

Computational Physics

Project 2

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Abstract

The goal of this project was to solve Schrödinger's equation to find the wave function of a system composed by an electron in a harmonic oscillator potential. The task could be written as a eigenvalue problem, so the implementation needed to solve the equation turned out to be the Jacobi method (or also known as Given's rotations) to diagonalize a matrix containing the discretization of the equation at different integration points. Once the matrix was written in a diagonal form I simply had to look at the lowest eigenstates and the relative eigenvectors, which gave the energy of the particle and the wave function respectively. We also had a closed form equation for the lowest eigenvalues to test our results with.

The second part of the project consisted in a similar problem, except that two interacting electron were to be considered inside the harmonic oscillator potential. By rewriting Schrödinger's equation in terms of the center of mass it turned out that very few changes to the previous problem were needed, namely the potential function.

On the programming side, the goal of the project was to implement and test diagonalization algorithms and to develop some reproducible tests and benchmarks for our algorithm, to show that it is working properly.

Analysis Of The Problem

The system we are considering is a 3-dimensional harmonic oscillator potential with spherical symmetry that acts on the electrons. The two particles interact with Coulomb's force. We will assume no angular components for the particles, only radial movement.

One Particle System Let's first consider the system with only one particle, the radial Schrödinger's equation is then:

$$-\frac{\hbar^2}{2m} \left(\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \right) \varphi(r) + V(r) \varphi(r) = E \varphi(r). \quad (1)$$

In this equation $\varphi(r)$ is the time independent wave function (note that since we are in spherical coordinates $r \in [0, \infty)$), the term $\left(\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \right) \varphi(r)$ is the spatial second order derivative written in spherical coordinates, $V(r)$ is our potential, in this case it's the harmonic oscillator potential, so $V(r) = \frac{1}{2} k r^2$ with $k = m\omega^2$ and E is the energy of the harmonic oscillator in three dimensions. We have a closed form for the energy $E(r)$ that are eigenstates of the above wave function.

$$E_n = \hbar\omega \left(2n + \frac{3}{2} \right),$$

with $n = 0, 1, 2, \dots$. To simplify equation 1 and to make the discretization easier we put $u(r) = r \varphi(r)$, this will also give boundary conditions that are easier to handle $u(0) = 0$ and $u(\infty) = 0$. So now the equation reads:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} u(r) + V(r) u(r) = E u(r). \quad (2)$$

The boundary conditions are $u(0) = 0$ and $u(\infty) = 0$. With the aim of getting to a dimensionless form for Schrödinger's equation we now introduce a dimensionless variable $\rho = (1/\alpha)r$ where α is a constant with dimension length and get:

$$-\frac{\hbar^2}{2m\alpha^2} \frac{d^2}{d\rho^2} u(\rho) + \frac{k}{2} \alpha^2 \rho^2 u(\rho) = E u(\rho). \quad (3)$$

We then multiply with $2m\alpha^2/\hbar^2$ on both sides and obtain

$$-\frac{d^2}{d\rho^2}u(\rho) + \frac{mk}{\hbar^2}\alpha^4\rho^2u(\rho) = \frac{2m\alpha^2}{\hbar^2}Eu(\rho). \quad (4)$$

The last step is to fix the constant α with a value that makes the potential coefficient equal to 1. So the choice is:

$$\alpha = \left(\frac{\hbar^2}{mk}\right)^{1/4}.$$

This simplification to a dimensionless equation however changed the energy eigenvalues of our original wave function. If we define

$$\lambda = \frac{2m\alpha^2}{\hbar^2}E,$$

we can rewrite Schrödinger's equation as

$$-\frac{d^2}{d\rho^2}u(\rho) + \rho^2u(\rho) = \lambda u(\rho). \quad (5)$$

This is the first equation to solve numerically. In three dimensions we still have a closed form for the eigenvalues: $\lambda_n = 3 + 4n$.

Two Particle System We will now study the system made up with two electrons in a harmonic oscillator well interacting with each other via a repulsive Coulomb interaction. If we consider no repulsive Coulomb interaction, we have the following Schrödinger equation in spherical coordinates, after having done some simplification as for the one particle system like setting $\psi(r_1, r_2) = r \varphi(r_1, r_2)$.

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr_1^2} - \frac{\hbar^2}{2m}\frac{d^2}{dr_2^2} + \frac{1}{2}kr_1^2 + \frac{1}{2}kr_2^2\right)\psi(r_1, r_2) = E\psi(r_1, r_2) \quad (6)$$

Note that we deal with a two-electron wave function $\psi(r_1, r_2)$ and two-electron energy E . This equation can be separated in two, by considering the relative coordinate $r = r_1 - r_2$ and the center-of-mass coordinate $R = 1/2(r_1 + r_2)$. The radial Schrödinger equation with the new coordinates now is:

$$\left(-\frac{\hbar^2}{m}\frac{d^2}{dr^2} - \frac{\hbar^2}{4m}\frac{d^2}{dR^2} + \frac{1}{4}kr^2 + kR^2\right)\psi(r, R) = E^{(2)}\psi(r, R). \quad (7)$$

We now make the ansatz for the wave function $\psi(r, R) = u(r)\phi(R)$ so we end up having two separated wave functions, the which energies sum up in the end. If we now add a Coulomb potential of the form $V(r_1, r_2) = \frac{\beta e^2}{|r_1 - r_2|} = \frac{\beta e^2}{r}$ only the relative equation will be changed, the center of mass will remain unchanged. The r -dependent Schrödinger equation becomes

$$\left(-\frac{\hbar^2}{m}\frac{d^2}{dr^2} + \frac{1}{4}kr^2 + \frac{\beta e^2}{r}\right)u(r) = E_r u(r). \quad (8)$$

We can immediately see that this equation is very similar to the one we had for the one particle system, so we proceed in simplifying the equation as before by introducing dimensionless variable $\rho = r/\alpha$.

$$-\frac{d^2}{d\rho^2}u(\rho) + \frac{1}{4}\frac{mk}{\hbar^2}\alpha^4\rho^2u(\rho) + \frac{m\alpha\beta e^2}{\rho\hbar^2}u(\rho) = \frac{m\alpha^2}{\hbar^2}E_r u(\rho). \quad (9)$$

We now define

$$\omega^2 = \frac{1}{4}\frac{mk}{\hbar^2}\alpha^4 \quad \alpha = \frac{\hbar^2}{m\beta e^2} \quad \lambda = \frac{m\alpha^2}{\hbar^2}E$$

The parameter ω gives us an idea of the strength of the oscillator potential, λ is the eigenvalue of the equation. We can now rewrite Schrödinger's equation in a very compact way:

$$-\frac{d^2}{d\rho^2}u(\rho) + \left(\omega^2\rho^2 + \frac{1}{\rho}\right)u(\rho) = \lambda u(\rho). \quad (10)$$

This analysis brought us to have two dimensionless differential equations. These represent two distinct systems and eigenvalue problems, however since their form is almost identical they can be solved with the same algorithm and the changes to the code required to compute the 2 particle system are really trivial.

Jacobi Method For Diagonalization

All the next steps are applicable to both equation 5 or 10, the only change will be the term of the potential. This type of problems can be written very easily in a matrix form.

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

Where \mathbf{u} is the eigenvector representing the wave function corresponding to its eigenvalue energy level, and \mathbf{A} is the matrix containing the discretized Schrödinger's equation. The first thing to do in order to compute the eigenvector of the equation is to discretize the second derivative in the standard way:

$$u'' = \frac{u(\rho + h) - 2u(\rho) + u(\rho - h)}{h^2} + O(h^2), \quad (11)$$

h will be our step-length. Since the domain is definend from 0 to $+\infty$ we need to set a maximum value for ρ . Since all the wave funnctions go to zero quite fast if we only want the first three eigenvalues a maximum value of $\rho_{max} = 7$ is enough. Since we require our matrix to be of a fixed size n_{step} our step length h is fixed by $h = \frac{\rho_{max}}{n_{step}}$. We can now write the Schrödinger equation in a discretized form (for simplicity we put $u_i = u(ih)$):

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + V_i u_i = \lambda u_i, \quad (12)$$

where V_i is the harmonic oscillator potential either alone or with the addition of the Coulomb potential term. If we trite out the matrix we can also add in the potential terms. The matrix is now a tridiagonal matrix, and the equation can be written:

$$\begin{pmatrix} \frac{2}{h^2} + V_1 & -\frac{1}{h^2} & 0 & 0 & \dots & 0 & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} + V_2 & -\frac{1}{h^2} & 0 & \dots & 0 & 0 \\ 0 & -\frac{1}{h^2} & \frac{2}{h^2} + V_3 & -\frac{1}{h^2} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \frac{2}{h^2} + V_{n_{step}-2} & -\frac{1}{h^2} \\ 0 & \dots & \dots & \dots & \dots & -\frac{1}{h^2} & \frac{2}{h^2} + V_{n_{step}-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ \dots \\ \dots \\ u_{n_{step}-1} \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \\ \dots \\ \dots \\ \dots \\ u_{n_{step}-1} \end{pmatrix}$$

An eigen value problem is very easy to solve if the matrix we are considering is diagonal. The elements on the diagonal are the the eigenvalue itself. The main idea is then to apply a sequence of rotations on the matrix in order to make it become diagonal.

$$(\mathbf{S}^T \mathbf{A} \mathbf{S})(\mathbf{S}^T) \mathbf{x} = \lambda (\mathbf{S}^T) \mathbf{x} \quad (13)$$

The matrix \mathbf{S} is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \cos(\theta) & \dots & \sin(\theta) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\sin(\theta) & \dots & \cos(\theta) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix}$$

The transformation is chosen by first searching through the matrix to find the largest non diagonal elements, then the angle θ is computed to make those elements become 0. Then the triple matrix product is computed. Since the transformation matrix is very simple a simple multiplication scheme can be derived. If we call $b_{i,j}$ the elements of the transformed matrix and let k, l be the indexes of the largest non diagonal element:

$$\begin{aligned} b_{ii} &= a_{ii} \quad \text{for } i \neq k, i \neq l \\ b_{ik} &= a_{ik} \cos \theta - a_{il} \sin \theta \\ b_{il} &= a_{il} \cos \theta + a_{ik} \sin \theta \\ b_{kk} &= a_{kk} \cos^2 \theta - 2a_{kl} \cos \theta \sin \theta + a_{ll} \sin^2 \theta \\ b_{ll} &= a_{ll} \cos^2 \theta + 2a_{kl} \cos \theta \sin \theta + a_{kk} \sin^2 \theta \\ b_{kl} &= (a_{kk} - a_{ll}) \cos \theta \sin \theta + a_{kl} (\cos^2 \theta - \sin^2 \theta) \end{aligned} \quad (14)$$

The last equation is the one that has to be set to zero to find the angle θ . It can be rewritten as $t^2 + 2\tau - 1 = 0$ where $t = \tan \theta$ and $\tau = \frac{a_{ll} - a_{kk}}{2a_{kl}}$. From the tangent we can easily get the cosine $\cos \theta = \frac{1}{\sqrt{1+t^2}}$ and sine

$\sin\theta = \tan\theta\cos\theta$. After every rotation we must rotate also the eigenvectors. From equation 13 we easily see how, if we start with an orthonormal basis of vectors, that could be for example the identity matrix if we consider every column as a vector, by applying these rotations we get to the eigenvectors we are looking for. The explicit for of these rotations reads:

$$\begin{aligned} u'_{ik} &= u_{ik}\cos\theta - u_{il}\sin\theta \\ u'_{il} &= u_{il}\cos\theta + u_{ik}\sin\theta \end{aligned} \quad (15)$$

To make the algorithm stop we must impose a threshold for the sum of all non diagonal matrix elements. We will set it to 10^{-8} which means that the matrix is not really diagonal, but all the elements non on the diagonal are very close to zero.

We have now a very easy to implement algorithm that I'll summarize here:

- choose a mesh size for the matrix n_{steps} , the maximum point of integration ρ_{max} and compute h
- create the starting $n \times n$ matrix containing the Schrödinger's equation and the identity matrix for the eigenvectors.
- find the largest non-diagonal element of the matrix a_{kl}
- compute the angle θ to set $a_{kl} = 0$
- apply the rotation scheme to the matrix
- apply the rotation scheme to the eigenvectors
- calculate the sum of all non diagonal methods and check if it's smaller then the required tolerance.

Implementation Tests

I wrote a `C++` code to solve this problem¹ and developed a function that given a certain matrix it returned it's diagonalization. While developing the code some tests were engineered to make sure that the program was working properly. The first test consisted in giving as an argument for the function a diagonal matrix. The function should then exit without doing anything.

Another test was to set up a very simple 2×2 matrix. This should be diagonalized in just one iteration by definition. If these requirements were not met, then it meant that the program wasn't working properly, and exited before starting doing the real problem calculation. As a further test we had the closed form for the eigenvalues of the one particle system to compare our results with, if some other values appeared, then there was an error in our code.

Numerical Results

After the algorithm had been implemented I ran some tests to find the three lowest energy wave functions on both systems. Here are the plots for the single electron system:

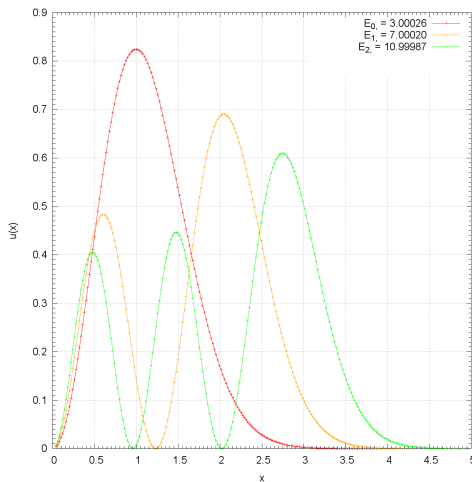


Figure 1: One particle system. Plot of the function $|u(x)|^2$ for the first three eigenvalues.

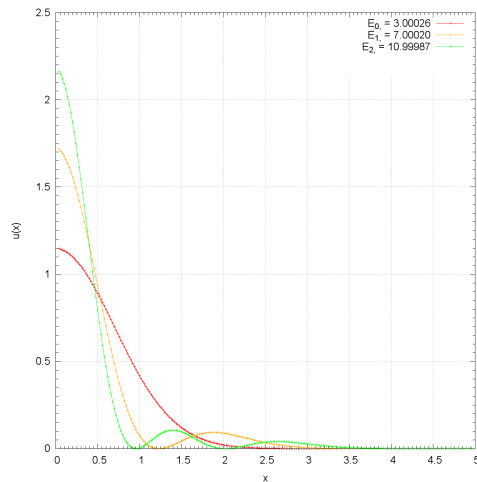


Figure 2: One particle system. Plot of the function $|\psi(x)|^2 = |u(x)|^2 / \rho^2$ for the first three eigenvalues

¹The full code can be found on GitHub <https://github.com/GioPede/FYS3150/blob/master/Project2/main.cpp>

And the ones for the two particle system with $\omega = 0.1$ for the relative distance only:

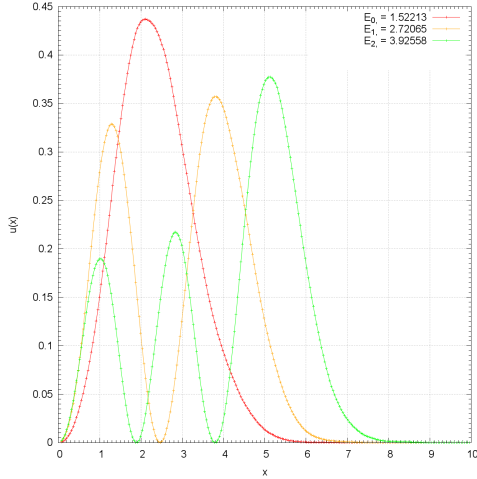


Figure 3: Two particle system. Plot of the function $|u(x)|^2$ for the first three eigenvalues.

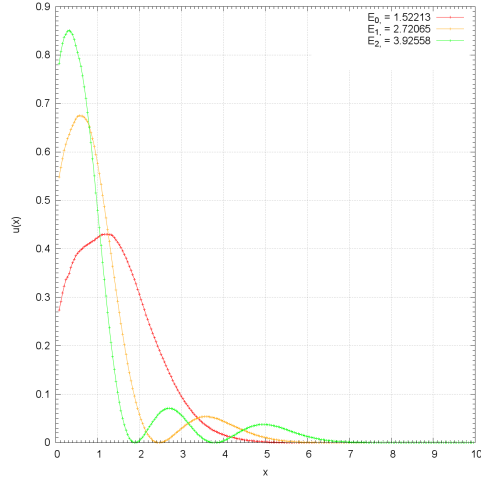


Figure 4: Two particle system. Plot of the function $|\psi(x)|^2 = |u(x)|^2 / \rho^2$ for the first three eigenvalues

The first two plots show also the values of the eigenvalues of which we had a closed form. They are equal to the ones that we were expecting with four leading digits:

	Expected	Numerical
E_0	3	3.00026
E_1	7	7.00020
E_2	11	10.99987

Figure 5: Expected eigenvalues and numerically found ones.

To achieve this precision however it was necessary to set up a matrix of dimension 250 in the interval $(0, 5)$ and due to its dimension it required approximately 40000 iterations, or in seconds $\simeq 2.5s$. This shows how inefficient this method is, especially because we had a very simple matrix as a tridiagonal one, so a better and specific algorithm could have been used, like Francis' algorithm or the QL algorithm with implicit shifts found on "Numerical Recipes" by William H. Press (function name `tqli`). I made an implementation of the latter and compared the number of total iterations along the matrix of the two methods. Here are the results:

Matrix Dimension	Jacobi Method	<code>tqli</code>
100	7052	12474
150	16905	27893
200	31015	48997
250	49559	75149

Figure 6: Total matrix iterations of the two methods related to the matrix dimension.

There is a clear difference between the two methods and the most of the difference is in the fact that the Jacobi method requires to look through the entire matrix at every step to look for the largest non-diagonal element, while the QL algorithm doesn't. This decreases the number of rotations needed but since each iteration requires approximately $2n$ operations for both methods and a single matrix sweep to look for the largest elements requires n^2 operations, the `tqli` is still much faster. For the same dimensionality of 250 it took for example 5.62586 seconds for the Jacobi method and 0.44235s for the QL algorithm; this algorithm took 5 seconds to diagonalize a matrix of size 1000, four times larger than the one other method could do in the same time. The Lanczos' method would be even faster at finding the eigenvalues, but it does not give any easy-to-implement algorithm for eigenvectors.

As a last test I ran a series of simulations for the two-particle system in which the parameter ω changed between the values 0.01, 0.5, 1 and 5. Here's a plot of the different wave functions.

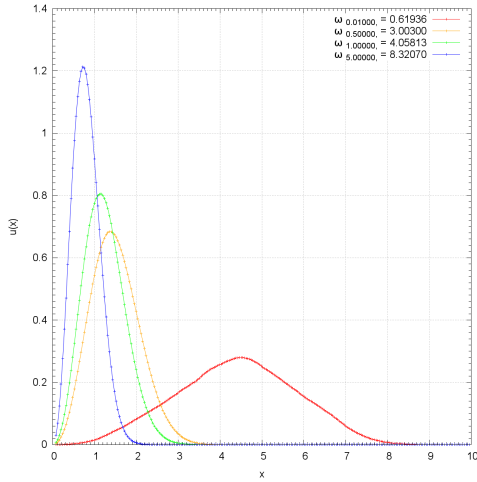


Figure 7: $|u(x)|^2$ for the two particle system with different values of ω

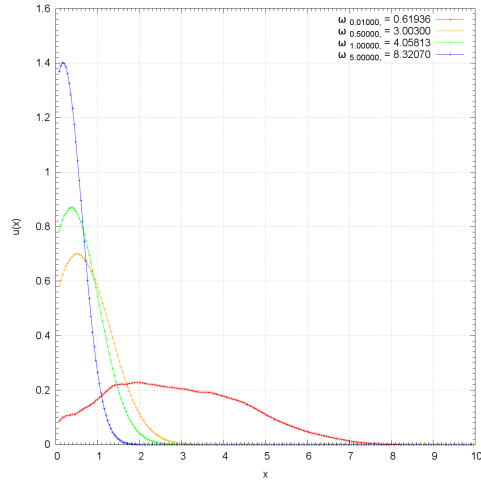


Figure 8: $|\psi(x)|^2$ for the two particle system with different values of ω

As expected if the harmonic oscillator is very weak if compared to the Coulomb interaction then the particle tend to be further apart, but if it gets stronger then the wave function tends to the one of the one particle system. The strangeness in slope for the $|\psi(x)|^2$ for $\omega = 0.01$ can be due to the roundoff errors and machine error.

Conclusions

This project showed that some quantum mechanical systems can be solved numerically by using diagonalization algorithms. The easiest to implement is the Jacobi algorithm which is very stable but requires a lot of computation time. The algorithm `tqli` found in literature applies more similarity transformation, but is much faster in computation time.

The application of these methods to Schrödinger equation in the case of a single electron in a three dimensional harmonic oscillator will give us the expected results, as it did for the two electron system when changing the value ω . The behavior of the probability density function which has been found is the one we expected, also in comparison with its corresponding one without the Coulomb interaction.