Lecture 2. Computational methods

Laplace approximation, Markov Chain Monte Carlo

20.03.2023-24.03.2023 Instructors: Alina Bazarova, Sebastian Starke, Steve Schmerler. Technical issues: Alexandre Strube

Why computational methods?

Recall that in our target formula for posterior $p(\theta \mid x) = \dfrac{p(\theta)p(x \mid \theta)}{\int_{\mathbb{R}} p(x)p(\theta \mid x)dx}$

where θ are our parameters the **integral** below can get **really nasty**!

BUT: this integral is just a constant! Rewrite $p(\theta \mid x) = \dfrac{1}{Z}p(x, \theta)$, where Z is just a normalising constant, although possibly varying over a large range.

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What to do?

Well, a fact of life: lots of things can be approximated with a normal distribution,

so why not!?

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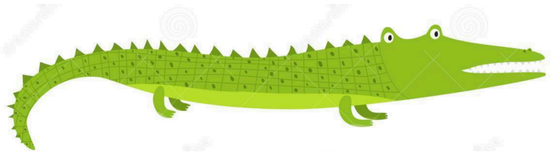

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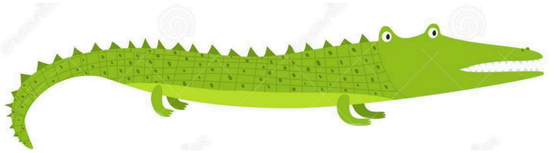

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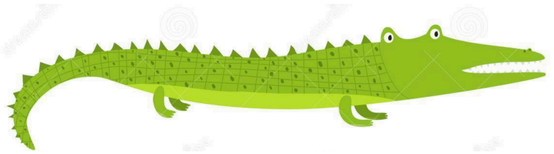

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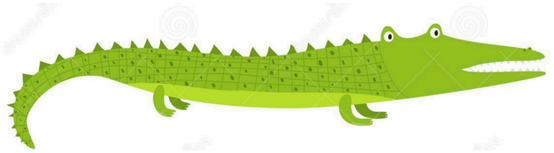

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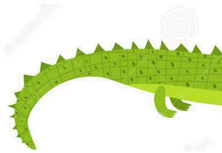
Note, that θ_{MAP} corresponds to **local maximum of the posterior**

Hence $\nabla f(\theta_{MAP}) = 0$ and the second term of the “crocodile" conveniently gets zeroed down:



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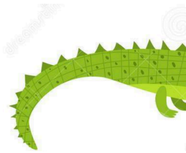

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Hence $\theta \mid X \sim N(\theta_{MAP}, -(\nabla^2 \ln p(X, \theta_{MAP}))^{-1})$

Laplace approximation 2. What is good about MAP?

1. **How to find MAP?** Iterative procedure, gradient ascent.

In **pymc3** function **find_map** which we already used in the first Jupyter notebook.

2. **How to find Hessian** $\nabla^2 \ln p(X, \theta)$**?**:

In **pymc3** function **find_hessian**

However with the large number of parameters this also becomes too computationally challenging, hence one needs another method

Jupyter notebook 2 Laplace approximation

Markov Chain Monte Carlo (MCMC) algorithm

Monte Carlo integration.

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(law of large numbers)

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This is called Markov chain Monte Carlo.

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Important: We can construct an MCMC algorithm which will have $p(\theta \mid X)$ as the stationary distribution!

Metropolis-Hastings sampler

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acceptance of candidate $Y = \theta_{t+1}$

rejection of all possible candidates Y

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Recall $\alpha(\theta, Y) = \min\left(1, \frac{\pi(Y)q(\theta \mid Y)}{\pi(\theta)q(Y \mid \theta)}\right)$, and hence

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Hence, once sample from stationary has been obtained, all subsequent samples are going to be from it. This means MCMC has converged. The period before convergence is called *burn-in*

Metropolis-Hastings: how it works in practice

1. Start at **current position** X.
2. Propose moving to a **new position** Y using proposal $q(Y \mid X)$
3. Accept/Reject the new position based on the position's adherence to the data and prior distributions using $\alpha(X, Y)$
    - If you accept: Move to the new position Y. Return to Step 1.
    - Else: Do not move to new position, stay at X. Return to Step 1.
4. After a large number of iterations, return **all accepted positions**.

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3. Proposal has to **explore the space efficiently**, sometimes it requires to perform experimentation and craftsmanship to construct a good one.

Jupyter notebook 2

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Single component MH and Gibbs sampler

Instead of updating θ *en bloc* it is often more convenient and computationally efficient to divide θ into components $\{\theta_1 \ldots \theta_h\}$ and update them one by one.

This means that instead of $q(Y \mid \theta)$ we will have $q(Y_i \mid \theta_{-i}, \theta_i)$, where $\theta_{-i} = \{\theta_1 \ldots \theta_{i-1}, \theta_{i+1} \ldots \theta_h\}$.

Acceptance probability will then be $\alpha(\theta_{-i}, \theta_i, Y_i) = \min\left(1, \frac{\pi(Y_i \mid \theta_{-i}) q(\theta_i \mid Y_i, \theta_{-i})}{\pi(\theta_i \mid \theta_{-i}) q(Y_i \mid \theta_i, \theta_{-i})}\right)$

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Gibbs sampling scheme

Assume we have data $X \sim p(X \mid \theta_1, \theta_2)$

1. Randomly initialize $\theta_1^{(0)}$ and sample $\theta_2^{(0)} \sim p(\theta_2 \mid X, \theta_1^{(0)})$

2. For step $t = 1, \ldots, T$

   (a) Sample $\theta_1^{(t)} \sim p(\theta_1 \mid X, \theta_2^{t-1})$

   (b) Sample $\theta_2^{(t)} \sim p(\theta_2 \mid X, \theta_1^{t-1})$