

Algorithms and techniques for matrix computations

Overview

- ❑ Matrices – a quick review
 - ❑ Matrix addition
 - ❑ Matrix multiplication
 - ❑ Matrix times vector
- ❑ BLAS – routines for matrices/vectors
 - ❑ different levels
 - ❑ naming conventions
 - ❑ calling BLAS from C programs

Matrices and Linear Equations

A close relationship:

A system of linear equations can be written in matrix form:

$$\mathbf{Ax} = \mathbf{b}$$

Matrix **A** holds the *a* constants

x is a vector of the unknowns

b is a vector of the *b* constants.

Systems of linear equations appear in almost all engineering problems

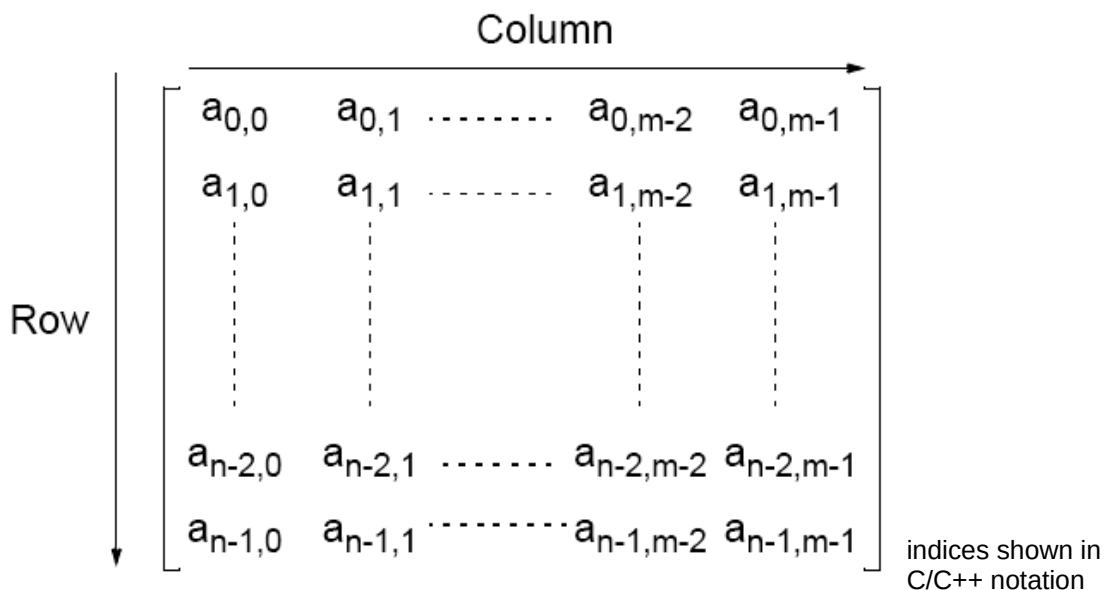
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Matrices — a review

An $n \times m$ matrix:



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Matrix Addition

Involves adding corresponding elements of each matrix to form the result matrix:

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

Given the elements of **A** as $a(i,j)$ and the elements of **B** as $b(i,j)$, each element of **C** is computed as

$$c_{i,j} = a_{i,j} + b_{i,j}$$
$$(0 \leq i < n, 0 \leq j < m)$$

Matrix Multiplication

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$$

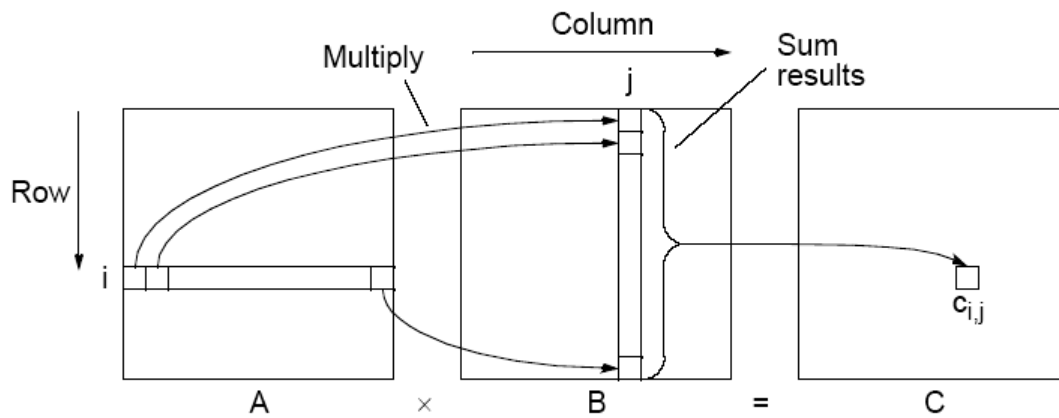
Multiplication of two matrices, **A** and **B**, produces the matrix **C** whose elements, $c(i,j)$ ($0 \leq i < n$, $0 \leq j < m$), are computed as follows:

$$c_{i,j} = \sum_{k=0}^{l-1} a_{i,k} b_{k,j}$$

where **A** is an $n \times l$ matrix and **B** is an $l \times m$ matrix.

Matrix multiplication

$$C = A \cdot B$$

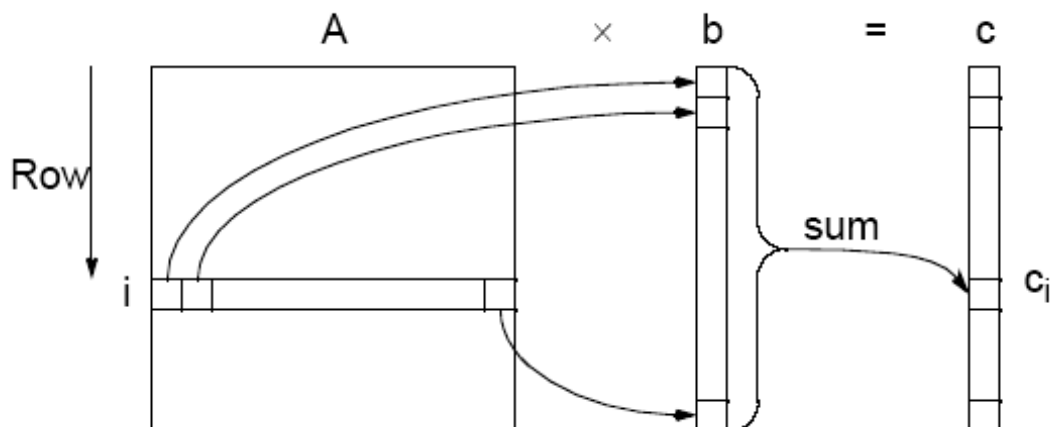


in vector notation: $c(i,j) = a(i) \cdot b(j)$

Matrix-Vector Multiplication

$$c = A \cdot b$$

Matrix-vector multiplication follows directly from the definition of matrix-matrix multiplication by making **B** an $n \times 1$ matrix (vector). Result an $n \times 1$ matrix (vector).



Using a library for matrices/vectors

Basic Linear Algebra Subroutines (BLAS)

- ❑ building blocks for linear algebra (de facto standard)
- ❑ started as a FORTRAN library (late 1970s)
- ❑ linear algebra engine in MATLAB, Python, R, Mathematica, . . .
- ❑ high performance when optimized for a specific system/architecture

BLAS levels

BLAS level 1 routines (1970s)

- ▶ vector operations, e.g.,

$$x^T y, \quad \|x\|_2, \quad x \leftarrow \alpha x, \quad y \leftarrow \alpha x + y$$

- ▶ use $O(n)$ operations for vectors of length n

BLAS level 2 routines (1980s)

- ▶ matrix-vector operations, e.g.,

$$y \leftarrow \alpha Ax + \beta y, \quad A \leftarrow \alpha xx^T + A, \quad x \leftarrow T^{-1}b, \quad T \text{ triangular}$$

- ▶ use $O(mn)$ operations for matrices of size $m \times n$

BLAS levels

BLAS level 3 routines (1980s)

- ▶ matrix-matrix routines, e.g.,

$$C \leftarrow \alpha AB + \beta C, \quad X \leftarrow T^{-1}B, \quad T \text{ triangular}$$

- ▶ use $O(n^3)$ operations for matrices of size $n \times n$

BLAS – what's in a name?

BLAS naming scheme

XYZZ

- ▶ First character X indicates data type (S, D, C, Z)
- ▶ BLAS level 1: letters YZZ indicate mathematical operation
- ▶ BLAS level 2+3: letters YY indicate matrix type
- ▶ BLAS level 2+3: letters ZZ indicate mathematical operation

Examples

- ▶ `dscal` — *double scale* ($x \leftarrow \alpha x$)
- ▶ `saxpy` — *single a x plus y* ($y \leftarrow \alpha x + y$)
- ▶ `dgemv` — *double general matrix-vector* ($y \leftarrow \alpha Ax + \beta y$)
- ▶ `dtrsv` — *double triangular solve vector* ($x \leftarrow T^{-1}x$)
- ▶ `ssymm` — *single symmetric matrix-matrix* ($C \leftarrow \alpha SB + \beta C$)

BLAS – memory & notations

- ▶ vectors and matrices are contiguous arrays
- ▶ matrices are stored in column-major ordering
- ▶ *stride* refers to distance between *consecutive* elements
- ▶ *leading dimension* (LDA) refers to distance between columns

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \end{bmatrix}, \quad \begin{bmatrix} * & * & * & * & * \\ * & * & A_{23} & A_{24} & A_{25} \\ * & * & A_{33} & A_{34} & A_{35} \\ * & * & * & * & * \end{bmatrix}$$

- ▶ *i*th column of *A* is a vector of length 4 with stride 1
- ▶ *i*th row of *A* is a vector of length 5 with stride 4
- ▶ (*A*₁₁, *A*₂₂, *A*₃₃, *A*₄₄) is a vector of length 4 with stride 5
- ▶ *A* is a matrix with 4 rows, 5 columns, stride 1, LDA 4
- ▶ *slice* (submatrix to the right) has 2 rows, 3 columns, stride 1, LDA 4

Calling (FORTRAN) BLAS from C

```
/* Prototype for BLAS dscal */
void dscal_(
    const int * n,           /* length of array */
    const double * a,        /* scalar a */
    double * x,              /* array x */
    const int * incx         /* array x, stride */
);

int main(void) {
    int i, incx, n;
    double a, x[5] = {2.0, 2.0, 2.0, 2.0, 2.0};

    /* Scale the vector x by 3.0 */
    n = 5; a = 3.0; incx = 1;
    dscal_(&n, &a, x, &incx);

    return 0;
}
```

CBLAS – BLAS in C

```
#include <stdio.h>
#if defined(__MACH__) && defined(__APPLE__)
#include <Accelerate/Accelerate.h>
#else
#include <cblas.h>
#endif

int main(void) {

    int i, incx, n;
    double a, x[5] = {2.0, 2.0, 2.0, 2.0, 2.0};

    /* Scale the vector x by 3.0 */
    n = 5; a = 3.0; incx = 1;
    cblas_dscal(n, a, x, incx);

    return 0;
}
```

BLAS or CBLAS – what to use?

- ❑ Calling (FORTRAN)-BLAS from C/C++ can be cumbersome
 - ❑ add a trailing “_” to routine name
 - ❑ all arguments have to be passed by address
- ❑ CBLAS is more convenient
 - ❑ just add a “cblas_” prefix to the routine name
 - ❑ all arguments have their natural type
 - ❑ there might be extra arguments, though
 - ❑ many CBLAS implementations call BLAS “under the hood”

BLAS or CBLAS – what to use?

- ❑ Some libraries implement a C interface with the original BLAS names – but C-style arguments
 - ❑ Intel MKL
 - ❑ Oracle Studio Performance Library
 - ❑ ...
- ❑ They might provide a CBLAS interface as well

Calling BLAS/CBLAS: some hints

Important things to have in mind:

- ❑ memory for matrices and vectors is expected to be contiguous, i.e. one large block, no holes – important when allocating memory
- ❑ check the access order of matrices, i.e. row-wise or column-wise, and adapt the corresponding parameters
- ❑ look carefully at parameters like 'leading dimension', etc, especially for non-square matrices

Dynamic allocation of matrices in C

- ❑ Many libraries that can handle matrices, like BLAS, require, that the memory is contiguous, i.e. allocated in one large block.
- ❑ On the next slides, you can find an implementation that does exactly that.

Allocating a matrix in C

```
// allocate a double-prec m x n matrix
double **
dmalloc_2d(int m, int n) {
    if (m <= 0 || n <= 0) return NULL;
    double **A = malloc(m * sizeof(double *));
    if (A == NULL) return NULL;
    A[0] = malloc(m*n*sizeof(double));
    if (A[0] == NULL) {
        free(A);
        return NULL;
    }
    for (i = 1; i < m; i++)
        A[i] = A[0] + i * n;
    return A;
}
```

De-allocating a matrix in C

```
// de-allocating memory, allocated with
// dmalloc_2d

void
dfree_2d(double **A) {
    free(A[0]);
    free(A);
}
```