<5장 의문점>

1. Convergence (수렴의 종류와 의미)

5.1 Definition. *Let $X_1, X_2, \ldots$ be a sequence of random variables and let X be another random variable. Let F_n denote the* CDF *of X_n and let F denote the* CDF *of X.*

꼭 이렇게 정의해야하나? 둘의 차이는 뭐지?

1. *X_n* **converges to** *X* **in probability**, *written $\boxed{X_n \xrightarrow{\text{P}} X}$, if, for every $\epsilon > 0$,*

$$\mathbb{P}(|X_n - X| > \epsilon) \to 0 \tag{5.1}$$

P(두값의 차이>1) → 0

〃 0.1 → 0

〃 0.01 → 0

두 값의 차이가 0에 가까워지면 ε보다 큰 확률이 0으로 수렴.

as $n \to \infty$.

2. *X_n* **converges to** *X* **in distribution**, *written $\boxed{X_n \rightsquigarrow X}$ if*

$$\lim_{n \to \infty} F_n(t) = F(t) \tag{5.2}$$

at all t for which F is continuous. 모든지점 t에 대해 CDF가 일치할 때

5.2 Definition. *X_n* **converges to** *X* **in quadratic mean** *(also called convergence in L_2), written $\boxed{X_n \xrightarrow{\text{qm}} X}$, if* 전체에 대한 Expectation?

$$\mathbb{E}(X_n - X)^2 \to 0 \tag{5.3}$$

as $n \to \infty$.

5.7.1 *Almost Sure and L_1 Convergence*

We say that X_n **converges almost surely to** X, written $X_n \xrightarrow{\text{as}} X$, if

$$\mathbb{P}(\{s : X_n(s) \to X(s)\}) = 1.$$ 모든 n에 대해 성립할 확률이 1인건가

We say that X_n **converges in** L_1 **to** X, written $X_n \xrightarrow{L_1} X$, if

$$\mathbb{E}|X_n - X| \to 0$$

as $n \to \infty$.

5.17 Theorem. *Let X_n and X be random variables. Then:*

(a) $X_n \xrightarrow{\text{as}} X$ *implies that* $X_n \xrightarrow{\text{P}} X$.

(b) $X_n \xrightarrow{\text{qm}} X$ *implies that* $X_n \xrightarrow{L_1} X$.

(c) $X_n \xrightarrow{L_1} X$ *implies that* $X_n \xrightarrow{\text{P}} X$.

almost surely > prob.

$L_2 > L_1 >$ prob.

2. Berry-Essèen's Inequality

5.11 Theorem (The Berry-Essèen Inequality). *Suppose that* $\mathbb{E}|X_1|^3 < \infty$. *Then*

$$\sup_z |\mathbb{P}(Z_n \le z) - \Phi(z)| \le \frac{33}{4} \frac{\mathbb{E}|X_1 - \mu|^3}{\sqrt{n}\sigma^3}. \tag{5.4}$$

http://individual.utoronto.ca/jordanbell/notes/berry-esseen.pdf

↳ 내가 건드릴 영역은 아닌듯 하다.

3. Delta Method

5.13 Theorem (The Delta Method). *Suppose that*

$$\frac{\sqrt{n}(Y_n - \mu)}{\sigma} \rightsquigarrow N(0,1)$$

and that g is a differentiable function such that $g'(\mu) \neq 0$. Then

$$\frac{\sqrt{n}(g(Y_n) - g(\mu))}{|g'(\mu)|\sigma} \rightsquigarrow N(0,1).$$

In other words,

$$Y_n \approx N\left(\mu, \frac{\sigma^2}{n}\right) \quad \text{implies that} \quad g(Y_n) \approx N\left(g(\mu), (g'(\mu))^2 \frac{\sigma^2}{n}\right).$$

Y_n과 μ 사이에 $\hat{\mu}$이 존재한다고 가정하고 $Y_n < \hat{\mu} < \mu$
g가 continuous 하다고 가정하면

$$g'(\hat{\mu}) = \frac{g(Y_n) - g(\mu)}{Y_n - \mu} \Rightarrow g(Y_n) = g(\mu) + g'(\hat{\mu})(Y_n - \mu)$$

$Y_n \xrightarrow{P} \mu$ 이면 $\hat{\mu} \xrightarrow{P} \mu$. Since g' is continuous, $g'(\mu) \xrightarrow{P} g'(\hat{\mu})$

$$g(Y_n) - g(\mu) = g'(\mu)(Y_n - \mu)$$

양변에 $\times \frac{\sqrt{n}}{\sigma}$

$$\frac{\sqrt{n}(g(Y_n) - g(\mu))}{\sigma} = g'(\mu)\frac{\sqrt{n}(Y_n - \mu)}{\sigma}$$

$$\frac{\sqrt{n}(Y_n - \mu)}{\sigma} \rightsquigarrow N(0,1)$$

Hence, $$\frac{\sqrt{n}(g(Y_n) - g(\mu))}{g'(\mu) \times \sigma} \rightsquigarrow N(0,1)$$

4. Strong Law of Large Numbers

5.18 Theorem (The Strong Law of Large Numbers). *Let* $X_1, X_2, \ldots$ *be* IID. *If* $\mu = \mathbb{E}|X_1| < \infty$ *then* $\overline{X}_n \xrightarrow{\text{as}} \mu$.

정근적으로

5.6 Theorem (The Weak Law of Large Numbers (WLLN)). [3]
If $X_1, \ldots, X_n$ *are* IID, *then* $\overline{X}_n \xrightarrow{\text{P}} \mu$.

A sequence X_n is **asymptotically uniformly integrable** if

$$\lim_{M\to\infty} \limsup_{n\to\infty} \mathbb{E}\left(|X_n| I(|X_n| > M)\right) = 0.$$

5. Central Limit Theorem 증명

MGF : $\psi_X(t) = \mathbb{E}e^{tX}$.

5.8 Theorem (The Central Limit Theorem (CLT)). *Let $X_1, \ldots, X_n$ be IID with mean μ and variance σ^2. Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$. Then*

$$Z_n \equiv \frac{\overline{X}_n - \mu}{\sqrt{\mathbb{V}(\overline{X}_n)}} = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \rightsquigarrow Z$$

$\mathbb{V}(\overline{X}_n) = \frac{\sigma^2}{n}$

$\frac{\frac{\overline{X}-\mu}{\sigma}}{\sqrt{n}} = \frac{\sqrt{n}(\overline{X}_n-\mu)}{\sigma}$

where $Z \sim N(0,1)$. In other words,

$$\lim_{n\to\infty} \mathbb{P}(Z_n \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

$\overline{X_n} = \frac{1}{n}\sum_{i=1}^n X_i$ $\qquad X_i \overset{iid}{\sim} (\mu, \sigma^2)$

$E[\overline{X_n}] = \frac{1}{n}\sum_{i=1}^n E[X_i] = \frac{1}{n} \times n \times \mu = \mu$

$Var(\overline{X_n}) = Var(\frac{1}{n}\sum_{i=1}^n X_i) = \frac{1}{n^2} Var(\sum_{i=1}^n X_i) = \frac{1}{n^2} \times n \times \sigma^2 = \frac{\sigma^2}{n}$

$Z_n = \frac{\sqrt{n}(\overline{X_n} - \mu)}{\sigma}$ 라고 했을 때 μ, σ^2는?

$E[Z_n] = 0$

$Var(Z_n) = \frac{n}{\sigma^2} Var(\overline{X_n} - \mu) = \frac{n}{\sigma^2} \times \frac{\sigma^2}{n} = 1 = E[Z_n^2] - E[Z_n]^2$

3.33 Theorem. *Let X and Y be random variables. If $\psi_X(t) = \psi_Y(t)$ for all t in an open interval around 0, then $X \overset{d}{=} Y$.*

↳ MGF가 같으면 같은 Distribution. ⇒ Z_n의 MGF가 Normal distribution의 MGF와 같은지 확인해보자

우선 MGF of unit Gaussian $\qquad X = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$E[e^{tX}] = \int e^{tx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$= \int \frac{1}{\sqrt{2\pi}} e^{(tx - \frac{1}{2}x^2)} dx = \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} \cdot e^{\frac{1}{2}t^2} dx = e^{\frac{1}{2}t^2}$

그렇다면 이번엔 Z_n 의 MGF를 구해보자.

$\psi_X(t) = \mathbb{E}e^{tX}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$ → Taylor Series of Exponential

3.31 Lemma. *Properties of the* MGF.

(1) If $Y = aX + b$, then $\psi_Y(t) = e^{bt}\psi_X(at)$.

(2) If $X_1, \ldots, X_n$ are independent and $Y = \sum_i X_i$, then $\psi_Y(t) = \prod_i \psi_i(t)$ where ψ_i is the MGF *of X_i.*

$$Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} = \frac{\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)\right)}{\sigma} = \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu)}{\sigma} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)$$

PROOF OF THE CENTRAL LIMIT THEOREM. Let $Y_i = (X_i - \mu)/\sigma$. Then, $Z_n = n^{-1/2}\sum_i Y_i$. Let $\psi(t)$ be the MGF of Y_i. The MGF of $\sum_i Y_i$ is $(\psi(t))^n$ 왜? and MGF of Z_n is $[\psi(t/\sqrt{n})]^n \equiv \xi_n(t)$. Now $\psi'(0) = \mathbb{E}(Y_1) = 0$, $\psi''(0) = \mathbb{E}(Y_1^2) = \mathbb{V}(Y_1) = 1$. So,

왜 1이지? e^0 이니까

$$Z_n = \frac{\sum_i Y_i}{\sqrt{n}}$$

$$\begin{aligned}
\psi(t) &= \psi(0) + t\psi'(0) + \frac{t^2}{2!}\psi''(0) + \frac{t^3}{3!}\psi'''(0) + \cdots \\
&= 1 + 0 + \frac{t^2}{2} + \frac{t^3}{3!}\psi'''(0) + \cdots \\
&= 1 + \frac{t^2}{2} + \frac{t^3}{3!}\psi'''(0) + \cdots
\end{aligned}$$

Now,

$$\begin{aligned}
\xi_n(t) &= \left[\psi\left(\frac{t}{\sqrt{n}}\right)\right]^n \\
&= \left[1 + \frac{t^2}{2n} + \frac{t^3}{3!n^{3/2}}\psi'''(0) + \cdots\right]^n \\
&= \left[1 + \frac{\frac{t^2}{2} + \frac{t^3}{3!n^{1/2}}\psi'''(0) + \cdots}{n}\right]^n \\
&\to e^{t^2/2}
\end{aligned}$$

unit gaussian과 일치

$$\left(\because \left(1 + \frac{a_n}{n}\right)^n \to e^a.\right)$$