Differential Cryptanalysis

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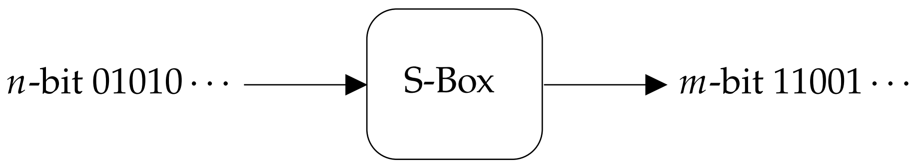
1 Definitions

1.1 S-Box



S-Box

Definition 1. Let $n, m \in \mathbb{Z}^+$. A function

$$S : \mathbb{F}_2^n \to \mathbb{F}_2^m$$

is a **S-Box**.

1.2 The XOR operation

$$\begin{array}{rccl} \oplus & : & \mathbb{F}_2 \times \mathbb{F}_2 & \longrightarrow & \mathbb{F}_2 \\ & & (x, y) & \longmapsto & z = x + y \bmod 2 \end{array}$$

x	y	$z = x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

Note (Thinking).

$$\begin{array}{rcccl} \oplus & : & \mathbb{F}_2 & \longrightarrow & [\mathbb{F}_2 \to \mathbb{F}_2] \\ & & x & \longmapsto & \oplus_x = \begin{cases} \mathrm{Id}(y) & : x = 0, \\ \neg(y) & : x = 1. \end{cases} \end{array}$$

$$\begin{array}{rccc} \oplus_n & : & \mathbb{F}_2^n \times \mathbb{F}_2^n & \longrightarrow & \mathbb{F}_2^n \\ & & \left(\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n\right) & \longmapsto & \{x_i \oplus y_i\}_{i=1}^n \end{array}$$

Note. We use the notation 0_n to denote $0_n = (0, \ldots, 0)$.

Note (Thinking)**.**

$$\begin{array}{rccc} \oplus_n & : & \mathbb{F}_2^n & \longrightarrow & [\mathbb{F}_2^n \to \mathbb{F}_2^n] \\ & & \{x_i\}_{i=1}^n & \longmapsto & (\oplus_n)_{\{x_i\}_{i=1}^n} = \{z_i\}_{i=1}^n, \text{ where } z_i = \begin{cases} \mathrm{Id}(y_i) & : x_i = 0, \\ \neg(y_i) & : x_i = 1. \end{cases} \end{array}$$

Proposition 1. *Let $X, Y, Z \in \mathbb{F}_2^n$. Then*

(1) $X \oplus_n Y = Y \oplus_n X$

(2) $(X \oplus_n Y) \oplus Z = X \oplus_n (Y \oplus Z)$

(3) $X \oplus_n 0_n = X = 0_n \oplus_n X$

(4) $X \oplus_n X = 0_n$

(5) $X \oplus_n Y = 0_n \implies X = Y$

(6) $A \oplus_n X = B \implies X = A \oplus_n B$

Note. By (4) and (5), we have $X \oplus_n Y = 0 \iff X = Y$.

Proof. PASS □

Remark 1.

- The binary operation $\oplus$ provides the structure of an **abelian group** on the set $\mathbb{F}_2^n$ with identity element 0_n.
- Because of the property (4)$X \oplus_n X = 0_n$, we see that *the inverse of any element is itself* with respect to the operation $\oplus$.

Definition 2. The **difference** of $X \in \mathbb{F}_2^n$ and $Y \in \mathbb{F}_2^n$ is defined as $X \oplus_n Y \in \mathbb{F}_2^n$.

2 Difference Set

2.1 Definition and Property

Difference Set of Bit-Sequence

Definition 3. Given $\alpha \in \mathbb{F}_2^n$, we define the **difference set** of α as follow:

$$\Delta_\alpha = \left\{(x_1, x_2) \in \mathbb{F}_2^n \times \mathbb{F}_2^n : x_1 \oplus x_2 = \alpha \in \mathbb{F}_2^n\right\} \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n.$$

Proposition 2. *For any $\alpha \in \mathbb{F}_2^n$ the set Δ_α contains 2^n elements and can be expressed as*

$$\Delta_\alpha = \left\{(x, x \oplus \alpha) : x \in \mathbb{F}_2^n\right\}.$$

Proof. Let

$$S := \left\{(x_1, x_2) \in \mathbb{F}_2^n \times \mathbb{F}_2^n : x_1 \oplus x_2 = \alpha \in \mathbb{F}_2^n\right\},$$
$$T := \left\{(x, x \oplus \alpha) : x \in \mathbb{F}_2^n\right\}.$$

We must show that $S = T$:

($S \subseteq T$) Let $(x, y) \in S$ then by definition $x \oplus y = \alpha$. Since $(x \oplus y = \alpha) \Rightarrow (y = x \oplus \alpha)$,

$$(x, y) = (x, x \oplus \alpha) \in T.$$

($T \subseteq S$) Let $(x, x \oplus \alpha) \in T$. Since

$$x \oplus (x \oplus \alpha) = \alpha,$$

$(x, x \oplus \alpha) \in S.$

□

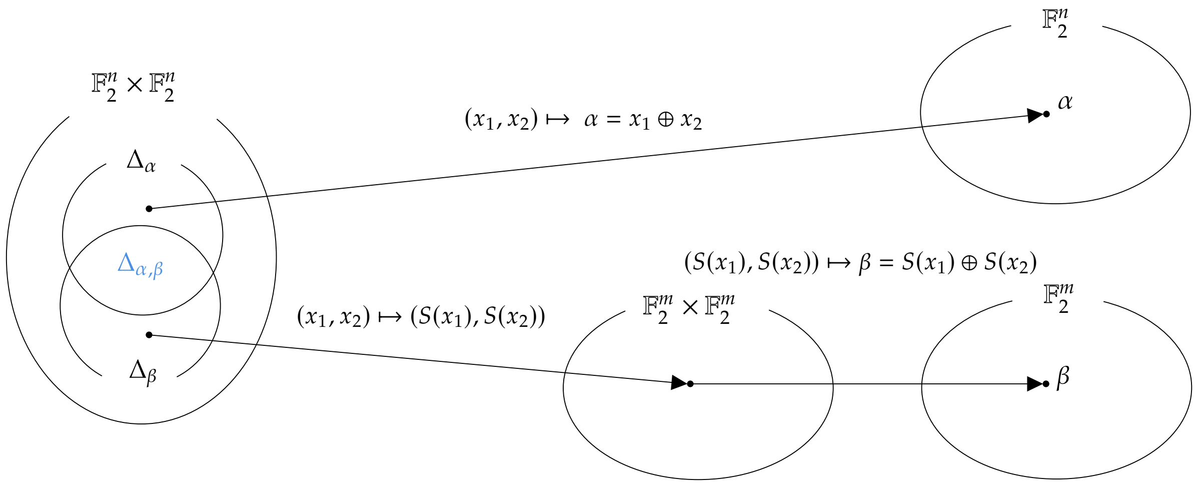
Corollary 2.1. *For any $\alpha \in \mathbb{F}_2^n$, we have $\Delta_\alpha \simeq \mathbb{F}_2^n$.*

Remark 2. Let us consider the case $\alpha = 0$ for the set Δ_α. When $\alpha = 0$ the difference set is

$$\Delta_0 = \left\{(x, x) : x \in \mathbb{F}_2^n\right\}$$

This set is often called the **diagonal** of $\mathbb{F}_2^n \times \mathbb{F}_2^n$.

2.2 Difference Sets of a S-BOX



Difference Set of a S-BOX

Let $S : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^m$ is a S-box. Let $\alpha \in \mathbb{F}_2^n$ and $\beta \in \mathbb{F}_2^m$. Consider

$$\Delta_\alpha = \left\{(x_1, x_2) : x_1 \oplus x_2 = \alpha \in \mathbb{F}_2^n\right\} \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n \quad \text{and}$$
$$\Delta_\beta = \left\{(x_1, x_2) : S(x_1) \oplus S(x_2) = \beta \in \mathbb{F}_2^m\right\} \subseteq \mathbb{F}_2^n \times \mathbb{F}_2^n.$$

We define the **difference set** of S with respect to α and β by

$$\Delta_{\alpha,\beta} = \Delta_\alpha \cap \Delta_\beta = \left\{(x_1, x_2) \in \mathbb{F}_2^n \times \mathbb{F}_2^n : x_1 \oplus x_2 = \alpha \in \mathbb{F}_2^n \text{ and } S(x_1) \oplus S(x_2) = \beta \in \mathbb{F}_2^m\right\}.$$

That is, $\Delta_{\alpha,\beta}$ is the set of ordered pairs of elements from $\mathbb{F}_2^n$ which have a difference of α and such that their images under S have a difference of β.

Remark 3.

- This can also written as

$$\Delta_{\alpha,\beta} = \left\{(x_1, x_2) \in \Delta_\alpha : (S(x_1), S(x_2)) \in \Delta_\beta\right\}.$$

- $\Delta_{\alpha,\beta}$ is always defined w.r.t. a given S-Box S. If we want to make this dependence explicit we can write $\Delta^S_{\alpha,\beta}$.

Cardinality of a Difference Set

We define $\delta_{\alpha,\beta}$ to be the cardinality of the finite set $\Delta_{\alpha,\beta}$, namely

$$\delta_{\alpha,\beta} := \left|\Delta_{\alpha,\beta}\right| \in \mathbb{Z}_{\geq 0}.$$

Proposition 3.

$$\delta_{\alpha,\beta} = \#\left\{x \in \mathbb{F}_2^n : S(x) \oplus S(x \oplus \alpha) = \beta \in \mathbb{F}_2^m\right\}.$$

Proof.

$$\begin{aligned}
\Delta_{\alpha,\beta} = \Delta_\alpha \cap \Delta_\beta &= \left\{(x_1, x_2) : x_1 \oplus x_2 = \alpha\right\} \cap \left\{(x_1, x_2) : S(x_1) \oplus S(x_2) = \beta\right\} \\
&= \left\{(x, x \oplus \alpha) : x \in \mathbb{F}_2^n\right\} \cap \left\{(x_1, x_2) : S(x_1) \oplus S(x_2) = \beta\right\} \\
&= \left\{x \in \mathbb{F}_2^n : S(x) \oplus S(x \oplus \alpha) = \beta \in \mathbb{F}_2^m\right\}.
\end{aligned}$$

□

Remark 4.

- When $\alpha = 0$ and $\beta = 0$ we have $\Delta_{0,0} = \Delta_0 = \left\{(x, x) : x \in \mathbb{F}_2^n\right\}$.
- In general when $\alpha = 0$ we find that $\Delta_{0,\beta} = \begin{cases} \Delta_0 & : \beta = 0 \\ \emptyset & : \beta \neq 0 \end{cases}$
- Since $|\Delta_0| = 2^n$ and $|\emptyset| = 0$, $\quad \delta_{0,\beta} = \begin{cases} 2^n & : \beta = 0 \\ 0 & : \beta \neq 0 \end{cases}$.

Proposition 4. *The integer $\delta_{\alpha,\beta} \in \mathbb{Z}_{\geq 0}$ is always even.*

Proof. Recall that 0 is even.

(Case I) When $\alpha = 0$, we saw either $\delta_{0,\beta} \in \{0, 2^n\}$ and these are even in either case.

(Case II) Suppose that $\alpha \neq 0$ and $\Delta_{\alpha,\beta} \neq \emptyset$. Let $(x_1, x_2) \in \Delta_{\alpha,\beta}$ then

$$(x_2, x_1) \in \Delta_{\alpha,\beta} \quad \text{and} \quad x_1 \neq x_2.$$

Therefore $(x_1, x_2) \neq (x_2, x_1)$. So if we pair (x_1, x_2) and (x_2, x_1), we can partition $\Delta_{\alpha,\beta}$ into subsets, each subset having cardinality of 2.

□

3 The DDT of a S-Box

3.1 Definition and Property

Differential Distribution Table

Let $S : \mathbb{F}_2^n \to \mathbb{F}_2^m$ be a S-Box. The **differential distribution table** (abbreviated DDT) of S is a table (or matrix) with 2^n-rows and 2^m-columns. We denote it by $\mathcal{D}_S$ or just by $\mathcal{D}$.

- The rows are indexed by the elements $\alpha \in \mathbb{F}_2^n = \{0, \ldots, 2^n - 1\}$.
- The columns are indexed by the elements $\beta \in \mathbb{F}_2^m = \{0, \ldots, 2^m - 1\}$.
- The entry at row index α and column index β is given by $\delta_{\alpha,\beta} = |\Delta_{\alpha,\beta}|$. That is,

$$\mathcal{D} = (\delta_{\alpha,\beta})_{2^n \times 2^m}.$$

Remark 5. The DDT of a S-Box is just table of all the possible integer values $\delta_{\alpha,\beta}$.

$\mathcal{D}$	0	1	$\cdots$	β	$\cdots$	$2^m - 1$
0	2^n	0		$\cdots$		0
1						
$\vdots$						
α				$\delta_{\alpha,\beta}$		
$\vdots$						
$2^n - 1$						

Proposition 5. *Let $S : \mathbb{F}_2^n \to \mathbb{F}_2^m$ be a S-Box with differential distribution table $\mathcal{D}$. The following properties hold for $\mathcal{D}$.*

(1) Every entry in $\mathcal{D}$ is a non-negative even integer between 0 and 2^n.

(2) The top-left entry of $\mathcal{D}$ is 2^n.

(3) The first row of $\mathcal{D}$ consists of all zeros except the first entry which is 2^n.

(4) If S is one-to-one then the first column of $\mathcal{D}$ consists all zeros except the first entry which is 2^n.

(5) The sum of the entries of each row is 2^n.

(6) If S is bijective then every row and column of the DDT add up to 2^n.

3.2 The DDT and Probabilities

When designing and analyzing attacks by differential cryptanalysis, we often want to translate the integer values from the DDT into probabilities.

Probability of DDT

Definition 4. Let the universe is $\Omega = \Delta_\alpha$ and the event $E = \Delta_{\alpha,\beta}$. We define probability $p_{\alpha,\beta}$ as follows:

$$p_{\alpha,\beta} = \frac{|E|}{|\Omega|} = \frac{\left|\Delta_{\alpha,\beta}\right|}{|\Delta_\alpha|} = \frac{\delta_{\alpha,\beta}}{2^n}.$$

Remark 6. Equivalently we can regard the universe $\Omega = \mathbb{F}_2^n$ and the event as the set

$$E = \left\{x \in \mathbb{F}_2^n : S(x) \oplus S(x \oplus \alpha) = \beta\right\}.$$

3.3 The DDT of a Linear S-Box

Linear S-Box

Definition 5. A S-Box $L : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^m$ is said to be **linear** if

$$x_1, x_2 \in \mathbb{F}_2^n \implies L(x_1 \oplus x_2) = L(x_1) \oplus L(x_2).$$

Remark 7.

- In the context of probabilities this states

$$p_{\alpha,\beta} = \begin{cases} 1 & : \beta = L(\alpha) \\ 0 & : \beta \neq L(\alpha) \end{cases}$$

- So the DDT of a linear S-Box is not interesting since every entry is either 0 or 2^n.

Proposition 6. *Let $L : \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^m$ be a linear S-Box. Let $\alpha \in \mathbb{F}_2^n$ and $\beta \in \mathbb{F}_2^n$. Then the difference sets of L and their cardinalities are given by*

$$\Delta_{\alpha,\beta} = \begin{cases} \Delta_\alpha & : \beta = L(\alpha) \\ \emptyset & : \beta \neq L(\alpha) \end{cases}, \quad \delta_{\alpha,\beta} = \begin{cases} 2^n & : \beta = L(\alpha) \\ 0 & : \beta \neq L(\alpha) \end{cases}.$$

That is, every row of the DDT of L consists of all zeros except one entry with a value of 2^n.

Proof. Let $\alpha \in \mathbb{F}_2^n$. Assume that $(x_1, x_2) \in \Delta_\alpha$, namely $x_1 \oplus x_2 = \alpha$ then

$$L(x_1) \oplus L(x_2) = L(x_1 \oplus x_2) = L(\alpha).$$

So $(x_1, x_2) \in \Delta_{\alpha,\beta} \iff \beta = L(\alpha)$. $\square$

3.4 The DDT of the XOR S-Box

- Let $k \in \mathbb{F}_2^n$ be fixed and define the bijective S-Box:

$$\begin{array}{rcccl} S & : & \mathbb{F}_2^n & \longrightarrow & \mathbb{F}_2^n \\ & & x & \longmapsto & x \oplus k \end{array}$$

- Note that $S(x_1 \oplus x_2) \neq S(x_1) \oplus S(x_2)$ when $k \neq 0_n$
- Therefore when $k \neq 0$ the S-Box S is non-linear
- Consider the equation for S
$$S(x) \oplus S(x \oplus \alpha) = \beta$$
- By the definition of S this is equivalent to

$$(x \oplus k) \oplus ((x \oplus \alpha) \oplus k) = \beta$$

- Which is equivalent to $\alpha = \beta$.
- We find the following properties hold for the XOR S-Box

$$\Delta_{\alpha,\beta} = \begin{cases} \Delta_\alpha & : \alpha = \beta \\ \emptyset & : \alpha \neq \beta \end{cases} \qquad \delta_{\alpha,\beta} = \begin{cases} 2^n & : \alpha = \beta \\ 0 & : \alpha \neq \beta \end{cases}$$

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