〈4장 문제〉 2, 5, 7

2. Let $X \sim \text{Poisson}(\lambda)$. Use Chebyshev's inequality to show that $\mathbb{P}(X \geq 2\lambda) \leq 1/\lambda$.

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$\mathbb{P}(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2} \quad \text{and} \quad \mathbb{P}(|Z| \geq k) \leq \frac{1}{k^2}$$

$\mu = \lambda$

양변에 $-\lambda$ 하자

$\sigma^2 = \lambda$ 일때, $P(X \geq 2\lambda) \leq 1/\lambda$ 인가?

$$P(|X-\mu| \geq \lambda) \leq \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

5. Prove Mill's inequality, Theorem 4.7. Hint. Note that $\mathbb{P}(|Z| > t) = 2\mathbb{P}(Z > t)$. Now write out what $\mathbb{P}(Z > t)$ means and note that $x/t > 1$ whenever $x > t$.

4.7 Theorem (Mill's Inequality). *Let $Z \sim N(0,1)$. Then,*

$$\mathbb{P}(|Z| > t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}.$$

$$P(|Z| > t) = 2P(Z > t)$$

cf) https://www.cymath.com/answer?q=integrate%20xe%5E(-x%5E2%2F2)

$$= \frac{2}{t} \times t \times P(Z > t)$$

$$= \frac{2}{t} \times t \times \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq \frac{2}{t} \times \int_t^\infty x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{t} \times \int_t^\infty x \cdot e^{-x^2/2}$$

검색해봄.

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{t} \times \left. -\frac{1}{e^{\frac{x^2}{2}}} \right|_t^\infty \qquad \left. -e^{-\frac{x^2}{2}} \right|_t^\infty = 0 - (-e^{-\frac{t^2}{2}}) = e^{-\frac{t^2}{2}}$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{1}{t} \times e^{-\frac{t^2}{2}} = \sqrt{\frac{2}{\pi}} \frac{e^{-t^2/2}}{t}$$

sample mean

7. Let $X_1, \ldots, X_n \sim N(0,1)$. Bound $\mathbb{P}(|\overline{X}_n| > t)$ using Mill's inequality, where $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$. Compare to the Chebyshev bound.

$$E[\bar{X}_n] = \mu = 0$$
$$V[\bar{X}_n] = \frac{\sigma^2}{n} = \frac{1}{n} = \left(\frac{1}{\sqrt{n}}\right)^2$$

i) Mill's inequality

$$P(|\bar{X}_n| > t) = P\left(\frac{|\bar{X}_n - 0|}{\frac{1}{\sqrt{n}}} > \frac{\frac{t}{1}}{\frac{1}{\sqrt{n}}}\right) = P(\sqrt{n}|\bar{X}_n| > \sqrt{n}\,t)$$
$$\leq \sqrt{\frac{2}{\pi}}\,\frac{e^{-nt^2/2}}{\sqrt{n}\,t}$$

ii) Chebyshev bound

$$P(|\bar{X}_n - \mu| \geq t) \leq \frac{\frac{1}{n}}{t^2} = \frac{1}{nt^2}$$

iii) 비교

$$\sqrt{\frac{2}{\pi}}\,\frac{e^{-nt^2/2}}{\cancel{\sqrt{n}}\,t} \quad vs \quad \frac{1}{nt^2}$$

$$\sqrt{\frac{2}{\pi}}\,e^{-\frac{nt^2}{2}} \quad vs \quad \frac{1}{t\sqrt{n}}$$

어떻게 비교하지?