Tutorial 4
11.02.20

1 Applications of Gaussian elimination

For this exercise, K is a field, and we consider an ambient linear space K^n for some $n \geq 2$. All vectors will be row vectors.

1. Let $V = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a linear subspace. Give an algorithm to compute a basis of V.
2. Let $W = \mathrm{Span}\{\mathbf{w}_1, \ldots, \mathbf{w}_e\}$ be another linear subspace. Give an algorithm to compute a basis of $V + W$.
3. Give an algorithm to compute a basis of $V \cap W$.

2 An algorithm for computing the characteristic polynomial

Let $A \in \mathcal{M}_n(\mathbb{K})$, the goal of the following method is to compute the characteristic polynomial of A in $O(n^3)$ time.

1. Let T be the transformation which acts on the left of a matrix A through $L_i \leftarrow L_i + \alpha L_j$, i.e., $T = I_n + \alpha E_{i,j}$. Here $E_{i,j}$ denotes an $n \times n$ matrix with 1 on the (i, j) position and 0s everywhere else. Describe the action of T^{-1} on the right of A in terms of column operations.
2. Using Question 1, show that one can find a matrix R such that

$$RAR^{-1} = \begin{bmatrix} a_{1,1} & a'_{1,2} & \cdots & a'_{1,n} \\ l_2 & a'_{2,2} & \ddots & a'_{2,n} \\ 0 & a'_{3,2} & \ddots & a'_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n,2} & \cdots & a'_{n,n} \end{bmatrix}.$$

   (Hint: perform row operations by multiplying on the left by some transformation matrices T_i and see what happens on the columns when you multiply on the right by T_i^{-1}).

3. Give an algorithm to compute the matrices R_n and M such that

$$R_n A R_n^{-1} = M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & \cdots & m_{1,n} \\ \ell_2 & m_{2,2} & m_{2,3} & \ddots & m_{2,n} \\ 0 & \ell_3 & m_{3,3} & \ddots & m_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \ell_n & m_{n,n} \end{bmatrix}$$

using $O(n^3)$ operations in $\mathbb{K}$.

Remark: such an "almost triangular" shape matrix is called an upper Hessenberg matrix, i.e., a matrix that has zero entries below the first subdiagonal. We have shown how to reduce any matrix into (upper) Hessenberg form.

4. Deduce an algorithm to compute the characteristic polynomial of A, with a complexity bound $O(n^3)$. Use the fact that two similar matrices have the same characteristic polynomial.

5. Could it be possible to find R such that $R^{-1}AR = M$ is upper triangular by (arbitrarily many) elementary operations in $\mathbb{K}$? If yes, explain how. If not, explain why.

3 A faster algorithm for characteristic polynomial

Let A be an $n \times n$ matrix. In this exercise, we will denote by n^ω the number of operations in K needed to multiply two n by n matrices with coefficients in K. You have seen in class that given a n by n matrix $M \in \mathcal{M}_n(K)$ (if you have not, take it as a fact), we can compute M^{-1} using $O(n^\omega \log n)$ operations in K (computing the inverse is asymptotically the same as multiplying).

1. Assume that v is a vector such that v, $Av, A^2v, \ldots, A^{n-1}v$ is a basis of K^n; then if B is the matrix with columns $v, Av, A^2v, \ldots, A^{n-1}v$, prove that $B^{-1}AB$ is a companion matrix, that is, a matrix of the following form

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 1 & 0 & \cdots & 0 & c_1 \\ 0 & 1 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{n-1} \end{bmatrix}.$$

2. If B is given, what is the cost of computing the characteristic polynomial of A using the previous question.

3. Explain how from an $n \times n$ matrix multiplication in time $O(n^\omega)$ we can deduce a $n \times m$ by $m \times k$ matrix multiplication algorithm in time $O(\max(n, m, k)^\omega)$

4. Define $w_0 = v, w_1 = (v, Av), w_2 = (v, Av, A^2v, A^3v), \ldots, w_k = (v, Av, A^2v, \ldots, A^{2^k-1}v)$

   Prove that w_k can be computed in time $O(kn^\omega)$ for $k < \log n$.

5. Under the assumption that v exists and that you know it, give a $O(n^\omega \log n)$ algorithm for computing the characteristic polynomial of a square matrix.

6. Does there always exist a v as in Question 1 ?