ON LEARNING FUNDAMENTAL CONCEPTS OF GROUP THEORY

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Abstract

The research reported in this paper explores the nature of student knowledge about group theory, and how an individual may develop an understanding of certain topics in this domain. As part of a long-term research and development project in learning and teaching undergraduate mathematics, this report is one of a series of papers on the abstract algebra component of that project.

The observations discussed here were collected during a six-week summer workshop where 24 high school teachers took a course in Abstract Algebra as part of their work. By comparing written samples, and student interviews with our own theoretical analysis, we attempt to describe ways in which these individuals seemed to be approaching the concepts of group, subgroup, coset, normality, and quotient group. The general pattern of learning that we infer here illustrates an action-process-object-schema framework for addressing these specific group theory issues. We make here only some quite general observations about learning these specific topics, the complex nature of "understanding", and the role of errors and misconceptions in light of an action-process-object-schema framework. Seen as research questions for further exploration, we expect these observations to inform our continuing investigations and those of other researchers.

We end the paper with a brief discussion of some pedagogical suggestions arising out of our considerations. We defer, however, a full consideration of instructional strategies and their effects on learning these topics to some future time when more extensive research can provide a more solid foundation for the design of specific pedagogies.

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Introduction

In this paper we hope to open a discussion concerning the nature of knowledge about abstract algebra, in particular group theory, and how an individual may develop an understanding of various topics in this domain. Our goal in making such a study is to eventually contribute to basic knowledge about human thinking as well as to serve the purposes of this specific area of mathematics. One way to do this is to introduce an increased degree of specificity to an analysis of student difficulties in understanding abstract concepts. Our present emphasis will be on interpreting the struggles of a class of in-service high school mathematics teachers as they tried to make sense out of a number of topics in group theory.

Of course, we are also interested in using these and other observations in the development of pedagogical strategies that can improve student success in learning abstract algebra. The work reported here is part of a long term research and development project in learning and teaching undergraduate mathematics.

We include, at the end, a brief discussion of some pedagogical suggestions arising out of our observations, but a full consideration of instructional strategies and their effect on learning this subject must await future investigations yet to be conducted. Nevertheless, we offer the current discussion as an opening to what we hope becomes a continuing investigation of this important area.

Why Group Theory?

Abstract algebra in general, and group theory in particular, presents a serious educational problem. Mathematics faculty and students generally consider it to be one of the most troublesome undergraduate subjects. It appears to give students a great deal of difficulty, both in terms of dealing with the content and the development of attitudes towards abstract mathematics. The literature contains some reports that support this judgement, such as Hart, in press and Selden & Selden, 1987.

In many colleges, abstract algebra is the first course for students in which they must go beyond learning "imitative behavior patterns" for mimicing the solution of a large number of variations on a small number of themes (problems). In such a course, students must come to grips with abstract concepts, work with important mathematical principles, and learn to write proofs. Although there are no formal studies, many students report that, after taking this course, they tended to turn off from abstract mathematics. Since a significant percentage of the student audience for abstract algebra consists of future mathematics teachers, it is particularly important that the profession of mathematics education develop effective pedagogical strategies for improving the attitude of high school mathematics teachers towards mathematical abstraction.

There is another reason, related to abstraction, for the importance of abstract algebra in general and quotient groups in particular. An individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples, indeed including groups consisting of undefined elements and a binary operation satisfying the axioms. Even if one begins with a very concrete group, the transition from the group to one of its quotients changes the nature of the elements and forces a student to deal with elements (e.g., cosets) that are, for her or him, undefined. This relationship between abstract groups and quotient groups has important historical antecedents (Nicholson, 1993)[1]

Objectives of this Study

Two related questions guide our continuing investigation. How may an individual learn certain topics in elementary group theory? And what relation does this have to understanding mathematics and abstraction in general?

Learning specific topics

Our goal in looking at how individuals learn specific topics is to see if it is possible to map a developmental sequence or genetic decomposition of various mathematical topics.

Our ultimate intention is not only to describe a sequence of mathematical ideas, but also to specify particular mental constructions which individuals may use in order to make sense out of a specific mathematical concept. We expect that such explanations will eventually point to effective instructional strategies for individual topics.

Our choice of topics had mainly to do with what appears to be a reasonable way to introduce students to the subject of abstract algebra. From the days of the new math, curriculum developers have felt that even young students could deal effectively with the basic idea of a group and its subgroups, including the prerequisite understanding of set and function (binary operation) on which these concepts are based. Since this seems to be a reasonable place to begin a course in abstract algebra, our study begins with an investigation of the nature of understanding these topics and how they may develop in an individual's mind.

The major difficulties in understanding group theory appear to begin with the concepts leading up to Lagrange's theorem and quotient groups — cosets, coset multiplication and normality. For this reason we chose to emphasize these topics in our present investigation.

[1] We are grateful to the editor for suggesting this point and the reference.

Understanding mathematics in general — a theoretical perspective

We might expect our investigations of individuals trying to make sense out of specific concepts to shed some light on general questions of learning. Just how complex are the concepts connected with groups in the mind of someone who is first beginning to understand them? How can we interpret and deal with difficulties that arise in the learning process?

In order to deal with questions such as these, a major tool with which this study is concerned is the theoretical perspective that we bring to our investigations. The questions we are asking, the methodology we use in asking them, and our interpretations of the students' responses are all driven by a constructivist approach, based on the ideas of Piaget (see, for example, Piaget, 1975) which we are trying to adapt to studies of advanced mathematical thinking.

We bring to this work a committment to the integration of theory, practice, and pedagogy which reflects the suggestions of educational researchers such as Erickson (1986) and Glaser (1968). Erickson in particular, advocates a strategy which integrates theory and data to produce "assertions" but not "proof" in the usual quantitative sense. The qualitative methods used here to observe the experiences of our participants are not intended to lead to firm conclusions but rather to interpretations based on an integration of theory and practice.

There are a number of theories of learning which have been applied to mathematics at the post-secondary level. These include epistemological obstacles (Bachelard, 1938 and Sierpiǹska, 1992), concept definition/image (Vinner, 1983), multiple representations (Kaput, 1987) and the operational/structural dichotomy (Sfard, 1992). As far as we know, none of these theoretical perspectives has been applied to abstract algebra. Our point of view is an attempt to extend the Piagetian concept of reflective abstraction to post-secondary mathematics (Dubinsky, 1991). It is quite different from these perspectives but has many points in common with them, especially with Sfard's theory.

The essence of our theoretical perspective is that an individual, disequilibrated by a percieved problem situation in a particular social context, will attempt to reequilibrate by assimilating the situation to existing schemas available to her or him, or, if necessary, use reflective abstraction to reconstruct those schemas at a higher level of sophistication. We have, in developing this theoretical perspective, attempted to analyze the constructions which may intervene. We find them to be mainly of four kinds — actions, processes, objects, and schemas.

An *action* is any repeatable physical or mental manipulation that transforms objects in some way. When the total action can take place entirely in the mind of an individual, or just be imagined as taking place, without necessarily running through all of the specific steps, we say that the action has been *interiorized* to become a *process*. It is then possible for the student to use the process to obtain new processes, for example by *coordinating* it with other processes; that is, to combine two or more processes, connecting "inputs" and "outputs" appropriately so that another process is formed. Also, a process may be *reversed* to obtain a new process. When it becomes possible for a process to be transformed by some action, then we say that it has been *encapsulated* to become an *object*.

One consequence of this theoretical perspective is that the role of symbols and other representational systems is subsidiary to the constructions of reflective abstraction. In particular, our view is that student difficulties with symbolism have their source in attempts to apply labels before objects have been constructed (through encapsulation). Once an object exists in an individual's mind, there is little difficulty in assigning a label to this object. The interpretation of a symbolic representation requires the individual to return from the object to the process from which it came.

There are many mathematical situations in which it is essential to be able to shift from an object back to a process. One of the tenets of our theory is that this can only be done by *de-encapsulating* the object, that is, to *go back to the process which was encapsulated in order to construct the object in the first place.*

A coherent set of processes and objects can be collected together and *thematized* to form an identifiable *schema.* In the theoretical perspective we use, schemas are the forms in which concepts exist in the mind of an individual. A schema can be used to deal with a problem situation by unpacking it and working with the individual processes and objects. A schema may also be treated as an object in that actions and processes may be applied to it.

It is important to note that this analysis is not independent of knowledge about a particular domain. At the same time that an individual struggles to reequilibrate, to interiorize, to encapsulate and de-encapsulate, he or she must also build a large amount of domain knowledge. This can happen at different rates with different topics.

For further discussion of our overall theoretical perspective along with some applications to other mathematical topics, the reader may consult Ayers et al, 1988, Breidenbach et al, 1991, Dubinsky, 1991 and Dubinsky & Harel, 1992.

A very important question for our long-term project is the extent to which these notions are useful in interpreting the data that we obtain and in pointing to effective instructional strategies. The present work is not intended to lead to any conclusions about pedagogical strategies that may follow from our observations. Rather, we only make some suggestions and raise some important questions related to the action-process-object-schema development here. We suspect that an overall curricular strategy may be needed to replace the traditional linear sequence.

Observing People Learn about Groups

The observations referred to here were collected from a group of 24 in-service high school mathematics teachers participating in a summer institute emphasizing novel approaches (often involving computers) to the teaching of mathematics. The curriculum included a course in abstract algebra. Some, but not all, of the participants had studied groups in their undergraduate days; in all cases, some years had passed since that time and there did not seem to be much residual knowledge. Thus we were looking at a population in the midst of an experience involving groups, but without a strong background in the subject.

The course used written material and worksheets which later became a textbook (Dubinsky and Leron, 1993.) The order in which topics were considered was fairly standard. Some specifics will be given below in the section on Interpretations to indicate what relation the observed developments might have had to the order in which the participants studied the topics in the course.

Near the end of the workshop, a written assessment (see Appendix I) was given to determine the participants' understanding of some of the material covered in the course. They were given two hours to answer the questions. The assessment was described to the participants as an "observation" designed to help the instructors assess the participant's learning. No grades were given in the course.

Based on the results of the written assessment, we selected 10 individuals and invited them to discuss their results in taped interviews. We interviewed participants who gave correct, partially correct, and incorrect answers on the assessment so that we could potentially access a range of understanding. We also favored individuals who appeared to be in the process of learning these ideas rather than only those who had clearly mastered the material or those who had obviously missed the point. Participants were asked the questions in Appendix II by faculty members of the institute and were prompted where appropriate for understanding that might not have been apparent on the written assessment. All interviews were completed before the results of the written assessment were discussed in class.

The transcribed comments of the participants were categorized in terms of certain group theory topics. In each category, our theoretical perspective was used to analyze the comments for ways in which the participants appeared to think about the specific topics. We feel that participant responses correspond well to the theoretical perspective described above and serve to illustrate some of the difficulties participants may have. But given the modest nature of the data and the complexity of the mathematical ideas, we make no claims that the theory has been validated or even confirmed by this study. Instead we offer the following discussion as one interpretation of how these students may have been learning about groups.

We offer here some comments which we see as plausible interpretations of the experiences of our high school teacher/participants (in the style of Erickson's "assertions"). We take their responses to be typical of students struggling with these concepts, but offer their experiences as examples rather than any sort of proof for our theory. Together, the examples and the theory form a preliminary interpretation of the written assessment and the interviews, and seem to us to reflect provocative questions to be studied by ourselves and others in continuing investigations.

For convenience of presentation, and perhaps to obtain additional insights, we have collected our discussion into two categories. The first category has to do with how our students may have been constructing a general concept of group — in terms of a specific group, and the notion of subgroup. The second category discusses a possible development of the mathematical ideas leading up to the concept of quotient group — the idea of a coset, mental construction of operations on cosets, and the concept of normality.

Developing the Concepts of Group and Subgroup

In this section we suggest that an individual's development of the concepts of group and subgroup may be synthesized simultaneously. Our observations are consistent with a progression in understanding that moves through various intermediate (and incomplete) ways of understanding groups and subgroups. That understanding may move from seeing groups and subgroups as primarily sets of discrete elements, to a stage where the operations as well as the group elements are incorporated into the necessary definition. Finally, a student may construct a thorough understanding of a group as an object to which actions can be applied.

Understanding Groups and Subgroups as Sets of Elements.

It appears possible that some students try to deal with problem situations involving a set and an operation by assimilating the situations to an existing set schema, ignoring the operation which is also present. We suggest that such a strategy may represent an early misconception of the concepts of group and subgroup.

<u>Groups as sets</u>

In the very first phase of learning the group concept, a student may interpret a group primarily in terms of its elements, that is, as a set. If the individual remains at this elementary understanding of groups, he or she may not distinguish a group by anything more than the number of elements in it.

One example of a student's response which may indicate a strong emphasis on groups as sets of elements occurred when Kim was asked if Z_6 were isomorphic to S_3?[2] Kim says the following[3]

[2] Z_3 is the group of elements $\{0,1,2\}$ with the operation of addition modulo 3; Z_6 is the group of elements $\{0,1,2,3,4,5\}$ with the operation of addition modulo 6; in general, Z_n is the group of elements $\{0,1,\ldots,n\}$ with the operation of addition modulo n.
S_3 is the symmetric group on $\{1,2,3\}$, i.e., the group of all permutations of 3 objects. In general, S_n is the group of all permutations of n objects.

[3] Names of students have been changed to avoid identification of individuals.

Kim: Probably so, S_3 has 6 elements in it and Z_6 has 6 elements in it, so without going through the whole procedure, I would say yes.

In addition to confusion about isomorphism, this student's understanding seems to emphasize the number of elements as a characterizing feature of a group.

Thus, it may be that Z_3 is considered to be any set with three elements that is known to be a group. For example, in the written assessment and the interview, another student, Cal, variously considers Z_3 to be the set $\{0,1,2\}, \{1,2,3\}, \{0,2,3\}$, or $\{0,2,4\}$.

Also consider Sue who answered Question 1(b) (on subgroups of Z_6) on the written assessment, (Appendix I), specifying a group by its elements; she wrote $\{1\ 0\}$ for a subgroup of Z_6 with two elements and $\{2\ 1\ 0\}$ for a subgroup of Z_6 with three elements.

At the earliest stages of understanding groups, the students may construct their own idea of group by considering familiar objects (elements of the group) and forming a process of associating these objects with each other in a set. Eventually, the students may encapsulate that process into an object which, for them, represents the group in question.

Subgroup as a subset

Understanding a subgroup as a subset is similar to understanding a group as a set. For a student at this stage, sometimes "being a subset", that is, having all its elements included in a bigger set, is sufficient to conclude the existence of a subgroup. In other cases students require that such subsets of elements share a common property.

In looking for subgroups of D_3, many students correctly mentioned the "rotations". Similarly, but incorrectly, some listed "the flips" as a subgroup[4]. Consider for example Cal who, in responding to Question 2(a) of the written assessment (Appendix I), listed the elements of D_3 as $\{R_0, R_1, R_2, D_1, D_2, D_3\}$ and identified the first three as the rotations and the second three as the flips. Then in responding to Question 2(c) he listed $\{R_0, R_1, R_2\}$ as a subgroup of D_3 isomorphic to Z_3 and in responding to Question 2(d) he listed $\{D_1, D_2, D_3\}$ as a subgroup of D_3 also isomorphic to Z_3. In all cases, he mentions the correct operation. Here is what happens when the interviewer asks Cal about his choice of $\{D_1, D_2, D_3\}$ as a subgroup[5]

I: And what about this one here? You want it isomorphic to Z_3. What you write here is $\{D_1, D_2, D_3\}$.

Cal: Yeah, I thought if you do them all...

I: The three flips.

Cal: Right.

I: You think it's a subgroup.

[4] D_3 is the group of symmetries of a triangle. In general, D_n is the group of all symmetries of a regular n-sided polygon.

[5] The designation **I:** refers to the interviewer in all of the interview excerpts in this paper, although it is not necessarily the same person each time.

Cal: Well, like you told me you have to have the same operation, it works on it the same as addition.

I: Well, that's not the point because it has to be a subgroup of this D_3. But is it a group at all under composition?

Cal: I thought it was. I didn't see anything that...I thought it was closed.

The interviewer then prompts Cal to see that the group properties are not satisfied by this subset. Thus we see that Cal appears to understand a subgroup as any coherent subset. The group operation is carried along, but it is not used.

Individuals who have not progressed beyond this point would probably have no difficulty in considering the even integers to be a subgroup of Z, but they might also think that the odd integers were a subgroup as well.[6]

This demonstrates a misconception caused by some students' efforts to construct a new concept (group) by relating it to a familiar concept (set). This is an example of reequilibration by assimilating the situation to existing available schemas before those schemas have been reconstructed to achieve a higher level of sophistication. It may happen that a student leaps over this step, or passes through it very quickly. But nevertheless, as we witnessed above, some students exhibited vestiges of this misconception after five weeks (approximately 50 contact hours) of instruction in group theory.

Seeing Groups and Subgroups as Sets with Operations.

As students encounter situations in which their current conception of groups as sets is inadequate, they may begin to include the group operation in their determination of groups and subgroups. A student may realize, from appropriate experiences, that a given set will have a number of properties, one of which is that a binary operation satisfying certain conditions can be constructed and associated with the set. We saw some evidence that could be interpreted in this way.

We were intentionally vague about the operation when we asked if Z_3 is a subgroup of Z_6 in the written assessment because we wanted to see what operation the student chose (of at least two possibilities) and what role the operation played for her or him. When we ask for an explanation of this answer in the interview, it is the interpretation which the student then gives that forms the object of our interest.

Consider, for example, the following exchange with Tim.

I: Is Z_3 a subgroup of Z_6?

Tim: Ok, I'll say yes.

I: Ok, and how do you check that to be sure?

...

Tim: Ok, let me take that back. It depends on the operation.

[6] Z is the group of all integers with the operation of addition.

One interpretation of this exchange is that, for Tim, the set is the predominant aspect of the group and the operation is secondary. But whether or not the emphasis of this conversation accurately represents Tim's complete understanding, it is apparent here that he at least exhibits some separation in these features of groups.

Groups as sets with operations

The group operation may be constructed on a set in a number of ways (a formula, a table, the operation induced by a group which contains the set). In general, the students we observed seemed to be quite comfortable with examples such as addition mod 3, group operation tables, and the operation of addition mod 6 induced on one or another of the subsets of Z_6. Making sense out of these various realizations of group operations requires that the student coordinate a number of general function concepts with ideas that arise in a study of groups.

Since an operation on a set is a function (both mathematically and psychogenetically) a student's function schema may intervene here with all of the power and difficulties related to the level of development of an individual's function schema (see Breidenbach et al, 1992 for a discussion of the psychogenesis of the function concept).

In any case, the students seemed to distinguish these operations individually and then organize them in some coherent manner. Initially the student may see that one operation is preferred (addition mod 3 in Z_3, composition of permutations in S_3, etc.) to others (e.g., the operation induced by Z_6 on $\{0, 1, 2\}$, or an arbitrary table.) At one point, Kim was discussing $\{0, 1, 2\}$ with the induced operation addition mod 6 because it is a subset of Z_6. In the middle of this discussion, she said the following.

> **Kim:** Ok, this is Z_3 I should be using addition mod 3.

A student may eventually encapsulate a set of objects and some operation on this set to form an object that is for her or him the conception of this particular group. After a de-encapsulation of the group object there may be other operations which could be applied to the set, but the one that came from the de-encapsulation may be preferred. For example, Lon's remark below could be interpreted this way. After some discussion regarding the operation induced by Z_6 on the subsets $\{0, 1, 2\}$ and $\{0, 2, 4\}$ Lon says:

> **Lon:** Z_3 wants to have as its operation addition mod 3 but the $\{0, 2, 4\}$ you mentioned is just fine and happy to have addition mod 6, to have it be a group.

Subgroups as sets with operations

As was the case with groups, the concept of subgroup may also become coordinated with an operation. At this point, for the student, a subgroup is a subset to which some operation has been attached, making it a group. Thus, in the following excerpt, it appears that for Z_3 to be a subgroup of Z_6, Ann only requires that Z_3 is a group (under addition mod 3) and that its elements are all in Z_6.

I: Is Z_3 a subgroup of Z_6?

Ann: Yes.

I: It is. Okay, do you want to write down the elements of Z_6 for instance.

Ann: Okay (writes, indicating the subset $\{0, 1, 2\}$.)

I: Okay, now you say that Z_3 is a subgroup of Z_6?

Ann: Uh-huh.

I: And how do you see that?

Ann: Well, if you make the table for addition, its gonna be closed and it, uhm, gets all the properties.... (The interviewer probes to see which operation Ann is using.)

Ann: Modular arithmetic.

I: Right, which modular arithmetic?

Ann: Well, a_3 (addition mod 3)

Cal puts it more succinctly in referring to Z_3 as a subroup of Z_6.

Cal: Well, it's a subset of Z_6 and a group in itself.

<u>The subgroup's operation is induced from the larger group</u>

Once students appreciate the role of the operation, they are able to understand that the subgroup operation must be the same as in the larger group. This requirement may appear to the student to be somewhat arbitrary and it might not relate to the restriction of the operation as a function on the larger group to the subset. For example, in the following excerpt we cannot be sure that May realizes *why* the operations have to be the same, or even that she knows why addition mod 6 occurs.

I: Is Z_3 a subgroup of Z_6?

May: Yes. Wait. (Pause) They're not even the same operation. It can't be.

I: Why not?

May: Because you'll get elements. Well, the definition says you have to be in the same operation, for one.

I: Ok, good point. If we're going to say that this is a subgroup, we're implying that its the same operation. What's wrong with the operation addition mod 6?

May: addition mod 6?

I: Z_6 is the group.

May: We'd be looking at something like that. Z_3 is not in this group, that's the thing that keeps sticking in my mind — they're not the same operation.

In this passage, May demonstrates a disequilibration that can occur when a conflict arises between the canonical operations on Z_6 and Z_3 on one hand, and the "same operation" requirement of the subgroup concept on the other.

Some students were more explicit about where the operation comes from when a subset is being considered. In the following comment, Sam apparently understands that the operation on the subset is the restriction of the operation in Z_6.

Sam: Z_3 is the set $\{0, 1, 2\}$ and Z_6 is the set $\{0, 1, 2, 3, 4, 5\}$. To be a subgroup, it would have to be closed under whatever operation you are using in Z_6.

We suggest that a student's concept of function must intervene when he or she considers the operation o on the set G. Rather than consider the static situation in which the subset H of G has an operation which happens to agree with o, the student may perform the (mental) action of restricting the function to (pairs of) elements of H. At this point the issue of H together with o being a group is raised and there must be a coordination of the emerging subgroup concept with the student's group concept and function concept.

Coordination of Groups and Subgroups

As we have already begun to see, the individual's development of the subgroup concept may be coordinated with the development of the concept of a group. We can see some indication of students' understanding of group and subgroup when they are asked to determine whether a specific group is a subgroup of another group. While such a decision may once have been made considering the elements only, when a student understands the role of operations, a different approach is used. It is possible that the student would consider a subset to be a subgroup if it is closed under the induced operation.

For Sam, closure is apparently enough.

I: Now would you please look at this: $\{0, 2, 4\}$, at this subset of Z_6. What can you say about it?

Sam: I don't know...It would be...a subgroup.

I: So why you're saying you don't know? You know. It is a subgroup of...?

Sam: Z_6.

I: ...of Z_6.

Sam: Because it would be closed under addition mod 6.

Ann also emphasizes closure, but she explicitly assumes that the group properties other than closure are inherited from the original group. This could be an example of awareness of the deficiency of relying on closure only and responding to disequilibrating experiences by looking for reasons to avoid reconstructions.

I: Suppose you look in Z_6 at the subset of elements consisting of 0, 2, and 4.

Ann: Okay.

I: Now that's a subset of Z_6, right?

Ann: Yeah.

I: Okay, uhm, would you say that subset is a subgroup?

Ann: Yes, I would that one.

I: Okay, I noticed you moved your fingers along each of the elements. What were you checking? What were you thinking about?

Ann: Closure.

I: You were thinking about closure.

Ann: Uh-hmmm.

I: Is closure all you need to be concerned with?

Ann: Well the other ones you told us were implied to the closure from Z_6.

Some students seemed to understand that in order for a subset to be a subgroup, all of the group properties, relative to the induced operation must be preserved. We observed Kim performing such a check as she determines that $\{0,2,4\}$ is a subgroup of Z_6.

Kim: If I'm using mod 6 and I add 2 and 4, I'll get 0, if I add 4 and 4, I'll get 2, so its closed. 0 would be an identity. If I added 2 to 2 I would get 0 — no, I would get 4 which is in the set so it is not the inverse. If I add 2 to 4, I'm going to get 0, so I'm going to have my inverse. So I'd say that's a group under addition mod 6, and therefore it's a subgroup.

She does not mention associativity at all. Lon, who does, may see that for the identity, it is only necessary that 0 be in the subset. But he may not yet have seen that the same is the case for inverses because he apparently checks, not that 4 is in the subset, which is enough, but that 4 is the inverse of 2, which is already known.

Lon: Let's see. Associativity: inherited. Identity: yeah, 0. Inverse: 2 and 4 are inverses of each other, and 0 is its own inverse.

Understanding Groups as Objects

Encapsulating a process into an object can be extremely difficult for students. It may be much delayed or even not occur at all in some instances. Moreover, we do not know very much about how to help bring this about (see for example Sfard, 1992, but also Ayers et al, 1988 and Breidenbach et al, 1991.) Our point of view is that when a student is in a situation in which applying actions are required, then he or she may tend to encapsulate processes in order to have objects to which the actions can be applied. In other words, trying to treat something as an object can lead to making it an object.

Our notion of action includes the determination that a certain property is satisfied. Trying to perform such an action can help with the formation of an object. There must be a "something" that posseses the property and thinking about this may lead to encapsulation.

Another kind of action that requires understanding of groups as objects is to see that two groups (and their operations) may be the same, that is, isomorphic in a naive sense. An awareness of group properties can lead to a reconstruction of the group concept so that a group is seen as an object. This reconstruction can make it possible, in some contexts, to consider two groups in which the sets and operations can be informally matched to be the same.

Properties of groups

Group features such as the order of the group, being cyclic or commutative or being a group of symmetries are examples of properties. It is likely at this stage that Lagrange's theorem is applied in checking for a subgroup, actually, showing that a subset is *not* a subgroup. Of course there is considerable experience suggesting that it is very hard for students to reach this stage and many of them, apparently, do not.

For example, some of our students, when asked to find a subgroup of order 4 of D_3, stated explicitly, that such a subgroup doesn't exist since 4 does not divide 6. Others, however, tried to create various possible subsets of the elements of D_3 such as $\{R_0, a_1, a_2, a_3\}$ (the identity and the three flips), or $\{[1,2,3],[3,2,1],[3,1,2],[2,1,3]\}$ (the identity and three other permutations), while still others simply listed set with four elements, not necessarily forming a group and not taken from D_3.

At this stage, also, when trying to determine whether one group is a subgroup of another, checking the group properties is simplified by the use of certain shortcuts. The student comes to realize and use the fact that of the four group properties, associativity is inherited, and that, in addition to closure, it is only necessary to check that the original identity is in the subset and that the inverse of every element of the subset is also in the subset.

Group as a generic object

The final step in the construction of the concept of a single group begins with the realization that other, apparently different, groups may be constructed, but they turn out not to be really different. At this point a developing (but still naive) conception of isomorphism may intervene and the student might construct the process of forming several specific groups and establishing isomorphisms between them (e.g., $\{0,1,2\}$ with addition mod 3 and $\{0,2,4\}$ with the operation induced by Z_6.). The encapsulation of this process would create an object which is the group in question.

None of the students that were interviewed made statements that could be interpreted as having taken this step. But we suspect that Lon is on the verge when he makes the following statement in reference to $\{0,2,4\}$ which he knows to be a subgroup of Z_6.

> **Lon:** As a matter of fact, it's Z_3, but, or, excuse me, it's isomorphic to Z_3.
>
> ...
>
> **Lon:** The problem here is that you have two groups that are isomorphic to each other, and yet one is a subgroup of a certain group, and the other is not.

With a little more thought, Lon may soon be able to resolve the "problem" by identifying all of the isomorphic realizations of Z_3.

Developing the Concepts of Coset and Normality

The mathematical development of the concepts discussed in this section begins with the formation of cosets followed by an attempt to introduce a binary operation on the set of cosets so as to obtain a new group, called the quotient group of G mod H — denoted G/H. This construction of a new group does not actually work in all cases because it depends on normality which is a property that a subgroup may or may not have in relation to the group.

It is generally agreed that the mathematics involved in these concepts is extremely difficult for students, and the students in this study were no exception. Although most students were able to construct the cosets and calculate their product in the relatively simple case of the subgroup of all multiples of 3 in Z_{18}, the number of successes fell off when it came to identifying the quotient group as isomorphic to a familiar group. The drop in performance was very sharp when it came to similar questions in D_3. Thus normality (which is really only an issue in the non-commutative D_3) did not appear to be well understood by very many students.

As we will see in the following discussion, the drop in performance that occured when the domain shifted from Z_{18} to D_3 was quite pronounced and became even sharper as the mathematical issues became more sophisticated.

Understanding Coset Formation

In the following discussion we would like to suggest that an individual's concept of formation of cosets could follow an action-process-object development which is dependent on the context.

The examples in this section are based on a generic group G, a subgroup H and the (left) cosets $aH, a \in G$ where $aH = \{ah : h \in H\}$

We made use of the two examples:
$G = Z_{18},\ H = \langle 3 \rangle$, and $G = D_3,\ H = \{I, R, R^2\}$ where $\langle 3 \rangle$ is the set of multiples of 3 in Z_{18}, and I, R, R^2 represent rotation by 0, 120, and 240 degrees respectively. In Z_{18} we used $a + H$ instead of aH for a coset. In discussing normality, we also considered right cosets.

<u>Coset formation as an action.</u>

Coset formation as an action is possible only in familiar situations and where explicit formulas are available.

For our students, the cosets of $\langle 3 \rangle$ in Z_{18} could usually be formed, but cosets in D_3 were more difficult. Often, students were be able to do the former but not the latter. For example, on Question 3 about Z_{18}, parts (a), (b), (c) were attempted by all but 3 students and part (d) was omitted by 4 students. On Question 4 about D_3, however, 8 participants omitted part (a) and 14 omitted part (b). (See Appendix I.)

Many of the students explained their computation of cosets in Z_{18} much as Cal did in the example below. For Cal, this computation is probably no more than an action, since he seems to require explicit listing of the individual elements at each point.

> **Cal:** Well, the number in front is what you add to each element inside the set. So zero added to these six elements would keep the same six. One [the number] added to each, which is in the first column, would give you the 1,4,7,10 and then you add 2 to these first the H which is 0 through 6,9,12,15. Then you add 2 to each and you get 2,5,8,11,14 and 17.

A possible misconception seen here is that the student may be confused about whether, in forming all cosets aH of a group G, a runs through each element of H or of G.

Coset formation as a process

The development of coset understanding from action to process may initially occur in familiar structures. As a student begins to interiorize cosets from actions to processes, familiar cosets such as those in Z_{18} may become processes, while for a complex coset, such as found in D_3, the student may have great difficulty even constructing an action.

At the transition, the gap between the student's ability to form cosets in Z_{18} and in D_3 widens. In the former case, the student who is thinking beyond formulas may begin to see patterns in sets such as

$$1 + \langle 3 \rangle = \{1, 4, 7, 10, 13, 16\}$$

There will be comments such as "The differences are all the same", "Every third one", "All that matters is where you start". In the following, Hal may be indicating an interiorization of the action by virtue of the pattern he expresses.

> **Hal:** Z_{18} subgroup H which is generated by the element 3. Ok, I interpreted this as $0 + H, 3 + \ldots$. Every third element beginning with 0. So $1 + H$ every third element beginning with 1 in Z_{18} and every third element beginning with 2. And that would generate all the elements that are in G.

We cannot be certain that Hal has really constructed a process for coset formation. In the written assessment, he was unable to do anything on Question 4 and, in the interview he required considerable prompting to construct the cosets in D_3 of the normal subgroup $\{I, R, R^2\}$. It may be that coset formation is still an action for him but, with the aid of formulas available in Z_{18} he is apparently starting to see patterns as he moves towards construction of a process for coset formation. The lack of simple formulas to apply might explain the delay in the case of D_3.

One thing to point out here is that understanding a simple concrete example, in this situation, does not seem to help much with extension to the general case.

Upon completion of interiorization, our theory suggests that constructing a coset would become a process that could be performed in a variety of situations. The student could think about doing it without actually making calculations. The student could not only form individual cosets, aH but could think of doing it for every $a \in G$. It is then necessary to decide that all cosets have been formed. The most obvious criterion is to stop when you start getting repetitions. Lon appears to be using such a method below:

> **I:** When you were talking about the elements in these cosets and stuff, what happened to $3 + H$ and $4 + H$?
>
> **Lon:** Okay, yeah. I should have said that $3 + H$, of course, which is a coset in its own right, is equal to the set $0 + H$ because you get the same members as if you had added 0. Same goes with 4,5,6 and so on.

Using some mathematical properties, some students will suggest shortcuts. This may be what Ann is doing in the following excerpt. She has just gone through the details of constructing the cosets in Z_{18} of the subgroup of multiples of 3. She seems to suggest that you stop making cosets when everything is used up and when you don't get new cosets.

> **Ann:** Well, it took care of all the numbers and you start going when you put 3(plus 18)H, addition 18H, you're gonna get the same set as you got in 0.

Coset as an object.

As a student begins to make the transition to coset as object, he or she may still understand a coset entirely in terms of the process of forming it. One indication of this might be a tendency not to use a simple name for a coset (such as $1 + H$), but to insist on writing out all of the elements of the set, or to use only descriptive names such as "the rotations", or "the flips".

In the written assessment, May computes the cosets in Z_{18}, listing each set and all of its elements. Then she assigns the names $0 + H,\ 1 + H, \ldots, 17 + H$, and indicates that each of them is equal to one of $0 + H,\ 1 + H,\ 2 + H$. In forming the operation table for the cosets, however, she does not use the labels but writes out all of the elements in every set.

Another possible indication of a not fully completed transition appeared in several interviews. When the students were asked for an explanation of coset multiplication which would require them to understand cosets as objects, some students responded to that request by first explaining only how the cosets are formed. It did not seem easy for them to talk about cosets in terms other than the process of forming them. Following is a typical example in which the interviewer asks Ann for an explanation of how she constructed the operation table for cosets of $\langle 3 \rangle$ in Z_{18}. Before talking about operations between cosets, Ann finds it necessary to talk about the cosets in terms of the process used to form them.

I: Can you explain to me how you got those uhm, uhm how you went about doing that, you know, setting up that table. How did you make it.

Ann: Well you just, if you take like 0 and add it to the sub-group, you get the same elements of course. But if you add one to each one of those, you get that set $\{1, 4, 7\}$ and so on. And if you add 2 to the H, you get those numbers there and you see that it encompasses all 18 numbers. So its like they're equivalent to like $\{1, 2, 3\}$ err $\{0, 1, 2\}$. I'd say mod 3, whatever you want to call it and that's how I'd set it up.

When the transition to coset as object is completed, the student may give symbolic names to cosets and use these names in working with coset multiplication. Where necessary, the student can de-encapsulate the name back to the set as process and use that process in making calculations. Consider, for example, Lee's response when asked to explain how she computes the operation table. It is a clear example of de-encapsulating an object (set) into the process from which it came (the elements in the set).

Lee: First of all, my $H + 0$ was the subgroup. And then $H + 1$ would be the subset that contains $\{1, 4, 7\}$. And then 2 would be 2 uh, each element added to 2.

I: Okay.

Lee: So what I did initially when I set up my operation table is to say that those are possible subsets. Ok, when I operated on... Anytime I'm operating on this set, it remains that set. But those numbers will generate, in Z_{18} under addition, they will have those elements contained in them.

It should be noted that on the written assessment, Lee's operation table is written entirely in terms of labels for the cosets — indeed, in all of Question 3, the only time she wrote a set was in specifying the identity in the quotient group $Z_{18}/\langle 3\rangle$, where she wrote $e(G/H) = H = \{0, 3, 6, 9, 12, 15\}$

We saw a similar set of responses from Lon. He and Lee were the only two students who succeeded completely on Question 4, regarding D_3.

Coset Multiplication

We see some parallels between the action-process-object development of coset formation which we just described and a similar construction for coset multiplication. Therefore we suggest that an individual's concept of coset multiplication may also follow an action-process-object development which is again dependent on the context.

Indeed, even more than with coset formation, coset multiplication is particularly dependent on the domain. It seems possible to understand coset multipication instrumentally in Z_{18} and be totally lost regarding the same operation in D_3.

In the written assessment, all but one of the students were able to compute the multiplication table for the cosets of $\langle 3\rangle$ in Z_{18}, but for D_3, although 14 students were able to find a subgroup which was normal, only three of them worked out the products of cosets.

It is difficult to attribute any general understanding of coset multiplication to students who are very comfortable in one situation and completely lost in the other. Nevertheless, in the case of adding cosets of $\langle 3 \rangle$ in Z_{18}, we can discern what may be a development. There appears to be not only a movement from action to process for the method of adding two cosets by adding all elements in one to all elements in the other (which is how it was initially defined in the course), but also for the construction of the method of adding two cosets by adding representatives (which is, for our students, a theorem).

In the following examples, again the group is Z_{18} and H is the subgroup $\langle 3 \rangle$ of multiples of 3.

Coset multiplication as an action

One straightforward type of action in mathematics is computation according to a formula. Often, a student may succeed with computations even before the concept behind it is fully understood. This is apparently the case for coset operations. For example, it seems that for Tim in the comment below, the result of adding (the coset operation in this case) $1 + H$ and $2 + H$ is $3 + H$ just because this doesn't contradict his prior knowledge that $1 + 2 = 3$.

> **Tim:** [In explaining why the sum of $1 + H$ and $2 + H$ is H]
> ...if you add what, $1+2$, then this would be what, Z_3 right? That would be 3 which in Z_3 would be 0 and that is why I put 0.

It seems analogous to children's ability to compute correctly the sum of "one-seventh" and "two-sevenths" or "one barbarow" and "two barbarows" without understanding of what the manipulated objects are.

As the participants in this study started to develop addition of cosets as an action, there were several variations of the statement, "Add everything in $a+H$ to everything in $b + H$." For example, Ann explains it as an action in a specific example. She is asked how she determined that adding $1 + H$ to itself gives $2 + H$.

> **Ann:** By adding them together element by element, you see those elements are congruent to the ones in the $2 +_{18} H$, like I take $1 + 1$ and you get 2, and then keep one going through the whole set, 1 plus 3 at 5, and so on...

Interiorization of the action to a process

In some cases, we can see what may be a movement towards constructing a process for coset multiplication that makes strong use of the specifics of addition in this case.

> **Lon:** We have to define this operation which I suppose, for want of a better word, would be called set addition. We are adding all the members of the one set to all the members of the other set, again adding addition mod 18, and your result will be the set of all possible answers that you can get from a member of the first set plus a member of the second.

Construction of the method of adding cosets by adding representatives — an encapsulation

Perhaps we can consider the method of adding two cosets by choosing a representative of each, adding the representatives and taking the coset of the sum as being essentially an encapsulation. That is, the process of adding all elements is encapsulated to the the single representative which is an object.

In the cases that we see here, the students all rely heavily on the specifics of the arithmetic in Z_{18}. They pick a very special representative and reason through the new method. Thus, the question of proving independence of representatives does not arise.

We can see this in Lon's continuation of his explanation.

Lon: So $1+H$ plus $0+H$ would give you — let's see, what's the easiest way to put this — every one of these members of $0+H$ is a quote, unquote multiple of, yeah, I can say multiple of 3. Okay, every one of the numbers here is congruent to 1 mod 3, and when I add... To be in this set you have to be congruent to 0 mod 3. To be in this one, you have to be congruent to 1 mod 3. And when you add two numbers like that, you have to get a result that's congruent to 1 mod 3. And all those numbers are the numbers in the set $1+H$, and no others.

De-encapsulation and addition by representatives

Some students showed an ability to go back and forth between the method of adding all elements of one set to all elements of the other and the method of adding representatives. This seems fairly clear in the following explanation by Gil. Asked to consider three examples of coset operation, he does the first by adding elements, the second by representatives, and the third by what could be a mixture of the two. It may be that his going back and forth only shows him in the midst of solidifying his encapsulation.

I: And what is the meaning of $1+H$ the operation $1+H$ gives you $2+H$?

Gil: This $1+H$ is the coset of H that is arrived at by adding 1 to the elements of H. So if $1+H$ would be this one $\{1,4,7,10,13,16\}$.

I: And now you have to make...

Gil: Right, and if I add that to this, it's like saying 1 plus every element in there, 4 plus every element in there, 7 plus every element in there, and so forth. And what I'm going to get as a result is a coset of H generated by 3 which is the same as $\{2,5,8,11,14,17\}$. Does that make sense to you?

I: I am trying to understand. Would you please do it once again — maybe take another element, $1+H$ operation $2+H$.

Gil: Okay, the $1+H$ and the $2+H$ — H operated on itself is going to give me the H back so I don't need to worry about that. So the 1 plus the 2 is the same as 3, but 3 — If I add a 3 to every element, I'd get the original set generated by 3. $0+3$ is 3, $3+3$ is 6, $6+3$ is 9, on down the line, $15+3$ is 18 which is 0 in Z_{18}.

I: And how do you add 0? For example, if we had $2+H+0+\ldots$ — excuse me — $2+H$ operation with $0+H$?

Gil: Okay, that would be taking one of the elements, or any element here and adding it to every element there, so if I take 2+ the 0, I am going to get a 2. And if I add

$5+0$ I am going to — or, excuse me, $2+0$ is 2, $5+0$ is 5, $8+0$ is 8, $11+0$ is 11, so I am re-identifying that same set,...

Most of the students interviewed made much less progress with cosets in D_3. In the case of Z_{18}, some students seemed to rely on the fact that 1 is first in $\{1, 4, 7, 10, 13, 16\}$ and 0 is first in $\{0, 3, 6, 9, 12, 15\}$ so that the addition of these two cosets can be obtained by adding the two first elements, $1 + 0 = 1$ and taking the coset which begins with 1. They had no such mechanical device in D_3 and several tried to find one. In general, they were not successful and the little progress they made required a considerable amount of prompting.

Constructing the Concept of Normality

Our students did not appear to have constructed very much of a viable coset concept to use in dealing with normality. Therefore we have only a few examples to illustrate a development which is mainly derived from the general theory with which we are working and our own understanding of normality. This development, we conjecture, begins with the formation of a left coset aH discussed above and extending this process to the analogous construction of a right coset Ha. Thereafter the student would be able to coordinate the two processes aH, Ha with equality to obtain $aH = Ha$. Then the process $aH = Ha$ can be encapsulated into an object which has a boolean value (true or false) that can vary with the element a of G. Thus the student could construct a function which assigns to each $a \in G$ the truth value of the assertion $aH = Ha$. This function can then be iterated over its domain and universal quantification can be applied to obtain a single value, $true$ or $false$. Finally, we would expect the student to thematize all of this into a schema which can be applied to any situation involving a group and a subgroup.

The examples we do have illustrate what may be an iteration of the boolean valued function over G and the formation of the schema for general application.

Constructing aH, Ha, and verifying their equality

We take the following description by Gil as an indication of the construction of the equality for a particular choice of a. He knows about doing it for different values of a, but he makes the error (repeated by several students) of using only the elements of H.

I: Okay, why do you think it is normal?

Gil: Because it ...

I: Just to make you relax, yes it is normal.

Gil: It is a subgroup of G and no matter what element of that subgroup I apply to the elements — for example, if I choose 3 — and this will work for every element, if I take 3 and I operate on H, that is the same as taking H and operating on 3 so it is kind of like a commutative type thing. If this is a normal subgroup, the best I understand it, any element a of H, operated on H is the same as H operated on a.

Iterating the boolean valued function $a \mapsto (aH = Ha)$ over all a in G

To confirm the normality, a student learns to check if the equality $(aH = Ha)$ holds over all a in G.

Since Kim had written the statement $aH = Ha$ in her response to Question 4(a) on the written assessment, the interviewer asked her to explain.

> **I:** What is, well H is clear because the problem says that is a subgroup. What is a?
>
> **Kim:** a is an element in D_3.
>
> **I:** An element in D_3, good. One particular element in D_3?
>
> **Kim:** No, I don't think so.
>
> **I:** No, what, which element? Because it doesn't say here, it just says $aH = Ha$.
>
> **Kim:** It could be any element. If any element does that, it is going to be normal.

Thus, in her last statement, Kim may be expressing a boolean valued function whose domain is D_3 ("any element") and value is the truth or falsity of the assertion $aH = Ha$ ("does that").

Thematizing the previous step into a schema that can be applied in any specific situation.

Gil states this clearly in describing how he checked that a subgroup is normal.

> **Gil:** To check if this is a normal subgroup, I operate this on that element and produce this (aH). What I am going to do now is to reorder these, and put this one first which means I am going to do this part of the composition first (Ha). So, and what I am doing is checking to see if that answer is the same as that answer $(aH = Ha)$, and if this is true in all cases, then this would be a normal subgroup.

(Note that he may or may not still think that "all cases" refers to H and not G.)

As with coset multiplication, there was confusion here in passing from normality of subgroups in Z_{18} to normality in D_3. In the former case, students were generally able to begin applying the definition and saw fairly quickly that it would always work because Z_{18} is commutative. In passing to D_3, however, the schema broke down completely for most students.

Interestingly, there is one example in which it went the other way. Hal had difficulty in checking the normality of $\langle 3 \rangle$ in Z_{18}, but eventually succeeded. Shortly thereafter, he was able to handle normality in D_3 without difficulty. Although, for most students, success with Z_{18} did not carry over to D_3, Hal seemed to be constructing sufficient understanding of normality by working through Z_{18} in the interview. He seemed able to subsequently extend the concept to D_3 as we talked to him.

Interpretations

In this section we summarize what we think may be going on as students try to learn group concepts. We only claim that the statements in this section are plausible explanations of the data we have gathered, and are generated with the help of our theoretical perspective and our own knowledge of the mathematics involved. Our main purpose in making any conclusions at all is to suggest pedagogical responses that may improve student learning about groups.

In this group of participants, the mathematical notions of group and subgroup appeared to develop in a somewhat parallel manner. It is the case that for each of these concepts, a psychogenesis can be presented as if it were an essentially linear development, but at many points one concept seems to be developed in concert with others. Often, progress in understanding one of the concepts appears to await developments in the others. There are a number of specific instances in which what is understood relative to one concept was used in constructing new understandings of another.

In addition, the students trying to understand groups and subgroups seem to need to make use of other mathematical concepts. The most important of these are the concept of set and the concept of function.

It is possible that the way in which understandings appear to develop might depend on the order in which the learner comes to interact with various concepts and explicit features of these interactions. To help clarify this possible relationship we include, in the following discussions of specific topics, some brief comments about how they occurred in the course we are considering.

Groups, Sets, and Functions

The development of the group concept appears to have a very primitive beginning, seemingly based entirely on the student's conception of a set. As the student gains experience, he or she may include a number of properties that this set can have. A binary operation is included among these properties, but at first, there may be no differentiation between properties.

We suggest that an important step in the development occurs when the student singles out the binary operation and focuses on its function aspect. Of course, a mathematical requirement here is that the student's function concept would have to encompass functions of two variables.

Apparently, the conclusion of this development is the encapsulation of two objects, a set and a function (binary operation) coordinated in a pair which may be the student's first real understanding of a group. Eventually, this will have to be reconstructed on a higher plane which includes the concept of isomorphism. Then the student can come to understand a group as an equivalence class of isomorphic pairs.

In the course, the participants worked with binary operations from the beginning. The emphasis was on the operation and not the domain set which is explicitly introduced later and discussed simultaneously with the function of two variables. The concept of group is introduced by providing the students with many (computer) experiences with examples of groups and testing properties at the end of which the formal definition is introduced.

Groups and Subgroups

We can conjecture that the connecting link between a group and a subgroup is the concept of function. After going through a number of developmental stages, the student's concept of subgroup may be solidly based on the notion of the restriction of a function to a subset of its domain.

We can see a strong parallelism in our psychogenesis of group and subgroup. For example, when the student's conception of group is a set with an undifferentiated collection of properties, he or she may think of a subgroup as nothing more than a subset of elements all sharing a common property—or as the subset of all elements which satisfy a certain condition. Focusing on the binary operation as a function appears to be important for the development of both concepts.

As with groups the participants' introduction to subgroups came initially through computer experiences with constructing examples and writing programs to test properties. The formal definition used was standard and does not explicitly mention the restriction of a function of two variables to a subset.

Concepts Leading to Quotient Groups

Again the participants' introduction to the concepts of coset, coset product, normality and quotient groups was by means of computer experiences followed by discussion followed by formal definitions. There was no instruction explicitly related to any action-process-object-schema development.

As we indicated in the beginning of this paper, it is generally believed (although we do not know of any formal studies) that most college students do not succeed in understanding the concept of quotient group. Indeed, major difficulties for students seem to begin appearing with the introduction of cosets. Things get worse with normality and, when it becomes time to construct quotient groups, a high percentage of students appear to be lost and may have "turned off" from the course. The students in this study were no exception. Their difficulties began to show on the written assessments and were even more evident in the interviews. The relevant scores on the written assessment are summarized in Table I which lists the numbers of students whose responses were mathematically correct, partially correct or incorrect on Questions 3 and 4 of the written assessment.

	Correct	Partially Correct	Incorrect
Question 3(a)	16	4	3
Question 3(b)	20	0	3
Question 3(c)	19	0	4
Question 3(d)	16	1	6
Question 4(a)	6	8	9
Question 4(b)	2	7	14

Table I. Scores on Cosets, Normality and Quotients

One interesting thing about the participants' responses is that on Question 3 which has to do with an easy, commutative group Z_{18}, their results are not too bad. Most students could work with the cosets, and form the quotient group (normality is not an issue here). But the scores dropped dramatically with the harder group D_3. Only about one fourth of the students could find both a normal subgroup and a non-normal one and most of them were completely lost in constructing the quotient group, even when they could find a normal subgroup.

The interviews provided a further indication of the difficulty that the students had with normality and quotient groups in the context of D_3. Five of the 10 interviewees made explicit statements such as "I don't know what a normal subgroup is" or "normality means that it is commutative.

It may be that lack of understanding of normality prevents students from beginning to think about coset operations and quotient groups in the non-commutative case. Note that, although mathematically a subgroup must be normal in order for the quotient group to exist, there is nothing to prevent a student from constructing, say, left cosets and starting to form a table of coset operations. The operation between two cosets A, B is defined by taking AB to be the set of all products xy of two elements x, y, with x taken from A and y taken from B. The result may not always be a coset and thus closure becomes an issue and the group axioms may not be satisfied. Although such matters were discussed in the course, these students, after struggling with normality, tended not to go on and try to make this construction.

Difficulty with normality may not, however, be the entire story. Of the 5 students who appeared to understand this concept, only one seemed comfortable in constructing the quotient group and two others were able to do it only with considerable prompting.

In interviewing these students, we were able to identify what appears to be fairly clear step-wise developments of the concepts of coset and coset operations. When these are described in terms of the construction of actions, processes, and objects, a major issue in this psychogenesis appears to be the encapsulation of the process of forming cosets into objects which are to be the elements of the quotient group. Again, the student's concept of function appears to be needed, but confusion seems to have occurred when he or she made an attempt to construct a binary operation on cosets *before* being able to manipulate these cosets as objects. An interesting point is that in some cases, formation of a coset did not guarantee that the student could deal with this set as an object. This suggests that the student's conception of sets may not have been adequate.

We were also able to identify some specific misconceptions that arose, particularly in connection with the concept of normality. There was a tendency to confuse normality and commutativity. Also, students appeared to be unclear about the relation between the property of a subgroup being normal and the question of whether the coset operation satisfies the group properties.

Complexity of Understanding

It is clear from our interview transcripts that an individual's understanding of the elementary concepts connected with mathematical groups is quite complex. The variety of interpretations, the errors, the misconceptions, the difficulties in passing from modular groups to permutations groups, the non-linearity of development of these concepts in an individual's mind — all these student reactions attest to the fact that constructing an understanding of even the very beginnings of abstract algebra is a major event in the cognitive development of a mathematics student. This is especially true if, as is often the case, abstract algebra is one of the first courses a student encounters which is not dominated by memorizing formulas and imitating the solutions of set-piece problems.

This complexity should not be surprising in light of what has been found by other researchers investigating much less sophisticated mathematical topics and even in areas completely unrelated to mathematics.

For example, Schoenfeld et al, 1990 and Kaput, 1992 have shown how complex are the very simple mathematical concepts of slope and linear function in the mind of a student. Piaget, 1977 has studied at the length the psychogenesis of these particular concepts and again we see that the development is long, laborious, and may be characterized by misunderstandings and nonlinearity. We should expect things to be even more complicated when the mathematics becomes more advanced.

Student Errors and Misconceptions

Researchers in mathematics education, such as Ferrini-Mundy & Gaudard, 1992 have suggested that the mistakes students make can provide windows through which we can observe the inner workings of a student's mind as he or she engages in the learning process. Several learning theories incorporate interpretations of errors. For example, the notion of epistemological obstacles introduced by Bachelard, 1938 (see also Sierpiǹska, 1992) suggests that these errors may represent stages through which a learner must pass, and that rather than eliminate them, we can only help a student avoid getting stuck. Similarly, the general theory of Piaget, 1975 includes the idea that concepts are constructed at a level that is adequate for dealing with a learner's current mathematical environment, but that when new phenomena are confronted, it is necessary to reconstruct concepts on a higher level. Thus, if the reconstruction is delayed, a student's conception of some mathematical notion may be adequate at one level, but erroneous at another.

We tend to agree with this more positive view of errors. Our observations here provide some support for the role errors and misconceptions can play in suggesting more positive and accepting responses to help improve learning. For example, in constructing the concept of subgroup, it is important for a student to compare the elements of two groups and check whether the elements of one are all included as elements of the other. It is not surprising, then, that some students will jump to the conclusion that one group is a subgroup of another based on the observation that the elements of one group are included in the other. We suggest that this "error" can be seen as an intermediate stage in the learning process. The need then is not to disabuse the student of this idea but to help her or him progress beyond it by considering the group operation as well as set membership.

More generally, consider the developmental steps that we have used to describe how various topics are being learned. On the one hand, it seems entirely reasonable to pass through these sequences. On the other hand, a student who happens to be at a particular stage (other than the "last") may seem to have a misconception if he or she is trying to deal with a situation that requires construction of the next step. Such a situation may be quite healthy and productive. Again, what is to be avoided is a student getting stopped at an intermediate developmental stage. This certainly appears to happen, and when it does, it really is a difficulty that needs to be overcome.

Pedagogical Suggestions

For many students, their early mathematical career consists of learning algorithms to solve repetitive problems. With abstract algebra, the abrupt change in mathematical style from learning algorithms to understanding concepts and the overall complexity of the subject imply that this course, above all, is not likely to succeed if taught in a traditional manner. Indeed, it may be the case that abstract algebra is not very successful at most universities. Although there are no studies, anecdotal evidence tends to support this supposition. Thus, we have a situation in which it may be important to consider alternative, innovative instructional strategies.

In this final section we consider the question of pedagogical strategies in light of the observations in this paper and related work reported in the literature. Our present work has not yet progressed to the point where we can do anything more than make a few general suggestions. Future studies and their reports will need to discuss actual design and implementation of instructional strategies as well as try to come to conclusions about their effectiveness.

Going through the Action-Process-Object-Schema steps

Once one has an idea of how concepts may be learned, the question becomes how the order of instruction should relate to such a path of cognitive development. We would like to emphasize that we are not suggesting that the concepts considered in this paper be taught in a linear sequence corresponding to the descriptions of development we have offered. The linear nature of our presentation is a convenience we have adopted in producing a written document. We do believe that the paths we have described may generally be followed in the sense that each of the steps are taken and something like the order given can be discerned. However the movement is almost surely not linear. One goal of research should be to find ways to accommodate this nonlinear movement in pedagogy. We consider this issue further in the section, Finding Alternatves to Linear Sequencing. In this section we consider the problem of individual steps.

How can we get students to take a specific step in the development of a given concept? In particular, what methods can be used to help students to interiorize actions, to construct processes, to reverse processes, to coordinate processes, to construct other processes, to encapsulate processes, to construct objects, and to thematize collections of processes and objects into schemas?

In many places, Piaget has emphasized that the major constructions in cognitive development cannot be accelerated by education (Piaget, 1972), but it is possible to enhance the experiential base of students to enrich the developments that do take place (Inhelder et al, 1974). Sfard, 1992 has discussed the exceptional difficulty of getting students to construct objects out of processes and even suggests the possibility that what we are calling encapsulation may be beyond the ability of some students.

On the other hand, we have experienced a certain amount of success in designing computer activities to directly foster students' constructing processes and objects and to help them move along paths suggested by research similar to the present study (Ayres et al, 1988, Breidenbach et al, 1991, Dubinsky, 1986, in press.) Therefore we would propose a similar approach for helping students understand groups.

In the case of thematizing a collection of processes and objects into a general schema, we feel that an essential requirement is that students reflect on the actions they are performing. Working together in cooperative teams, especially when students are talking about how to implement specific mathematical concepts on the computer, can provide opportunities for discussion and reflection.

Misconceptions and Getting Stuck

How can we help students avoid misconceptions and errors such as those discussed in this paper, or at the very least recognize that they have fallen into them?

In our view, the "antidote" to a misconception is a disequilibrating contradiction between a conception and experience. Unfortunately, since one tends to interpret experience according to one's knowledge (including misconceptions), an individual can often appear to ignore such contradictions.

Again, we suggest team work and computer activities. Students seem to be more likely to think seriously about (rather than just accept) contradictions presented by their peers than those presented by their teachers (Vidakovic, 1993). As for the computer, it appears to have much more power to shake up a students' thought processes than does any human.

Regarding the latter, we can offer one piece of anecdotal evidence from a course in elementary discrete mathematics. Students always have difficulty accepting the idea that the boolean value of $P \Longrightarrow Q$ is true when P and Q are both false. Our experience in using computers in Discrete Mathematics (Baxter et al, 1988) is that when such an expression is entered in the computer and it returns the value true, the student is more likely to go through a learning disequilibration/reequilibration cycle than when such an assertion is presented in class.

Meeting Prerequisites

We feel quite firmly that students' conceptions of sets and functions play an important role in learning the group concepts we have considered. We believe that there are several essential requirements here. Students must understand sets, not only as processes (of putting elements together), but as objects which can themselves be elements of sets and can be acted upon by functions. Students must understand functions as processes — in particular, the restriction of the function's process to a subset of its domain — in order to be able to construct the idea of the operation induced by a group on a subset. Students must construct their understanding of properties of functions such as one-to-one and onto in order to work with permutations. Finally, in order to study examples of groups such as permutation groups and automorphism groups, students must be able to encapsulate the process of a function in order to see it as an object.

Students who have not constructed relatively powerful conceptions of sets and functions may have a great deal of trouble with many group concepts. This may suggest that care should be taken to see that these prerequisites are met, either before the course begins, at the beginning of the course, before group theory is studied, or interwined with the early study of group concepts.

Our suggestion of an emphasis on a strong foundation of set and function concepts is based on our interpretations of the observations in this paper and on one mathematical formulation of the group concept. There are, however, other ways of thinking about groups and it could turn out that these suggest alternative emphases for building foundations.

Finding Alternatives to Linear Sequencing

Many years ago, Jerome Bruner proposed, for mathematics curricula, that a spiral curriculum should replace the simple, but simplistic linear sequence in which a complex topic is broken up into a logically coherent sequence of small units. Starting with the proposition that "any topic can be taught in an intellectually honest manner, at any age", Bruner suggested that full-blown mathematical concepts be taught at very early ages, albeit in naive forms. Then, in subsequent years, the topics should be revisited and considered repeatedly at successively higher levels of sophistication (Bruner, 1964). This philosophy was one of the driving principles of the "new math" curriculum.

We believe very strongly that an alternative to linear sequencing must be found. Many students *do* come to abstract algebra courses without strong conceptions of functions and some are even ill-equipped to deal with sets. Perhaps abstract algebra is where they are supposed to strengthen their conceptions of these topics. Just as generals seem to always plan to fight the previous war, students appear to always be learning the material of the previous course. It may not be so bad if we can arrange matters so that a student is no worse than "off by one", throughout her or his school life.

We feel, however, that evidence of how people learn mathematics from this and other studies suggests that Bruner's spiral curriculum may not be the last word on the subject. For example, in all of the various excerpts we have given from the student Kim, one can see that this individual appears to be, during a very short period (just a few days at most) at several different levels of development simultaneously. Neither a linear nor spiral sequence may be appropriate for such a student.

The issue is not just a question of prerequisites. We have seen how students may *simultaneously* be constructing their understanding of groups and subgroups. This makes no sense *logically*. You can't know what a subgroup is until you know what a group is. Perhaps not. Perhaps, one begins with an incomplete understanding of the concept of group, with some solid mathematical knowledge, some misconceptions, and many gaps. Then, in trying to make sense out of examples and properties of subgroups (appearing in an environment constructed by the teacher), the student enhances her or his concept of group. This continues with isomorphism, normality, quotient group, and so on.

One alternative that has been suggested is called *learning by successive refinement*, a method common in software development (Wirth, 1971). In this approach, the student is always dealing with the whole except that he or she goes through a sequence of simplified versions of it, each one a bit more complex and more like the final version than the previous one. The successive refinements can take place along various dimensions such as degree of formality, language, global vs. local, or degree of generality. (See Leron, 1987.)

Another possibilty is the *microworld* which is a software environment that establishes its own complete and consistent environment that represents a non-trivial piece of knowledge and is attractive to the intended audience. (See Papert, 1980.) The use of microworlds does not prescribe the interaction with the learner, but is built into the constraints of the system. The basic metaphor is that children learn by living in this world much as the children observed by Piaget were learning by interacting with the "real" world. They build mental models, make conjectures, try them out, and refine them when they are found wanting.

One might even consider a curricular point of view that is different from all of these and may be seen as something like an *holistic spray.* That is, using computers, and anything else available, students might be thrust into an environment which contains as much as posible about the group concept. The idea is that everything is sprayed at them in an holistic manner. Each individual (or team) tries to make sense out of the situation — that is, they try to do the problems that the teacher askes them to solve, or to answer the questions which the teacher or fellow students ask. In this way the students enhance their understanding of one or another concept bit by bit. They keep coming at it, always trying to make more sense, always learning a little more, and sometimes feeling a great deal of frustration. And it is the role of the teacher, not to eliminate this frustration, but to help students learn to manage it, and use it as a hammer to smash their own ignorance.

It could be that such a curricular design may be more in consonant with how people can learn mathematics. It seems that the data and interpretations in this paper are not inconsistent with such a view. In the future, we hope to report on teaching experiments based on such a philosophy so that we may be able to see if this is a direction in which learning mathematical concepts can be improved.

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Appendix I: Written Assessment Questions

Abstract Algebra Observation, IFSMACSE, July 26, 1991

NAME:__________________________________

We are interested in looking into the kinds of understanding of abstract algebra you have gained from this course so far. The results will be used only for improving our method of teaching, not for grading. Please make an effort to answer the best you can. Thank you.

Question 1

(a) Consider the group $[Z_5 - \{0\}, *_5]$, consisting of the set $\{1, 2, 3, 4\}$ and the operation of multiplication mod 5. Please add one or two sentences explaining your answers to the following questions.

i. What is the identity element of this group?

ii. What is the inverse of 3 in this group?

(b) Consider the group Z_6, with addition mod 6. Give an example of each of the following. Please explain your answers.

i. A subgroup of Z_6 having 2 elements.

ii. A subgroup of Z_6 having 3 elements.

iii. A subset of Z_6 which is not a subgroup.

(c) Is Z_3 a subgroup of Z_6? Please explain your answer.

Question 2

Consider the group D_3 of all the symmetries of an equilateral triangle, with the operation of composition. (Reminder: A symmetry of the triangle is a rigid transformation that maps it onto itself. Composing two such transformations means "doing them one after the other".)

(a) How many elements does D_3 have? Please list them all.

(b) For each of the elements of D_3, please write down its inverse.

(c) Give an example of a subgroup of D_3 (other than the whole of D_3 or just $\{e\}$). What familiar group is this subgroup isomorphic to (i.e., the same except for renaming)?

(d) Can you find a subgroup of D which is isomorphic to Z_2? To Z_3? To Z_4?

Question 3

Consider the group $G = Z_{18}$, its subgroup $H = \langle 3 \rangle = \{0, 3, 6, 9, 12, 15\}$, and the quotient group G/H (this is the group of cosets, written in ISETL as `GmodH`.) The following questions all relate to this particular example.

(a) How many elements does G/H have? Please list them all. (You may wish to give short names to the elements of G/H.)

(b) Write down the operation table of G/H.

(c) What is the identity element of G/H? What is the inverse of the coset of 2?

(d) Find a familiar group which is isomorphic to G/H.

Question 4

Consider again the group D_3 of all the symmetries of an equilateral triangle (from Question 2), with the operation of composition.

(a) Find a normal subgroup N of D_3 and a subgroup H which is not normal. Justify your answers briefly.

(b) Describe what the quotient group of D_3/N looks like, where N is the normal subgroup you found in (a).

Appendix II: Interview Questions

The questions refer to their responses to the Written Assessment questions. Show the interviewee the appropriate response that he or she wrote.

Question 1

This question refers to their response on Question 1 (c) of the Written Assessment. Give the student an opportunity to confirm or change the response.

Is Z_3 a subgroup of Z_6?

If the answer is no *then ask the following question.*

What about the subset $\{0, 2, 4\}$?

If the response to this question is yes *then ask the student about the conflict between this answer and the original answer.*

If the response to the original question was yes *then ask the student for details of the subgroup. This could lead to questions about the induced operation if the subgroup offered has for its set the subset* $\{0, 1, 2\}$.

Question 2

Is Z_6 isomorphic to S_3?

Question 3

This is about the student's response to Question 2. We are interested in whether the student uses geometrical transformations of an equilateral triangle or analytic formulas for permutations of three objects in doing this problem. Whichever the student has used, ask about the other method.

Which of these two methods do you think is better? Why?

Question 4

This is about the student's response to Question 3(b) on the Written Assessment.

(a) Explain how you constructed the table.

(b) In particular, how did you get your answer to $(1+H)(1+H)$, $(1+H)(2+H)$, $(2+H)(0+H)$?

(c) Is H a normal subgroup?

Question 5

This is about the student's response on Question 4 of the Written Assessment. If the student did not have much success with part (a), give her or him an opportunity to try it again. If the student is unable to find a normal subgroup, the interviewer should provide one, and let the student proceed with part (b). The following questions/suggestions could be given as prompts if necessary.

(a) How many elements does D_3/N have?

(b) List the elements in D_3/N.

(c) Form the multiplication table for D_3/N.

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