

Hamiltonian Zoo

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No Matter Where They're From

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I. MODEL HAMILTONIANS

1. Coulomb

$$H_{\text{Coulomb}} = \frac{q_1 q_2}{4\pi\epsilon_0 r} \quad (1)$$

A. Electronic Structure

1. Molecular Hamiltonian

$$H_{\text{BO}} = -\sum_A \frac{1}{2M_A} \nabla_A^2 - \sum_i \frac{1}{2} \nabla_i^2 - \sum_{iA} \frac{Z_A}{r_{iA}} + \sum_{i>j} \frac{1}{r_{ij}} + \sum_{B>A} \frac{Z_A Z_B}{R_{AB}} \quad (2)$$

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2. Born-Oppenheimer Model

3. Watson Model

B. Fine and Hyperfine Structure

1. Stark

2. Zeeman

$$H_{\text{Zeeman}} = -\frac{\mu_B (g_l \vec{L} + g_s \vec{S})}{\hbar} \cdot \vec{B} \quad (3)$$

3. Complete magnetic dipole interaction

$$H_{\text{Magnetic Dipole}} = \frac{g_I \mu_N \mu_B \mu_0}{2\pi} \left(\frac{1}{L_z} \sum_i \frac{\hat{l}_{zi}}{r_i^3} \vec{I} \cdot \vec{L} + \frac{2g_s}{S_z} \sum_i \frac{\hat{s}_{zi}}{r_i^3} (3(\vec{I} \cdot \hat{r})(\vec{S} \cdot \hat{r}) - \vec{I} \cdot \vec{S}) + \frac{g_s}{3S_z} \sum_i \hat{s}_{zi} \delta^3(\vec{r}_i) \vec{I} \cdot \vec{S} \right) \quad (4)$$

4. Frosch-Foley

$$H_{\text{Frosch-Foley},1} = \frac{e\hbar}{2mc} \frac{2\mu \cdot \mathbf{L}}{r_1^3} + \frac{i2\mu_0}{\hbar} \mathbf{S} \cdot [\mathbf{p} \times \mathbf{A}] + \frac{2e^2\hbar}{4m^2c^3} \mathbf{S} \cdot [\epsilon' \times \mathbf{A}]. \quad (5)$$

$$H_{\text{Frosch-Foley},2} = aI_zL_z + b\vec{I} \cdot \vec{S} + cI_zS_z, \quad (6)$$

where

$$a \equiv 2g_I\mu_0\mu_N \sum_i \left\langle \frac{1}{r_i^3} \right\rangle, \quad (7)$$

$$b \equiv 2g_I\mu_0\mu_N \sum_i \left\langle \frac{2e^2mc^2\mathcal{E}_r}{(E + eV + mc^2)^2} \frac{(1/2 \sin^2 \chi + \cos^2 \chi)}{r_i^2} - \frac{1}{2} \left(\frac{3 \cos^2 \chi - 1}{r_i^3} \right) \right\rangle, \text{ and} \quad (8)$$

$$c \equiv 2g_I\mu_0\mu_N \sum_i \left\langle \frac{2e^2mc^2\mathcal{E}_r}{(E + eV + mc^2)^2} \frac{(1/2 \sin^2 \chi - \cos^2 \chi)}{r_i^2} + \frac{3}{2} \left(\frac{3 \cos^2 \chi - 1}{r_i^3} \right) \right\rangle. \quad (9)$$

5. *Spin-orbit*

6. *Orbit-orbit*

7. *Spin-other-orbit (Gaunt)*

8. *Fermi Contact*

C. Special Relativity

1. *Dirac*

$$H = \beta mc^2 - eV + c\boldsymbol{\alpha} \cdot \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right). \quad (10)$$

2. *Klein-Gordon*

3. *Darwin ($1e^-$)*

$$H_{\text{Darwin-1}} = \alpha_{\text{FS}}^2 \frac{\pi Z}{2} \sum_i \delta(\vec{r}_i) \quad (11)$$

4. *Darwin ($2e^-$)*

$$H_{\text{Darwin-2}} = \alpha_{\text{FS}}^2 \pi \sum_{i < j} \delta(\vec{r}_{ij}) \quad (12)$$

5. *Mass Velocity*

$$H_{\text{Mass-Velocity}} = -\alpha_{\text{FS}}^2 \frac{1}{8} \sum_i \nabla_i^4. \quad (13)$$

6. *Breit-Pauli*

7. *Dirac-Coulomb-Breit*

D. Quantum Electrodynamics

1. *Aracki-Sucher*

$$H_{\text{Aracki-Sucher}} = -\alpha_{\text{FS}}^3 \frac{7}{6\pi} \sum_{i>j} \lim_{a \rightarrow 0} \left(\frac{\theta(r_{ij} - a)}{r_{ij}^3} + 4\pi (\gamma + \ln a) \delta(\vec{r}_{ij}) \right) \quad (14)$$

2. *One-Loop*

$$H_{1\text{-loop}} = \alpha_{\text{FS}}^4 \pi Z^2 \left(\frac{427}{96} - \ln 2 \right) \sum_i \delta(\vec{r}_i) \quad (15)$$

3. *Schwinger-Dyson*

E. Nuclear Motion

$$H_{\text{Nuc}} = -\frac{\hbar^2}{2} \sum_{i=1}^N \sum_{\alpha=1}^3 \frac{1}{M_i} \frac{\partial^2}{\partial R_{i\alpha}^2} + V(\mathbf{R}_1, \dots, \mathbf{R}_N) \quad (16)$$

1. *Rigid Rotor Harmonic Oscillator*

2. *Centrifugal distortion*

3. *Centrifugal distortion*

F. Long-range Interactions

1. *van der Waals*

2. *Lenard-Jones*

3. *Casimir-Polder*

4. *Resonance dipole-dipole*

5. *Meath*

6. *Axilrod-Teller*

G. Hubbard Models

1. *Non-interacting Hubbard model*

$$H = \sum_{\mathbf{k}} \sum_{\sigma} (\varepsilon_{\mathbf{k}} - \mu) n_{\mathbf{k},\sigma} \quad (17)$$

2. *1D Hydrogen chain Hubbard model*

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} (\hat{c}_{i,\sigma}^{\dagger} \hat{c}_{j,\sigma} + \hat{c}_{j,\sigma}^{\dagger} \hat{c}_{i,\sigma}) + U \sum_{i=1}^N \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}, \quad t > 0, U > 0 \quad (18)$$

3. 1D Hydrogen chain Hubbard model with μ term

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} (\hat{c}_{i,\sigma}^{\dagger} \hat{c}_{j,\sigma} + \hat{c}_{j,\sigma}^{\dagger} \hat{c}_{i,\sigma}) + U \sum_{i=1}^N \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} - \mu \sum_i (n_{i\uparrow} + n_{i\downarrow}) \quad (19)$$

4. With particle-hole symmetry

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} (\hat{c}_{i,\sigma}^{\dagger} \hat{c}_{j,\sigma} + \hat{c}_{j,\sigma}^{\dagger} \hat{c}_{i,\sigma}) + U \sum_{i=1}^N \left(\hat{n}_{i\uparrow} - \frac{1}{2} \right) \left(\hat{n}_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i (n_{i\uparrow} + n_{i\downarrow}) \quad (20)$$

5. Bose-Hubbard (Boson Hubbard)

$$H = -t \sum_{\langle i,j \rangle} \hat{b}_i^{\dagger} \hat{b}_j + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i \quad (21)$$

6. Fermi-Hubbard (Fermi Hubbard)

7. Bose-Fermi-Hubbard

8. Jaynes-Cummings-Hubbard

9. Tavis-Cummings-Hubbard

10. Periodic Anderson Model

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} (\hat{c}_{i,\sigma}^{\dagger} \hat{c}_{j,\sigma} + \hat{c}_{j,\sigma}^{\dagger} \hat{c}_{i,\sigma}) + V \sum_{\langle i,j \rangle} \sum_{\sigma} (c_{j\sigma}^{\dagger} d_{i\sigma} + d_{i\sigma}^{\dagger} c_{j\sigma}) + U \sum_{i=1}^N \left(\hat{n}_{di\uparrow} - \frac{1}{2} \right) \left(\hat{n}_{di\downarrow} - \frac{1}{2} \right) - \mu \sum_i (n_{di\uparrow} + n_{di\downarrow} + n_{ci\uparrow} + n_{ci\downarrow}), \quad (22)$$

where $d_{i\sigma}$ is the annihilation operator for an electron in the localized band at site i , $c_{i\sigma}$ is the annihilation operator for an electron in the conduction band at site i , V is termed the hybridization and is the transition amplitude of an electron moving from the localized band at site i to the conduction band at site i or vice versa.

H. Disorder Models

1. Tight-binding Model with Onsite Disorder

$$H = \sum_i \omega_i \hat{c}_i^\dagger \hat{c}_i + t \sum_{\langle i,j \rangle} \hat{c}_i^\dagger \hat{c}_j, \quad (23)$$

where ω_i is drawn from a uniform distribution of width W , the disorder strength, and the angular brackets indicate a sum over nearest-neighbours only.

2. Tight-binding Model with Continuous Off-diagonal Disorder

3. Tight-binding Model with Binary Off-diagonal Disorder

4. Long-range Disorder Models

I. Mean-field Hamiltonians

1. Clean Bose-Hubbard

$$H_{\text{MF}} = \sum_i \left[-\mu \hat{n}_i + \frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - zt (\psi^* \hat{b}_i + \psi \hat{b}_i^\dagger) + zt \psi^* \psi \right] \quad (24)$$

J. Spin lattices

1. XY model

$$H = -J \sum_i \left(\frac{1+\gamma}{2} \sigma_{x,i} \sigma_{x,i+1} + \frac{1-\gamma}{2} \sigma_{y,i} \sigma_{y,i+1} \right) - h \sum_i \sigma_{z,i} \quad (25)$$

$$H = -J \sum_i \left(\frac{1+\gamma}{2} x_i x_{i+1} + \frac{1-\gamma}{2} y_i y_{i+1} \right) - h \sum_i z_i \quad (26)$$

K. Open Quantum Systems

1. *Rabi model*

2. *Spin-Boson*

3. *Feynman-Vernon*

4. *Leggett-Caldeira*

5. *Ishizaki-Fleming*

6. *Jaynes-Cummings*

L. Models of Superconductivity

1. *BCS*

$$H_{\text{BCS}} = \sum_i \epsilon_i n_i + g \sum_{ij} c_{i\uparrow}^\dagger c_{-i\downarrow}^\dagger c_{-j\downarrow} c_{j\uparrow} \quad (27)$$

2. *Richardson-Gaudin*

3. *Exactly solvable pairing Hamiltonians*

SU(2), Rank 1 algebra

$$H_{\text{SU}(2)} = \sum_i \epsilon_i n_i - g \sum_{ij} P_i^+ P_j \quad (28)$$

SO(5), Rank 2 algebra

$$H_{\text{SO}(5)} = \sum_i \epsilon_i n_i - g \sum_{ijk} P_{ik}^+ P_{jk} \quad (29)$$

SO(8), Rank 4 algebra

$$H_{\text{SO}(8)} = \sum_i \epsilon_i n_i - g_T \sum_{ijk} P_{ik}^+ P_{jk} - g_S \sum_{ijk} D_{ik}^+ D_{jk} \quad (30)$$

4. *t*-*J* model

M. Models of Superfluidity

1. *2D p-wave Fermi superfluid*

$$H_{2DFS} = \sum_i \epsilon_i a_i^\dagger a_i + \frac{i\Delta_i}{2} (a_i^\dagger a_{-i}^\dagger + \text{H.c.}) \quad (31)$$

N. Spin models

1. *Ising*

2. *Transverse Ising*

3. *XY*

4. *XYZ*

O. Heisenberg

P. J1-J2 Model

Q. Majumdar-Ghosh

R. AKLT Model

S. Kitaev Models

1. *Toric Code*

2. *Ocko-Yoshida*

3. *Honeycomb Model*

$$H_{\text{Honeycomb}} = -J_x \sum_{x\text{-links}} x_i x_j - J_y \sum_{y\text{-links}} y_i y_j - J_z \sum_{z\text{-links}} z_i z_j \quad (32)$$

T. Miscellaneous

1. *Fertitta-Booth auxiliary Hamiltonian*

$$H_{\text{FB}} = \sum_{x=0}^{N_{\text{cells}}-1} \sum_{\alpha}^{N_{\text{Impurities}}} \sum_k^{N_{\text{aux}}} \nu_{\alpha k} \left(\hat{c}_{\alpha+x}^{\dagger} \hat{c}_{k+x} + \hat{c}_{k+x}^{\dagger} \hat{c}_{\alpha+x} \right) + \varepsilon_k \hat{c}_{k+x}^{\dagger} \hat{c}_{k+x} \quad (33)$$

Matrix Zoo

U. 2-level systems (spin-1/2 particles)

$$z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (34)$$

$$A_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, A_3 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_4 \equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad (35)$$

$$s_+ \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{x + iy}{2} \quad (36)$$

$$s_- \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{x - iy}{2} \quad (37)$$

$$s_\alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{\mathbb{1} + z}{2} \equiv b \quad (38)$$

$$s_\beta \equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\mathbb{1} - z}{2} \quad (39)$$

V. 3-level systems (spin-1 particles)

$$\begin{aligned} z_3 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & x_3 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & y_3 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} & s_{3+} &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & s_{3-} &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & x_3^2 &\equiv \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} & y_3^2 &\equiv \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ \lambda_1 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &\equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &\equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda_7 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\ \lambda_8 &\equiv \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} & s_3^1 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^2 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^3 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$z_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, x_3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, y_3 \equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad (40)$$

$$\lambda_1 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(x_3 + z_3 x_3 + x_3 z_3), \lambda_2 \equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(y_3 + y_3 z_3 + z_3 y_3) \quad (41)$$

$$\lambda_3 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2\mathbb{1} + \frac{1}{2}(z_3 - \frac{3}{2}x_3^2 - \frac{3}{2}y_3^2), \lambda_4 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{1}{2}(x_3^2 - y_3^2) \quad (42)$$

$$\lambda_5 \equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \frac{1}{2}(x_3 y_3 + y_3 x_3), \lambda_6 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{2}(x_3 - z_3 x_3 - x_3 z_3) \quad (43)$$

$$\lambda_7 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{1}{2}(y_3 - y_3 z_3 - z_3 y_3), \lambda_8 \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{1}{\sqrt{3}}(-2\mathbb{1} + \frac{3}{2}(z_3 + \frac{1}{2}x_3^2 + \frac{1}{2}y_3^2)) \quad (44)$$

$$B_1 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_2 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_4 \equiv \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_5 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (45)$$

$$B_6 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, B_7 \equiv \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, B_8 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, B_9 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad (46)$$

$$s_{3+} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \frac{x_3 + iy_3}{2} \quad (47)$$

$$s_{3-} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \frac{x_3 - iy_3}{2} \quad (48)$$

$$s_{3,1} \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}z_3^2 + \frac{1}{4}i(y_3 x_3 - x_3 y_3) = \frac{z_3}{2}(\mathbb{1} + z_3) \quad (49)$$

$$s_{3,2} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(x_3^2 + y_3^2) - \frac{1}{2}z_3^2 \quad (50)$$

$$s_{3,3} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}z_3^2 - \frac{1}{4}i(y_3 x_3 - x_3 y_3) = \frac{z_3}{2}(z_3 - \mathbb{1}) \quad (51)$$

$$x_3^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, y_3^2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (52)$$

$$x_3 y_3 + y_3 x_3 = \begin{pmatrix} 0 & 0 & -2i \\ 0 & 0 & 0 \\ 2i & 0 & 0 \end{pmatrix}, y_3 z_3 + z_3 y_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, z_3 x_3 + x_3 z_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad (53)$$

$$|1\rangle\langle 2| \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} (\lambda_6 + i\lambda_7) \quad (54)$$

$$\lambda_7 \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (55)$$

$$\lambda_7 \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 \lambda_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_7 \lambda_3 + \lambda_3 \lambda_7 = \lambda_1 \quad (56)$$

$$\lambda_7 \lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = -\lambda_6 \lambda_1, \lambda_2 \lambda_7 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\lambda_1 \lambda_6, \lambda_7 \lambda_2 + \lambda_2 \lambda_7 = -\lambda_4, \lambda_6 \lambda_1 + \lambda_1 \lambda_6 = \lambda_4 \quad (57)$$

$$\lambda_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = |\lambda_1| \quad (58)$$

$$\lambda_5 \lambda_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_2 \lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 \lambda_2 + \lambda_2 \lambda_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \lambda_6 \quad (59)$$

$$\lambda_2^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = |\lambda_3| \quad (60)$$

$$y_3 x_3 = \begin{pmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{pmatrix} = \lambda_5 - iz_3 \quad (61)$$

W. 4-level systems (spin- $3/2$ particles)

$$\begin{aligned}
z_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & x_3 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & y_3 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} & s_{3+} &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & s_{3-} &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & x_3^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} & y_3^2 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
\lambda_1 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &\equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &\equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda_7 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
\lambda_8 &\equiv \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} & s_3^1 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^2 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^3 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$s_{4,1} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (z_2 + z_1 z_2 + z_1 + \mathbb{1}) \quad (72)$$

$$s_{4,2} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (\mathbb{1} + z_1 - z_2 - z_1 z_2) \quad (73)$$

$$s_{4,3} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (\mathbb{1} - z_1 + z_2 - z_1 z_2) \quad (74)$$

$$s_{4,4} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{4} (\mathbb{1} - z_1 - z_2 + z_1 z_2) \quad (75)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} (\mathbb{1} - z_1 + z_1 z_2 + z_2) \quad (76)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} (\mathbb{1} - z_2 + z_1 z_2 + z_1) \quad (77)$$

$$|10\rangle\langle 01| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (x_1 x_2 - i y_1 x_2 + i x_1 y_2 + y_1 y_2) \quad (78)$$

$$|01\rangle\langle 10| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (x_1 x_2 - i x_1 y_2 + i y_1 x_2 + y_1 y_2) \quad (79)$$

$$\text{CNOT} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{2} (\mathbb{1} + z_1 + x_2 - z_1 x_2) \quad (80)$$

X. 5-level systems (spin-2 particles)

$$\begin{aligned}
z_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & x_3 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & y_3 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} & s_{3+} &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & s_{3-} &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & x_3^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} & y_3^2 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
\lambda_1 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &\equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &\equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda_7 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
\lambda_8 &\equiv \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} & s_3^1 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^2 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^3 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$s_{5,1} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}A_D^1 + \sqrt{\frac{1}{12}}\Lambda_D^2 + \sqrt{\frac{1}{24}}\Lambda_D^3 + \sqrt{\frac{1}{40}}A_D^4 + \frac{1}{5}\mathbb{1} \quad (94)$$

$$s_{5,2} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -\frac{1}{2}A_D^1 + \sqrt{\frac{1}{12}}\Lambda_D^2 + \sqrt{\frac{1}{24}}\Lambda_D^3 + \sqrt{\frac{1}{40}}A_D^4 + \frac{1}{5}\mathbb{1} \quad (95)$$

$$s_{5,3} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -\sqrt{\frac{1}{3}}\Lambda_D^2 + \sqrt{\frac{1}{24}}\Lambda_D^3 + \sqrt{\frac{1}{40}}A_D^4 + \frac{1}{5}\mathbb{1} \quad (96)$$

$$s_{5,4} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = -\sqrt{\frac{3}{8}}\Lambda_D^3 + \sqrt{\frac{1}{40}}A_D^4 + \frac{1}{5}\mathbb{1} \quad (97)$$

$$s_{5,5} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = -\sqrt{\frac{2}{5}}A_D^4 + \frac{1}{5}\mathbb{1} \quad (98)$$

Y. 6-level systems (spin-3/2 particles)

$$\begin{aligned}
z_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} & x_3 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & y_3 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} & s_{3+} &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} & s_{3-} &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & x_3^2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} & y_3^2 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
\lambda_1 &\equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &\equiv \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_4 &\equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &\equiv \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda_7 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
\lambda_8 &\equiv \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix} & s_3^1 &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^2 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & s_3^3 &\equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$|0_b 1_t\rangle\langle 1_b 0_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (x\lambda_1 + iy\lambda_1 - ix\lambda_2 + y\lambda_2) \quad (120)$$

$$|1_b 0_t\rangle\langle 0_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (x\lambda_1 - iy\lambda_1 + ix\lambda_2 + y\lambda_2) \quad (121)$$

$$|0_b 1_t\rangle\langle 1_b 0_t| + |1_b 0_t\rangle\langle 0_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} (x\lambda_1 + y\lambda_2) \quad (122)$$

$$|1_b 1_t\rangle\langle 0_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (iy\lambda_3 - x\lambda_3) + \sqrt{\frac{1}{48}} (x\lambda_8 - iy\lambda_8) + \frac{1}{6} (x - y) \quad (123)$$

$$|1_b 1_t\rangle\langle 0_b 1_t| + |0_b 1_t\rangle\langle 1_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} (\lambda_5 y + \lambda_4 x) \quad (124)$$

$$|1_b 0_t\rangle\langle 1_b 2_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \frac{1}{4} (\lambda_6 - z\lambda_6 + i\lambda_7 - iz\lambda_7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = |1_b 1_t\rangle\langle 1_b 2_t| \quad (125)$$

$$|1_b 0_t\rangle\langle 1_b 2_t| + |1_b 2_t\rangle\langle 1_b 0_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = \frac{1}{2} (\lambda_6 - \lambda_6 z) \quad (126)$$

$$|0_b 1_t\rangle\langle 0_b 0_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (z\lambda_1 + \lambda_1 - iz\lambda_2 - i\lambda_2) \quad (127)$$

$$|0_b 0_t\rangle\langle 0_b 1_t| \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (z\lambda_1 + \lambda_1 + iz\lambda_2 + i\lambda_2) \quad (128)$$

$$|0_b 1_t\rangle\langle 0_b 0_t| + |0_b 0_t\rangle\langle 0_b 1_t| \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} (z\lambda_1 + \lambda_1) \quad (129)$$

$$|0_b 2_t\rangle\langle 1_b 0_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (x\lambda_4 + iy\lambda_4 - ix\lambda_5 + y\lambda_5) \quad (130)$$

$$|0_b 2_t\rangle\langle 1_b 0_t| + |1_b 0_t\rangle\langle 0_b 2_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} (x\lambda_4 + y\lambda_5) \quad (131)$$

$$|0_b 2_t\rangle\langle 0_b 0_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4} (z\lambda_4 + \lambda_4 - iz\lambda_5 - i\lambda_5) \quad (132)$$

$$|0_b 2_t\rangle\langle 0_b 0_t| + |0_b 0_t\rangle\langle 0_b 2_t| \equiv \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} (z\lambda_4 + \lambda_4) \quad (133)$$

$$|1_b 1_t\rangle\langle 1_b 2_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(\lambda_6 + i\lambda_7 - z\lambda_6 - iz\lambda_7) \quad (134)$$

$$|1_b 1_t\rangle\langle 1_b 2_t| + |1_b 2_t\rangle\langle 1_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \frac{1}{2}(\lambda_6 - z\lambda_6) \quad (135)$$

$$|1_{t,1}\rangle\langle 2_{t,1}|x_2 + |2_{t,1}\rangle\langle 1_{t,1}|x_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = \lambda_6 x_2 = -\frac{1}{2}z\lambda_1 + \frac{1}{2}\lambda_1 - \frac{1}{\sqrt{3}}x\lambda_8 + \frac{1}{3}x \neq x_2\lambda_6 \quad (136)$$

$$x_1|1_{t,2}\rangle\langle 2_{t,2}| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}x|0_b 1_t\rangle\langle 0_b 2_t| + |0_b 2_t\rangle\langle 0_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}(\lambda_6 + z\lambda_6)(\lambda_6 + i\lambda_7) \quad (137)$$

$$x_1|1_{t,2}\rangle\langle 2_{t,2}| + x_1|2_{t,2}\rangle\langle 1_{t,2}| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} = x\lambda_6 \quad (138)$$

$$|0_b 2_t\rangle\langle 1_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(x\lambda_6 + iy\lambda_6 - ix\lambda_7 + y\lambda_7) = \frac{1}{4}(\lambda_6 + \lambda_6 z + i\lambda_7 + i\lambda_7 z) \neq \frac{1}{4}(\lambda_6 + z\lambda_6 + i\lambda_7 + iz\lambda_7) \quad (139)$$

$$|0_b 1_t\rangle\langle 0_b 2_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4}(\lambda_6 + z\lambda_6 + i\lambda_7 + iz\lambda_7) \neq \frac{1}{4}(\lambda_6 + \lambda_6 z + i\lambda_7 + i\lambda_7 z) \quad (140)$$

$$|0_b 1_t\rangle\langle 0_b 2_t| + |0_b 2_t\rangle\langle 0_b 1_t| \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} (\lambda_6 + z\lambda_6) \quad (141)$$

II. OTHER HAMILTONIANS

A. Feynman Hamiltonian

Wavefunction Zoo

B. Electronic Structure Ansätze

1. Hartree Product

2. Configuration Interaction

3. Coupled Cluster

CC(n)

Bruckner-CC(n)

EOM-CC(n)

EOM-IP-CC(n)

FS-CC(n)

DLPNO-CC(n)

MR-EOM

4. Geminals

AP1roG/pCCD

APIG

APSetG

APG

Potential Zoo

C. Diatomics

Harmonic Oscillator

Morse

Lenard-Jones

Morse/Long-range

Tiemann

Morse-Rosen

D. Triatomics

Jensen

Schwenke

PolyMLR

E. Torsion Potentials

Functional Zoo

Function Zoo

Master Equation Zoo

Particle Zoo

https://en.wikipedia.org/wiki/Particle_zoo

Complexity Zoo (founded by Scott Aaronson of UWaterloo)

https://complexityzoo.uwaterloo.ca/Complexity_Zoo

Algorithms Zoo (founded by Stephen Jordan of NIST)

<https://math.nist.gov/quantum/zoo/>