_z^T$$

Rigid Body Velocity

For detailed interpretation, refer to Chapter 2.4 of "A Mathematical Introduction to Robotic Manipulation"

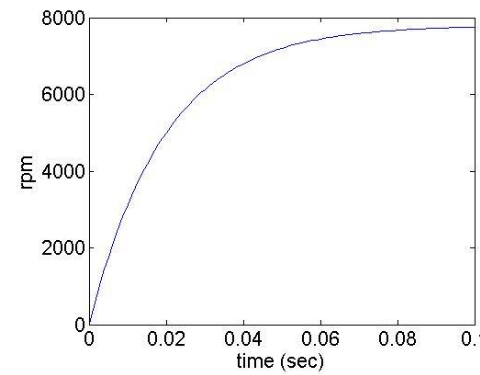
- General rigid body transformation:
  - $g_{ab} = \begin{bmatrix} \boldsymbol{R}_{ab} & \boldsymbol{p}_{ab} \\ 0 & 1 \end{bmatrix}$
- Velocity of a point viewed in the spatial frame:
  - $\boldsymbol{v}_p^a = \dot{g}_{ab} g_{ab}^{-1} \boldsymbol{p}^a = \boldsymbol{\omega}_{ab}^a \times \boldsymbol{p}^a - \boldsymbol{\omega}_{ab}^a \times \boldsymbol{p}_{ab} + \dot{\boldsymbol{p}}_{ab}$
- Velocity of a point viewed in the body frame:
  - $\boldsymbol{v}_p^b = g_{ab}^{-1} \boldsymbol{v}_p^a = \boldsymbol{\omega}_{ab}^b \times \boldsymbol{p}^b + \boldsymbol{R}_{ab}^T \cdot \dot{\boldsymbol{p}}_{ab}$

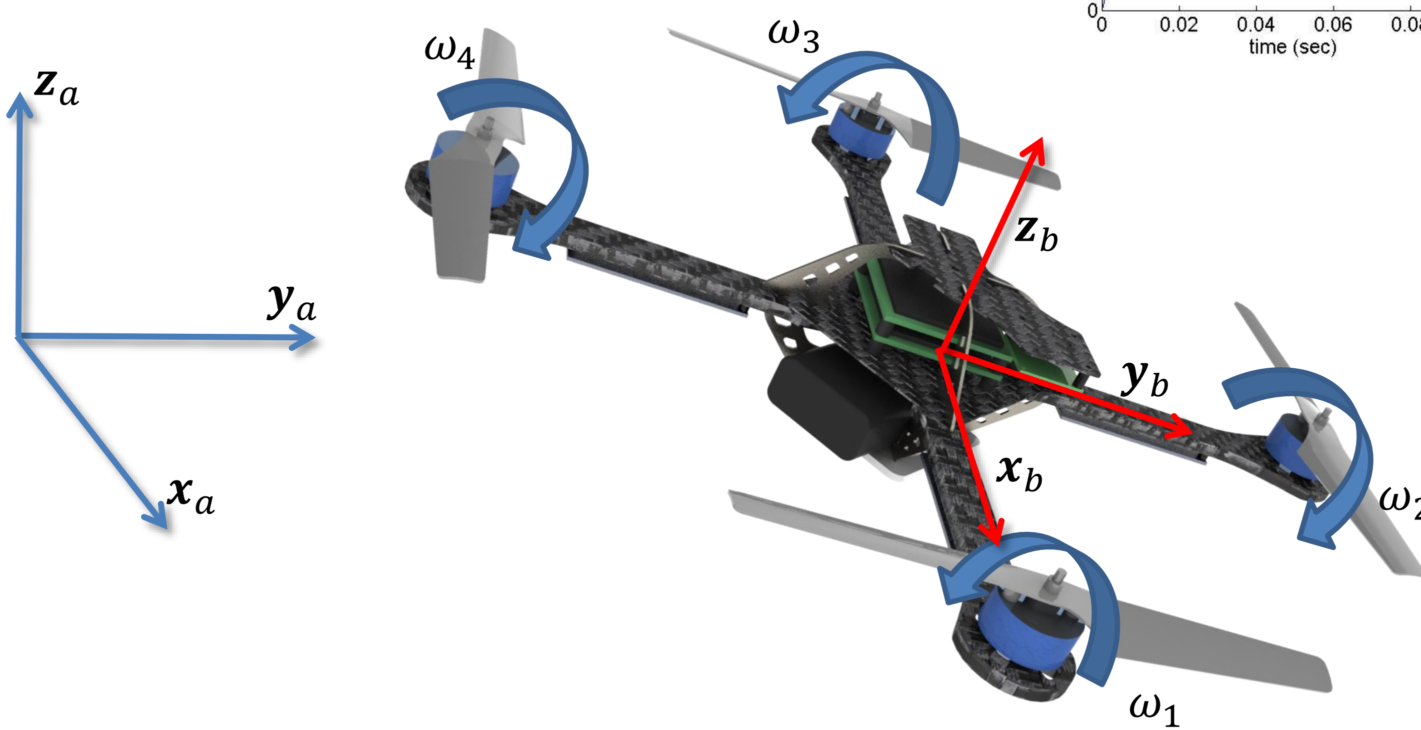


Quadrotor Dynamics

Quadrotor Dynamics

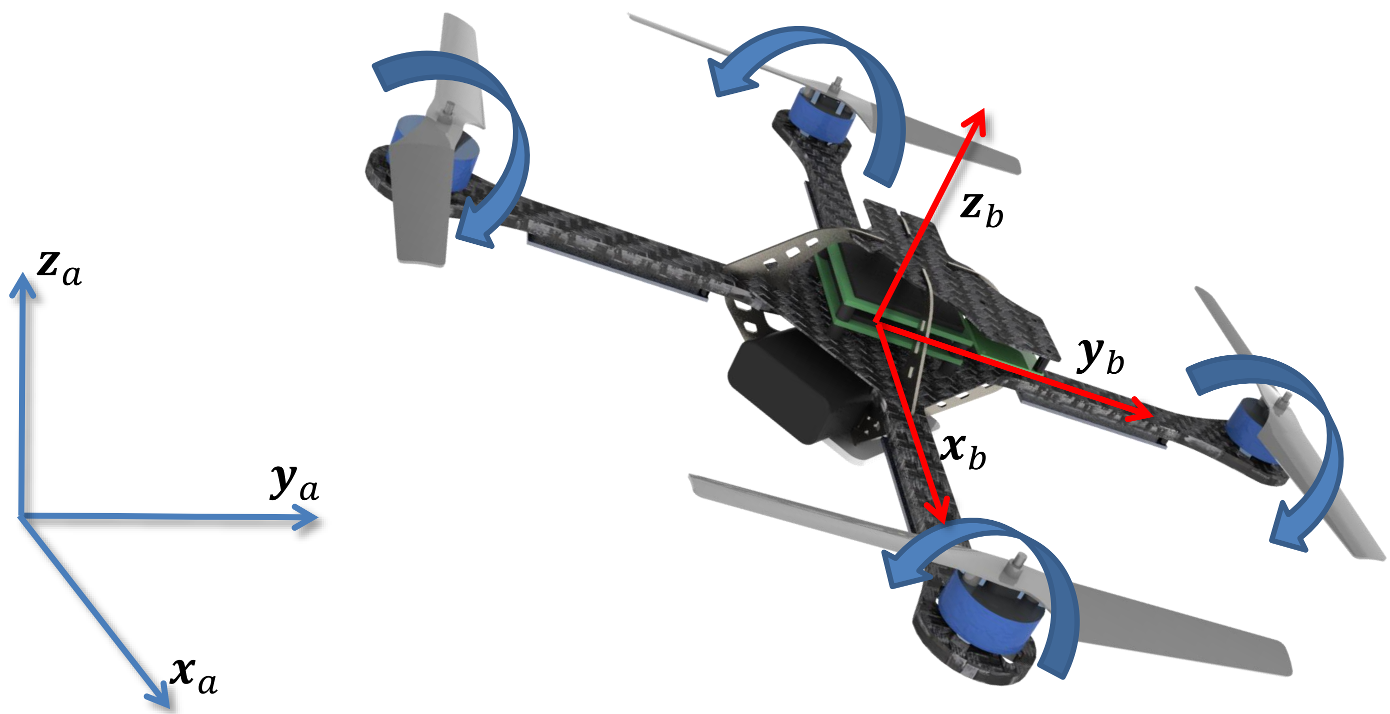
- Motor model: $\dot{\omega}_i = k_m(\omega_i^{des} - \omega_i)$
- Thrust from individual motor: $F_i = k_F \omega_i^2$
- Moment from individual motor: $M_i = k_M \omega_i^2$





Quadrotor Dynamics

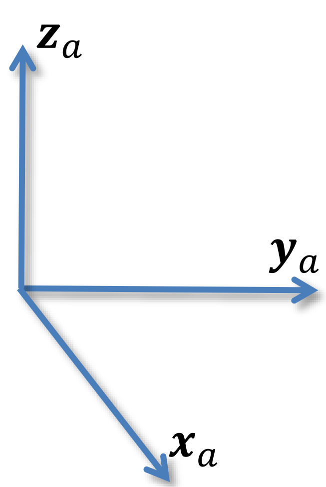
- Z-X-Y Euler Angles: $\boldsymbol{R}_{ab} = \boldsymbol{R}_z(\psi) \cdot \boldsymbol{R}_x(\phi) \cdot \boldsymbol{R}_y(\theta)$
- Sequence of three rotations about body-fixed axes
- What are the singularities?

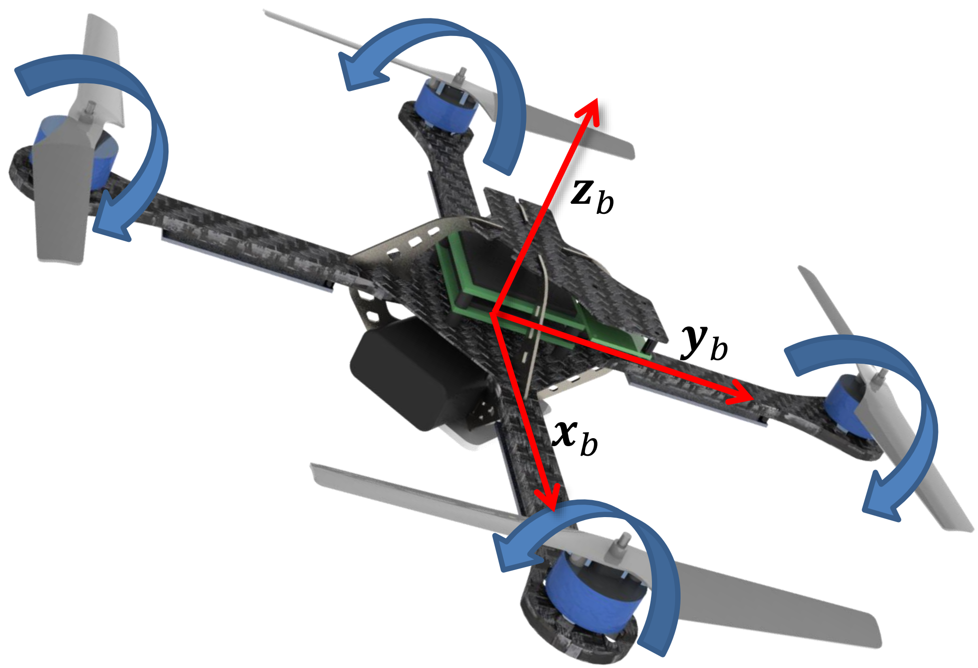


Quadrotor Dynamics

- $$\boldsymbol{R}_{ab} = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\psi c\theta s\phi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$
- $$\boxed{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}} = \begin{bmatrix} c\theta & 0 & -c\phi s\theta \\ 0 & 1 & s\phi \\ s\theta & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

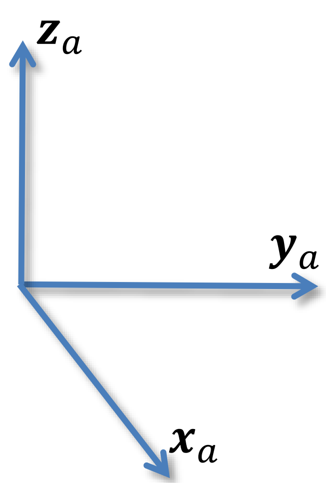
Instantaneous body angular velocity.

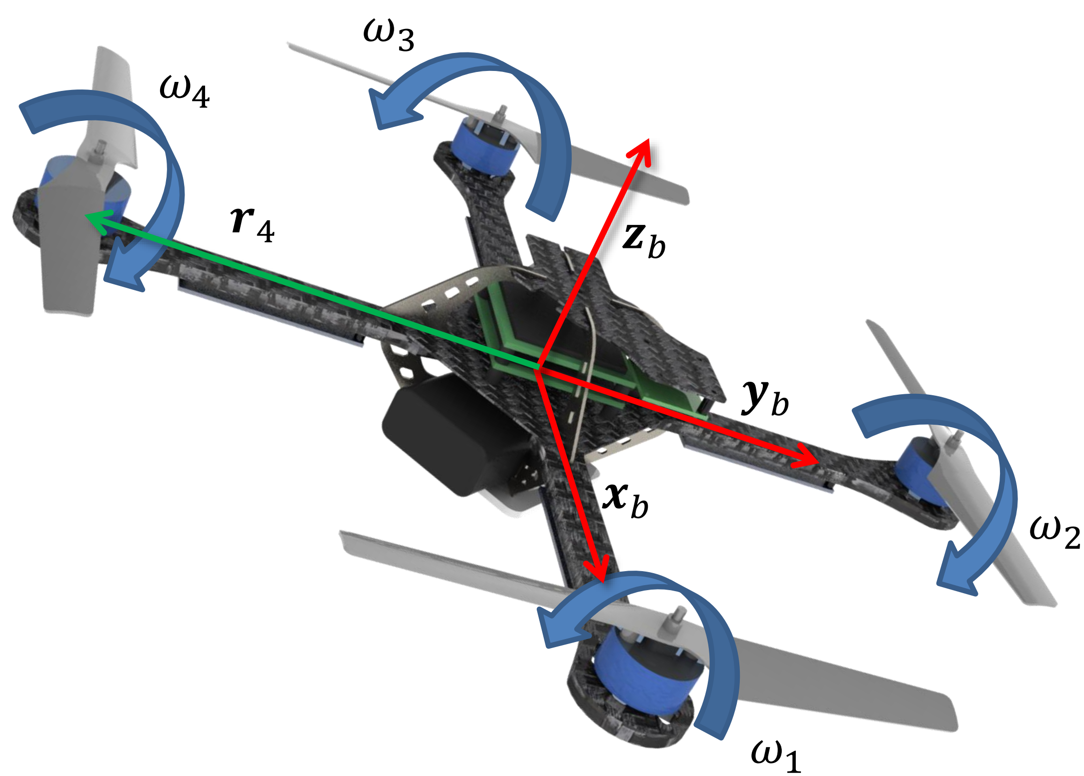




Quadrotor Dynamics

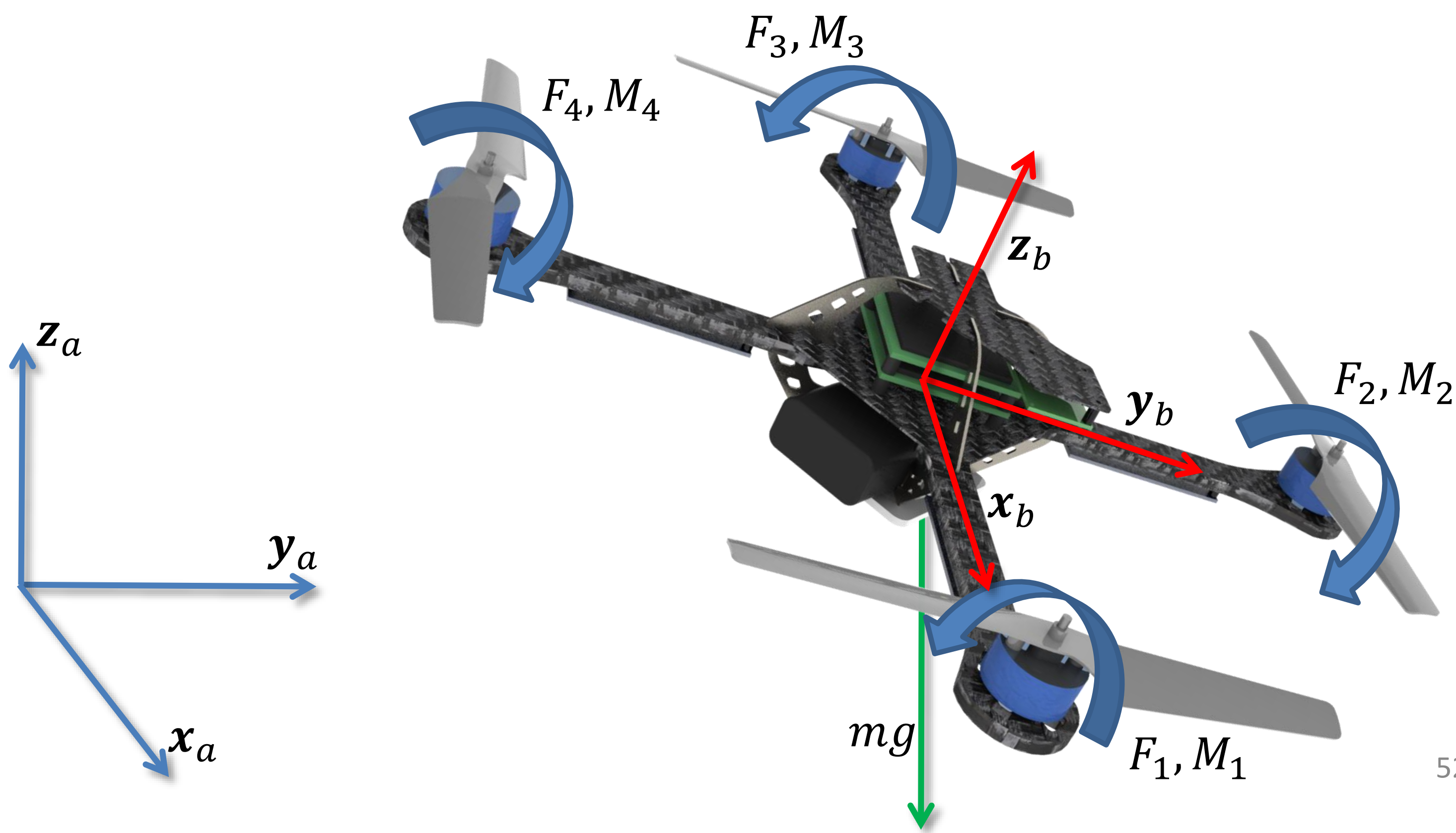
- $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \boldsymbol{F}_4 - mg\boldsymbol{z}_a$
- $\boldsymbol{M} = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2 + \boldsymbol{r}_3 \times \boldsymbol{F}_3 + \boldsymbol{r}_4 \times \boldsymbol{F}_4 + \boldsymbol{M}_1 + \boldsymbol{M}_2 + \boldsymbol{M}_3 + \boldsymbol{M}_4$
- $\boldsymbol{F}_i = [0, 0, F_i]^T$
- $\boldsymbol{M}_i = [0, 0, \pm M_i]^T$



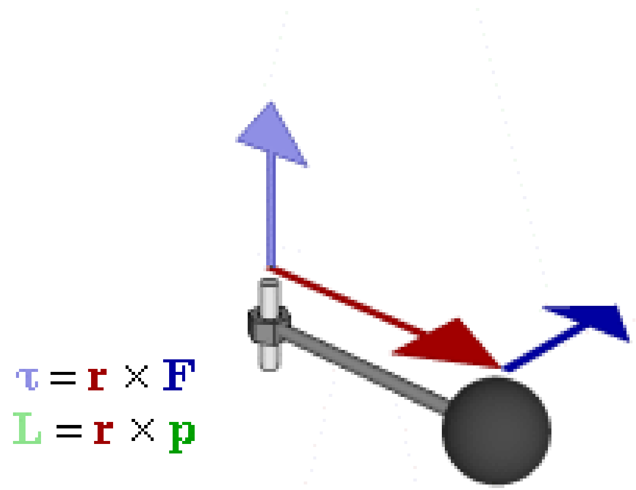


Newton-Euler Equations

- Newton Equation: $m\ddot{\boldsymbol{p}}^a = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \boldsymbol{R}_{ab} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$



Newton-Euler Equations

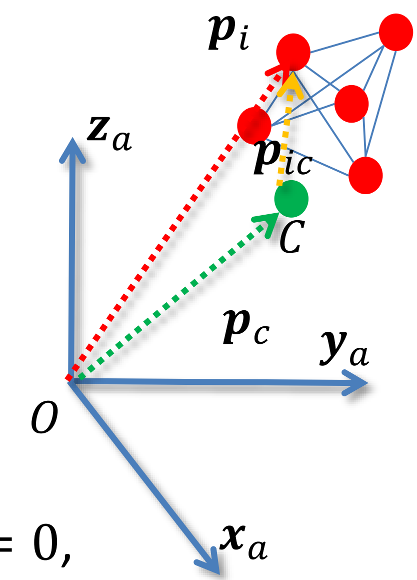


Relationship between force (F), torque/moment of force(τ), momentum (p), and angular momentum (L) vectors in a rotating system. r is the position vector.

Newton-Euler Equations

- The rigid body as a collection of particles
  - Center of mass (CoM): $\boldsymbol{p}_c$
  - Position of the i-th particle to CoM: $\boldsymbol{p}_{ic} = \boldsymbol{p}_i - \boldsymbol{p}_c$
  - Velocity of the i-th particle to CoM: $\boldsymbol{v}_{ic} = \dot{\boldsymbol{p}}_i - \dot{\boldsymbol{p}}_c = \boldsymbol{v}_i - \boldsymbol{v}_c$
  - Angular momentum of the i-th particle:

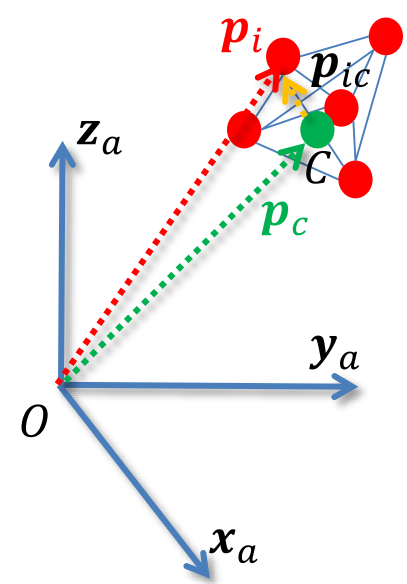
$$\boldsymbol{H}_i = \boldsymbol{p}_{ic} \times m_i \boldsymbol{v}_i$$



- Angular momentum of the rigid body:
  - $\boldsymbol{H} = \sum \boldsymbol{p}_{ic} \times m_i \boldsymbol{v}_i$
  - Since: $\sum m_i \boldsymbol{p}_{ic} = \sum m_i (\boldsymbol{p}_i - \boldsymbol{p}_c) = \sum m_i \boldsymbol{p}_i - \boldsymbol{p}_c \sum m_i = 0$,
  - We have: $\sum \boldsymbol{p}_{ic} \times m_i \boldsymbol{v}_c = (\sum m_i \boldsymbol{p}_{ic}) \times \boldsymbol{v}_c = 0$
  - Therefore: $\boldsymbol{H} = \sum \boldsymbol{p}_{ic} \times m_i \boldsymbol{v}_i - \sum \boldsymbol{p}_{ic} \times m_i \boldsymbol{v}_c = \sum \boldsymbol{p}_{ic} \times m_i \boldsymbol{v}_{ic}$
  - Since: $\boldsymbol{v}_{ic} = \boldsymbol{\omega} \times \boldsymbol{p}_{ic}$,
  - We have: $\boldsymbol{H} = \sum \boldsymbol{p}_{ic} \times (\boldsymbol{\omega} \times m_i \boldsymbol{p}_{ic}) = -\sum \boldsymbol{p}_{ic} \times (m_i \boldsymbol{p}_{ic} \times \boldsymbol{\omega})$

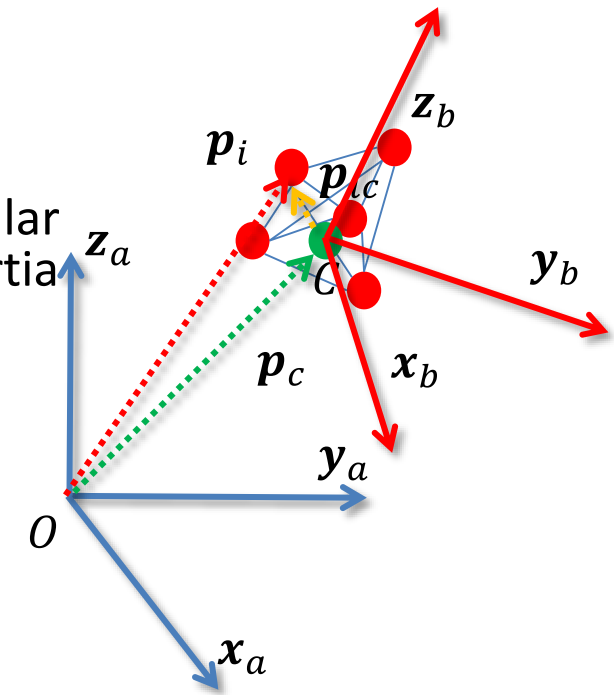
Newton-Euler Equations

- Rotational dynamics
  - Angular momentum: $\boldsymbol{H} = \sum \boldsymbol{p}_{ic} \times m_i \boldsymbol{v}_i$
  - Take the derivative: $\dot{\boldsymbol{H}} = \sum \dot{\boldsymbol{p}}_{ic} \times m_i \boldsymbol{v}_i + \sum \boldsymbol{p}_{ic} \times m_i \dot{\boldsymbol{v}}_i$
  - Since $\sum \dot{\boldsymbol{p}}_{ic} \times m_i \boldsymbol{v}_i = \sum \boldsymbol{v}_i \times m_i \boldsymbol{v}_i - \boldsymbol{v}_c \times m_i \boldsymbol{v}_i = \sum -\boldsymbol{v}_c \times m_i \boldsymbol{v}_i = -\boldsymbol{v}_c \times \frac{d}{dt} \sum m_i \boldsymbol{p}_i = -\boldsymbol{v}_c \times \frac{d}{dt} \boldsymbol{p}_c \sum m_i = 0$
  - We have $\dot{\boldsymbol{H}} = \sum \boldsymbol{p}_{ic} \times m_i \dot{\boldsymbol{v}}_i$
  - Referring to Newton's second law: $\boldsymbol{F}_i + \sum_{i \neq j} \boldsymbol{F}_{ij} = m_i \dot{\boldsymbol{v}}_i$
  - $\dot{\boldsymbol{H}} = \sum \boldsymbol{p}_{ic} \times m_i \dot{\boldsymbol{v}}_i = \sum \boldsymbol{p}_{ic} \times (\boldsymbol{F}_i + \sum_{i \neq j} \boldsymbol{F}_{ij}) = \sum \boldsymbol{p}_{ic} \times \boldsymbol{F}_i$
  - We also know that the external moment: $\boldsymbol{M} = \sum \boldsymbol{p}_{ic} \times \boldsymbol{F}_i$
  - We have the rotational dynamics: $\boldsymbol{M} = \dot{\boldsymbol{H}}$



Newton-Euler Equations

- Finishing the work on rotational dynamics
  - Given: $\boldsymbol{H} = -\sum \boldsymbol{p}_{ic} \times (m_i \boldsymbol{p}_{ic} \times \boldsymbol{\omega})$
  - And using the fact: $(\boldsymbol{Ra}) \times (\boldsymbol{Rb}) = \boldsymbol{R}(\boldsymbol{a} \times \boldsymbol{b})$
    - $\boldsymbol{R}$: rotation matrix
    - $\boldsymbol{a}, \boldsymbol{b}$: vectors
  - We can transform the representation of the angular momentum to the body frame with constant inertia matrix:



$$\boldsymbol{H} = -\sum \boldsymbol{p}_{ic} \times (m_i \boldsymbol{p}_{ic} \times \boldsymbol{\omega})$$

$$= -\sum \boldsymbol{R}_{ab}\boldsymbol{p}_{ic}^b \times \left(m_i \boldsymbol{R}_{ab}\boldsymbol{p}_{ic}^b \times \boldsymbol{R}_{ab}\boldsymbol{\omega}^b\right)$$

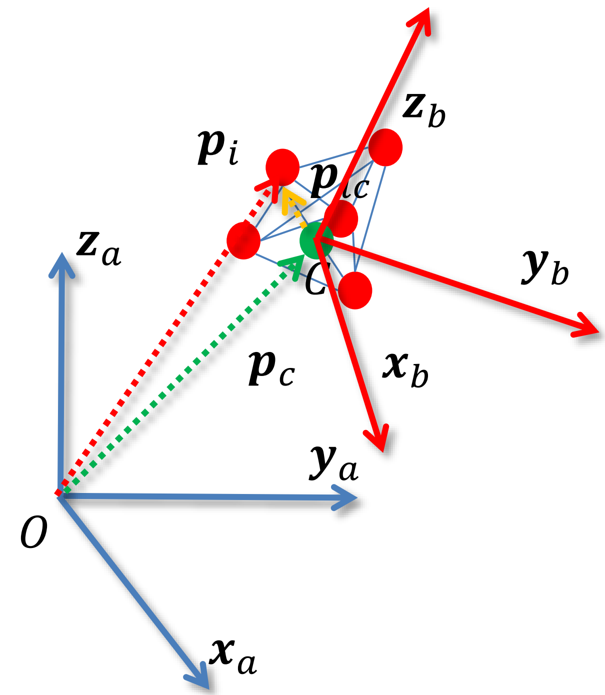
$$= -\boldsymbol{R}_{ab} \sum \boldsymbol{p}_{ic}^b \times \left(m_i \boldsymbol{p}_{ic}^b \times \boldsymbol{\omega}^b\right)$$

$$= -\boldsymbol{R}_{ab} \sum m_i \cdot \boldsymbol{p}_{ic}^b \times \left(\widehat{\boldsymbol{p}}_{ic}^b \cdot \boldsymbol{\omega}^b\right)$$

$$= \boldsymbol{R}_{ab}(-\sum m_i \cdot \widehat{\boldsymbol{p}}_{ic}^b \cdot \widehat{\boldsymbol{p}}_{ic}^b) \cdot \boldsymbol{\omega}^b = \boldsymbol{R}_{ab}(\boldsymbol{I}^b \boldsymbol{\omega}^b)$$

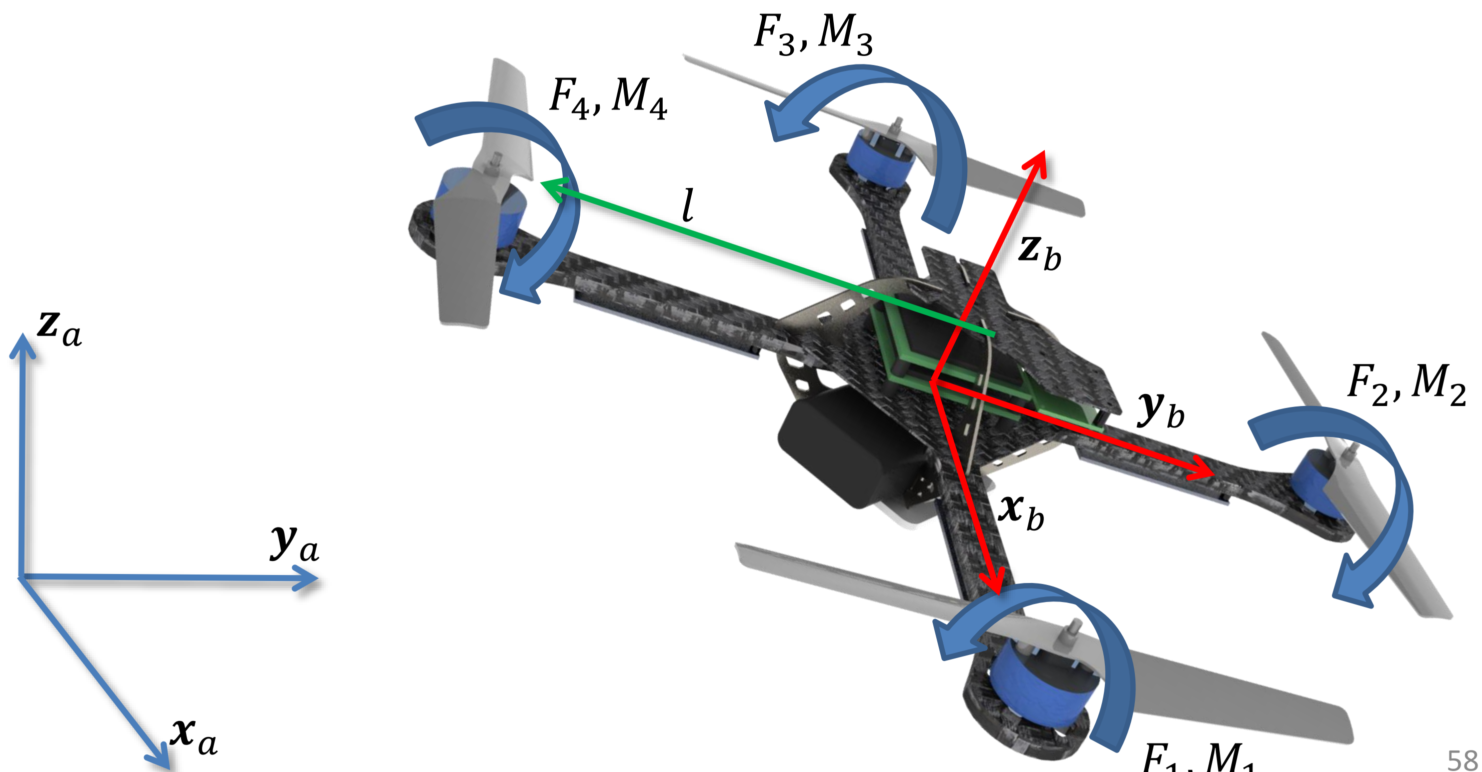
Newton-Euler Equations

- Finishing the work on rotational dynamics
  - Given $\boldsymbol{H} = \boldsymbol{R}_{ab}(\boldsymbol{I}^b\boldsymbol{\omega}^b)$
  - Take the derivative: $\dot{\boldsymbol{H}} = \dot{\boldsymbol{R}}_{ab}\boldsymbol{I}^b\boldsymbol{\omega}^b + \boldsymbol{R}_{ab}\boldsymbol{I}^b\dot{\boldsymbol{\omega}}^b = \boldsymbol{R}_{ab}\widehat{\boldsymbol{\omega}}^b\boldsymbol{I}^b\boldsymbol{\omega}^b + \boldsymbol{R}_{ab}\boldsymbol{I}^b\dot{\omega}^b = \boldsymbol{R}_{ab}(\boldsymbol{\omega}^b \times \left(\boldsymbol{I}^b\boldsymbol{\omega}^b\right) + I^b\dot{\boldsymbol{\omega}}^b)$
  - Also transform the moment into body frame: $\boldsymbol{M} = \boldsymbol{R}_{ab}\boldsymbol{M}^b$
  - Finally: $\boldsymbol{M}^b = \boldsymbol{\omega}^b \times \left(\boldsymbol{I}^b\boldsymbol{\omega}^b\right) + I^b\dot{\boldsymbol{\omega}}^b$



Newton-Euler Equations

- Euler Equation: $\boldsymbol{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \boldsymbol{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$

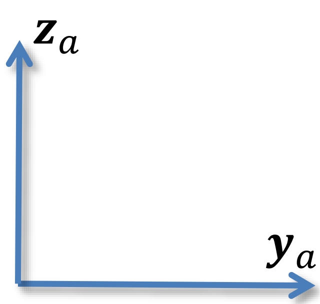


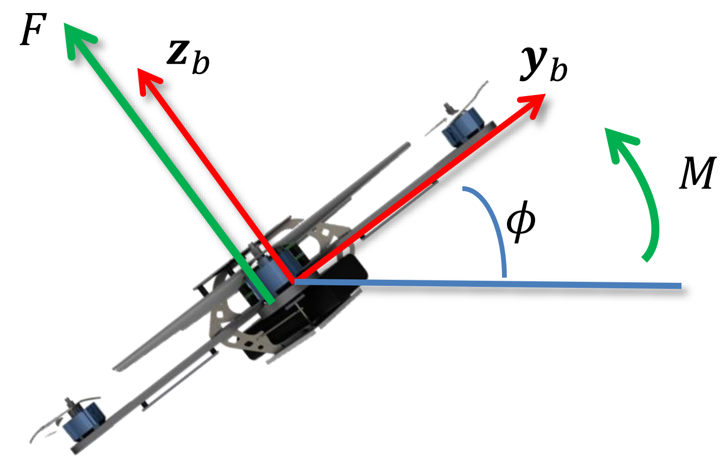
Quadrotor Dynamics

- Motor model: $\dot{\omega}_i = k_m(\omega_i^{des} - \omega_i)$
- Thrust from individual motor: $F_i = k_F \omega_i^2$
- Moment from individual motor: $M_i = k_M \omega_i^2$
- Newton Equation: $m\ddot{\boldsymbol{p}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \boldsymbol{R} \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$
- Euler Equation: $\boldsymbol{I} \cdot \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \boldsymbol{I} \cdot \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} l(F_2 - F_4) \\ l(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix}$

A Planar Quadrotor

- $$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} F \\ M \end{bmatrix}$$





ω_1

Assignment

- Chapter 2.1-2.4 of “A Mathematical Introduction to Robotic Manipulation”
- Paper Reading: “The GRASP Multiple Micro-UAV Test Bed”, Nathan Michael, Daniel Mellinger, Quentin Lindsey, and Vijay Kumar.

Next Lecture…

- Control basics
- Quadrotor control
- Trajectory generation

Introducing Quaternion

- Rotation matrix $\boldsymbol{R} \in SO(3)$
  - No singularity
  - Redundant parameters
  - Kinematics: $\dot{\boldsymbol{R}} = \boldsymbol{R}\widehat{\boldsymbol{\omega}}$, $\boldsymbol{\omega}$ is the body angular rate
- ZYX Euler angle
  - Singular at roll angle of 90 degrees
  - Minimum number of parameters
  - Kinematics: $\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \frac{\cos(\psi)}{\cos(\theta)} & \frac{\sin(\psi)}{\cos(\theta)} & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ \cos(\psi)\tan(\theta) & \sin(\psi)\tan(\theta) & 1 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$
- Is there a *singularity free* parameterization that with *reduced* parameters? YES → quaternion

Introducing Quaternion

- Recall a rotation matrix $\boldsymbol{R}$ can be decomposed as a rotation axis $\boldsymbol{u}$ of angle θ by $\boldsymbol{R} = e^{\hat{\boldsymbol{u}}\theta} = \boldsymbol{I} + \hat{\boldsymbol{u}} \sin\theta + \hat{\boldsymbol{u}}^2(1 - \cos\theta)$
- Any vector $\boldsymbol{p}$, after rotation $\boldsymbol{R}$, is $\boldsymbol{p}' = \boldsymbol{R}\boldsymbol{p}$

$$\Rightarrow \begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \boldsymbol{R} \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix}$$

A representation of rotation

$$\Rightarrow \begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} u \end{bmatrix} \circ \begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix} \circ \begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \boldsymbol{u} \end{bmatrix}; \forall \boldsymbol{p}, \boldsymbol{u}, \theta$$

A new binary operation

where

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \circ \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Definitions of Quaternion

- Quaternion

$$\boldsymbol{Q} \stackrel{\text{def}}{=} q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} \stackrel{\text{def}}{=} q_0 + \boldsymbol{q} \stackrel{\text{def}}{=} \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix}$$

where $\boldsymbol{i} \circ \boldsymbol{i} = \boldsymbol{j} \circ \boldsymbol{j} = \boldsymbol{k} \circ \boldsymbol{k} \stackrel{\text{def}}{=} -1; \boldsymbol{i} \circ \boldsymbol{j} \stackrel{\text{def}}{=} \boldsymbol{k}; \boldsymbol{j} \circ \boldsymbol{k} \stackrel{\text{def}}{=} \boldsymbol{i}; \boldsymbol{k} \circ \boldsymbol{i} \stackrel{\text{def}}{=} \boldsymbol{j}$

- Conjugate

$$\overline{\boldsymbol{Q}} \stackrel{\text{def}}{=} q_0 - q_1\boldsymbol{i} - q_2\boldsymbol{j} - q_3\boldsymbol{k} \stackrel{\text{def}}{=} q_0 - \boldsymbol{q} \stackrel{\text{def}}{=} \begin{bmatrix} q_0 \\ -\boldsymbol{q} \end{bmatrix}$$

- Norm $\|\boldsymbol{Q}\| \stackrel{\text{def}}{=} \boldsymbol{Q} \circ \overline{\boldsymbol{Q}}$

- Inverse $\boldsymbol{Q}^{-1} \stackrel{\text{def}}{=} \dfrac{\overline{\boldsymbol{Q}}}{\|\boldsymbol{Q}\|}$

Properties of Quaternion

- Property 1

$$\boldsymbol{A} = a_0 + \boldsymbol{a}, \quad \boldsymbol{B} = b_0 + \boldsymbol{b}$$

particularly

$$\boldsymbol{A} \circ \boldsymbol{B} = (a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b}) + (a_0 \boldsymbol{b} + \boldsymbol{a} b_0 + \boldsymbol{a} \times \boldsymbol{b})$$

$$(0 + \boldsymbol{a}) \circ (0 + \boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{b}$$

- Property 2 $\quad (\boldsymbol{A} \circ \boldsymbol{B}) \circ \boldsymbol{C} = \boldsymbol{A} \circ (\boldsymbol{B} \circ \boldsymbol{C})$
- Property 3 $\quad \overline{\boldsymbol{A} \circ \boldsymbol{B}} = \overline{\boldsymbol{B}} \circ \overline{\boldsymbol{A}}$
- Property 4 $\quad \boldsymbol{A} = a_0 + a_1 \boldsymbol{i} + a_2 \boldsymbol{j} + a_3 \boldsymbol{k}$
  $\boldsymbol{B} = b_0 + b_1 \boldsymbol{i} + b_2 \boldsymbol{j} + b_3 \boldsymbol{k}$

Exactly what we need for representing rotations!!

$$\boxed{\boldsymbol{A} \circ \boldsymbol{B} = \underbrace{\begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}}_{\mathcal{L}(\boldsymbol{A})} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix}}_{\mathcal{R}(\boldsymbol{B})} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}$$

Properties of Quaternion

- Product of Quaternions $\boldsymbol{A} \circ \boldsymbol{B} = \mathcal{L}(\boldsymbol{A}) \begin{bmatrix} b_0 \\ \boldsymbol{b} \end{bmatrix} = \mathcal{R}(\boldsymbol{B}) \begin{bmatrix} a_0 \\ \boldsymbol{a} \end{bmatrix}$

$$\mathcal{L}(\boldsymbol{A}) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} = \begin{bmatrix} a_0 & -\boldsymbol{a}^T \\ \boldsymbol{a} & a_0 \boldsymbol{I}_3 + \hat{\boldsymbol{a}} \end{bmatrix}$$

$$\mathcal{R}(\boldsymbol{B}) = \begin{bmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & b_3 & -b_2 \\ b_2 & -b_3 & b_0 & b_1 \\ b_3 & b_2 & -b_1 & b_0 \end{bmatrix} = \begin{bmatrix} b_0 & -\boldsymbol{b}^T \\ \boldsymbol{b} & b_0 \boldsymbol{I}_3 - \hat{\boldsymbol{a}} \end{bmatrix}$$

Unit Quaternion and Rotation

Recall

$$\underbrace{\begin{bmatrix} 0 \\ \boldsymbol{p}' \end{bmatrix}}_{\boldsymbol{P}'} = \underbrace{\begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\boldsymbol{u} \end{bmatrix}}_{\boldsymbol{Q}} \circ \underbrace{\begin{bmatrix} 0 \\ \boldsymbol{p} \end{bmatrix}}_{\boldsymbol{P}} \circ \underbrace{\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\boldsymbol{u} \end{bmatrix}}_{\overline{\boldsymbol{Q}}}$$

$$\boldsymbol{P}' \stackrel{\text{def}}{=} 0 + \boldsymbol{p}'; \boldsymbol{P} \stackrel{\text{def}}{=} 0 + \boldsymbol{p}; \quad \boxed{\boldsymbol{Q} \stackrel{\text{def}}{=} \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{u}}$$

If using quaternion Q to represent a rotation:

Basic principle: $\boldsymbol{P}' = \boldsymbol{Q} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}$

- $\boldsymbol{P}', \boldsymbol{P}$ are respectively the coordinates of the post-rotation vector and prior-rotation vector, both in the same frame.
- $\boldsymbol{Q}$ is the rotation quaternion associated with the rotation

Unit Quaternion and Rotation

Basic principle: $\boldsymbol{P'} = \boldsymbol{Q} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}$

Property 1 Constant vector： $\boldsymbol{P}^b = \overline{\boldsymbol{Q}} \circ \boldsymbol{P} \circ \boldsymbol{R}$

- $\boldsymbol{P}^b, \boldsymbol{P}$ are respectively the coordinates in the body frame and the world frame, of the same vector
- Can be interpreted as that the vector is rotating in the opposite direction, then call the first result

Unit Quaternion and Rotation

Basic principle: $\boldsymbol{P}' = \boldsymbol{Q} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}$

Property 2 Two sequent rotations

- Case 1
  $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ are the two rotation quaternions where the rotation axes are both represented in the initial frame

$$\left.\begin{aligned} \boldsymbol{P}' &= \boldsymbol{Q}_1 \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}_1 \\ \boldsymbol{P}'' &= \boldsymbol{Q}_2 \circ \boldsymbol{P}' \circ \overline{\boldsymbol{Q}}_2 = \underbrace{\boldsymbol{Q}_2 \circ \boldsymbol{Q}_1}_{\boldsymbol{Q}} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}_1 \circ \overline{\boldsymbol{Q}}_2 \end{aligned}\right\} \Rightarrow \boldsymbol{Q} = \boldsymbol{Q}_2 \circ \boldsymbol{Q}_1$$

- Case 2
  $\boldsymbol{Q}_1, \boldsymbol{Q}_2$ are the two rotation quaternions where the rotation axis of $\boldsymbol{Q}_2$ is represented in the frame obtained by performing $\boldsymbol{Q}_1$.

$$\boldsymbol{P}' = \boldsymbol{Q}_1 \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}_1$$

$$\boldsymbol{P}'' = \left(\boldsymbol{Q}_1 \circ \boldsymbol{Q}_2 \circ \overline{\boldsymbol{Q}}_1\right) \circ \boldsymbol{P}' \circ \overline{\boldsymbol{Q}_1 \circ \boldsymbol{Q}_2 \circ \overline{\boldsymbol{Q}}_1} = \left(\boldsymbol{Q}_1 \circ \boldsymbol{Q}_2 \circ \overline{\boldsymbol{Q}}_1\right) \circ \boldsymbol{Q}_1 \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}_1 \circ \overline{\boldsymbol{Q}_1 \circ \boldsymbol{Q}_2 \circ \overline{\boldsymbol{Q}}_1} = \underbrace{\boldsymbol{Q}_1 \circ \boldsymbol{Q}_2}_{\boldsymbol{Q}} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}_1 \circ \boldsymbol{Q}_2}$$

$$\Rightarrow \boldsymbol{Q} = \boldsymbol{Q}_1 \circ \boldsymbol{Q}_2$$

Quaternion Kinematics

Recall

$$\boldsymbol{P}' = \boldsymbol{Q} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}$$

Taking derivative yields

$$\dot{\boldsymbol{P}}' = \dot{\boldsymbol{Q}} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}} + \boldsymbol{Q} \circ \boldsymbol{P} \circ \dot{\overline{\boldsymbol{Q}}}$$

$$= \dot{\boldsymbol{Q}} \circ \underbrace{\overline{\boldsymbol{Q}} \circ \boldsymbol{P}' \circ \boldsymbol{Q}}_{\boldsymbol{P}} \circ \overline{\boldsymbol{Q}} + \boldsymbol{Q} \circ \underbrace{\overline{\boldsymbol{Q}} \circ \boldsymbol{P}' \circ \boldsymbol{Q}}_{\boldsymbol{P}} \circ \dot{\overline{\boldsymbol{Q}}}$$

$$= \dot{\boldsymbol{Q}} \circ \overline{\boldsymbol{Q}} \circ \boldsymbol{P}' + \boldsymbol{P}' \circ \boldsymbol{Q} \circ \dot{\overline{\boldsymbol{Q}}}$$

As $\boldsymbol{Q} \circ \overline{\boldsymbol{Q}} = 1$

Taking derivative yields

$$\dot{\boldsymbol{Q}} \circ \overline{\boldsymbol{Q}} + \boldsymbol{Q} \circ \dot{\overline{\boldsymbol{Q}}} = 0 \Rightarrow \left.\begin{aligned} \boldsymbol{Q} \circ \dot{\overline{\boldsymbol{Q}}} &= -\dot{\boldsymbol{Q}} \circ \overline{\boldsymbol{Q}} \\ \boldsymbol{Q} \circ \dot{\overline{\boldsymbol{Q}}} &= \overline{\dot{\boldsymbol{Q}} \circ \overline{\boldsymbol{Q}}} \end{aligned}\right\} \dot{\boldsymbol{Q}} \circ \overline{\boldsymbol{Q}} = 0 + \boldsymbol{a}; \boldsymbol{Q} \circ \dot{\overline{\boldsymbol{Q}}} = 0 - \boldsymbol{a}$$

Notice

$$\dot{\boldsymbol{P}}' = (0 + \boldsymbol{a}) \circ \boldsymbol{P}' + \boldsymbol{P}' \circ (0 - \boldsymbol{a})$$

Quaternion Kinematics

$$\dot{\boldsymbol{P}}' = (0+\boldsymbol{a})\circ\boldsymbol{P}' + \boldsymbol{P}'\circ(0-\boldsymbol{a})$$

$$= \begin{bmatrix}0\\ \boldsymbol{a}\end{bmatrix}\circ\begin{bmatrix}0\\ \boldsymbol{p}'\end{bmatrix} + \begin{bmatrix}0\\ \boldsymbol{p}'\end{bmatrix}\circ\begin{bmatrix}0\\ -\boldsymbol{a}\end{bmatrix}$$

$$= \begin{bmatrix}0\\ \boldsymbol{a}\times\boldsymbol{p}'\end{bmatrix} - \boldsymbol{a}\cdot\boldsymbol{p}' + \begin{bmatrix}0\\ \boldsymbol{p}'\times(-\boldsymbol{a})\end{bmatrix} - \boldsymbol{p}'\cdot(-\boldsymbol{a})$$

$$\boxed{\begin{bmatrix}0\\ \dot{\boldsymbol{p}}'\end{bmatrix} = \begin{bmatrix}0\\ 2\boldsymbol{a}\times\boldsymbol{p}'\end{bmatrix}}$$

Recall $\dot{\boldsymbol{p}}' = \boldsymbol{\omega}\times\boldsymbol{p}'$, $\boldsymbol{\omega}$ is angular velocity vector represented in the world frame

Then $2\boldsymbol{a} = \boldsymbol{\omega} \quad \Rightarrow \boldsymbol{a} = \frac{1}{2}\boldsymbol{\omega}$

$$\Rightarrow \dot{\boldsymbol{Q}}\circ\overline{\boldsymbol{Q}} = \begin{bmatrix}0\\ \boldsymbol{a}\end{bmatrix} = \frac{1}{2}\underbrace{\begin{bmatrix}0\\ \boldsymbol{\omega}\end{bmatrix}}_{\boldsymbol{\Omega}} \qquad \boxed{\Rightarrow \dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{\Omega}\circ\boldsymbol{Q}}$$

$$\boldsymbol{Q} = \begin{bmatrix}q_0\\ \boldsymbol{q}\end{bmatrix} \Rightarrow \dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{\Omega}\circ\boldsymbol{Q} = \mathcal{L}(\boldsymbol{\Omega})\boldsymbol{Q} = \frac{1}{2}\begin{bmatrix}0 & -\boldsymbol{\omega}^T\\ \boldsymbol{\omega} & \widehat{\boldsymbol{\omega}}\end{bmatrix}\boldsymbol{Q}$$

Quaternion Kinematics

$$\dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{\Omega} \circ \boldsymbol{Q}$$

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 \\ \boldsymbol{\omega} \end{bmatrix}$$

$\boldsymbol{\omega}$ is angular velocity represented in the world frame

– What if $\boldsymbol{\omega}$ is represented in the body frame (i.e. $\boldsymbol{\omega}^b$)?

$$\boldsymbol{\Omega}^b = \begin{bmatrix} 0 \\ \boldsymbol{\omega}^b \end{bmatrix}$$ is represented in body frame

$$\Rightarrow \quad \boldsymbol{\Omega}^b = \overline{\boldsymbol{Q}} \circ \boldsymbol{\Omega} \circ \boldsymbol{Q}$$

$$\Rightarrow \quad \boxed{\dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{Q} \circ \boldsymbol{\Omega}^b}$$

$$\boldsymbol{Q} = \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix}$$

$$\Rightarrow \dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{Q} \circ \boldsymbol{\Omega}^b = \mathcal{R}(\boldsymbol{\Omega}^b)\boldsymbol{Q} = \frac{1}{2}\begin{bmatrix} 0 & -{\boldsymbol{\omega}^b}^T \\ \boldsymbol{\omega}^b & -\widehat{\boldsymbol{\omega}^b} \end{bmatrix}\boldsymbol{Q}$$

Unit Quaternion and Rotation

- Rotation to quaternion: $\boldsymbol{Q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\boldsymbol{u} \end{bmatrix}$, where $\boldsymbol{R} = e^{\hat{\boldsymbol{u}}\theta}$
- Quaternion to rotation
- $\boldsymbol{R} = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_0q_2+q_1q_3) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(-q_0q_1+q_2q_3) \\ 2(-q_0q_2+q_1q_3) & 2(q_0q_1+q_2q_3) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$
- Rotating a vector: $\boldsymbol{P}' = \boldsymbol{Q} \circ \boldsymbol{P} \circ \overline{\boldsymbol{Q}}$
- Rotating a frame: $\boldsymbol{P}^b = \overline{\boldsymbol{Q}} \circ \boldsymbol{P} \circ \boldsymbol{Q}$
- Sequential rotation (extrinsic): $\boldsymbol{Q} = \boldsymbol{Q}_n \ldots \circ \boldsymbol{Q}_2 \circ \boldsymbol{Q}_1$
- Sequential rotation (intrinsic): $\boldsymbol{Q} = \boldsymbol{Q}_1 \circ \boldsymbol{Q}_2 \ldots \circ \boldsymbol{Q}_n$
- Kinematics under spatial frame: $\dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{\Omega} \circ \boldsymbol{Q} = \frac{1}{2}\begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & \hat{\boldsymbol{\omega}} \end{bmatrix}\boldsymbol{Q}$
- Kinematics under body frame: $\dot{\boldsymbol{Q}} = \frac{1}{2}\boldsymbol{Q} \circ \boldsymbol{\Omega} = \frac{1}{2}\begin{bmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -\hat{\boldsymbol{\omega}} \end{bmatrix}\boldsymbol{Q}$