ELEC 5660 **Name:** ____________________
Midterm
March 9, 2021
Time Limit: 120 Minutes **Student ID:** ____________________

Please write down your answer on blank sheets clearly.
Please write down the steps to get the answer. A single final result gets no marks.

Question	Points	Score
1	21	
2	25	
3	21	
4	33	
Total:	100	

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1. Given elementary rotation matrices:

$$\mathbf{R_x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}, \quad \mathbf{R_y}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \mathbf{R_z}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

   (a) (3 points) Write down the rotation matrix from the body frame to the world frame in the $Z-Y-X$ Euler angle representation ($\mathbf{R_{wb}}(\phi, \theta, \psi) = \mathbf{R_z}(\psi) \cdot \mathbf{R_y}(\theta) \cdot \mathbf{R_x}(\phi)$).

   (b) (4 points) What conditions can cause the singularity of the Euler angle representation? Explain by providing the rotation matrices under the conditions.

   (c) (6 points) The special euclidean group is denoted as

$$SE(3) = \{(\mathbf{p}, \mathbf{R}) : \mathbf{p} \in \mathbb{R}^3, \mathbf{R} \in SO(3)\}$$

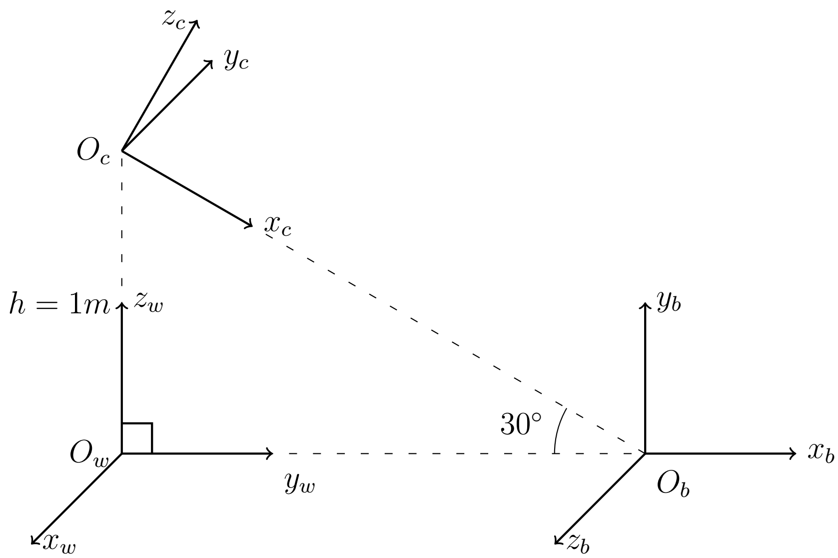
   Show that $g \in SE(3)$ is a rigid body transformation.

   (d) (8 points) Consider the homogeneous coordinate, where the transformation consists of a rotation $\mathbf{R_{wb}}$ and a translation $\mathbf{t_{wb}}$ from body frame to the world frame $\mathbf{T_{wb}}$. $\mathbf{T_{wb}}$ can be written in the block representation:

$$\mathbf{T_{wb}} = \begin{bmatrix} \mathbf{R_{wb}} & \mathbf{t_{wb}} \\ \mathbf{0} & 1 \end{bmatrix}.$$

   Write down the twist $\mathbf{T_{wb}}^{-1} \cdot \dot{\mathbf{T}}_{\mathbf{wb}}$ into a similar block representation. For the blocks that are not constant, what are the physical meanings of them?

2. Consider a scenario with three frames w, b and c.The construction of the coordinate systems are shown below, note how frames are formulated. x_c, z_c, x_b, y_b, y_w and z_w are all in the same plane. x_w, z_b are parallel and in the same direction. y_c is parallel to x_w and z_b but in the opposite direction.



(a) (10 points) Write down the homogeneous transformation from frame b to frame c, $\mathbf{T_{bc}}$ and the homogeneous transformation from frame c to frame b, $\mathbf{T_{cb}}$ respectively. (For a point with homogeneous representation in the frame m, $\mathbf{p^m}$ and homogeneous representation in the frame n, $\mathbf{p^n}$, the following formula holds: $\mathbf{p^m} = \mathbf{T_{mn}} \cdot \mathbf{p^n}$.)

(b) (15 points) Suppose that frame b and c are fixed together (there is no relative movement between frame b and c). Consider rotating frame b and c about an axis $\mathbf{v^w} = [1, 1, 0]^T$ at 3 rad/sec (The rotating direction can be defined using right-hand rule). Derive the instantaneous angular velocities $\omega^{\mathbf{w}}_{\mathbf{wb}}$ and $\omega^{\mathbf{c}}_{\mathbf{wc}}$. ($\omega^{\mathbf{w}}_{\mathbf{wb}}$ is the angular velocity of frame b with respect to frame w view in frame w and $\omega^{\mathbf{c}}_{\mathbf{wc}}$ is the angular velocity of frame c with respect to frame w view in frame c).

3. A 1-D quintic polynomial trajectory consists of two segments is represented as:

$$f(t) = \begin{cases} f_1(t) \doteq \sum_{i=0}^{5} p_{1,i} (t - T_0)^i & T_0 \le t \le T_1 \\ f_2(t) \doteq \sum_{i=0}^{5} p_{2,i} (t - T_1)^i & T_1 \le t \le T_2 \end{cases} \tag{1}$$

$\mathbf{p}_k = [p_{k,0}, p_{k,1}, \cdots, p_{k,5}]^T, k = 1, 2$ is the vector of polynomial coefficients. We want to minimize jerk (3th order derivative) of the trajectory to ensure its smoothness.

(a) (7 points) Write down the cost function of one segment for generating minimum jerk trajectory. The cost function can be expressed in matrix form $\mathbf{p}_k^T \mathbf{Q}_k \mathbf{p}_k$. Please show the entries of $\mathbf{Q}_1$. You may write down the whole matrix directly, or derive the general formula for each entry q_{ij}, and write down the range of i and j.

(b) (7 points) Between the two segments, we should ensure that their derivatives are continuous. Please write down the continuity constraints of position, velocity and acceleration in the matrix form as $\mathbf{A}\left[\mathbf{p}_1^T, \mathbf{p}_2^T\right]^T = \mathbf{0}$. Similarly, you can write down the whole matrix or just show the general formula of the entries of $\mathbf{A}$, a_{ij}, and write down the range of i and j.

(c) (7 points) Suppose we want the final ($t = T_2$) position and velocity of the trajectory to be p_f and v_f. Show the entries of $\mathbf{B}$, which is in the derivative constraints' formulation, $\mathbf{B}\mathbf{p}_2 = [p_f, v_f]^T$. Similarly, you can write down the whole matrix or just show the general formula of the entries b_{ij} and write down the range of i and j.

4. Consider a robot moving in a bounded grid-based discretized environment shown in Fig. 1. White grids are traversable space, and gray grids represent obstacles. The green and red grid represent the start and goal points. A robot is only able to move in horizontal and vertical directions (from one grid to its four neighboring grids). The cost of one horizontal / vertical movement is 1. Each grid is denoted by a coordinate (x, y) as shown in the figure. Here we want to apply some path search algorithms to find a path between the start and goal points. Each grid that is free is associated with a graph node and undirected edges are connected to its four neighbors **if the neighbor is free and within the bound**.

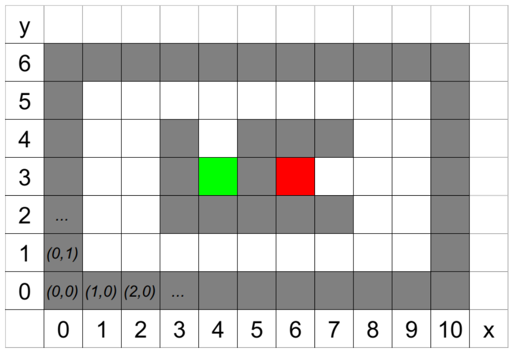


Figure 1: A discretized environment representation

(a) (6 points) Perform a path search from the start point to the goal point using Dijkstra's algorithm. In the search process, several nodes are expanded/visited starting from the green grid, until the red grid is reached. Write down the coordinates of those expanded nodes **in sequence**.

In the containers that store all the graph nodes to be visited, you may find some nodes are associated with the same score function value. If this happens, you can assume that those nodes are first sorted to increasing x coordinate and then increasing y coordinate, i.e., the one with smaller x coordinate is expanded first. Then if x is the same, the one with smaller y coordinate is expanded first.

(b) (6 points) Repeat the previous problem using the A* search algorithm and write down the coordinates of those expanded nodes in sequence. Suppose that an Manhattan heuristic is used, i.e., estimated cost at a point (x_1, y_1) is the Manhattan distance to the goal point (x_g, y_g) (red grid): $|x_1 - x_g| + |y_1 - y_g|$.

(c) (6 points) By comparing the results of (a) and (b), what are the advantages of the A* algorithm over the Dijkstra's algorithm? Why does A* algorithm have these advantages?

(d) (10 points) Based on (b), now we want to modify the strategy of expanding the nodes to further accelerate the searching. We still use an Manhattan heuristic but weight it by a *tie breaker*, which means the Manhattan distance is multiplied by a factor 1.0001.

- Again, derive and write down the coordinates of those expanded nodes in sequence.
- Compare the result with that of (b), describe your finding and explain it.
- Meanwhile, you may notice that there is a problem in the modified heuristic. What is the problem and what will it cause?

(e) (5 points) Please describe a sampling-based method to solve this problem.