<4장 의문점>

1. Markov's Inequality

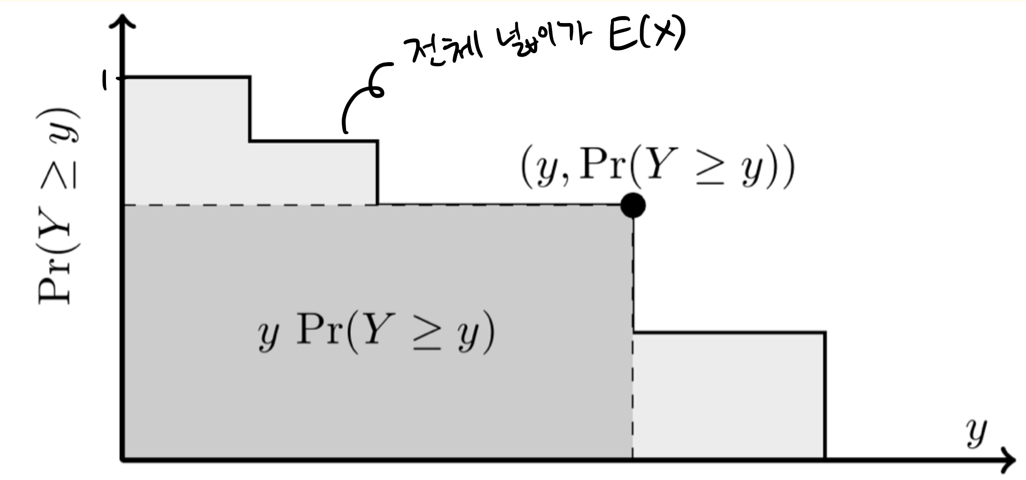
4.1 Theorem (Markov's inequality). *Let X be a non-negative random variable and suppose that $\mathbb{E}(X)$ exists. For any $t > 0$,* $x > t$

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}. \tag{4.1}$$

$$E(X) = \int_0^\infty x \cdot p(x)\,dx \geq \int_t^\infty x \cdot p(x)\,dx \geq \int_t^\infty t \cdot p(x)\,dx = t\int_t^\infty p(x)\,dx$$

$$E(X) \geq t \cdot \int_t^\infty p(x)\,dx = t \cdot P(X > t)$$

$$\Rightarrow P(X > t) \leq \frac{E(X)}{t}$$



https://cheng2zhang.wordpress.com/2015/12/02/markovs-inequality/

2. Chebyshev's Inequality

4.2 Theorem (Chebyshev's inequality). *Let $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathbb{V}(X)$. Then,*

$$\mathbb{P}(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2} \quad \text{and} \quad \mathbb{P}(|Z| \geq k) \leq \frac{1}{k^2} \tag{4.2}$$

where $Z = (X-\mu)/\sigma$. In particular, $\mathbb{P}(|Z| > 2) \leq 1/4$ and $\mathbb{P}(|Z| > 3) \leq 1/9$.

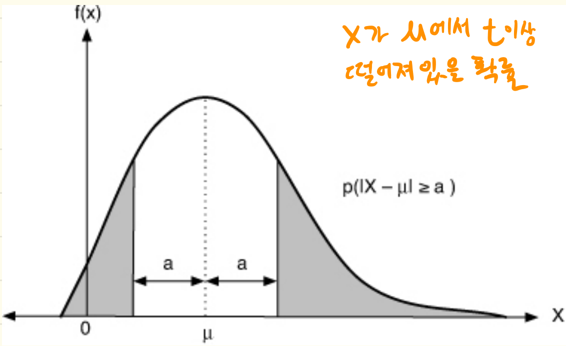
우선 $p(x \geq t) \leq \frac{E(x)}{t}$ 니까 $p(x-\mu \geq t) \leq \frac{E(x-\mu)}{t}$

분산의 정의

$$p(|x-\mu| \geq t) = p(|x-\mu|^2 \geq t^2) \leq \frac{E(|x-\mu|^2)}{t^2} = \frac{\sigma^2}{t^2}$$

$$\Rightarrow p(|x-\mu| \geq t) = \frac{\sigma^2}{t^2}$$

또한 $p\left(\frac{|x-\mu|}{\sigma} \geq t\right) \leq \frac{E\left(\frac{|x-\mu|^2}{\sigma^2}\right)}{t^2} = \frac{1}{t^2}$



여태까지 이런걸 측정하는 분포들은 다 Gaussian 이었는데 신기하군

http://www.informit.com/articles/article.aspx?p=1863432&seqNum=7

3. Hoeffding's Inequality

4.4 Theorem (Hoeffding's Inequality). *Let $Y_1, \ldots, Y_n$ be independent observations such that* t는 무슨 의미를 가지지?
$\mathbb{E}(Y_i) = 0$ *and* $a_i \leq Y_i \leq b_i$. *Let* $\epsilon > 0$. *Then, for any* $t > 0$, 음?!

$$\mathbb{P}\left(\sum_{i=1}^{n} Y_i \geq \epsilon\right) \leq e^{-t\epsilon} \prod_{i=1}^{n} e^{t^2(b_i - a_i)^2/8}. \tag{4.3}$$

4. Mill's Inequality

4.7 Theorem (Mill's Inequality). *Let $Z \sim N(0,1)$. Then,*

$$\mathbb{P}(|Z| > t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}.$$

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For $Z \sim \mathcal{N}(0,1)$ (i.e. a standard Normal distirbution) and $t > 0$ we have

$P(|Z| > t) = P(Z^- < -t) + P(Z^+ > t)$

where Z^- are negative values of Z and Z^+ are positive values of Z.

Now, the distribution of Z is symmetric around 0, so $P(Z^- < -t) = P(Z^+ > t)$, leading to

$P(|Z| > t) = 2P(Z^+ > t)$

As part of the inequalities involved in proving Markov's inequality (see third slide in here http://www.math.leidenuniv.nl/~gugushvilis/STAN4.pdf), we have (for probability density function $f(x)$)

$\int_t^\infty x f(x) dx \geq t \int_t^\infty f(x) dx = tP(x > t)$

substituting the values with our standard Normal distribution, we have

$\frac{1}{\sqrt{2\pi}} \int_t^\infty x e^{-\frac{x^2}{2}} dx \geq \frac{t}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{x^2}{2}} dx = tP(Z^+ > t) = \frac{t}{2} P(|Z| > t)$

Integrating the left hand side results in

$\frac{1}{\sqrt{2\pi}} \int_t^\infty x e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} [-e^{-\frac{x^2}{2}}]_t^\infty = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

Therefore

$\frac{t}{2} P(|Z| > t) \leq \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

나중에 혼자 해보기

leading to

$P(|Z| > t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{t^2}{2}}}{t}$

5. Cauchy-Schwarz Inequality

4.8 Theorem (Cauchy-Schwartz inequality). *If X and Y have finite variances then*

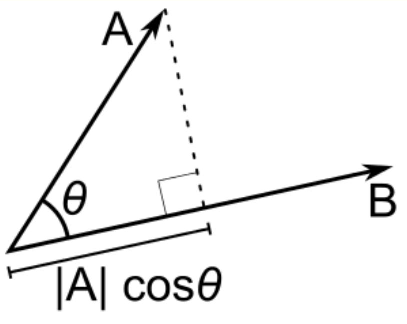
$$\mathbb{E}|XY| \le \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}. \tag{4.5}$$

벡터로 생각하면 Non-zero vector $\vec{x}, \vec{y} \in R^n$

$$|\vec{x} \cdot \vec{y}| \le \|\vec{x}\| \times \|\vec{y}\|$$

$\|\vec{x}\|$는 벡터의 길이

dot product   length × length



근데 Expectation 상으로는 감이 잘 안오네..

$$V(X) = E[X^2] - E[X]^2$$
$$E[X^2] = V(X) + E[X]^2$$

$$E[|XY|] \le \sqrt{E(X^2)E(Y^2)}$$

$$E[X^2Y^2] \le E[X^2]E[Y^2]$$

6. Jensen's Inequality

https://youtu.be/10xgmpG_uTs