Lecture 7

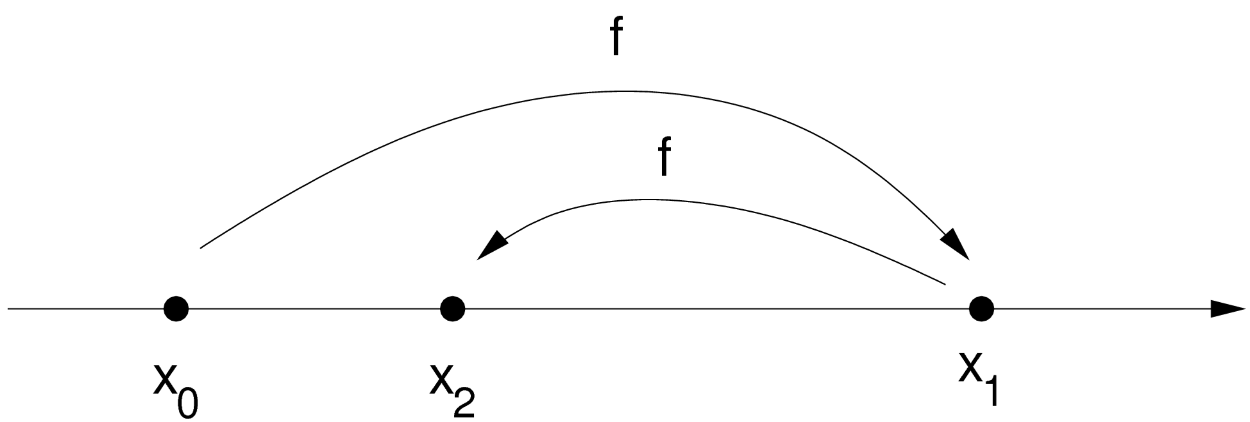
- One dimensional maps
  - Preliminaries
  - Fixed points, stability and cobwebs
  - Logistic map
    - Period doubling
    - Chaos
    - Intermittency
    - Liapunov exponents
    - Universality (qualitative, quantitative)
    - (Renormalization as a way to understand universality)
  - Summary

One Dimensional Maps

- "New" class of dynamical systems in which time is discrete -> difference equations, recursion relations, iterated maps or **maps**
- 1d map: $x_{n+1} = f(x_n)$ ("map" usually refers to the function and the equation ...)
- Orbit: sequence $x_0, x_1, x_2, \ldots$
- Why maps?
  - Tools to analyze differential Eq's (Poincare map, Lorenz map, ...)
  - Models of natural phenomena (digital electronics, economics and finance, certain animal populations ...)
  - Simple examples of chaos

One Dimensional Maps

- Why can 1d maps exhibit much “richer” dynamical behaviour then 1d continuous systems?



- We'll see later that 1d maps can exhibit:
  - Fixed points, oscillations and even chaos

Fixed Points

- Fixed point: $x^* = f(x^*)$
- Stability?
  - Consider nearby orbit $x_n = x^* + \eta_n$ Is it attracted or repelled from x*?

$$x_{n+1} = x^* + \eta_{n+1} = f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2)$$

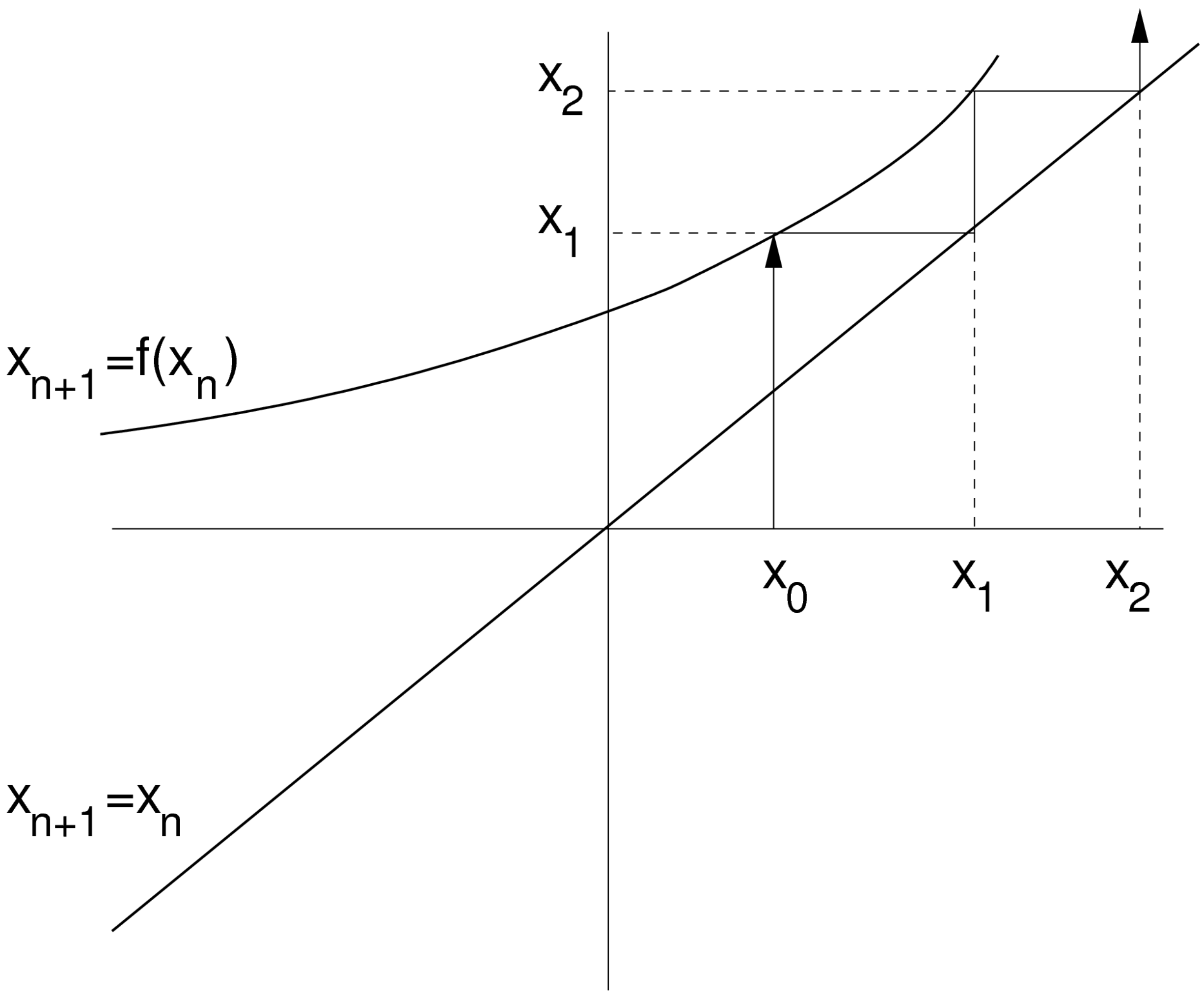
$$\longrightarrow \quad \eta_{n+1} = f'(x^*)\eta_n + O(\eta_n^2)$$

  - Neglect O(η^2) terms -> linearized map with eigenvalue/multiplier λ=f'(x*)

$$\longrightarrow \quad \eta_n = \lambda^n \eta_0$$

  - |f'(x*)|<1 -> **linearly stable**, =1 **marginal**, >1 **unstable**
  - f'(x*)=0 -> **superstable** $\eta_n \propto \eta_0^{(2^n)}$

Cobwebs



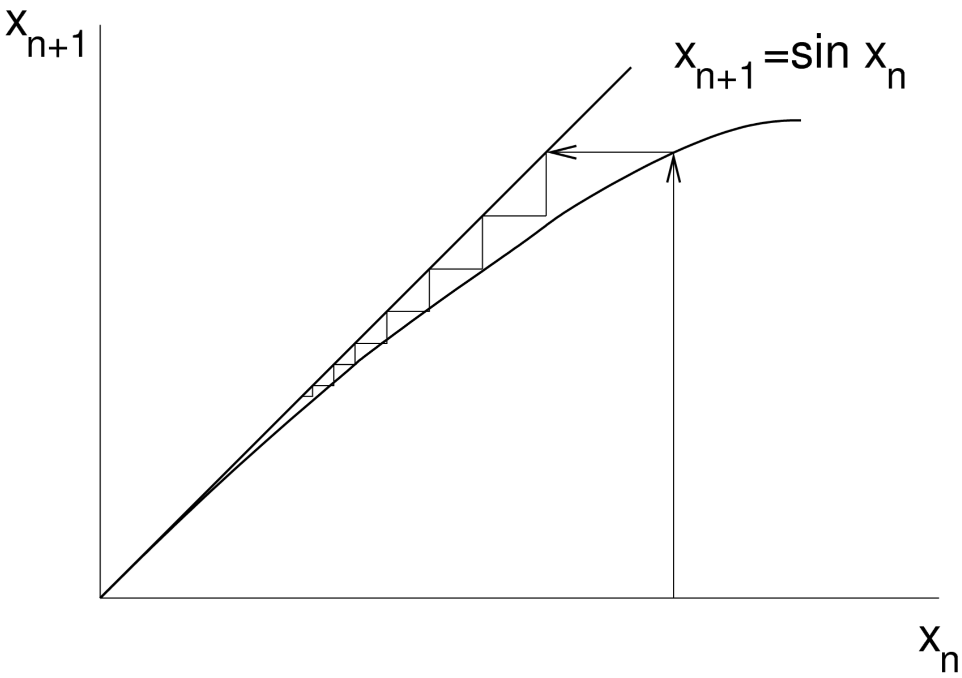
Examples

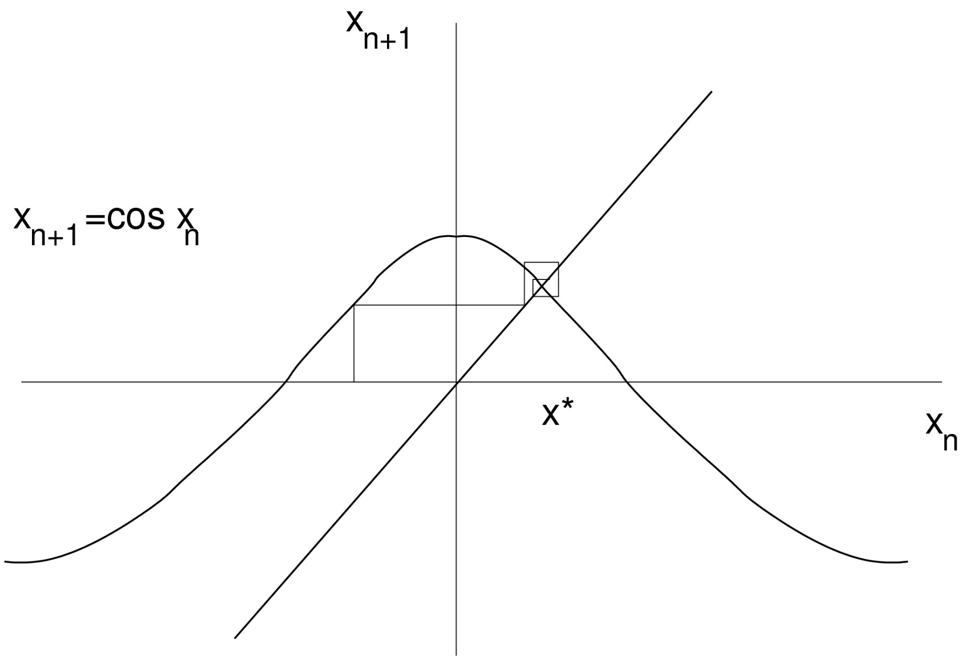
- Let's have a look at

$$x_{n+1} = \sin x_n \qquad\qquad x_{n+1} = \cos x_n$$

$$x^* = 0 \qquad\qquad x^* = 0.739\ldots$$

$$\lambda = f'(0) = \cos(0) = 1 \qquad\qquad \lambda = -\sin(0.739\ldots), 0 > \lambda > -1$$



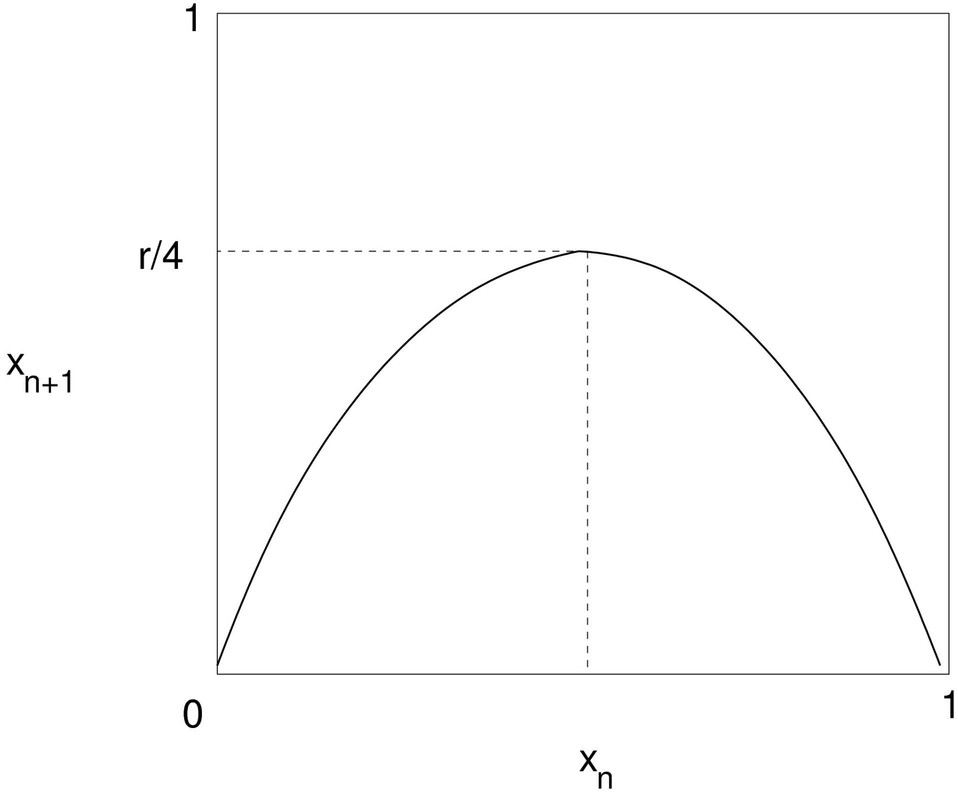


Logistic Map

- Analogue of logistic eq. for population growth

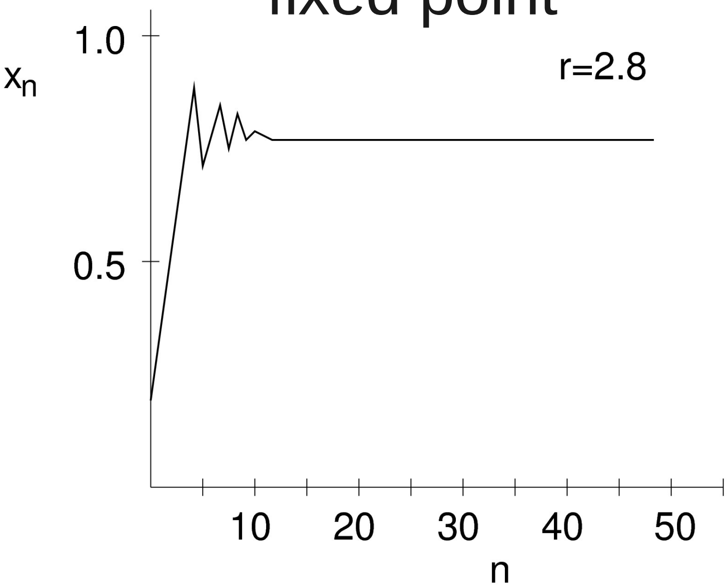
$$x_{n+1} = r\,x_n(1 - x_n)$$

  - x_n ... population in nth generation
  - r ... growth rate, consider 0<=r<=4



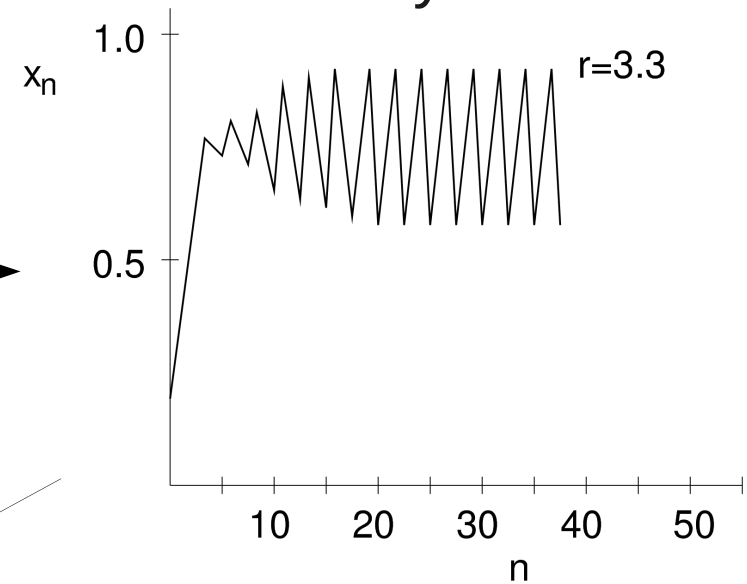
Numerics

fixed point



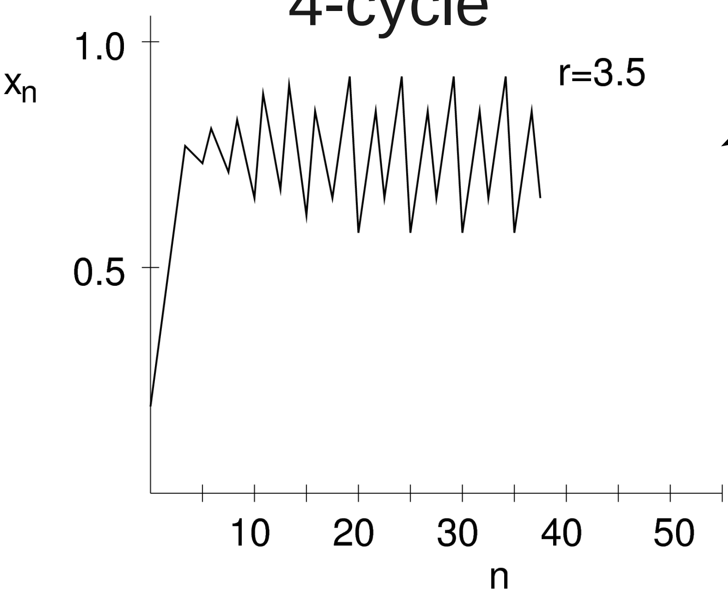
Period
Doubling

2-cycle



Period
Doubling

4-cycle



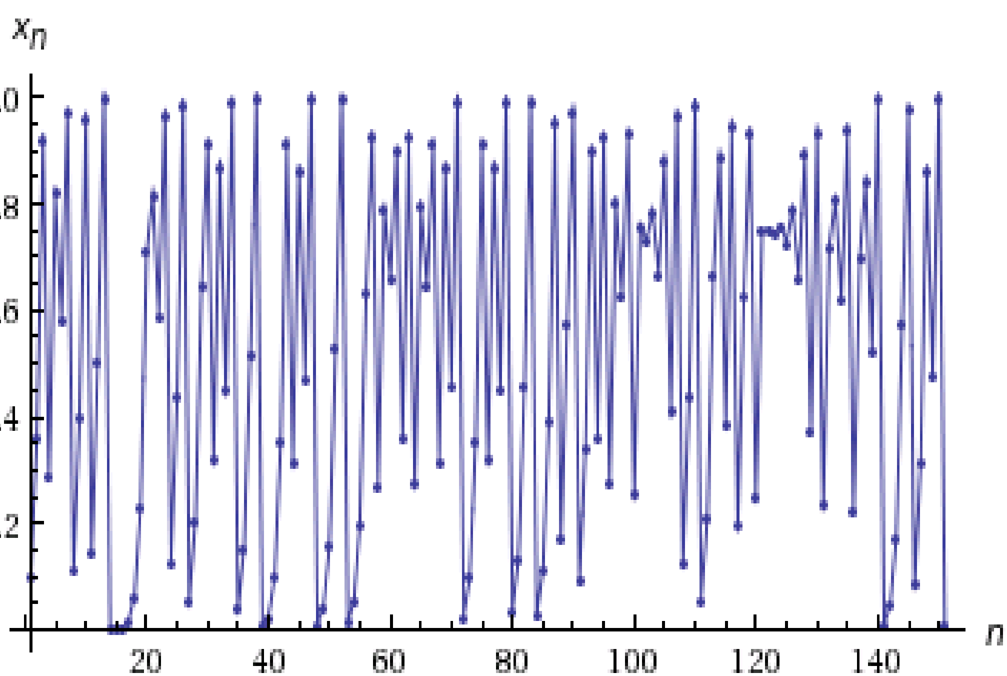
Further Period
Doublings to 8, 16, 32, ... cycles

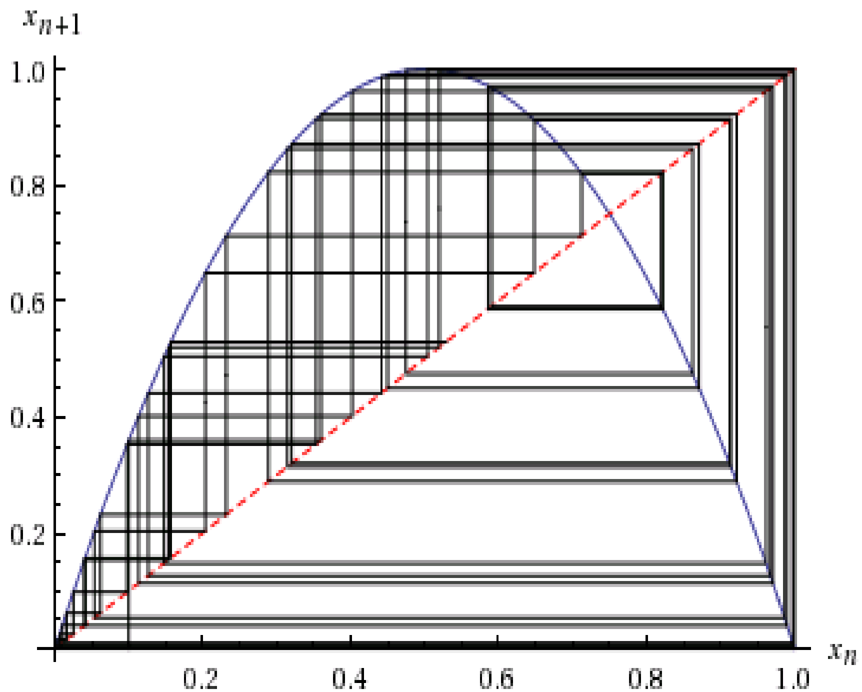
Period Doubling

- r_n ... value of r where 2^n-cycle is born
  - $r_1=3$ 2-cycle
  - $r_2=3.449...$ 4-cycle
  - $r_3=3.54409...$ 8-cycle
  - $r_4=3.5644...$ 16-cycle
  - $r_{inf}=3.569946...$ infinite cycle
- Distances between successive bifurcations become smaller and smaller ... geometric convergence
- What about $r>r_{inf}$?

Chaos ...

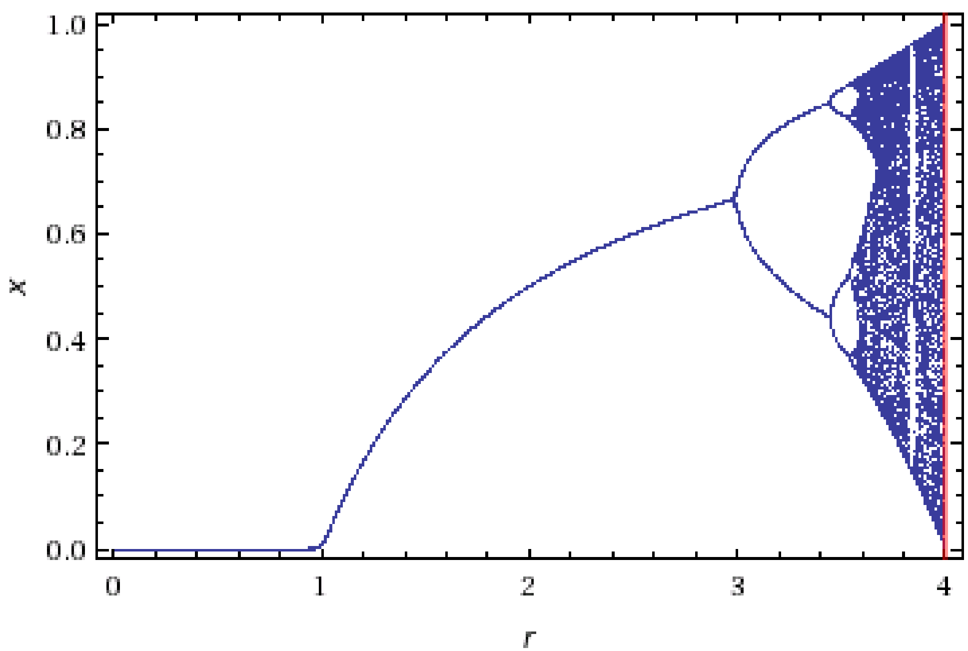
- For example r=3.9 – aperiodic irregular dynamics similar to what we have seen for continuous systems
- However ... not all $r > r_{inf}$ have chaotic behaviour!





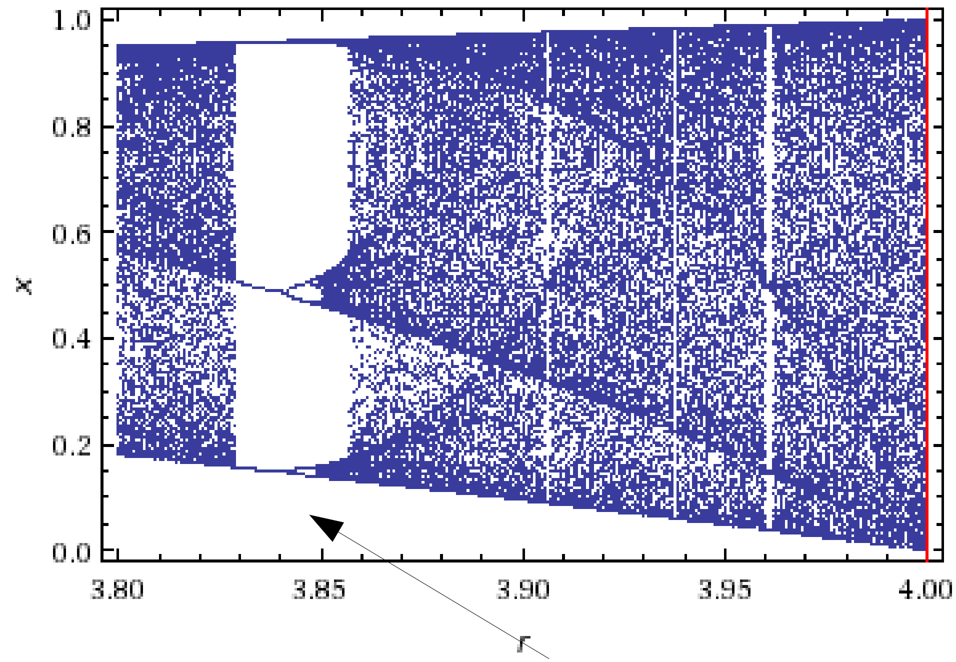
(lines successively connect the first 50 iterates and the dashed line $y = x$)

Bifurcation Diagram



(iterates 100 through 150 for each *r*)

Zoomed in:



(iterates 300 through 450 for each *r*)

stable 3-cycle

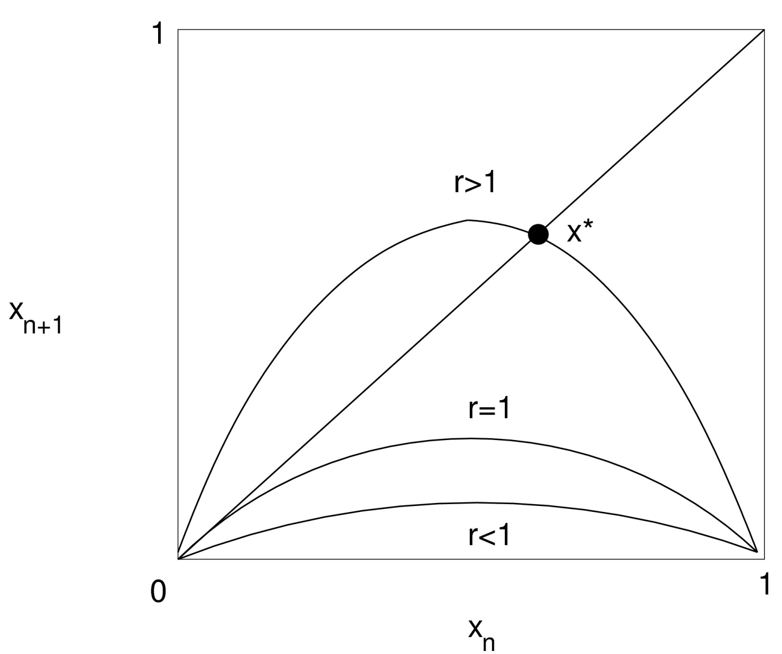
- For $r > r_{inf}$ diagram shows mixture of order and chaos, periodic windows separate chaotic regions
- Blow-up of parts appear similar to larger diagram ...

Logistic Map -- Analysis

- Fixed points and stability $x_{n+1}=r\,x_n(1-x_n)$

  $$x^*=r\,x^*(1-x^*) \longrightarrow x^*=0 \;\vee\; x^*=1-1/r$$

- Stability $f'(x^*)=r-2\mathrm{rx}^*$
  - f'(0)=r -> origin is stable for r<1
  - f'(x*)=2-r -> stable for -1<2-r<1, i.e. unstable for r>3

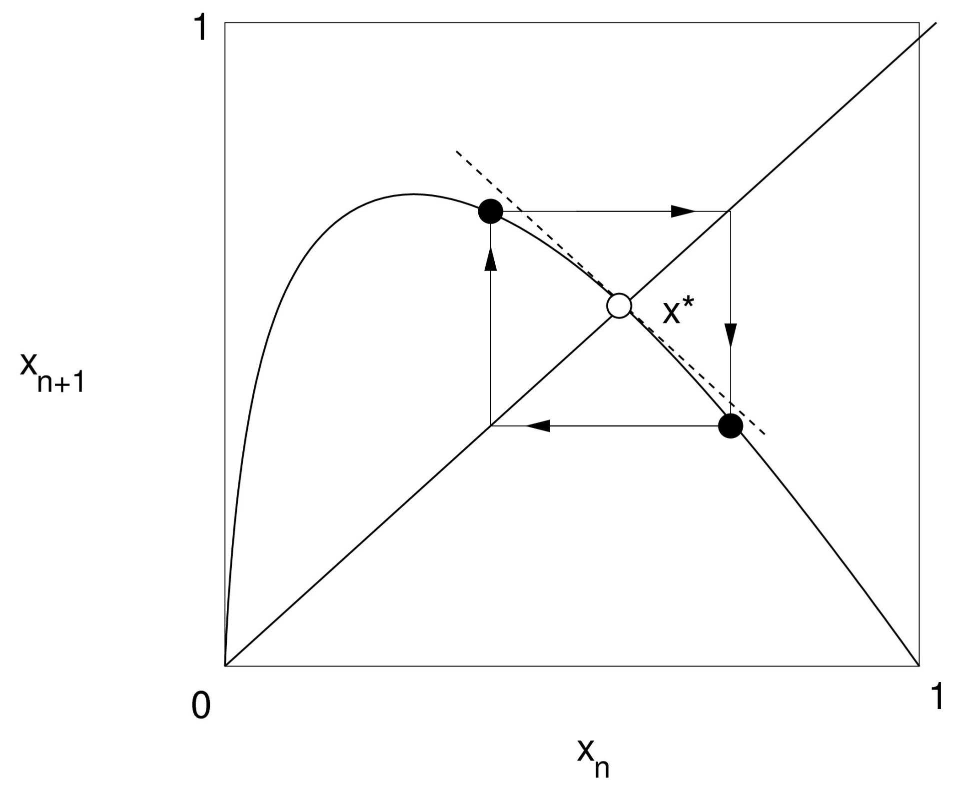


- Small r -> origin only FP
- Increasing r -> parabola “grows”, becomes tangential to diagonal
- r>1 parabola intersects diagonal in a second FP, origin loses stab.

  -> **transcritical bifurcation**
- Larger r -> slope at x* becomes steeper and f'(x*)=-1 for r=3

  -> **flip bifurcation**

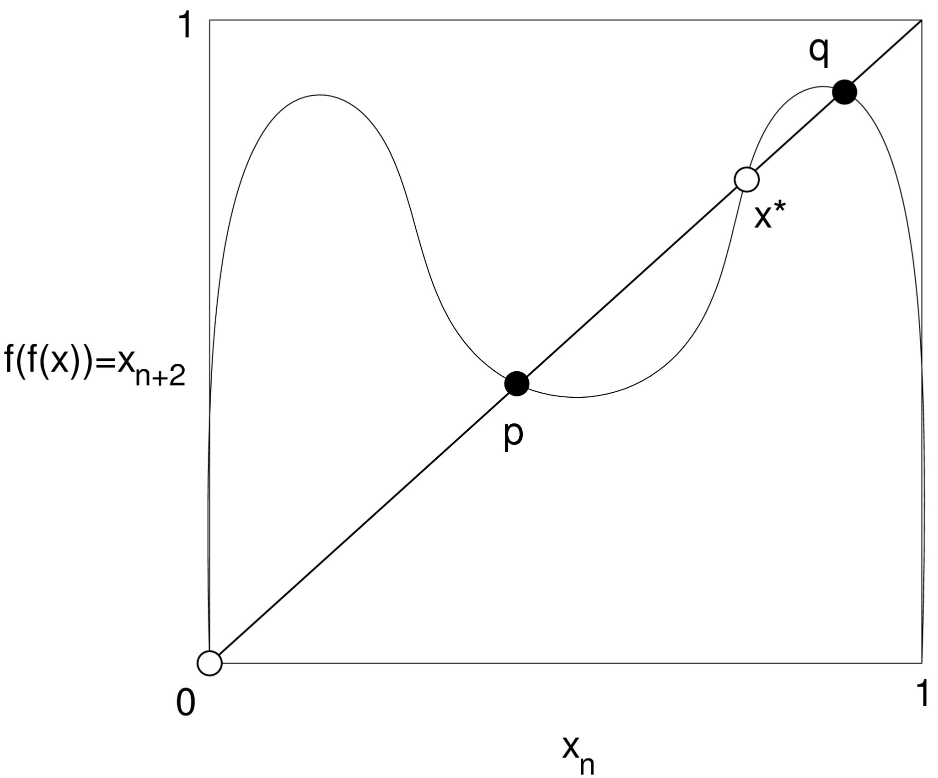
Flip Bifurcations and Period Doubling



- Local picture near a FP with f'(x*)=-1, if f is concave a small stable 2-cycle appears
- This is an example of a supercritical flip bifurcation

Analyzing 2-cycles

- 2-cycle exists if there exist p and q with (p!=q) and f(p)=q and f(q)=p
- ... or: p is a fixed point of second iterate p=f(f(p))



$$f^2(x) = r(rx(1-x))(1-rx(1-x))$$

Solve:

$$f^2(x) - x = 0$$

Factor out x and x-(1-1/r) ...

$$p,q = \frac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

Exists for r>3 and bifurcates cont. from 1-1/r at r=3

Stability of 2-cycles

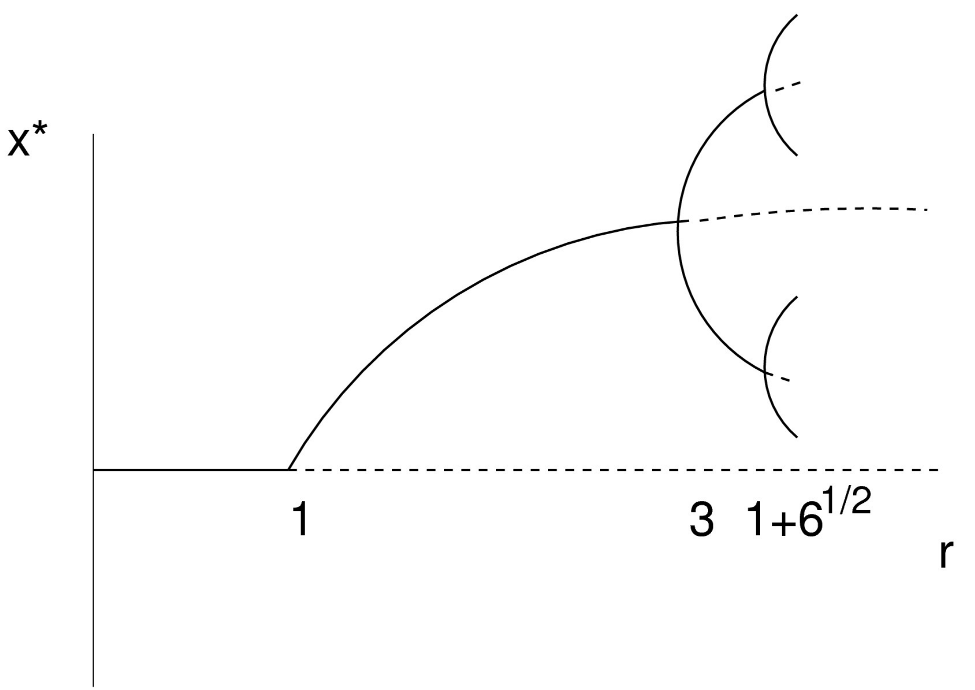
- Calculate multiplier of second iterate f(f(x))

$$\lambda = d/dx\left(f\left(f\left(x\right)\right)\right)_{x^*=p} = f'\left(f\left(p\right)\right)f'\left(p\right) = f'\left(p\right)f'\left(q\right)$$

$$\lambda = r\left(1-2\mathrm{q}\right)r\left(1-2\mathrm{p}\right)$$

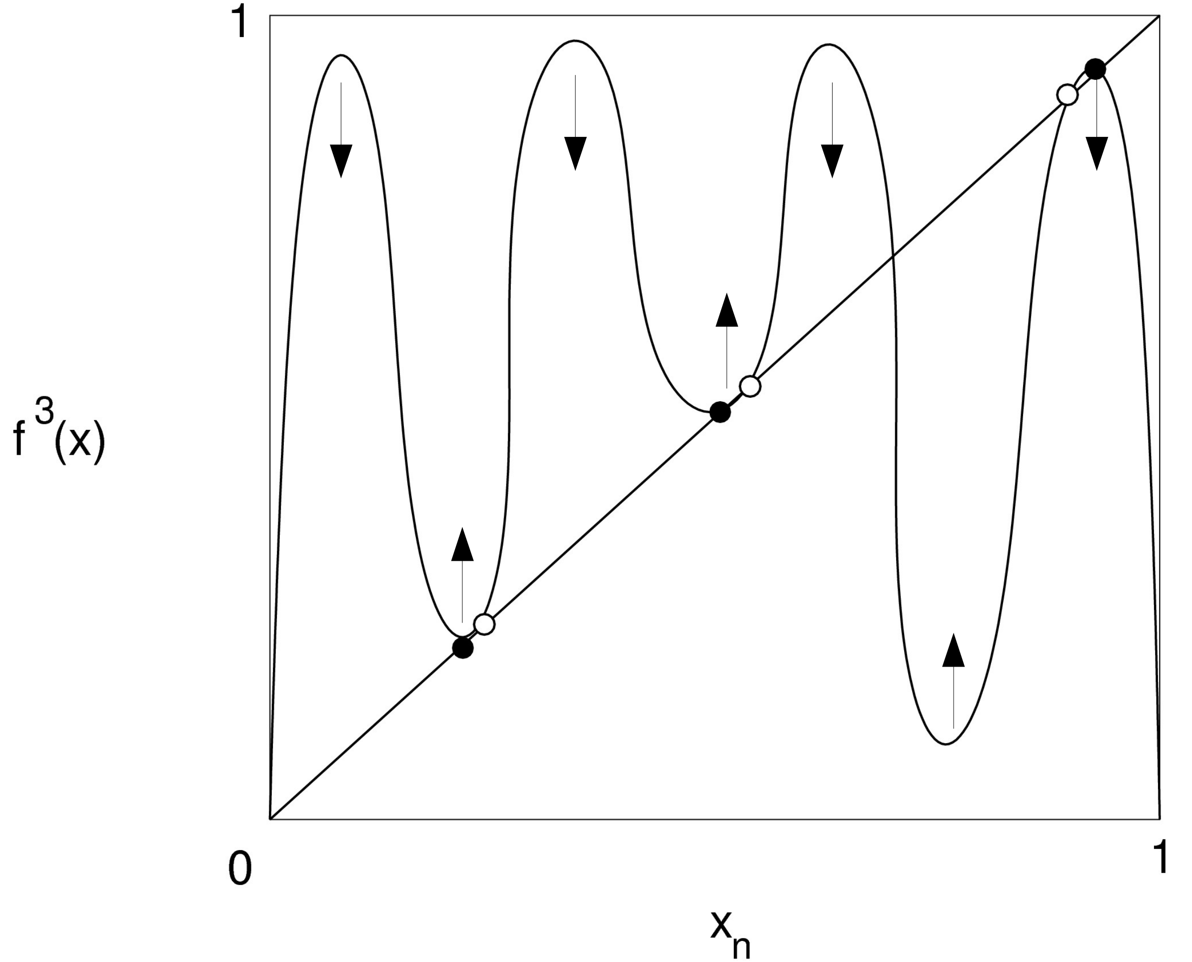
$$\lambda = 4+2\mathrm{r}-r^2$$

$$\longrightarrow \quad \left|4+2\mathrm{r}-r^2\right|<1 \quad \longrightarrow \quad 3<r<1+\sqrt{6}$$



Understanding Periodic Windows

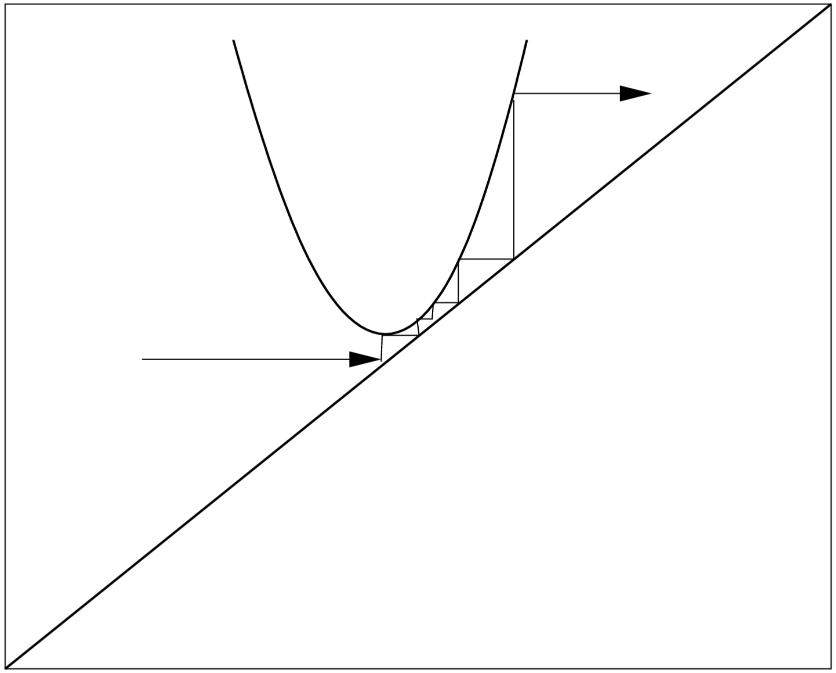
- Birth of a stable 3-cycle, $f^3(p)=p$ (8th degree)



Tangent bifurcation at $r=1+8^{1/2}$

Intermittency

- Just below period-3 window trajectories show intervals of period three behaviour interspersed with intervals of chaotic behaviour
- “Ghost” of a 3-cycle ...



Intermittency

- Intermittency is common in systems in which transition to chaos occurs via a saddle node bifurcation of cycles
- In experimental systems (e.g. laser systems):
  - Appears as nearly periodic motion interrupted by occasional irregular bursts which are statistically distributed
  - Bursts become more and more frequent ...
  - **Intermittency route to chaos**

Liapunov Exponents

- Consider x_0 and $x_0+\delta_0$. δ_n is separation after n iterations. If $|\delta_n|=|\delta_0| \exp(n\lambda)$ -> λ is Liapunov exponent
- More precisely: $\delta_n = f^n(x_0+\delta_0) - f^n(x_0)$

$$\lambda \approx 1/n \ln|\delta_n/\delta_0|$$
$$= 1/n \ln\left|\frac{f^n(x_0+\delta_0)-f^n(x_0)}{\delta_0}\right|$$
$$\approx 1/n \ln|(f^n)'(x_0)| = 1/n \ln\left|\prod_{i=0}^{n-1} f'(x_i)\right|$$
$$= 1/n \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

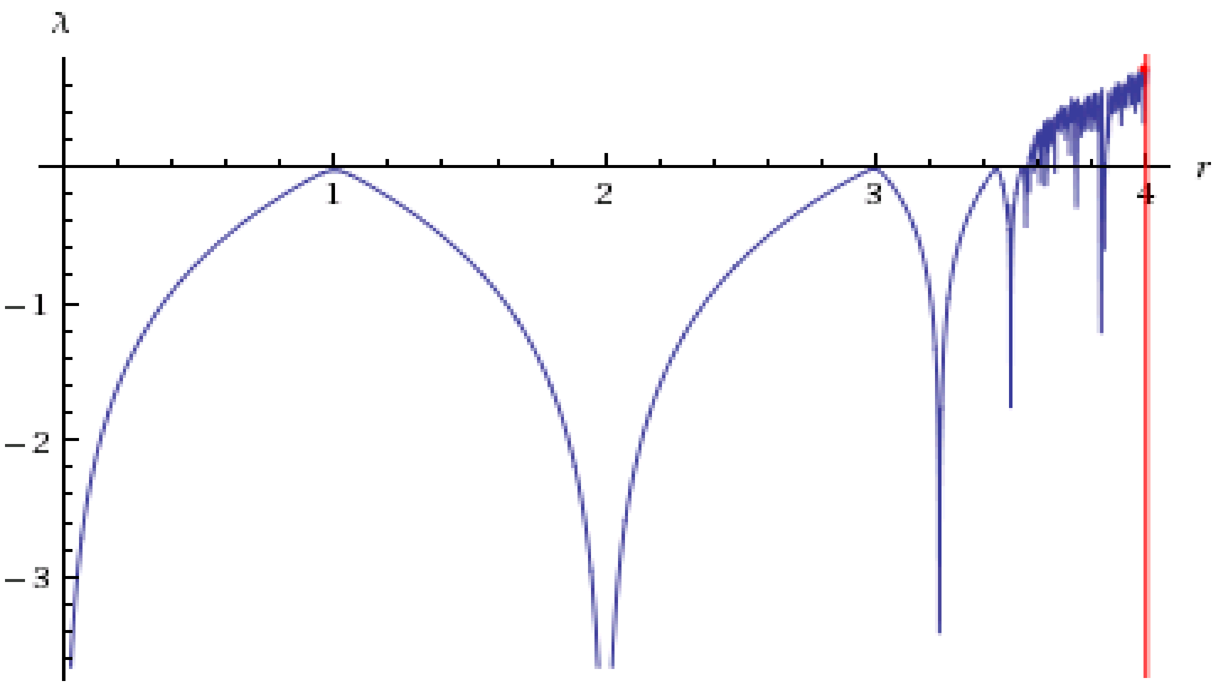
Liapunov Exponents

- So, for an orbit starting at x_0 we define

$$\lambda = \lim_{n\to\infty} 1/n \sum_{i=0}^{n-1} \ln|f'(x_i)|$$

- Stable fixed points and cycles: $\lambda<0$

superstable: $\lambda=-\infty$ chaotic attractors: $\lambda>0$



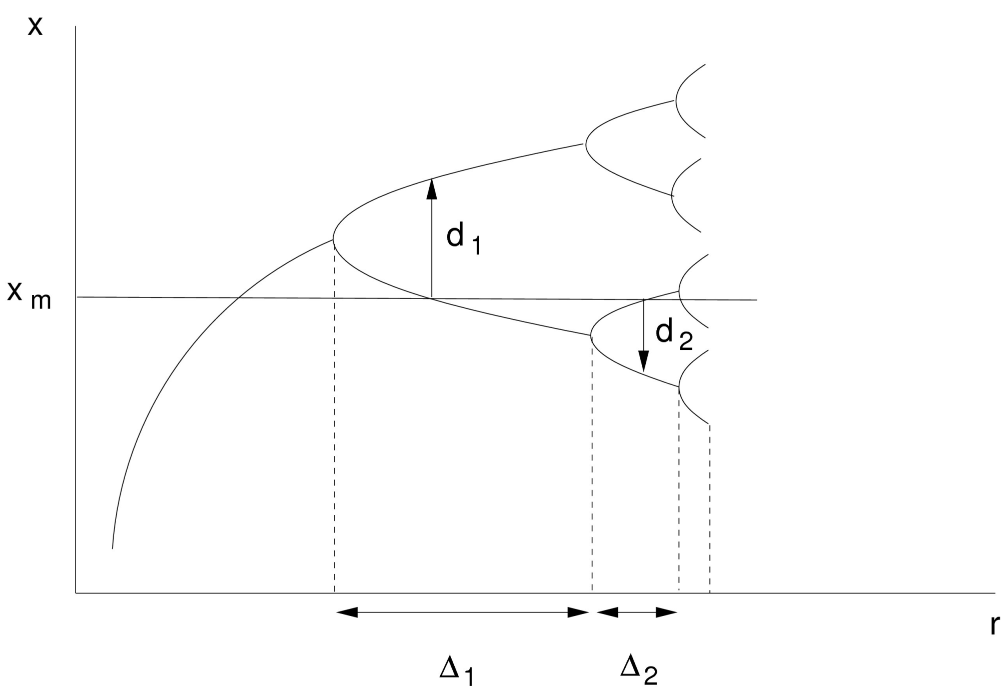
Liapunov spectrum of the logistic map

Qualitative Universality

- For various unimodal maps (e.g. $x_{n+1} = r \sin \pi x_n$) bifurcation diagrams look rather “similar”
- Metropolis et al. (1973): $x_{n+1} = r f(x_n), \;\; f(0) = f(1) = 0$
  - As r is varied the sequence in which stable periodic solutions appear when r is varied is always the same
  - So called “U-sequence” up to period 6: 1,2,2*2,6,5,3,2*3,5,6,4,6,5,6
  - Has e.g. been found in experiments with the Belousov-Zhabotinski reaction in a continuously stirred flow reactor

Quantitative Universality -- Feigenbaum

- Quantify bifurcation diagrams in some way
  - r-direction $\Delta_n = r_n - r_{n-1}$
  - x-direction d_n



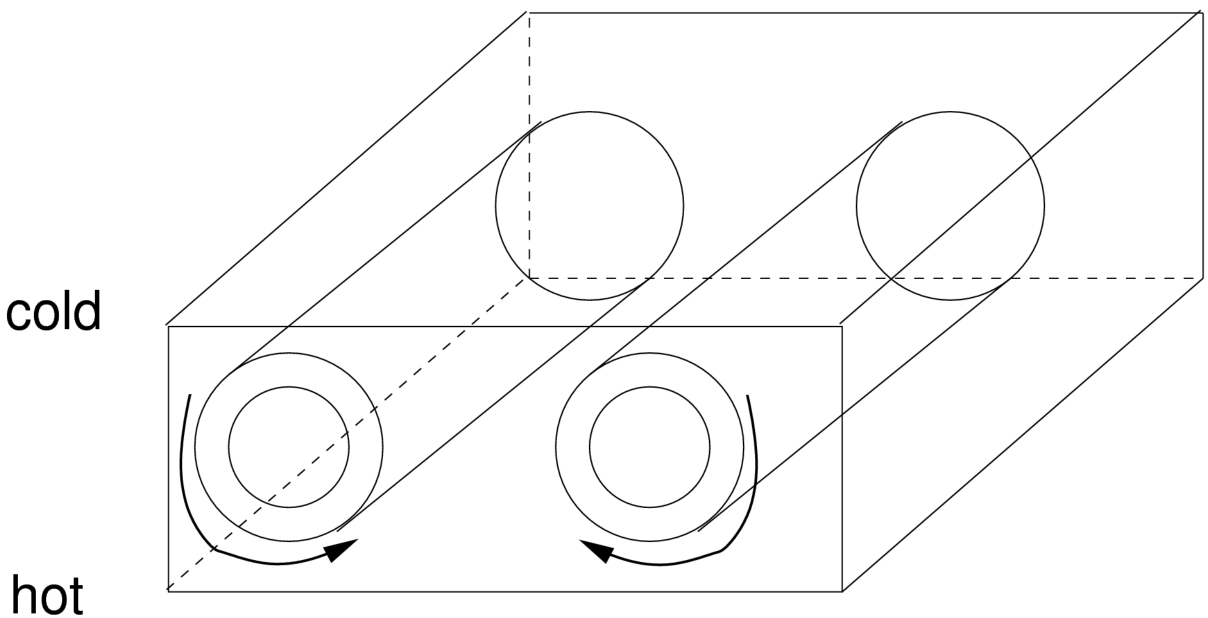
$$\delta = \lim_{n \to \infty} \frac{\Delta_n}{\Delta_{n+1}} = 4.669 \ldots$$

$$\alpha = \lim_{n \to \infty} \frac{d_n}{d_{n+1}} = -2.5029 \ldots$$

Both α and δ are **universal**, i.e. independent of precise form of the map f

Experimental Tests

- E.g.: Libchaber (1982)
  - Box with liquid mercury, heated from below
  - Control parameter is Rayleigh number R (measure for temperature gradient)



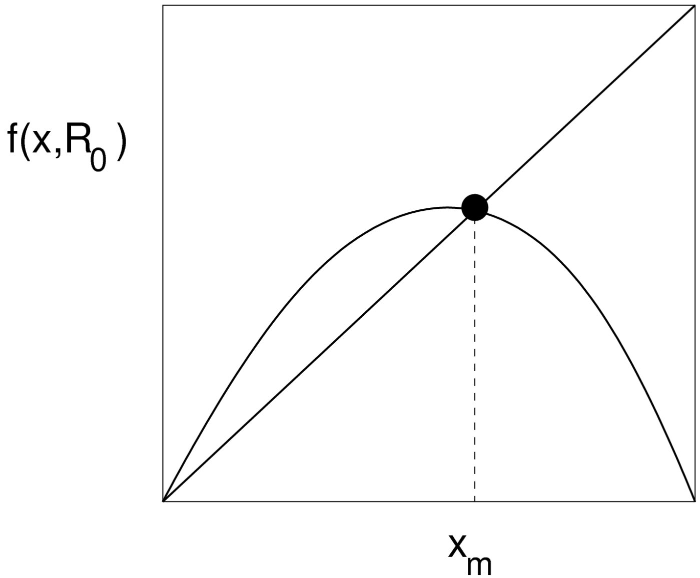
- $R<R_c$: conduction without convection
- $R>R_c$: convection occurs, rolls appear, rolls straight, motion steady
- Larger R: another instab., temperature waves along roles (magnetic field used)
- Measured δ~4.4(1)

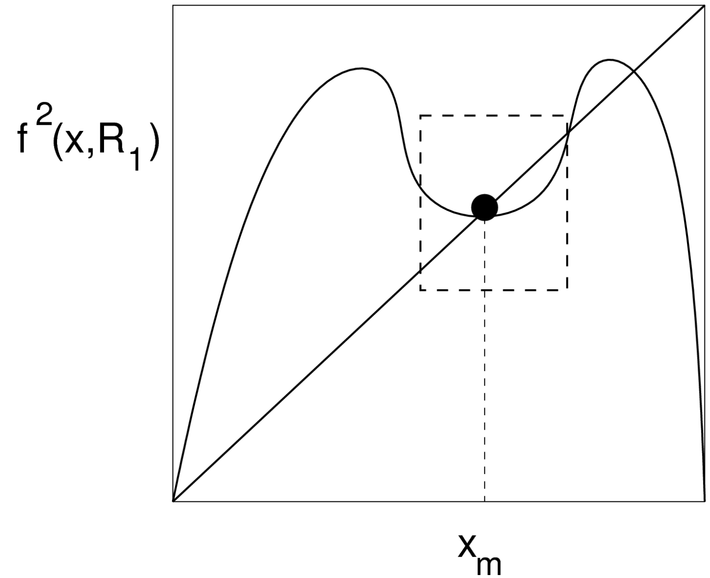
Feigenbaum's Renormalization Theory

- $f(x,r)$ ... unimodal map which undergoes period doubling route to chaos as r increases;
  - x_m maximum of f,
  - r_n ... value of r at which 2^n-cycle is born
  - R_n... value at which 2^n-cycle becomes superstable
- Turns out superstable cycle always contains x_m as one of its points
- Exploit self-similarity of figtree
- Compare f with its second iterate; then “renormalize” one map into the other

Renormalization (2)

- Compare: $f(x, R_0)$ and $f^2(x, R_1)$
  - same stability properties and x_m superstable FP for both of them !





  - First step: shift origin by x_m and subtract x_m from f

Renormalization (3)

- Second step: rescale by alpha: $x'=\alpha x$

$$f^2(x, R_1) \rightarrow \alpha f^2(x/\alpha, R_1)$$

- Use local resemblance of f and f^2 near x_m:

$$f(x, R_0) \approx \alpha f^2(x/\alpha, R_1)$$

$$f^2(x/\alpha, R_1) \approx \alpha f^4(x/\alpha^{2,} R_2) \quad \text{i.e.} \quad f(x, R_0) \approx \alpha^2 f^4(x/\alpha^{2,} R_2)$$

$$\longrightarrow \quad f(x, R_0) \approx \alpha^n f^{2^n}(x/\alpha^n, R_n)$$

- Feigenbaum found numerically:

$$\lim_{n\rightarrow\infty} \alpha^n f^{(2^n)}(x/\alpha^n, R_n) = g_0(x) \quad \text{is universal with a superstable FP}$$

Renormalization (4)

- We might construct other such functions by starting with some R_i (not R_0) -> universal functions $g^i(x)$ with superstable 2^i-cycle
- Most interesting is the one starting at R_∞

$$f(x, R_\infty) \approx \alpha f^2(x/\alpha, R_\infty) \quad \text{or}$$

$$g(x) = \alpha g^2(x/\alpha)$$

- This is a functional equation for g
  - Boundary conditions: g'(0)=0 (shifted maxima), g(0) =1 (defines x-scale)
  - Note: $g(0) = \alpha g(g(0))$ and $g(0) = 1$ ➔ $\alpha = 1/g(1)$

Renormalization (5)

- For an approximate solution expand g as a polynomial $g(x)=1+c_2x^2+c_4x^4+\ldots$
  - Compare matching powers of x -> system of equations for coefficients
  - Feigenbaum (1979): seven term expansion yielded α~-2.5029
- Calculation of δ is much harder ...

Summary

- Maps, stability of fixed points and arguments using cobweb diagrams
- Logistic map
  - Period doubling route to chaos
  - Intermittency
  - Liapunov exponents
  - Universality
    - Qualitative
    - Quantitative