Exposure & CVA for Large Portfolios of Vanilla Swaps: The Thin-Out Optimization

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Abstract

In this article we present an efficient optimization for calculating the exposure and CVA for large portfolios of vanilla swaps. It is based on a "thin-out" procedure applied to fixed payment streams, which reduces a very frequent stream of payments to a much less frequent one. The procedure requires careful handling of the path-dependence that arises from the floating legs of the swaps. We compute the exposure and CVA for a large portfolio of fixed-for-floating swaps, and find that our approximation reduces the computation time for the portfolio to that of a single swap, with a roughly annual schedule. Moreover, the approximation maintains a particularly high accuracy. Our technique is entirely model independent and can be applied to various instruments such as FX-forwards, cross-currency swaps etc.

1 Introduction

In this paper, we address the calculation of counterparty exposure and credit valuation adjustment (CVA) for large portfolios of vanilla swaps.

In the past, banks had only to calculate the present value (PV) of their portfolios; there was almost no need to calculate the exposure to their counterparties. Since the recent global financial crisis, however, this is no longer the case. Indeed, various banking supervision authorities now require the fair valuation of counterparty exposure and CVA estimates. The literature on this subject is quite rich, see [3, 4, 6, 7, 8, 9, 10], and references therein.

Financial institutions often face computational problems in their exposure calculations, even for vanilla portfolios, which typically contain thousands or even millions of swaps. The PV calculation of such portfolios is based on yield curve analytics and thus is very fast. This is not the case for the portfolio *exposure*, however, which represents conditional expectations of future payments with respect to the observation date.

This approach to the exposure calculation is consistent with the CVA, and is known as the *modeling* approach. First formulated in Cesari et al. [7], it assumes the existence of an arbitrage-free model with a certain measure, and associates the exposure with a *future price* delivered by the model. The exposure calculation requires tools for computing conditional expectations—either analytics or, more generally, Monte Carlo methods equipped with a regression algorithm (see [1] for a detailed review).

To the best of our knowledge, possible optimizations of the future price calculations for large portfolios of swaps has not been extensively studied, although we note the work of Brigo-Masetti [5] in relation to this. They approximated moments of the portfolio of swaps under quite restrictive assumptions (e.g., unique floating rate sequence) using the volatilities and correlations of a corresponding BGM model. Applying moment matching techniques, they approximated the portfolio process using lognormal and shifted-lognormal models.

We go further and produce a much more accurate approximation of the exposures. Our algorithm is *model independent*, and is applicable to any portfolio of swaps containing arbitrary fixed and floating payments (Libors). This includes plain vanilla interest rate swaps, cross-currency basis swaps, FX-forwards, as well as semi-exotic structures such as those with amortizing notionals and so on. The only restriction is related to floating rate payments: they are required to be set in-advance and in their own currency. Quanto payments and Libors set in-arrears for example, are excluded from our algorithm.

We approximate the initial vanilla portfolio by a *single fixed payment stream* (PS) with relatively rare payments. For example, annual or even bi-annual frequency delivers excellent results. In other words, our algorithm *thins-out* the initial, possibly very frequent, payments in the portfolio. Since the effective payment stream depends on the initial yield curve, but not on any model parameters (volatilities, mean-reversions etc.), the approximation is model-independent.

We equip the resulting payment stream with certain path-dependent adjustments on exposure observation dates. They are related to the floating rates set before a given observation date, but paid after. Perhaps surprisingly, they contribute quite importantly to the approximation accuracy.

Having approximated the portfolio with a single fixed payment stream and certain adjustments, extremely rapid future price calculations are possible using Monte Carlo simulation. This approach is important if we want to aggregate exotics into the future price calculation; see [2] for the algorithmic exposure calculation using backward Monte Carlo pricing of arbitrarily complicated (exotic) instruments.

This article is organized as follows. In the next section, we introduce simple fixed payment streams and discuss how they split into past and future components, relative to a given observation date. In §3, we describe a thin-out procedure applied to a given payment stream. This is the main tool which will allow for a fast, yet accurate, approximation of the exposures. In §4, we consider a portfolio of swaps and introduce the notion of a "superswap," the aggregation of all payments into a single fixed payment stream, and its exposure. We also discuss how to decompose the floating rate payments. In §5, we apply the thin-out procedure to the superswap in two slightly different ways to give approximate expressions for the future price. In §6, we show some numerical examples and briefly discuss different choices of the set of thin-out dates. We conclude in §7.

We note that, although we only consider the single currency case here, our methods apply equally well to a portfolio of multi-currency swaps. In this more general case, we simply separate out the payments into different currencies, and apply the thin-out procedure to each.

2 Notation and Definitions

Following the modeling approach of Cesari et al. [7], we assume an arbitrage-free model having today's yield curve and calibrated to the swaption market, with numeraire N and expectation operator $\mathbb{E}$ in the corresponding measure.

Denote by $P(t,T)$ a zero-coupon bond with maturity T as seen from time t, and by $L(t,T)$ a Libor rate for the interval $[t,T]$. Then

$$L(t,T) = \frac{1}{\delta_{tT}} \left(\frac{1}{P(t,T)} - 1 \right), \tag{1}$$

where δ_{tT} is its day-count fraction. In the case of multiple Libor curves, we use *deterministic* spreads (adjustments) over the *theoretical* Libors $L(t,T)$. We will also denote the deterministic discount factors as $D(t) = P(0,t)$.

A schedule is an ordered vector of dates t_i, which we refer to as $\mathbf{t} = \{t_i\}$. (Vector quantities will be denoted in bold.) Given such a schedule, we introduce a fixed *payment stream*

$$\mathcal{S}_{\mathbf{t},\mathbf{A}},$$

which signifies a sequence of payments $\mathbf{A} = \{A_i\}$ on $\mathbf{t} = \{t_i\}$.

The PS satisfies the elementary arithmetic rules of a vector space

$$\alpha\, \mathcal{S}_{\mathbf{t},\mathbf{A}} = \mathcal{S}_{\mathbf{t},\alpha\,\mathbf{A}},$$

$$\mathcal{S}_{\mathbf{t}^{(1)},\mathbf{A}^{(1)}} + \mathcal{S}_{\mathbf{t}^{(2)},\mathbf{A}^{(2)}} = \mathcal{S}_{\mathbf{t},\mathbf{A}},$$

where $\mathbf{t}$ is a sorted union $\mathbf{t}^{(1)} \cup \mathbf{t}^{(2)}$ and $\mathbf{A}$ is the set of corresponding payment amounts calculated as follows. If element t_i belongs to both $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$, say, $t_i = t_k^{(1)} = t_j^{(2)}$, the corresponding amount is a sum $A_i = A_k^{(1)} + A_j^{(2)}$, otherwise, $A_i = A_k^{(n)}$ for $t_i = t_k^{(n)}$.

For an observation date τ, we *observe* the PS by taking

$$\mathcal{S}_{\mathbf{t},\mathbf{A}}(\tau) = \sum_{t_i<\tau} A_i\,\mathfrak{I}(t_i,\tau) + \sum_{t_i\geq\tau} A_i\,P(\tau,t_i).$$

The second term contains payments *on and after* the observation date, which are naturally seen at τ as zero-coupon bonds. The first term, which contains payments *before* the observation date, is a little trickier to handle: the payments are somehow “invested forward” from t_i to τ, into a term $\mathfrak{I}(t_i,\tau)$, such that

$$N(t_i)\,\mathbb{E}\left[\frac{\mathfrak{I}(t_i,\tau)}{N(\tau)}\;\middle|\;\mathcal{F}_{t_i}\right] = 1. \tag{2}$$

Such forward investments are, of course, not unique. We could, for example, invest in the numeraire or a zero-coupon bond:

$$\mathfrak{I}(t_i,\tau) = \begin{cases} \frac{N(\tau)}{N(t_i)} & \text{numeraire,} \\ \frac{1}{P(t_i,\tau)} & \text{zero-coupon bond.} \end{cases} \tag{3}$$

Note that (2) is satisfied by both types of forward investment.

3 The “Thin-Out” Procedure

Our main technical tool, which we call the *thin-out* procedure, is applied to a fixed PS $\mathcal{S}_{\mathbf{t},\mathbf{A}}$, given a set $\mathbf{T} = \{T_j\}$ of thin-out dates. These are expected to be (much) less dense than the initial payment schedule $\mathbf{t}$. The result of the procedure is a reduced or “thined-out” PS $\mathcal{S}_{\mathbf{T},\mathbf{B}}$, in which the following algorithm is used to compute the new amounts $\mathbf{B} = \{B_j\}$.

The idea is based on *splitting* the payments A_i between adjacent thin-out dates. Let us start first with a single payment $A = 1$ at t with two bounding thin-out dates, $T_1 < t < T_2$. We split the unit payment between two dates as follows:

$$1 \text{ at } t \;\begin{array}{l} \nearrow \;\frac{D(t)}{D(T_1)}\,\frac{T_2-t}{T_2-T_1} \text{ at } T_1, \\[2em] \searrow \;\frac{D(t)}{D(T_2)}\,\frac{t-T_1}{T_2-T_1} \text{ at } T_2. \end{array}$$

This corresponds to a linear arbitrage-free interpolation in maturity for zero-coupon bonds as seen at time $\tau < T_1$:

$$P(\tau,t) \simeq P(\tau,T_1)\,\frac{D(t)}{D(T_1)}\,\frac{T_2-t}{T_2-T_1} + P(\tau,T_2)\,\frac{D(t)}{D(T_2)}\,\frac{t-T_1}{T_2-T_1}.$$

Thus, the PS $\mathcal{S}_{t,1}$ gives rise to a reduced PS $\mathcal{S}_{\{T_1,T_2\},\{B_1,B_2\}}$ with new amounts

$$\begin{aligned} B_1 &= \frac{D(T)}{D(T_1)}\frac{T_2-t}{T_2-T_1}, \\ B_2 &= \frac{D(T)}{D(T_2)}\frac{t-T_1}{T_2-T_1}. \end{aligned}$$

Generalizing to an arbitrary PS $\mathcal{S}_{\mathbf{t},\mathbf{A}}$, the reduced version $\mathcal{S}_{\mathbf{T},\mathbf{B}}$ will consist of the new amounts

$$\begin{aligned} B_0 &= \sum_i \mathbb{1}_{t_i \le T_0} A_i \frac{D(t_i)}{D(T_0)}, \\ B_j &= \sum_i \mathbb{1}_{T_j < t_i \le T_{j+1}} A_i \frac{D(t_i)}{D(T_j)}\frac{T_{j+1}-t_i}{T_{j+1}-T_j} \\ &\quad + \sum_i \mathbb{1}_{T_{j-1} < t_i \le T_j} A_i \frac{D(t_i)}{D(T_j)}\frac{t_i-T_{j-1}}{T_j-T_{j-1}} \qquad \text{for} \qquad j=1,\cdots,N-1, \\ B_N &= \sum_i \mathbb{1}_{t_i > T_N} A_i \frac{D(t_i)}{D(T_N)}, \end{aligned} \tag{4}$$

where the index i runs over all payments in the initial PS $\mathcal{S}_{\mathbf{t},\mathbf{A}}$.

It is important to note that the thin-out procedure preserves the present value of the PS. Indeed, it is easy to show with (4) that

$$\text{PV}\left(\mathcal{S}_{\mathbf{T},\mathbf{B}}\right) = \mathcal{S}_{\mathbf{T},\mathbf{B}}(0) = \sum_j B_j D(T_j) = \sum_i A_i D(t_i).$$

Finally, since the thin-out amounts B_j are independent of all model parameters, the procedure is entirely *model-independent*.

4 The Superswap

A natural operation for a portfolio of vanilla swaps is the aggregation of all payments into a *superswap*. Given the potentially very large size of the portfolio, we will have frequent (e.g., daily) payments of fixed and floating rates. We suppose that at each payment date t_i we pay/receive

$$F_i + \sum_m n_i^{(m)} L_i^{(m)}, \tag{5}$$

where F_i is a deterministic amount (number), aggregated from fixed payments or spreads, and $n_i^{(m)}$ is the number of Libors $L_i^{(m)}$, aggregated from the floating payments.

Note that the Libors $L_i^{(m)}$ can have different start (fixing) dates $s_i^{(m)}$, but we assume that all the end dates fall on t_i, i.e. $L_i^{(m)} = L\left(s_i^{(m)}, t_i\right)$. As already mentioned in the introduction, this means that we deal with standard floating payments fixed at the beginning, and paid at the end, of their respective periods.

In the case of multiple Libor curves, we use *deterministic* spreads (adjustments) over the *theoretical* Libors $L(t,T)$ given by (1) calculated given a model (and its yield curve). In this way, the deterministic spread represents a *fixed* payment and is aggregated in the fixed part of our general payment given by (5).

The superswap future price for an observation date τ is the conditional expectation of all payments after τ, given by

$$V(\tau) = N(\tau)\,\mathbb{E}\left[\sum_i \mathbb{1}_{t_i>\tau}\frac{F_i + \sum_m n_i^{(m)} L_i^{(m)}}{N(t_i)}\;\middle|\;\mathcal{F}_\tau\right].$$

Although the majority of Libors will have start dates after the observation date, some of them will have been fixed *before* it; they form a “path-dependent” tail, which complicates the writing of the superswap and its future price in terms of a fixed payment stream.

4.1 The Libor Payments

Let $L(t,T)$ be a Libor rate for the interval $[t,T]$, as in (1), and let us fix an observation date τ. Then the Libor payment $\mathcal{P}_{L(t,T)}$ at T, as seen on the observation date τ looks as follows (with the dcf, for simplicity, equal to δ_{tT}):

$$\mathcal{P}_{L(t,T)}(\tau) = \begin{cases} P(\tau,t) - P(\tau,T), & \tau < t, \\ \frac{P(\tau,T)}{P(t,T)} - P(\tau,T), & t < \tau < T. \end{cases}$$

If the observation date is prior to the Libor start date, then the payment can be expressed in terms of a PS

$$\mathcal{P}_{L(t,T)} = \mathcal{S}_{\{t,T\},\{1,-1\}}, \qquad \tau < t.$$

However, for the case in which the Libor has been fixed but not yet paid, the payment cannot be exactly expressed in terms of a PS. In this case, we approximate by taking

$$\mathcal{P}_{L(t,T)}(\tau) \simeq \mathfrak{I}(t,\tau) - P(\tau,T), \qquad t < \tau < T,$$

with the forward investment chosen as in (3). Although the numeraire investment is faster in the Monte Carlo context, it is less accurate than the zero-coupon bond investment, so we will always use the latter in our numerical experiments. Due to the relatively short time span $T-t$ (typically about 3–6M), this approximation is fairly robust. It allows us to “forget” about the payment time T, giving a uniform treatment of all superswap payments in terms of fixed payment streams.

We now apply the thin-out procedure of §3 to these payment streams, and proceed with different approximations of the portfolio future price for given observation dates τ_n.

5 Approximations

Define a *master PS* as the fixed payment stream representation of the superswap, i.e. a sum of fixed payments and Libor ones:

$$\mathcal{S}_{\mathbf{t}',\mathbf{A}} = \mathcal{S}_{\mathbf{t},\mathbf{F}} + \sum_{m,i} n_i^{(m)} \mathcal{P}_{L_i^{(m)}}, \tag{6}$$

where $\mathbf{t}'$ is a sorted union of payments dates $\mathbf{t}$ and all Libor dates. The amounts $\mathbf{F} = \{F_i\}$ are fixed amounts, and the Libor payment streams are given by

$$\mathcal{P}_{L_i^{(m)}} = \mathcal{P}_{L\left(s_i^{(m)}, t_i\right)} = \mathcal{S}_{\{s_i^{(m)}, t_i\}, \{1, -1\}},$$

where $s_i^{(m)}$ is starting date of Libor $L_i^{(m)}$.

5.1 Full PS Approximation

We call our first approximation a *full PS* one. For a given superswap, its input consists of a master PS $\mathcal{S}_{\mathbf{t}', \mathbf{A}}$, as in (6), and a set of observation dates $\{\tau_n\}$. For each observation date we first truncate the master PS to those payments on or after τ_n, and calculate its τ_n-observation

$$\sum_i \mathbb{1}_{\tau_n \le t'_i} A_i \, P(\tau_n, t'_i),$$

where index i runs over master PS elements.

Secondly, we prepare a *past PS*

$$\mathcal{S}_{\mathbf{s}^{(n)}, \mathbf{A}^{(n)}}$$

depending on the observation date τ_n. The payment dates of this PS are the start dates $\mathbf{s}^{(n)}$ of those Libors which started before τ_n but will be paid after it. That is, the start dates of those Libor payments

$$N \, L\left(s, t\right),$$

with notional N, for which the start and end dates bound the observation date, $s < \tau_n \le t$. For each such Libor, we add a payment of N at time s to the past PS. This represents the *first* part of the Libor payment $\{1, -1\}$; the second part is already included in the truncated master PS.

We then calculate the τ_n-observation of the past PS by

$$\sum_j A_j^{(n)} \, \mathfrak{I}\left(s_j^{(n)}, \tau_n\right),$$

where index j runs over the past PS elements. Putting everything together, the superswap future price is

$$V(\tau_n) \simeq \sum_i \mathbb{1}_{\tau_n \le t'_i} A_i \, P(\tau_n, t'_i) + \sum_j A_j^{(n)} \, \mathfrak{I}\left(s_j^{(n)}, \tau_n\right).$$

We note that the past PS represents a “path-dependent” tail.

Clearly the full PS approximation preserves the PV of the exposures. It is a “mild” approximation—we only approximate the path-dependent tails—and does not give a big performance advantage: we still need to calculate the zero-coupon bonds for the truncated master PS.

Next we will apply the thin-out procedure to obtain a massive performance increase.

5.2 Thin-Out PS Approximations

The *thin-out PS* approximation starts with a given set $\mathbf{T} = \{T_i\}$ of thin-out dates which cover the payment dates interval $[t_0, t_N]$, but are less frequent. Its input is taken from the full PS section above: the master PS $\mathcal{S}_{\mathbf{t}',\mathbf{A}}$, and the past PS $\mathcal{S}_{\mathbf{s}^{(n)},\mathbf{A}^{(n)}}$, for each observation date τ_n.

We start by calculating a *reduced master PS* $\mathcal{S}_{\mathbf{T},\mathbf{B}}$ with dates $\mathbf{T}$ and amounts $\mathbf{B}$ by the thin-out procedure (4). As for the full PS approximation, this is truncated to those payments on or after τ_n, and we take its τ_n-observation

$$\mathcal{S}_{\mathbf{T},\mathbf{B}}(\tau_n) = \sum_i \mathbb{1}_{\tau_n \le T_i} B_i \, P(\tau_n, T_i),$$

where index i runs over reduced master PS elements.

Then, for each past PS $\mathcal{S}_{\mathbf{s}^{(n)},\mathbf{A}^{(n)}}$ corresponding to observation date τ_n, we apply the thin-out procedure (4) to obtain the reduced past PS $\mathcal{S}_{\mathbf{T}^{(n)},\mathbf{B}^{(n)}}$. The past thin-out dates $\mathbf{T}^{(n)}$ include the *global* thin-out dates $\mathbf{T}$ which lie between the first past date $s_0^{(n)}$ and the observation date. One can also add the first and last past PS dates $\mathbf{s}^{(n)}$ to the obtained past thin-out dates. Note that this option is used in our numerical experiments.

The reduced past PS τ_n-observation is

$$\mathcal{S}_{\mathbf{T}^{(n)},\mathbf{B}^{(n)}}(\tau_n) = \sum_j B_j^{(n)} \, \mathfrak{I}\left(T_j^{(n)}, \tau_n\right).$$

where index j runs over the reduced past PS elements.

Although the thin-out procedure preserves the PV of a fixed PS, our truncation of the reduced master PS slightly breaks it. As a byproduct of the reduced past PS calculation, we thus also compute a PV adjustment payment of J_n at τ_n, which serves to restore the PV of the future price.

The resulting future price for the thin-out PS approximations is

$$V(\tau_n) \simeq \sum_i \mathbb{1}_{\tau_n \le T_i} B_i \, P\left(\tau_n, T_i\right) + \sum_j B_j^{(n)} \, \mathfrak{I}\left(T_j^{(n)}, \tau_n\right) + J_n,$$

where the PV adjustment

$$J_n = V(0) - \sum_i \mathbb{1}_{\tau_n \le T_i} B_i \, D(T_i) - \sum_j B_j^{(n)} \, D\left(T_j^{(n)}\right)$$

depends only on the yield curve D.

The thin-out PS approximation gives a large performance advantage since the initial portfolio is approximated by a PS with possibly quite infrequent payments (in our experiments, annual frequency delivers excellent results).

The "path-dependent" tail $\mathcal{S}_{\mathbf{T}^{(n)},\mathbf{B}^{(n)}}(\tau_n)$ can be further removed by modifying the PV adjuster J_n. This gives rise to the *forward thin-out PS* approximation

$$V(\tau_n) \simeq \sum_i \mathbb{1}_{\tau_n \le T_i} B_i \, P\left(\tau_n, T_i\right) + J_n,$$

where now

$$J_n = V(0) - \sum_i \mathbb{1}_{\tau_n \le T_i} B_i \, D(T_i).$$

We note, however, that this modification spoils the approximation accuracy, as we will see in our numerical experiments.

6 Numerical Methods and Experiments

As a general rule, we can use either Monte Carlo simulations or analytic results to compute the exposures and CVA for our portfolio of swaps. The advantages of using a Monte Carlo simulation are the following:

- It becomes practical to aggregate the exposures of vanillas and exotics, as well as exposures from different asset classes.
- It permits the coupling to credit processes and the use of arbitrary collateral.
- It is fast for the thin-out approximations, although can be very slow for the exact and full PS methods.

At any rate, the Monte Carlo simulation should be equipped with either a generic backward Monte Carlo procedure to compute conditional expectations, or analytics for zero-coupon bonds. The former requires an algorithm to perform backwards propagation through the union of observation and payment dates (with possibly daily frequency for the full PS methods), whereas the latter involves the calculation of all zero-coupon bonds corresponding to future payments, for each observation date.

The main advantage of using analytic methods is that they are generally very fast. In the case at hand, however, they can only be applied to the forward thin-out method, since the others exhibit path-dependence. Moreover, such methods imply

- Future price calculations for a single asset class only, and excluding any additional exotic structures.
- A restriction to using only simple collateral.

All the following results have been obtained using a low-discrepancy Monte Carlo simulation with analytics for zero-coupon bonds.

We consider two distinct portfolios, containing 100 and 1000 single-currency fixed-for-floating IR swaps, all starting at $t = 0$, and with tenors out to 20Y, and take the observation dates at 1M intervals, out to 20Y. The tenors, the fixed and floating leg frequencies, the swap rates and the notionals are all randomized.

We consider the Hull-White one-factor model with volatility = 1%, mean reversion = 4% and an initial constant yield = 1%. Note that for other models, including the LMM, we find much the same qualitative behavior.

The CVA is calculated as

$$\sum_n (\tau_n - \tau_{n-1}) h\, e^{-h\tau_n}\, \mathbb{E}\left[\frac{\max\left(V(\tau_n), 0\right)}{N(\tau_n)}\right],$$

where $h = 0.5$% is the hazard rate.

We calculate the CVA via a Monte Carlo simulation using 2500, 5000 and 10000 paths with, in each case, 16 runs using different seeds. The mean values are taken over the 16 runs, together with the standard deviations. The timings are the mean time, in seconds, taken to calculate the CVA over the 16 runs, *after* the superswap PSs have been computed.

To calculate the future price distributions, we use a Monte Carlo simulation with 10000 paths, and a single run with a fixed seed. The exposures are plotted as the CDF of $V(\tau_n)$ for $\tau_n =$ 5Y, 10Y and 15Y.

6.1 Choice of Thin-Out Dates

Before showing our results, we will briefly discuss possible choices for the set of thin-out dates. An obvious one is to take equally-spaced dates at, say, 6M, 1Y and 2Y intervals, all out to 20Y.

For our 1000-swap portfolio, the master PS histogram is shown in Figure 1 below, with the corresponding reduced master PS, in which we have taken equally spaced thin-out dates at 1Y intervals. Note that the master PS payments are denser closer to the origin, but this is just an artifact of how we have set up our portfolio of swaps: the tenors out to 20Y are randomized, so all swaps contribute payments at small times, but only some contribute payments at large times.

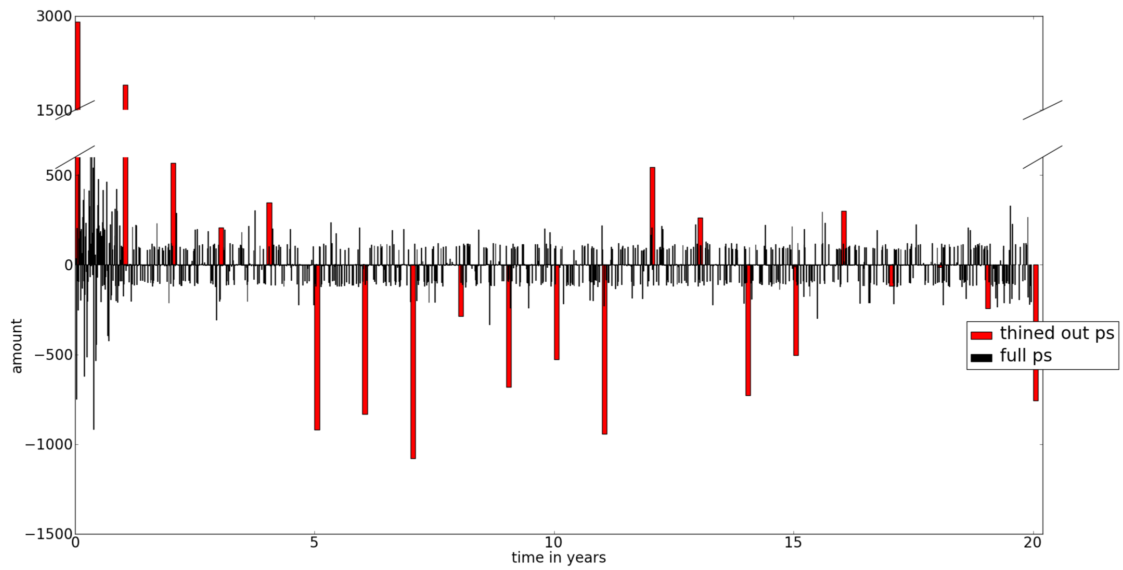

Figure 1: Superswap PS for a portfolio of 1000 swaps and 1Y thin-out dates at equally spaced intervals.

Other choices of thin-out dates are of course possible, the natural question being whether or not there is a way of specifying the "optimal" set. For instance, another choice is to set the thin-out dates to be those dates of the master PS for which the (absolute values of the) payment amounts are largest. That is, to try to capture the *density* of the master PS.

For our specific portfolios of swaps, we find the following choice provides a slightly better thin-out approximation to the exact CVA, relative to the simple choice of equally-spaced dates: for a given number of thin-out dates, we split the master PS into that number of equally spaced intervals; and, within each interval, we set the thin-out date to be that date for which the (absolute value of the) master PS amount is largest.

For this slightly different set of thin-out dates, starting with a 1Y interval, the reduced master PS is shown in Figure 2 below. All the following results make use of this procedure, with initial thin-out intervals of 6M, 1Y and 2Y.

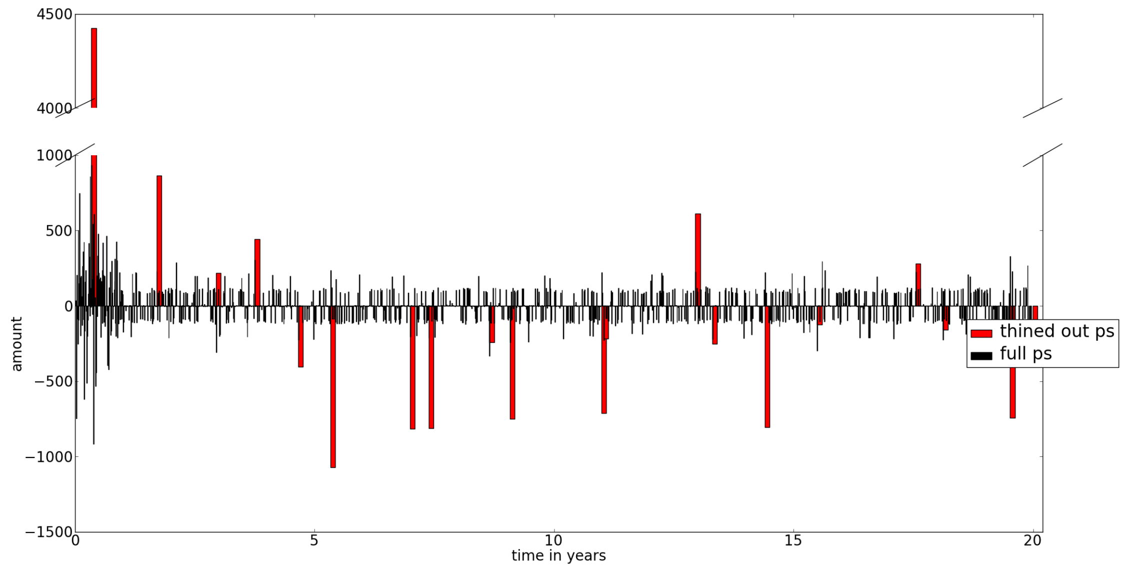

Figure 2: Superswap PS for a portfolio of 1000 swaps and 1Y thin-out intervals, with thin-out dates set according to the maximum amount in each such interval.

6.2 CVA

The CVA results for the Hull-White model, with parameters as detailed above, are shown in Tables 1 and 2.

		2500 paths		5000 paths		10000 paths	
Method		CVA	Time	CVA	Time	CVA	Time
exact		1.6340 ± 0.0040	6.51	1.6329 ± 0.0024	12.46	1.6339 ± 0.0018	24.85
full PS		1.6411 ± 0.0040	5.17	1.6400 ± 0.0024	10.17	1.6409 ± 0.0018	20.56
thin-out PS	6M	1.6389 ± 0.0039	0.35	1.6378 ± 0.0024	0.69	1.6387 ± 0.0018	1.42
	1Y	1.6240 ± 0.0039	0.22	1.6229 ± 0.0024	0.44	1.6239 ± 0.0018	0.85
	2Y	1.6215 ± 0.0038	0.15	1.6204 ± 0.0024	0.29	1.6213 ± 0.0019	0.59
thin-out fwd PS	6M	1.5314 ± 0.0036	0.26	1.5303 ± 0.0022	0.52	1.5311 ± 0.0017	1.06
	1Y	1.5145 ± 0.0036	0.15	1.5135 ± 0.0022	0.29	1.5143 ± 0.0016	0.58
	2Y	1.5029 ± 0.0035	0.09	1.5019 ± 0.0022	0.17	1.5028 ± 0.0017	0.35

Table 1: CVA for a portfolio of 100 swaps.

Method		2500 paths CVA	Time	5000 paths CVA	Time	10000 paths CVA	Time
exact		9.7405 ± 0.0207	35.32	9.7349 ± 0.0157	57.27	9.7378 ± 0.0081	102.22
full PS		9.7588 ± 0.0207	13.70	9.7532 ± 0.0157	27.19	9.7560 ± 0.0081	54.66
thin-out PS	6M	9.7406 ± 0.0207	0.38	9.7350 ± 0.0157	0.75	9.7378 ± 0.0081	1.50
	1Y	9.7275 ± 0.0206	0.23	9.7220 ± 0.0157	0.44	9.7247 ± 0.0081	0.89
	2Y	9.7382 ± 0.0206	0.16	9.7327 ± 0.0158	0.30	9.7355 ± 0.0082	0.61
thin-out fwd PS	6M	9.3010 ± 0.0197	0.28	9.2956 ± 0.0150	0.55	9.2982 ± 0.0078	1.11
	1Y	9.3072 ± 0.0198	0.15	9.3018 ± 0.0150	0.29	9.3045 ± 0.0078	0.59
	2Y	9.3337 ± 0.0198	0.09	9.3283 ± 0.0151	0.17	9.3309 ± 0.0078	0.36

Table 2: CVA for a portfolio of 1000 swaps.

Clearly, the exact calculation is prohibitively slow. The full PS approximation is only slightly faster, but a *huge* increase in speed can be seen when using the thin-out approximations. The thin-out PS approximation is extremely accurate for 6M and 1Y intervals, although less frequent thin-out dates show a corresponding loss of accuracy. The dependence on thin-out date frequency is not simply linear, however, since we have not chosen equally spaced thin-out dates.

The thin-out forward PS approximation is not particularly good, although is the fastest of the methods. Indeed, as we have mentioned above, we can use purely analytic results with this method, and should thus expect possible further increases in speed. However, the accuracy is not nearly so good, relative to the thin-out PS method.

Although we have shown only the results for the Hull-White model here, similar qualitative comments can be made for the LMM case.

6.3 Future Price Distributions

The future price distributions for the Hull-White model, with parameters as above, are shown in Figures 3-8.

Clearly, the thin-out procedure provides a very accurate approximation of the exact future price distributions at all time horizons. The approximate distributions show little deviation from the exact results, even for very infrequent thin-out dates.

Similar qualitative comments can again be made for the LMM case.

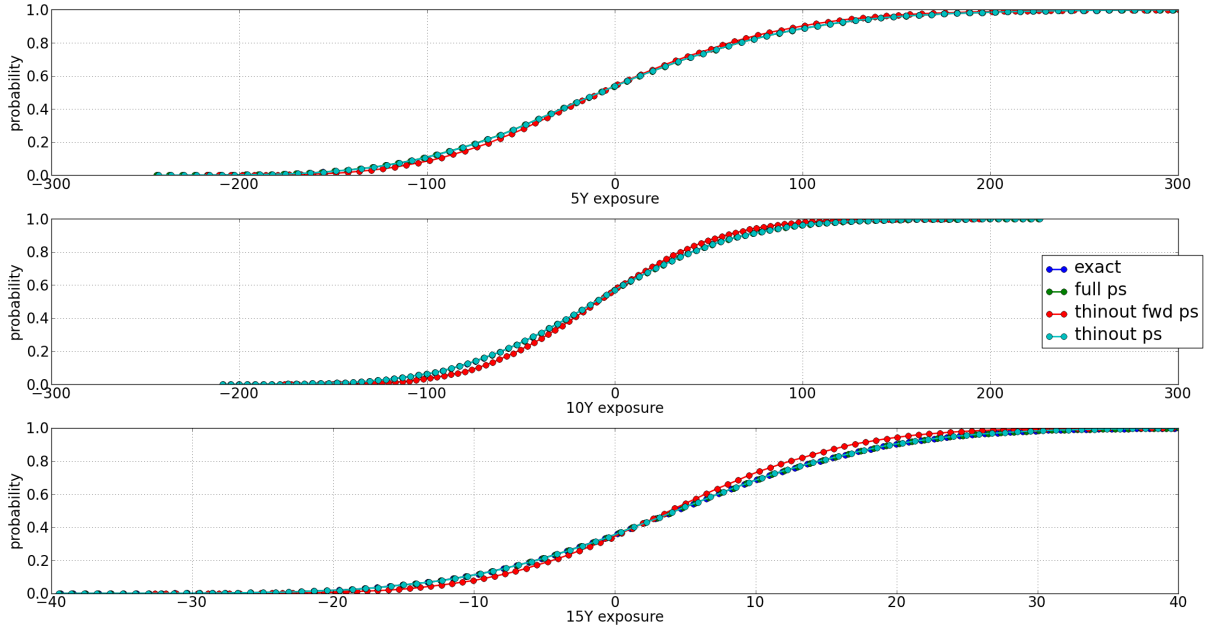

Figure 3: Future price distributions for a portfolio of 100 swaps at 5Y, 10Y and 15Y for 6M thin-out intervals.

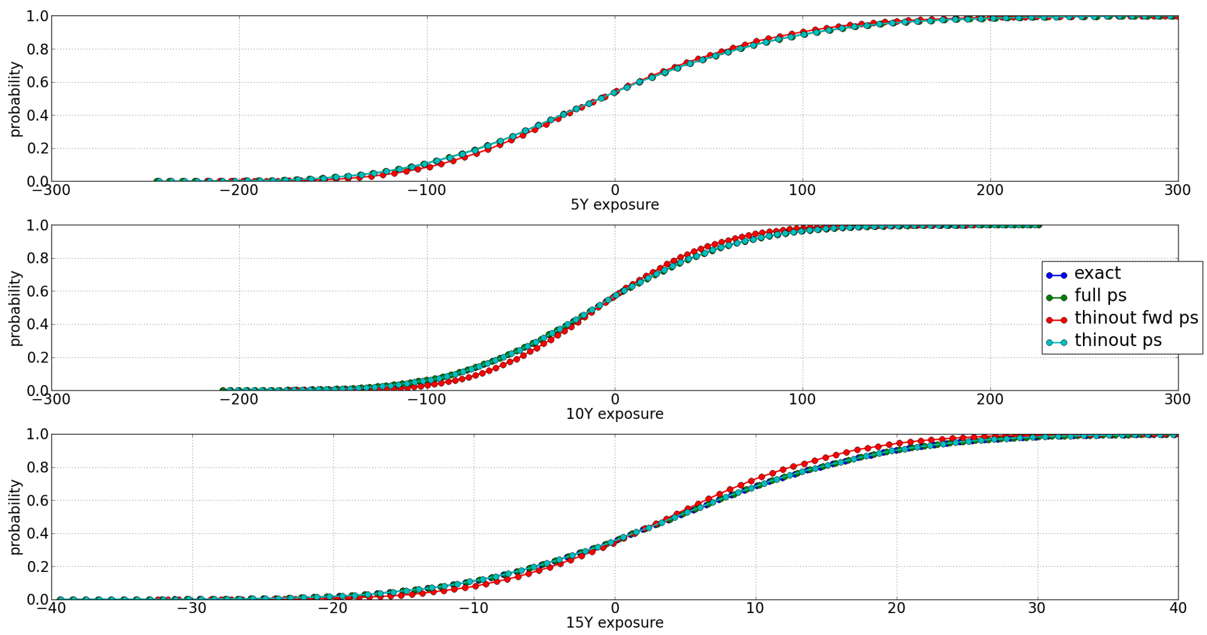

Figure 4: Future price distributions for a portfolio of 100 swaps at 5Y, 10Y and 15Y for 1Y thin-out intervals.

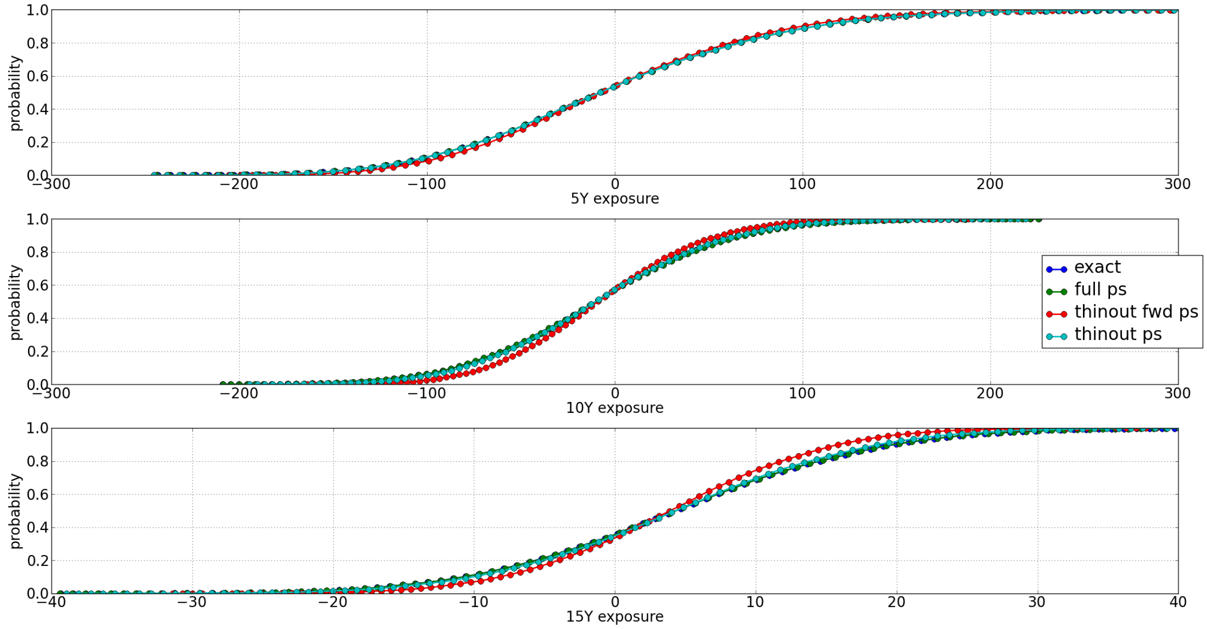

Figure 5: Future price distributions for a portfolio of 100 swaps at 5Y, 10Y and 15Y for 2Y thin-out intervals.

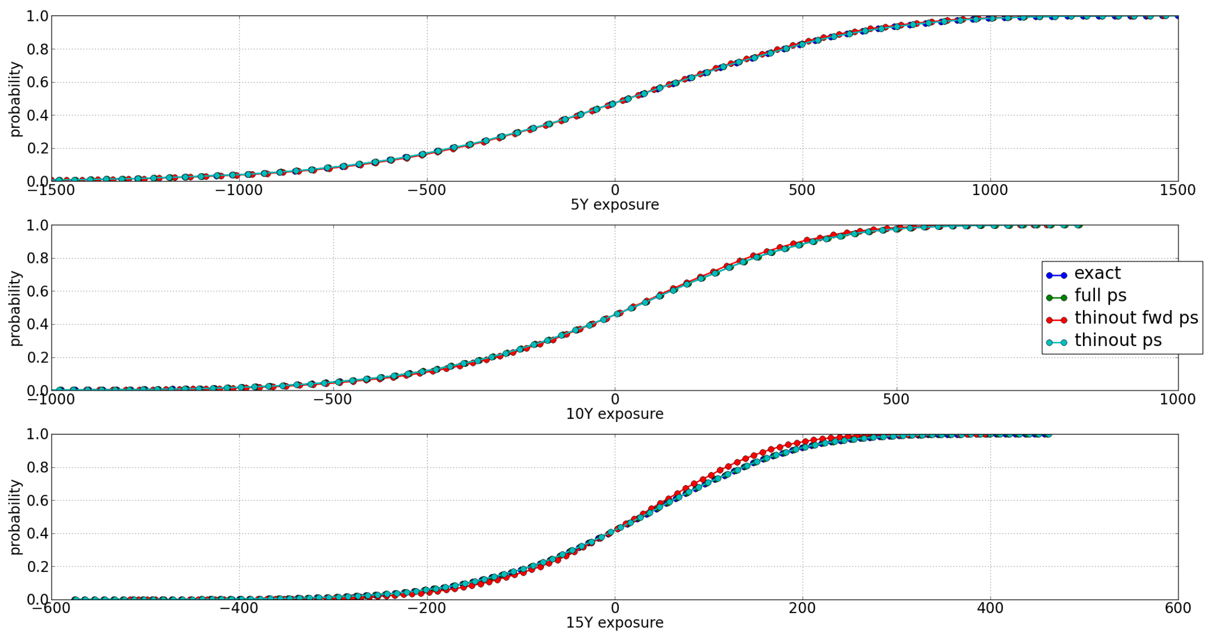

Figure 6: Future price distributions for a portfolio of 1000 swaps at 5Y, 10Y and 15Y for 6M thin-out intervals.

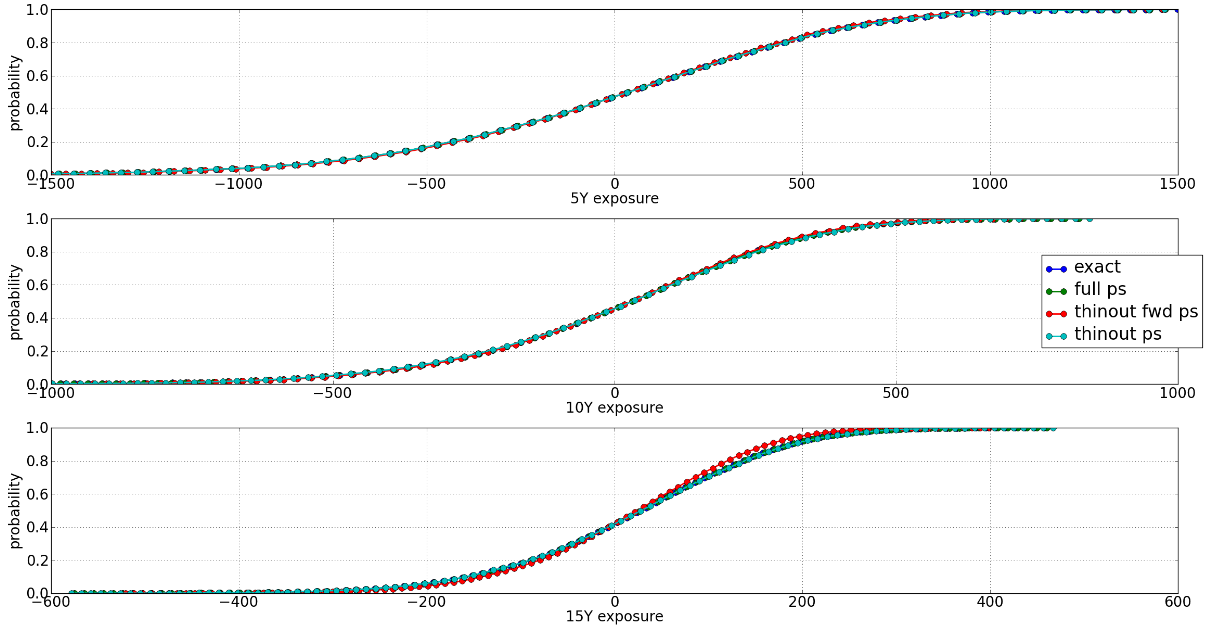

Figure 7: Future price distributions for a portfolio of 1000 swaps at 5Y, 10Y and 15Y for 1Y thin-out intervals.

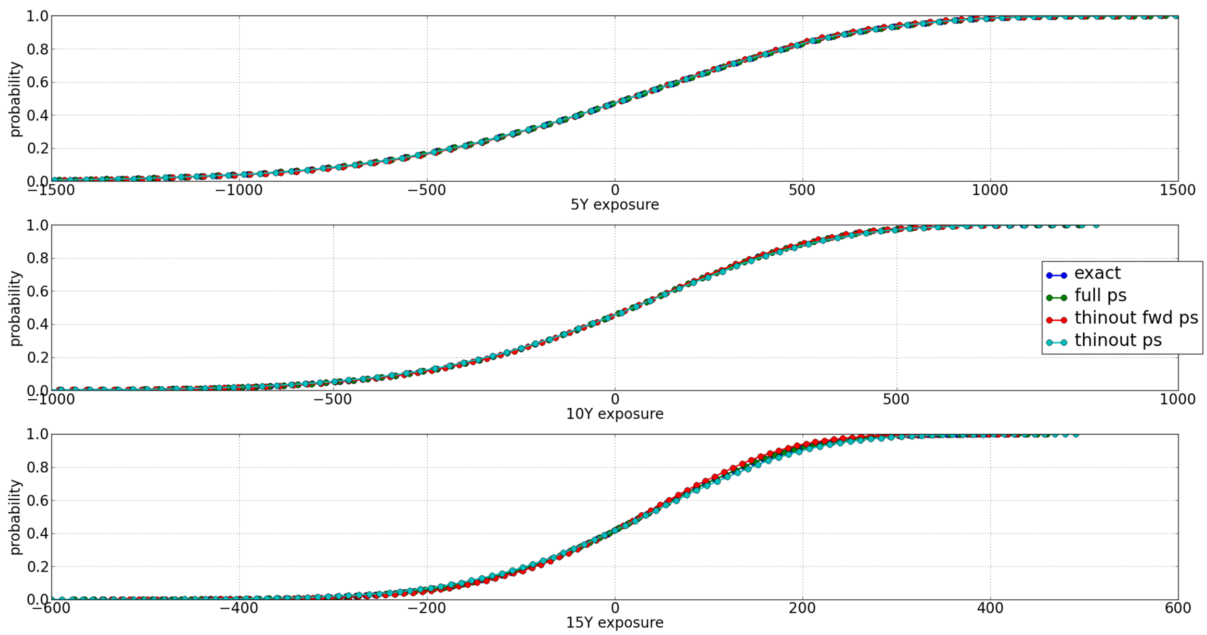

Figure 8: Future price distributions for a portfolio of 1000 swaps at 5Y, 10Y and 15Y for 2Y thin-out intervals.

7 Conclusion

We have presented an efficient optimization for calculating the exposure of a large portfolios of IR swaps. It is based on a thin-out procedure of payment streams and requires careful handling of the path-dependent streams that come from the floating rate payments.

The methods can be easily applied to the simpler case of a large portfolio of FX-forwards and also be generalized to multi-currency structures containing, for example, cross-currency swaps.

We have described various approximations to use in the future price calculations. The first, our "full PS" approximation, is very close to the exact result—and this indeed justifies our approximation of Libor payments in terms of fixed PSs—but is not much faster.

Our "thin-out PS" and "forward thin-out PS" approximations, on the other hand, exhibit a massive acceleration in the future price calculation times. Moreover, the former is very accurate, relative to the exact results, even when using infrequent thin-out intervals such as 1-2Y. The forward approximation is less accurate, but can be used with analytic methods.

The thin-out approximation clearly scales extremely well. For an arbitrarily large portfolio of swaps, once the thined-out payment streams are calculated, the exposure calculation will be as fast as that for a single swap with roughly annual payments.

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References

[1] L. Andersen and V. Piterbarg. *Interest Rate Modeling*. Atlantic Financial Press, 2010.

[2] A. Antonov, S. Issakov, and S. Mechkov. Algorithmic Exposure and CVA for Exotic Derivatives. Numerix working paper, available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1960773, 2011.

[3] D. Brigo and A. Capponi. Bilateral counterparty risk with application to CDSs". *Risk*, 2010. Available at http://www.risk.net/digital_assets/859/brigo_technical.pdf.

[4] D. Brigo, A. Capponi, P. A., and V. Papatheodorou. Collateral Margining in Arbitrage-Free Counterparty Valuation Adjustment including Re-Hypotecation and Netting". Available at http://arxiv.org/pdf/1101.3926.pdf, 2011.

[5] D. Brigo and M. Masetti. A Formula for Interest Rate Swaps Valuation under Counterparty Risk in presence of Netting Agreements. Available at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=717344, 2005.

[6] E. Canabarro and D. Duffie. *Measuring and Marking Counterparty Risk*, chapter 9. Euromoney Books, 2003. Available at http://darrellduffie-com.gowest1.gowesthosting.com/uploads/surveys/DuffieCanabarro2004.pdf.

[7] G. Cesari, J. Aquilina, N. Charpillon, Z. Filipovic, G. Lee, and I. Manda. *Modelling, Pricing, and Hedging Counterparty Credit Exposure: A Technical Guide*. Springer-Verlag, 2010.

[8] J. Gregory. *Counterparty Credit Risk: The new challenge for global financial markets*. Wiley Finance, 2010.

[9] M. Pykhtin, editor. *Counterparty Credit Risk Modelling: Risk Management, Pricing and Regulation*. Risk Books, 2005.

[10] M. Pykhtin and D. Rosen. Pricing counterparty risk at the trade level and credit valuation adjustment allocations. *The Journal of Credit Risk*, 6:3–38, 2010. Available at http://www.risk.net/digital_assets/4557/v6n4a1.pdf.