# Introduction to Aerial Robotics
## Lecture 9

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# Outline

- Extended Kalman Filter
- Augmented State Extended Kalman Filter
- Particle Filter

# The Kalman Filter

# Bayes' Filter

State

Control input

- **Prior:** $p(x_0)$
- **Process model:** $f(x_t \mid x_{t-1}, u_t)$
- **Measurement model:** $g(z_t \mid x_t)$

Measurement

- **Prediction step:**
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int f(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$
- **Update step:**
- $p(x_t \mid z_{1:t}, u_{1:t}) = \dfrac{g(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t})}{\int g(z_t \mid x_t')\, p(x_t' \mid z_{1:t-1}, u_{1:t})\, dx_t'}$

# Assumptions

- The prior state of the robot is represented by a Gaussian distribution
  - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- The process model $f(x_t \mid x_{t-1}, u_t)$ is linear with additive Gaussian white noise
  - $x_t = A_t\, x_{t-1} + B_t\, u_t + n_t$
  - $n_t \sim N(0, Q_t)$
- The measurement model $g(z_t \mid x_t)$ is linear with additive Gaussian white noise
  - $z_t = C_t\, x_t + v_t$
  - $v_t \sim N(0, R_t)$

# Kalman Filter

- Prior:
  - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- Transition model:
  - $x_t = A_t\ x_{t-1} + B_t\ u_t + n_t$
  - $n_t \sim N(0, Q_t)$
- Measurement model:
  - $z_t = C_t\ x_t + v_t$
  - $v_t \sim N(0, R_t)$

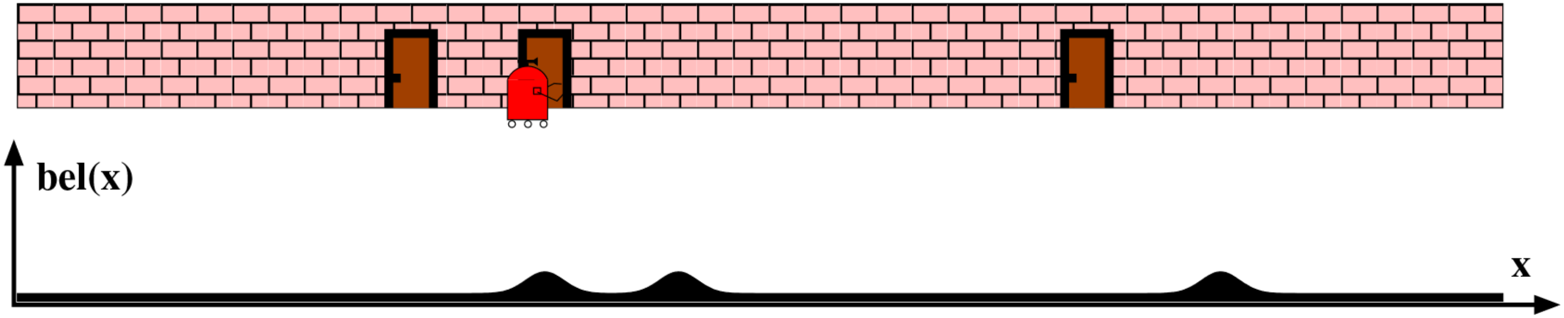
- Prior:
  - $\mu_{t-1}, \Sigma_{t-1}$
- Prediction:
  - $\bar{\mu}_t = A_t\ \mu_{t-1} + B_t\ u_t$
  - $\bar{\Sigma}_t = A_t\ \Sigma_{t-1}\ A_t^T + Q_t$
- Update:
  - $\mu_t = \bar{\mu}_t + K_t(z_t - C_t\ \bar{\mu}_t)$
  - $\Sigma_t = \bar{\Sigma}_t - K_t\ C_t\ \bar{\Sigma}_t$
  - $K_t = \bar{\Sigma}_t\ C_t^T\ (C_t\ \bar{\Sigma}_t\ C_t^T + R_t)^{-1}$

# Continuous Dynamics

- Can convert continuous time systems
- $\dot{x} = f(x, u, n) = A\,x + B\,u + U\,n$
- Into discrete time systems using one-step Euler integration
- $x_t = F\,x_{t-1} + G\,u_t + V\,n_t$
- $F = (I + \delta t\,A), G = \delta t\,B,\ V = \delta t\,U$
- This will introduce some error, but the observations can help correct it
- Prediction:
  - $\bar{\mu}_t = F\,\mu_{t-1} + G\,u_t$
  - $\bar{\Sigma}_t = F\,\Sigma_{t-1} F^T + V\,Q\,V^T$

# Kalman Filter Discussion

- **Advantages:**
  - Simple
  - Purely matrix operations
    - Computationally efficient, even for high dimensional systems
- **Disadvantages:**
  - Assumes everything is linear and Gaussian
  - Unimodal distribution
    - Cannot handle multiple hypotheses



# Extended (to handle nonlinear systems) Kalman Filter

# Assumptions for EKF

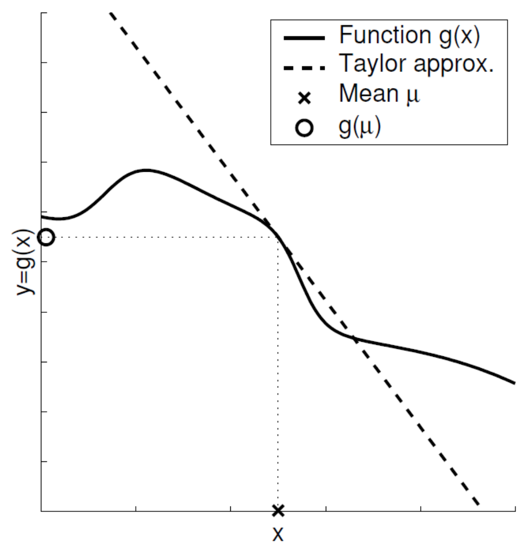
- The prior state of the robot is represented by a Gaussian distribution
  - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- The continuous time process model is:
  - $\dot{x} = f(x, u, n)$
  - $n_t \sim N(0, Q_t)$ is Gaussian white noise
- The measurement model is:
  - $z = g(x, v)$
  - $v_t \sim N(0, R_t)$ is Gaussian white noise

# Prediction

- Process model is nonlinear
- Need to convert the continuous dynamics to a discrete time system
- Look over a finite time interval $\tau = (t', t)$, where $t - t' = \delta t$
  - $t' \rightarrow t - 1$, $\bar{t}$ is an infinitesimal step before $t$
- Options:
  - Integrate the process model over the time horizon $\tau$
    - $x_{\bar{t}} = \Phi(\bar{t}; x_{t-1}, u, n)$
    - Difficult to do in general
  - Use numerical integration
    - One-step Euler integration

# Prediction – Linearization

- Linearize the dynamics about $x = \mu_{t-1},\ u = u_t,\ n = 0$
  - $\dot{x} \approx f(\mu_{t-1}, u_t, 0) + \left.\frac{\partial f}{\partial x}\right|_{\mu_{t-1},u_t,0} (x - \mu_{t-1}) + \left.\frac{\partial f}{\partial u}\right|_{\mu_{t-1},u_t,0} (u - u_t) + \left.\frac{\partial f}{\partial n}\right|_{\mu_{t-1},u_t,0} (n - 0)$
- Let:
  - $A_t = \left.\frac{\partial f}{\partial x}\right|_{\mu_{t-1},u_t,0}$
  - $B_t = \left.\frac{\partial f}{\partial u}\right|_{\mu_{t-1},u_t,0}$
  - $U_t = \left.\frac{\partial f}{\partial n}\right|_{\mu_{t-1},u_t,0}$




- Linear dynamics:
  - $\dot{x} \approx f(\mu_{t-1}, u_t, 0) + A_t(x - \mu_{t-1}) + B_t(u - u_t) + U_t(n - 0)$

# Prediction – Discrete Time

- One-step Euler integration
  - $x_{\bar{t}} \approx x_{t-1} + f(x_{t-1}, u_t, n_t)\,\delta t$
  - $\approx x_{t-1} + \delta t\, f(\mu_{t-1}, u_t, 0) + \delta t\, A_t\,(x_{t-1} - \mu_{t-1}) + \delta t\, B_t(u_t - u_t)$
    $+ \delta t\, U_t(n_t - 0)$
  - $\approx x_{t-1} + \delta t\, f(\mu_{t-1}, u_t, 0) + \delta t\, A_t\,(x_{t-1} - \mu_{t-1}) + \delta t\, U_t(n_t - 0)$
  - $\approx (I + \delta t\, A_t)\, x_{t-1} + \delta t\, U_t\, n_t + \delta t(f(\mu_{t-1}, u_t, 0) - A_t\, \mu_{t-1})$
  - $\approx F_t\, x_{t-1} + V_t\, n_t + \delta t(f(\mu_{t-1}, u_t, 0) - A_t\, \mu_{t-1})$
- Prediction:
  - $\bar{\mu}_t = \mu_{t-1} + \delta t\, f(\mu_{t-1}, u_t, 0)$
  - $\bar{\Sigma}_t = F_t\, \Sigma_{t-1}\, F_t^T + V_t\, Q_t\, V_t^T$

# Update – Linearization

- Linearize the measurement model about $x = \bar{\mu}_t,\ v = 0$
  - $g(x, v) \approx g(\bar{\mu}_t, 0) + \left.\frac{\partial g}{\partial x}\right|_{\bar{\mu}_t,0} (x - \bar{\mu}_t) + \left.\frac{\partial g}{\partial v}\right|_{\bar{\mu}_t,0} (v - 0)$
- Let:
  - $C_t = \left.\frac{\partial g}{\partial x}\right|_{\bar{\mu}_t,0}$
  - $W_t = \left.\frac{\partial g}{\partial v}\right|_{\bar{\mu}_t,0}$
- Linear observation model:
  - $z_t = g(x_t, v_t) \approx g(\bar{\mu}_t, 0) + C_t\,(x_t - \bar{\mu}_t) + W_t\, v_t$

# Update

- Follow the same derivation as the Kalman Filter
- $\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_t & W_t \end{bmatrix} \begin{bmatrix} x_{\bar{t}} \\ v_t \end{bmatrix} + \begin{bmatrix} 0 \\ g(\bar{\mu}_t, 0) - C_t \bar{\mu}_t \end{bmatrix}$
- Mean:
  - $E[X_t] = E[\bar{X}_t] = \bar{\mu}_t$
  - $E[Z_t] = E[C_t \bar{X}_t + W_t V_t + g(\bar{\mu}_t, 0) - C_t \bar{\mu}_t]$
  - $\qquad = C_t \bar{\mu}_t + g(\bar{\mu}_t, 0) - C_t \bar{\mu}_t$
  - $\qquad = g(\bar{\mu}_t, 0)$

# Update

- Follow the same derivation as the Kalman Filter
- $\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_t & W_t \end{bmatrix} \begin{bmatrix} x_{\bar{t}} \\ v_t \end{bmatrix} + \begin{bmatrix} 0 \\ g(\bar{\mu}_t, 0) - C_t \bar{\mu}_t \end{bmatrix}$

- Covariance:
  - $\Sigma = \begin{bmatrix} I & 0 \\ C_t & W_t \end{bmatrix} \begin{bmatrix} \bar{\Sigma}_t & 0 \\ 0 & R_t \end{bmatrix} \begin{bmatrix} I & C_t^T \\ 0 & W_t^T \end{bmatrix}$
  - $= \begin{bmatrix} \bar{\Sigma}_t & \bar{\Sigma}_t C_t^T \\ C_t \bar{\Sigma}_t & C_t \bar{\Sigma}_t C_t^T + W_t R_t W_t^T \end{bmatrix}$

# Update

- Recall that for a multivariate Guassian $Y = \begin{bmatrix} X \\ Z \end{bmatrix}$ with mean $\mu = \begin{bmatrix} \mu_X \\ \mu_Z \end{bmatrix}$ and covariance $\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XZ} \\ \Sigma_{ZX} & \Sigma_{ZZ} \end{bmatrix}$
- The conditional density $f_{X|Z}(x \mid Z = z)$ is Gaussian with
  - $\mu_{X|Z} = \mu_X + \Sigma_{XZ}\Sigma_{ZZ}^{-1}(z - \mu_Z)$
  - $\Sigma_{X|Z} = \Sigma_{XX} - \Sigma_{XZ}\,\Sigma_{ZZ}^{-1}\,\Sigma_{ZX}$
- Result:
  - $\mu_t = \bar{\mu}_t + K_t\,(z_t - g(\bar{\mu}_t, 0))$
  - $\Sigma_t = \bar{\Sigma}_t - K_t\,C_t\,\bar{\Sigma}_t$
  - $K_t = \bar{\Sigma}_t\,C_t^T\,(C_t\,\bar{\Sigma}_t\,C_t^T + W_t\,R\,W_t^T)^{-1}$

# Extended Kalman Filter

- Prediction step:
  - $\bar{\mu}_t = \mu_{t-1} + \delta t \, f(\mu_{t-1}, u_t, 0)$
  - $\bar{\Sigma}_t = F_t \, \Sigma_{t-1} \, F_t^T + V_t \, Q_t \, V_t^T$

  $$\left.\begin{array}{l} - \; \dot{x} = f(x, u, n) \\ - \; n_t \sim N(0, Q_t) \end{array}\right\} \text{Assumptions}$$

  $$\left.\begin{array}{l} - \; A_t = \left.\frac{\partial f}{\partial x}\right|_{\mu_{t-1}, u_t, 0} \\ - \; U_t = \left.\frac{\partial f}{\partial n}\right|_{\mu_{t-1}, u_t, 0} \end{array}\right\} \text{Linearization}$$

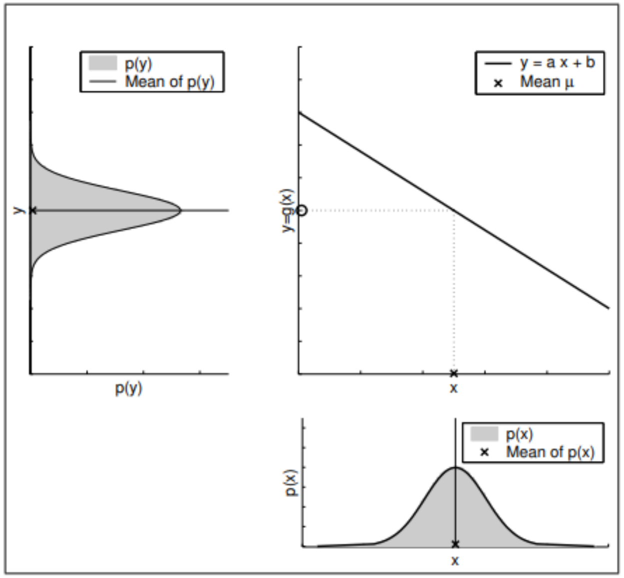
  $$\left.\begin{array}{l} - \; F_t = I + \delta t \, A_t \\ - \; V_t = \delta t \, U_t \end{array}\right\} \text{Discretization}$$

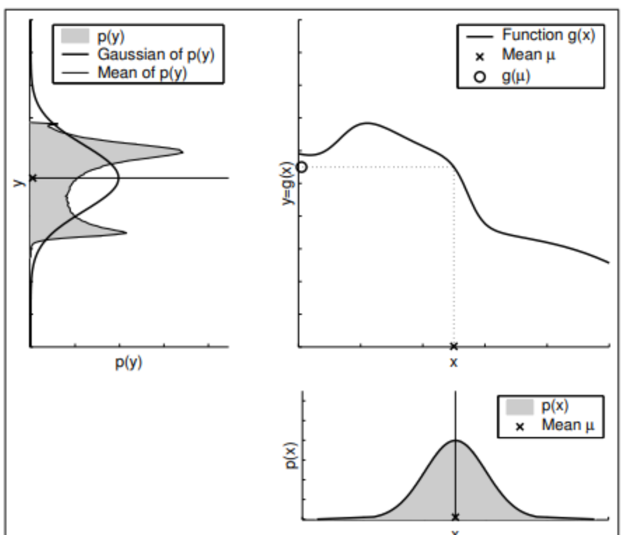
- Update step:
  - $\mu_t = \bar{\mu}_t + K_t \, (z_t - g(\bar{\mu}_t, 0))$
  - $\Sigma_t = \bar{\Sigma}_t - K_t \, C_t \, \bar{\Sigma}_t$
  - $K_t = \bar{\Sigma}_t \, C_t^T \, (C_t \, \bar{\Sigma}_t \, C_t^T + W_t \, R \, W_t^T)^{-1}$

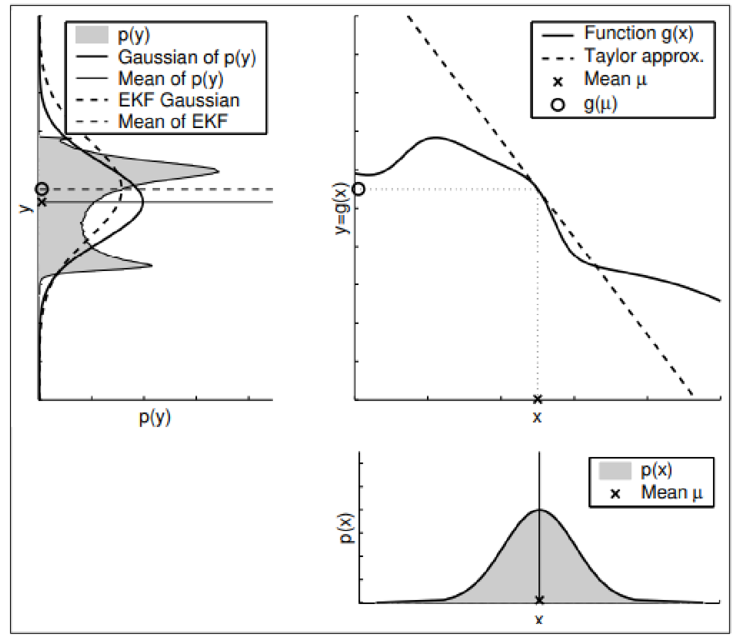
  $$\left.\begin{array}{l} - \; z_t = g(x_t, v_t) \\ - \; v_t \sim N(0, R_t) \end{array}\right\} \text{Assumptions}$$

  $$\left.\begin{array}{l} - \; C_t = \left.\frac{\partial g}{\partial x}\right|_{\bar{\mu}_t, 0} \\ - \; W_t = \left.\frac{\partial g}{\partial v}\right|_{\bar{\mu}_t, 0} \end{array}\right\} \text{Linearization}$$

# More on Linearization

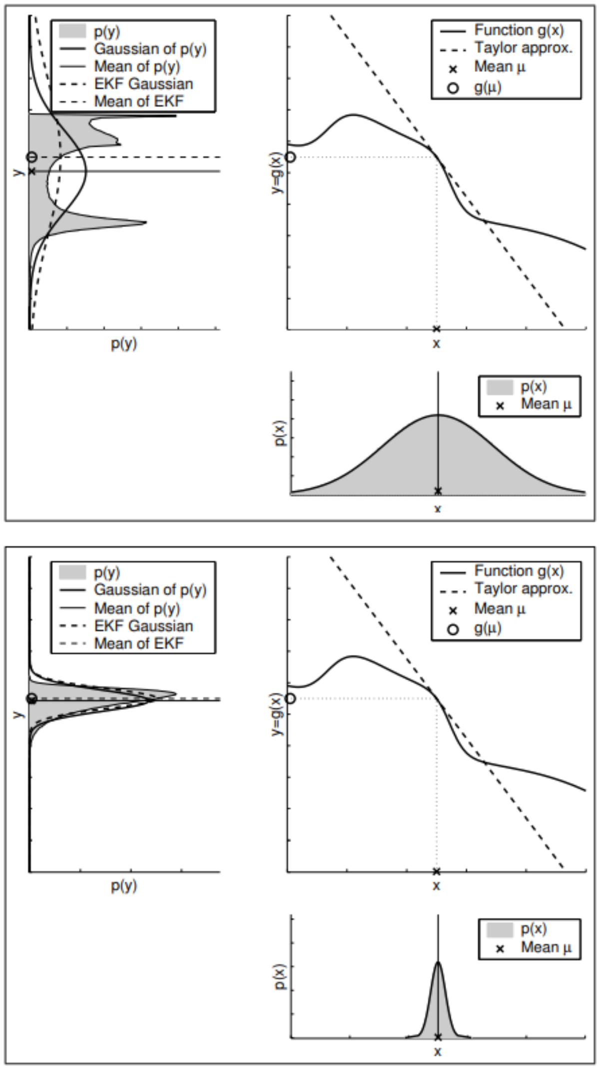


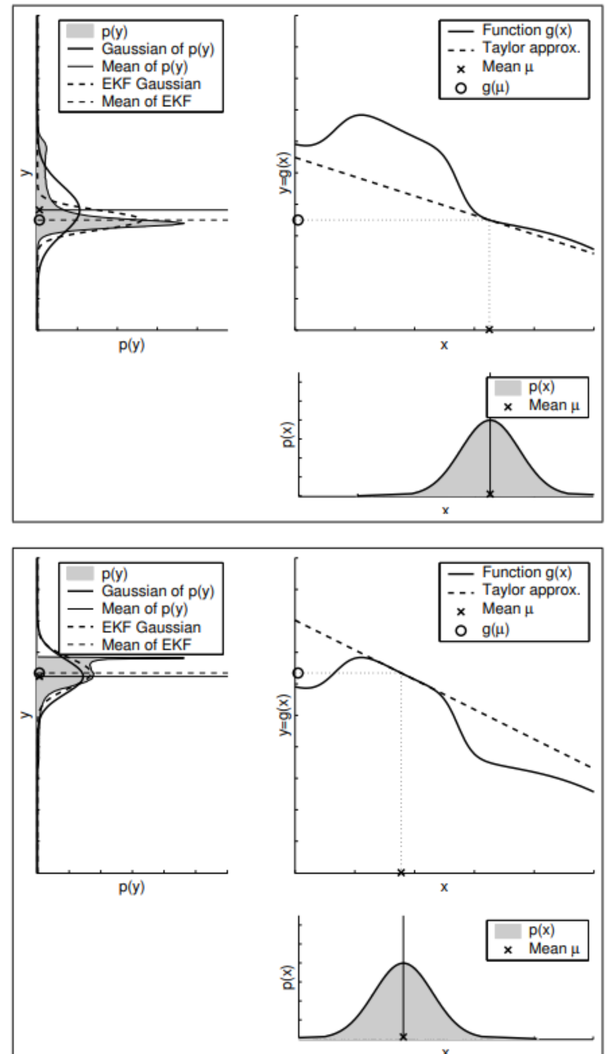




EKF aims to generate Gaussian Approximation of the random variable under nonlinear function

# More on Linearization



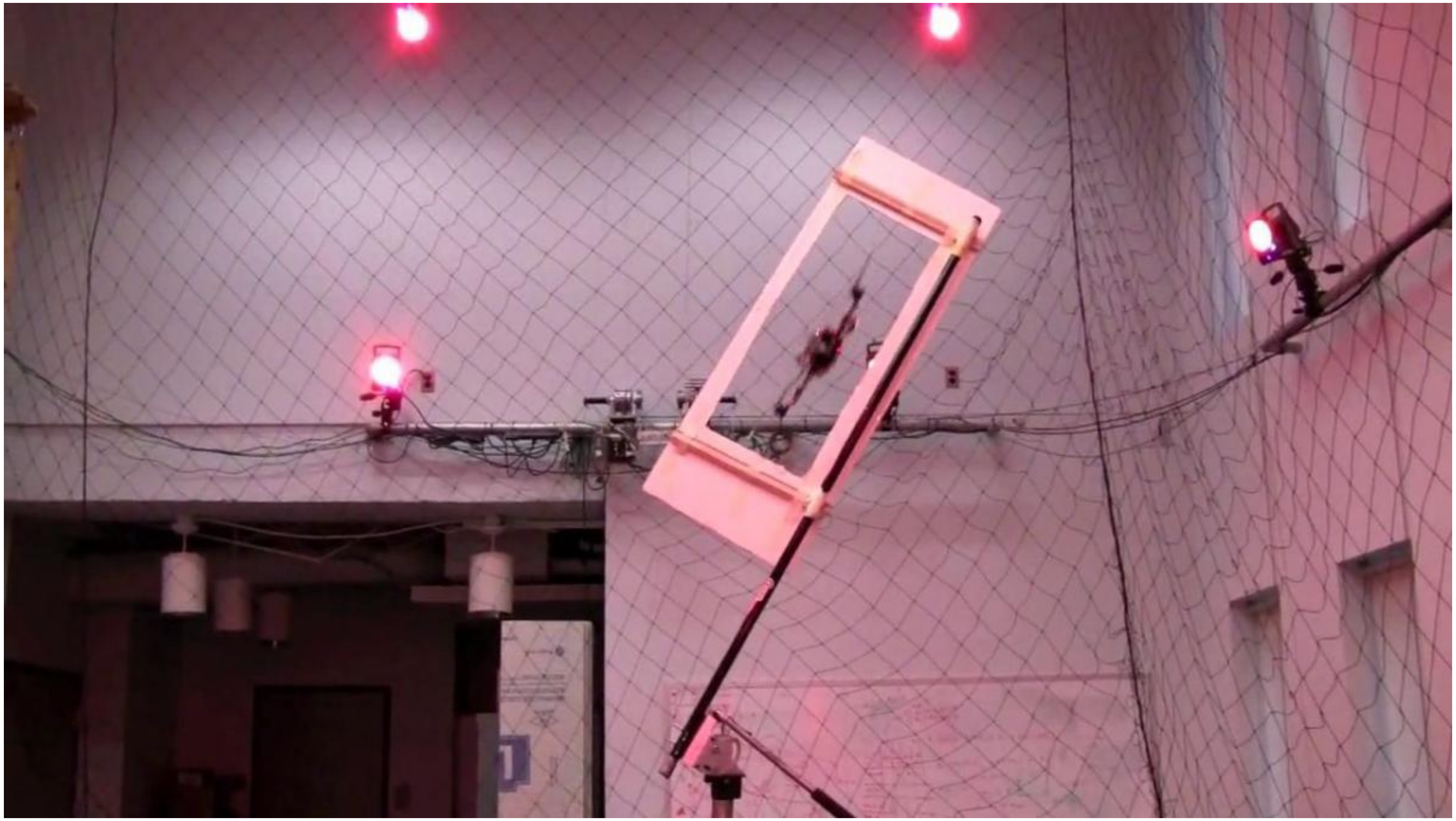

Different uncertainties of the random variable




Different nonlinearities of the function

# Example Problem

# Quadrotor with a Good Velocity Sensor

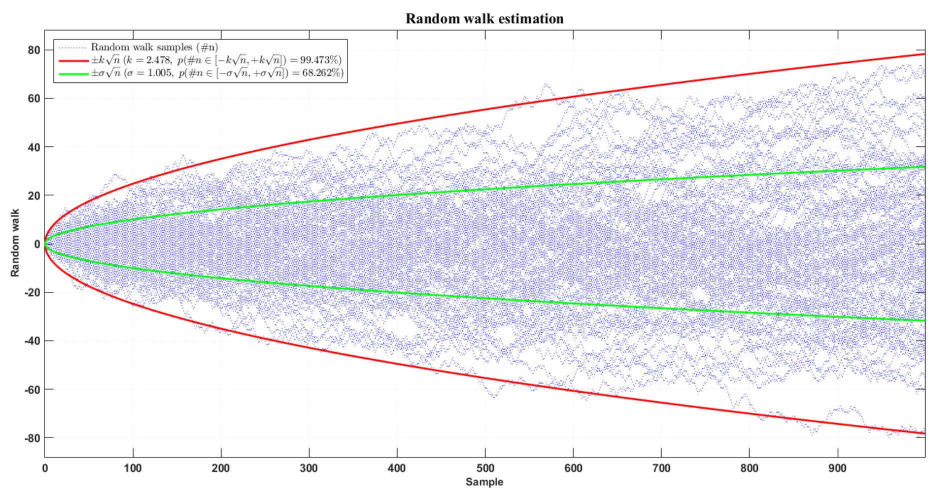


# State

- Can estimate the commanded linear velocity using the motion tracking system and the angular velocity using a gyroscope
- $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{b}_g \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{orientation} \\ \text{gyroscope bias} \end{bmatrix} \in \mathbf{R}^9$
- Use Z-X-Y Euler angle parameterization of $SO(3)$ for orientation
  - $\mathbf{q} = [\phi, \theta, \psi]^T = [\text{roll}, \text{pitch}, \text{yaw}]^T$
  - $\boldsymbol{R} = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\psi c\theta s\phi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$
  - Quaternion-based rotation representation is also possible and is better, refer to your L2 supplementary slides for details

# Process Model

- **Assumption:** the motion tracking system gives a noisy estimate of the linear velocity
  - $\mathbf{v}_m = \dot{\mathbf{p}} + \mathbf{n}_v$
- **Assumption:** the gyroscope gives a noisy estimate of the angular velocity
  - $\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_g + \mathbf{n}_g$
- **Assumption:** the drift in the gyroscope bias is described by a Gaussian, white noise process
  - $\dot{\mathbf{b}}_g = \mathbf{n}_{bg}$
  - $\mathbf{n}_{bg} \sim N(0, Q_g)$



# Process Model

- $\boldsymbol{\omega}_m$ is in the body frame, $\mathbf{q}$ is in the world frame
- **Recall:** the angular velocity in the body frame is given by
- $\boldsymbol{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -c\phi s\theta \\ 0 & 1 & s\phi \\ s\theta & 0 & c\phi c\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = G(\mathbf{q})\dot{\mathbf{q}}$
- Use mocap and gyroscope measurements as process input $u = [\mathbf{v}_m, \boldsymbol{\omega}_m]$
- Process model:
- $\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v}_m - \mathbf{n}_v \\ G(\mathbf{x}_2)^{-1}(\boldsymbol{\omega}_m - \mathbf{x}_3 - \mathbf{n}_g) \\ \mathbf{n}_{bg} \end{bmatrix}$
- How to obtain the covariance matrix for $\mathbf{n}_g$ and $\mathbf{n}_{bg}$?
  - Recall the definition of diagnostic and causal information (L8)
  - Sensor characterization using specialized setup

# Measurement Model

- Use a camera to measure the pose of the robot
- Use theory of projective geometry
- Can estimate the position and orientation of the robot using a minimum of 4 features on the ground plane, e.g., utilizing markers
  - Can recover $\mathbf{q}$ from the rotation matrix $\boldsymbol{R}$
- $$\mathbf{z} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} + \mathbf{v}$$
  $$= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{b}_g \end{bmatrix} + \mathbf{v}$$
  $$= C\,\mathbf{x} + \mathbf{v}$$
- How to obtain the covariance matrix for $\mathbf{v}$?
  - Utilize your evaluation results from Project 2 Phase 1 w.r.t. mocap

# Quadrotor with a Good Acceleration Sensor

# State

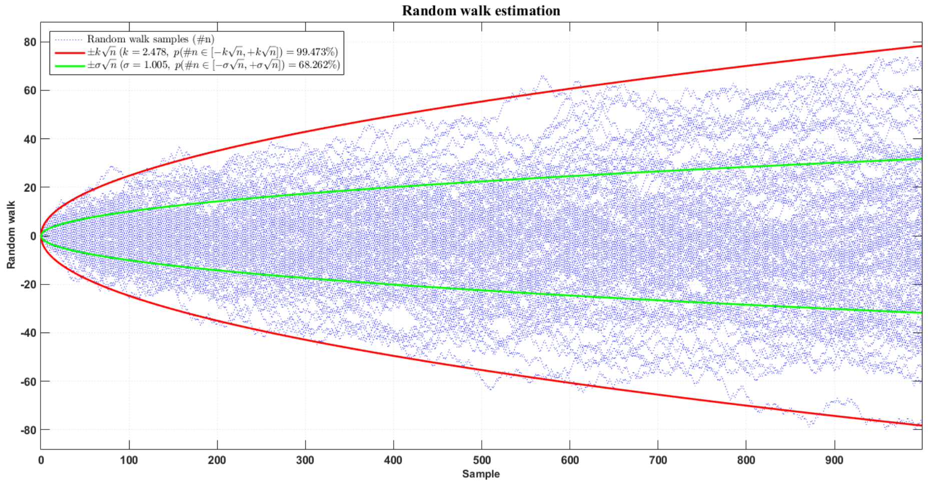
- Can estimate the commanded linear acceleration and angular velocity using the IMU

- $$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \dot{\mathbf{p}} \\ \mathbf{b}_g \\ \mathbf{b}_a \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{orientation} \\ \text{linear velocity} \\ \text{gyroscope bias} \\ \text{accelerometer bias} \end{bmatrix} \in \mathbf{R}^{15}$$

- Use Z-X-Y Euler angle parameterization of $SO(3)$ for orientation
  - $\mathbf{q} = [\phi, \theta, \psi]^T = [\text{roll}, \text{pitch}, \text{yaw}]^T$
  - $$\boldsymbol{R} = \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\phi s\theta & c\phi c\psi & s\psi s\theta - c\psi c\theta s\phi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix}$$

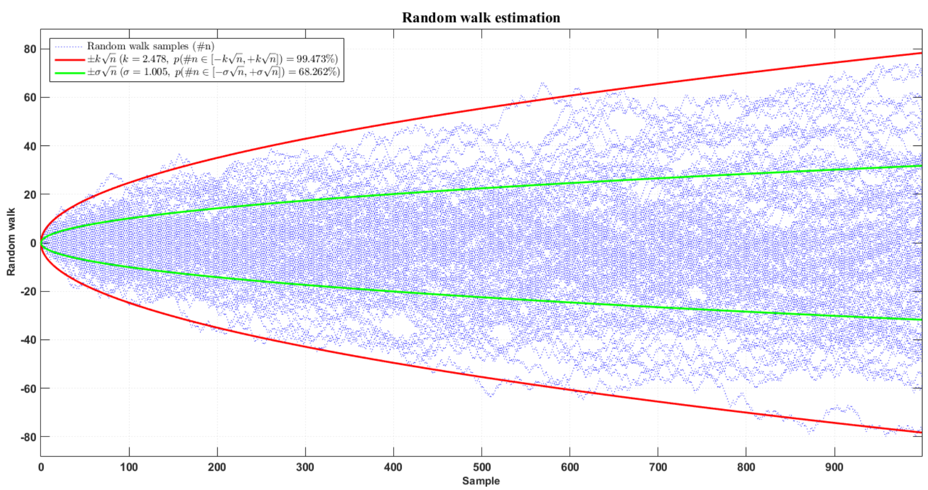
# Process Model – Gyroscope

- **Assumption:** the gyroscope gives a noisy estimate of the angular velocity
  - $\boldsymbol{\omega}_m = \boldsymbol{\omega} + \mathbf{b}_g + \mathbf{n}_g$
- **Assumption:** the drift in the gyroscope bias is described by a Gaussian, white noise process
  - $\dot{\mathbf{b}}_g = \mathbf{n}_{bg}$
  - $\mathbf{n}_{bg} \sim N(0, Q_g)$



# Process Model – Accelerometer

- **Assumption:** the accelerometer gives a noisy estimate of the linear acceleration
  - $\mathbf{a}_m = \boldsymbol{R}(\mathbf{q})^T(\ddot{\mathbf{p}} - \mathbf{g}) + \mathbf{b}_a + \mathbf{n}_a$
- **Assumption:** the drift in the accelerometer bias is described by a Gaussian, white noise process
  - $\dot{\mathbf{b}}_a = \mathbf{n}_{ba}$
  - $\mathbf{n}_{ba} \sim N(0, Q_a)$



# Process Model

- Process model:
- $\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_3 \\ G(\mathbf{x}_2)^{-1}(\omega_m - \mathbf{x}_4 - \mathbf{n}_g) \\ \mathbf{g} + R(\mathbf{x}_2)(\mathbf{a}_m - \mathbf{x}_5 - \mathbf{n}_a) \\ \mathbf{n}_{bg} \\ \mathbf{n}_{ba} \end{bmatrix}$

# Measurement Model

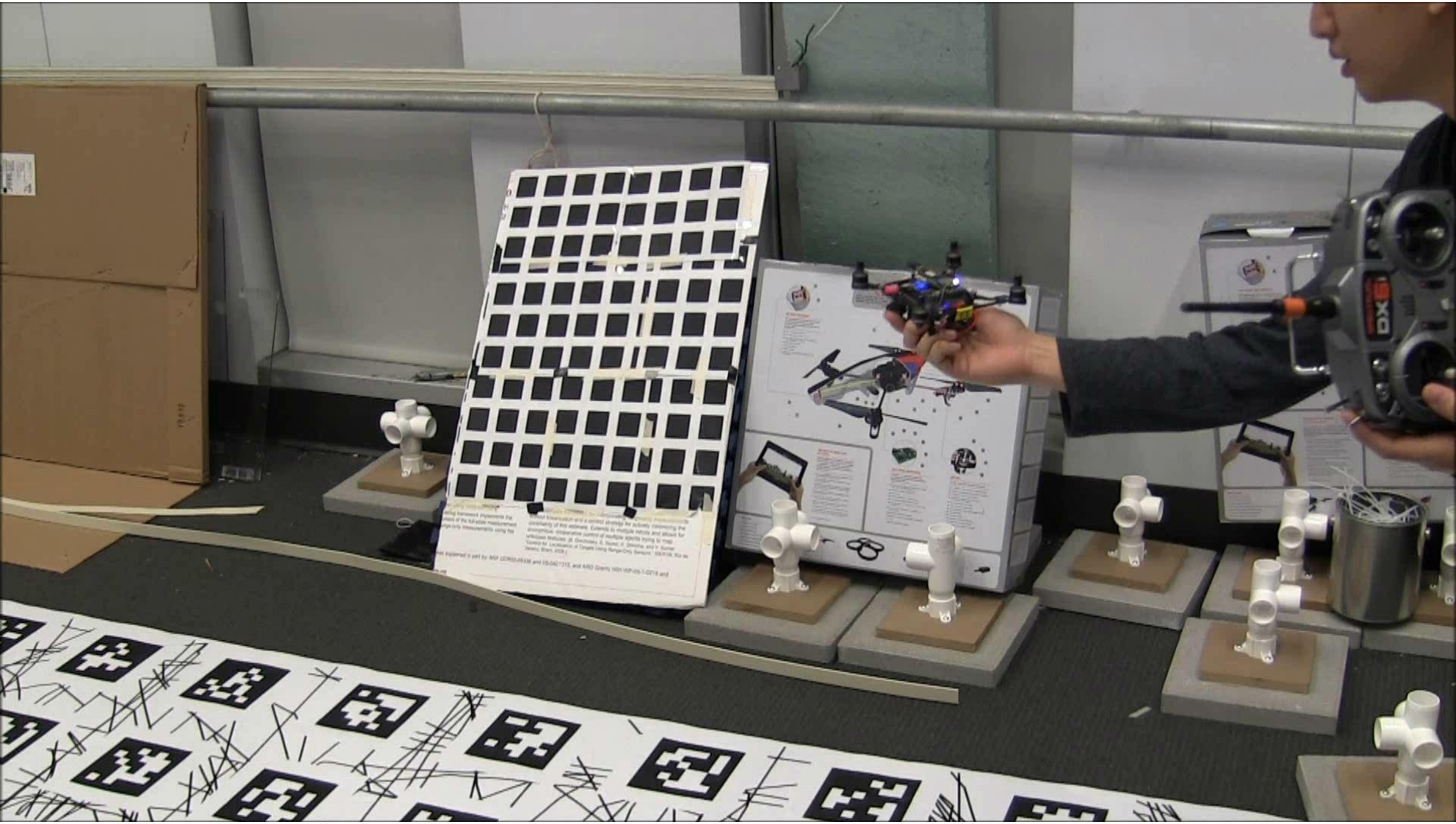
- Use a camera to measure:
  - The pose of the robot (using markers)
- $\mathbf{z} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \end{bmatrix} + \mathbf{v}$

$$= \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \dot{\mathbf{p}} \\ \mathbf{b}_g \\ \mathbf{b}_a \end{bmatrix} + \mathbf{v}$$

$$= C\,\mathbf{x} + \mathbf{v}$$

# Measurement Model

- Use a camera to measure:
  - The pose of the robot (using markers)
  - The body frame linear velocity (using optical flow)
  - Multi-sensor fusion, robust to single sensor failure
  - May also split it into two measurement models
- $$\begin{aligned}\mathbf{z} &= \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \boldsymbol{R}(\mathbf{q})^T \dot{\mathbf{p}} \end{bmatrix} + \mathbf{v} = \\ &= \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \boldsymbol{R}(\mathbf{x}_2)^T \mathbf{x}_3 \end{bmatrix} + \mathbf{v} \\ &= g(\mathbf{x}, \mathbf{v})\end{aligned}$$



# Augmented State EKF for Fusing Relative Measurements

# Motivation

- In some situations, only relative state measurements are available.
  - Project 2 Phase 2: keyframe-based visual odomerty
  - Relative pose between current frame and the latest keyframe
- The measurement depends on the current state and a previous state.
- The measurement model is:
  - $z_{t|t_i} = g(x_t, x_{t_i}, v_{t|t_i})$
  - $v_{t|t_i} \sim N(0, R_t)$ is Gaussian white noise
  - Where $t_i$ is the time that the keyframe is generated
- However, this violates the Markov assumption. How to deal with it?

# Augmented State and Covariance

- Copy the part of the original state $\mathbf{x} \in \mathbb{R}^n$ affected by the measurements ($\mathbf{x}_{t_i}^i \in \mathbb{R}^{n_i}, n_i < n$) and augment the original state, where $t_i$ is the timestamp of the previous state. There are at total $m$ augmented states.

- The full state vector with $i$ augmented states
  - $\check{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_{t_1}^1 \\ \vdots \\ \mathbf{x}_{t_m}^m \end{bmatrix}$

- The full covariance matrix with $i$ augmented states
  - $\check{\boldsymbol{\Sigma}} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathbf{x}\mathbf{x}} & \boldsymbol{\Sigma}^{\mathbf{x}\mathbf{x}_{t_1}^1} & \cdots & \boldsymbol{\Sigma}^{\mathbf{x}\mathbf{x}_{t_m}^m} \\ \boldsymbol{\Sigma}^{\mathbf{x}_{t_1}^1\mathbf{x}} & \boldsymbol{\Sigma}^{\mathbf{x}_{t_1}^1\mathbf{x}_{t_1}^1} & & \boldsymbol{\Sigma}^{\mathbf{x}_{t_1}^1\mathbf{x}_{t_m}^m} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Sigma}^{\mathbf{x}_{t_m}^m\mathbf{x}} & \boldsymbol{\Sigma}^{\mathbf{x}_{t_m}^m\mathbf{x}_{t_1}^1} & \cdots & \boldsymbol{\Sigma}^{\mathbf{x}_{t_m}^m\mathbf{x}_{t_m}^m} \end{bmatrix}$

# State Augmentation and Removal

- Binary selection matrix $\mathbf{B}_i$ to select part of the original state
  - $\mathbf{x}_{t_i}^i = \mathbf{B}_i\mathbf{x}$
- State augmentation operator $\mathbf{M}^+$
  - $m$ augmented states already exists, adding the $m+1$ augmented state
  - $\mathbf{M}^+ = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times\sum_{k=1}^m n_k} \\ \mathbf{0}_{\sum_{k=1}^m n_k \times n} & \mathbf{I}_{\sum_{k=1}^m n_k} \\ \mathbf{B}_{m+1} & \mathbf{0}_{n_{m+1}\times\sum_{k=1}^m n_k} \end{bmatrix}$
- State removal operator $\mathbf{M}^-$ of an augmented state $\mathbf{x}_{t_j}^j$
  - $\mathbf{M}^- = \begin{bmatrix} \mathbf{I}_{\boldsymbol{a}} & \mathbf{0}_{a\times n_j} & \mathbf{0}_{a\times b} \\ \mathbf{0}_{b\times a} & \mathbf{0}_{b\times n_j} & \mathbf{I}_{\boldsymbol{b}} \end{bmatrix}$
  - $a = n + \sum_{k=1}^{j-1} n_k$
  - $\mathrm{b} = \sum_{k=j+1}^{m} n_k$
- The updated state vector and covariance matrix
  - $\check{\mathbf{x}}^{\pm} = \mathbf{M}^{\pm}\check{\mathbf{x}}$
  - $\check{\boldsymbol{\Sigma}}^{\pm} = \mathbf{M}^{\pm}\check{\boldsymbol{\Sigma}}\mathbf{M}^{\pm T}$

# Prediction for Augmented State EKF

- For the system
  - $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{n})$
  - $\boldsymbol{n}_t \sim N(\mathbf{0}, \boldsymbol{Q}_t)$ is Gaussian white noise

- Recall the prediction of the original EKF:
  - $\bar{\mathbf{x}}_t = \mathbf{x}_{t-1} + \delta t\, f(\mathbf{x}_{t-1}, \boldsymbol{u}_t, \mathbf{0})$
  - $\bar{\boldsymbol{\Sigma}}_t = \boldsymbol{F}_t \boldsymbol{\Sigma}_{t-1} \boldsymbol{F}_t^T + \boldsymbol{V}_t\, \boldsymbol{Q}_t\, \boldsymbol{V}_t^T$

# Prediction for Augmented State EKF

- Prediction only affect the main state, by separating the main state and the augmented states:
  - $\check{\mathbf{x}}_{t-1} = \begin{bmatrix} \mathbf{x}_{t-1} \\ \mathbf{x}^{\mathrm{Aug}}_{t-1} \end{bmatrix}$
  - $\check{\boldsymbol{\Sigma}}_{t-1} = \begin{bmatrix} \boldsymbol{\Sigma}^{\mathbf{xx}}_{t-1} & \boldsymbol{\Sigma}^{\mathbf{x}\mathbf{x}^{\mathrm{Aug}}}_{t-1} \\ \boldsymbol{\Sigma}^{\mathbf{x}^{\mathrm{Aug}}\mathbf{x}}_{t-1} & \boldsymbol{\Sigma}^{\mathbf{x}^{\mathrm{Aug}}\mathbf{x}^{\mathrm{Aug}}}_{t-1} \end{bmatrix}$

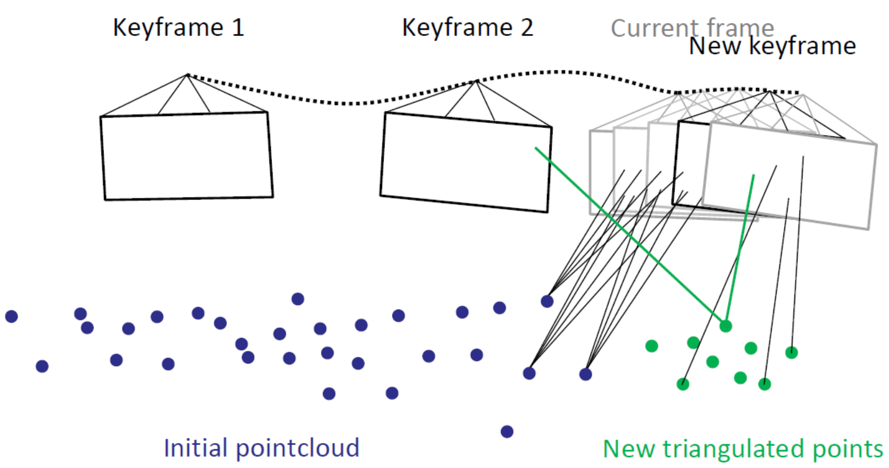
- Using the result of the prediction of the main state:
  - $\bar{\check{\mathbf{x}}}_t = \begin{bmatrix} \bar{\mathbf{x}}_t \\ \mathbf{x}^{\mathrm{Aug}}_{t-1} \end{bmatrix}$ (augmented states remain unchanged during prediction)
  - $\bar{\check{\boldsymbol{\Sigma}}}_t = \begin{bmatrix} \bar{\boldsymbol{\Sigma}}^{\mathbf{xx}}_t & \boldsymbol{F}_t \boldsymbol{\Sigma}^{\mathbf{x}\mathbf{x}^{\mathrm{Aug}}}_{t-1} \\ \boldsymbol{\Sigma}^{\mathbf{x}^{\mathrm{Aug}}\mathbf{x}}_{t-1} \boldsymbol{F}_t^{\,T} & \boldsymbol{\Sigma}^{\mathbf{x}^{\mathrm{Aug}}\mathbf{x}^{\mathrm{Aug}}}_{t-1} \end{bmatrix}$

# Update for Augmented State EKF

- For a relative measurement at $t$ with respect to $t_i$
  - $\mathbf{z}_{t|t_i} = g(\mathbf{x}_t, \mathbf{x}_{t_i}, \boldsymbol{v}_{t|t_i})$
  - $\boldsymbol{v}_{t|t_i} \sim N(\mathbf{0}, \mathbf{R}_t)$ is Gaussian white noise
- After linearization
  - $\mathbf{z}_{t|t_i} \approx g\left(\bar{\mathbf{x}}_t, \mathbf{x}_{t_i}, 0\right) + \boldsymbol{C}_t(\check{\mathbf{x}}_t - \bar{\check{\mathbf{x}}}_t) + \mathbf{W}_t \boldsymbol{v}_{t|t_i}$
  - $\boldsymbol{C}_t = \left[\frac{\partial g}{\partial \mathbf{x}_t}\Big|_{\bar{\mathbf{x}}_t}, \mathbf{0}, \frac{\partial g}{\partial \mathbf{x}_{t_i}}\Big|_{\mathbf{x}_{t_i}}, \mathbf{0}\right]$
- Update as the original EKF
  - $\boldsymbol{K}_t = \bar{\check{\boldsymbol{\Sigma}}}_t \boldsymbol{C}_t^T \left(\boldsymbol{C}_t \bar{\check{\boldsymbol{\Sigma}}}_t \boldsymbol{C}_t^T + \mathbf{W}_t \mathbf{R}_t \mathbf{W}_t^T\right)^{-1}$
  - $\check{\mathbf{x}}_t = \bar{\check{\mathbf{x}}}_t + \boldsymbol{K}_t(\mathbf{z}_{t|t_i} - g(\mathbf{x}_t, \mathbf{x}_{t_i}, \mathbf{0}))$
  - $\check{\boldsymbol{\Sigma}}_t = \bar{\check{\boldsymbol{\Sigma}}}_t - \boldsymbol{K}_t \boldsymbol{C}_t \bar{\check{\boldsymbol{\Sigma}}}_t$

# Example Problem

# Quadrotor with a Good Acceleration Sensor and Keyframe-based Visual Odometry



# State

- To fuse the relative pose measurement from single-keyframe visual odometry, the state is augmented as:
- $\check{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \\ \mathbf{x}_{\mathrm{K}_1} \\ \mathbf{x}_{\mathrm{K}_2} \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \mathbf{q} \\ \dot{\mathbf{p}} \\ \mathbf{b}_g \\ \mathbf{b}_{\boldsymbol{a}} \\ \mathbf{p}_{\mathrm{K}} \\ \mathbf{q}_{\mathrm{K}} \end{bmatrix} = \begin{bmatrix} \text{position} \\ \text{orientation} \\ \text{linear velocity} \\ \text{gyroscope bias} \\ \text{accelerometer bias} \\ \text{keyframe position} \\ \text{keyframe orientation} \end{bmatrix} \in \mathbf{R}^{21}$

# Process Model

- The process model is:
- $$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_3 \\ G(\mathbf{x}_2)^{-1}\left(\omega_m - \mathbf{x}_4 - \mathbf{n}_g\right) \\ \mathbf{g} + \boldsymbol{R}(\mathbf{x}_2)(\mathbf{a}_m - \mathbf{x}_5 - \mathbf{n}_a) \\ \mathbf{n}_{bg} \\ \mathbf{n}_{ba} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$
- The main state evolve as normal, while the augmented states stay unchanged

# Measurement Model

- The measurement model of the relative transform w.r.t. the keyframe (of the visual odometry) is
- $\mathbf{z}_{t|t_i} = \begin{bmatrix} \mathbf{R}(\mathbf{q}_{\mathrm{K}})^T(\mathbf{p} - \mathbf{p}_{\mathrm{K}}) \\ \mathrm{euler}(\mathbf{R}(\mathbf{q}_{\mathrm{K}})^T\mathbf{R}(\mathbf{q})) \end{bmatrix} + \boldsymbol{v}_{t|t_i} = \begin{bmatrix} \mathbf{R}\left(\mathbf{x}_{\mathrm{K}_2}\right)^T(\mathbf{x}_1 - \mathbf{x}_{\mathrm{K}_1}) \\ \mathrm{euler}\left(\mathbf{R}\left(\mathbf{x}_{\mathrm{K}_2}\right)^T\mathbf{R}(\mathbf{x}_2)\right) \end{bmatrix} + \boldsymbol{v}_{t|t_i}$
- The relative position $\mathbf{R}(\mathbf{q}_{\mathrm{K}})^T(\mathbf{p} - \mathbf{p}_{\mathrm{K}})$ is expressed in the camera frame associated with the keyframe, and so is the relative rotation $\mathbf{R}(\mathbf{q}_{\mathrm{K}})^T\mathbf{R}(\mathbf{q})$
- The function $\mathrm{euler}(\mathbf{R})$ converts a rotation matrix into Euler angles
- Linearization is left for your own exercise

# Changing Keyframe

- When the keyframe is changed, the augmented state of the old keyframe is removed:
  - $\mathbf{M}^{-} = \begin{bmatrix} \mathbf{I}_{15} & \mathbf{0}_{15\times 6} \\ \mathbf{0}_{6\times 15} & \mathbf{0}_{6\times 6} \end{bmatrix}$
  - $\check{\mathbf{x}}^{-} = \mathbf{M}^{-}\check{\mathbf{x}}$
  - $\check{\boldsymbol{\Sigma}}^{-} = \check{\boldsymbol{\Sigma}} = \mathbf{M}^{-}\check{\boldsymbol{\Sigma}}\mathbf{M}^{-T}$

- Then the augmented states of the new keyframe is added:
  - $\mathbf{M}^{+} = \begin{bmatrix} \mathbf{I}_{15} \\ \mathbf{I}_{6\times 15} \end{bmatrix}$
  - $\check{\mathbf{x}}_{\text{new}} = \mathbf{M}^{+}\check{\mathbf{x}}^{-}$
  - $\check{\boldsymbol{\Sigma}}_{\text{new}} = \check{\boldsymbol{\Sigma}}^{+} = \mathbf{M}^{+}\check{\boldsymbol{\Sigma}}^{-}\mathbf{M}^{+T}$

# Recap

# Bayes' Filter

State

Control input

- **Prior:** $p(x_0)$
- **Process model:** $f(x_t \mid x_{t-1}, u_t)$
- **Measurement model:** $g(z_t \mid x_t)$

Measurement

- **Prediction step:**
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int f(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$
- **Update step:**
- $p(x_t \mid z_{1:t}, u_{1:t}) = \dfrac{g(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t})}{\int g(z_t \mid x'_t)\, p(x'_t \mid z_{1:t-1}, u_{1:t})\, dx'_t}$

# Assumptions

- The prior state of the robot is represented by a Gaussian distribution
  - $p(x_0) \sim N(\mu_0, \Sigma_0)$
- The continuous time process model is:
  - $\dot{x} = f(x, u, n)$
  - $n_t \sim N(0, Q_t)$ is Gaussian white noise
- The observation model is:
  - $z = h(x, v)$
  - $v_t \sim N(0, R_t)$ is Gaussian white noise

# Extended Kalman Filter

- Prediction step:
  - $\bar{\mu}_t = \mu_{t-1} + \delta t \, f(\mu_{t-1}, u_t, 0)$
  - $\bar{\Sigma}_t = F_t \, \Sigma_{t-1} \, F_t^T + V_t \, Q_t \, V_t^T$

  - $\dot{x} = f(x, u, n)$
  - $n_t \sim N(0, Q_t)$

  Assumptions

  - $A_t = \left.\frac{\partial f}{\partial x}\right|_{\mu_{t-1}, u_t, 0}$
  - $U_t = \left.\frac{\partial f}{\partial n}\right|_{\mu_{t-1}, u_t, 0}$

  Linearization

  - $F_t = I + \delta t \, A_t$
  - $V_t = \delta t \, U_t$

  Discretization

- Update step:
  - $\mu_t = \bar{\mu}_t + K_t \, (z_t - g(\bar{\mu}_t, 0))$
  - $\Sigma_t = \bar{\Sigma}_t - K_t \, C_t \, \bar{\Sigma}_t$
  - $K_t = \bar{\Sigma}_t \, C_t^T \, (C_t \, \bar{\Sigma}_t \, C_t^T + W_t \, R \, W_t^T)^{-1}$

  - $z_t = g(x_t, v_t)$
  - $v_t \sim N(0, R_t)$

  Assumptions

  - $C_t = \left.\frac{\partial g}{\partial x}\right|_{\bar{\mu}_t, 0}$
  - $W_t = \left.\frac{\partial g}{\partial v}\right|_{\bar{\mu}_t, 0}$

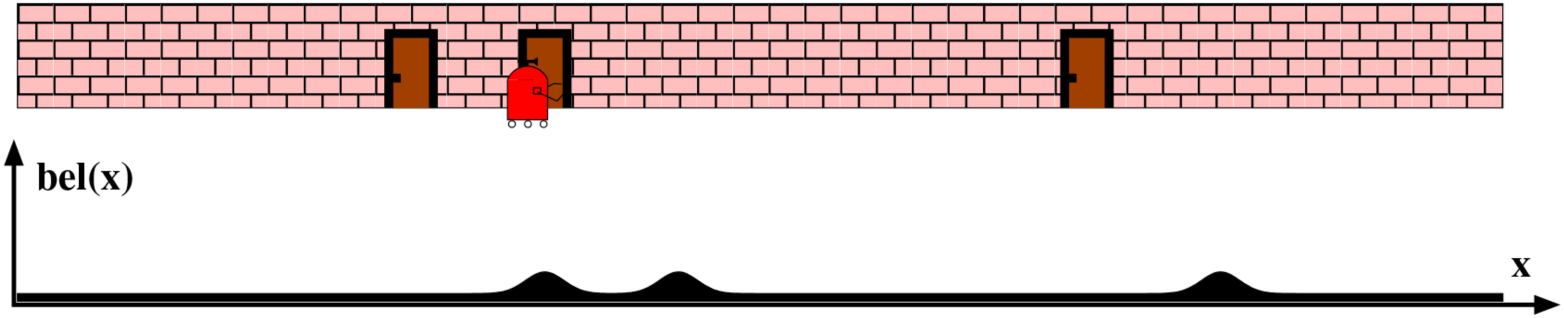
  Linearization

# Applications

- EKF is widely used in following applications
  - Pose estimation
  - Parameter estimation
  - Map building
  - Simultaneous localization and mapping (SLAM)
  - Feature tracking
  - Target tracking

# EKF Discussion

- **Advantages:**
  - Simple
  - Computationally efficient, even for high dimensional systems
  - Works with generic process and observation models
- **Disadvantages:**
  - Must calculate the Jacobian of the process and observation models
  - No guarantee of global convergence
  - Unimodal distribution



# Particle Filter



Modified from slides from Sebastian Thrun, Wolfram Burgard, and Dieter Fox

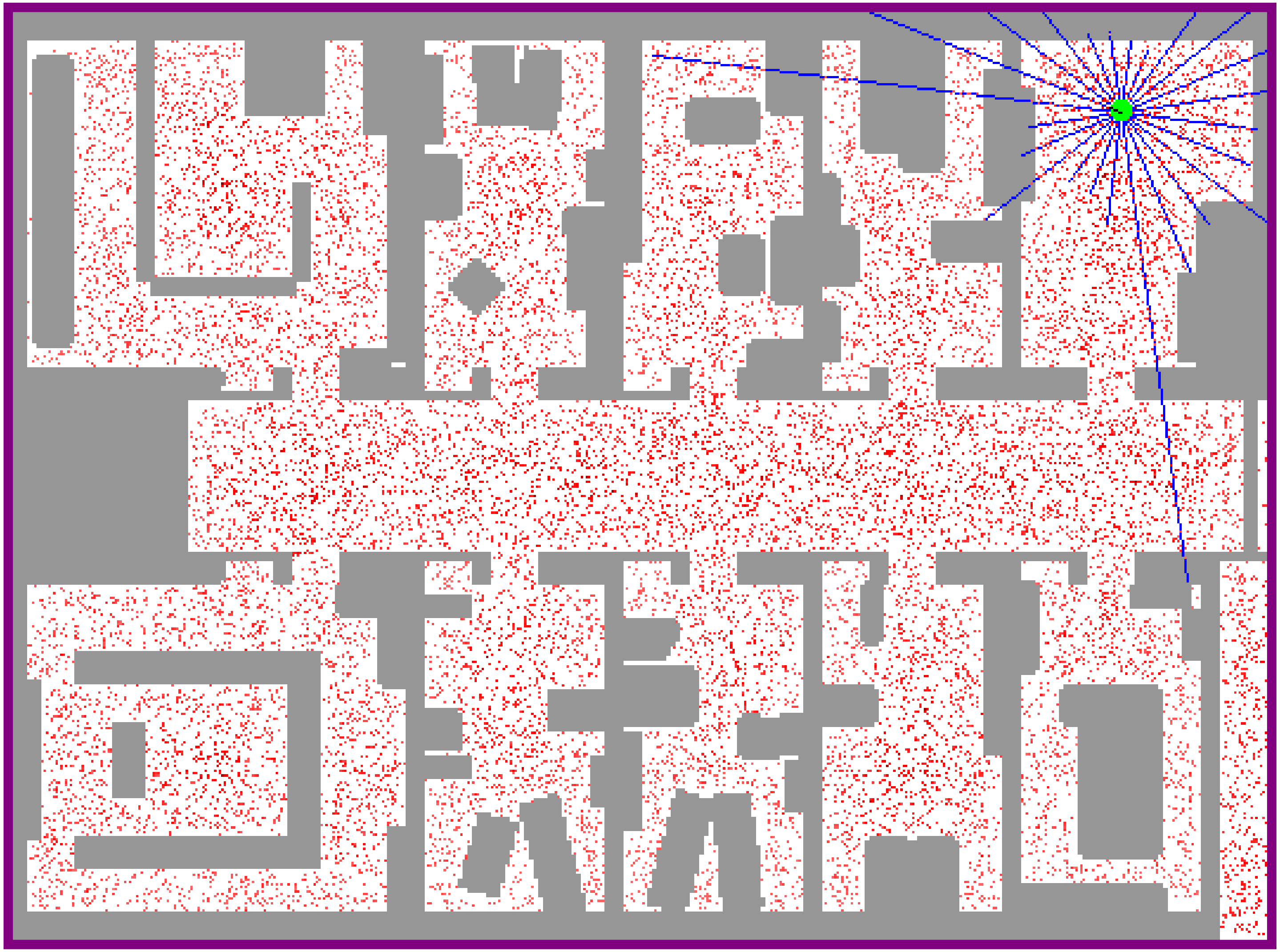
# Particle Filters

- Represent belief by random samples
- Estimation of non-Gaussian, nonlinear processes

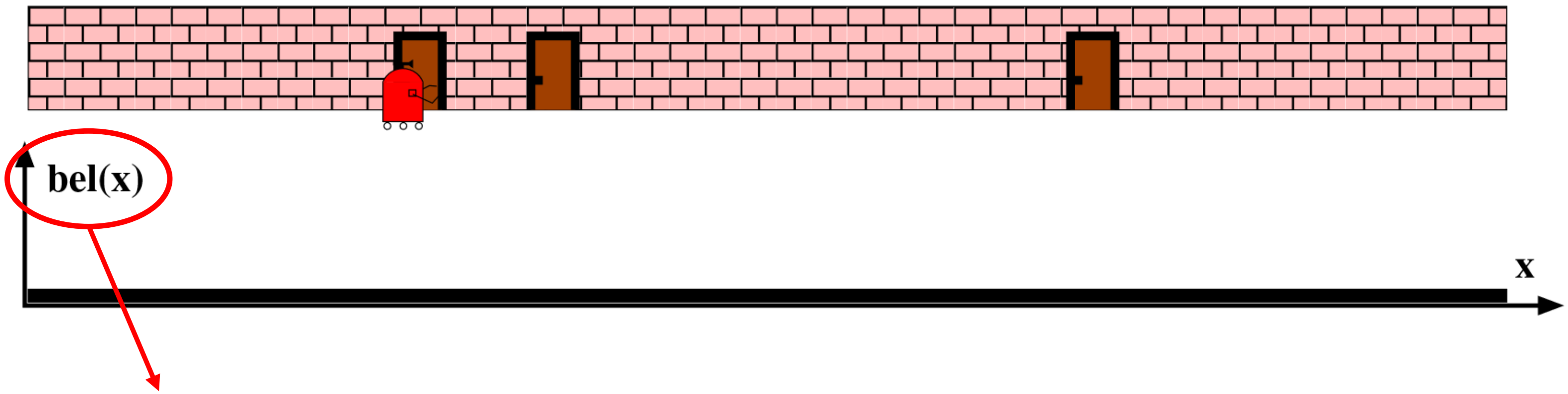
- Monte Carlo filter, Survival of the fittest, Condensation, Bootstrap filter, Particle filter

- Filtering: [Rubin, 88], [Gordon et al., 93], [Kitagawa 96]
- Computer vision: [Isard and Blake 96, 98]
- Dynamic Bayesian Networks: [Kanazawa et al., 95]

# Sample-based Localization (sonar)

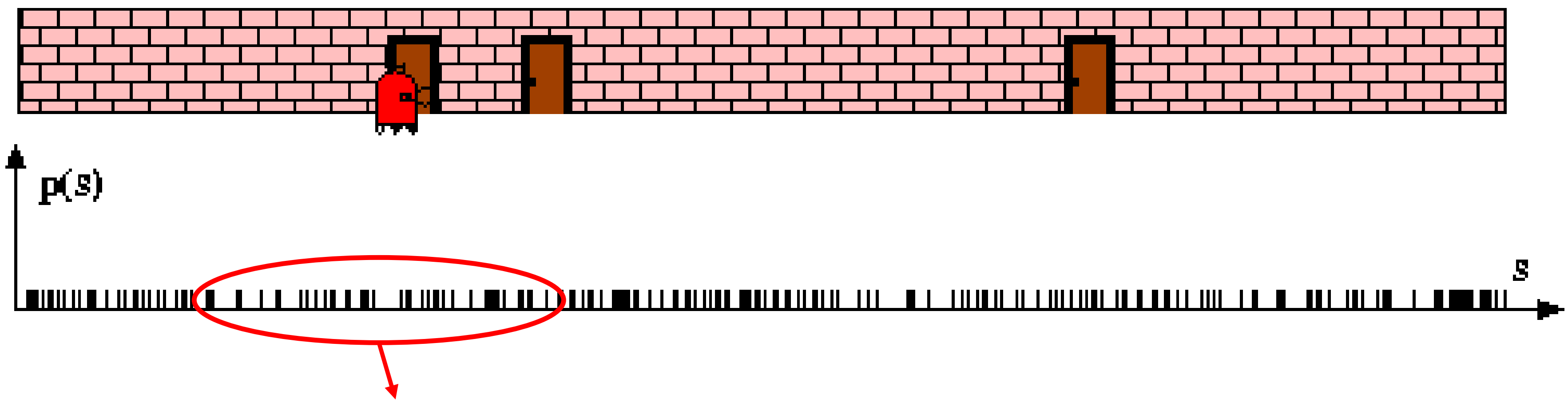


# Particle Filters




Use “belief” (or probability distribution) to represent robot state (pose)

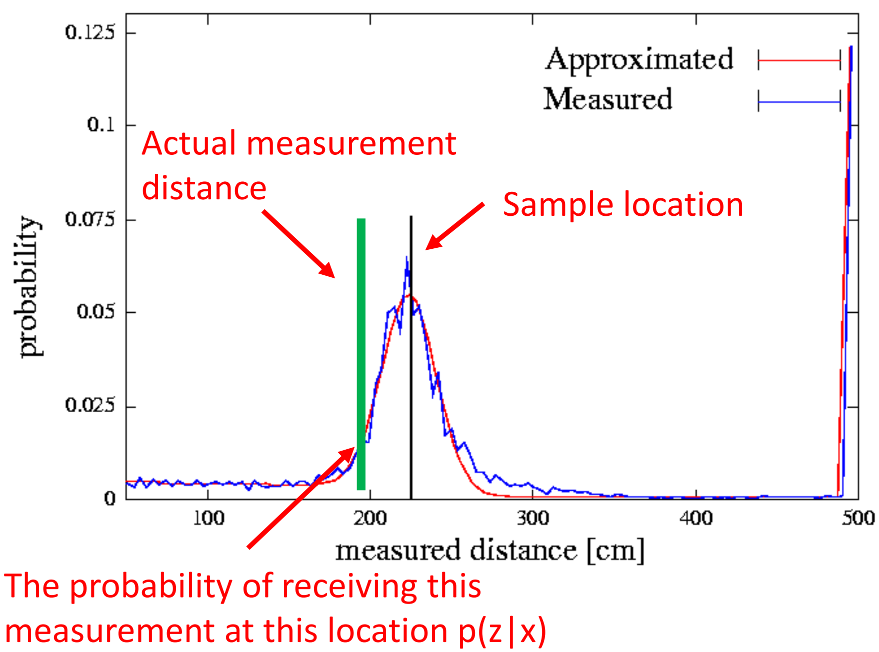
# Particle Filters



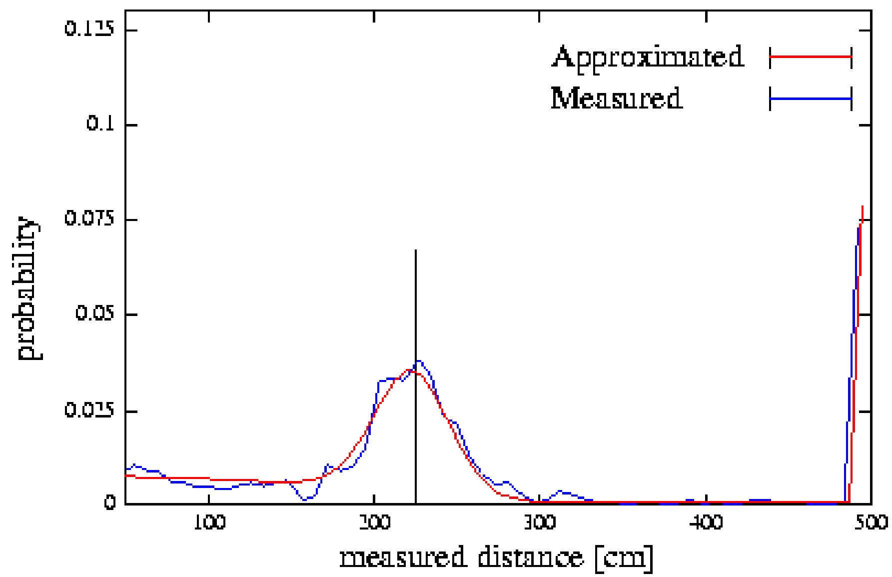

Use samples to represent the belief

# Measurement Model: Door Proximity Sensor

$$p(z|x)$$




**Laser sensor**

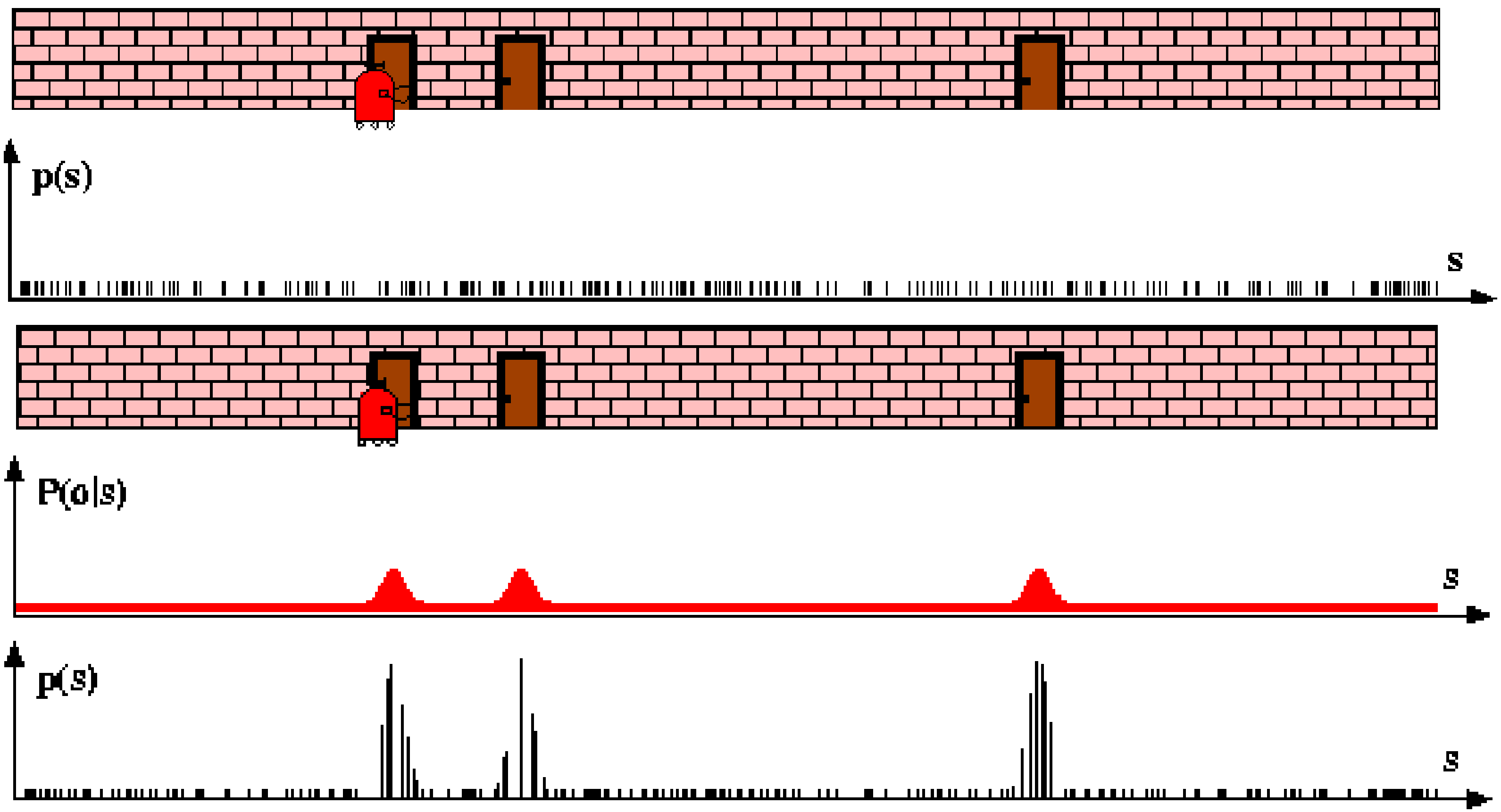



**Sonar sensor**

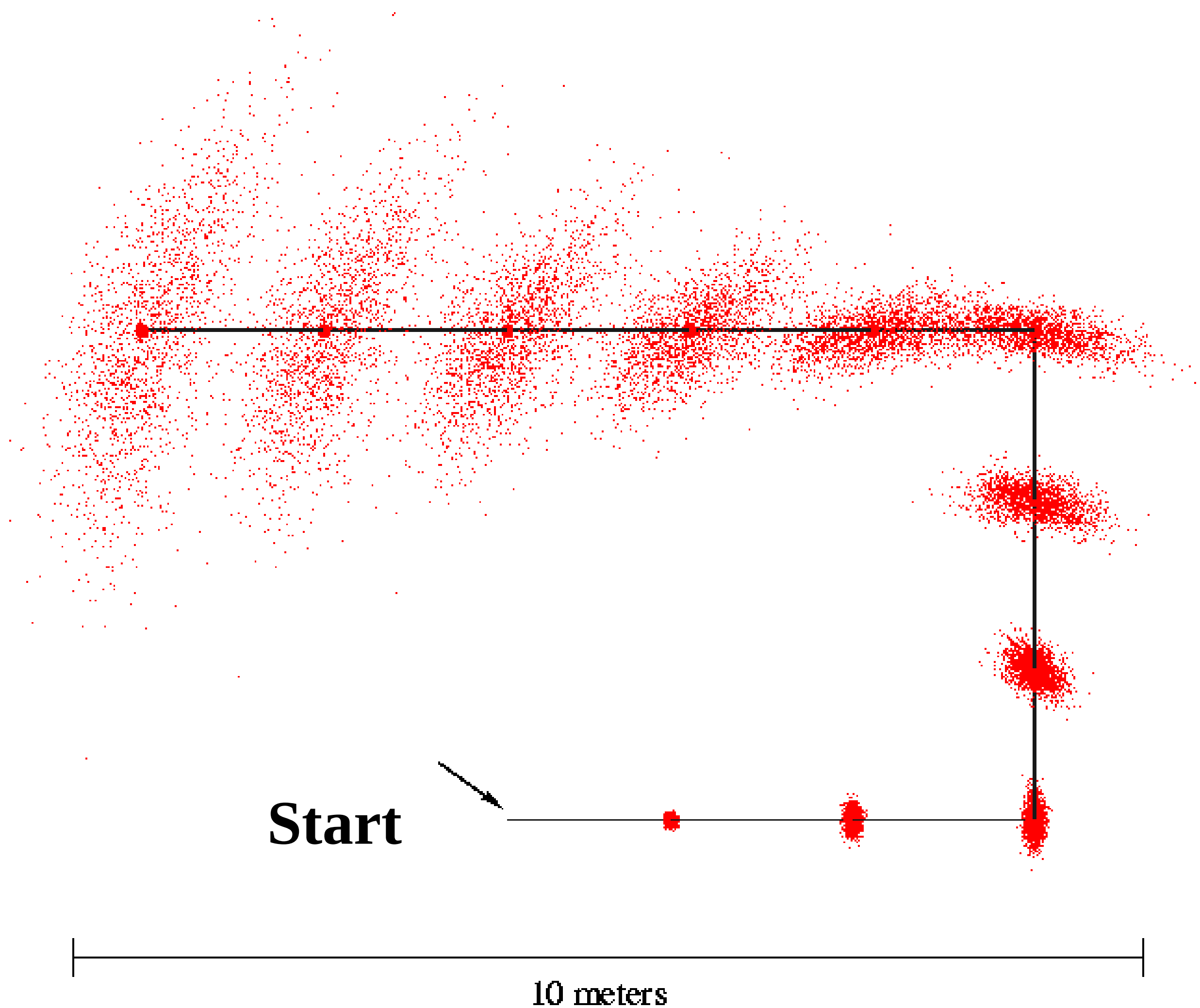
# Sensor Information: Importance Sampling

$$w_t \leftarrow \frac{\alpha\, p(z_t|x_t)\, Bel^-(x_t)}{Bel^-(x_t)} = \alpha\, p(z_t|x_t)$$

$$Bel(x_t) \leftarrow \alpha\, p(z_t|x_t)\, Bel^-(x_t)$$

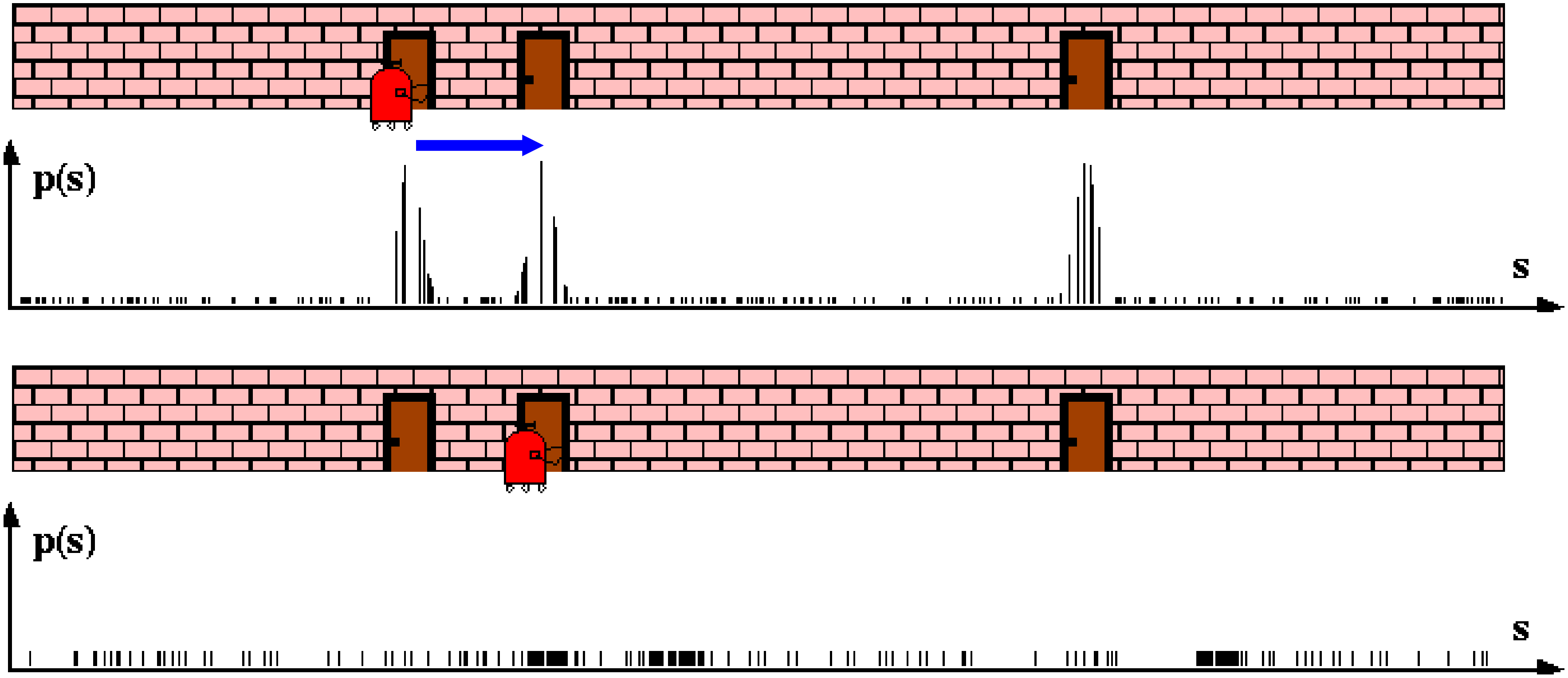


# Process (Motion) Model



# Resampling and Robot Motion

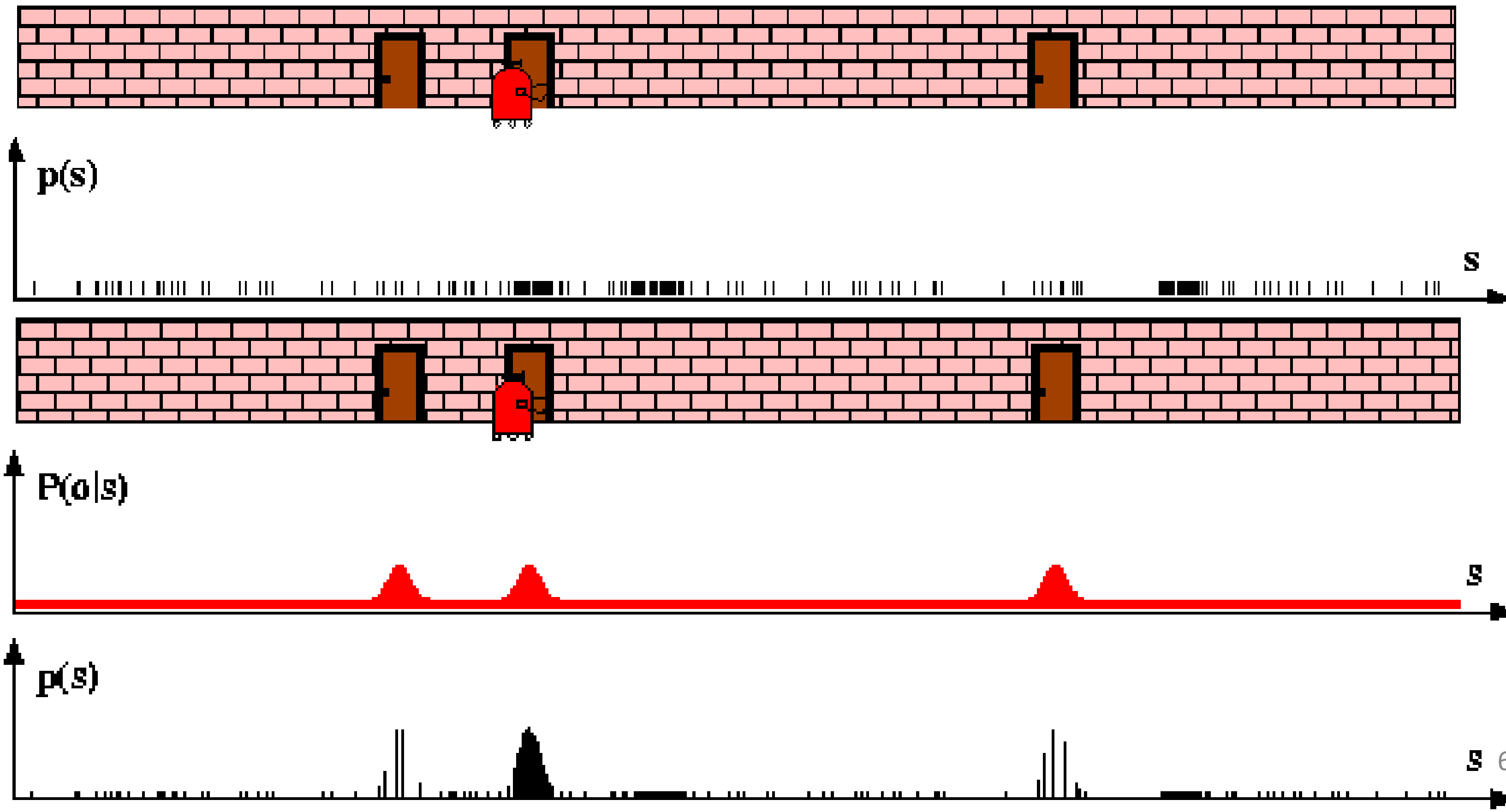
$$Bel^{-}(x_{t+1}) \leftarrow \int p(x_{t+1}|x_{t,}u_{t+1})Bel(x_t)\,\mathrm{d}x_t$$



# Sensor Information: Importance Sampling

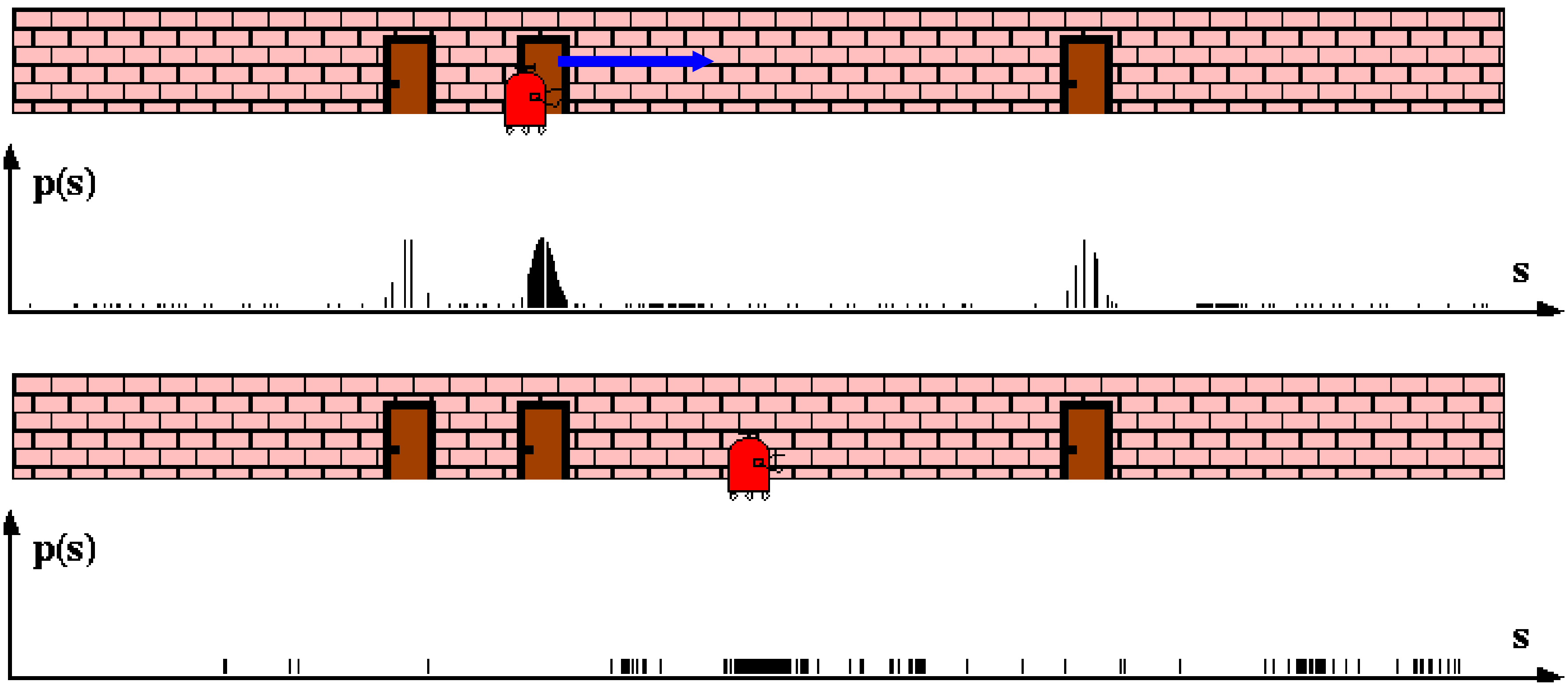
$$w_{t+1} \leftarrow \frac{\alpha\, p(z_{t+1}|x_{t+1})\, Bel^{-}(x_{t+1})}{Bel^{-}(x_{t+1})} = \alpha\, p(z_{t+1}|x_{t+1})$$

$$Bel(x_{t+1}) \leftarrow \alpha\, p(z_{t+1}|x_{t+1})\, Bel^{-}(x_{t+1})$$



# Resampling and Robot Motion

$$Bel^{-}(x_{t+2}) \leftarrow \int p(x_{t+2}|x_{t+1,}u_{t+1})Bel(x_{t+1})\,\mathrm{d}x_{t+1}$$



# Particle Filter

- Algorithm **particle_filter** $(S_{t-1}, u_t, z_t)$:
- $S_t = \emptyset, \quad \eta = 0$
- **For** $i = 1 \dots n$ *Generate new samples*
  - Sample index $j(i)$ from the discrete distribution given by $w_{t-1}$
  - Sample $x_t^i$ from $p(x_t | x_{t-1}, u_t)$ using $x_{t-1}^{j(i)}$ and $u_t$
  - $w_t^i = p(z_t | x_t^i)$ *Compute importance weight*
  - $\eta = \eta + w_t^i$ *Update* normalization factor
  - $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ *Insert*
- **For** $i = 1 \dots n$
  - $w_t^i = w_t^i / \eta$ *Normalize weights*

# Particle Filter as Bayes' Filter

- **Prior:** $p(x_0)$
- **Process model:** $f(x_t \mid x_{t-1}, u_t)$
- **Measurement model:** $g(z_t \mid x_t)$

- **Prediction step:**
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int f(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$

draw $x_{t-1}^i$ from $Bel(x_{t-1})$

draw $x_t^i$ from $p(x_t \mid x_{t-1}^i, u_t)$

importance factor for $x_t^i$: $w_t^i \propto p(z_t \mid x_t)$

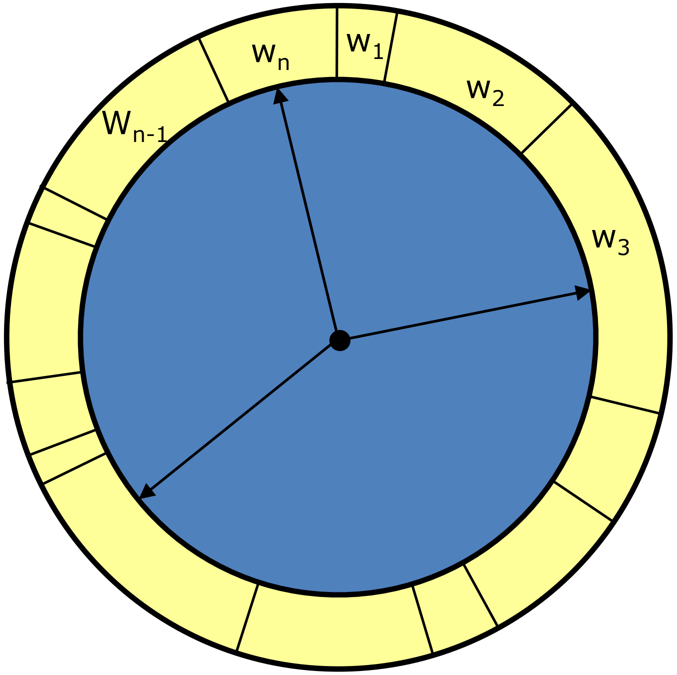
- **Update step:**
- $p(x_t \mid z_{1:t}, u_{1:t}) = \dfrac{g(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t})}{\int g(z_t \mid x_t')\, p(x_t' \mid z_{1:t-1}, u_{1:t})\, dx_t'}$

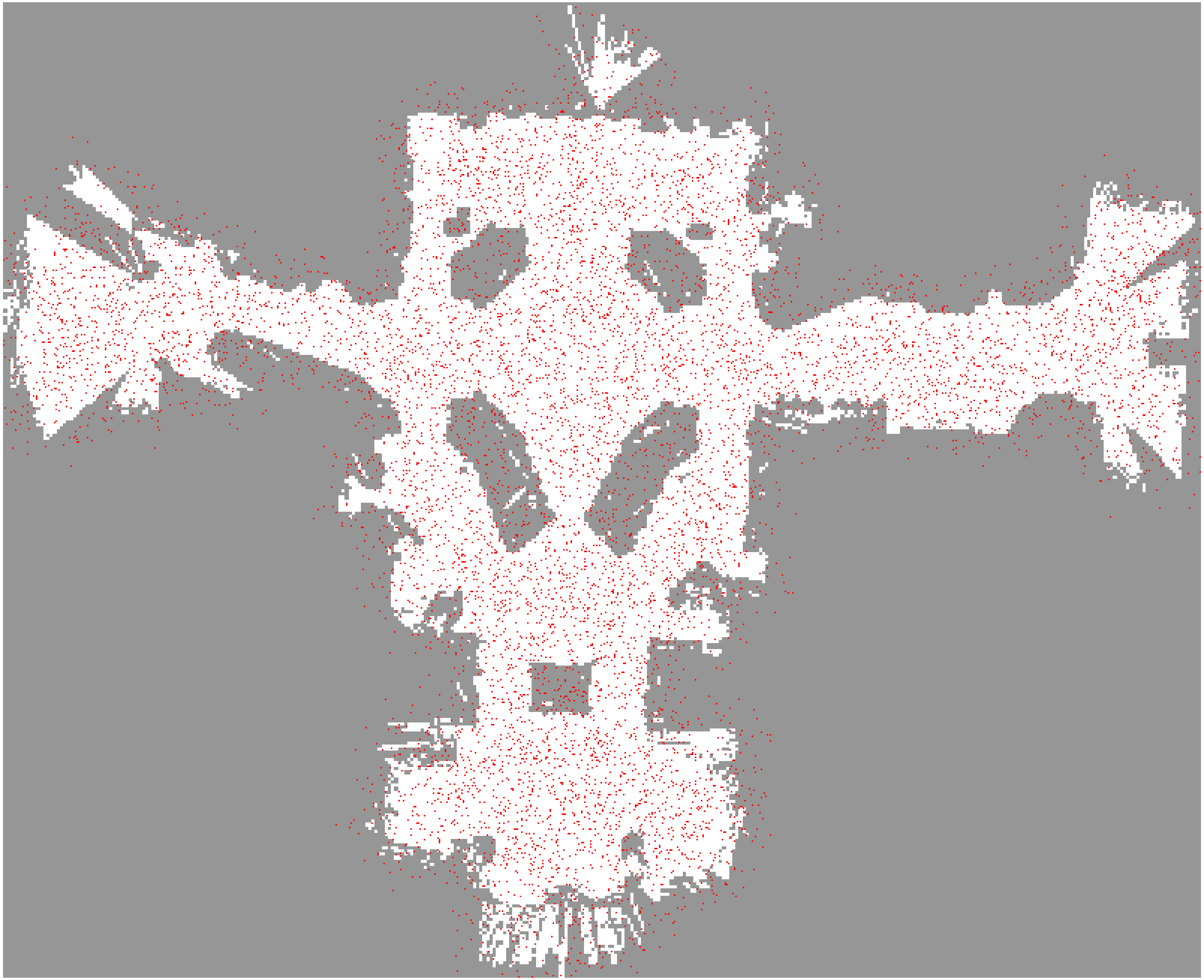
# Resampling

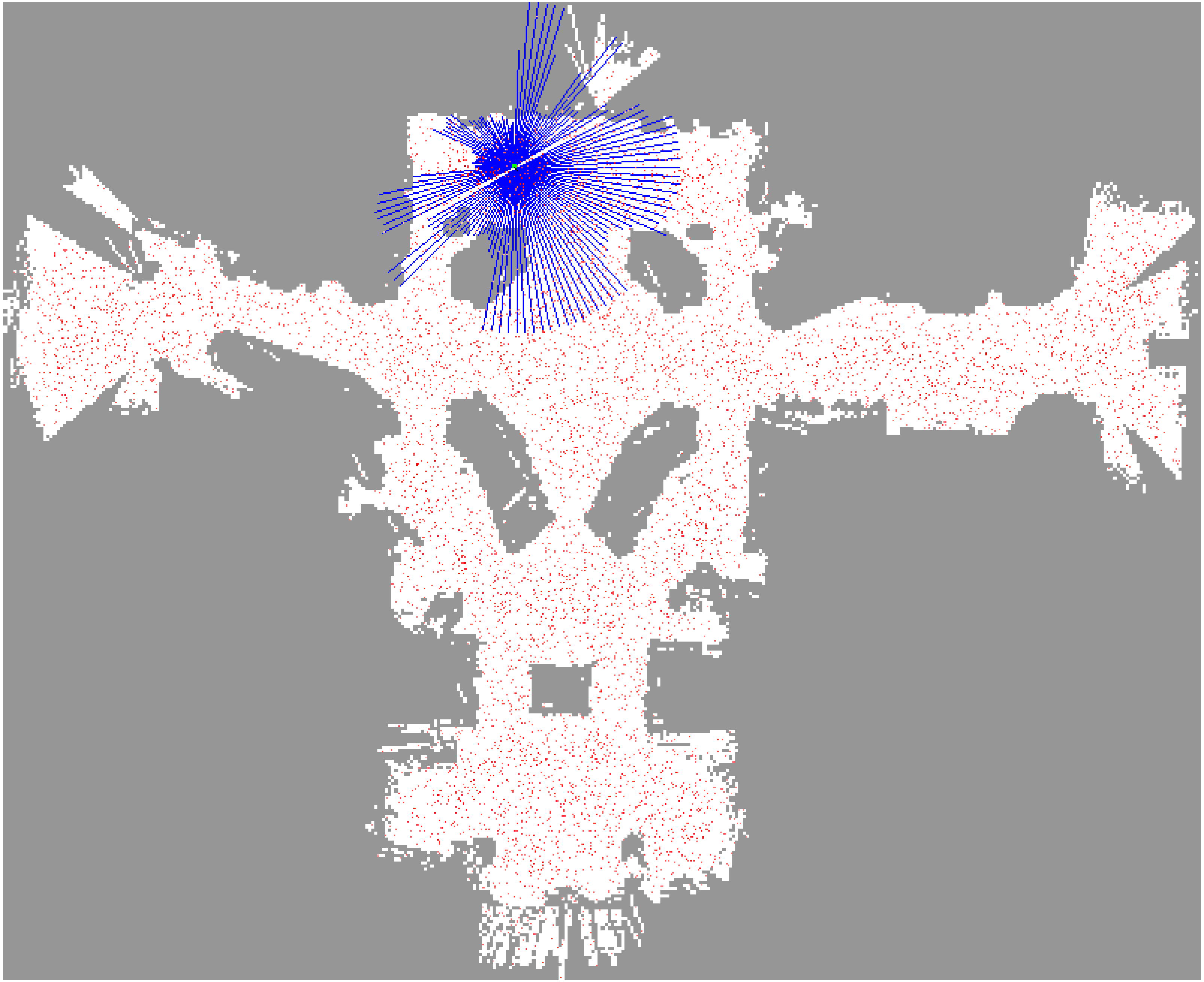
- Algorithm **particle_filter** $(S_{t-1}, u_t, z_t)$:
- $S_t = \emptyset, \quad \eta = 0$
- **For** $i = 1 \ldots n$ *Generate new samples*
  - Sample index $j(i)$ from the discrete distribution given by $w_{t-1}$ **How?**
  - Sample $x_t^i$ from $p(x_t | x_{t-1}, u_t)$ using $x_{t-1}^{j(i)}$ and $u_t$
  - $w_t^i = p(z_t | x_t^i)$ *Compute importance weight*
  - $\eta = \eta + w_t^i$ *Update* normalization factor
  - $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ *Insert*
- **For** $i = 1 \ldots n$
  - $w_t^i = w_t^i / \eta$ *Normalize weights*

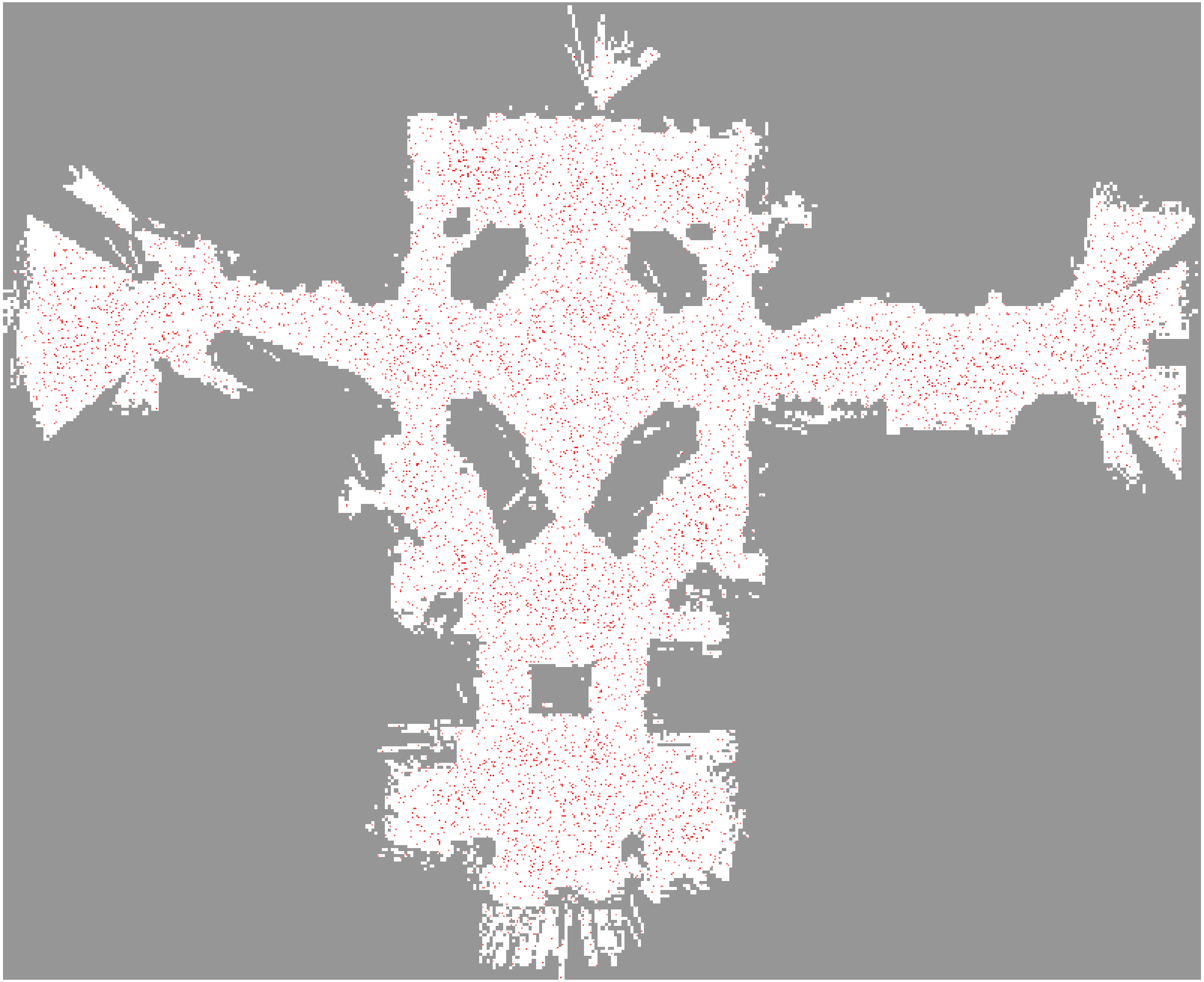
# Resampling

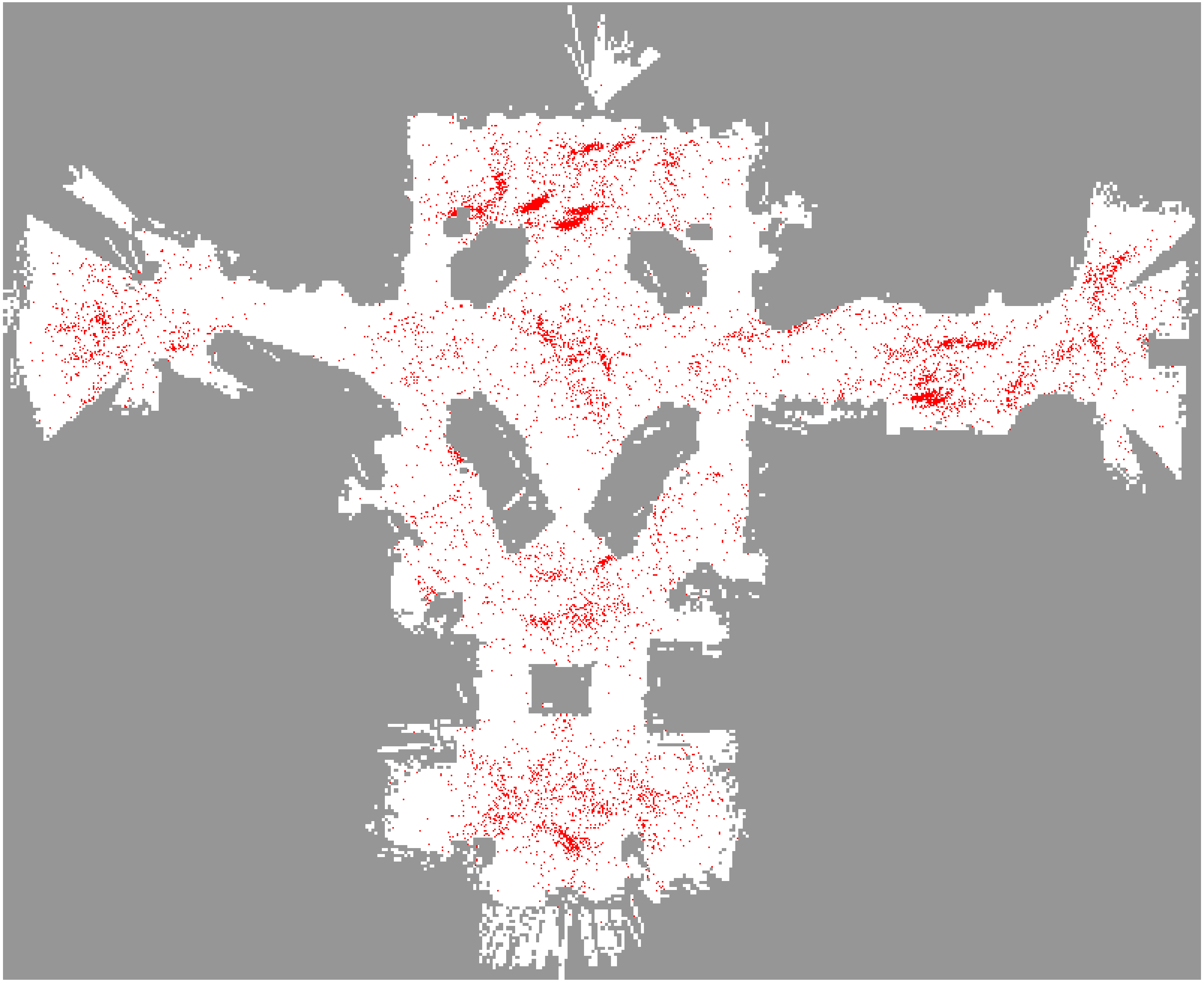
- **Given**: Set $S$ of weighted samples
- **Wanted** : Random sample, where the probability of drawing $x_t^i$ is given by $w_t^i$
- Typically done *n* times with replacement to generate new sample set $S'$ with uniform weight, but different density

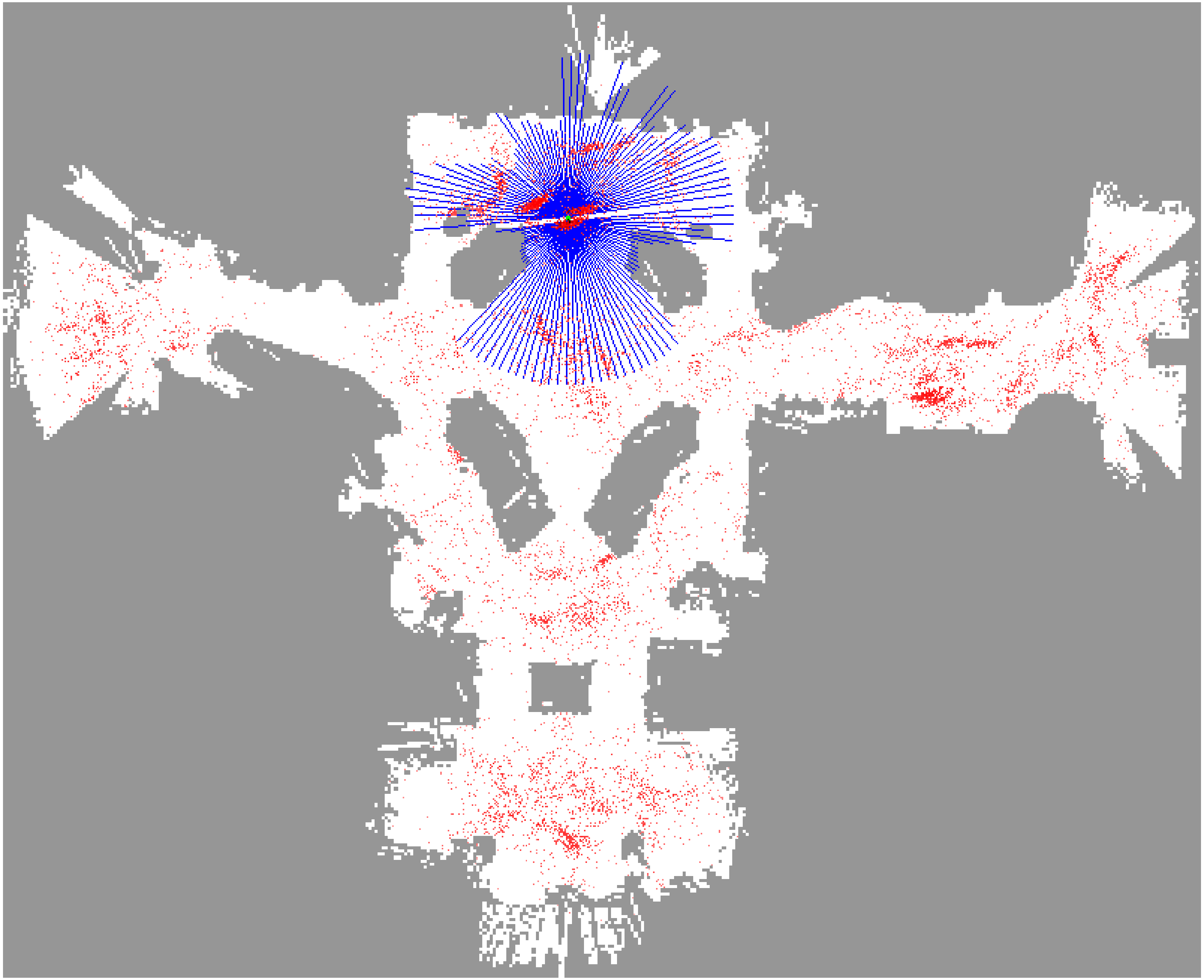


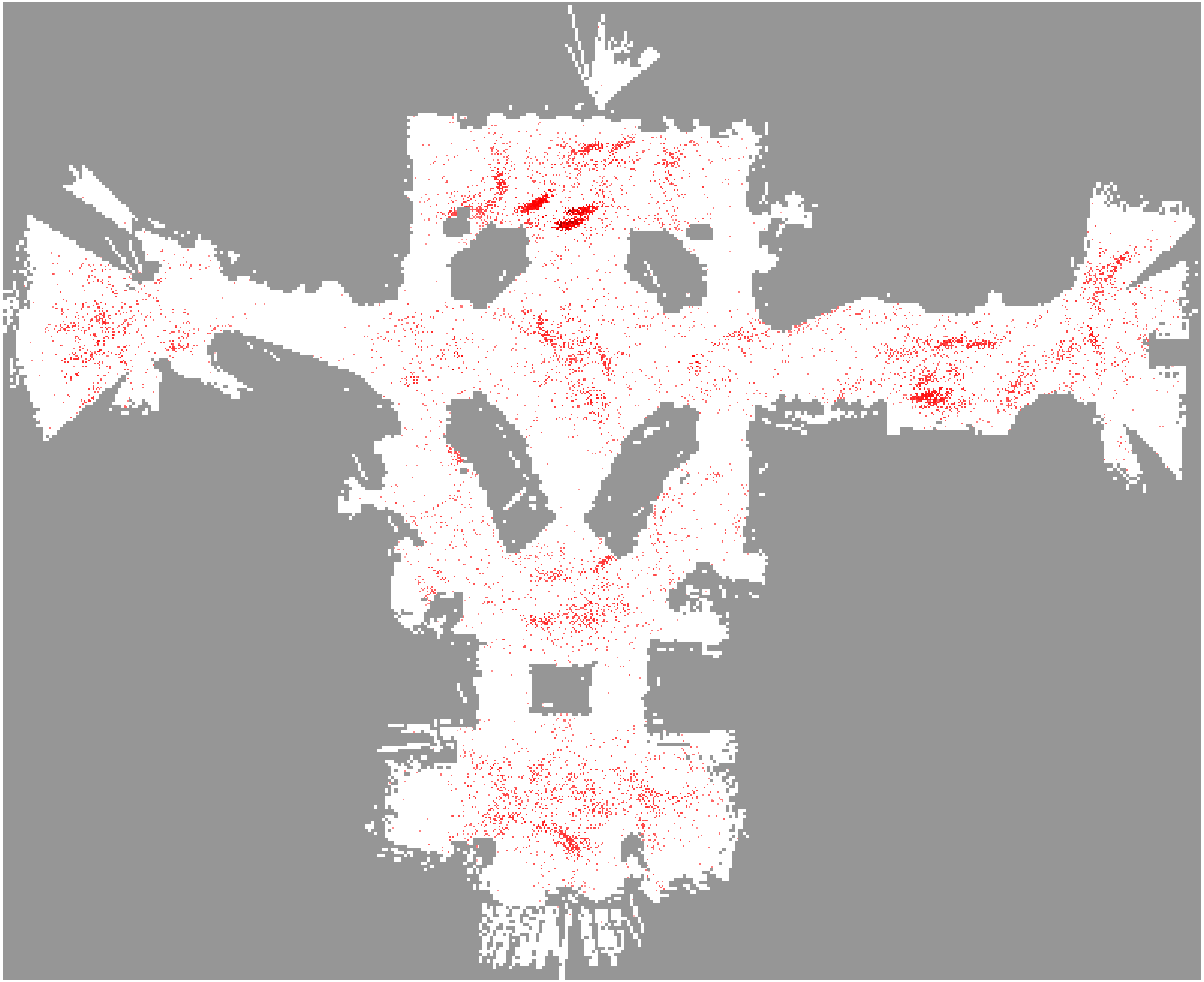


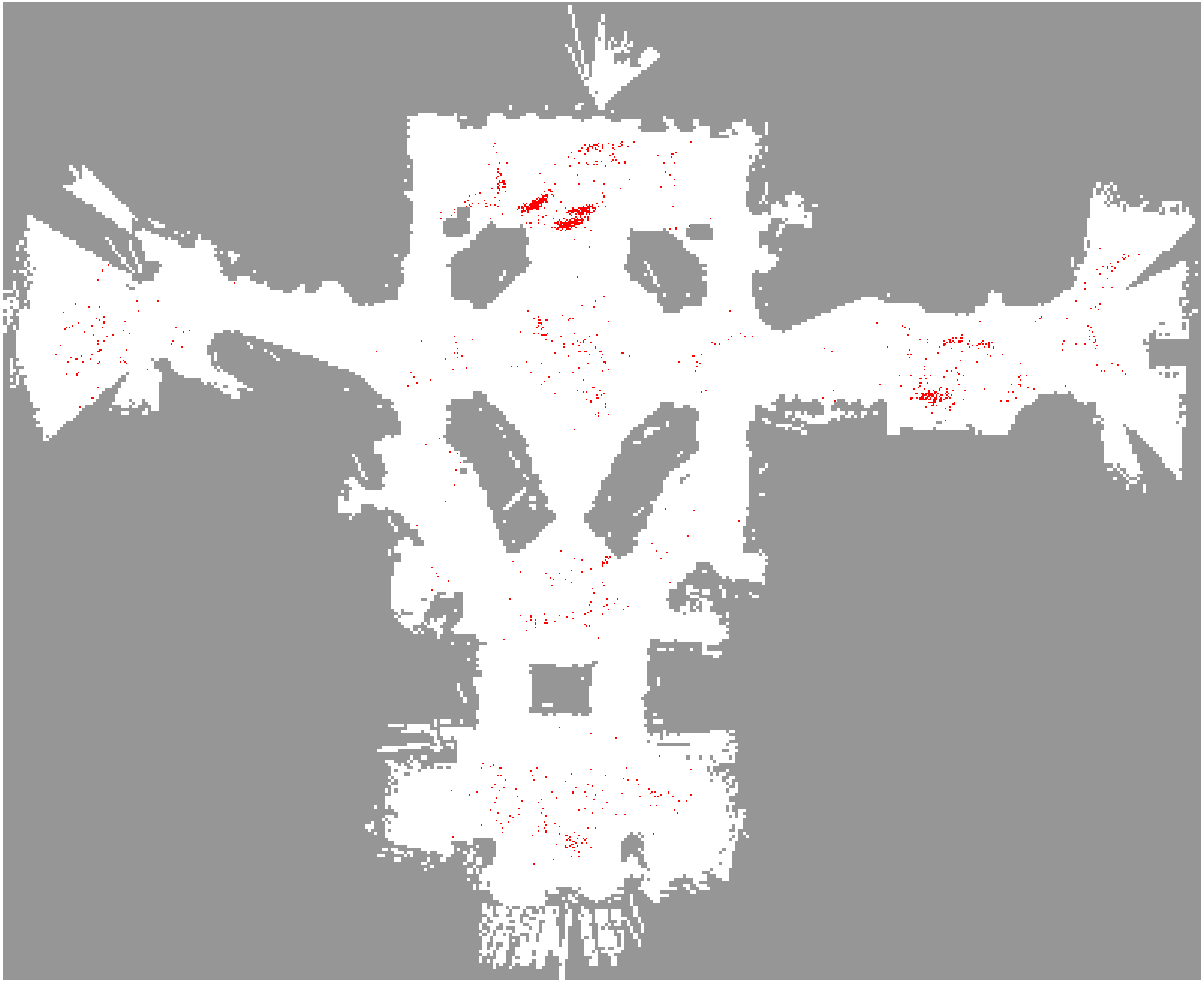


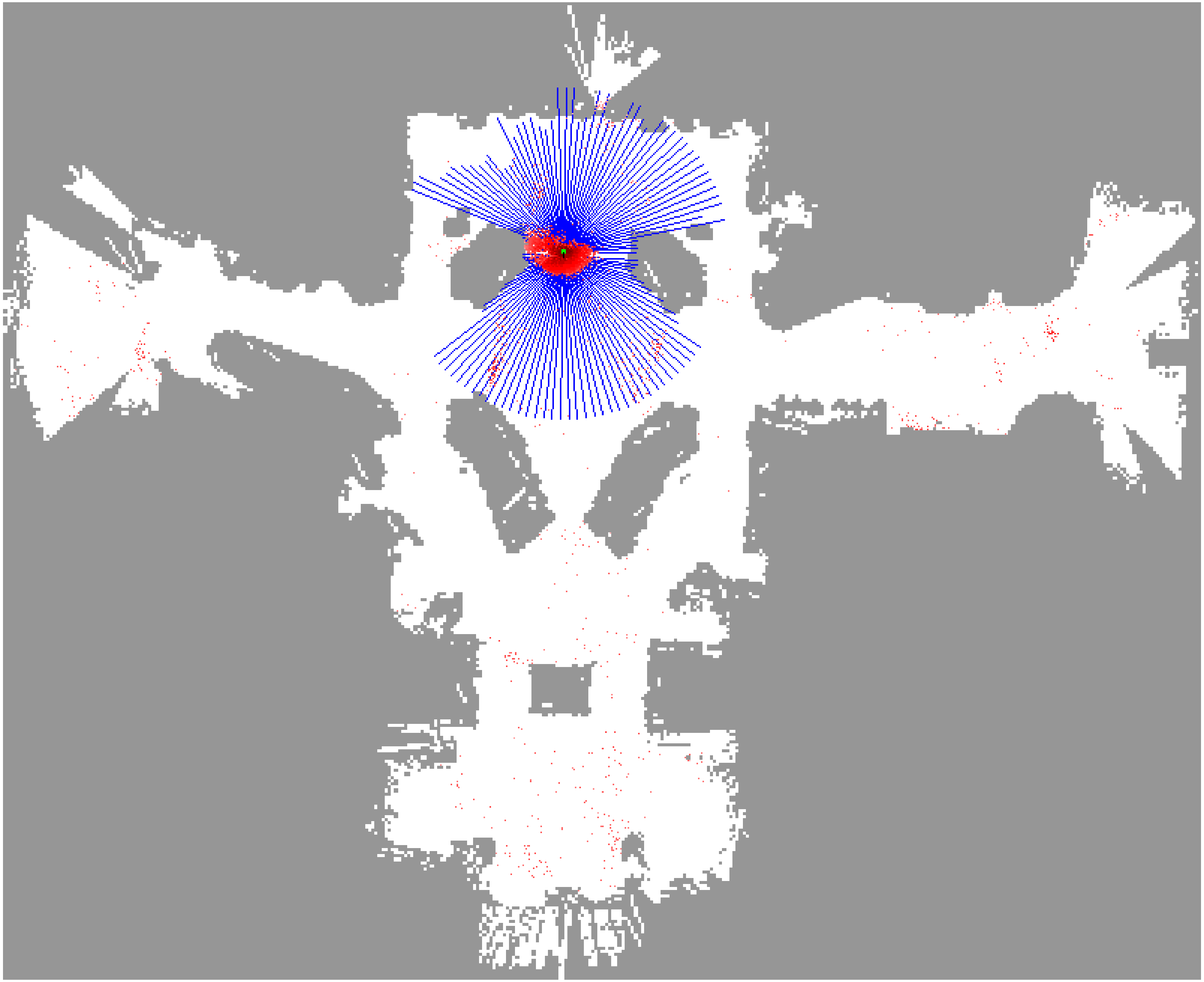


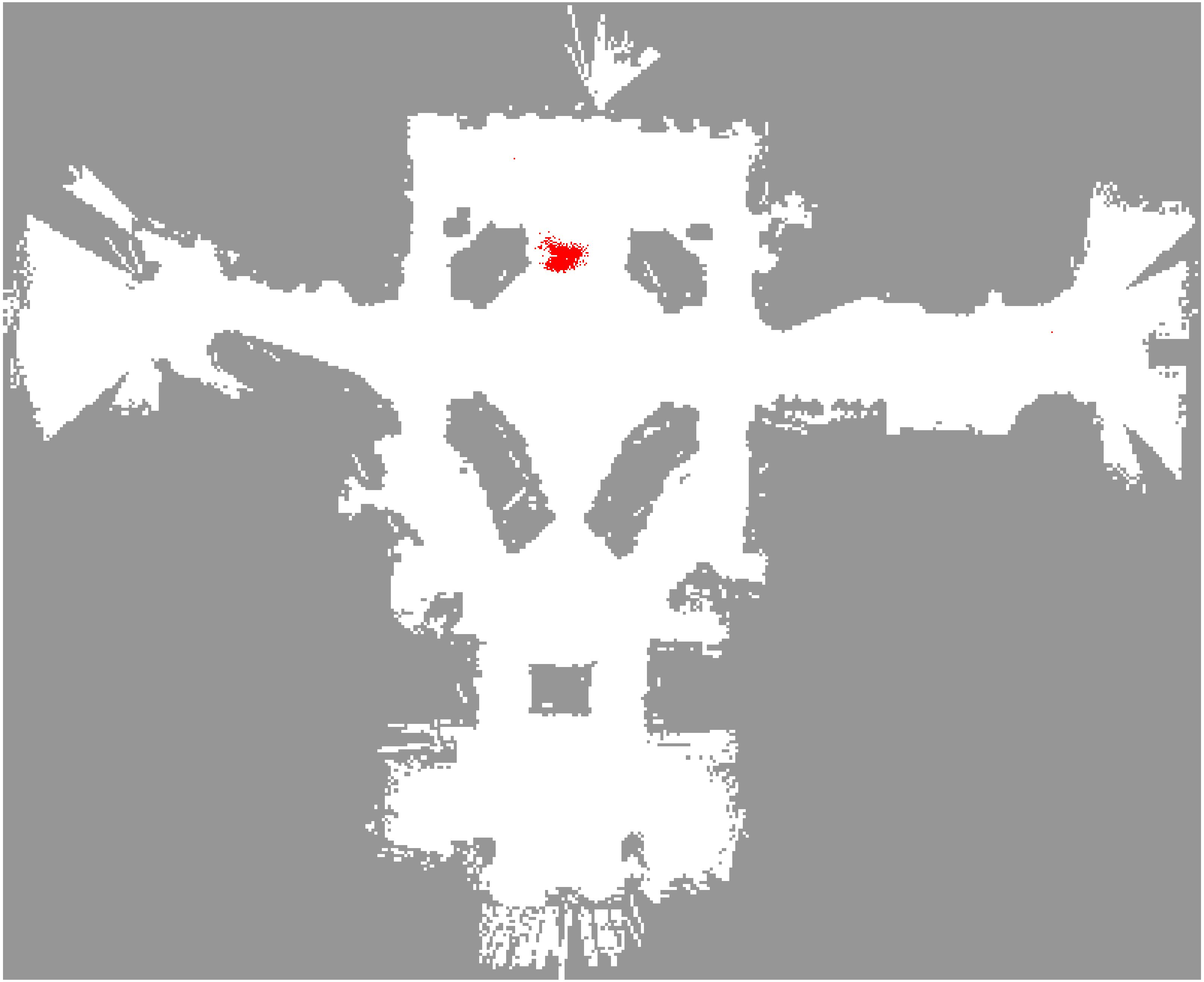


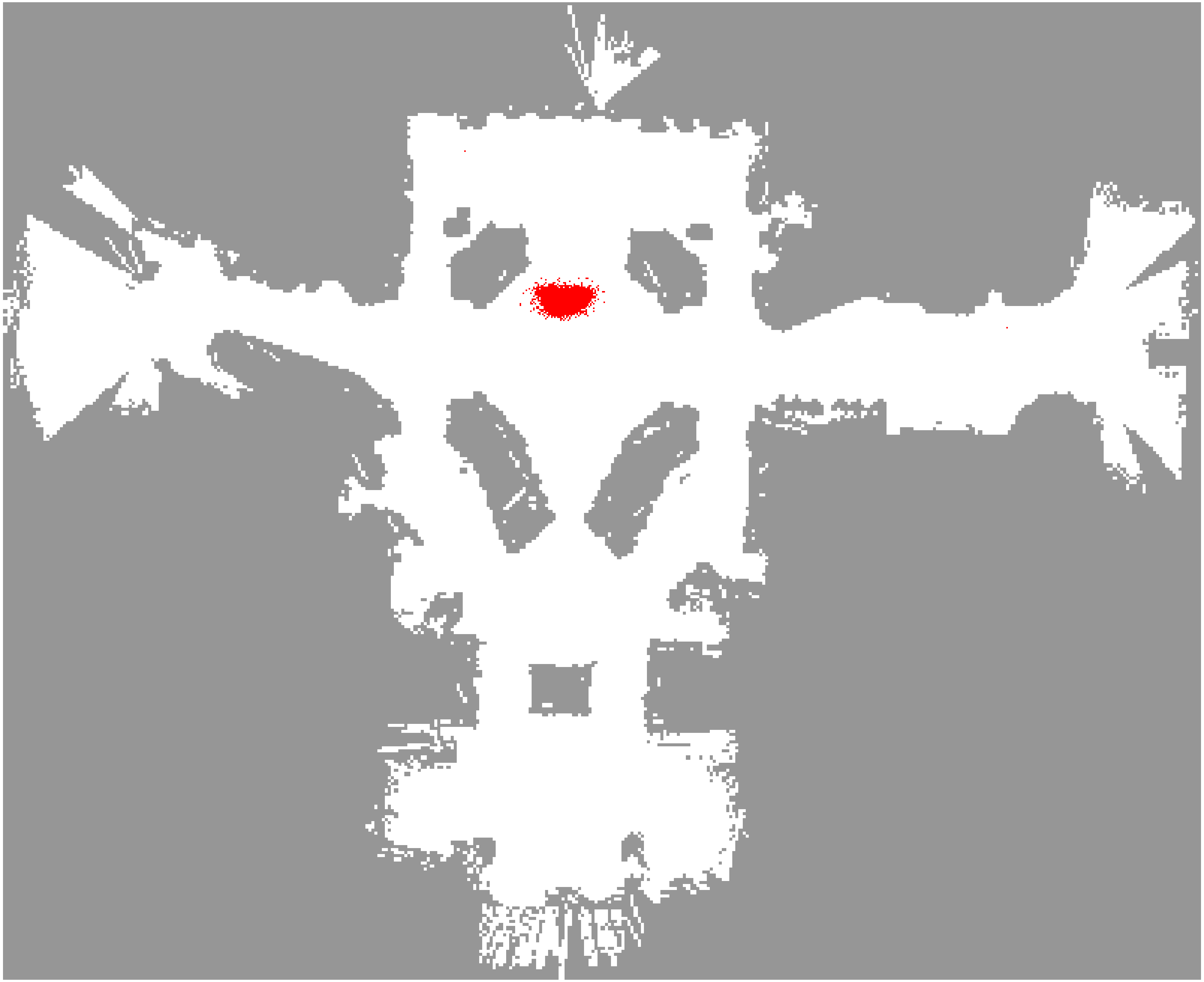


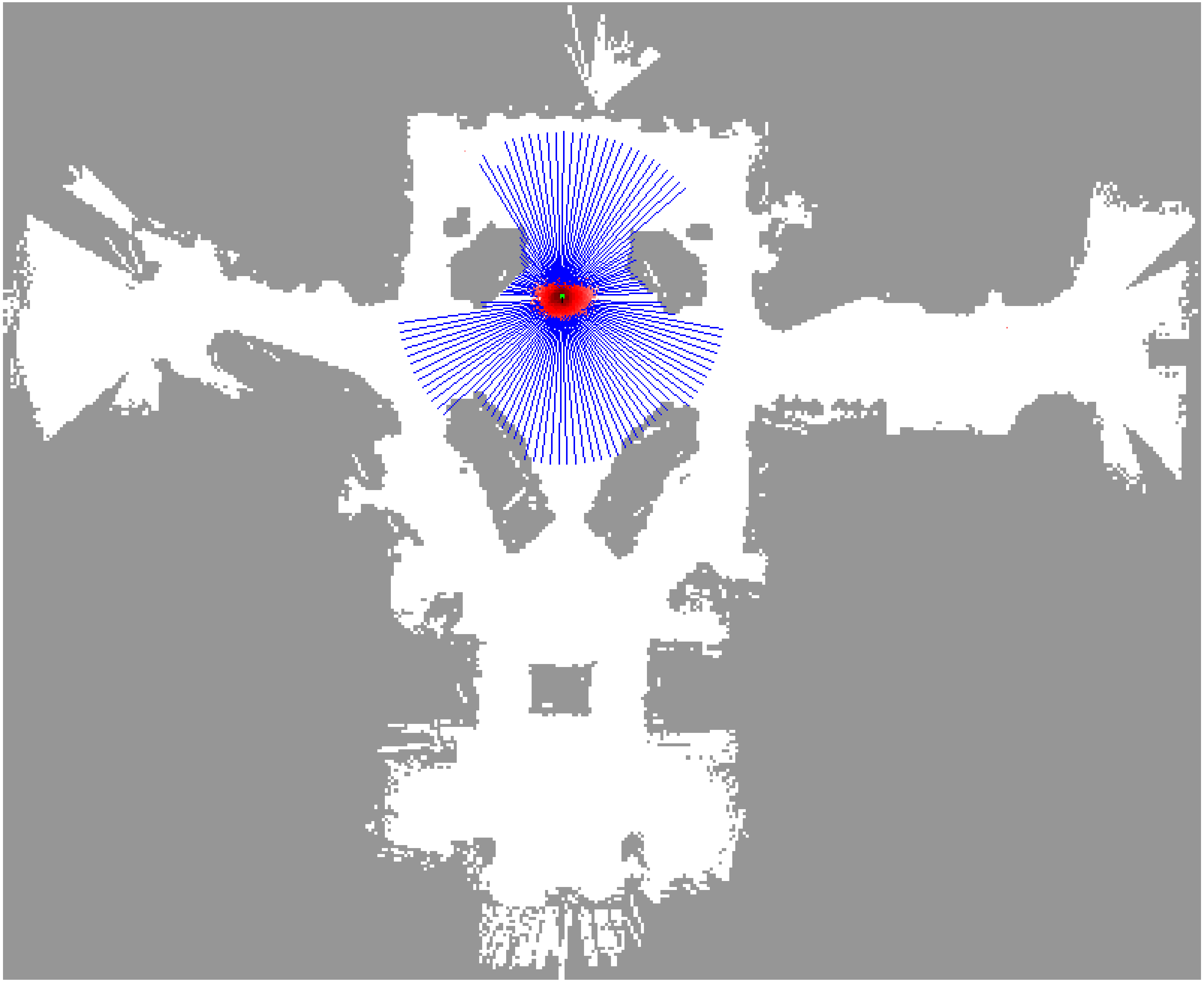


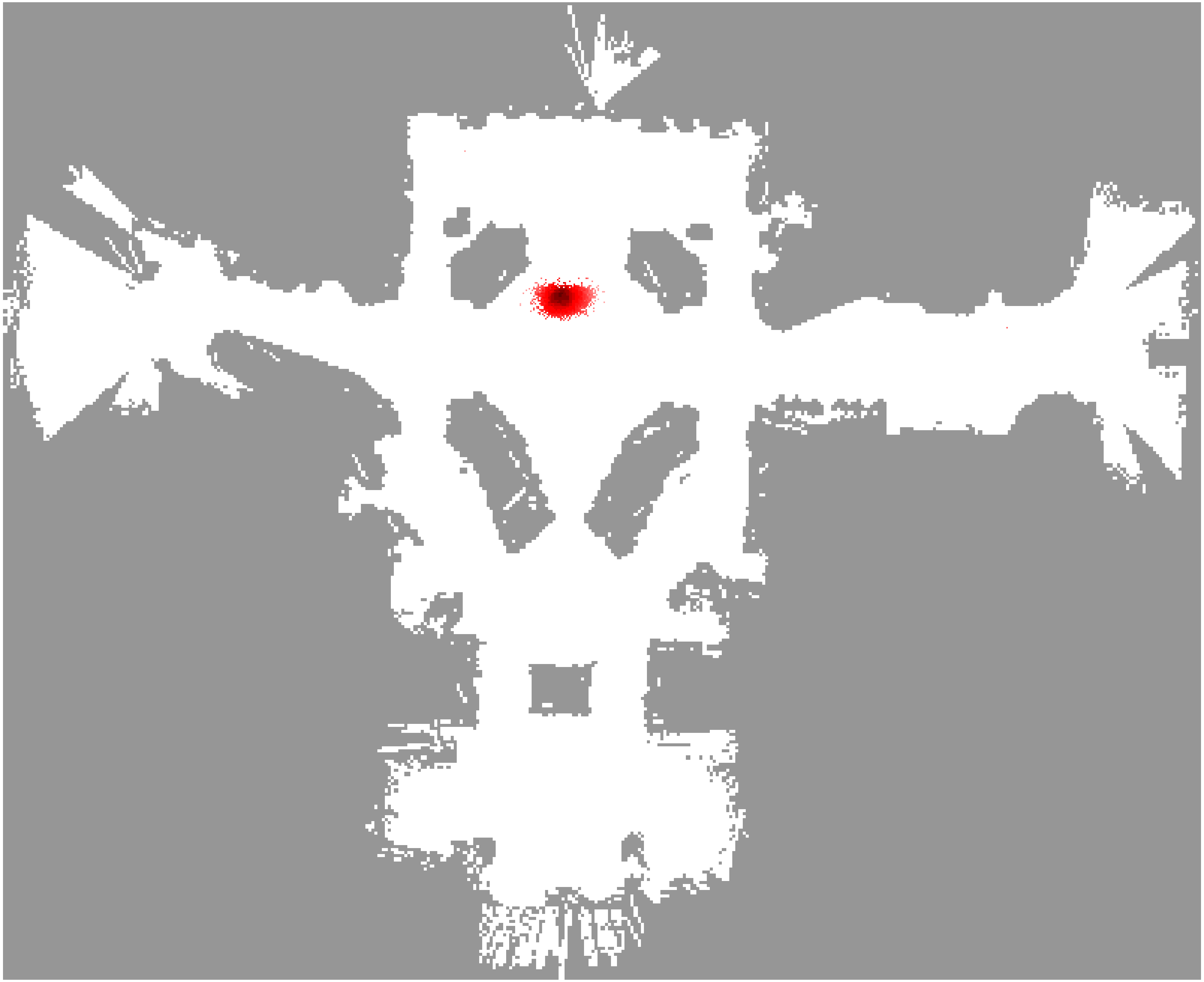


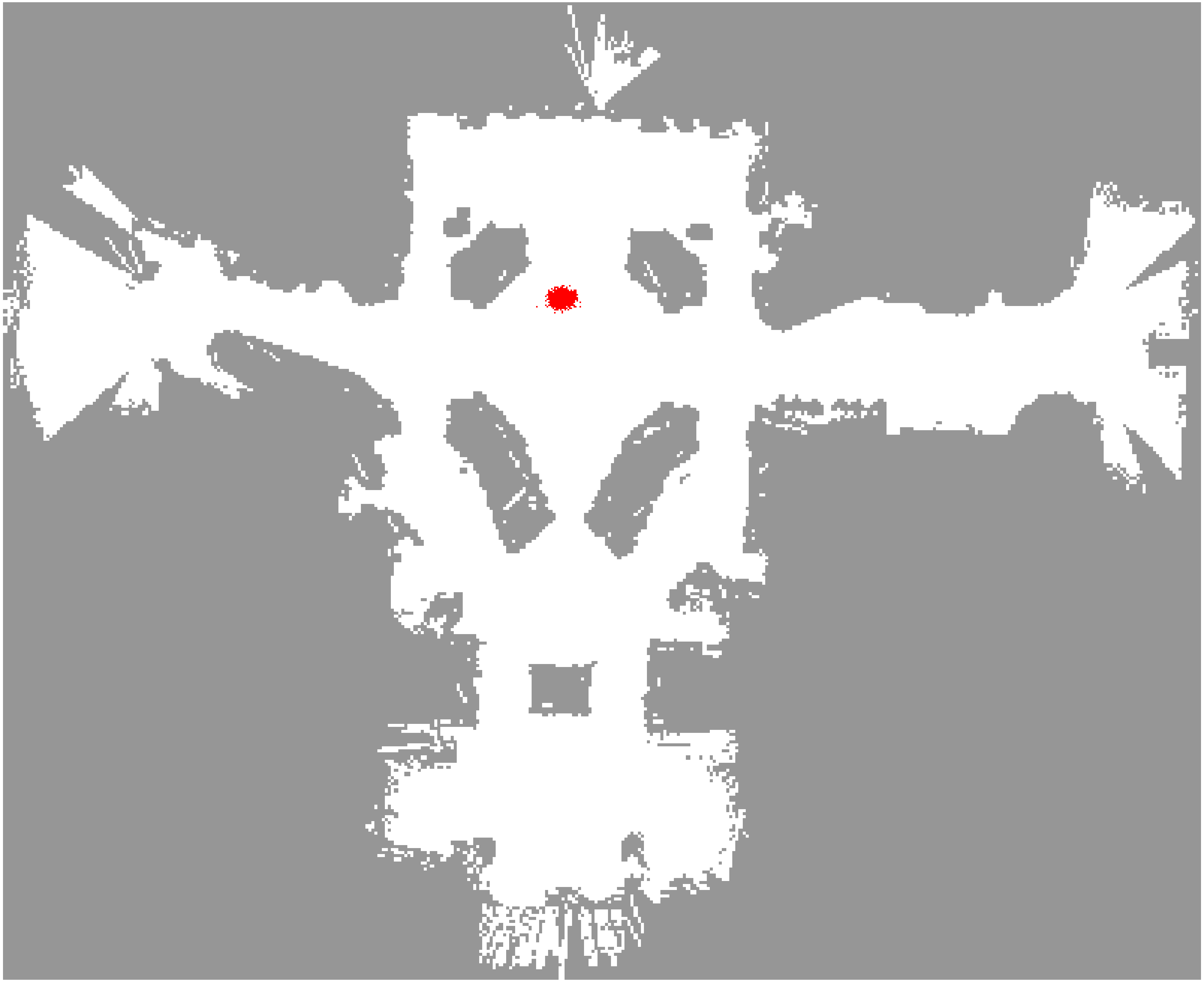


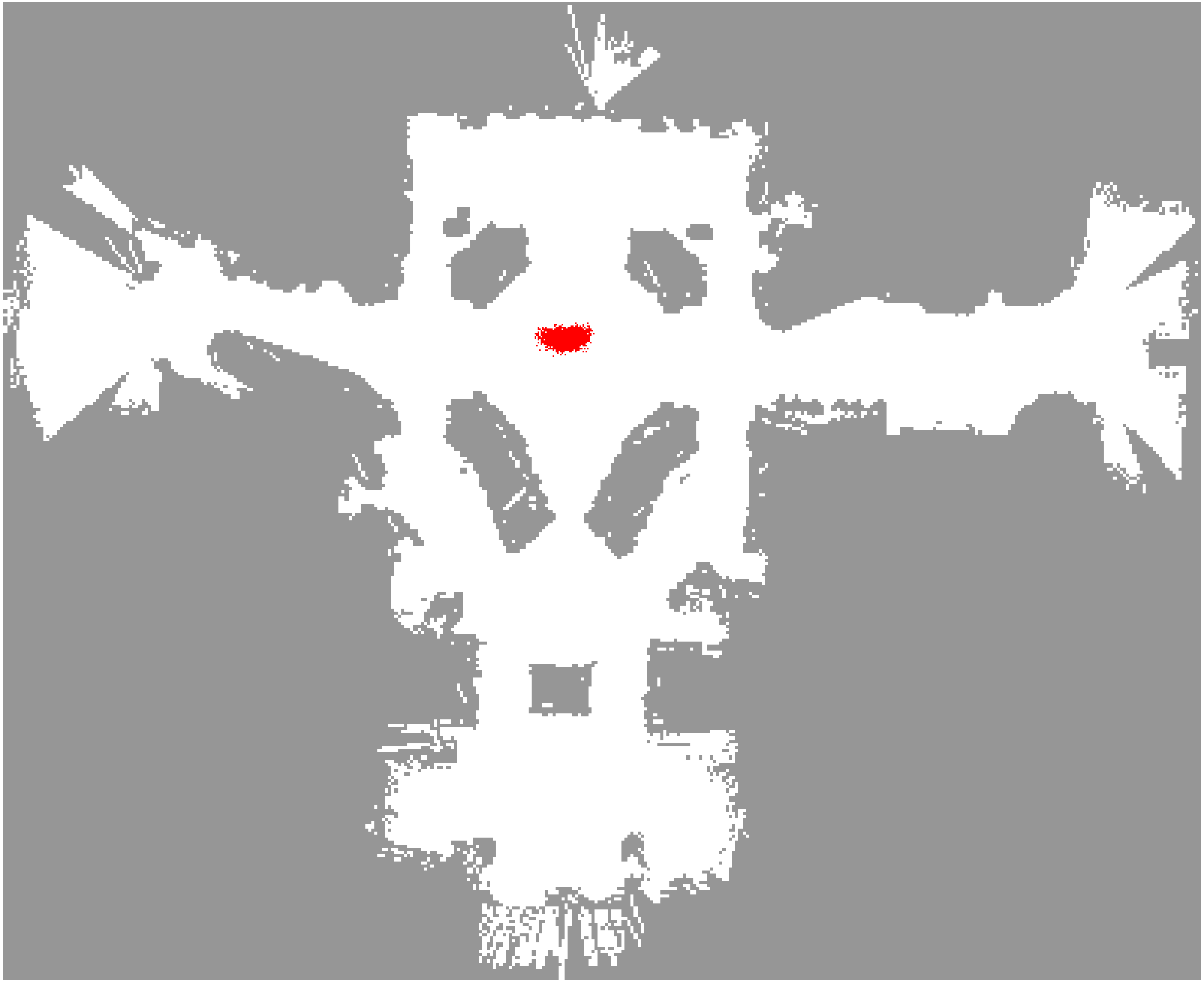


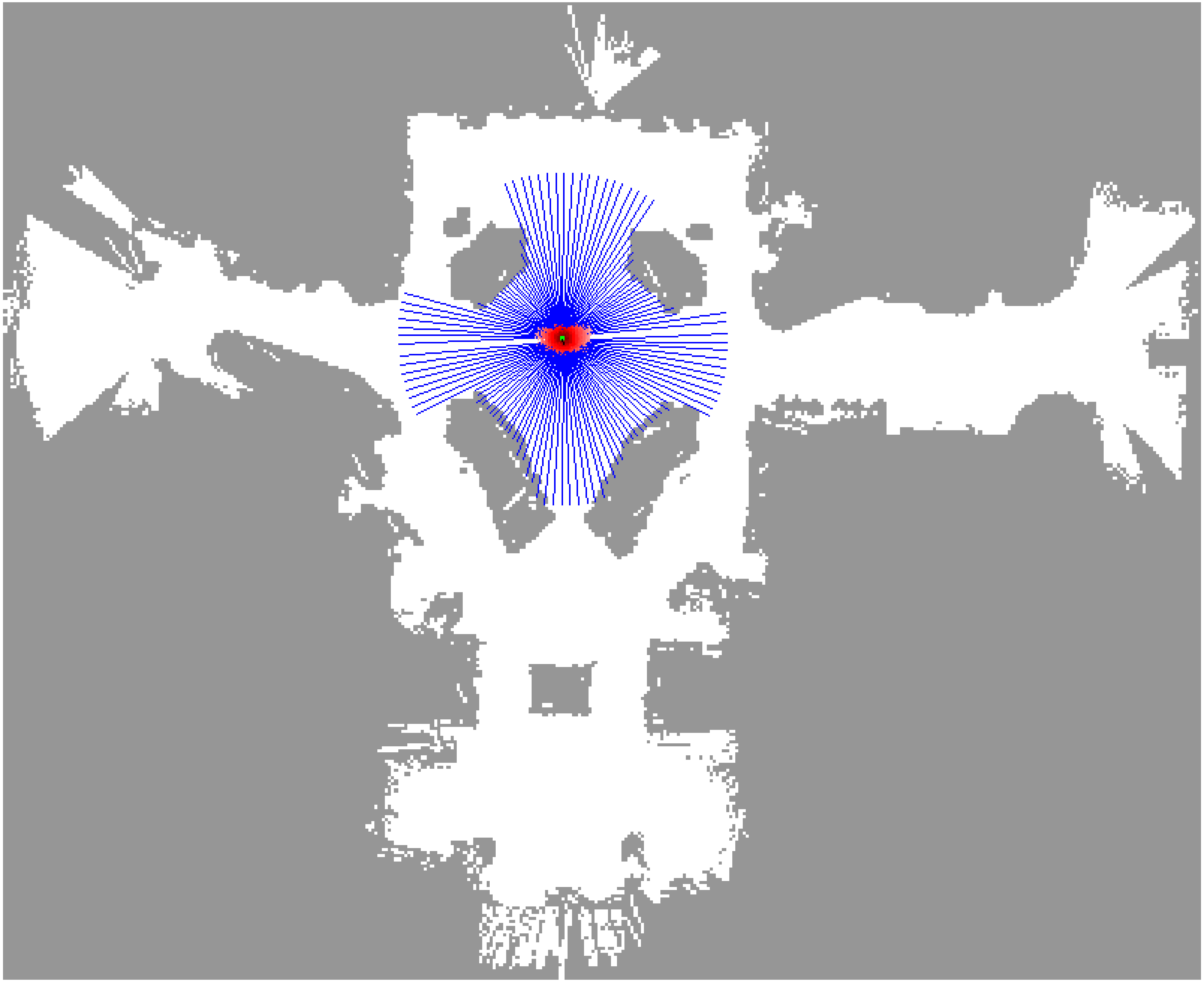


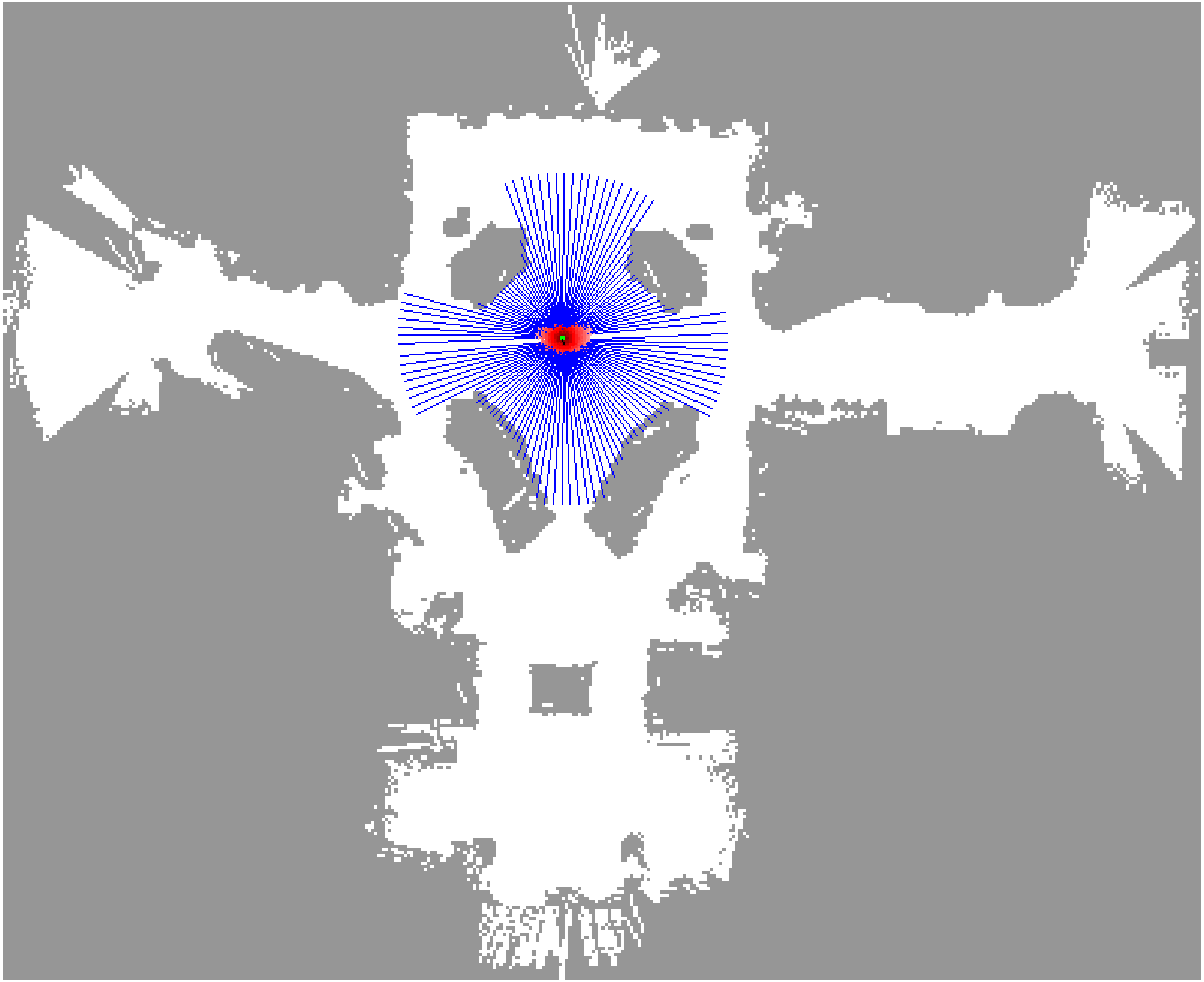




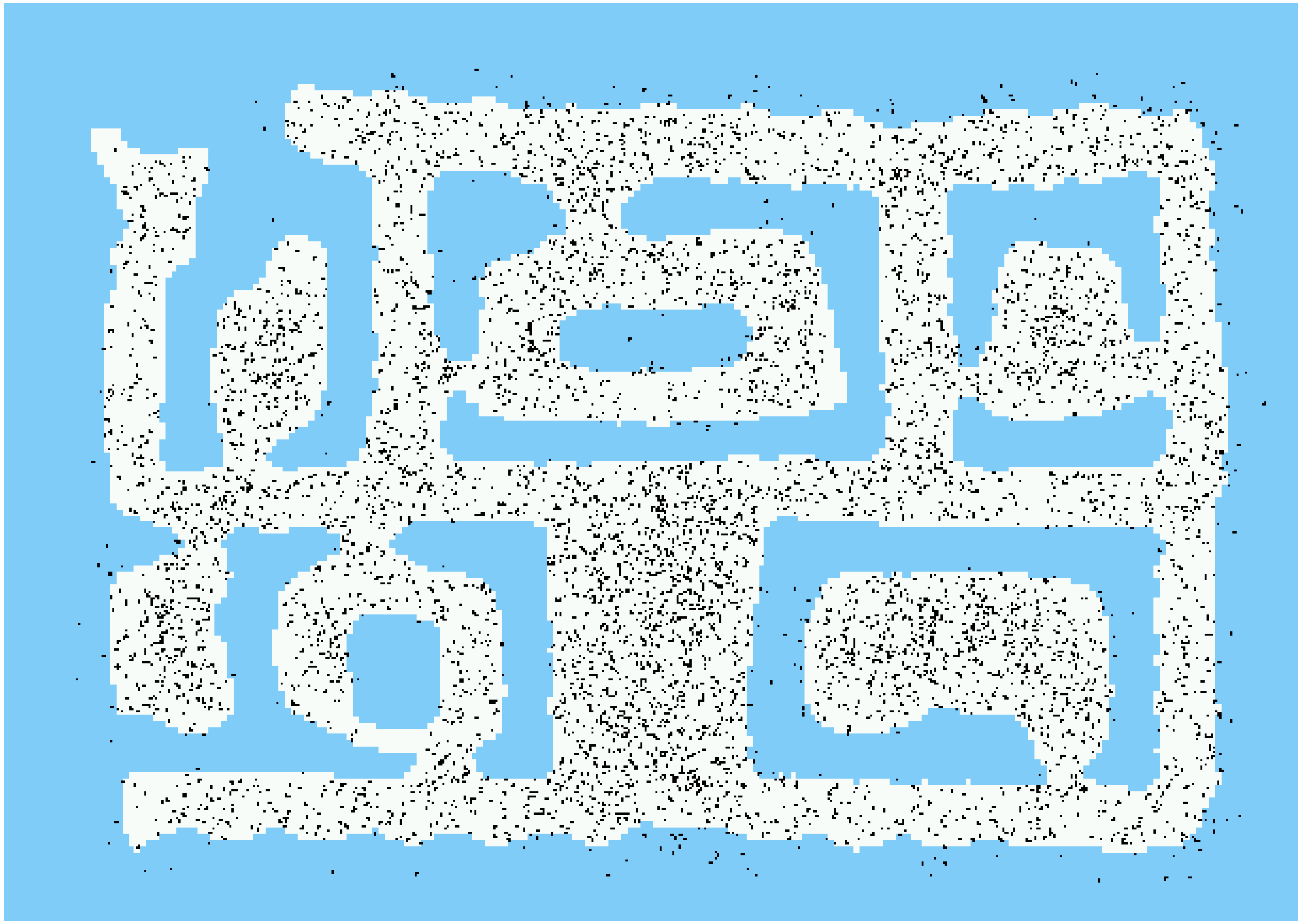




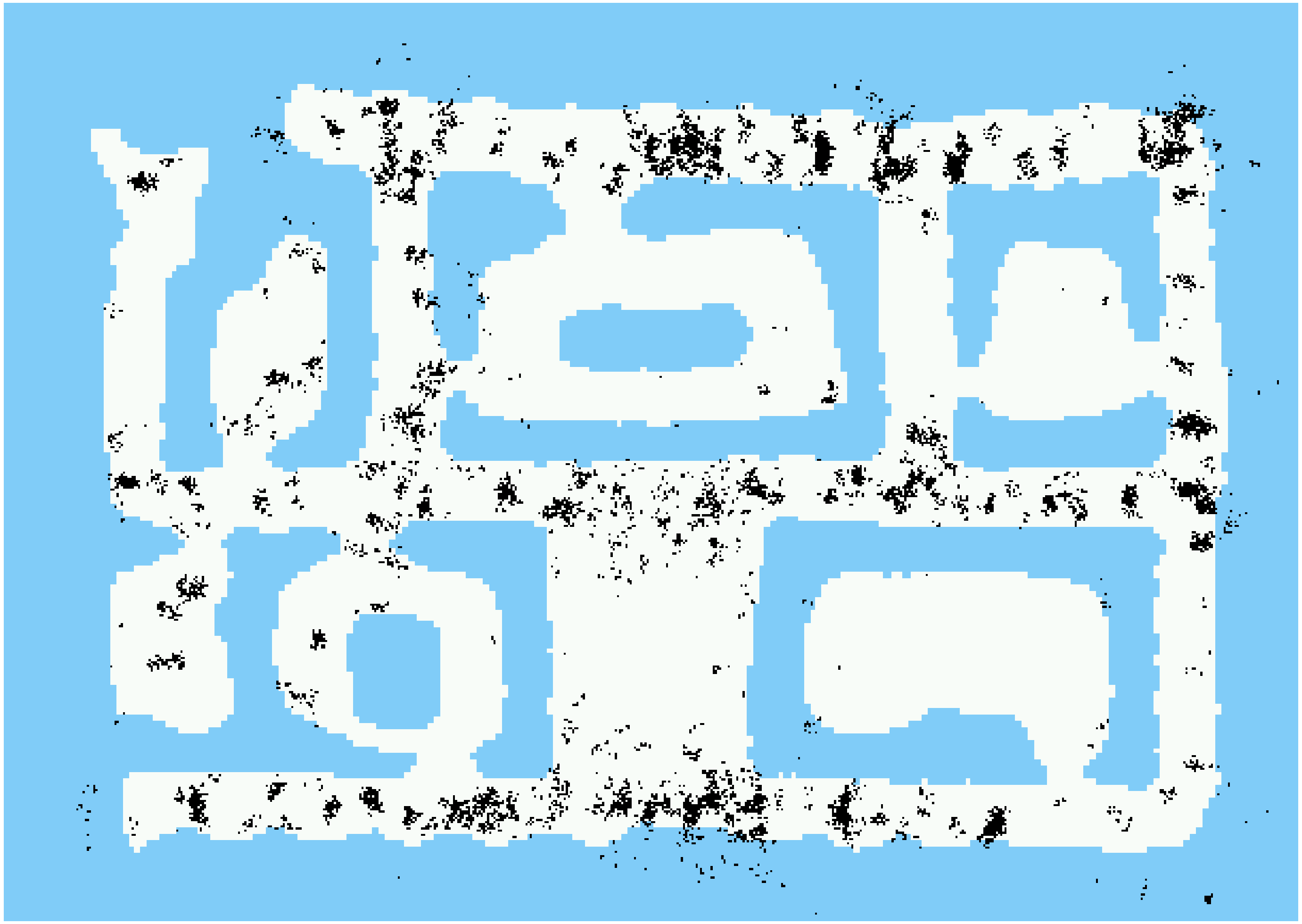




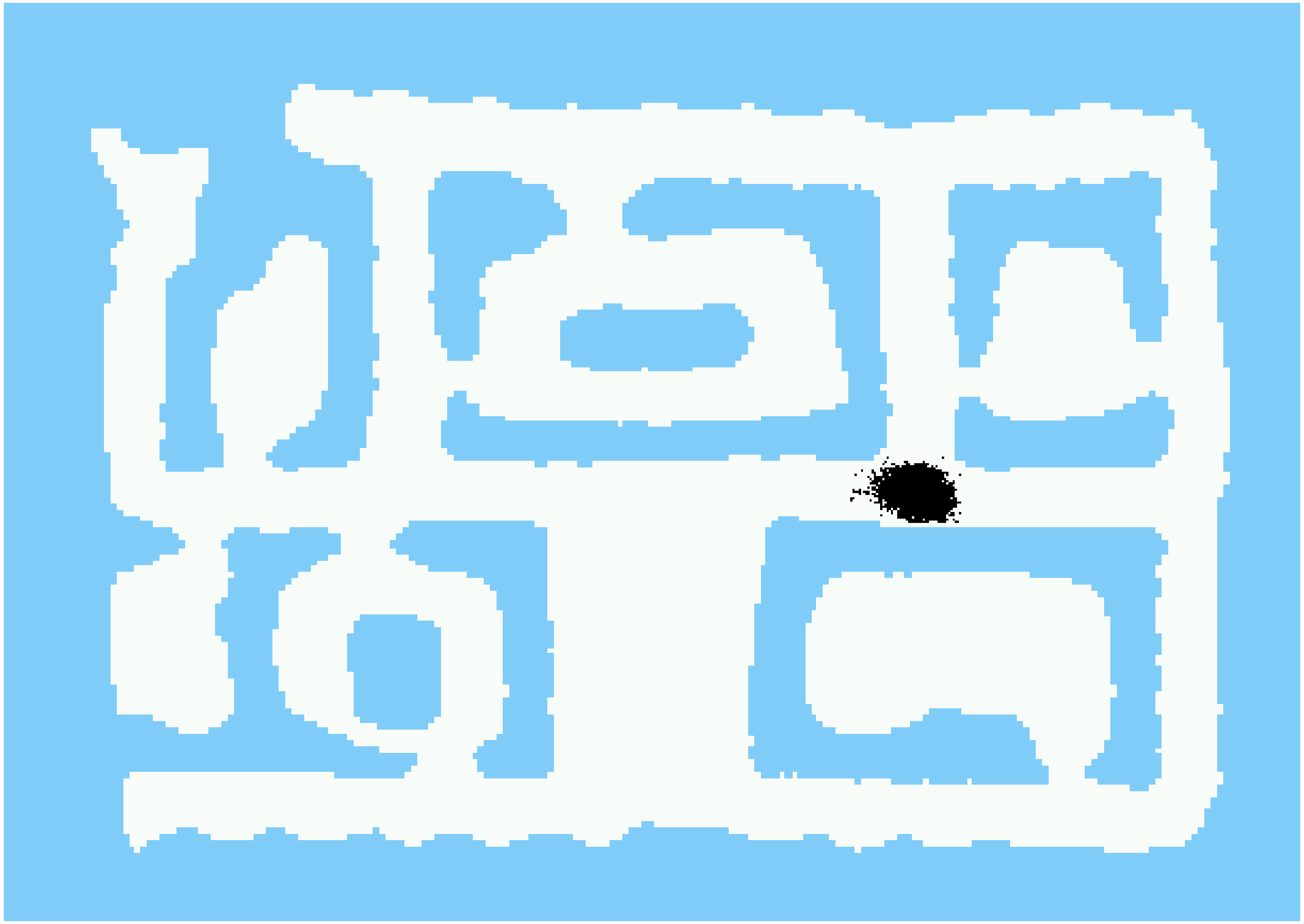
# Initial Distribution



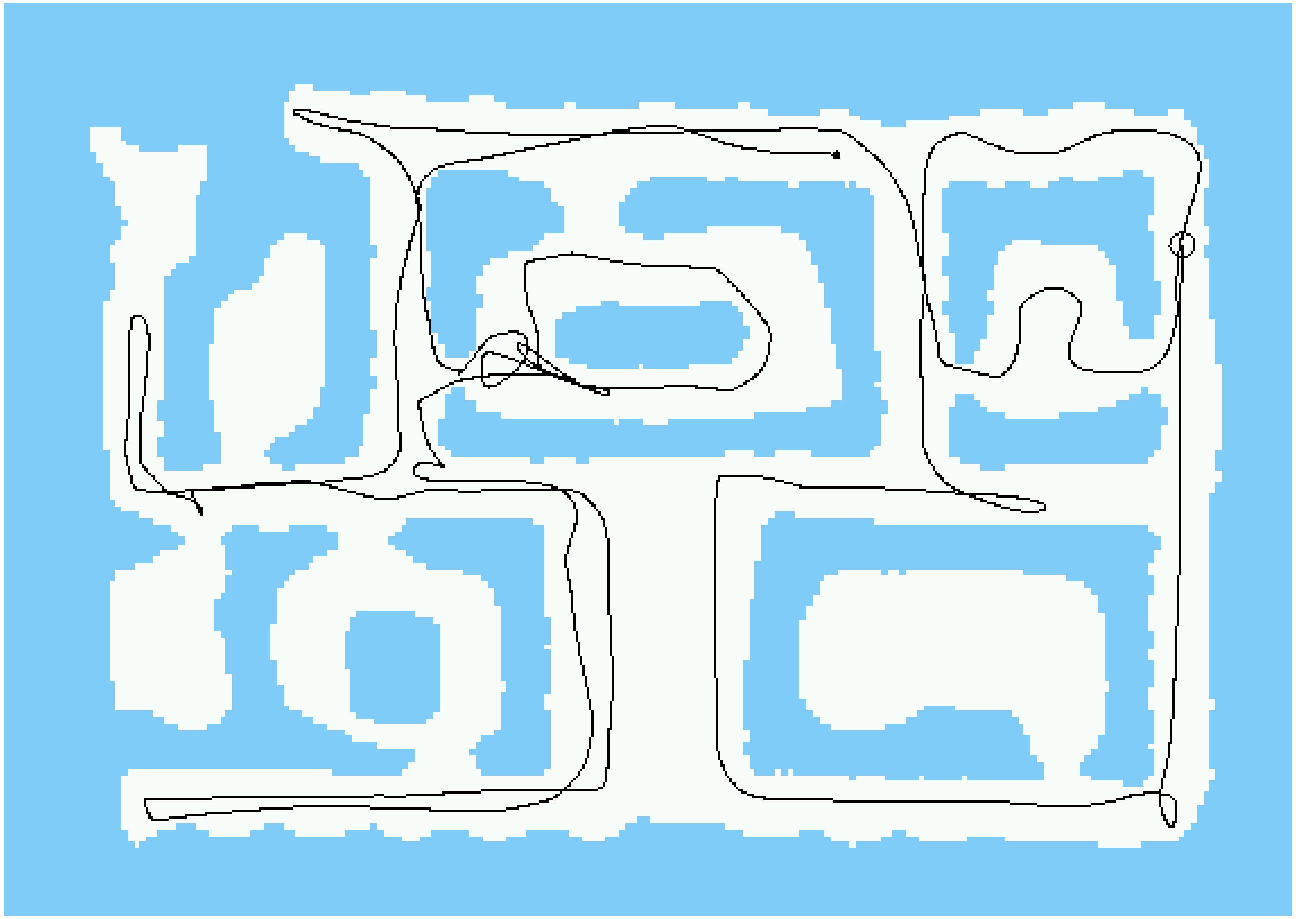
# After Incorporating 10 Sonar Scans



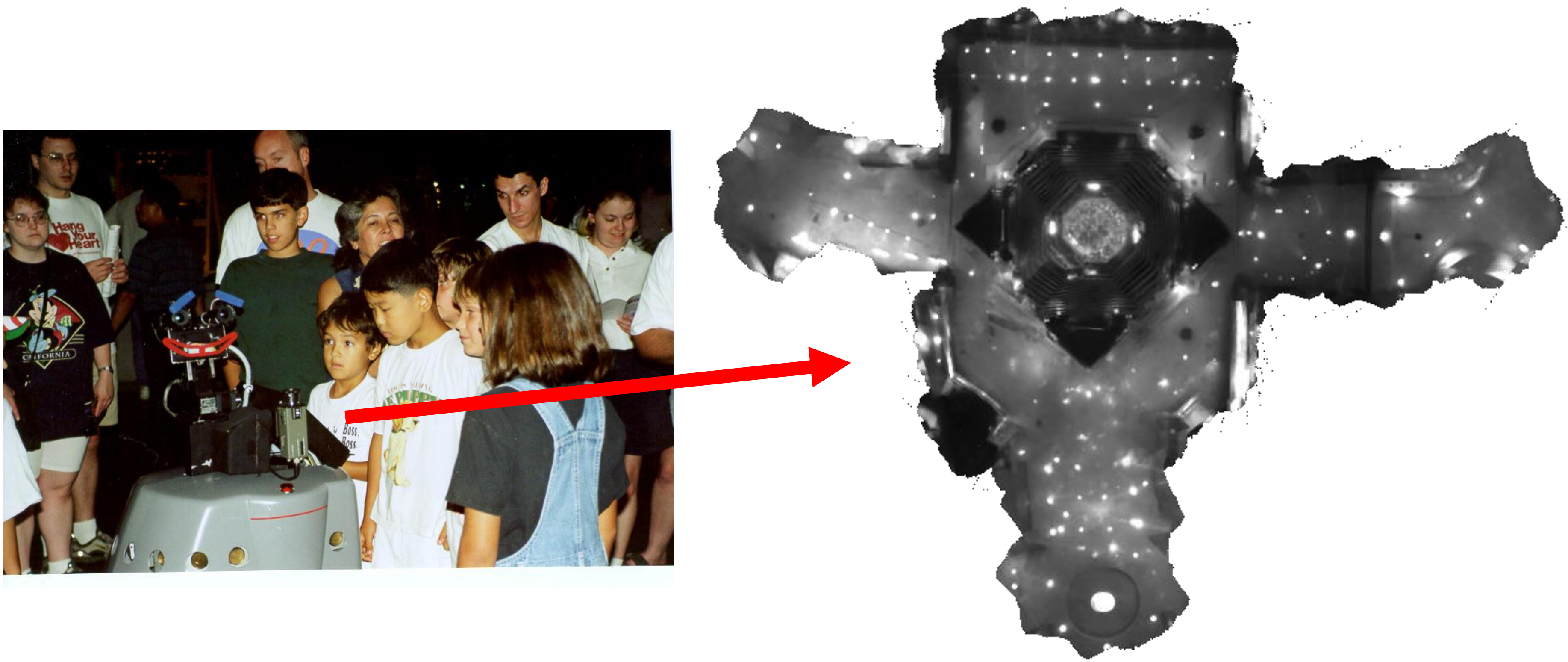
# After Incorporating 65 Sonar Scans



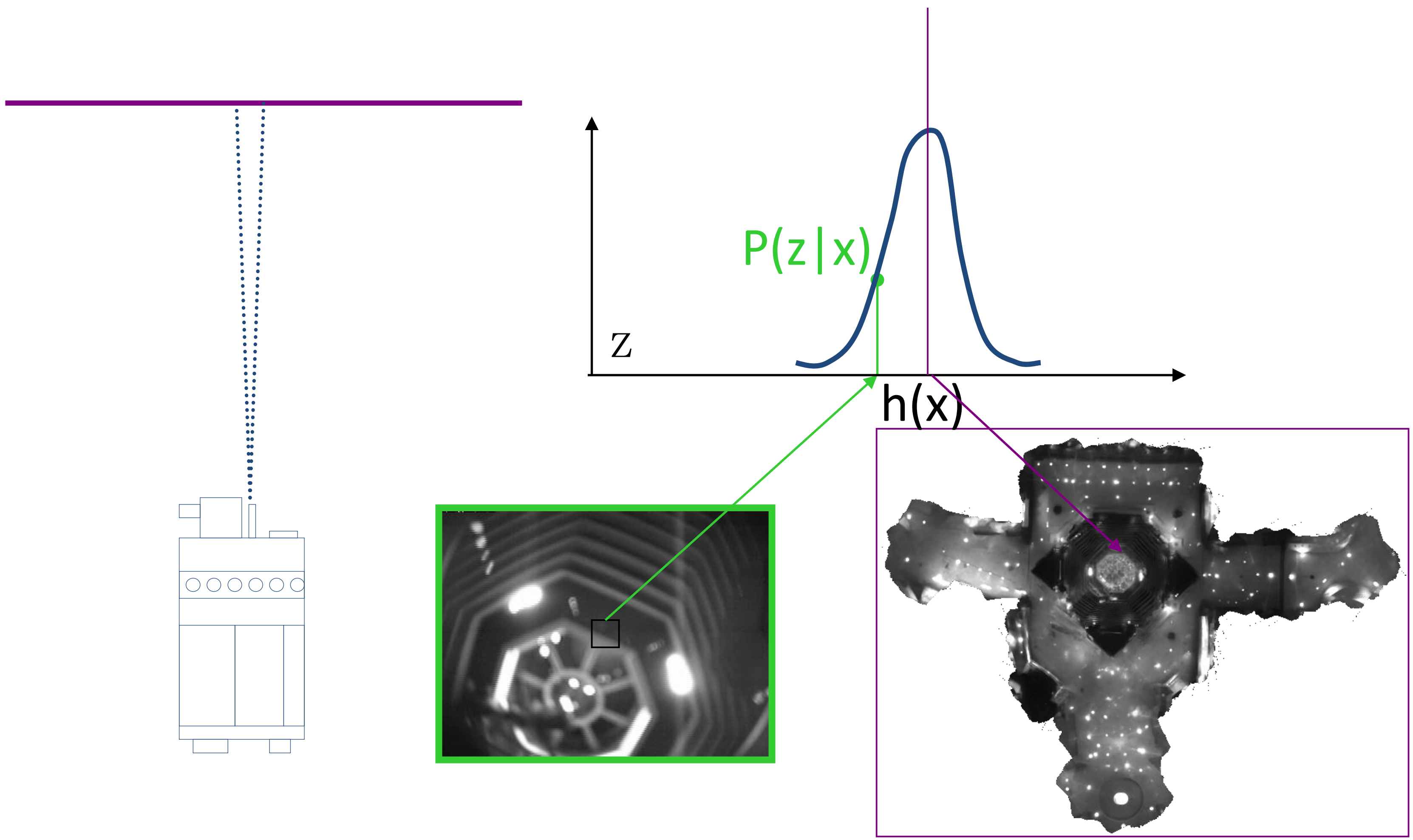
# Estimated Path



# Using Ceiling Maps for Localization

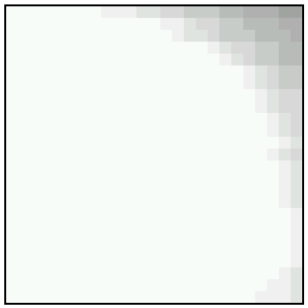


# Vision-Based Localization

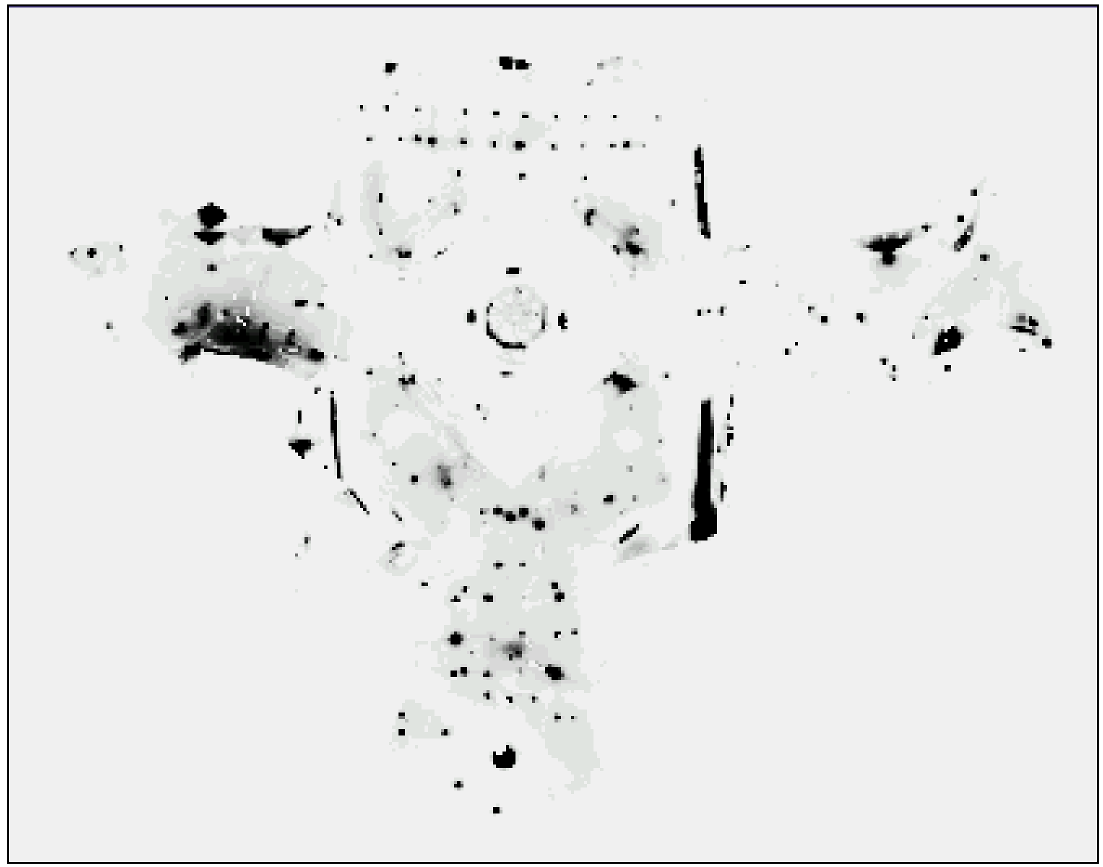


# Under a Light

Measurement z:

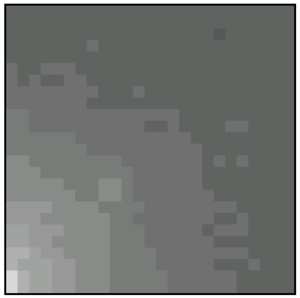


$P(z|x)$:

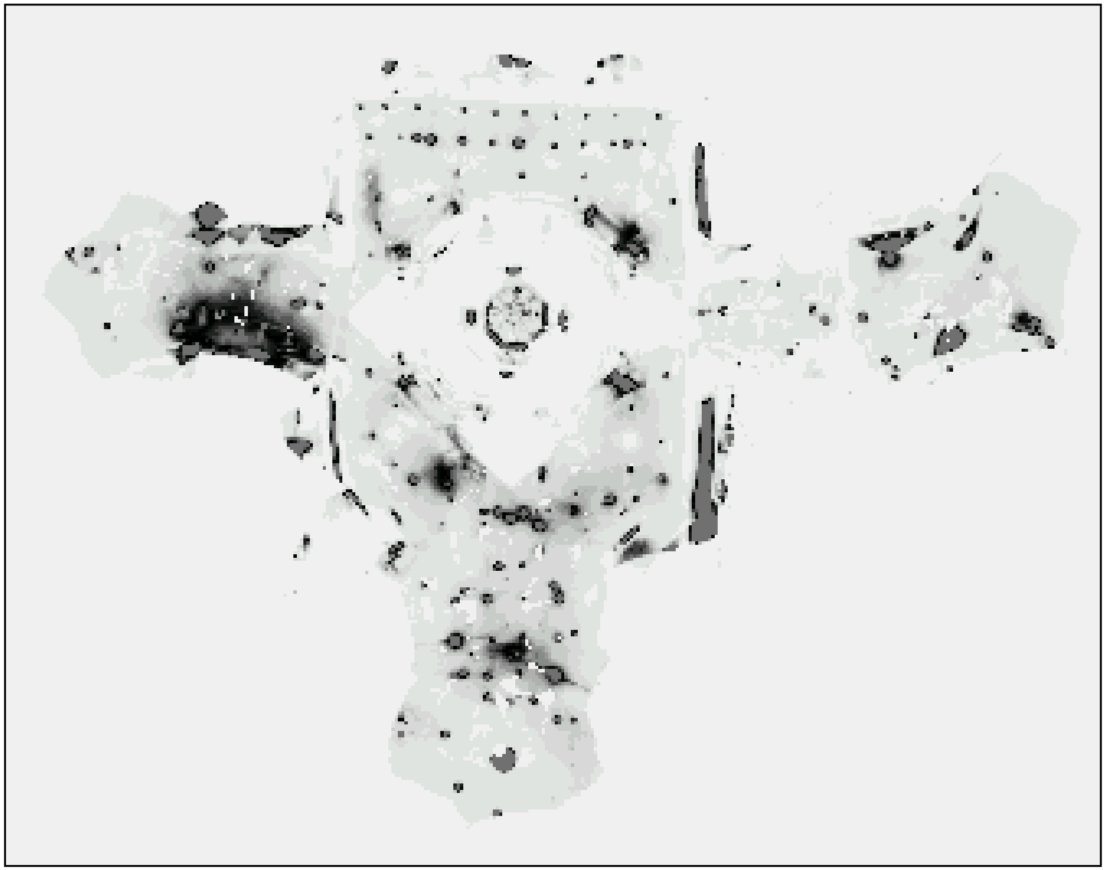


# Next to a Light

Measurement z:



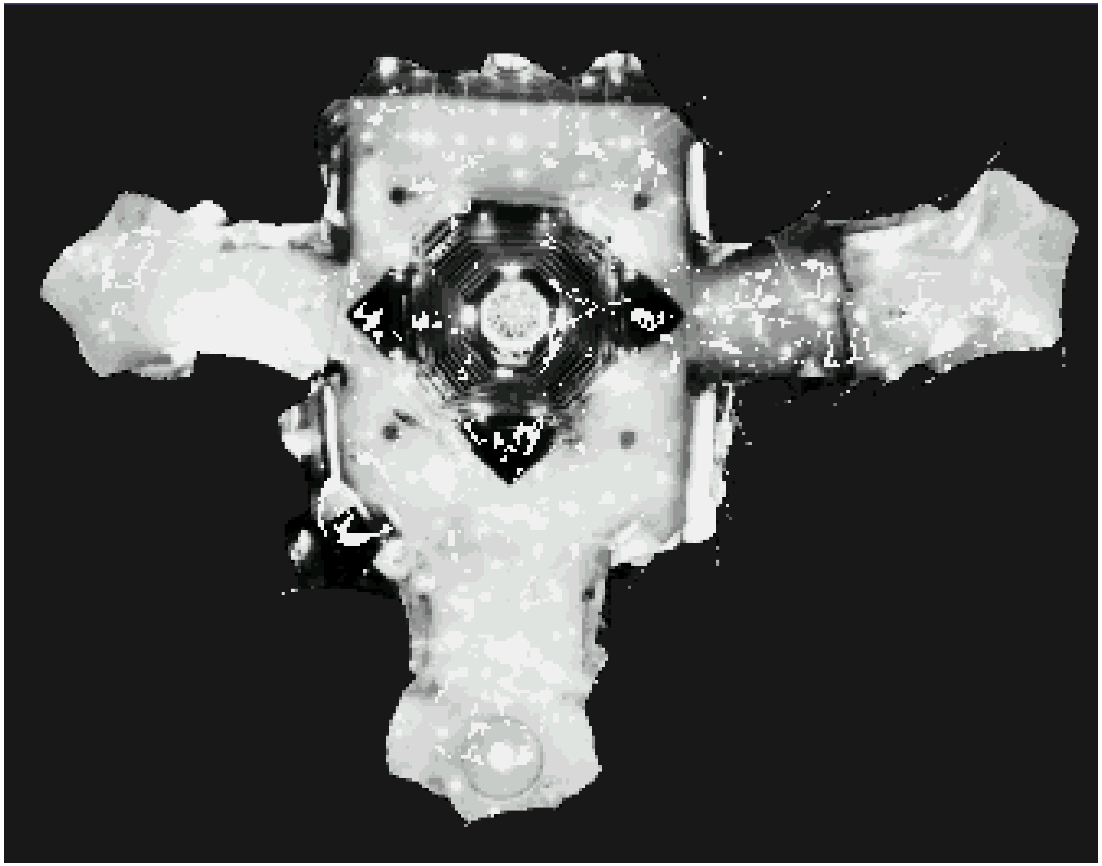
$P(z|x)$:



# Elsewhere

Measurement z:

$P(z|x)$:



# Limitations

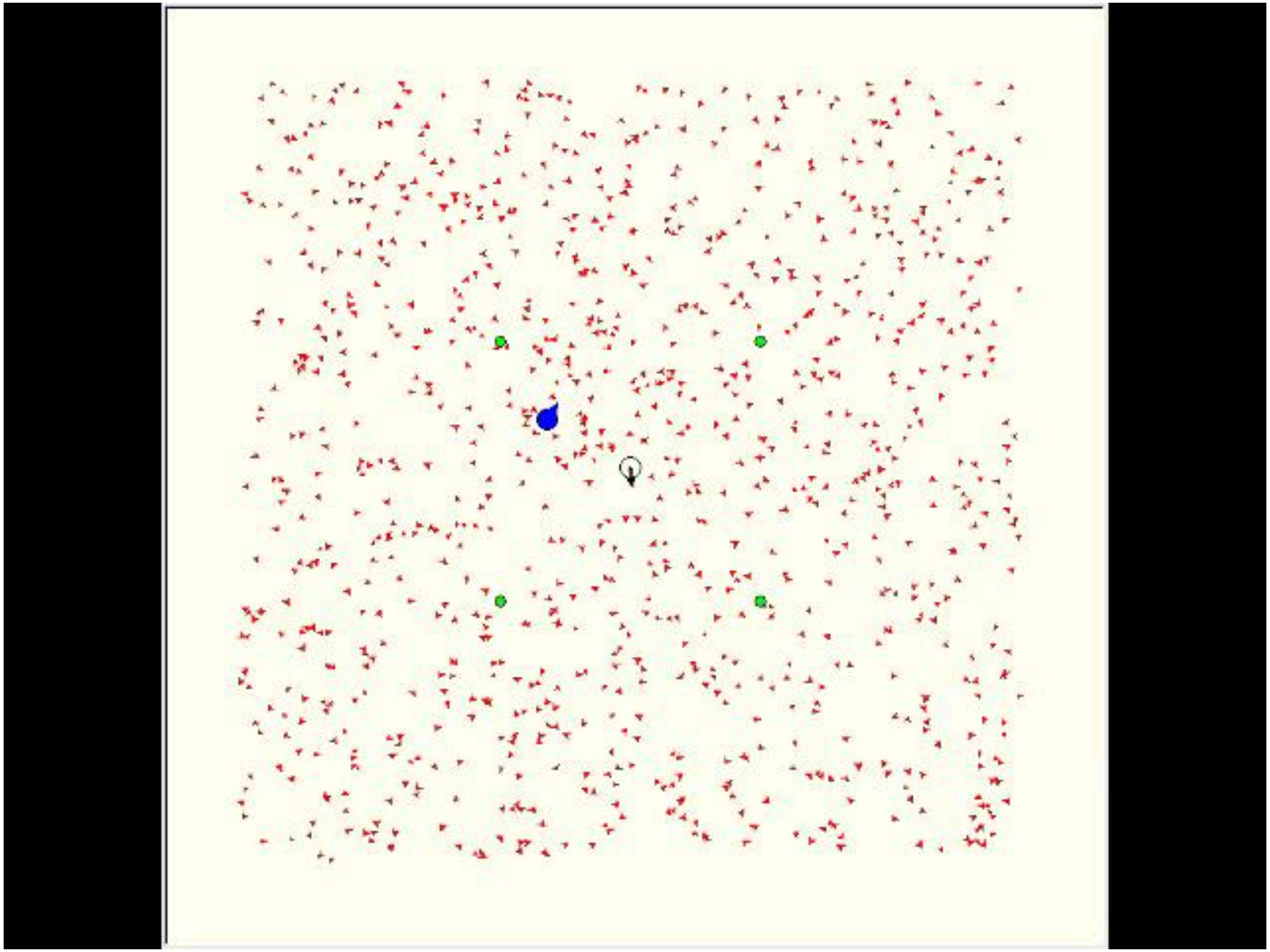
- The approach described so far is able to
  - track the pose of a mobile robot and to
  - globally localize the robot

- How can we deal with the kidnapped robot problem?

# The Kidnapped Robot Problem

- Randomly insert samples (the robot can be teleported at any point in time)
- Insert random samples proportional to the average likelihood of the particles (the robot has been teleported with higher probability when the likelihood of its observations drops)

# The Kidnapped Robot Problem



# Summary

- Particle filters are an implementation of recursive Bayesian filtering
- They represent the posterior by a set of weighted samples.
- In the context of localization, the particles are propagated according to the motion model.
- They are then weighted according to the likelihood of the observations.
- In a re-sampling step, new particles are drawn with a probability proportional to the likelihood of the observation.

# Summary

- Pros:
  - able to represent arbitrary distribution
  - Able to handle nonlinear systems without linearization
- Cons:
  - May need lots of particles to represent high dimensional state space, computational complexity increases significantly w.r.t state dimension
  - Particle degeneracy problem
- Applications:
  - Widely used for low dimensional problems: robot pose tracking, target tracking, etc.
  - Used to be popular for SLAM, but not anymore

# Reading

- "Probabilistic Robotics", Sebastian Thrun, Wolfram Burgard, and Dieter Fox, Chapter 2, Chapter 3

# Logistics

- Project 2, phase 2 due today: 04/13
- Project 3, phase 1 is released, due in one week
- Project 4 released for those of you who choose to do the physical labs
- Lab tutorial this week