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Models and Theories of Mathematical Understanding: Comparing Pirie and Kieren's Model of the Growth of Mathematical Understanding and APOS Theory

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ABSTRACT. The search for a meaningful cognitive description of understanding has ensued for the past half a century. Within the past three decades, new and integrative perspectives have grown out of Richard Skemp's distinctions between instrumental and relational understanding. The growth of these perspectives, up until 1987, was documented by Tom Schroeder in his PME synthesis of the work on understanding resulting from Richard Skemp's instrumental/relational contrasts. Since 1987, the work on understanding has progressed and this paper examines the new, more recent theoretical frameworks of understanding which have arisen from these roots. This paper focuses on two theoretical frameworks, Pirie and Kieren's model of the growth of mathematical understanding and Dubinsky's APOS theory, and discusses other contemporary theoretical frameworks such as the work by Cornu and Sierpinska on cognitive or epistemological obstacles, the investigations into concept definition and concept image by Vinner and Tall, Kaput's explorations of multiple representations, and Sfard's distinctions between operational and structural conceptions. Besides explicating the definitions of understanding proposed by these two frameworks, the discussion addresses their elements and constructs as well as their linkages to historical and recent characterizations of understanding. The paper then argues why Pirie and Kieren's model and APOS theory satisfy the Schoenfeld (2000) criteria for classification as a theory and finally concludes with discussions of a variety of interconnections between these two theories as well as the elements which make them distinct from each other such as their origins, organizations, relationships to other frameworks, and implications of the two theories for both assessment and pedagogical practices.

This paper opens with a brief history of the search to obtain a clearly defined conceptualization of the meaning of the term "understanding" consistent with the

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As is the case with any review and synthesis, this paper can only draw upon the author's interpretations of published snapshots of these evolving theories. As a result, the interpretations and conclusions drawn herein address a temporal look at continuously changing theories by providing a state of the theory look to each theory's accumulated and self-supported published writings.

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work of Schroeder (1987). Historically, various characterizations have associated understanding with knowledge, linkages built from mathematical operations, an act, or simply a grasping of meaning. The discussion contrasts the thinking prior to Skemp's influential work distinguishing the difference between instrumental and relational understanding with the various theoretical frameworks that have since developed. In particular, a brief description of the viewpoints taken by Brownell and Polya as well as their contemporaries provides a backdrop for examining the movement to distinguish between understanding and knowledge. This chain culminates in an abridged discussion of four recent theoretical frameworks addressing the development and acquisition of mathematical understanding thereby setting a stage for sections describing Pirie and Kieren's model of the growth of mathematical understanding and Dubinsky's APOS theory.

In brief, Pirie and Kieren (1991a) have aligned themselves with Schoenfeld's (1989) assessment of understanding as an unstable and retrogressive organic element and consider understanding as a whole, dynamic, layered but non-linear, never-ending process of growth. They reject the notion of the growth of understanding as a monotonically increasing function and consider it as a dynamic, organizing, and reorganizing process (Pirie & Kieren, 1992b, 1994b). On the other hand, Dubinsky (1991) aligns APOS theory with the Piagetian perspective that *reflective abstraction* is the key to cognitive development of logico-mathematical concepts.[1] From the perspective of APOS theory, understanding is a never-ending process of iterative schema construction through reflective abstraction, a cognitive process where physical or mental actions are reconstructed and reorganized by the learner on a higher plane of thought and thereby become understood by that learner (Ayers, Davis, Dubinsky, & Lewin, 1988). In addition to expounding these ideas, this discussion addresses the elements and constructs of the theories. With such components in mind, brief illustrations identify how their respective theories are related to both historical and recent characterizations of understanding.

The paper concludes with a discussion of a variety of interconnections between Pirie and Kieren's model of understanding and APOS theory. Both of these theories have constructivist origins but also contain elements that differentiate them. The focus then turns to the organizational structures of the two theories with particular attention paid to the elements and constructs of each the two theories. Next, the discussion expounds on the linkages to recent theoretical frameworks of understanding as well as a connection between Pirie and Kieren's model of understanding with a particular refinement of APOS theory defined by Clark et al. (1997). The final element of this section compares and contrasts the implications of the two theories for both assessment and pedagogical practices.

1. A brief history of "understanding"

Even though the term "understanding" has been freely used in mathematics education literature, the search for a concise definition of "understanding" has been going on for years. In particular, Brownell and Sims (1946) felt mathematical understanding was a difficult concept to define and stated, "A technically exact definition

[1] A logico-mathematical concept is one where the physical properties of objects have been abstracted and integrated into a learner's mental framework through physical experience. Analysis of this experience takes place through thought processes about the physical experience rather than manipulation of the physical objects from which the concept was derived.

of 'understand' or 'understanding' is not easily found or formulated" (p. 163). However, many writers have operated under the assumption that a well-defined notion of understanding existed, thereby causing entanglements with philosophy when they abandoned speaking of understanding in an ideal sense and attempted to explicate its meaning (Sierpinska, 1990b). These difficulties, according to Sierpinska (1990a), arose from the mathematics education community's inability to distinguish between knowledge and understanding prior to Skemp's (1976) famous paper on "Instrumental and relational understanding." In particular, it was only until a 1978 reprint in the *Arithmetic Teacher* that Skemp's distinction between knowledge and understanding drew the attention of the U. S. mathematics education community.

1.1. "Understanding" prior to 1978. Prior to Skemp's influential paper, U. S. researchers generally identified understanding with knowledge. Understanding became equated with the development of connections in the context of performing algorithmic operations and problem solving (Brownell, 1945; Brownell & Sims, 1946; Fehr, 1955; Polya, 1945; Van Engen, 1949; Wertheimer, 1959). For instance, Brownell and Sims (1946) characterized understanding as (a) an ability to act, feel, or think intelligently with respect to a situation; (b) varying with respect to degree of definiteness and completeness; (c) varying with respect to the problem situation presented; (d) requiring connections to real-world experiences and the inherent symbols; (e) requiring verbalizations although they may contain little meaning; (f) developing from varied experiences rather than repetitive; (g) influenced by the methods employed by the teacher; and (h) inferred from observations of actions and verbalizations. Polya (1962), on the other hand, identified understanding as complementary to problem solving as indicated in the following quotation:

> One should try to understand everything; isolated facts by collating them with related facts, the newly discovered through its connections with the already assimilated, the unfamiliar by analogy with the accustomed, special results through generalization, general results by means of suitable specialization, complex situations by dissecting them into their constituent parts, and details by comprehending them within a total picture. (p. 23)

This understanding, according to Polya (1962), cannot be classified as either present or not present because it is more a matter of extent rather than presence. In particular, Polya (1962) identified four levels of understanding a mathematical rule: (a) "mechanical"—a memorized method that can be applied correctly, (b) "inductive"—acceptance that explorations with simple cases extend to complex cases, (c) "rational"—acceptance of the proof of the rule as demonstrated by someone else, and (d) "intuitive"—personal conviction as to the truth beyond any doubt. Such levels characterize understanding as knowledge associated with mathematical rules. Similar characterizations occur in the writings of other researchers during this period. For instance, Flavell (1977) spoke of number knowledge or understanding in his discussions of number conservation whereas Davis (1978) argued understanding's dependence on whether the knowledge involved concepts, generalizations, procedures or number facts. Lehman (1977) equated understanding with three types of knowledge—applications, meanings, and logical relationships.

	Concrete	Iconic	Symbolic
Relational			
Instrumental			
Logical			

TABLE 1. Tall's categories of understanding

1.2. "Understanding" after 1978. Skemp's (1976) work distinguished understanding from knowledge and emphasized categories of mathematical understanding (Byers & Erlwanger, 1985). In particular, Skemp (1976) classified *relational* understanding as knowing what to do and why it should be done and *instrumental* understanding as having rules without the reasons. Each of these understandings has its own set of advantages. Instrumental understanding, according to Skemp (1976), tends to enable easy recall, to promote more tangible and immediate rewards, and to provide quick access to answers. Relational understanding, on the other hand, provides avenues for more efficient transfer, for extracting information from a learner's memory, for having such understanding as a goal in to itself, and for propagating the growth of understanding. Later, the classification evolved to include a third category denoted as *logical*—organization as in formal proof (1979) and finally a fourth category identified as *symbolic*—connection of symbolism and notation to associated ideas (Skemp, 1982), thereby creating four categories of understanding—relational, instrumental, logical, and symbolic—each subdivided into reflective and intuitive subcategories.

The categories of relational and instrumental understanding spawned a variety of other categorizations, namely (a) procedural and conceptual, (b) concrete and symbolic, and (c) intuitive and formal (Ball, 1991; Herscovics & Bergeron, 1988; Hiebert & Lefevre, 1986; Hiebert & Wearne, 1986; Nesher, 1986; Ohlsson, Ernest, & Rees, 1992; Resnick & Omanson, 1987; Schroeder, 1987). Byers and Herscovics (1977), combining the ideas of Bruner and Skemp, developed a tetrahedral classification of understanding with the following categories: instrumental, relational, intuitive, and formal. Tall (1978) suggested a matrix of categories (see Table 1) as characterizing understanding.

Other more general views propose that understanding is the development of connections between ideas, facts, or procedures (Burton, 1984; Davis, 1984; Ginsburg et al., 1992; Greeno, 1977; Hiebert, 1986; Hiebert & Carpenter, 1992; Janvier, 1987; McLellan & Dewey, 1895; Michener, 1978; Nickerson, 1985; Ohlsson, 1988). Forming a network of these connections provides a structure for situating new information by recognizing similarities, differences, inclusive relationships, and transference relationships between models. Thus, the development of understanding is a process of connecting representations to a structured and cohesive network. The connection process requires the recognition of relationships between the piece of knowledge and the elements of the network as well as the structure as a whole.

1.3. Recent views of "Understanding". Even though researchers now separate understanding from knowledge, evidence exists that the mathematics education community has not reached unilateral agreement as to the meaning of "understanding" since various authors approach it from diverse viewpoints (Schroeder, 1987). In particular, recent constructivist conceptualizations of understanding have been proposed in addition to Pirie and Kieren's model of the growth of mathematical understanding and Dubinsky's APOS theory. These include frameworks such as *cognitive* or *epistemological* obstacles (Bachelard, 1938; Cornu, 1991; Sierpinska, 1990b), *concept definition* and *concept image* (Davis & Vinner, 1986; Tall, 1989, 1991; Tall & Vinner, 1981; Vinner, 1983, 1991), *multiple representations* (Kaput, 1985, 1987a, 1987b, 1989a, 1989b) and a dichotomy between *operational* and *structural* conceptions (Sfard, 1991, 1992, 1994). Many of these characterizations have common elements especially since most derive from an underlying constructivist perspective that a learner's understanding is built by forming mental objects and making connections among them.

1.3.1. *Understanding as overcoming cognitive obstacles.* The concept of *cognitive* obstacles helps identify difficulties students encounter as they engage in the learning enterprise and therefore becomes useful in constructing better strategies for teaching (Cornu, 1991). *Cognitive* obstacles, first defined by Bachelard (1938), are classified as *genetic* or *psychological* obstacles, *didactical* obstacles, or *epistemological* obstacles depending on if they arose because of personal development, the teaching practice, or the nature of the mathematical concepts (Cornu, 1991). In particular, epistemological obstacles contain two essential attributes: (a) they are unavoidable as one constructs understandings of some mathematical concepts, and (b) the historical development of the concept reflects their existence (Cornu, 1991).

Sierpinska (1990b) drew her conceptualization of understanding from Lindsay, Husserl, Dilthey, Dewey, and Ricoeur and regarded "understanding as an act, but an act involved in a process of interpretation being a developing dialectic between more and more elaborate guesses and validations of these guesses" (p. 26). From this perspective, understanding derives its basis from the learner's ideologies, predispositions, preconceptions, convictions, and unperceived schemes of thought. This foundation may contain factors that act as obstacles to the further construction of understanding. Surmounting such an obstacle requires the learner to experience a mental conflict that calls into question the learner's convictions. Additionally, Sierpinska (1987) commented "if the presence of an epistemological obstacle in a student is linked with a conviction of some kind then overcoming this obstacle does not consist in replacing this conviction by an opposite one. This would be falling into the dual obstacle" (p. 374). As a result, overcoming an obstacle means that the learner must divest held convictions and analyze those beliefs from an external reference point. In doing this, the learner can recognize the tacit assumptions responsible for the cognitive dissonance and then evaluate alternative hypotheses. This evaluation requires the learner to identify the objects associated with a concept, identify common and disparate properties of the objects, generalize the scope of an concept's application, and finally synthesize the relationships between properties, facts, and objects to organize them into a consistent whole.

Not every act of understanding corresponds to an act of overcoming an epistemological obstacle but in general these can be equated. For instance, Sierpinska (1990a) stated "overcoming epistemological obstacles and understanding are two complementary pictures of the unknown reality of the important qualitative changes in the human mind. This suggests a postulate for epistemological analyses of mathematical concepts: they should contain both the 'positive' and the 'negative' pictures, the epistemological obstacles and the conditions of understanding" (p. 28). Thus, in Sierpinska's eyes, the use of an epistemological analysis of a mathematical concept helps determine a student's held understanding by making the observer aware of the various ways of perceiving a concept and the potential pitfalls inherent to them. The development of understanding is describable in many cases as the conscious awareness of an obstacle and this awareness leads to new ways of knowing. These new ways of knowing can result in the unfortunate acquisition of new epistemological obstacles. In particular, the act of overcoming an epistemological obstacle can merely open the learner to a larger domain containing additional epistemological obstacles. From this perspective, epistemological obstacles act as the dual to understanding since epistemological obstacles focus backwards on the errors and understanding looks forward to the new ways of knowing (Sierpinska, 1990b). Gauging the depth of understanding is accomplished through the identification of the number and quality of acts of understanding achieved or the number of epistemological obstacles overcome. These viewpoints provide complementary pictures of the qualitative changes in the mind as a learner interacts with concepts.

1.3.2. *Understanding as generating concept images and concept definitions.* According to Vinner (1991), learner's acquire concepts when they construct a *concept image*—the collection of mental pictures, representations, and related properties ascribed to a concept. Tall and Vinner (1981) write:

> We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes ... As the concept image develops it need not be coherent at all times ... We call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion. (p. 152)

Evidently, a concept image differs from a concept's formal definition, if one exists, since a concept image exemplifies the way a particular concept becomes viewed by an individual (Davis & Vinner, 1986). The concept image involves the various linkages of the concept to other associated knowledge structures, exemplars, prototypical examples, and processes. As a result, the concept image is the overall cognitive structure constructed by a learner; however, in different contexts distinct components of this concept image come to the foreground. These excited portions of the concept image comprise the *evoked concept image* which consists of a proper subset of the concept image. This distinction between the image and the evoked image permits one to explain how students can respond inconsistently, providing evidence of understanding in one circumstance and a lack of understanding in another. A learner's description of his or her understandings may supply other discrepancies. In particular, any concept image has a related *concept definition*—the form or words used by a student to specify the concept. This concept definition, however, can differ from the formal mathematical definition of a concept since the concept definition is an individualized characterization of the concept.

Construction of these concept images occurs as a learner encounters new information and faces the consolidation of this information into the already present cognitive structure. The process of incorporation relates to the Piagetian notion of transition and in particular, assimilation and accommodation (Tall, 1991). *Assimilation* involves the taking in of new data and forming linkages between this new information and the original structure. In contrast, *accommodation* reorganizes part or the whole of the individual's cognitive structure. Underlying assimilation and accommodation are two essential mechanisms of cognitive development: generalization and abstraction. In mathematics, *generalization* typically refers to the process of applying an argument to a broader context; however, the type of generalization employed by a learner is dependent on the learner's already present cognitive structure. *Abstraction*, on the other hand, occurs when the learner focuses on the properties of an object and then considers those properties in isolation from the object from which they were derived. In this case, the structure of properties becomes an entity unto itself having application in other related domains.

Harel and Tall (1991) identified three types of generalization: expansive generalization, reconstructive generalization, and disjunctive generalization. *Expansive generalization* refers to the learner expanding the range of applicability of an existing schema[2] without reconstructing it. In other words, the learner sees previously applied methods as special cases of a new generalized procedure. In this case, the scope of application broadens without restructuring the internal cognitive structure even though the previous elements of the cognitive structure become substructures under the expanded scheme of application. *Reconstructive generalization* on the other hand occurs when the learner "reconstructs an existing schema in order to widen its applicability" (Harel & Tall, 1991, p. 38). That is, the learner searches for a new structure that takes initially isolated but related procedures or concepts and organizes the simpler elements as special cases under a more general case. *Disjunctive generalization* is perhaps the most worrisome from the prospective of teaching. In this type of generalization, the learner expands the scope of applicability by constructing a new, disjoint schema unconnected to previously studied concepts or procedures that could be considered special cases of this new schema. When speaking of disjunctive generalization, Tall (1991) stated "It is a generalization in the sense that the student may now be able to operate on a broader range of examples, but it is likely to be of little lasting value to the student as it simply adds to the number of disconnected pieces of information in the student's mind without improving the student's grasp of the broader abstract implications" (p. 12). As a result, disjunctive generalization encumbers the learner with an additional burden that can result in failure.

Abstraction differs from generalization in terms of the cognitive focus of the learner. Rather than extending ideas from one context to another, abstraction occurs when the learner focuses on the underlying structure of the contexts and extrapolates the common qualities or features. This process culminates in the construction of a set of axioms. As a result, abstraction requires a massive mental reconstruction in order to build the properties of the abstract object (Tall, 1991). Once the properties are abstracted, their application to a new context requires reconstructive generalization since "the abstracted properties are reconstructions of the original properties, now applied in a broader domain" (Harel & Tall, 1991, p. 39). In particular, extending the range of possible applications provides the opportunity for expansive generalization to transpire since the learner can relate the *abstract theory* (i.e. the axioms and their consequences) to elements in another cognitive structure. There is a problematic element, similar to compartmentalized

[2]Here, a schema refers to a structure developed by the learner to organize the various concepts, procedures, etc. linked to an endeavor such as solving linear equations in one variable.

knowledge from disjunctive generalization, associated with abstraction. If a learner encounters a limited set of examples, they may contain properties which do not hold for the entire class of objects. As a result, the learner must return, perhaps repeatedly, to reconstruct the abstract object and eliminate the nonessential properties. Even though these mechanisms may be problematic at times, their application permits the learner to build more extensive and possibly better interconnected, except in the case of disjunctive generalization, concept images.

1.3.3. *Understanding as operating with multiple representations.* According to Kaput (1989a), cognitive power exists in multiple, linked representations. These provide redundancy while permitting the learner to suppress some aspects of complex ideas and emphasize others. Facility with these representations and their linkages permits the learner to understand complex ideas in new ways and effectively apply them (Kaput, 1989a). The term *representation* is a trans-theoretic term for it encompasses a variety of characterizations: *cognitive* and *perceptual*, *computer*, *explanatory*, *mathematical*, and *symbolic* (Kaput, 1985, 1987b). In general, a *representation system* (or *symbol system*) aids in the instantiation of mathematical objects, relations, and processes by creating an environment in which shared cultural or linguistic artifacts can be expressed amongst the community (Kaput, 1989a). Such a representation system involves: (a) two worlds, the represented and the representing; (b) the elements of the represented world being represented; (c) the elements of the representing world doing the representing; and (d) the correspondence that affixes the connection between the two worlds (Kaput, 1985).

Mathematical representations or *mathematical symbol systems* are a special representation system where the represented world is a mathematical structure and the representing world is a *symbol scheme*[3] containing special correspondences (Kaput, 1987b). In particular, mathematical symbol systems, similarly to natural language and pictorial systems, manage the influx of experience by breaking it into chunks, assigning symbols to those chunks, and coordinating these notations in a milieu devoid of the original nuances and complex referents (Kaput, 1987b). The mathematical symbol system and its connections then can form a structure which acts as a symbol system used to represent another symbol system thereby exhibiting a self-similarity under magnification.

Kaput (1989a) contends that a learner's development and expression of mathematical meaning can be viewed from the lens of the construction of notational forms and structures in mental representations where a *mental representation* is the means by which an individual organizes and coordinates the flow of experience. In particular, Kaput (1987a) stated "The fundamental premise is that the root phenomena of mathematics learning and application are concerned with representation and symbolization because these are at the heart of the content of mathematics and are simultaneously at the heart of the cognitions associated with mathematical activity" (p. 22). In this view, the growth of mathematical meaning is the construction and utilization of representations and symbolization.

[3]"A symbol scheme is a concretely realizable collection of characters together with more or less explicit rules for identifying and combining them" (Kaput, 1987b). In essence, a symbol scheme is an abstract means of representing a more complex concepts using quasi-realistic symbols to delineate typically intangible objects. For instance, the Hindu-Arabic numbers with their concatenations would satisfy the definition of a symbol scheme. Similarly, the coordinate axes with their syntactical rules would also be considered a symbol scheme.

As a learner constructs personal meaning, negotiation occurs between two separate worlds: *physical operations* which are observable and *mental operations* which are hypothetical. In particular, the development of understanding is the movement from operating in the world of physical operations to operating in the world of mental operations. In order to accomplish this, the learner must employ "(i) deliberate, active interpretation (or 'reading'), and (ii) the less active, less consciously controlled and less serially organized processes of having mental phenomena evoked by physical material" (Kaput, 1992, p. 522). Underlying the physical operations are the notations used to display the operations. A *notation system*[4] is separate from any particular physical representation (thereby differentiating it from a symbol scheme which must be linked to two worlds: the represented and the representing) and contains a set of rules that define its objects and the allowable actions upon those objects.

With this in mind, one can recognize that most true mathematical activity involves the *coordination of* and *translation between* different notation systems (Kaput, 1992). Given two notation systems with their sets of rules defining the objects of the system, the allowable actions upon those objects, and in the case of symbol schemes the correspondent physical mediums in which the systems may be instantiated, a learner can (i) negotiate between the two notation systems, (ii) integrate cognitions, (iii) transform objects within a particular representation, or (iv) cognize about a notational system. The first two mechanisms comprise *referential extensions* since they "horizontally translate" between either notational systems or mathematical structures. In the first case, meaning arises from identifying the connected components of different representational systems through translation and in the second, through the construction and testing of mathematical models, which amount to translation and coordination of cognitive organizations.

The latter two mechanisms amount to *consolidations* whereby the learner engages in "vertical growth" by transforming actions at one level into objects and relations capable of being operated upon at another level. The third mechanism produces meaning through pattern and syntax learning through transformations within a particular notation system that may or may not contain references to external meanings. This growth is a reorganization of the notational system creating a hierarchical structure albeit within the original notation system. In contrast, the last mechanism, cognition about a notational system, yields mathematical meaning by reifying[5] actions, procedures, and concepts into phenomenological objects that can potentially act as the basis for new actions, procedures, and concepts as a higher level of organization (Kaput, 1992). These mechanisms produce meaning and therefore develop understanding for the learner by either creating new linkages between representation systems or reorganizing elements within a representation system.

1.3.4. *Understanding as constructing operational and structural conceptions.* Sfard (1991) defines the building blocks of mathematics as two entities: *concept* and *conception.* A concept refers to an official mathematically defined idea whereas

[4]Familiar notation systems, such as numeration systems and algebraic notation systems for one or several variables, typically include textual elements but they can also include strictly pictorial elements correspondent to Dienes blocks, Cuisenaire rods, fraction bars, etc.

[5]Reification, a Piagetian construct used also by Sfard and APOS theory, is the ability of the learner to envision, almost simultaneously, the results of processes as permanent objects inseparable from the underlying processes from which they arose.

a conception involves a cluster of the learner's internal representations and linkages caused by the concept. These two definitions are related to Vinner's (1991) discussion of formal concept definition and the description by Tall and Vinner (1981) of a concept image. Sfard (1991) asserts that mathematical concepts inhabit a duality of conception for they can be viewed as static, instantaneous and integrative—*structural* or dynamic, sequential, and detailed—*operational*.

An operational conception, although difficult to describe, concerns processes, algorithms, and actions which occur at the physical or mental level. A *structural* conception on the other hand is more abstract, more integrated, and less detailed than an operational conception. In particular, a structural conception is somehow isomorphic to the ability to "see" advanced mathematical constructs which are not physical entities but rather abstract mental organizations perceivable in only one's mind's eye (Sfard, 1991). This capability of seeing the invisible objects that form the mathematical concept draws one into the world of visualization. Sfard proposes that structural conceptions receive support from compact and integrative mental images rather than from verbal representations which require serial processing. These mental images permit the learner to make the abstract ideas more tangible and conceive them as almost physical entities where the operations upon them occur entirely in the mind's eye. In addition, such visualization empowers the learner to develop a holistic view of the concept thereby allowing observations from multiple perspectives while preserving the identity of and relationships within the concept.

From Sfard's perspective, understanding reaches beyond an ability to solve problems or prove theorems (Sfard, 1994). In essence, she equates understanding with the construction of links between symbols and the development of a structural conception. However, according to Sfard (1991), "*there is a deep ontological gap between operational and structural conceptions*" (p. 4). Even though a gap exists, operational and structural conceptions are not mutually exclusive. In particular, they are complementary in the sense that they are two views of the same mathematical concept and are inseparable since the concept harbors both operational and structural elements. The operational conception views the concept as a process and the structural conception equates the concept with a static object transcendent of its process roots. However, for conceptual development, both of these conceptions are necessary. For instance, Sfard (1991) states "Indeed, in order to speak about mathematical objects, we must be able to deal with *products* of some processes without bothering with the processes themselves ... It seems, therefore, that the structural approach should be regarded as the more advanced stage of conceptual development. In other words, we have good reasons to expect that *in the process of concept formation, operational conceptions would precede the structural*" (p. 10). This statement does not imply that structural conceptions can only develop after the construction of operational conceptions, rather that in general this is the natural path for development. In other words, as one views the historical development of many mathematical concepts, society moved through a series of stages culminating in a structural conception. In particular, Sfard (1991) used a historical analysis of concept formation to identify three distinct stages in the process: the generation of a process from already familiar objects, the emergent recognition of the processes as autonomous entities, and the ability to conceive the new entity as a synthesized, object-like structure.

These three stages are classified as *interiorization*, *condensation*, and *reification*, respectively. Interiorization occurs as a learner becomes familiar with the processes which eventually can be reified into a mathematical object. Interiorization in this context remains similar to the mechanism described by Piaget (1970) for it essentially comprises a movement from conceptions based upon physical operations to those founded on mental representations of the processes. The everyday meaning of condensation is similar to its meaning in this context. Rather than working through a long sequence of related but distinct mental processes, condensation enables the learner to conceive of a sequence as a single process relating input and output without the intervening steps. The attachment of a name to the condensed sequence gives birth to a new concept which remains affixed to a process orientation until it becomes reified.

Reification, according to Sfard and Linchevski (1994), is ultimately responsible for the development of mathematical objects. In essence, reification is a quantum leap from conceiving the new entity as tightly connected to a process to conceiving the notion of the entity as an object which itself can be acted upon. As a result, reification is an ontological shift on the part of the learner (Sfard, 1991). This shift permits the capability to see something familiar from an entirely different perspective that detaches the condensed sequence from the originating sequence. The ensuing structure, although invariably connected to the processes it exemplifies, can now be viewed as a static object in the mind's eye. In addition, the new entity begins to draw its meaning from its membership not in the realm of processes but rather as a member of a category of abstract objects which enhances the scope of applications. In particular, Sfard (1991) declares:

> At some point, this category rather than any kind of concrete construction becomes the ultimate base for claims on the new object's existence. A person can investigate general properties of such category and various relations between its representatives. He or she can solve problems involving finding all the instances of the category which fulfill a given condition. Processes can be performed in which the new-born object is an input. New mathematical objects may now be constructed out of the present one. (p. 20)

Once a process has been reified, it yields an object upon which a higher level process can act. This process can then become interiorized and the entire cycle repeated. As a result, the three-phase system of interiorization, condensation, and reification is generally hierarchical and repetitive.

The development of understanding in this view is the capability to break free from one's own constrained thinking. In doing so, the learner gains the capability to perceive a process no longer as a sequence of physical acts which have been interiorized and condensed but rather as an object. This object along with the other objects in the mathematical universe and the operations potentially performed upon them, according to Sfard (1994), receive meaning though metaphorical reflection. In particular, Sfard (1994) aligns herself with Lakoff and Johnson's thesis that "metaphors constitute the universe of abstract ideas, that they create rather than reflect it, that they are the source of our understanding, imagination, and reasoning" (p. 47). The metaphor therefore provides meaning to an abstract concept since it provides a figurative projection of operations performed in a physical reality into the world of ideas. This characterization remains true even when the concept appears far removed from physical reality since underlying the concept is a long chain of metaphors eventually rooted in actions performed in a physical reality. As a result, it is Sfard's contention that reification, the transition from an operational to a structural mode of thinking, accounts for the construction of mathematical concepts and this fabrication corresponds to the birth of a metaphor. This metaphor brings the mathematical object into being and thereby deepens the learner's understanding of the mathematical universe.

2. Pirie and Kieren's Model of Understanding

Pirie and Kieren consider their work on the growth of mathematical understanding to provide some answers to the cogent questions raised by Sierpinska (1990b): "Q1. Is understanding an act, an emotional experience, an intellectual process, or a way of knowing? ... Q3. Are there levels, degrees or rather kinds of understanding? ... Q5. What are the conditions for understanding as an act to occur? ... Q7. How do we come to understand? ... Q8. Can understanding be measured and how?" (p. 24). Pirie and Kieren's initial definition of mathematical understanding evolved from Glasersfeld's constructivist definition of understanding (Kieren, 1990). In particular, Glasersfeld (1987) proposed the following definition of understanding:

> The experiencing organism now turns into a builder of cognitive structures, intended to solve such problems as the organism perceives or conceives ... among which is the never ending problem of consistent organizations [of such structures] that we call understanding. (p. 7)

Glasersfeld perceived understanding as the continual process of organizing one's knowledge structures. Using this definition as an advance organizer, Pirie and Kieren (1991a) began to develop their theoretical position concerning mathematical understanding. They characterize mathematical understanding as the following:

> Mathematical understanding can be characterized as leveled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without. (Pirie & Kieren, 1989, p. 8)

It is the purpose of the following discussion to explicate Pirie and Kieren's model more precisely.

2.1. Elements of Pirie and Kieren's Model of the Growth of Mathematical Understanding. Using the above definition, Pirie and Kieren conceptualize their model of the growth of mathematical understanding as containing the eight potential layers shown in Figure 1. The process of coming to understand begins the core of the model called the *primitive knowing*[6] layer. Primitive connotes a starting place rather than low level mathematics. The core's content is all the information brought to the learning situation by the student. These contents have been discussed under various names: "intuitive knowledge" (Leinhardt, 1988),

[6] The primitive knowing layer was initially referred to as "primitive doing" or "doing" in articles prior to 1991.

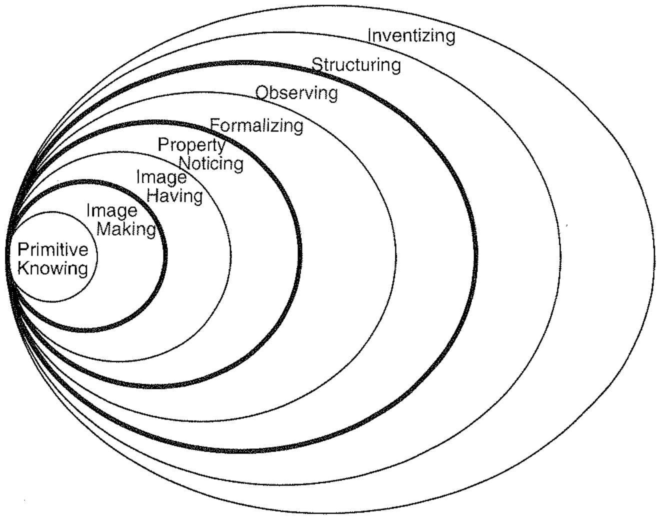

FIGURE 1. A diagrammatic representation of the model for growth of mathematical understanding

"situated" knowledge (Brown, Collins, & Duguid, 1989), and "prior" or "informal" knowledge (Saxe, 1988). For a particular concept such as fractions, one can surmise that in tracing the growth of mathematical understanding a learner comes to the learning situation with a host of information which may or may not inform the growth of understanding. Resnick and Omanson (1987) noted that students entered a mapping instruction[7] with mental representations of block subtraction attached to a rich knowledge base associated with subtraction. Here the learner brings both understandings of subtraction beyond the primitive knowing layer and other understandings supportive of continued growth. In contrast, Hiebert and Wearne (1986), in examining decimal learning, found students to perceive decimal symbols as part of a new symbol system accompanied by a new set of rules thus minimizing the links to previously learned material. In turn, the core's under-utilized links can impact the process of understanding.

At the second layer, called *image making*, the learner is able to make distinctions based on previous abilities and knowledge. These images are not necessarily "pictorial representations" but rather convey the meaning of any kind of mental image. The actions at this layer involve the learner doing, either mentally or physically, something to gain an idea about a concept. For instance, the learner while engaged in folding or cutting activities may develop an image of fractions as things gained from cutting something into equal smaller pieces. As a result, the actions at this layer involve the development of connections between referents and symbols as described by Wearne and Hiebert (1988), Greeno (1991) and Brownell (1945) as the learner employs fraction language to discuss and record the actions. At the next layer, called *image having*, single-activity associated images are replaced by

[7] In this case, the mapping instruction involved three components: (1) learning and practicing subtraction with both base 10 blocks and symbols, (2) practicing symbol recording without physically moving the blocks, and (3) doing the written symbol manipulation without the presence of the blocks.

a mental picture. The development of these mental pictures, or more precisely, mental process-oriented images, frees the learner's mathematics from the need to perform particular physical actions (Pirie & Kieren, 1992b). These mental images have been discussed under the guise of several names: "concept image" (Davis & Vinner, 1986), "frames" and "knowledge representation structures" (Davis, 1984), and "students' alternative frameworks" (Driver & Easley, 1978). The freedom to imagine a concept unconstrained by the physical processes which elicited the image is useful to the growth of mathematical knowing since the learner begins to recognize obvious global properties of the inspected mathematical images.

At the fourth layer referred to as *property noticing*, the learner can examine a mental image and determine the various attributes associated with the image. Besides noticing the properties internal to a specific image, the learner is capable of noticing the distinctions, combinations or connections between multiple mental images. These properties combine to construct evolving definitions that may identify particular characteristics while ignoring other elements of the concept. According to Pirie and Kieren (1991a), the difference between image having and property noticing is the ability to notice a connection between images and explain how to verify the connection. These connections, according to Michener (1978), arise from the exploration and manipulation of a concept at many levels such as "examining relevant examples, perturbing settings and statements and fiddling around numerically and pictorially" (p. 373). It may be at this property noticing layer that the learner notices commonalties of various images and develops a concept definition (Tall & Vinner, 1981) built upon the interplay between multiple linked images rather than disconnected images. In reference to understanding fractions, the actions of the learner reveal a recognition that equivalent fractions are generated by multiplying numerator and denominator by the same factor. The verbalization associated with this layer would consist of producing a series of equivalent fractions such as

$$2/3 = 4/6 = 6/9 = \ldots \text{ (Pirie \& Kieren, 1991a, p. 2)}$$

The difference between the property noticing layer's actions and those of image having layer is a capability to notice equivalencies and explain the necessary techniques for developing them.

At the fifth layer of understanding, called *formalizing*, the learner is able to cognize about the properties to abstract common qualities from classes of images. At this layer the learner has class-like mental objects built from the noticed properties, the abstraction of common qualities, and abandonment of the origins of one's mental action (Pirie & Kieren, 1989). Description of these class-like mental objects results in the production of full mathematical definitions. The language used to describe a concept does not have to be formal mathematical language; however, the general descriptions provided by students must be equivalent to the appropriate mathematical definition. It may be at this layer that Michener's description (Michener, 1978) of the first of a three-phase passage to full understanding comes into play. This first stage consists of the learner gaining familiarity with the concept and its neighboring concepts. Acquisition of the definitions occurs but the concern is with the particular concept and the scope of connections between it and other concepts remains minimal and local. With respect to fractions, the learner can now speak of fractions as a class of formal objects unconnected to specific examples and represent this class in terms of a/b at the formalizing layer. The learner also can view fractions as a set of numbers or general entities no longer action-oriented.

The following layer of understanding, *observing*, entails the ability to consider and reference one's own formal thinking. Beyond the learner engaging in metacognition, the learner is also able to observe, structure and organize personal thought processes as well as recognize the ramifications of thought processes. At this layer, the learner can produce verbalizations concerning cognitions about the formalized concept. This description echoes the second stage proposed by Michener (1978) in which the learner gains an overall conceptualization of the subject matter and its development. In particular, the concern resides on "items and relations within the representation spaces and the theory as a whole; it is more global in outlook" (Michener, 1978, p. 376). The learner combines the definitions, examples, theorems, and demonstrations to identify the essential components, connecting ideas, and the means for traversing between those ideas. For the case of fractions, the learner has progressed to the production of verbalizations concerning cognitions connected with fractions at the organizing layer. Here, the learner might observe that: "There can be no smallest half fraction" (Pirie & Kieren, 1992b, p. 247). Such an observation is different from the property noticing layer recognition, any half fraction can be made smaller because it can be folded in half, or the image having layer realization of many folds produces small pieces.

Once one is capable of organizing one's formal observations, the natural expectation is to determine if the formal observations are true. The learner after gaining such awareness can then explain the interrelationships of these observations by an axiomatic system (Pirie & Kieren, 1989). This layer is called *structuring.* At this layer, the learner's understandings transcend a particular topic for the understanding inhabits a larger structure. This structuring layer appears well correlated with the third stage described by Michener (1978). In this third stage, the learner begins to see relationships between several subjects, address certain questions about the underlying ideas, axioms and examples, relate these underlying ideas across multiple domains, and perceive the interconnectedness of several theories. Therefore at the structuring layer, fractions are conceived as beyond the physical entities associated with the image making layer, the action-oriented equivalencies associated with the property noticing layer, and the resultant of formal algorithms associated with the formalizing layer. The learner would now be able to conceive proofs of properties associated with fractions such as the closure of half fractions under addition where the addition of fractions is viewed as a logical property following from other logical properties (Pirie & Kieren, 1991a, 1992b).

The outermost ring of the Pirie and Kieren's model of mathematical understanding is called *inventizing.* Originally referred to as *inventing*, this layer's name changed to the present term to distinguish the activities associated with this layer and the inventions that can occur at lower layers of understanding (Pirie & Kieren, 1994b). As a result, the use of inventizing does not imply that one cannot invent at other layers, but rather is used to indicate the ability to break free of a structured knowledge which represents complete understanding and to create totally new questions which will result in the development of a new concept. At this layer, the learner's mathematical understanding is unbounded, imaginative and reaches beyond the current structure to contemplate the question of "what if?" This questioning results in a learner's use of structured knowledge as primitive knowing when investigating beyond the initial domain of exploration. For example, extension of the fraction notation a/b for $a + bi$ to $a/b/c/d$ for $a + bi + cj + dk$ took the mathematician Hamilton from a structured understanding of complex numbers into a new system called the quaternions (Pirie & Kieren, 1991a).

2.2. Constructs of Pirie and Kieren's Model of Understanding. Pirie and Kieren's model of understanding contains an inherent dynamism which is apparent in several components. In particular, the core of the model, primitive knowing, has an underlying dynamic quality. Pirie and Kieren (1990), for instance, characterize this movement in the following statement:

> One obvious consequence of this model is that, outer levels grow recursively from the inner ones, but knowing at an outer level allows for and indeed retains the inner levels. Outer levels embed and enfold the inner ones. In fact this [is] a relativity theory of understanding and therefore a particular feature of primitive doing [referred to in later papers as primitive knowing] is that observers can consider what they wish as the focus of this level. For example, one could observe a person at the inventing [later referred to as inventizing] level as having their entire previous understanding as a new primitive doing [primitive knowing]. A main consequence of this line of thinking is that, to an observer, understanding has a fractal quality. One could look at the understanding of a person "inside" primitive doing [primitive knowing] and observe the same leveled structure. (p. 5)

From this, one can see that Pirie and Kieren view the inner core, called primitive knowing, as composed of complete models similar to the whole. This property gives the inner core the attributed *fractal characteristic* evidenced in Figure 2. This nesting points to the importance of the information contained in the inner core since that information constrains one's knowing at outer layers (Pirie & Kieren, 1989). As a result, this core knowledge can either beneficially aid a student in understanding a concept or hinder the student from understanding by acting as an obstacle (Mack, 1990; Resnick et al., 1989).

The most critical feature of Pirie and Kieren's model of understanding is the dynamic process of *folding back*. When one encounters a problem whose solution is not immediately attainable, one must see the necessity to fold back to an inner layer to extend one's current, inadequate understanding. The process of folding back to an inner layer finds one examining the layer's understandings in a manner different from the actions originally displayed when operating at the layer. This difference is both qualitatively and intently different due to the motivation associated with the folding back and the developed understandings of the outer rings (Pirie & Kieren, 1991a, 1992a). As a result, the extension of one's understanding is not simply a product of generalizing a given layer's activities nor a consequence of reflectively abstracting one's understanding to attain a new outer layer, but more usually the extension occurs by folding back to recursively reconstruct and reorganize one's inner layer knowledge, and so further extend outer layer understanding. Such a process is similar, according to Pirie and Kieren (1992b), to the reconstruction of an existing schema described both by Sfard (1991) and Harel and Tall (1991).

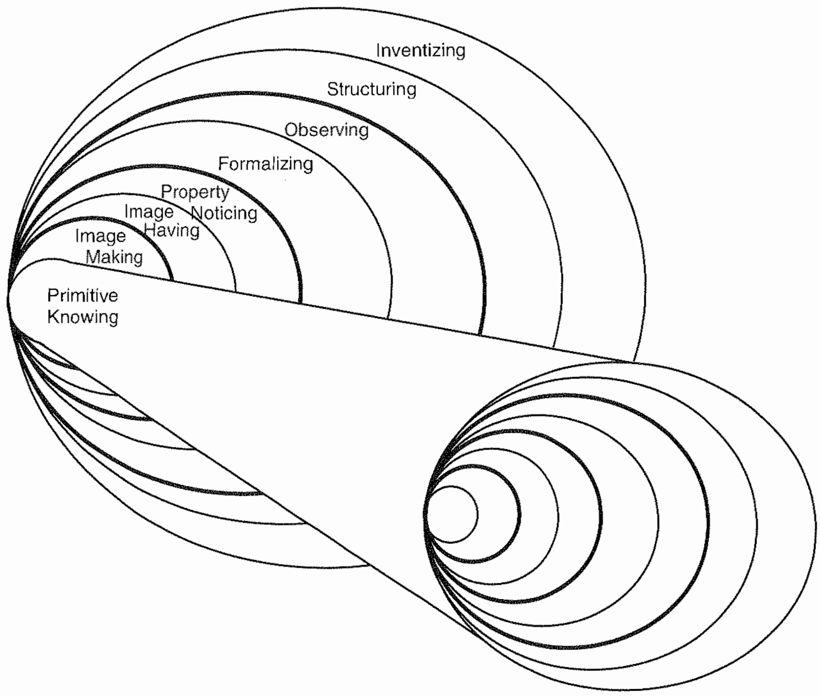

FIGURE 2. A diagrammatic representation of the model for growth of mathematical understanding illustrating the self-similar nature of the inner core called primitive knowing

Another construct of the model identifies the *complementarity of a process and a form-oriented action.* Each of the layers, beyond the primitive knowing layer, contains a complementarity of form and process as identified in Figure 3. Pirie and Kieren contend that one must exhibit the form-oriented action to be fully functioning at a layer (Kieren, 1990; Pirie & Kieren, 1990, 1994b). This form-oriented action acts as a demonstration to an external agent attempting to determine the layer of understanding at which a learner is operating. Therefore, the absence of a layer's displayed complementary action does not demonstrate that the learner is operating at a particular layer. Pirie and Kieren (1991a) extended these notions and re-labeled the diagrams shown in Figure 4 to allow easier discussion of the blended, laminar layers and the acting and expressing complements therein.

In particular, Pirie and Kieren (1994b) assert that if students perform only actions without the correspondent expression then their understandings are inhibited from movement to the next layer.

The image making layer is composed of two complementary elements called *image doing* and *image reviewing.* The image doing learner sees previous work as complete and does not return to it; whereas, an image reviewing learner involves the constructive alteration of previous behavior without necessarily noticing a pattern (Pirie & Kieren, 1991a). Image doing initially may appear ill-defined since engagement in any activity appears to be image doing. However, image doing, according

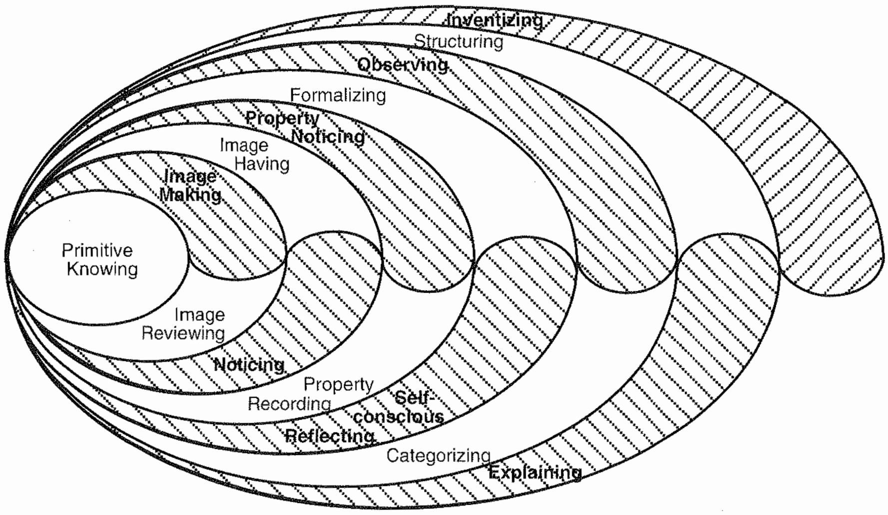

FIGURE 3. The within level complementarities

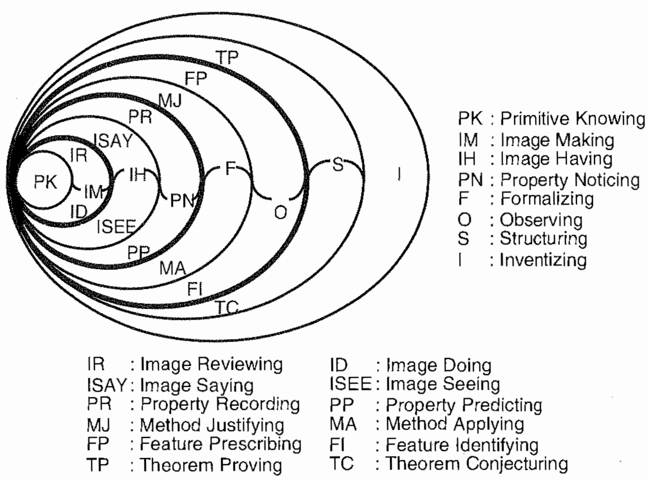

FIGURE 4. Rings with acting and expressing complements identified

to Pirie and Kieren (1994b), consists of only potentially fruitful actions linked to intentional making of some sort of images for a concept.

The image having layer has two complementary elements, *image seeing* and *image saying*. The *image seeing* act has collected together previously encountered examples and has a pattern; whereas, the *image saying* behavior articulates the pattern associated with the image (Pirie & Kieren, 1991a). In particular, when engaged in image seeing, a learner identifies a discrepant element as uncorrelated with the learner's mental image without being able to articulate why. On the other hand, image saying involves the learner in both articulating both the image and the reason the discrepant element does not fit the image (Pirie & Kieren, 1994b). At the property noticing layer, the two complementarities are *property predicting* and *property recording*. The *property predicting* act relates the image to a property noticed by the learner and *property recording* is an act of incorporating into the learner's cognitive structure the noticed property as something that exists and seems to work. According to Pirie and Kieren (1994b), both the image having and the property noticing layers have a particular characteristic that distinguishes them from other layers. At these two layers, the "acting" notions produce temporal understandings in the sense that they will diminish over time if not coordinated with their complementarity "expressing" notions. As a result, the lack of an "expressing" activity seems to obstruct movement beyond their previous images and therefore to higher layers in their model (Pirie & Kieren, 1994b).

At the formalizing layer, *method applying* and *method justifying* are the two complementarities whereas the observing layer contains the complementarities of *feature identifying* and *feature describing* (Pirie & Kieren, 1994b). The last layer to contain complementarities is the structuring layer and it contains *theorem conjecturing* and *theorem proving* (Pirie & Kieren, 1994b). The complementarities for these last three layers of their model are defined without any further description other than the provision of illustrative terms in Pirie and Kieren (1994b).

The last construct of the model is the darker rings called "*don't need*" boundaries. These boundaries signify movement of the learner to a more elaborate and stable understanding which does not necessarily require the elements of the lower layers (Pirie & Kieren, 1992b). For example, once students have moved to the image having layer, it is no longer necessary to elicit examples of image making or elements from the primitive knowing layer. A person at the image having layer is at a qualitatively different layer of understanding when actively involved in *image seeing* and *image saying* activities for the learner does not necessarily see a mathematical object as the result of a doing activity, but rather as an entity with identifiable features (Pirie & Kieren, 1991a). Movement from the image making layer to the image having layer involves a qualitative change in the associated thinking processes. The learner has moved from layers associated with unselfconscious knowing to conscious thinking. Therefore, moving over a "don't need" boundary signifies a major qualitative change in one's understanding. Similar qualitative changes have occurred when one achieves either the formalizing or structuring layers since there is no longer a need for an image or a concrete meaning for the mathematical activity. However, the overcoming of a "don't need" boundary does not imply that a learner may never pass back into that lower level understanding. In fact, these "don't need" boundaries are typically crossed back over during times of folding back in order to reorganize and reconstruct lower level understandings in order to expand outer level understandings.

3. Dubinsky's APOS Theory

Piaget's proposal of the process of *reflective abstraction* as the key to the construction of logico-mathematical concepts influenced Dubinsky's development of the Action-Process-Object-Schema theory (APOS theory). In this theory, the development of understanding " ... begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas" (Asiala et al., 1996, p. 8). The mechanism

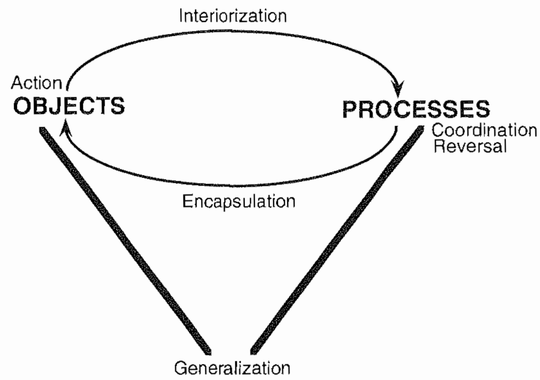

FIGURE 5. Schemas and their construction

of constructing these schemas, reflective abstraction, is the heart of APOS theory since it separates the properties connected to a concept and identifies the salient elements which comprise the concept apart from the context. In particular, reflective abstraction extends the construction of connections between abstracted concepts and builds a structure out of the related abstractions.

According to Dubinsky (1991), five different kinds of Piagetian constructions are essential to developing abstract mathematical concepts—*generalization*, *interiorization*, *encapsulation*, *coordination* and *reversal*. The fifth construction, *reversal*, considered to be crucial to advanced mathematical thinking from the APOS perspective was not part of Piaget's description of reflective abstraction although it was contained in his writings (Dubinsky, 1991). The ensuing sections describe APOS theory's components, define its various constructions, and relate the components and constructions.

3.1. Elements of APOS theory. Dubinsky proposes that a schema is more than a static entity since it remains inseparable from its own dynamic, continuous evolution. Figure 5 depicts the structure of a schema and identifies the influential constructions.

The primary building block of understanding is *action* (similar to Piaget's *action schemes*). Asiala et al. (1996) identify that understanding of a mathematical concept originates through the manipulation of previously constructed mental or physical objects to form actions. An action equates with any repeatable physical or mental operation that transforms either a physical or mental object in some manner. As a result, the actions tend to be algorithmic in nature and externally driven (Clark et al., 1997). In reference to the function concept, a learner may conceive a function as involving the plugging of numbers into an algebraic expression and calculating the resultant values. Such a conceptualization appears restricted and static since it remains inexorably linked to individualized evaluations that occur one at a time.

Even though an action conception appears limited in scope, it is a necessary element in the construction of understanding since actions are done on the mathematical objects within a learner's realm of experience. As an action becomes *interiorized* through a sequence of repeating the action and reflecting upon it, the action no longer remains driven by external influences since it becomes an internal construct called a *process* (similar to Piaget's *operations*). Attainment of this process conception indicates the learner can reflect on the process, describe it, and even reverse the steps of transformation without resorting to external stimuli (Asiala et al., 1996). In particular, the learner now can use this process to obtain new processes either through *coordination* or *reversal*. The coordination of multiple related processes results in the construction of new processes. Many times this is necessary as a learner encounters elements of a new topic and recognizes underlying structures that permit the application of several processes developed in a different context. Additionally, situations exist where the learner encounters topics which require the composition of seemingly unrelated processes to build a more complex structure (Dubinsky, 1991). Reversal, on the other hand, permits the learner to conceive of a new process that undoes the sequence of transformations comprising the initial process. Essentially, reversal is the construction of a process which countermands an internalized process. For instance with respect to the function concept, a learner can link two or more processes together to produce a composite function thereby coordinating several processes into a singular process or reverse a function to obtain the inverse of the function. In addition, the strengthening of a process conception permits greater access for the learner to notions such as one-to-one and onto.

When a learner can reflect on a process and transform it by some action, then the process is considered to have been *encapsulated* to become an *object*. According to Sfard and Linchevski (1994), the process of reification is virtually synonymous with encapsulation. Once encapsulated, the object exists in an individual's mind necessitating the assignment of a label to the object (Dubinsky et al., 1994). The resultant label allows the learner to both name the object and connect that name with the process from which the object was constructed. This dual vision of the object is essential because the learner needs to be able to *de-encapsulate* an object thereby returning to the process in a form prior to its encapsulation. De-encapsulation enables the learner to use the properties inherent in the object to perform new manipulations upon it. For instance, with respect to functions, a learner must be facile in encapsulating processes into objects and de-encapsulating objects into processes when considering manipulations such as adding, multiplying, or creating sets of functions (Asiala et al., 1996).

The final element of the APOS theory, the *schema*, is a collection of processes and objects (Dubinsky et al., 1994). This collection may be more or less coherent but the learner utilizes it to organize, understand, and make sense of observed phenomena or concepts. As a result, a schema generally contains other subordinate schemas to span across a particular domain. The term schema has strong connections to Piaget's use of the term *schemata* which in turn relates to Tall and Vinner's descriptions of a concept image (Asiala et al., 1996). However, the schemas designated by Dubinsky et al. (1994) correspond to Piaget's thematized schematas which indicates that the collection coalesced into an object upon which actions can take place. In particular, these schemas are the mental representations of concepts as they exist in the learner's mind. As a result, Dubinsky et al. (1994) point out that "A schema can be used to deal with a problem situation by unpacking it and working with the individual processes and objects. A schema may also be treated as an object in that actions and processes may be applied to it" (p. 271). This ability to conceive a schema as an object upon which actions and processes can transform the schema infers a fractal quality with schemas containing other schemas as elements.

In particular, schemas are organizing structures that incorporate the actions, processes, objects and other schemas that a learner invokes to deal with a new mathematical problem situation. Constructing such structures requires a mechanism called *generalization* which permits a wider scope of schema utilization also referred to as *expansive generalization* by Harel and Tall (1991). In particular, the generalization of an already constructed schema occurs when the learner enlarges the venue of applicability without changing the structure of the schema. Piaget referred to such a construction as a reproductive or generalized assimilation and called the generalization *extensional* (Dubinsky, 1991). Such generalization is the simplest and most familiar form of reflective abstraction since it involves the application of an already existent schema to a new set of objects.

More recently, Clark et al. (1997), while investigating student construction of the chain rule understanding, attempted to utilize this notion of schema to negotiate between members of the RUMEC research team explanations of the exhibited student understandings. However, it became apparent during their attempt to classify student responses that the descriptions of actions, processes, and objects were insufficient in this context. In response, Clark et al. (1997) state "In the course of these negotiations, we realized that Actions, Processes, and Objects alone were insufficient to describe the student understanding of the chain rule" (p. 353). In essence, for the concept of chain rule that incorporates a variety of sub-concepts, APOS theory did not provide a sufficiently robust explanation for all the various nuances encountered, since it does not result from the encapsulation of a single process.

Clark et al. (1997) then returned to the writings of Piaget and Garcia to examine a triad mechanism useful in the description of schema development. This mechanism defines three particular stages in the development of a concept: *Intra*, *Inter*, and *Trans*. "The Intra stage is characterized by the focus on a single object in isolation from other actions, processes, or objects. The Inter stage is characterized by recognizing relationships between different actions, processes, objects and/or schemas. We find it useful to call a collection at the Inter stage of development a pre-schema. Finally the Trans stage is characterized by the construction of coherent structure underlying some of the relationships discovered in the Inter stage of development" (Clark et al., 1997, pp. 353–354). This refinement of the concept of schema permitted Clark et al. (1997) to explain the actions and statements of students with respect to the chain rule as well as McDonald, Mathews and Strobel (2000) to study the cognitive development of the concept of sequence. Consequently, the notion of schema continues to be reexamined and redeveloped as the theory evolves.

3.2. Constructs of APOS Theory. The construction of understanding, from the APOS perspective, passes through several stages driven by external cues which then become internalized, reflected upon and eventually organized. In particular, Asiala et al. (1996) characterized the development of mathematical knowledge from the perspective of APOS theory in the following manner:

> An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problems situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes

> and objects and organizing these in schemas to use in dealing with the situations. (p. 7)

The act of reflection, a construct essential to APOS theory, is the learner's conscious attention to the operations being performed. This reflection plays an integral role in learning and knowing since it involves reaching beyond contemplation on the particular performance of techniques and algorithms no matter how complicated they may be (Asiala et al., 1996). In particular, reflection provides the learner with an awareness of how procedures work, a feeling for the results without physically performing the operations, an ability to analyze and manipulate variant algorithms, and a capability to see relationships and organize experience. This reflection is an integral part of reflective abstraction that consists of drawing properties for situations by paying conscious attention to the actions, interiorizing those actions into processes, encapsulating the processes into objects and finally organizing related actions, processes, objects into mental entities called schemas. In particular, the reflection on a schema with intention to transform it extends a learner's understanding by yielding an additional means of constructing an object (Cottrill et al., 1996). Thus, APOS theory accounts for the construction of objects from two different sources—encapsulation of processes and *reflections* upon schemas.

Another essential construct of APOS theory related to schemas is a theoretical description called a *genetic decomposition*. The fact that schemas are mental entities engenders one of the inherent difficulties of examining a learner's schema. In order to overcome this, although not in entirety, Dubinsky and others utilize genetic decompositions of concepts to characterize the linkages and representations within a concept. A genetic decomposition of a concept derives from three sources: psychological data, Piagetian ideas of concept formation, and understandings of the concept held by mathematicians (Dubinsky, 1991). The psychological data draw from observations of students in the midst of learning a concept. Piaget's ideas of successive refinements influence the construction and revision of the modeling genetic decomposition. These refinements engender from reflections on the learning experiments which guide the adjustments to the model thereby better accommodating new and relevant phenomena to produce a richer and more representative model. As the model requires less modification to account for the data generated from a teaching experiment, greater evidence exists that the model is descriptive. Lastly, mathematical descriptions of the concept are essential since the genetic decomposition must make sense from a mathematical perspective. However according to Dubinsky (1991), the genetic decomposition does not necessarily reflect how a particular mathematician would analyze the concept to formulate a method for teaching the concept. As a result, a genetic decomposition is an idealized characterization of the mathematically expected representations, linkages, objects, processes, and actions generally ascribed to the concept. In addition, the genetic decomposition provides a possible path for a learner's concept formation; however, it may not be representative of the path taken by all students (Dubinsky, 1991).

Asiala et al. (1996) assert "Our tentative understanding suggests that an individual's schema for a concept includes her or his version of the concept that is described by the genetic decomposition, as well as other concepts that are perceived to be linked to the concept in the context of problem situations" (p. 12). As a result, a learner's schema may or may not represent the whole or even a part of the genetic decomposition. This schema may lack essential elements or contain elements not considered mathematically connected to the concept. However, as Dubinsky (1991) pointed out, "It is not possible to observe directly any of a subject's schemas or their objects and processes. We can only infer them from our observation of individuals who may or may not bring them to bear on problems—situations in which the subject is seeking a solution or trying to understand a phenomenon. But these very acts or recognizing and solving problems, of asking new questions and creating new problems are the means (in our opinion, essentially the only means) by which a subject constructs new mathematical knowledge" (p. 103). As a result, attempting to uncover a learner's schema by presenting new tasks, observing the outward displays, and making inferences about the internal actions, processes, objects, and schemas from them appears futile. The futility arises from the possibility that the engagement with a task encourages the reorganization of thinking. However, consistent responses across multiple tasks indicate the learner assimilated a portion of a schema exemplified in a genetic decomposition.

4. Examining shared and distinctive elements

Pirie and Kieren's model of the growth of understanding and Dubinsky's APOS theory hold many elements that are isomorphic to one another and a few where they diverge. For instance, one can examine why Pirie and Kieren's model of the growth of mathematical understanding and Dubinsky's APOS theory satisfy the criteria associated with the status of a theory, compare the origins of the two theories, relate their organizational structures, examine the linkages to alternative theories of understanding, and discuss the implementation of the theories with respect to assessment and instructional practices. As a result, the following sections further identify these comparable qualities as well as the qualities which make the two theories distinct.

4.1. Why do these two frameworks satisfy the criteria for a theory? Prior to turning to the question of whether Pirie and Kieren's model and APOS theory truly satisfy criteria for a theory, a few terms need to be clarified. A *model* is a representation useful only to the degree in which it describes the linkages between represented objects and organizes a structure to help us understand those objects. In general, a model may systematically obscure particular features in an attempt to simplify relationships thereby leaving much of a situation unrepresented. Consequently, the model provides a working description of reality focused on the objects and their relationships without claiming absolute truth. Similarly, a *theory* does not claim absolute truth and its explanations focus on the level of mechanism. That is, it focuses on overarching meta-relationships rather than represented objects in a particular situation while still grappling with the working description of complex phenomena. Lastly, the term *theoretical framework* which has been used in this paper is a descriptive of either a model or theory which has not been scrutinized according to the criteria of Schoenfeld (2000).

So, what are Schoenfeld's criteria? According to Schoenfeld (1998), three criteria defined whether a theoretical framework could be classified as a theory: explanatory power, predictive power, and scope. These three criteria have been expanded in Schoenfeld (2000) to include: (a) descriptive power, (b) explanatory power, (c) scope, (d) predictive power, (e) rigor and specificity, (f) falsifiability, (g) replicability, and (h) multiple sources of evidence ("triangulation"). As each requirement is defined, a discussion will present how Pirie and Kieren's model of the growth of mathematical understanding and Dubinsky's APOS theory satisfy the criteria.

4.1.1. *Descriptive Power.* Descriptive power involves the capacity of the theoretical framework to capture the essential features under investigation in ways that permit faithful examination of the phenomena being described (Schoenfeld, 1998). Clearly, both Pirie and Kieren's model of the growth of understanding and Dubinsky's APOS theory satisfy this first criteria. Pirie and Kieren utilize interview transcripts and graphical images to characterize the movement of a student between levels of their model to identify the types of understandings utilized to answer various questions during the interview process. Similarly, Dubinsky's APOS theory uses both written and oral data collection techniques to form a descriptive document characterizing the student's "achieved" level of understanding. In this sense, both theories amply provide sufficient description for which a reader can interpret the theory and recognize the data correspondent to the conclusions. Both of these theories provide the observer of students' external actions or utterances with a means of collecting, organizing, and analyzing those observations. At the same time, they both are incomplete, leaving peripheral elements unexplained to characterize the focal issues.

4.1.2. *Explanatory Power.* Explanatory power refers to a framework's ability to explain mechanisms—descriptions of how and why things fit together and work. According to Schoenfeld (2000), the explanations provide the underlying reasons for why a student can or cannot perform a particular task. In essence, a theory must contain precise, descriptive terms which indicate the important objects of the theory, their interrelationships, and the reasons particular things are possible or not. Dubinsky and McDonald (2001) argue that the explanatory power of APOS theory resides in the theory's ability to point to particular mental constructions of actions, processes, objects, and/or schemas that one student appears to have made whereas the other has not. In a similar fashion, Pirie and Kieren's theory allows them to explain differences in student performance based upon the level of understanding correspondent to prior concepts. For instance, in Pirie and Kieren (1992a), they differentiate between the performance of students based upon their ability to work with "thirds" based upon the students' external actions and utterances providing indication of differences in the layer of operation. In particular, their theory points to developed images, noticed relationships between images, formalized descriptions of those relationships, etc. as explanations for variances in student performance.

4.1.3. *Scope.* Scope involves the range of phenomena attended to by the theory. Essentially, a comprehensive theory must apply to a broad range of phenomena rather than a localized concept. APOS theory has been widely employed by Dubinsky and members of his RUMEC team across topics such as: functions, abstract algebra (binary operations, groups, subgroups, cosets, normality, quotient groups), discrete mathematics (induction, permutations, symmetries, existential and universal quantifiers), calculus (limits, chain rule, derivative, infinite sequences), statistics (mean, standard deviation, central limit theorem), elementary number theory (place value in base n, divisibility, multiples, converting between bases), and fractions. The number of concepts mentioned clearly points to the applicability of APOS theory to a broad range of phenomena typically linked to undergraduate mathematics, although a few of the topics have been investigated with younger children (Dubinsky & McDonald, 2001). In contrast, Pirie and Kieren's model of the growth of understanding has generally focused on the development of understandings in younger children. Consequently, the range of phenomena for which Pirie and Kieren's theory has been applied is smaller than that of APOS theory. Concepts such as fractions, quadratic functions, and other middle school content have been the focus of their investigations with a study of geometry learning in Pirie and Kieren (1991a). However, others have employed Pirie and Kieren's model of understanding with respect to calculus (Meel, 1995), abstract algebra knowledge (Grinevitch, in preparation), and teacher interventions (Towers, Martin, & Pirie, 2000) thereby raising the possibility that Pirie and Kieren's model may be applied to a broader range of phenomena than middle school mathematical concepts.

4.1.4. *Predictive Power.* Predictive power is not at the level of those made in the physical sciences but rather refers to the ability of the theory to provide reasonable predictions as to the observed actions and utterances based upon prior information. In essence, predictive power permits the researcher to anticipate responses, based upon prior knowledge, before the participant actually responds. Pirie and Kieren's model of the growth of understanding draws upon the weight of prior information elicited from a student in order to identify the images and structures built by a student. Using this knowledge, Pirie and Kieren's theory both suggests potential actions and utterances of a student with respect to a new task based upon their experience with prior tasks. For instance, in Pirie and Kieren (1992a), they state the following about a student: "Sandy, a gifted eight year old, who had shown formalized understanding with respect to 'half fractions,' appeared to apply a formal method to generate new fractions" (p. 515). As students utilized their knowledge of "half fractions" and attempted to extrapolate this to "third fractions," it was clear that Pirie and Kieren's theory permitted the teacher to anticipate the potential images developed and how the students would interact with those newly-developed images. Further instances of making testable predictions of student responses were found in Towers, Martin, and Pirie (2000) with respect to teacher interventions. In a similar fashion, APOS theory permits the development of predictions. According to Dubinsky and McDonald (2001), APOS theory provides the opportunity to make testable predictions "that if a particular collection of actions, processes, objects and schemas are constructed in a certain manner by a student, then this individual will likely be successful using certain mathematical concepts and in certain problem situations" (p. 4). The *genetic decompositions* of concepts employed by APOS theory both provide descriptive information as well as a means of generating hypotheses about how the learning takes place and what elements interact with the development of an individual's understanding of a particular concept.

4.1.5. *Rigor and specificity.* Rigor and specificity refer to the ability of a theory to clearly identify the elements inherent to the theory and the mechanisms which connect them. Explicitly, rigor and specificity are concerned with the terms and relationships of theory being well-defined, that is, if you were interviewing a student could you easily detect that they were operating at a particular layer in Pirie and Kieren's model or with a particular conception in APOS theory. Earlier in this paper the elements and constructs of both Pirie and Kieren's model and Dubinsky's APOS theory were defined. These characterizations not only provided the descriptions of the elements and their linkages but identified their relationships to other related perspectives. For examples of student data and a means of interpreting the conversations, examine the Pirie and Kieren (1994b) description of Teresa or Julie working with fractions, the Pirie and Kieren (1992) description of Sandy, the section 7.1.4 description of Ada in Brown et al. (1997) who was in flux between action and process conception of a group, or the chapter 7 characterization in Cottrill (1999) describing students, such as Tim, Al, Ray, Peg, and Jack, from the perspective of operating at either the Intra, Inter, or Trans level.

4.1.6. *Falsifiability.* Falsifiability is needed by any theory. In practice, falsifiability means that the predictions made and elements defined by theory must hold up to empirical examination. In fact, both Pirie and Kieren's model and Dubinsky's APOS theory do not claim their frameworks are truth but serve as a descriptive language needing scrutinization and testing. For instance, Dubinsky and McDonald (2001) state:

> We do not think that a theory of learning is a statement of truth and although it may not be an approximation to what is really happening when an individual tries to learn one or another concept in mathematics, this is not our focus. Rather we concentrate on how a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in students who are trying to construct their understandings of mathematical concepts and by suggesting directions for pedagogy that can help in this learning process. (p. 1)

Similarly, Pirie (1988) states:

> In all actuality, we can never fully comprehend "understanding" itself. As Piaget (1980) claims for all knowledge, with each step that we take forward in order to bring us nearer to our goal, the goal itself recedes and the successive models that we create can be no more than approximations, that can never reach the goal, which will always continue to posses undiscovered properties. What we can, however, do is attempt to categorize, partition and elaborate component facets of understanding in such a way as to give ourselves deeper insights into the thinking of children. (p. 2)

Consequently, neither Pirie and Kieren's model or APOS theory claim to be truth but rather claim to provide a language for the mechanisms necessary to describe development in a variety of settings.

From this perspective, both Pirie and Kieren and their colleagues and Dubinsky and his RUMEC colleagues have continued to empirically test the applicability of their respective frameworks to a variety of mathematical concepts and under a variety of settings. As a new concept is explored or a new setting is examined, the researchers continually refine and adapt their frameworks to better explain the phenomena thereby transcending prior descriptions while maintaining compatibility with prior assertions. In this manner, these frameworks provide dynamic lenses for viewing situations and the interactions contained within those situations. Consequently, any challenges to their veracity must be focused on a framework's inner consistency and its ability to interpret situations under those constraints.

4.1.7. *Replicability.* Replicability is closely tied to rigor and specificity. According to Schoenfeld (2000) "There are two related sets of issues: (1) Will the same thing happen if the circumstances are repeated? (2) Will others, once appropriately trained, see the same thing in the data" (p. 648)? In particular, replicability is concerned with a framework's ability to describe similar behaviors in similar manners as well as different groups of researchers trained in a particular framework seeing and describing similar events with similar language. That is, the framework must be defined well enough that others can look at the same data and reach the same conclusions.

Generally, replicability is concerned with the ability of a researcher to examine the procedures and constructs, implement a study using those procedures and constructs, and interpret the data in similar fashions. Discussion of the application of both Pirie and Kieren's model and Dubinsky's APOS theory to the analysis of particular mathematical concepts clearly indicate their respective data collection methods, employed theoretical structures, mode of analysis, and general results or parsed transcripts. By being able to review the language and work of the participants in these studies, one can gain a vision of the interaction of the theories and the data along with a confidence of the repeatability of such methods to other participants and situations. However, the added condition of repeatability should not be construed with certainty. Pirie and Kieren's model and Dubinsky's APOS theory provide a range of possible outcomes assuming a student interprets questions in manners anticipated by the researchers (see section 4.5.2 on *Instructional practices* for further discussion).

4.1.8. *Triangulation.* The last criterion identified by Schoenfeld (2000) was that of triangulation. That is, a theory must not be developed based upon a limited set of experiences but rather must utilize information collected in a variety of settings (classroom interviews, individuals working on tasks, pairs working together, etc.). In doing so, the theory can be examined for internal consistencies as well as oversensitivity to local factors not influential to the general case. In addition, triangulation refers to the theory's ability to utilize multiple sources of information when analyzing a particular concept as well as being informed from a broad range of phenomena. As previously argued, both Pirie and Kieren and Dubinsky fine-tuned their theories over a wide spectrum of phenomena and in a variety of settings by gathering evidence on students from a variety of sources. With respect to the use of internal triangulation, Pirie and Kieren's model focuses on interview data collected on an individual basis, as part of pairs or groups, and as part of a class. Interpretations concerning a student's placement with respect to the theory also come from the learner's interaction with multiple tasks designed to help the researcher observe the changes in mathematical knowing. On the other hand, Dubinsky's APOS theory utilizes both written work and interview data to coalesce a vision of the actions, processes, objects and/or schemas developed by a student. Using this two-pronged approach, data collection methods used by APOS researchers afford the researcher with a global overview from written work and then the ability to focus on particular "bits" of knowledge during the interviews.

This discussion has provided a means of validating the assertion that Pirie and Kieren's model of the growth of understanding and Dubinsky's APOS theory satisfy the criteria set forth by Schoenfeld (2000). The rest of this section will further identify the qualities which connect the two theories as well as those qualities which make them distinct.

4.2. Origins of the theories. Both Pirie and Kieren's model of the growth of understanding and Dubinsky's APOS theory arise from a constructivist point of view. Pirie and Kieren's model arose from Glasersfeld's perception that understanding is a continual process of organizing one's knowledge structures whereas Piaget's discussions of reflective abstraction drove much of the construction of the APOS theoretical framework. According to each theory, the learner actively participates in the enterprise of constructing understanding from the elements and situations provided in mathematical problems. This construction, according to both Pirie and Kieren's model and APOS theory, is a process where learners continually engage in the act of organizing their own knowledge structures (Ayers et al., 1988; Pirie & Kieren, 1991a). As a result, both theories describe understanding as a never-ending process. In addition to this bond, the ties between the theories do not end with their constructivist roots. Both theories developed out of a tradition of significant observation and interaction with students engaged with particular mathematical content. It is with this respect that the two theories are both similar and different.

The primary origin of Pirie and Kieren's model rests in the observation of the teaching and learning of middle and high school level mathematics such as fractions and quadratic functions (Pirie & Kieren, 1989, 1991a). Such observations occurred as part of whole class "teaching experiments" with Tom Kieren acting as the teacher and students were involved in a constructivist environment (Pirie & Kieren, 1994a). Students similar to "Sandy" (Pirie & Kieren, 1992b) were selected from the larger learning environment for individualized interview sessions. Concurrent with the interviews, class activities were audio or videotaped while research assistants tracked the work of subgroups of students, augmenting transcripts with field notes, and collected student work (Pirie & Kieren, 1992a). These various artifacts were studied for the presence of activities which allowed Pirie and Kieren to test, elaborate, and develop constructs and properties of their theory.

Pirie and Kieren (1992b) describe an overall learning environment in which students are engaged for a couple of weeks with the area of fractions using the "half family" and paper-folding explorations. An excerpt from an interview with Sandy then is used to illustrate the various interviewer questions and Sandy's responses. One element to be noticed is that the interviewer's questions were not fixed but rather reacted to Sandy's responses while posing new questions, elaborating on old ones, or achieving understanding of the student's responses. Using the transcript, the researchers proceeded to produce a fine-grained map of the growth of Sandy's understanding as it is observed during the interview. The phrase "as it is observed" is of particular importance since Pirie and Kieren (1994) state "we make no claims as to what might have gone on 'in the students' heads'" (p. 182). As a result, all inferences about what images were tapped to answer questions, how Sandy returned to previous ways of knowing in order to extend understandings, and the stability of the interviewer's interpretations of Sandy's responses (for further discussion, see Pirie and Kieren, 1992b, pp. 249–255) were mediated by the learner's reactions to the questions and triangulated against the learner's other responses.

Integral to the overall process of theory validation is a reflection on the teaching components utilized to spur students to actively engage with the content and personally explore the interconnections of content. However, the results of designed teacher interventions, even those with a specific intent, depend on the actions and interpretations of the students (Pirie & Kieren, 1992a). In fact, students may interpret a teacher's comments in unintended ways causing unexpected responses.

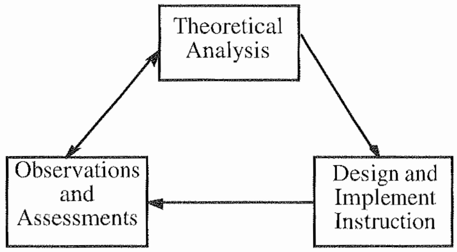

FIGURE 6. The APOS Cyclical Research Framework

Consequently, the mere examination of what a student does in the face of a mathematical task is not sufficient (Pirie & Kieren, 1992a). For example, questions posed by the interviewer or teacher need to be analyzed with respect to the student's response rather than the intended effect. Additionally, the interviewer or teacher must recognize that students can understand things at many levels and therefore investigators should expect students to respond at various levels at various points throughout a class or interview.

In contrast to Pirie and Kieren's focus on the growth of student understanding of middle school mathematics, APOS theory developed out of the observation of students working with undergraduate abstract algebra, functions and calculus, discrete mathematics, and elementary number theory (Dubinsky & McDonald, 2001). This development, according to Asiala et al. (1996), continues to involve a cycle of research (see Figure 6) beginning with a theoretical analysis of the elements perceived as necessary for the development of conceptual understanding, the design of instructional activities to compel students to make the requisite constructions, the observation and assessment of student constructions, and the reflections on the theory's ability to capture the nuances of those constructions. The particular composition of this research cycle involves the blend of (a) observing students in the process of learning mathematical concepts in order to identify the development of conceptual structures, (b) analyzing the data in light of the underlying APOS theory to generate a genetic decomposition indicative of one possible way a student might construct the concept, (c) designing instructional activities to help move students to make those constructions and thereby make the particular reflective abstractions indicated by the genetic decomposition, and (d) repeating the process after reflection on the genetic decomposition and the instructional treatment (Dubinsky, 1991). This research cycle continues to spiral upon itself, combined with explicit reexamination of the theoretical analysis' ability to describe the various nuances uncovered during observation and assessment, until stability is reached between the theoretical analysis and its ability to explicate what it means to learn a particular concept and how the suggested instructional treatments help students succeed in that learning (Dubinsky, 1994).

Integral to such a research cycle is the concerted examination of student learning in classroom situations where in many cases the *ACE Teaching Cycle* (described later in this paper) was employed. For example, Asiala et al. (1996) describes a data collection focused on student responses to written instruments and individualized interviews. The interviews were audio-taped, transcribed, and correlated with ancillary documentation produced by the student (Asiala et al., 1996). These data were then used to "evaluate the usefulness of our models of cognition by asking whether the mental constructions proposed in the initial analyses appeared to be made by the students. Second, ... [to] evaluate the mathematical performance of the students on tasks related to the concepts" (Asiala et al., 1998, p. 15). In effect, a concept's genetic decomposition was analyzed in light of the various responses to the instruments and interviews in order to determine the efficacy of the model to detect and explain differences in understanding by pointing to specific mental constructions of actions, processes, objects and/or schemas held by the learner.

For instance, the APOS theory-related research focuses on the ability of the genetic decomposition to describe potential mental constructions as the learner moves through various stages of cognitive development. In Cottrill et al. (1996), the researchers describe the evolution of a genetic decomposition of the concept of limit. The resulting decomposition that resulted was:

- The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.
- Interiorization of the action of Step 1 to a single process in which $f(x)$ approaches L as x approaches a.
- Encapsulate the process of Step 2 so that, for example, in talking about combination properties of limits, the limit process becomes an object to which actions (e.g., determine if a certain property holds) can be applied.
- Reconstruct the process of Step 2 in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \epsilon$.
- Apply a quantification scheme to connect the reconstructed process of the previous step to obtain the formal definition of limit ...
- A completed ϵ–δ conception applied to specific situations. (Cottrill et al., 1996, pp. 174–175)

The researchers then implemented a nearly seven-week series of instructional activities engaging calculus students with tasks involving both computer and group work. These activities were designed to help the students make the mental constructions defined in the above genetic decomposition and included computer investigations into approximations, graphical representations of the limit concept, computer-aided constructions of a value approaching a limit, computer-based construction of the limit concept, and ϵ–δ windows.

Drawing from interviews of selected students, the genetic decomposition was revised to better reflect the constructions being observed. One should note, however, that similar to Pirie and Kieren's belief concerning the intractability of observing the internal cognitions of a learner, the founders of APOS theory agree with Glasersfeld's radical constructivist[8] position, according to Asiala et al. (1996), and states that "it is impossible for one individual to really know what is going on in the mind of another individual" (p. 29). As a result, the revised genetic decomposition leads to continued observation and revisitation to the overall theory and either its

[8]Glasersfeld disagreed with the Piagetian assertion that an individual's concepts of reality could be mirrored and instead supported the idea that reality lies beyond an observer's scope of experiential justification. Consequently, he believed that even though one can choose well-defined tasks or experiences, the constructive process achieves no prescribed ends and that it is untenable that there exists some perfect internal representation or match against which one can test a learner's understandings, i.e., constructed reality.

revision and refinement or in rare cases, an enhancement of the theory to further explicate observed phenomena not describable in terms of the original theory. In particular, Cottrill et al. (1996), noted that an even more preliminary step to the genetic decomposition involving the action of evaluating f at a single point near or equal to a was needed. Also, Steps 2 and 3 of the original genetic decomposition were revised to allow for the possibility of multiple processes leading to the construction of a coordinated schema which could then be acted upon and eventually encapsulated. Using this revised genetic decomposition, Cottrill et al. (1996) were able to reconcile their observations drawn from the interview data and make attributions for differences in student performance as well as provide pedagogical suggestions for increased attention to domain and range processes and how to use the function to coordinate them.

Clearly, Pirie and Kieren's model and Dubinsky's APOS theory are grounded in a great deal of observation and teacher interactions with students actually doing mathematics. They both grew from similar constructivist roots and this underlying foundation has resulted in considerable commonalities between the theoretical frameworks. For example, the search for a means of describing mathematical understanding, and ability to detect fine-grained, observable differences between students. However, the development of APOS theory from investigations into undergraduate mathematical content and APOS researchers' usage of genetic decompositions indicates an emphasis on the *mathematical* portion of mathematical understanding. In contrast, Pirie and Kieren focused on the *understanding* element by relegating the mathematical content to background. Even though the foci are different, both theories are sensitive to the mathematical actions of students and integrate on-going observations of student actions in context to have the researchers reevaluate and revise their theories thereby better describing the learner's constructions.

4.3. Organization and mechanisms of the theories. Pirie and Kieren organized their model into eight layers—Primitive Knowing, Image Making, Image Having, Property Noticing, Formalizing, Observing, Structuring, and Inventizing—with each layer delineating a qualitative change in growth of a learner's understanding. In addition, the model contains a construct referred to as the "don't need boundaries." When a learner passes a "don't need" boundary, the learner exhibits the capability "to work with notions that are no longer obviously tied to previous forms of understanding, but these previous forms are embedded in the new layer of understanding and readily accessible if needed" (Pirie & Kieren, 1994b, p. 172). As a result, the layer closest to an inner "don't need" boundary indicates qualitatively different understanding in comparison to the previous layer whereas the layer closest to an outer "don't need" boundary involves taking those new understandings and coordinating them into richer relationships. The sections between two successive "don't need" boundaries behave as units which first involves the construction of a new way of conceiving and then the organization of these new conceptions into a form capable of being transformed into a higher layer of conception.

The three "don't need" boundaries partition Pirie and Kieren's eight layered model of understanding into four units that have strong connections to elements of Dubinsky's APOS theory. In particular, the first "don't need" boundary occurs between the *image making* and the *image having* layers since a learner gains an image of a concept and does not need external actions or the specific instances of image making (Pirie & Kieren, 1994b). This description is similar to the differences between the action and process conceptions described by Dubinsky et al. (1994) where a *process* conception indicates the learner internalized the external actions thereby allowing the coordination and reversal of the process to form new mental processes. This first "don't need" boundary in Pirie and Kieren's model of understanding separates the external actions and the newly developing internal processes that the learner built upon these actions.

The next "don't need" boundary in Pirie and Kieren's model of understanding comes between the *property noticing* and *formalizing* layers. In this instance, the learner moves from having images of a concept to coalescing these into a formal mathematical idea (Pirie & Kieren, 1994b). In particular, overcoming this boundary indicates the learner's diminished reliance on images and permits the learner to conceive of the mathematical ideas as a class-like mental object (Pirie & Kieren, 1992). In APOS theory, movement from a *process* conception to an *object* conception occurs "[w]hen an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations (whether they be actions or processes) can act on it, and is able to actually construct such transformations, then he or she is thinking of the process as an object. In this case, we say that the process has been encapsulated to an object" (Asiala et al., 1996, p. 11). Thus, advancement to an object conception involves coalescing a concept's processes into an object no longer requiring the underlying processes but retaining them in the sense of Sfard's operational/structural dichotomy.

The last "don't need" boundary described by Pirie and Kieren resides between the *observing* and *structuring* layers of their model. Transcending this boundary, according to Pirie and Kieren (1994b), indicates that the learner has developed a mathematical structure and does not require the meaning brought to it by inner layers of understanding. Similarly, the movement between an *object* conception and a *schema* conception involves the learner in constructing a coherent structure of actions, processes, objects and subordinate schemas that exist in the mind of the learner. This structure permits the learner to deal with problem situations by allowing the schema to be unpacked and work with the underlying objects and processes which it encompasses (Dubinsky et al., 1994). Thus, the "don't need" boundaries of Pirie and Kieren's model of understanding partition their model into four overarching units similar to the four levels of Dubinsky's APOS theory. The primitive knowing and image making layers correspond to the action conception, image having and property noticing layers corresponds to the processes conception, formalizing and observing layers necessitate an object conception, and lastly structuring and inventizing layers organize a structure similar to a schema conception.

In addition, both theories contain a comparable mechanism for the extension of understanding. Pirie and Kieren's model of understanding provides folding back as a generative mechanism that corresponds to a learner's return to an inner layer of understanding to augment, restructure, or expand the inner layer's inadequate understandings (Pirie & Kieren, 1994b). In the case of folding back, this adjustment to the inner layer understandings receives guidance from the experiences and understandings constructed as part of the outer layers. *Inadequate* can mean two different things: incomplete or insufficient. *Incomplete* implies that additional elements must be added to a lower layer of understanding to support the development of understanding at a higher layer. This type of folding back tends to occur when a learner has not fully reached the formalizing layer and must restructure the lower layers to incorporate new images necessary to build a formalized understanding. *Insufficient* suggests the scope of understanding at a lower layer needs expanding to inform the extension of understanding to a new set of circumstances. For instance, a student may have a formalized understanding of adding pairs of fraction using a personally generated cross multiplication trick but when faced with having to add three fractions, the student may fold back to image making to create a method to accomplish this (Pirie & Kieren, 1992). In this case, such activity consists of expanding the scope of the understanding to handle addition of three fractions but this expansion receives direction from the formalized understandings already present thereby suggesting possible viable strategies for the expansion.

This latter description corresponds to Dubinsky's portrayal of *de-encapsulation* which is a return "*to the process which was encapsulated in order to construct the object in the first place*" (Dubinsky, 1994, p. 271). In particular, de-encapsulation permits the learner to return to the process origins of objects, coordinate those processes to form new processes that can themselves be encapsulated into new objects. As a result, de-encapsulation serves the generative role of returning to previous ways of thinking to provide a foundation for the extension of previous conceptualizations. For instance, to understand addition of functions, Dubinsky (1991) asserts that the learner must de-encapsulate the function objects to their process-oriented roots, those processes must be coordinated through point-wise addition, and then the resulting process encapsulated to form a new object. As a result, the de-encapsulation mechanism of Dubinsky's APOS theory permits the extension of understanding by expanding the scope of previously held understandings similar to the folding back mechanism of Pirie and Kieren's model of understanding.

4.4. Linkages to other theoretical frameworks. Pirie and Kieren's model of the growth of understanding, Dubinsky's APOS theory, and the other contemporary theoretical frameworks[9] of understanding all form different but compatible lenses for the development of understanding. In particular, each framework has been developed to provide researchers and teachers means of observing understanding as an ongoing process in which interpretation is predicated upon the student's personal knowledge structures, the social dynamics of the learning situation in which the learning occurs, and the constraints on the display of those understandings due to the nature of the environment. Given the content and setting in which researchers attuned their particular framework, each focused on different components and uncovered specific connections between those components. This yields a collection of theoretical frameworks where each provides a portrait of students engaged in the development of understanding; but as is the case with any portrait, certain features of that development are highlighted and others are de-emphasized. In this next section, the goal will be to further explicate the relationships, if any, that Pirie and Kieren's model of the growth of understanding and Dubinsky's APOS theory have with the theoretical frameworks associated with *cognitive* or *epistemological* obstacles, *concept definition* and *concept image*, *multiple representations*, and *operational* and *structural* entities.

[9]The continued use of "theoretical framework" in reference to the other recent work on understanding is not meant to convey that they do not satisfy the requirements identified by Schoenfeld (2000). It is quite conceivable that some if not all do satisfy these requirements but this paper has not focused itself on attempting to argue this status for them.

4.4.1. *Errors, misconceptions, and epistemological obstacles.* Researchers attribute errors and misconceptions to extrapolations based on insufficient examples, ambiguous examples, missing linkages, or even partial linkages between relevant pieces of knowledge such as conceptual and procedural knowledge (Hatano, 1988; Hiebert & Carpenter, 1992; Hiebert & Wearne, 1986; Mansfield & Happs, 1992; Silver, 1986; Tall, 1989; VanLehn, 1986). According to Ferrini-Mundy and Gaudard (1992), these errors act as windows for observing the inner workings of a learner's mind and correspond to elements in the theoretical framework of epistemological obstacles previously described in this paper. Both Pirie and Kieren's model of understanding and Dubinsky's APOS theory contain components that account for the development and surmounting of misconceptions and epistemological obstacles.

In particular, Pirie and Kieren account for the possible development of misconceptions when discussing movements within their model. Such movement requires the construction of additional connections eventually reaching the structured layer where the network has become a stable entity. In the process of moving from the image having layer to the property noticing layer and then the formalizing layer, the learner proceeds through a process of internalizing images, recognizing properties associated with groups of images and integrating the various recognized properties into formalized mental objects that have no internal contradictions (Pirie & Kieren, 1991a). A learner moving from image having to formalizing does not have to overcome a misconception in making such a transition since the potentially troublesome images may not have been constructed. However, engagement in property noticing on a limited domain of images can result in the extrapolation of connections from the evoked images which may be incomplete in the domain of all images associated with a topic. This construction, although not necessarily encountered, can result in the development of a troublesome image considered a misconception. In the case that such a misconception arises, the movement from property noticing to formalizing signifies the learner has overcome some of the property-based misconceptions by recognizing inconsistencies with established connections, developing correct connections, forging elaborate networks of connections, and finally constructing mental objects.

Many of the same mechanisms for developing and overcoming misconceptions identified in the discussion of Pirie and Kieren's model also exist in APOS theory. For instance, Dubinsky et al. (1994) mention that "the general theory of Piaget (1975) includes the idea that concepts are constructed at a layer that is adequate for dealing with a learner's current mathematical environment, but that when new phenomena are confronted, it is necessary to reconstruct concepts on a higher layer. Thus, if the construction is delayed, a student's conception of some mathematical notion may be adequate at one layer, but erroneous at another" (p. 295). A misconception, from Dubinsky's perspective, is an understanding that has not reconciled itself with a broader context but remains reasonable in a narrower scope. Thus, Dubinsky (1991) uses a "lack of understanding" to mean that the student has not constructed an appropriate schema for the examined concept. However, overcoming an epistemological obstacle equates with a learner interacting with a disequilibrating environment, reflecting on held understandings, and reorganizing those understandings to integrate the discrepant elements of the perplexing situation.

4.4.2. *Connections to concept images.* As already mentioned, both Pirie and Kieren's model of understanding and Dubinsky's APOS theory have characteristics associated with *concept image*. In particular, Pirie and Kieren (1992a), although not directly referring to the term *concept image*, state that "young children use their own created mental objects, some of which do have a figural quality but others of which are abstract, to build their mathematical knowledge" (p. 505). This figural quality particularly arises at the image having and property noticing layers although existent at other layers. In addition, such a description echoes the definition of concept image as a collection of mental pictures, representations, and related properties ascribed to a concept by a learner. In fact, Pirie and Kieren (1994b) commented that their use of the word "image" in labeling two of their layers was due to "evidence at these levels is frequently based upon pictorial representation" (p. 166); however, understanding at those levels are not restricted to only this mode of expression but can include mental imagery which is defined by the reasonably established concept of mental objects. On the other hand, the word "idea" could have been used but it was thought that "image" was less ambiguous and carried many of the connotations sought by the theory.

In addition, Pirie and Kieren (1992b) draw connections to the work by Harel and Tall (1991) concerning generalization. As Harel and Tall (1991) detail different student mental constructions, they tacitly use phrases such as "the subject reconstructs an existing scheme" and Pirie and Kieren (1992b) connect this explicitly to one of the essential features of their theory, namely folding back. Also of interest is a belief that disjunctive generalization integrates with Pirie and Kieren's (1992b) general idea of disconnected understanding as well as that generic abstraction "accords well with our theory that understanding grows by noticing properties of one's images and by making observations on one's formalizing" (p. 245). Consequently, the images and processes connected within the theoretical framework of concept images have strong ties to elements of Pirie and Kieren's model.

The theoretical framework of concept definition/image, according to Dubinsky (1991), explains why understanding fails to develop whereas reflective abstraction describes the essential elements for the development of understanding. However, Asiala et al. (1996) assert that the notion of concept image provides explanations beyond why understanding did not develop. They consider the concept image to correspond with Piaget's schemata which is closely linked to APOS theory's use of schema. According to Dubinsky (1996), "A schema is a more or less coherent collection of objects and processes. A subject's tendency to invoke a schema in order to understand, deal with, organize, or make sense out of a perceived problem situation is his or her knowledge of an individual concept in mathematics. Thus an individual will have a vast array of schemas" (p. 102). As a result, a concept image and a schema serve the same purpose of organizing and structuring held understandings concerning a concept. Additionally, the last two sentences quoted from Dubinsky resonate with Tall and Vinner's description of evoked concept images which involve the portions of a concept image activated by particular stimuli. In fact, Tall and Vinner (1981) assert that "different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole" (p. 152). Thus, a strong connection between concept images and schemas exists.

4.4.3. *Connections to multiple representations and operational and structural conceptions.* Pirie and Kieren do not make any specific reference to the ideas of multiple representations as described by Kaput. However, a discussion of Sfard's duality of operational and structural conceptions occurs in some papers by Pirie and Kieren. For instance, Pirie (1988) in a paper briefly surveying several interpretations of understanding, i.e., instrumental, rational, intuitive, constructed, formalized, etc. mentioned that according to Sfard, an operational conception is the first step in the acquisition of a new mathematical idea. This assertion accounted for the linkage of an initial operative notion of a concept with constructing a beginning understanding. A broader discussion appears in Pirie and Kieren (1992b), where they identified Sfard's work as "interested in understanding at the level of algorithmic thinking" (p. 244). This characterization indicates a belief that Sfard contends the acquisition of mathematical ideas comes from initially operationally derived notions which in turn coalesce through interiorization, condensation, and reification into a structural view. The linkage of Sfard's work to Pirie and Kieren's model of understanding focuses on Sfard's concern for the formalizing activity in mathematical understanding. Sfard (1991) implies that the reification process, i.e., objectifying an operational notion, involves a difficult change on the part of the learner. It is Pirie and Kieren's belief that their theory provides description of several actions and mechanisms that are integral to this change (Pirie & Kieren, 1992b). However, Pirie and Kieren's (1992) belief in the non-linearity of the growth of understanding separates them from Sfard.

Similar to Pirie and Kieren's model of understanding, Dubinsky's APOS theory draws no explicit connections to Kaput's theoretical framework of multiple representations. However, Dubinsky (1994) mentions that he approaches the idea of multiple representations in a different manner than other researchers. Rather than using tools provided in some computer environment, Dubinsky devises investigations where students engage with the computer in developing their own tools for comparing different representations of a concept. Additionally, Dubinsky's APOS theory connects with Sfard's operational and structural duality (Dubinsky et al., 1994). Sfard and Linchevski (1994) expound upon this linkage when they ascribe Dubinsky's APOS theory, as described in Dubinsky (1991), as providing a closely related model to their process-object duality of mathematical concepts. In particular, they assert that the main connection between these two frameworks is the Piagetian description of reflective abstraction.

In addition, the term "reification" and Dubinsky's use of "encapsulation" have similar meanings since both arise from Piaget's description of the operational origins of mathematical notions (Sfard, 1994). In particular, Sfard (1991) states "The pioneering work in this field has been done by Piaget, who wrote in his book on genetic epistemology (1970, p. 16): 'the [mathematical] abstraction is drawn not from the object that is acted upon, but from the action itself. It seems to me that this is the basis of logical and mathematical abstraction'" (p. 17). Consequently, Sfard contends that both Dubinsky's work and her own work further elaborated Piaget's original ideas and that her characterization of a dichotomy between operational and structural conceptions provides a broader conjecture for the duality of mathematical thinking.

4.4.4. *Linkages of Pirie and Kieren's model of understanding to a refinement of APOS theory.* The refinement of the description of schema development provided by Clark et al. (1997) reveals an interesting connection to some layers of Pirie and Kieren's model of understanding. In particular, Clark et al. (1997) identified three stages in the development of a concept: *Intra*, *Inter*, and *Trans* previously described in Section 3.1. At the *Intra* stage, a learner has a collection of rules but has not recognized relationships between them. This is similar to the image having layer of Pirie and Kieren's model of understanding which is described as a level where images of a concept have been built from single activities but these images, although isolated, form a foundation for mathematical knowing (Pirie & Kieren, 1991a). The *Inter* stage occurs when the learner can organize the various cases encountered and recognize the relationships between these cases (Clark et al., 1997). This stage is isomorphic to the property noticing layer that indicates the learner can examine held images for specific and relevant properties to note their distinctions, combinations and connections and predict their actualization (Pirie & Kieren, 1991a). Attainment of the *Trans* stage indicates the learner has constructed an underlying structure for a particular concept. It is at this point that "The elements in the schema must move from being described essentially by a list to being described by a single rule" (Clark et al., 1997, p. 360). This description is similar to the formalizing layer which involves the learner in conscious contemplation of the noticed properties to abstract common qualities from them thereby possibly constructing a full mathematical definition to accompany the new class-like mental object (Pirie & Kieren, 1991a). Thus, each of the stages in this refinement of APOS theory reasonably correlate with the three layers of Pirie and Kieren's model of understanding: image having, property noticing and formalizing.

4.5. Assessment and instructional implementations. Both Pirie and Kieren's model of understanding and Dubinsky's APOS theory gather information on student understanding and question how to use this data to adjust instructional practices to promote the growth of understanding. The next two sections will attempt to provide information about the implications these theories have with respect to assessment and their ramifications for instruction.

4.5.1. *Assessment of understanding.* Pirie and Kieren (1989, 1992b), in concert with the opinions expressed by Pirie and Schwarzenberger (1988) as well as Simon (1993), recognize the utility of interviews to track a student's movements through the layers of understanding. In particular, Pirie and Kieren believe a written instrument, especially a multiple-choice exam, does not completely expose student understandings since they posit student understandings can only be inferred and not measured. They contend that written instruments are less facile to delve into the growth of student understanding since written instruments in their view provide static captions of student external demonstrations. As a result, they consider interviews as the primary means of uncovering a learner's changing understandings. These interviews permit one to make illations about a learner's awareness of relationships between concepts, ability to adapt procedures to novel situations, possession of generic examples, and fluency with language and symbolism.

Drawing inferences about movements between layers of understanding can only ensue after the learner performs actions and verbalizations (Kieren, 1990; Pirie &

Kieren, 1990). These form-oriented actions[10] and verbalizations demonstrate to an external agent the learner's operation at a level of understanding. Without such demonstration, the external agent might think the understanding has not been achieved. Integral to the establishment of a learner's level of understanding are situations that require the learner to act and verbalize in reference to unique and complex situations. These circumstances help exhibit a learner's present level of understanding and promote the development of understanding when instigated in an interview situation. The interview questions act as teaching tools and become a useful diagnostic instrument since they will either provoke understanding to an outer layer, invoke folding back, or validate the learner's understanding. Even though an investigator may wish to categorize questions, categorization can only occur after the hearer responds to a question demonstrating the effect of the question on the learner's changing understanding. In particular, a question which is provocative points a learner to an outer layer enabling continued development of understanding. In-vocative questions cause a learner to fold back to an earlier layer to continue progressing. Validating questions provide evidence that a learner is operating at a specific layer of understanding by encouraging the learner to demonstrate, either in verbal or symbolic forms, current mathematical actions (Pirie & Kieren, 1991a).

Thus, Pirie and Kieren's use of pro-vocative, in-vocative, and validating questions coincides with Glasersfeld's assertion that even though one might be "studying the construction of mathematical reality by individuals with in the space of their experience. In this construction, although there may be well-defined tasks or spaces for experience, there are no prescribed ends toward which this construction strives" (Steffe & Kieren, 1994, p. 721). As a result, Pirie and Kieren (1990, 1991a) take the position that one cannot give a student understanding rather the student will create her or his own understanding. In addition, Pirie and Kieren (1994b) state "Each person's mathematical understanding is unique. Indeed, since we believe that all knowledge is personally constructed and organized, students in any environment will construct understanding in some form" (p. 526). As a result, the interview process becomes a teaching tool and an opportunity to map the external evidences exhibited by a learner and examine how the learner responds to a variety of questions.

In contrast, the assessment of understanding from the perspective of Dubinsky's APOS theory is less radical constructivist in nature. Rather than relying entirely on interviews, the application of APOS theory according to Asiala et al. (1996) typically utilizes two kinds of tools: written instruments and in-depth interviews. These written instruments include fairly standard items, typically open-ended in nature, focused on the content and the student responses to them receive relatively traditional analysis. These written instruments provide a broad scoped look at what students may or may not be learning and indications about possible mental constructions. From such a broad position, the interviews then narrow in on the displayed understandings by gathering additional information. In particular, Asiala et al. (1996) point out that "the interviews are far more valuable than written assessment instruments used alone ... [since] for one student the written work may

[10]A form-oriented action is a learner's external demonstration helpful in indicating the layer of understanding at which a learner is potentially operating. See Section 2.2 for an enhanced discussion.

appear essentially correct while the [interview] transcript reveals little understanding, while for another student the reverse may be true" (p. 25). As a result, the interviews become tools that permit one to observe the external signs of internal understandings; however, all the data collected in reference to a learner is aggregated to gain a better perspective on the held understandings.

The design and use of the interview questions are another place where the practices of APOS theory significantly diverge from the interview practices associated with Pirie and Kieren's model of understanding. According to Asiala et al. (1996), the interview questions emerge from the genetic decomposition previously developed by the interviewer. Following the interview stage, the analysis of the data feeds into a loop of categorizing student responses, referencing them with respect to the proposed mental constructions defined in the genetic decomposition, and examining the genetic decomposition in light of the data. In particular, Brown et al. (1997) state "student responses are compared to find very fine mathematical points which some students seem to understand (or operations that some can perform) but others cannot. Then we try to find some explanation for the difference in terms of some construction of actions, processes, objects and/or schemas. If we can find an explanation that seems to work, then it is used to revise the genetic decomposition" (p. 189). Thus, the interviews serve a dual purpose of providing insight into the students' mathematical constructions, the mathematics being learned and used, and the descriptive and explanatory powers of the genetic decomposition.

4.5.2. *Instructional practices.* Pirie and Kieren (1994b) note that insights garnered from their model of understanding are useful in the planning and engaging in mathematical lessons as well as making observations about curriculum development. Drawing from their beliefs about students' development of mathematical understanding, they construct a variety of instructional practices that relate the classroom environment to the promotion of growth of understanding. Underlying these practices are a set of beliefs which drive the implementation and interactions of the teacher with the students.

In particular, Pirie and Kieren (1992a) identify four critical tenets teachers must hold when creating a constructivist environment for encouraging mathematical learning and understanding:

> 1) Although a teacher may have the intention to move students toward particular mathematics learning goals, she will be well aware that such progress may not be achieved by some of the students and may not be achieved as expected by others . . .
> 2) In creating an environment or providing opportunities for children to modify their mathematical understanding, the teacher will act upon the belief that there are different pathways to similar mathematical understanding . . .
> 3) The teacher will be aware that different people will hold different mathematical understandings . . .
> 4) The teacher will know that for any topic there are different levels of understanding, but that these are never achieved "once and for all." (pp. 507–508)

The first tenet suggests that a constructivist teacher must be both aware and reactive to the ever-evolving classroom environment. This continual restructuring of the classroom is informed by observations of individual student fabrications of understandings as well as the whole classes' constructions. As a result, a teacher must, according to Kieren (1997), place new emphases:

(1) on listening to rather than simply listening for;
(2) on acting with the students in doing mathematics rather than simply showing students how to do mathematics;
(3) on establishing effective discourse of mathematical argument or mathematical conversation rather than simply the discourse of telling, interrogating, and evaluating;
(4) on the mechanism of students' mathematical thinking rather than simply on students' answers;
(5) on the teacher and students as fully implicated by their actions each in learning of the other; and
(6) on the • teacher [sic] as co-developer of a lived mathematics curriculum not just a recipient of or conduit for a predecided curriculum. (p. 33)

These emphases transform the classroom environment and allow the teacher to focus on the understandings constructed by the individual students while gaining awareness necessary to plan additional classroom experiences.

The second tenet implies that a teacher must recognize there is no specific form or sequence of instruction since no unique or best path to the construction of understanding exists. However, Pirie and Martin (1997) identify a particular effective sequence for teaching the concept of linear equations. This sequence, although presented as applicable across a classroom, involved the teacher with the understandings held by the individual students in the class. Initially, the teacher provided advance organizers to bring the elements of primitive knowing to the forefront of the students' minds and prepare the groundwork for the development of understanding. After this, the teacher presented a task requiring investigation leading students into explorations that would expose the students' thinking, cause reflection, and guide future thinking. The teacher then made constant entreaties for the students to fold back and reorganize lower levels of understanding without inappropriately pushing students. Even though described as a sequence, each student receives different questions and probes; however, this overall instructional practice appears successful (Pirie & Martin, 1997).

The third tenet proposes that the teacher must consider that each individual's understandings are mediated by the internal understandings held by the individual. As a result, the teacher cannot assume that understanding of a certain topic can be transmitted to or gained by students. The reason for this is, according to Pirie and Kieren (1992a), "An understanding of a topic is not an acquisition. Understanding is an ongoing process which is by nature unique to that student" (p. 508). Thus, Pirie and Kieren appear to be stating that the teacher cannot desire to impart a particular idealized version of understanding to students.

The last tenet involves the responses of the teacher regarding observations drawn in the classroom. Even though outward evidences appear to indicate multiple students hold similar understandings, each student has unique understandings. As a result, the teacher must validate each student's level of understanding and compare the levels of understanding across students. Additionally, instruction should not focus solely on the ways of thinking correspondent to a single layer since the normal growth process requires students to systematically fold back to earlier layers of understanding. As a result, the learning environment should be constructed to promote folding back in hopes of advancing students' growth of understanding.

Underlying this entire discussion has been a belief that "teachers cannot give students understanding. Only the student can build her own understanding. The role of the teacher is to provoke and enable this growth" (Pirie & Kieren, 1991a, p. 5). The provocation and enabling of growth reaches beyond simply asking students to work on high level mathematics and includes the generation of opportunities to promote understanding. The three types of questions, provocative, invocative, and validating, are an integral part of this undertaking. By using these questioning techniques, the teacher can address the individualized understandings of a student by provoking movement to an outer layer of understanding, by invoking folding back to a previous level of understanding, and by encouraging students to validate their own reasoning (Pirie & Kieren, 1990).

The instructional approach for fostering conceptual thinking connected with Dubinky's APOS theory differs from the organization provided by Pirie and Kieren although it contains many of the same features. According to Cottrill et al. (1996), "The main contribution it [the APOS theoretical perspective] makes to the design of instruction is to suggest specific mental constructions that can be made in learning the material. The instruction focuses on getting students to make these constructions" (p. 169). As a result, the genetic decomposition represents a possible path a learner may take to the development of conceptual understanding as well as a guide for the development of any instructional activity. Specifically, the genetic decomposition provides the teacher with a general path which may lead the student to construct appropriate understandings. However, Dubinsky et al. (1994) do not consider the development of an instructional sequence as a simplistic, linear sequence where a complex topic is dissected into a logically coherent sequence of small components. In effect, Asiala et al. (1996) point out that according to APOS theory "the growth of understanding is highly non-linear with starts and stops; the student develops partial understandings, repeatedly returns to the same piece of knowledge, and periodically summarizes and ties ideas together" (p. 13). Thus, pedagogical practices need to account for a learner's simultaneously construction of understanding of multiple concepts with one concept subordinate to another. In order to accomplish this, instructional experiences from the APOS perspective employ the holistic spray.

> The idea [of holistic spray] is that everything is sprayed at them in a holistic manner. Each individual (or team) tries to make sense out of the situation—that is, they try to do the problems that the teacher asks them to solve, or to answer questions which the teacher or fellow students ask. In this way the students enhance their understandings of one or another concept bit by bit. They keep coming at it, always trying to make more sense, always learning a little more, and sometimes feeling a great deal of frustration. And it is the role of the teacher, not to eliminate this frustration, but to help students learn to manage it, and to use it as a hammer to smash their own ignorance. (Dubinsky et al., 1994, p. 300)

An integral part of this strategy involves the engagement of students in intentionally disequilibrating environments integrating as much as possible about the concept under study. Of particular importance is the social aspect of the learning situation (Asiala et al., 1996). Rather than the environment supporting individualized exploration, the individuals comprising a cooperative learning group each bring their own perspective to the material presented and therefore construct their own versions of understanding. However, in order for the group to efficiently communicate, members must mediate their understandings to the group during discussion. Therefore, according to Dubinsky's perspectives from APOS theory, the social context in which the learning takes place enhances the individualized construction of understanding.

One particular instantiation of the above theoretical pedagogical approach has been coined by APOS researchers as the *ACE Teaching Cycle* which integrates activities, class discussions, and exercises. The design of the pedagogical approach focuses on eliciting the specific mental constructions suggested by the theoretical analysis. In order to accomplish this, the activities involve cooperative learning groups engaged in the exploration of mathematical concepts using computer assignments as a means of building a knowledge base. In particular, Asiala et al. (1996) stated "It is important to note that there are major differences between these computer activities and the kind of activities used in 'discovery learning.' While some computer activities may involve an element of discovery, their primary goal is to provide students with an experience base rather than to lead them to correct answers" (p. 14). In effect, the computer activities serve as a means of helping students make sense of different portions of the whole concept thereby incrementally setting the stage for enhancing the student's understanding through personal reflection and group discussion. In addition, the computer activities bring to the foreground the components needed to make the theoretical analysis' requisite mental constructions.

To further build upon this foundation laid by the computer activities, the classroom discussions reflect on the computer activities and involve students, working in teams, with paper-and-pencil tasks (Asiala et al., 1996). The instructor actively engages groups in discussion of the computer experiences and how the paper-and-pencil work integrates with those experiences and occasionally providing definitions, explanations, and integrating overviews to move the discussion forward. Finally, in the *ACE Teaching Cycle*, the exercises assigned after the conclusion of the computer activities and classroom discussion are considered relatively traditional but with the intention to "reinforce the ideas they [the students] have constructed, to use the mathematics they have learned and on occasion, to begin thinking about situations that will be studied later" (Asiala et al., 1996, p. 14). One major difference from the exercises contained in a traditional text and the texts born out of the APOS theory is that in traditional texts the exercises provide template problems which correspond to the examples and theory presented in a particular chapter. This template, from the perspective of APOS theory, circumvents the disequilibrium and formation of rich mental constructions considered necessary for the building of understanding. As a result, APOS theory-based texts incorporate the exercises after the computer activities and discussion of the mathematics contained therein. The texts do not provide "worked examples" forcing students to investigate the ideas underlying an activity, reflect upon the computer activities, and integrate those with class discussions to look for linkages and solution paths.

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