# ELEC 5660 Midterm

TANG Jiawei
20672550

1. (a) $R_{wb}(\phi, \theta, \psi) = R_z(\psi) R_y(\theta) R_x(\phi)$

$\psi \ \theta \ \phi \ (z \ y \ x)$

$$= \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta \\ \cos\theta\sin\psi & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

(2) If $\sin\theta = 1$ or $\sin\theta = -1$, the singularity of Euler angle representation will happen.

① $\sin\theta = 1$

$$R_{wb}(\phi, \theta, \psi) = \begin{bmatrix} 0 & \sin(\phi - \psi) & \cos(\phi - \psi) \\ 0 & \cos(\phi - \psi) & -\sin(\phi - \psi) \\ -1 & 0 & 0 \end{bmatrix}$$

② $\sin\theta = -1$

$$R_{wb}(\phi, \theta, \psi) = \begin{bmatrix} 0 & -\cos(\psi + \phi) & -\cos(\psi + \phi) \\ 0 & \cos(\psi + \phi) & -\sin(\psi + \phi) \\ 1 & 0 & 0 \end{bmatrix}$$

(c) Denote $g \in SE(3)$

① Denote $T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$. $g$

for $p_b, p_q \in \mathbb{R}^4$.

$$g(p_b - p_q) = T(p_b - p_q)$$

$$\|g(p_b - p_q)\| = \sqrt{(p_b - p_q)^T T^T T (p_b - p_q)}$$

$$= \sqrt{(p_b - p_q)^T (p_b - p_q)} = \|p_b - p_q\|$$

such that the length is preserved.

② For cross product, we need to prove

$T([v,1]^T \times [w,1]^T) = T[v,1]^T \times T[w,1]^T$.

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ 1 \end{bmatrix} \times \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix}$$

$$(Rv + t) \times (Rw + t)$$

since $R_{ab}(v \times w) = R_{ab}(v) \times R_{ab}(w)$

we can also prove that $T$ preserves cross product

QED.

(d) $T_{wb} = \begin{bmatrix} R_{wb} & t_{wb} \\ 0 & 1 \end{bmatrix}$

$$\dot{T}_{wb} = \begin{bmatrix} \dot{R}_{wb} & \dot{t}_{wb} \\ 0 & 0 \end{bmatrix} \qquad T_{wb}^{-1} = \begin{bmatrix} R_{wb}^T & -R_{wb}^T t_{wb} \\ 0 & 1 \end{bmatrix}$$

Hence
$$T_{wb}^{-1} \dot{T}_{wb} = \begin{bmatrix} R_{wb}^T & -R_{wb}^T t_{wb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R}_{wb} & \dot{t}_{wb} \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} R_{wb}^T \dot{R}_{wb} & R_{wb}^T \dot{t}_{wb} - R_{wb}^T t_{wb} \\ 0 & 0 \end{bmatrix}$$

The left top block is the instantaneous body angular velocity

The right top block is the instantaneous body linear velocity

2. (a) for $T_{bc}$

$$t_{bc} = [-\sqrt{3},\ 1,\ 0]^T$$

Denote $\phi$ for x axis, $\theta$ for y axis, $\psi$ for z axis

$$\phi = -90^\circ,\quad \psi = -30^\circ,\quad \theta = 0^\circ$$

$$R_{bc} = R_y(0)\,R_z(-30^\circ)\,R_x(-90^\circ)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore T_{bc} = \begin{bmatrix} R_{bc} & t_{bc} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & -\sqrt{3} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{cb} = \begin{bmatrix} R_{bc}^T & -R_{bc}^T t_{bc} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

(3)(a) $f(t) = \sum_i p_i t^i$ , to minimize jerk, we consider $f^{(3)}(t)$

$$f^{(3)}(t) = \sum_{i \geq 3} i(i-1)(i-2)\, t^{i-3} p_i$$

$$(f^{(3)}(t))^2 = \sum_{i \geq 3,\, j \geq 3} i(i-1)(i-2)\, j(j-1)(j-2)\, t^{i+j-6} p_i p_j$$

$$J(T) = \int_0^T (f^{(3)}(t))^2 = \sum_{i \geq 3,\, j \geq 3} \frac{i(i-1)(i-2)\, j(j-1)(j-2)}{i+j-5} T^{i+j-5} p_i p_j$$

$$= \begin{bmatrix} \vdots \\ p_i \\ \vdots \end{bmatrix}^T \begin{bmatrix} & \vdots & \\ \cdots & \frac{i(i-1)(i-2)\, j(j-1)(j-2)}{i+j-5} T^{i+j-5} & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ p_j \\ \vdots \end{bmatrix}$$

$$= p_k^T Q_k p_k$$

Hence $$Q_1 = \begin{bmatrix} & \vdots & \\ \cdots & \frac{i(i-1)(i-2)\, j(j-1)(j-2)}{i+j-5} T_1^{i+j-5} & \cdots \\ & \vdots & \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & T_1 & T_1^2 & T_1^3 & T_1^4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2T_1 & 3T_1^2 & 4T_1^3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6T_1 & 12T_1^2 & 0 & 0 & -2 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow \text{position} \\ \rightarrow \text{velocity} \\ \rightarrow \text{acceleration} \end{array}$$

(c)

$$B = \begin{bmatrix} 1 & T_2 & T_2^2 & T_2^3 & T_2^4 \\ 0 & T_2 & 2T_2 & 3T_2^2 & 4T_2^3 \end{bmatrix} \begin{array}{l} \rightarrow \text{position} \\ \rightarrow \text{velocity} \end{array}$$

4(a) Dijkstra's

(4, 3) → (4, 4) → (4, 5) → (3, 5) → (5, 5) →
(2, 5) → (6, 5) → (1, 5) → (2, 4) → (7, 5) →
(1, 4) → (2, 3) → (8, 5) → (1, 3) → (2, 2) →
(8, 4) → (9, 5) → (1, 2) → (2, 1) → (8, 3) → (9, 4) →
(1, 1) → (3, 1) → (7, 3) → (8, 2) → (9, 3) →
(4, 1) → (6, 3) (Find the goal)

(b) $A^*$

(4, 3) → (4, 4) → (4, 5) → (5, 5) → (6, 5) → (3, 5) →
(7, 5) → (2, 5) → (8, 5) → (2, 4) → (8, 4) → (2, 3) →
(8, 3) → (7, 3) → (6, 3) (find the goal)

(c) $A^*$ can reduce the number of visited note during the searching task, the advantange is from that the heuristic funtion provides a searching direction to speed up the searching process.

(d) ① (4, 3) → (4, 4) → (4, 5) → (5, 5) → (6, 5) →

(7, 5) → (8, 5) → (8, 4) → (8, 3) → (7, 3) →

(6, 3) (find the goal)

②
This method can find the path faster than $A^*$ method by ignoring unnecessary searching

③ The heuristic of each step is larger than the real cost, so the result may not be optimal. for map is larger than 10000. this method will be fail.

(e) Sampling-based planning algorithm finds path by sampling random points in the environment. Heuristic are used to maximize the exploration of the space and bias the direction of search. Once the path between goal and start point is found, the search task is finished.