MATHEMATICS OF DEEP LEARNING

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Lecture 11: Optimization Landscapes: spin glasses, tensors and deep networks

OBJECTIVES LECTURE 11

- Landscape of Spherical Spin Glasses

- Landscape of Tensor Decomposition and Spiked Tensor Model.

- Landscape of Neural Network Optimization

SPHERICAL SPIN GLASSES

- Consider the Hamiltonian of the Spherical Spin Glass of order p with N particles:

$$H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1,\dots,i_p=1}^{N} J_{i_1,\dots i_p}\sigma_{i_1}\dots\sigma_{i_p}\ ,$$

$$\sigma \in S^{N-1}(\sqrt{N})\ ,\ J_{i_1,\dots,i_p} \sim \mathcal{N}(0,1)\ .$$

- $H_{N,p}$ is the (only) Gaussian Process on the sphere whose covariance is given by

$$\mathbb{E}[H_{N,p}(\sigma)H_{N,p}(\sigma')] = N\left(\underbrace{N^{-1}\langle\sigma,\sigma'\rangle}_{\text{overlap}}\right)^p\ .$$

COMPLEXITY OF RANDOM HIGH-DIMENSIONAL FUNCTION

- In this case, the landscape is random. We want to understand notions of average-case complexity of this landscape.
- Consider the Random variable

$$\mathrm{Crt}_{N,k}(B) = \sum_{\sigma;\nabla H(\sigma)=0} \mathbf{1}\{H_{N,p}(\sigma) \in NB\}\mathbf{1}\{\mathrm{ind}[\nabla^2 H_{N,p}(\sigma)] = k\},$$

$$B \subset \mathbb{R} \qquad \mathrm{ind}[\nabla^2 H] = \text{ number of negative eigenvalues of } \nabla^2 H$$

  - it counts the number of configurations where $H_{N,p}$ has a critical point of a given index, at a given energy value.

COMPLEXITY OF SPHERICAL SPIN GLASS

- The first-order measure of complexity is to understand $\mathbb{E}\mathrm{Crt}_{N,k}(B)$ as $N \to \infty$, for fixed k, B and p.

- [Auffinger, Ben Arous & Cerny'10] exploited the Gaussian structure to prove

Theorem: [ABC'10]

$$\mathbb{E}[\mathrm{Crt}_{N,k}(B)] = 2\sqrt{\frac{2}{p}}(p-1)^{N/2}\mathbb{E}^N_{\mathrm{GOE}}\left\{e^{-N\frac{p-2}{2p}(\lambda^N_k)^2}\mathbf{1}\left[\lambda^N_k \in \sqrt{\frac{p}{2p-2}}B\right]\right\}$$

  - *GOE* is the Gaussian Orthogonal Ensemble, a random matrix where entries are independent Gaussians.
  - The joint distribution of eigenvalues is well-known and studied.

COMPLEXITY OF SPHERICAL SPIN GLASS

- Using this result, we obtain the following asymptotic behavior:

Theorem [ABC'10]: For all $p \geq 2$,

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathrm{Crt}_{N,k}(u) = \Theta_{k,p}(u)\,,$$

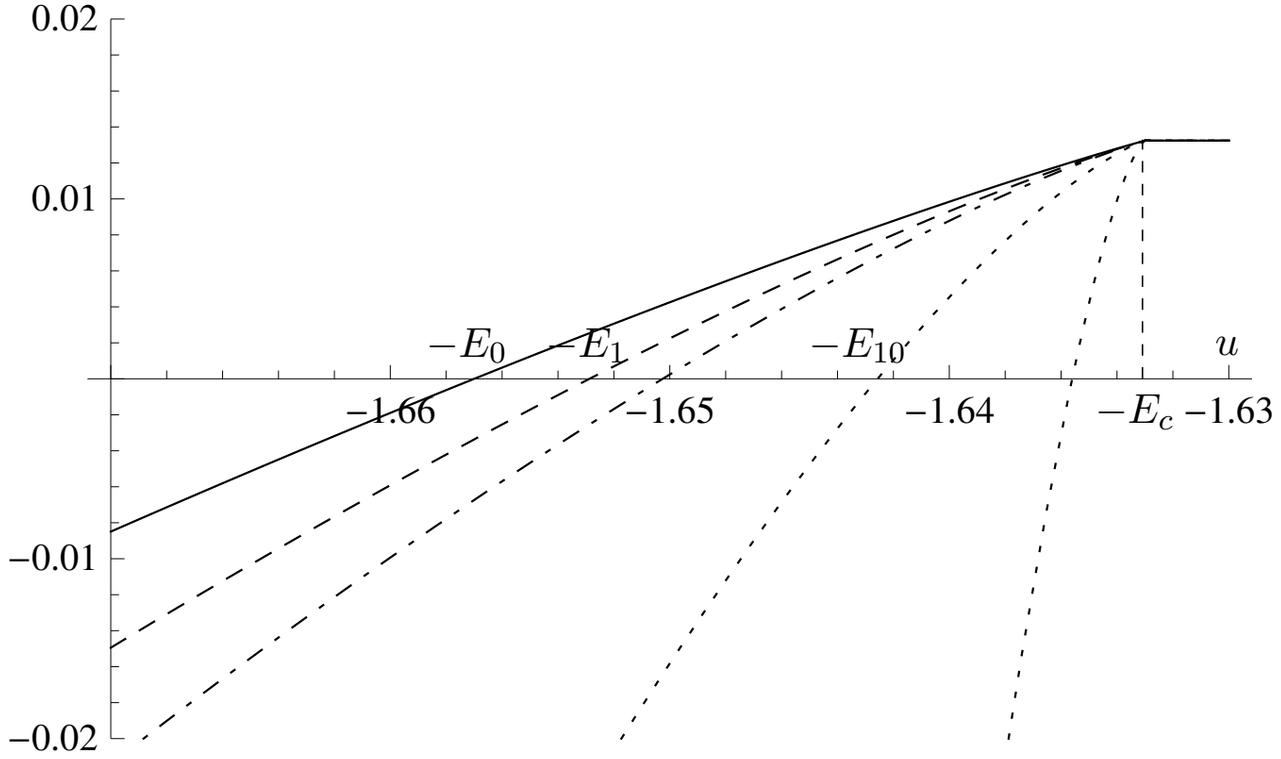
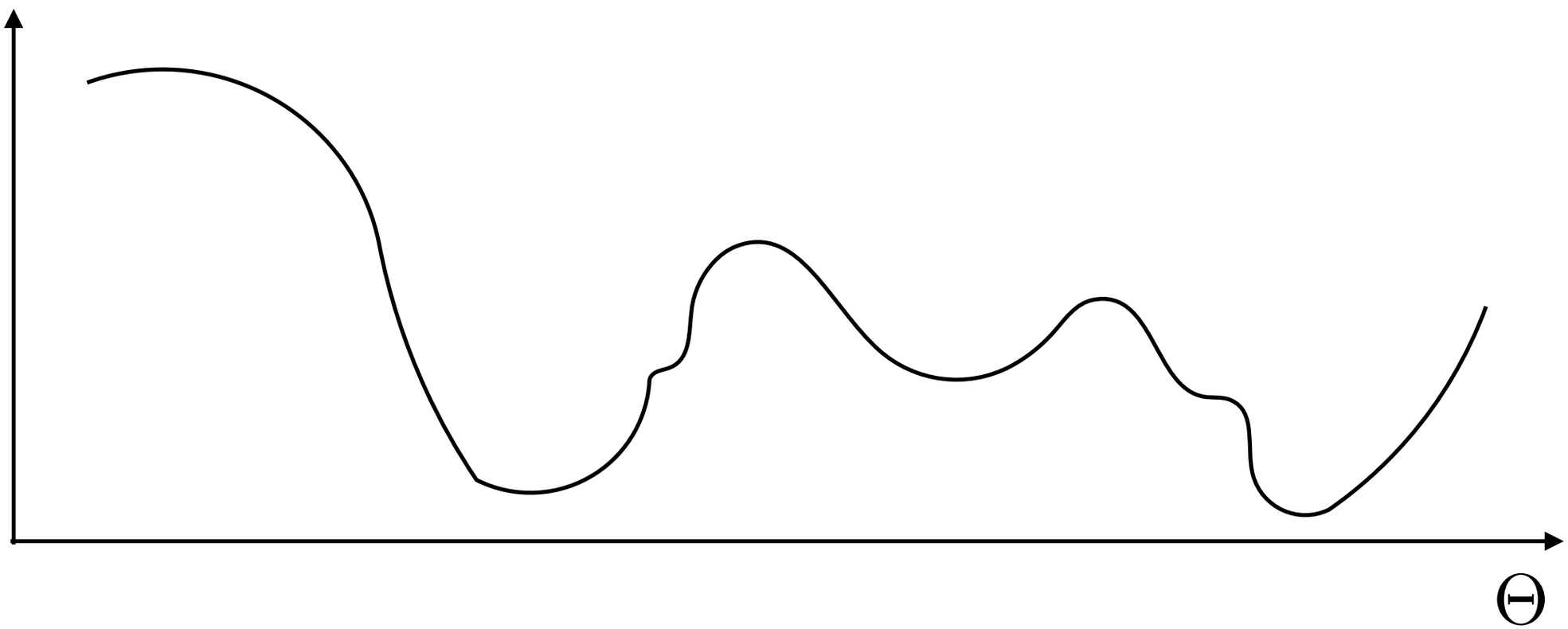


FIGURE 1. The functions $\Theta_{k,p}$ for $p = 3$ and $k = 0$ (solid), $k = 1$ (dashed), $k = 2$ (dash-dotted), $k = 10$, $k = 100$ (both dotted). All these functions agree for $u \geq -E_\infty$.

SPHERICAL SPIN GLASS

- [Auffinger et al '10] [Auffinger, Ben Arous'13], obtained a complete description of the behavior of critical points of spherical spin glasses.

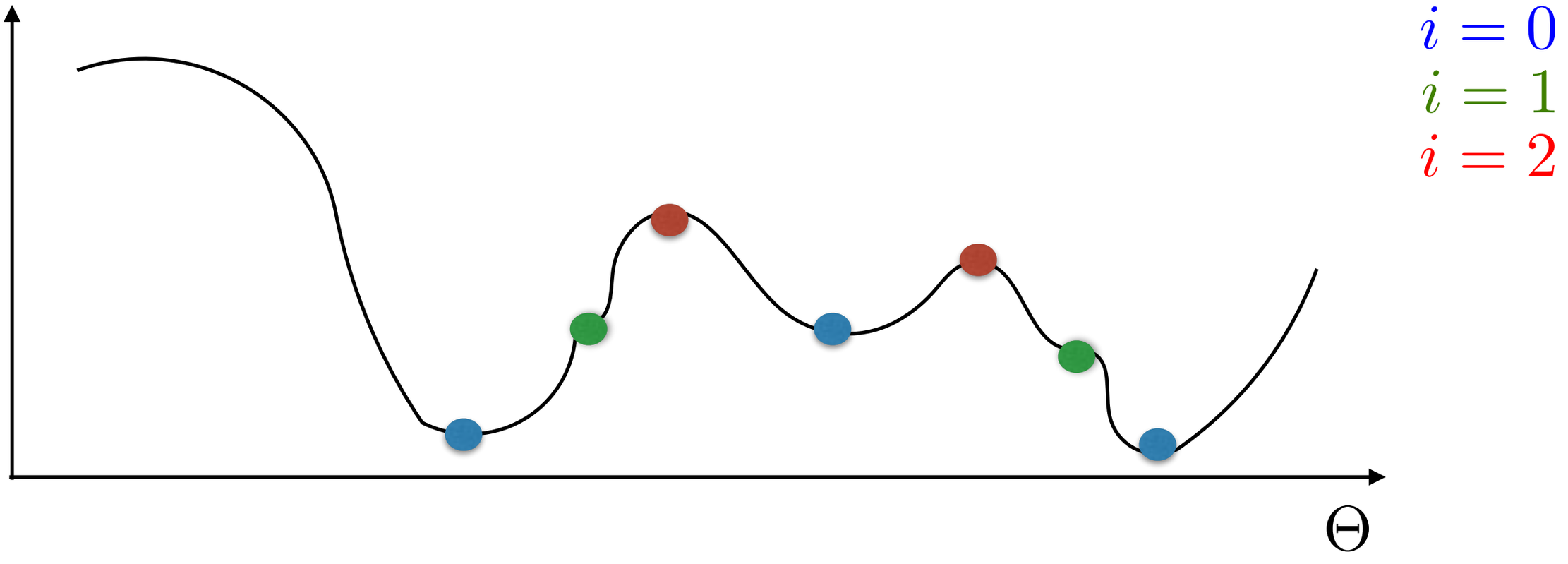
- In particular, critical points (ratio of negative to positive eigenvalues of the Hessian) occur at different energy bands:



SPHERICAL SPIN GLASS

- This result uncovers an important aspect of the landscape: layered structure of critical points according to their *index*.

- In particular, index of critical points (ratio of negative to positive eigenvalues of the Hessian) occur at different energy bands:



SPHERICAL SPIN GLASSES

- Let $E_k(p)$ be such that $\Theta_{k,p}(-E_k(p)) = 0$.
- All critical points of $H_{N,p}$ of finite index concentrate in the band $(-NE_0(p), -NE_\infty(p))$

Theorem 2.14. *Let for an integer $k \geq 0$ and $\varepsilon > 0$, $B_{N,k}(\varepsilon)$ be the event "there is a critical value of index k of the Hamiltonian $H_{N,p}$ above the level $-N(E_\infty(p) - \varepsilon)$", that is $B_{N,k}(\varepsilon) = \{\mathrm{Crt}_{N,k}((-E_\infty(p) + \varepsilon, \infty)) > 0\}$. Then for all $k \geq 0$ and $\varepsilon > 0$,*

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(B_{N,k}(\varepsilon)) < 0. \tag{2.25}$$

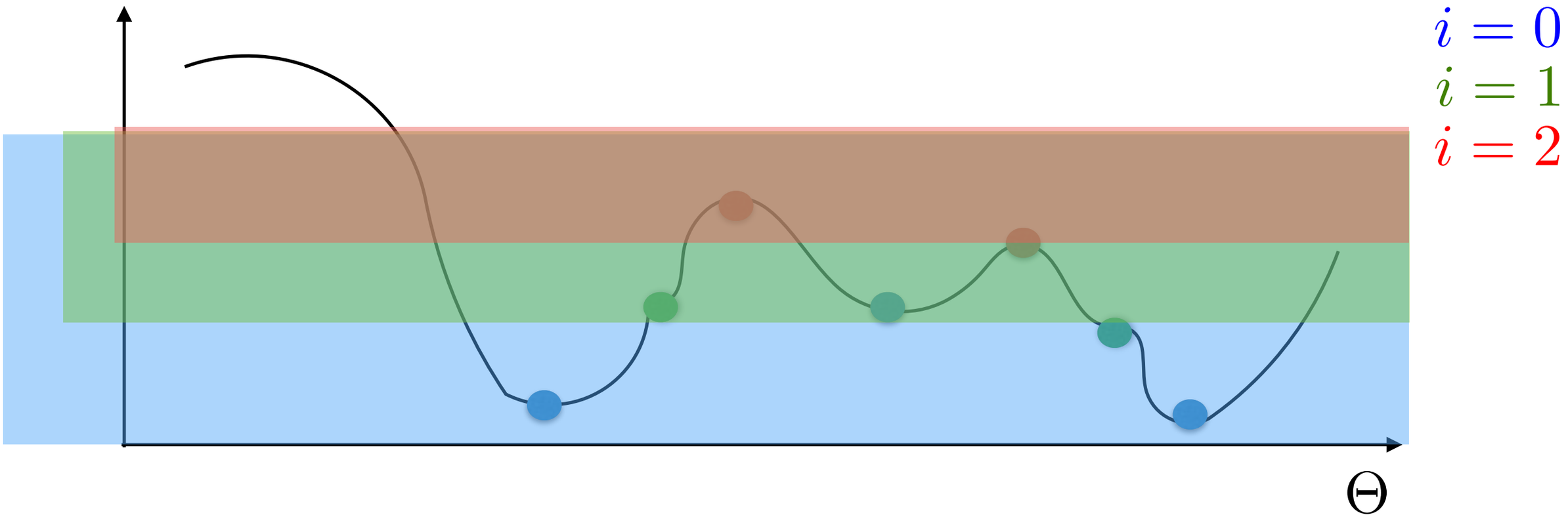
- They concentrate according to their index:

Theorem 2.15. *For $k \geq 0$ and $\varepsilon > 0$, let $A_{N,k}(\varepsilon)$ to be the event "there is a critical value of the Hamiltonian $H_{N,p}$ below the level $-N(E_k(p) + \varepsilon)$ and with index larger or equal to k", that is $A_{N,k}(\varepsilon) = \{\sum_{i=k}^{\infty} \mathrm{Crt}_{N,i}(-E_k(p) - \varepsilon) > 0\}$. Then for all $k \geq 0$ and $\varepsilon > 0$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}(A_{N,k}(\varepsilon)) < 0. \tag{2.26}$$

SPHERICAL SPIN GLASSES

- As $N \to \infty$, the number of local minima dominate the rest of the indices.
- Moreover, in the energy range $(-NE_k(p), -NE_{k+1}(p))$, we only find critical points with index $\leq k$
- In particular, what is the behavior of the energy gap to move from one local minima to the next?



SPHERICAL SPIN GLASSES

- These results describe the *average* number of critical points at in terms of energy value and index.
- But do they capture the *typical* behavior of a given realization of the random function $H_{N,p}$?
- This requires control not only of $\log \mathbb{E}\mathrm{Crt}_{N,k}$, but rather of $\mathbb{E} \log \mathrm{Crt}_{N,k}$

- In statistical physics, this is referred as the *quenched entropy,* as opposed to the *annealed entropy*.

THE KAC-RICE FORMULA

- The key mathematical tool behind these results is the so-called *Kac-Rice formula*.
- It is a general tool to compute the expected number of points on a manifold that satisfy certain random functional constraints. Given $P : \mathbb{R}^d \to \mathbb{R}^d$, $Q : \mathbb{R}^d \to \mathbb{R}^k$, B open set in $\mathbb{R}^k$ the Kac-Rice formula counts the expected number of points $x \in \mathbb{R}^d$ that satisfy $P(x) = 0$, $Q(x) \in B$.
- When $P(x) = \nabla f(x), Q(x) = \{\nabla^2 f(x), x\}$, $B = \{A; 0 \succeq A\} \otimes E, (E \subset S^{d-1})$

Kac-Rice lemma: Let f be a random function defined on the unit sphere S^{d-1} and $E \subset S^{d-1}$. Denote by M_f the set of all local maxima of f. Then under appropriate regularity

$$\mathbb{E}\{M_f \cap E\} = \int_x \mathbb{E}\left[|\det(\nabla^2 f)|\mathbf{1}(0 \succeq \nabla^2 f) \cdot \mathbf{1}(x \in E)|\nabla f = 0\right] \rho_{\nabla f(x)}(0) dx\,.$$

NOISE + SIGNAL LANDSCAPE MODEL

- What is the effect of adding a deterministic smooth function on top of the previous random model?

- How to move from average complexity measures to typical complexity?

NOISE + SIGNAL LANDSCAPE MODEL

- [Ros et al.'18] consider the modified loss

$$H_{p,q,r}(\sigma) = -\sum_{i_1,\ldots,i_p=1}^{N} J_{i_1,\ldots,i_p}\sigma_{i_1}\ldots\sigma_{i_p} - rNf_q(N^{-1}\langle\sigma, v_0\rangle)$$

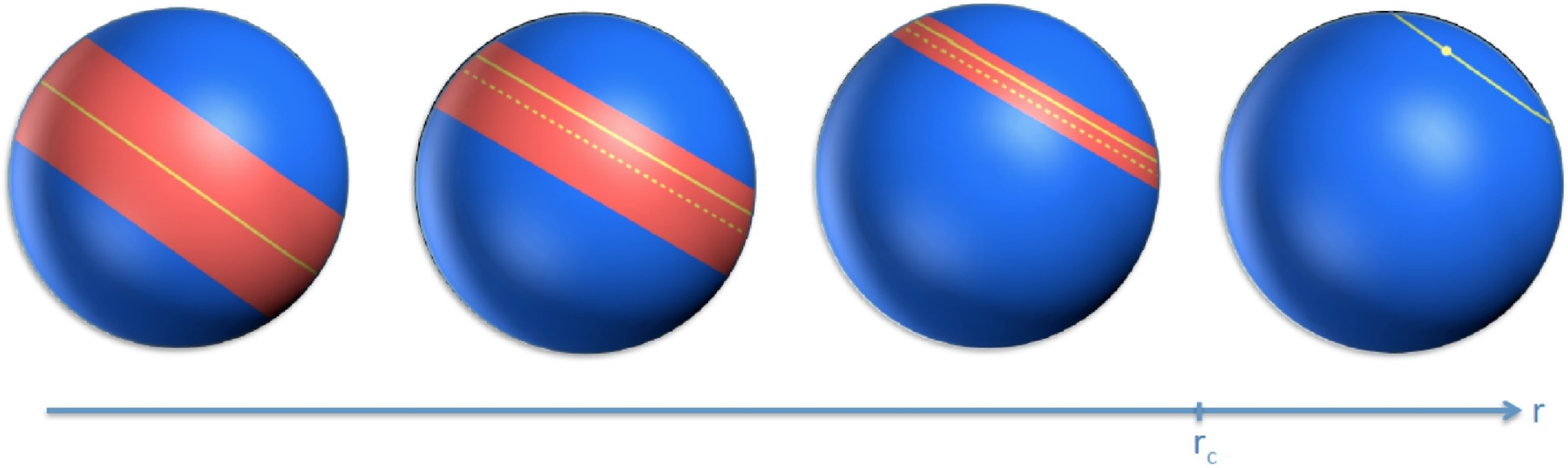
- First term is random as before
- Deterministic term: $f_q : [0,1] \to \mathbb{R}$ monotonic increasing, $s.t.\, \partial^r f(0) = 0 \,\forall\, r < q\,, \partial^q f(0) \neq 0\,.$
- It favors a fixed configuration v_0. r controls the SNR.
- If $f_q = \frac{x^q}{q}, q = p$, this is the *spiked tensor model*.

NOISE + SIGNAL LANDSCAPE MODEL

- There is a tension between the random and deterministic components.
- The random component contains exponential number of critical points (from previous analysis), whereas the deterministic component has only one global minima.
- The energy landscape is fundamentally controlled by both the SNR parameter r and by the order q.
- How does the qualitative behavior depend upon r and q?

NOISE + SIGNAL LANDSCAPE MODEL

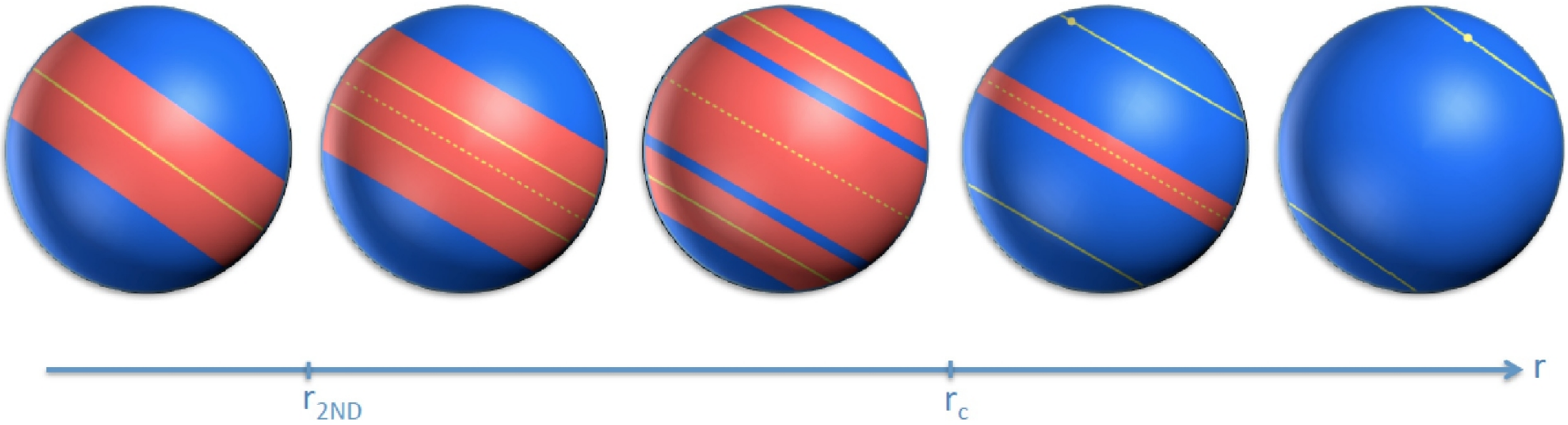
- First case: $q=1$



- bulk of local minima in red strip, deepest minima in yellow strip.
- At $r = r_c$ we encounter a phase-transition: only one minima survives, although not exactly at $\sigma = v_0$.

NOISE + SIGNAL LANDSCAPE MODEL

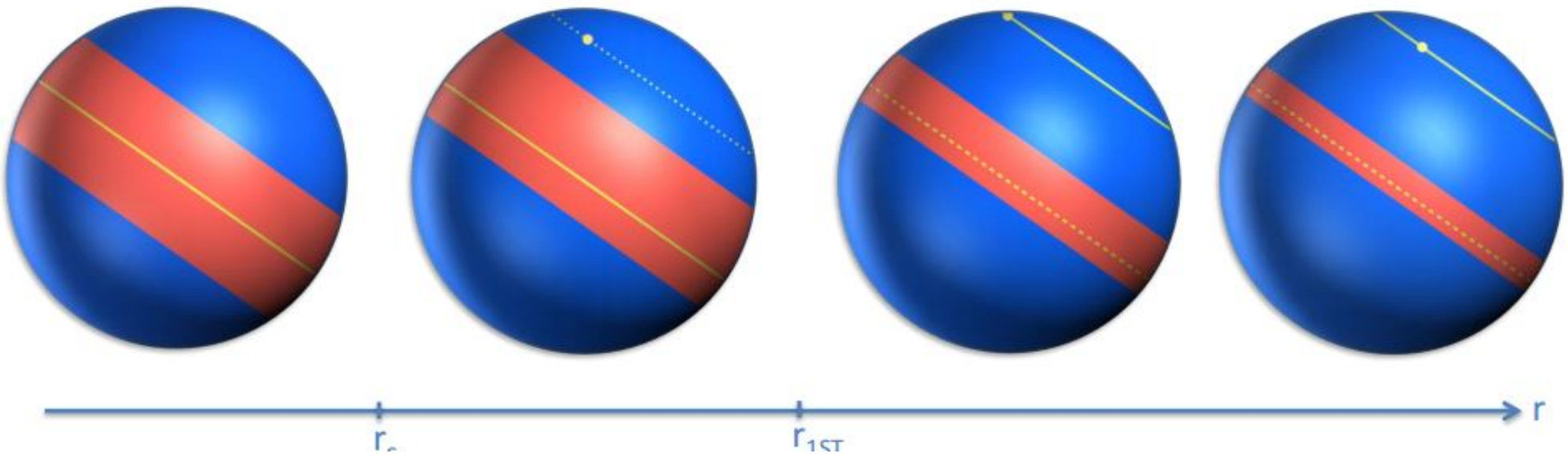
- Second case, $q=2$:

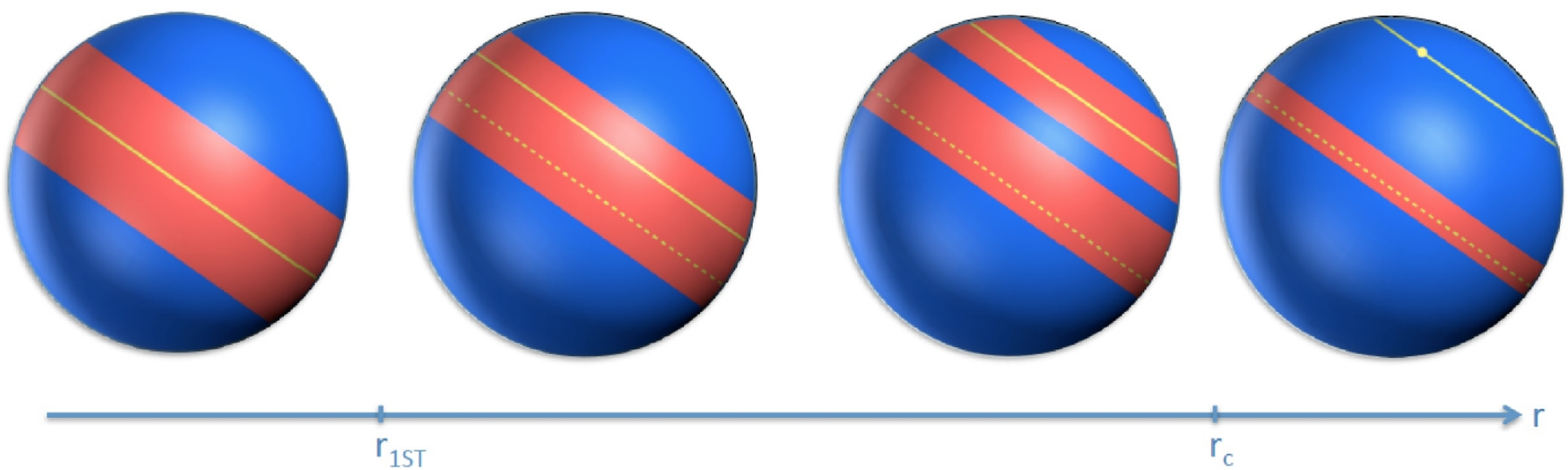


- Now there are two transitions: first the deepest local minima splits and moves towards poles, breaking the bulk of local minima in three bands.
- Next, the bands closer to poles turn into two isolated global minima, and equator band eventually disappears.

NOISE + SIGNAL LANDSCAPE MODEL

- General case, $q>2$: Now the deterministic term has no influence on Gradient and Hessian at equator.





THE REPLICA METHOD

- These results are obtained qualitatively using tools from Statistical Physics.
- One remarkable tool is the *replica method*.
- It starts by considering the system free energy at temperature β

$$f = -\frac{1}{\beta N}\mathbb{E}\log Z\,, \qquad Z = \sum_{\sigma}\exp(-\beta H(\sigma))\,.$$

- As the temperature $\beta \to 0$, the partition function is dominated by states close to global minima.
- But, how to estimate the expectation $\mathbb{E}\log Z$?

THE REPLICA METHOD

- The replica method computes instead the moments $\mathbb{E}Z^r$, $r \in \mathbb{N}$ and considers the analytic continuation

$$\mathbb{E}\log Z = \lim_{r \to 0} \frac{Z^r - 1}{r} .$$

- The moment $\mathbb{E}Z^r$ can be expressed in terms of r *replicas* $\sigma^{(1)}, \ldots, \sigma^{(r)}$:

$$\mathbb{E}[Z^r] = \sum_{\sigma^{(j)}} \mathbb{E}\exp\left(-\beta \sum_{j=1}^{r} H(\sigma^{(j)})\right) .$$

- The *Replica Symmetry Breaking* (RSB) formalism studies the geometry of critical points by studying how replicas overlap.

THE LANDSCAPE OF TENSOR DECOMPOSITION

- Consider a 4-th order tensor T of the form

$$T = \sum_{i=1}^{n} a_i \otimes a_i \otimes a_i \otimes a_i \ , \quad a_i \in \mathbb{R}^d \, .$$

- Focus on the case where rank $n \gg d$.
- Goal: recover components $a_1 \dots a_n$ from T.
- This is in general NP-hard.
- However, for a given problem distribution, we can consider the average-case recovery.

THE LANDSCAPE OF TENSOR DECOMPOSITION

- Ge and Ma focus on the Gaussian case $a_i \sim \mathcal{N}(0, I)$, using the loss

$$\max F(x) = \sum_{i,j,k,l=1}^{d} T_{i,j,k,l} x_i x_j x_k x_l = \langle T, x^{\otimes 4} \rangle_F = \sum_{i=1}^{d} \langle a_i, x \rangle^4 \,,\; x \in S^{d-1}.$$

- This objective finds the direction that mostly correlates with T
- Another random polynomial on the sphere, but now coefficients are not iid Gaussian.
- For $n \ll d^2$, global maxima of F are located around $\pm \frac{1}{\sqrt{d}} a_i$.

THE LANDSCAPE OF TENSOR DECOMPOSITION

- Ge and Ma use Kac-Rice formula to attack the over complete case:

Theorem [Ge & Ma]: Let $\epsilon, \zeta \in (0, 1/3)$, d sufficiently large, and assume $d^{1+\epsilon} < n < d^{2-\epsilon}$. Then whp the superlevel set $\{x \in S^{d-1}; F(x) \geq 3(1+\zeta)n\}$ contains exactly $2n$ local maxima, each of which is close to one of $\pm \frac{1}{\sqrt{d}} a_i$.

- No spurious local maxima in this super-level set.
- A random initialization on the sphere does not suffice:

$$\mathbb{E}[\langle a, x \rangle^4] = 3\,, a \sim \mathcal{N}(0, I), x \sim \mathrm{Unif}(S^{d-1})\,.$$

- The proof builds an estimator for the Kac-Rice formula.
- Open: spurious local maxima outside this level set?

SPIKED TENSOR MODEL

- A related problem is the recovery of a rank-one tensor

$$U = u^{\otimes k}, u \in \mathbb{R}^d$$

from noisy measurements: $Y = \lambda u^{\otimes k} + \frac{1}{\sqrt{2d}} W$

$W \in (\mathbb{R}^d)^{\otimes k}, W = \sum_{\pi \in S_d} G^\pi/(k!), G_{i_1,\ldots,i_k} \sim \mathcal{N}(0,1).$

- Similarly as before, we consider the (ML) objective

$$\max F(x) = \langle Y, x^{\otimes k} \rangle_F, s.t.\, x \in S^{d-1}.$$

- λ is the Signal-to-Noise ratio of the problem.
- Solving this objective is generally NP-hard for k>2.

SPIKED TENSOR MODEL

- In [Ben Arous, Mei, Montanari and Mica'18], the authors study the landscape of optimization for a typical realization of Y.

- Using again the Kac-Rice formula, they compute the expected number of critical points of the likelihood objective, as a function of SNR λ , order k and dimension d.

- The expected # of local maxima with $F(x) \approx s$, $\langle x, u \rangle \approx m$ is $\exp\{dS_0(m, s) + o(d)\}$.

SPIKED TENSOR MODEL

- In particular, the expected bulk of (exponential) local maxima are in the annulus $|\langle u, x\rangle| \leq \Theta(\lambda^{-1/(k-2)})$.
- This suggests that local ascent methods need to be initialized outside this annulus: $x_0\ s.t.\ |\langle u, x_0\rangle| > C\lambda^{-1/(k-2)}$
- But random initialization results in $|\langle u, x_0\rangle| = \Theta(d^{-1/2})$, thus one can escape local maxima provided

$$\lambda \geq Cn^{\frac{k-2}{2}}\ .$$

- This rate matches the best known polynomial time algorithm (power iteration algorithm by [Montanari et al.'14]).

FROM TENSORS TO NEURAL NETWORKS

- in [Mondelli & Montanari,'18], the authors reduce the learning of a 2-layer neural network with Gaussian input to the problem of tensor decomposition.

- in [Cohen, Shashua et al], network architecture (width and depth) is related to certain forms of tensor decomposition (parafac vs hierarchical). Depth provides more efficient tensor approximation.

DEEP NETWORKS AND SPIN GLASSES

- See also:
  - *"The effect of Gradient Noise on the Energy Landscape of Deep Networks"*, Chaudhari & Soatto. They study exterior magnitude field and its associated smoothing annealing schemes to reduce number of critical points.
  - *"Explorations on high dimensional landscapes"*, Sagun, Guney, Ben Arous, LeCun. Study the existence of a narrow band containing the bulk of the critical points of deep energy landscapes in the high-dimensional setting.

OPTIMIZATION LANDSCAPE OF NEURAL NETWORKS

- Relatively recent field, but many papers already.
  - Rate of papers increasing.
- Most papers focus on the deterministic aspect of the landscape: Given an input data distribution $(X, Y) \sim P$ and model architecture $\Phi(x; \theta)$, $\theta \in \Theta$, we consider the landscape

$$F(\theta) = \mathbb{E}_P[\ell(\Phi(X; \theta), Y)]$$

- This framework contains both population and empirical risk, by considering either P or $\hat{P} = \frac{1}{L}\sum_{l=1}^{L} \delta_{(X_l, Y_l)}$.
- Here, the *mean* landscape is expected to be of similar complexity than the *empirical* landscape for large L.

RELATED WORK

- Models from Statistical physics have been considered as possible approximations [Dauphin et al.'14, Choromanska et al.'15, Segun et al.'15]
- Tensor factorization models capture some of the non convexity essence [Anandukar et al'15, Cohen et al. '15, Haeffele et al.'15]
- [Shafran and Shamir,'15] studies bassins of attraction in neural networks in the overparametrized regime.
- [Soudry'16, Song et al'16] study Empirical Risk Minimization in two-layer ReLU networks, also in the over-parametrized regime.

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- [Tian'17] studies learning dynamics in a gaussian generative setting.
- [Chaudhari et al'17]: Studies local smoothing of energy landscape using the local entropy method from statistical physics.
- [Pennington & Bahri'17]: Hessian Analysis using Random Matrix Th.
- [Soltanolkotabi, Javanmard & Lee'17]: layer-wise quadratic NNs.

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- We can perturb any convex function in such a way it is no longer convex, but such that gradient descent still converges.
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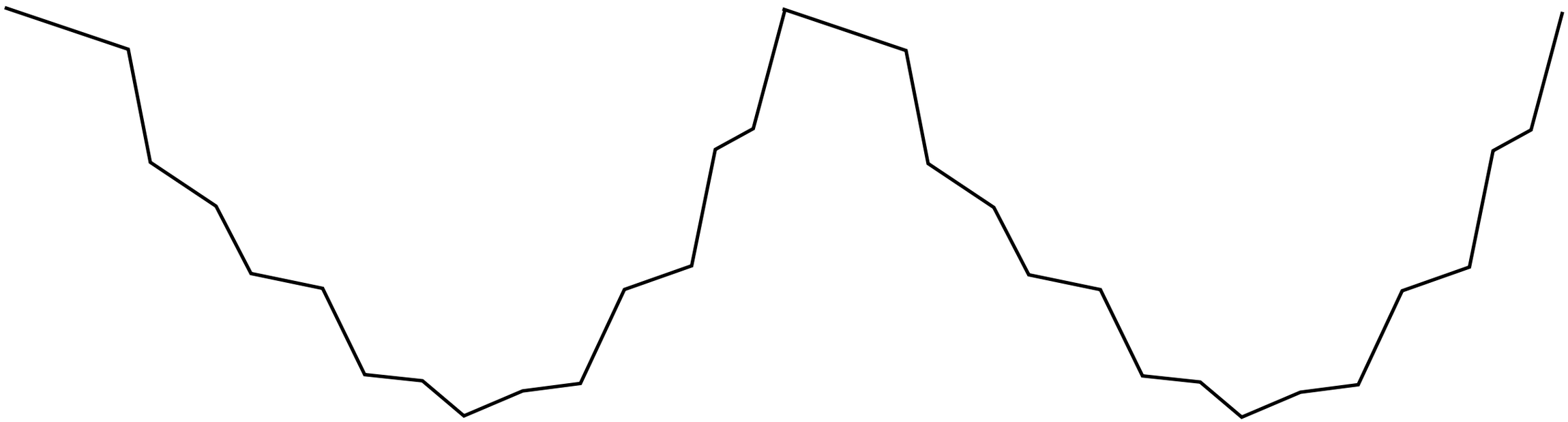
NON-CONVEXITY ≠ NOT OPTIMIZABLE

- We can perturb any convex function in such a way it is no longer convex, but such that gradient descent still converges.
- E.g. quasi-convex functions.
- In particular, deep models have internal symmetries.

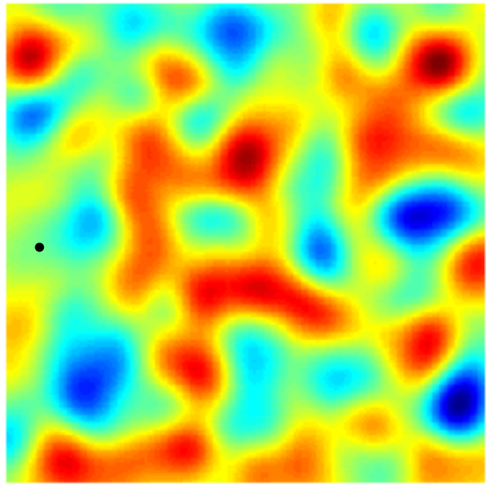
$$F(\theta) = F(g.\theta) \ , \ g \in G \text{ compact.}$$



ANALYSIS OF NON-CONVEX LOSS SURFACES

- Given loss $E(\theta)$, $\theta \in \mathbb{R}^d$, we consider its representation in terms of level sets:

$$E(\theta) = \int_0^\infty \mathbf{1}(\theta \in \Omega_u) du \ , \ \Omega_u = \{y \in \mathbb{R}^d \ ; \ E(y) \leq u\} \ .$$



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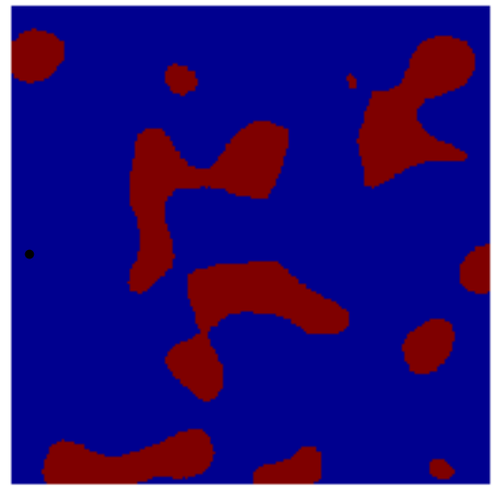

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- In particular, we ask how connected they are, i.e. how many connected components N_u at each energy level u ?

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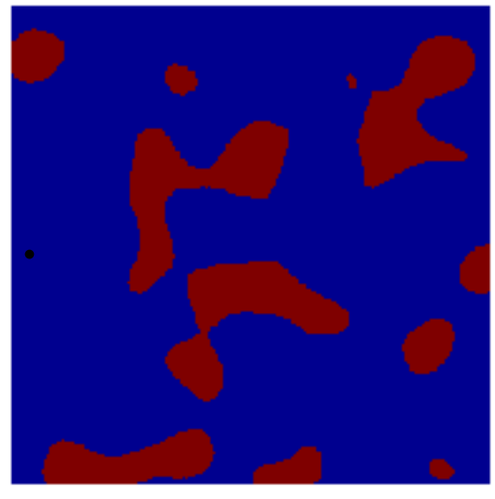

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- In particular, we ask how connected they are, i.e. how many connected components N_u at each energy level u ?

- Related to presence of poor local minima:

Proposition: If $N_u = 1$ for all u then E has no poor local minima.

(i.e. no local minima y^* s.t. $E(y^*) > \min_y E(y)$)

SPURIOUS VALLEYS

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- **Definition:** A *valley* is a connected component of the sublevel set Ω_u.

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SPURIOUS VALLEYS

- More generally, we are interested in certifying existence of descent paths.
- **Definition:** A *valley* is a connected component of the sublevel set Ω_u.

  Definition: A *spurious valley* is a connected component of the sublevel set Ω_u that does not contain a global minima.
- If a loss has no spurious valley, then one can continuously move from any point in parameter space to a global minima without increasing the loss:

  Given any initial parameter $\theta_0 \in \Theta$,
  $\exists$ continuous path $\theta : t \in [0,1] \mapsto \theta(t) \in \Theta$ st:
  $\theta(0) = \theta_0$, $\theta(1) \in \arg\min_{\theta} E(\theta)$, and
  $t \mapsto E(\theta(t))$ is non-increasing.

LINEAR VS NON-LINEAR DEEP MODELS

- Some authors have considered linear "deep" models as a first step towards understanding nonlinear deep models:

$$E(W_1, \ldots, W_K) = \mathbb{E}_{(X,Y)\sim P} \|W_K \ldots W_1 X - Y\|^2 \ .$$

$$X \in \mathbb{R}^n \ , \ Y \in \mathbb{R}^m \ , \ W_k \in \mathbb{R}^{n_k \times n_{k-1}} \ .$$

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Theorem: [**Kawaguchi'16**] If $\Sigma = \mathbb{E}(XX^T)$ and $\mathbb{E}(XY^T)$ are full-rank and Σ has distinct eigenvalues, then $E(\Theta)$ has no poor local minima.

- studying critical points.
- later generalized in [Hardt & Ma'16, Lu & Kawaguchi'17]

LINEAR VS NON-LINEAR DEEP MODELS

$$E(W_1, \dots, W_K) = \mathbb{E}_{(X,Y)\sim P} \|W_K \dots W_1 X - Y\|^2 \ .$$

Proposition: [BF'16]

1. If $n_k > \min(n, m)$, $0 < k < K$, then $N_u = 1$ for all u.

2. (2-layer case, ridge regression)
   $E(W_1, W_2) = \mathbb{E}_{(X,Y)\sim P} \|W_2 W_1 X - Y\|^2 + \lambda(\|W_1\|^2 + \|W_2\|^2)$
   satisfies $N_u = 1 \ \forall \ u$ if $n_1 > \min(n, m)$.

- We pay extra redundancy price to get simple topology.

LINEAR VS NON-LINEAR DEEP MODELS

$$E(W_1, \dots, W_K) = \mathbb{E}_{(X,Y)\sim P} \|W_K \dots W_1 X - Y\|^2 \ .$$

Proposition: [BF'16]

1. If $n_k > \min(n, m)$, $0 < k < K$, then $N_u = 1$ for all u.

2. (2-layer case, ridge regression)
   $E(W_1, W_2) = \mathbb{E}_{(X,Y)\sim P} \|W_2 W_1 X - Y\|^2 + \lambda(\|W_1\|^2 + \|W_2\|^2)$
   satisfies $N_u = 1 \ \forall \ u$ if $n_1 > \min(n, m)$.

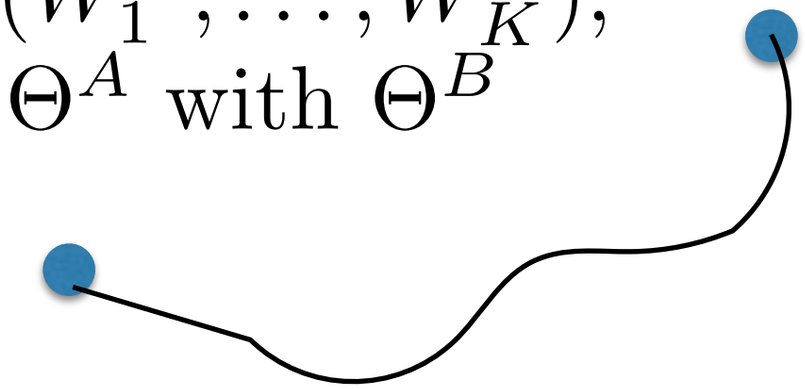
- We pay extra redundancy price to get simple topology.
- This simple topology is an "artifact" of the linearity of the network:

Proposition: [BF'16] For any architecture (choice of internal dimensions), there exists a distribution $P_{(X,Y)}$ such that $N_u > 1$ in the ReLU $\rho(z) = \max(0, z)$ case.

PROOF SKETCH

- Goal:

Given $\Theta^A = (W_1^A, \ldots, W_K^A)$ and $\Theta^B = (W_1^B, \ldots, W_K^B)$, we construct a path $\gamma(t)$ that connects Θ^A with Θ^B st $E(\gamma(t)) \leq \max(E(\Theta^A), E(\Theta^B))$.



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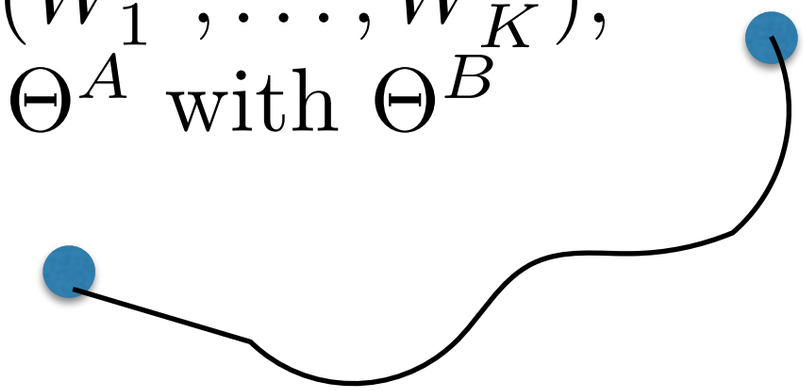

➤ Main idea:

1. Induction on K.
2. *Lift* the parameter space to $\widetilde{W} = W_1 W_2$: the problem is convex $\Rightarrow$ there exists a (linear) path $\widetilde{\gamma}(t)$ that connects Θ^A and Θ^B.
3. Write the path in terms of original coordinates by *factorizing* $\widetilde{\gamma}(t)$.

➤ Simple fact:
If $M_0, M_1 \in \mathbb{R}^{n \times n'}$ with $n' > n$,
then there exists a path $t : [0,1] \to \gamma(t)$
with $\gamma(0) = M_0$, $\gamma(1) = M_1$ and
$M_0, M_1 \in \text{span}(\gamma(t))$ for all $t \in (0,1)$.

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[with L. Venturi, A. Bandeira, '17]

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- Construct paths on the Grassmanian manifold of linear subspaces
- Generalizes best known results for multilinear case (no assumptions on covariance).

BETWEEN LINEAR AND RELU: POLYNOMIAL NETS

- Quadratic nonlinearities $\rho(z) = z^2$ are a simple extension of the linear case, by lifting or "kernelizing":

$$\rho(Wx) = \mathcal{A}_W X \ , \ X = xx^T \ , \ \mathcal{A}_W = (W_k W_k^T)_{k \leq M} \ .$$

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- No poor local minima with much better bounds in the scalar output two-layer case:

Theorem: [BBV'18] The two-layer quadratic network regression $\mathbb{E}_{(X,Y)\sim P}|U(WX)^2 - Y|^2$ has no spurious valleys if $M > 2N$.

ASYMPTOTIC CONNECTEDNESS OF RELU

- Good behavior is recovered with nonlinear ReLU networks, provided they are sufficiently overparametrized:

- Setup: two-layer ReLU network:

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- Overparametrisation "wipes-out" local minima (and group symmetries).

- The bound is cursed by dimensionality, ie exponential in n .

- Result is based on local linearization of the ReLU kernel (hence exponential price).

KERNELS ARE BACK?

$$\Phi(x;\Theta) = W_k \rho(W_{k-1} \dots \rho(W_1 X))) \ , \ \Theta = (W_1, \dots W_k) \ ,$$

- The underlying technique we described consists in "convexifying" the problem, by mapping *neural* parameters Θ to *canonical* parameters $\beta = \mathcal{A}(\Theta)$:

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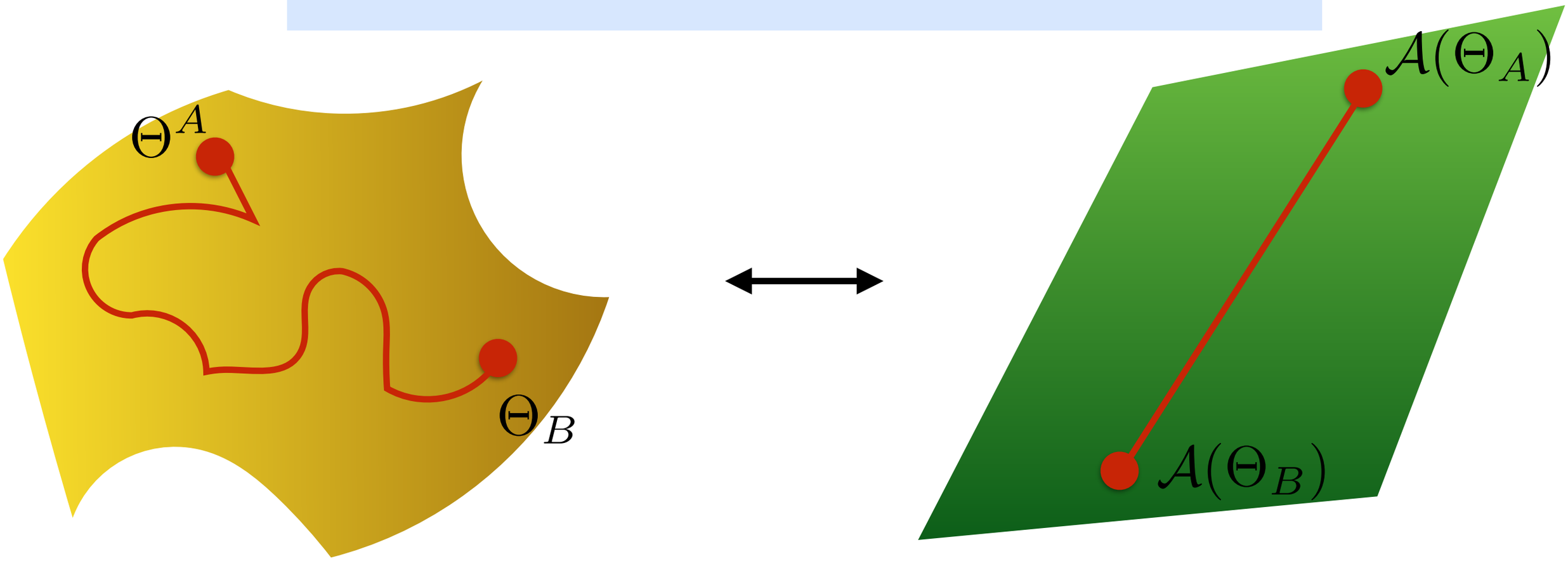

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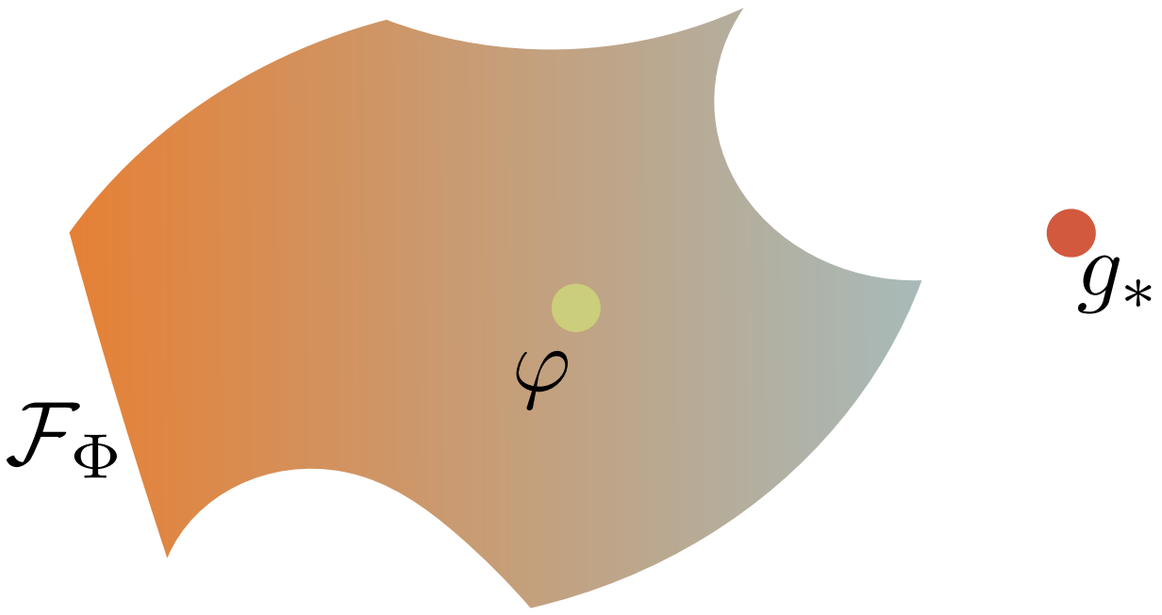
Theorem: [BVV'18] If $\dim\{\mathcal{A}(w),\, w \in \mathbb{R}^n\} = q < \infty$, then $L(U,W) = \mathbb{E}\|U\rho(WX) - Y\|^2$ has no spurious valley if $M \geq q$.

- This includes Empirical Risk Minimization (since RKHS is only queried on finite # of datapoints), and polynomial activations.
- See [Bietti&Mairal'17,Zhang et al'17, Bach'17] for related work.

PARAMETRIC VS MANIFOLD OPTIMIZATION

- This suggests thinking about the problem in the functional space generated by the model:

$$\mathcal{F}_\Phi = \{\varphi : \mathbb{R}^n \to \mathbb{R}^m \,; \varphi(x) = \Phi(x; \Theta) \text{ for some } \Theta\} \,.$$



$$\min_{\varphi \in \mathcal{F}_\Phi} \|\varphi - g_*\|_p$$

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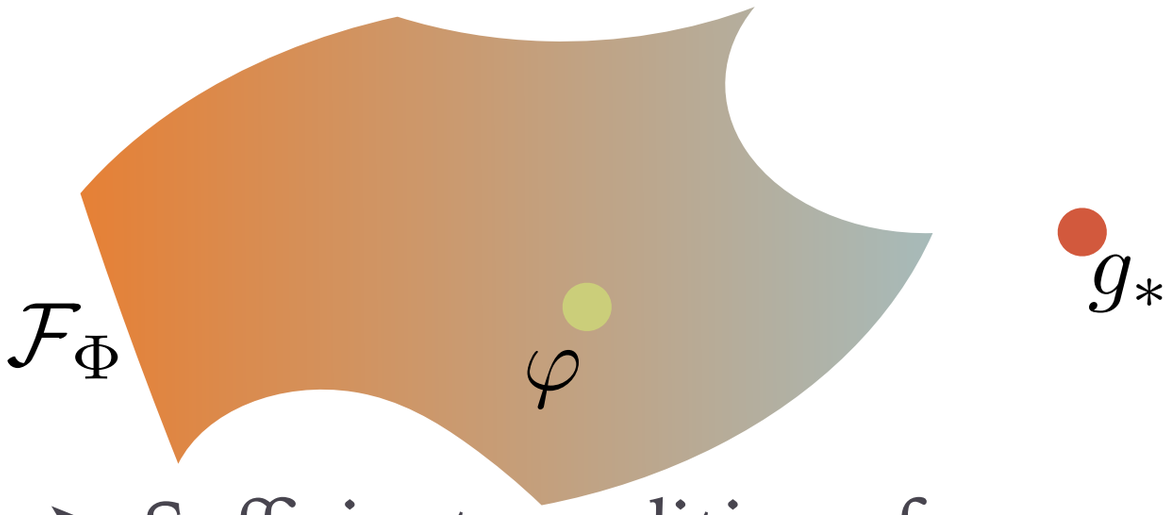

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  - $\mathcal{F}_\Phi$ convex and Θ sufficiently large so that we can move freely within.
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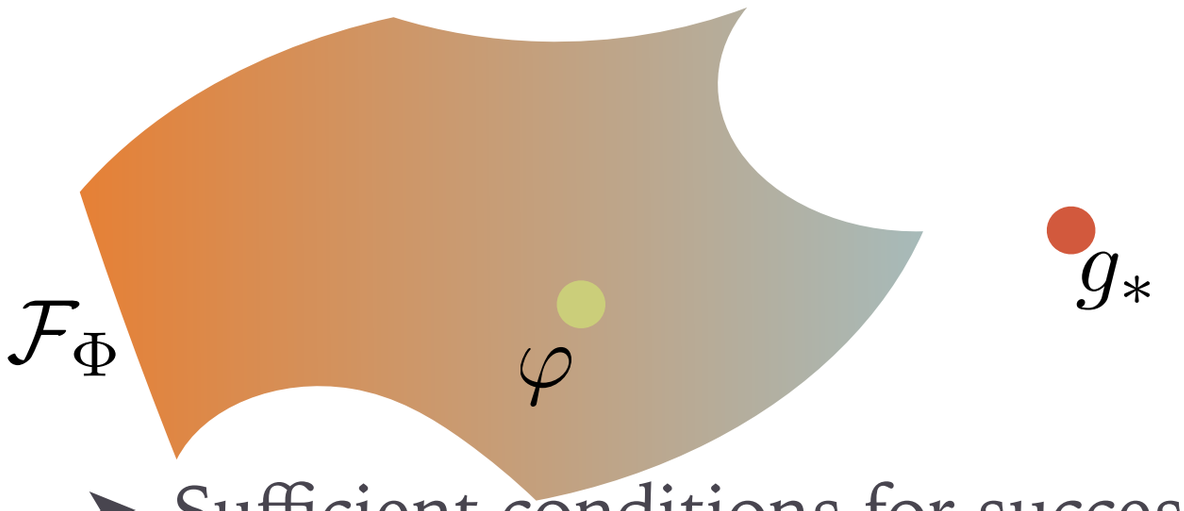

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- What happens when the model is not sufficiently overparametrised?

FROM SIMPLE LANDSCAPES TO ENERGY BARRIER?

- Does a similar macroscopic picture arise in our setting?
- Given $\rho(z)$ homogeneous, assume
  - $\tilde{\rho}(\langle w, X\rangle) = \langle A_w, \psi(X)\rangle$, with $\dim(\psi(X)) = f(N)$.
- Define

$$\beta(M,N) = \inf_{S;\dim(S)=f^{-1}(M)} \inf_{\substack{U\in\mathbb{R}^{m\times M}\\ W\in\mathbb{R}^{M\times f^{-1}(M)}}} \sup_{\substack{\mathbb{E}\|Z\|\le N-f^{-1}(M),\\ P_S Z=0}} \mathbb{E}\|U\rho(WP_S X+Z)-Y\|^2$$

  - Best loss obtained by first projecting the data onto the best possible subspace of dimension $f^{-1}(M)$and adding bounded noise in the complement.
  - $\beta(M,N)$ decreases with M and $\beta(f(N),N) = \min_{U,W} E(U,W)$.

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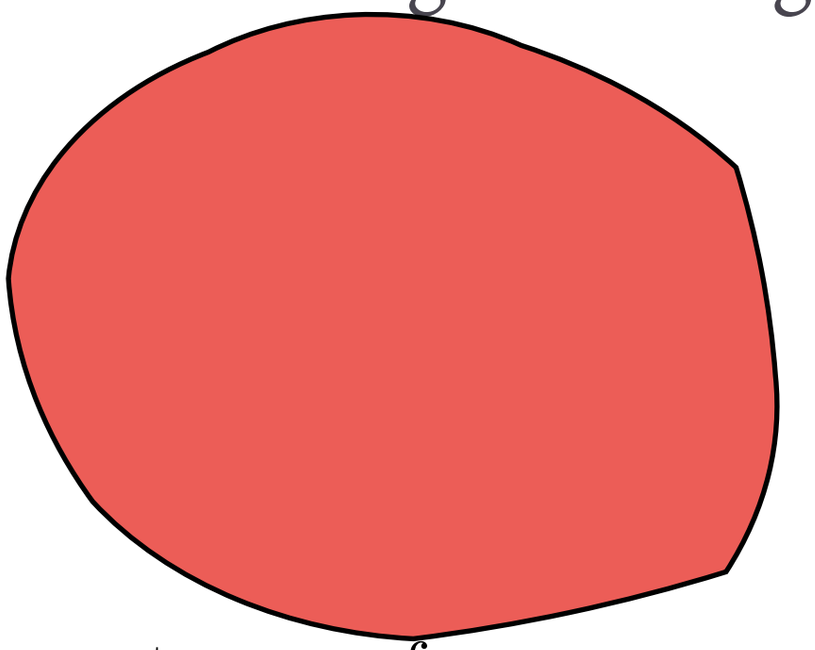
Conjecture [LBB'18]: The loss $L(U,W) = \mathbb{E}\|U\rho(WX)-Y\|^2$ has no poor local minima above the energy barrier $\beta(M,N)$.

FROM TOPOLOGY TO GEOMETRY

- The next question we are interested in is conditioning for descent.
- Even if level sets are connected, how easy it is to navigate through them?
- How "large" and regular are they?



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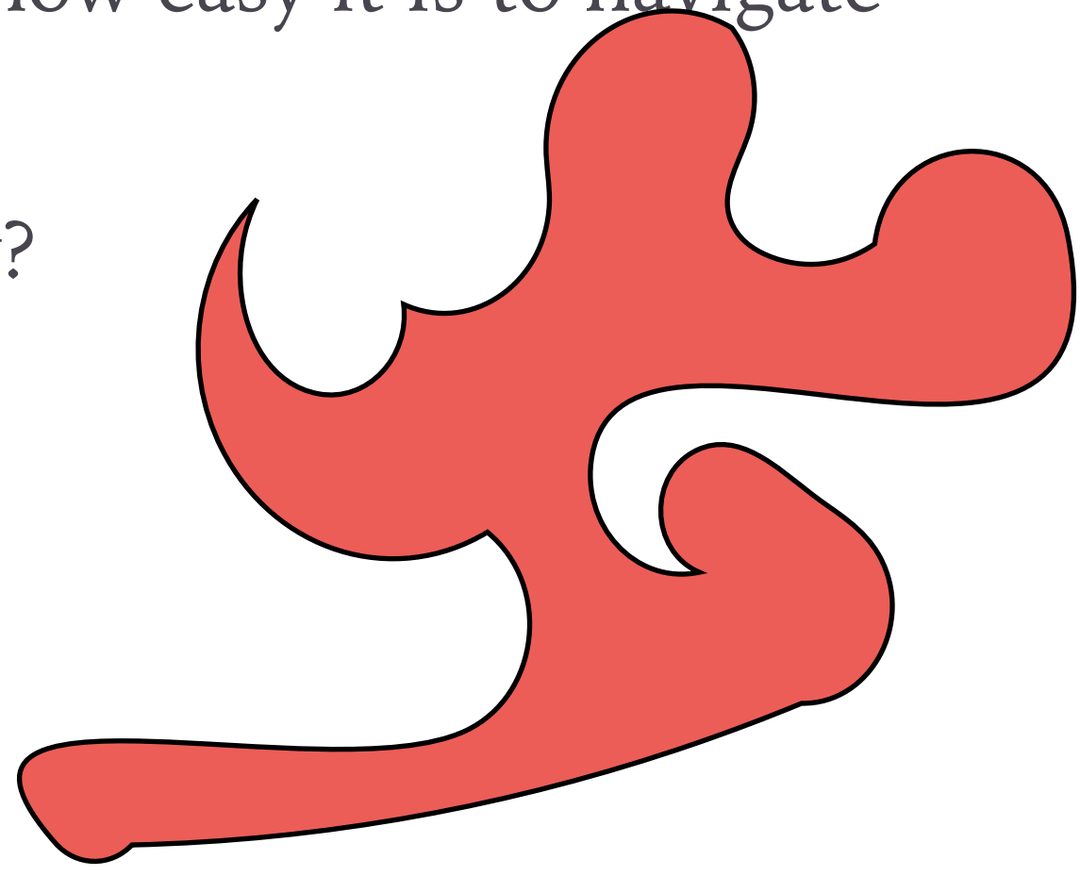

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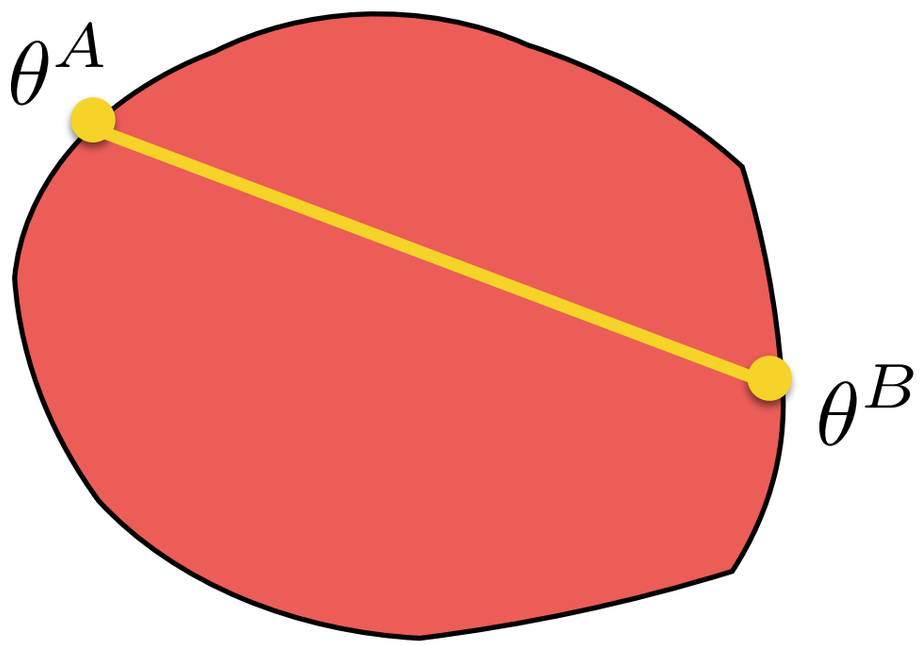
- Even if level sets are connected, how easy it is to navigate through them?
- We estimate level set geodesics and measure their length.



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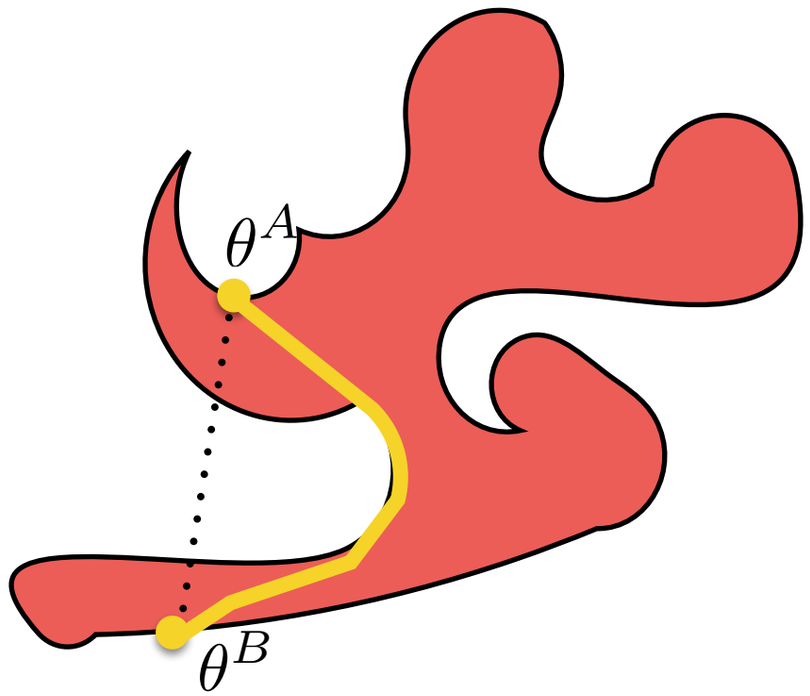

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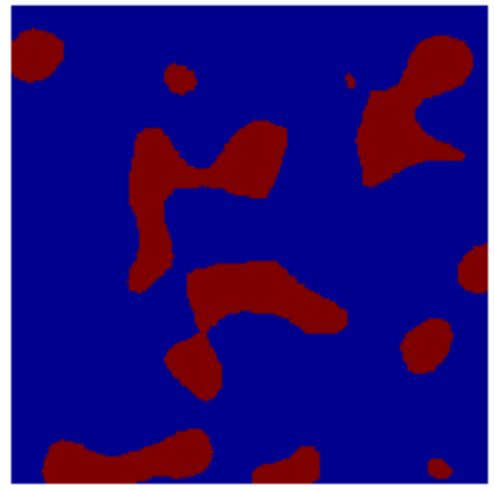
FINDING CONNECTED COMPONENTS



- Suppose $\theta_1,\ \theta_2$ are such that $E(\theta_1) = E(\theta_2) = u_0$
  - They are in the same connected component of Ω_{u_0} iff

    there is a path $\gamma(t),\ \gamma(0) = \theta_1, \gamma(1) = \theta_2$ such that
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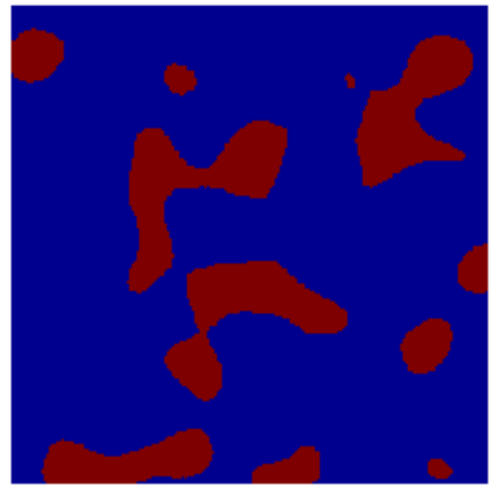

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θ_1

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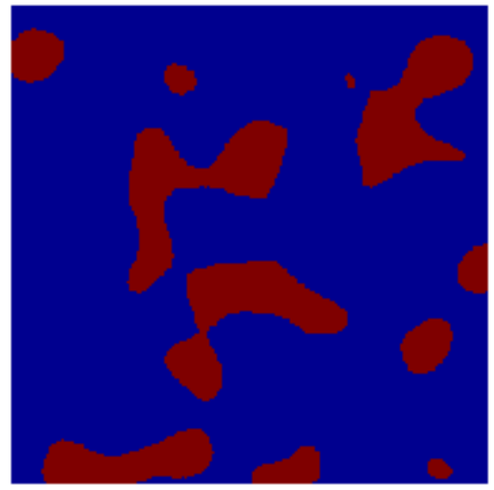

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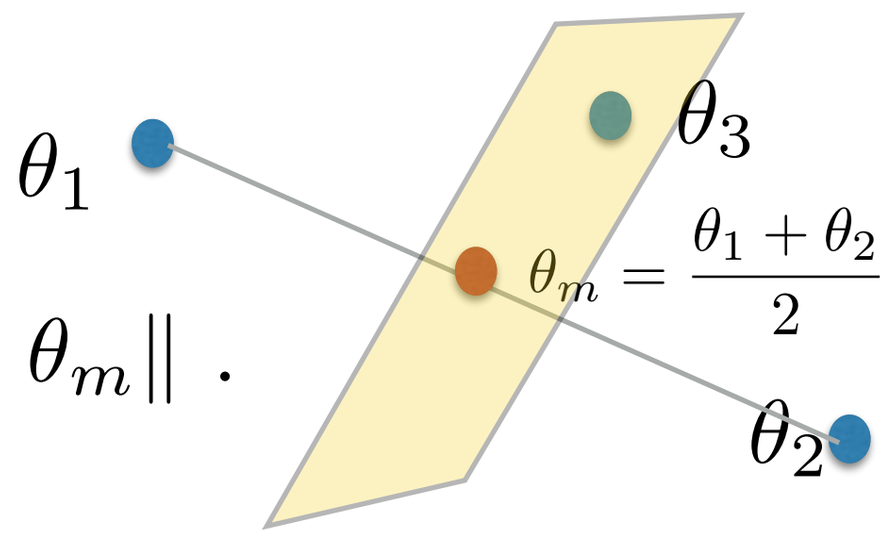

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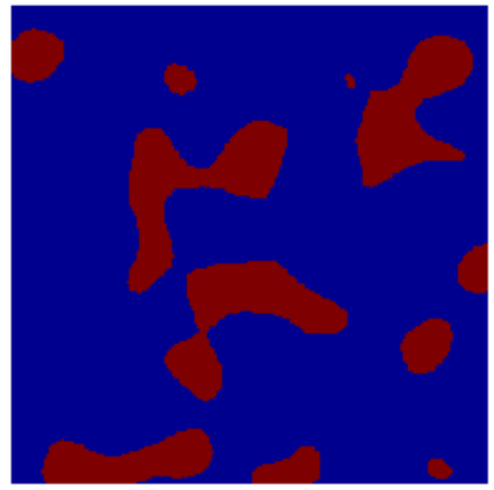

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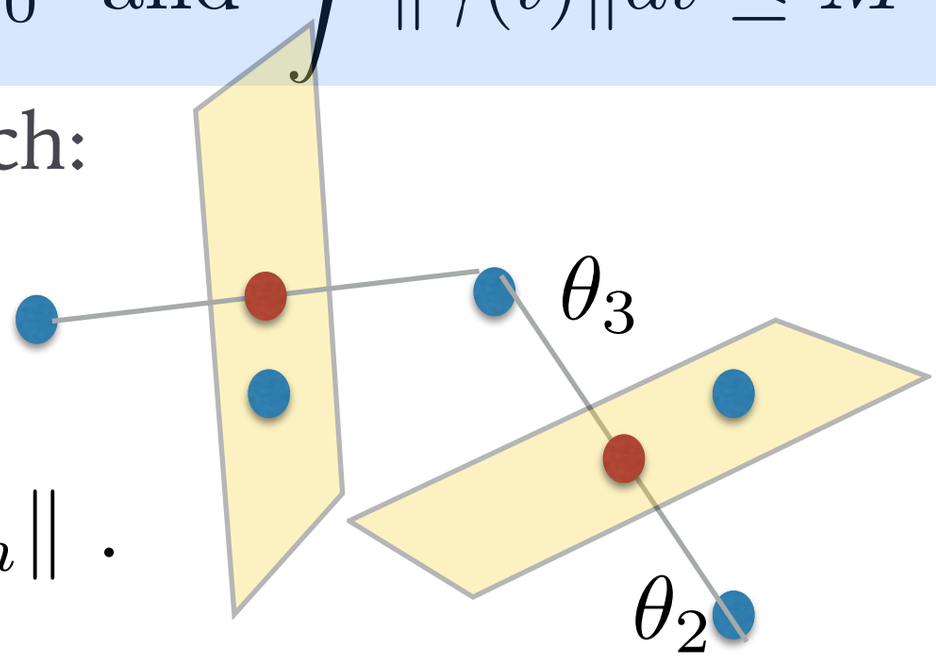

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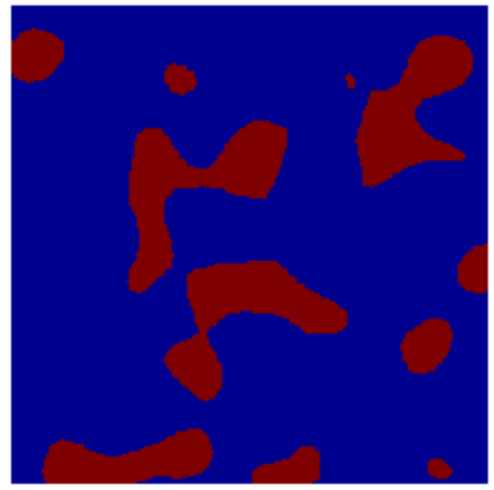

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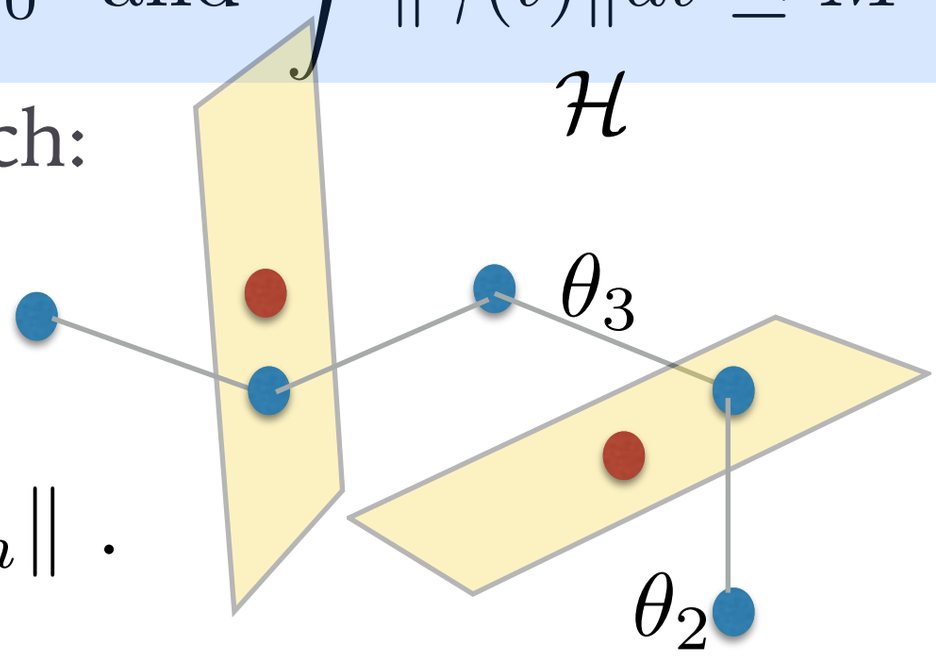
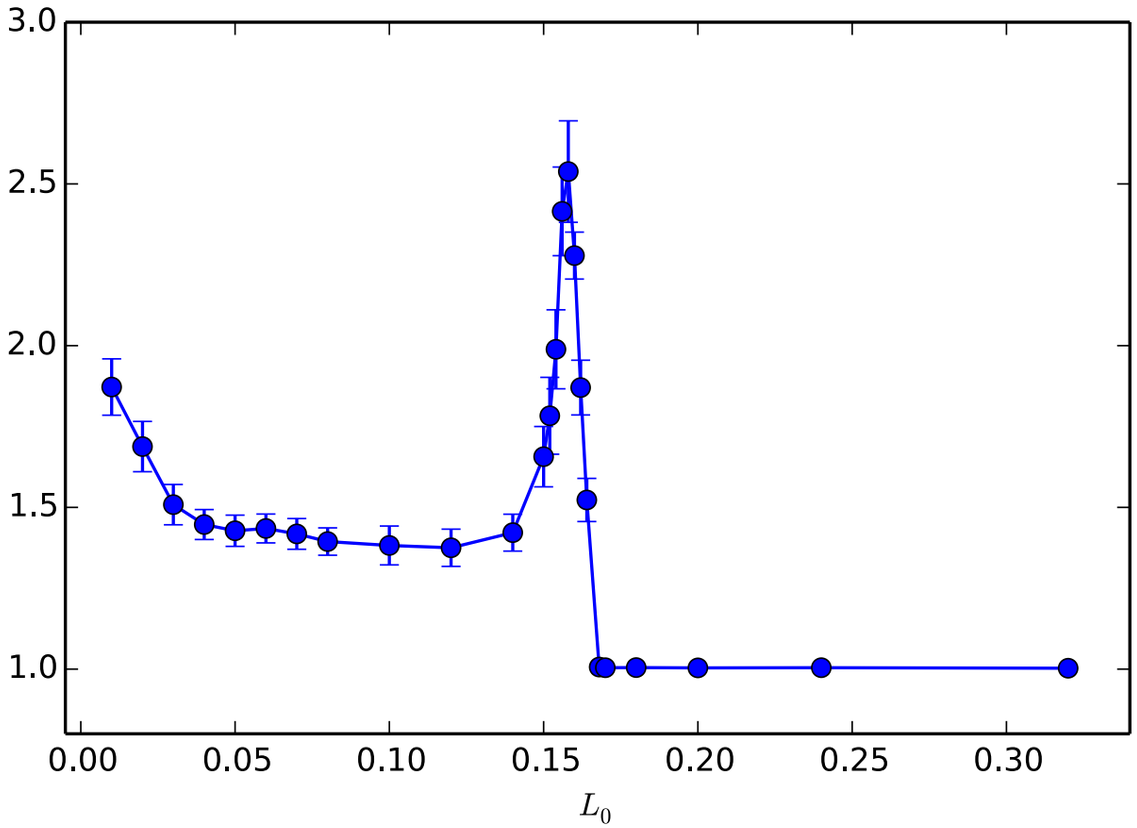
$$\theta_m = \frac{\theta_1 + \theta_2}{2}$$

$$\theta_3 = \arg \min_{\theta \in \mathcal{H};\ E(\theta) \leq u_0} \|\theta - \theta_m\|\ .$$

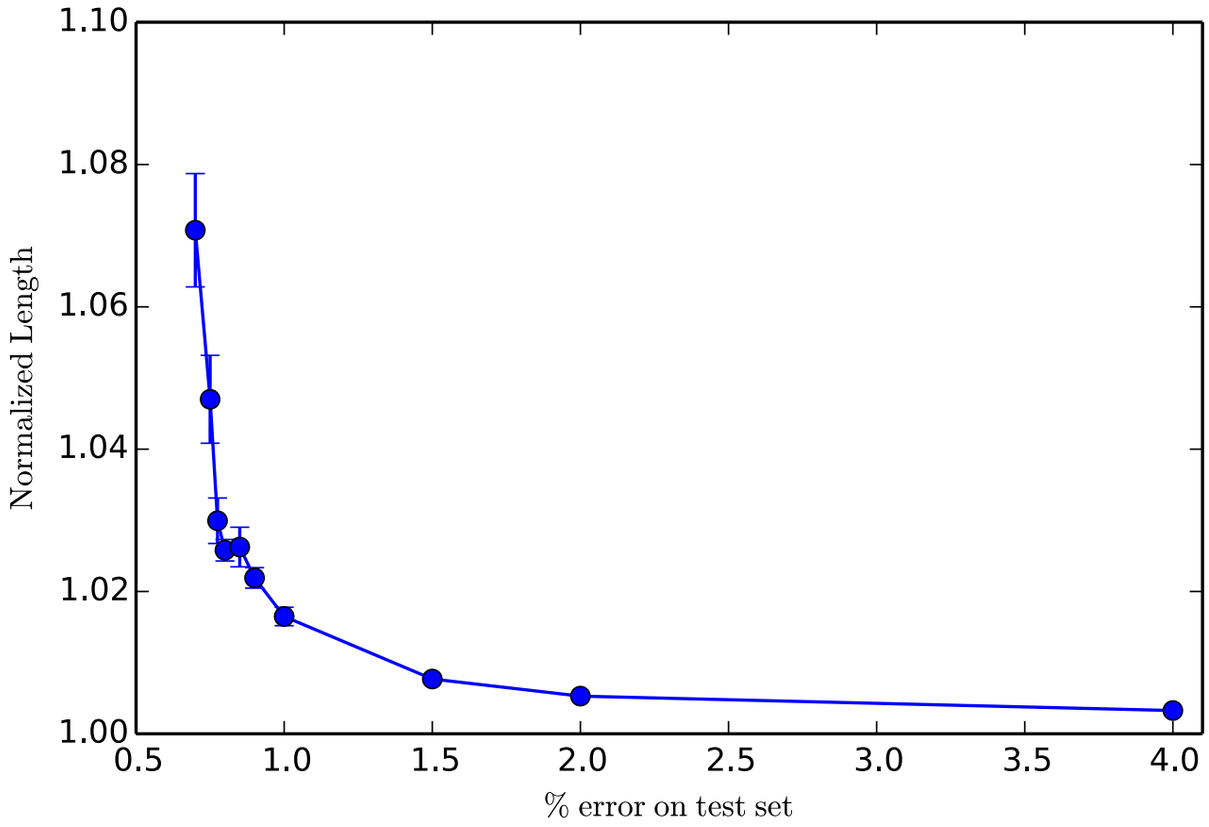


NUMERICAL EXPERIMENTS

- Compute length of geodesic in Ω_u obtained by the algorithm and normalize it by the Euclidean distance. Measure of curviness of level sets.



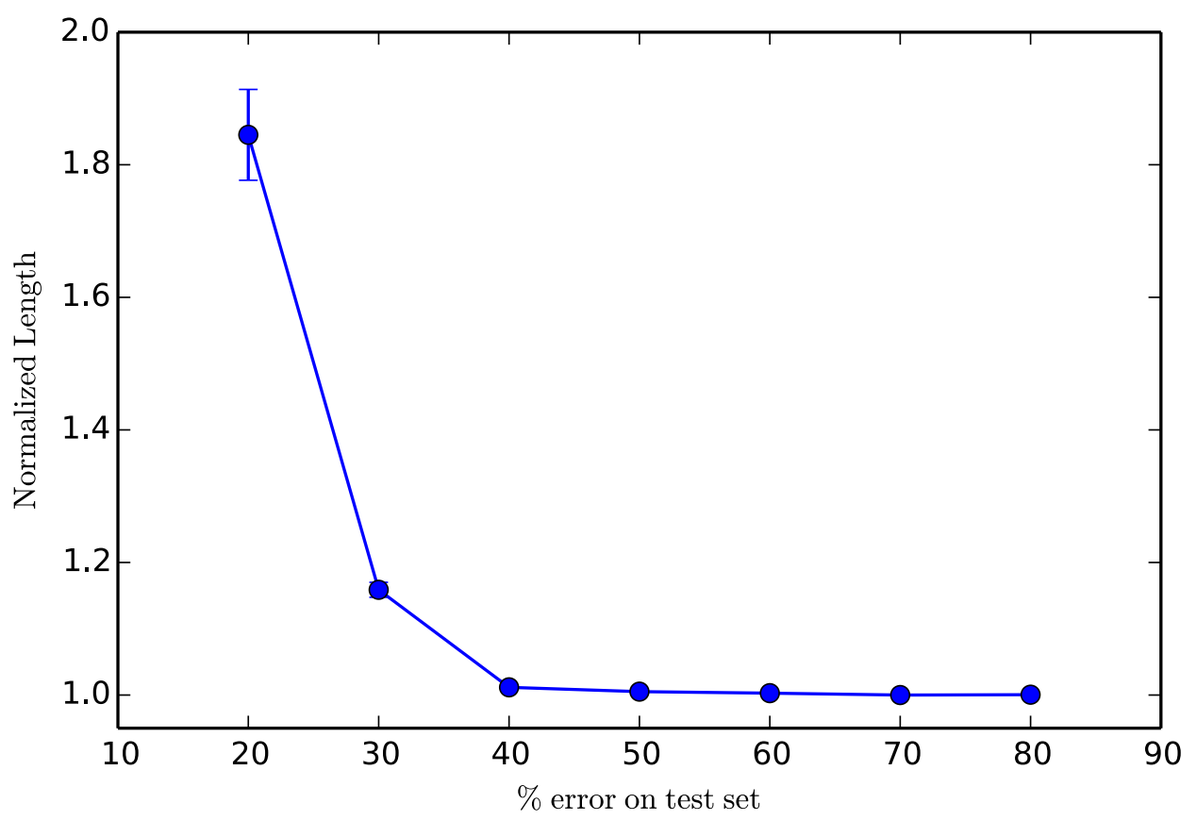
cubic polynomial



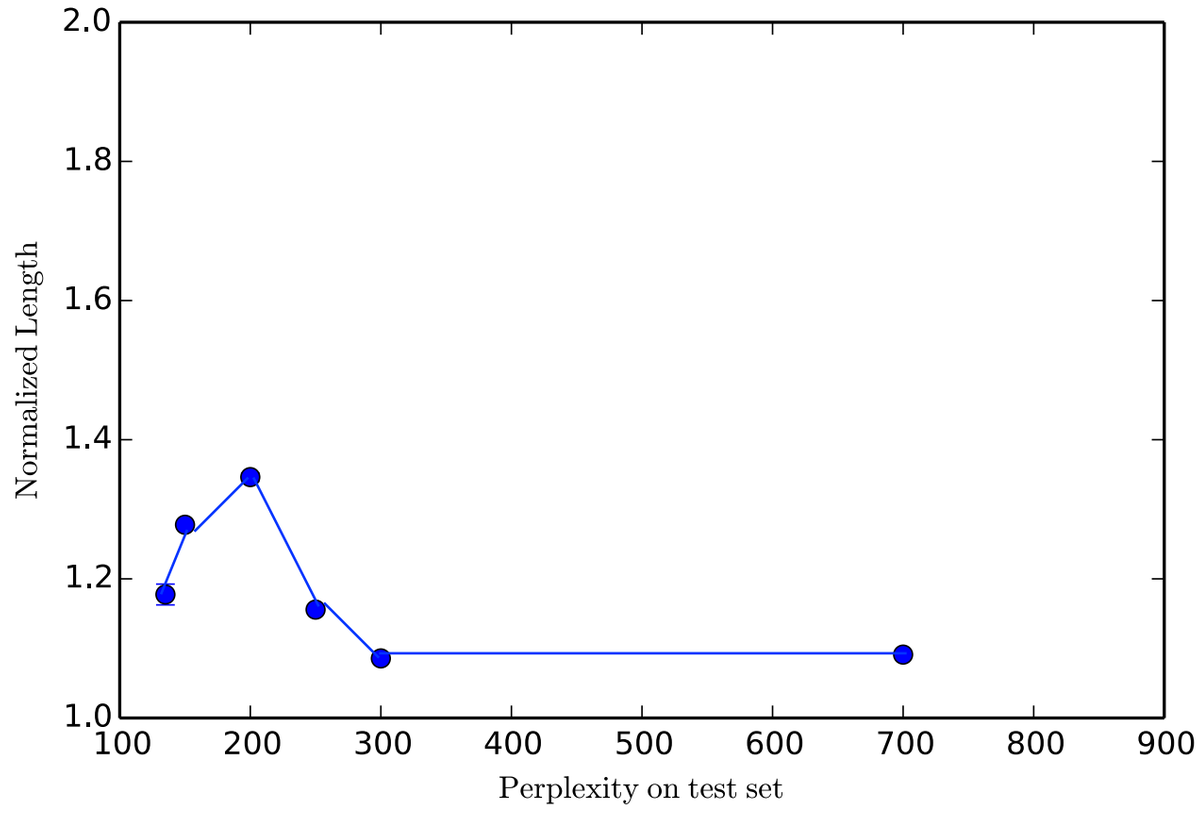
CNN/MNIST

NUMERICAL EXPERIMENTS

- Compute length of geodesic in Ω_u obtained by the algorithm and normalize it by the Euclidean distance. Measure of curviness of level sets.



CNN/CIFAR-10



LSTM/Penn

FOLLOW-UPS

- recent icml paper
-

ANALYSIS AND PERSPECTIVES

- #of components does not increase: no detected poor local minima so far when using typical datasets and typical architectures (at energy levels explored by SGD).
- Level sets become more irregular as energy decreases.
- Presence of “energy barrier”? extend to truncated Taylor?
- Kernels are back? CNN RKHS
- Open: “sweet spot” between overparametrisation and overfitting?
- Open: Robustness to noise in specification of activation function. Connection with Stochastic Gradient descent?