Note on a geometric approach to total variation filter

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1 Definitions

Definition 1.1. (Total Variation) We call *total variation* the function $TV : \mathbb{R}^n \to \mathbb{R}^n$:

$$TV(\mathbf{y}) = \sum_{i<n} |y_{i+1} - y_i|$$

Definition 1.2. (Total Variation Filter) Given $\lambda \in \mathbb{R}$, $\lambda \geq 0$ and $\mathbf{y} \in \mathbb{R}^n$, let $C_{\mathbf{y},\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ be the function:

$$C_{\mathbf{y},\lambda}(\mathbf{x}) = \|\mathbf{x} - \mathbf{y}\|_2^2 + TV(\mathbf{x})$$

The problem of minimising $C_{\mathbf{y},\lambda}$ is called *total varition filter*.

Since $C_{\mathbf{y},\lambda}$ is convex and non-negative it admits a finite unique global minimum, which will be denoted by $\hat{\mathbf{y}}$ in the following.

2 Equivalence of total variation filter and taut string problem

Let $\mathbf{1} \in \mathbb{R}^n$ be the vector of all 1.

The following fact is an immediate consequence of the definition of TV.

Lemma 2.1. *For any* $\mathbf{x} \in \mathbb{R}^n, \delta \in \mathbb{R}$

$$TV(\mathbf{x} + \delta\mathbf{1}) = TV(\mathbf{x})$$

Lemma 2.2. $\sum_j y_j = \sum_j \hat{y}_j$

Proof. For any $\delta \in \mathbb{R}$ let: $f(\delta) = \|y - \hat{y} \ + \delta\mathbf{1}\|_2^2 + \lambda TV(\hat{y} + \delta\mathbf{1})$. By lemma 2.1, $f(\delta) = \|\hat{y} \ + \delta\mathbf{1} - y\|_2^2 + \lambda TV(\hat{y})$. Hence f is differentiable and $f'(0) = 2\sum_j(\hat{y}_j - y_j)$. But the definition of $\hat{y}$ implies that 0 is a minimum of f, so $f'(0) = 0$. $\square$

Let $\mathbf{s}, \hat{\mathbf{s}} \in R^n$ be the *partial sums* of respectively $\mathbf{y}$ and $\hat{\mathbf{y}}$ starting from 0. In other words: $s_i = \sum_{j \le i} y_j$ and $\hat{s}_i = \sum_{j \le i} \hat{y}_j$.

Lemma 2.3. *For any $j \in [1, n]$:*

1. $\hat{y}_j > \hat{y}_{j+1} \Rightarrow \hat{s}_j = s_j - \lambda/2$
2. $\hat{y}_j < \hat{y}_{j+1} \Rightarrow \hat{s}_j = s_j + \lambda/2$
3. $\hat{y}_j = \hat{y}_{j+1} \Rightarrow |\hat{s}_j - s_j| < \lambda/2$

Proof. For any index $j \in [1, n]$, define $\mathbf{u}^j \in \mathbb{R}^n$ be such that $u_i^j = 1$ for $i \le j$ and $\mathbf{u}_i^j = 0$ for $i > j$ and let $f(\delta) = C_{\mathbf{y},\lambda}^{TVF}(\hat{\mathbf{y}} + \delta \mathbf{u}^j) - C_{\mathbf{y},\lambda}^{TVF}(\hat{\mathbf{y}})$.

It is useful to decompose f as $f_E + f_{TV}$, where $f_E(\delta) = \|\hat{\mathbf{y}} + \delta \mathbf{u}^j - \mathbf{y}\|_2^2 - \|\hat{\mathbf{y}} - \mathbf{y}\|_2^2$ and $f_{TV}(\delta) = \lambda(TV(\hat{\mathbf{y}} + \delta \mathbf{u}^j) - TV(\hat{\mathbf{y}}))$.

We have $f(0) = 0$ and $f(\delta) > 0$ for $\delta \neq 0$ from the definition of $\hat{\mathbf{y}}$.

To prove 1), assume $\hat{\mathbf{y}}(j) > \hat{\mathbf{y}}(j+1)$ for some j.

Observe that, for sufficiently small δ, it is $TV(\hat{\mathbf{y}} + \delta \mathbf{u}^j) = TV(\hat{\mathbf{y}}) + \delta$ and $f_{TV}(\delta) = \lambda\delta$. Therefore there exist an open interval I containing 0 such that f restricted to I can be written as: $f(\delta) = f_E(\delta) + \lambda\delta$ and f is differentiable in I. Since 0 is a minimum for f we must have $f'(0) = f'_E + \lambda = 2(\hat{s}_j - s_j) + \lambda = 0$, wihch means $\hat{s}_j = s_j - \lambda/2$ and the first point is proved.

The proof of 2) is entirely symmetric.

To prove 3) we follow a similar argument. Assume $\hat{y}_j = \hat{y}_{j+1}$.

First of all, observe that in this case: $f_{TV}(\delta) = \lambda|\delta|$. Moreover, by simple algebra, $f_E(\delta) = 2\delta(\mathbf{u}^j, \hat{\mathbf{y}} - \mathbf{y}) + \delta^2\|\mathbf{u}^j\|_2^2 = 2(\hat{s}_j - s_j)\delta + C\delta^2$, where we put $C = \|u\|_2^2$ and used $(,)$ to indicate the dot product in $\mathbb{R}^n$.

Now, let us use the sign function `sign`, to write $f(\delta) = |\delta|(\lambda + 2(\hat{s}_j - s_j)\texttt{sign}(\delta) + C|\delta|)$, where recall $C > 0$. For this quantity to be positive for arbitrary $\delta > 0$ it must necessarily be $\lambda + 2(\hat{s}_j - s_j) > 0 \Rightarrow \hat{s}_j - s_j \ge -\lambda/2$. Similarly, considering negative δ, it is necessary that $\hat{s}_j - s_j \le \lambda/2$. These two inequalities finally prove point 3). □

We are now able to prove an equivalence between *total variation filter* and a seemingly unrelated geometric problem:

Definition 2.1. *The taut string problem* For $\mathbf{y} \in R^n$ and $\lambda \in \mathbb{R}^n$ with $\delta \ge 0$, let $\mathbf{s}$ be the partial sum vector of $\mathbf{y}$. The *taut string* problem consists in determining an $\mathbf{t} \in \mathbb{R}^n$ such that:

- $t_j \in [s_j - \lambda/2, s_j + \lambda/2]$, for $j < n$
- $t_n = s_n$
- The total length of the chain of segments in $\mathbb{R}^2$ connecting points

$$(0,0), (1, t_1), \ldots, (n, t_n)$$

  is minimum.

The following is a simple geometric fact characterising the $\mathbf{t}$ in terms of its local properties:

Lemma 2.4. *A vector* $\mathbf{t}$ *satisfying* $t_i \in [s_i - \lambda/2, s_i + \lambda/2]$ *for each* i *solves the taut string problem for* $\mathbf{y}$ *and* λ *if and only if for all* i *in*$[1, n]$*:*

- $(t_{i-1} + t_{i+1})/2 < t_i \Rightarrow t_i = s_i - \lambda/2$
- $(t_{i-1} + t_{i+1})/2 > t_i \Rightarrow t_i = s_i + \lambda/2$

where $\mathbf{s}$ *is the vector of partial sums of* y *and we conventionally set* $t_0 = 0$

Proof. (sketch) Since the taut string problem is convex, a solution is the global optimum if and only if it is a local minimum in each coordinates. That means that $\mathbf{t}$ is a solution of the taut string problem if and only if changing anyone of its components increases the overall length of the string. In other words, let $f_i(x) = |\overline{t_{i-1}x}| + |\overline{xt_{i+1}}|$, then $\mathbf{t}$ is the taut string solution iff, for every i, the minimum of f_i over the admissible interval $[s_i - \lambda/2, s_i + \lambda/2]$ is t_i. But it is easy to see that f_i is minimised at $m = (t_{i-1} + t_{i+1})/2$ and that $f(x)$ increases as $|x - m|$ increases; as a consequence, f_i having a minimum at t_i over its admissible interval is equivalent to conditions 1, 2. $\square$

Theorem 2.5. *(Equivalence of total variation filter and taut string) Let* $\hat{\mathbf{y}}$ *be the solution of the total variation filter with parameters* $\mathbf{y}$ *and* λ*. Then* $\hat{\mathbf{s}}$*, the vector of partial sum of* $\hat{\mathbf{y}}$*, is the solution of the taut string problem of parameters* $\mathbf{y}$ *and* λ

Proof. Let $\hat{\mathbf{y}}$ be the total variation filter solution and $\hat{\mathbf{s}}$ its partial sums vector. We will be done if we prove that $\hat{\mathbf{s}}$ satisfies conditions 1 and 2 of Lemma 2.4. Assume that $\hat{s}_i > (\hat{s}_{i-1}+\hat{s}_{i+1})/2$. Note that $(\hat{s}_{i-1}+\hat{s}_{i+1})/2 = \hat{s}_{i-1}+(\hat{y}_i+\hat{y}_{i+1})/2$ therefore we derive $\hat{y}_{i+1} < \hat{y}_i$ and, by Lemma 2.3, $\hat{s}_i = s_i - \lambda/2$. Symmetric argument prove that $\hat{s}_i < (\hat{s}_{i-1} + \hat{s}_{i+1})/2$ implies $\hat{s}_i = s_i + \lambda/2$ $\square$

3 Linear time solution for the taut string problem

We will sketch a algorithm for solving the taut string problem of worst case linear time complexity.

Algorithm 1: Update majorant geodesic paths

Input: s_i current median point of bounding tube, N input size
Data:

- **z**: rightmost point in common to geodesics path to $s_i + \lambda$ and $s_i - \lambda$
- $\mathbf{Q}_+, \mathbf{Q}_-$: queue containing geodesic paths from z to respectively $s_i + \lambda, s_i - \lambda$

$s_+ :=$ upper extreme of bounding tube above s_i
while $\texttt{size}(Q) \geq 1$ **do**
    if $\angle Q_+[-2]Q_+[-1]s_+$ *is concave* **then**
        $\texttt{pop\_back}(Q_+)$
    end
end
if $\texttt{size}(Q_+) == 1$ **then**
    while $\texttt{size}(Q_-) > 1$ **do**
        if $\angle Q_-[0]Q_-[1]s_+$ *is convex* **then**
            $\texttt{pop\_front}(Q_-)$
            $z := \texttt{front}(Q_-)$
            Append z to output sequence
        end
    end
    $Q_+ = \{z, s_+\}$
else
    $\texttt{push\_back}(Q_+, s_+)$
end

The idea of the solution is to sweep through the bounding tube median points, maintaining at each step the geodesic path to the upper/lower extremal point of the bounding tube.

More precisely, the algorithm maintains the following information updated at each step. Let s_i^+, s_i^- be the points in the bounding tube with x-coordinate equal i having respectively maximum, minimum y-coordinate. Let $g(x, y)$ denote the geodesic path connecting points x, y in the bounding tube, i.e. all the extremal of the segments constituting the geodesic polygonal going from x to y, and let $\mathbf{0}$ denote the origin of the cartesian frame.

The key observation is the following:

Theorem 3.1. *For any $i \in [1, n]$, let z is the rightmost point in the intersection of $g(\mathbf{0}, s_i^+)$ and $g(\mathbf{0}, s_i^-)$. Then:*

- *for any $j \geq i$, $g(\mathbf{0}, s_j^+)$ contains $g(\mathbf{0}, z)$ followed by $g(z, s_j^+)$ and $g(\mathbf{0}, s_j^-)$ contains $g(\mathbf{0}, z)$ followed by $g(z, s_j^-)$*
- *$g(z, s_i^+), g(z, s_i^-)$ are respectively convex, concave.*
- *$g(z, s_i^+) \cap g(z, s_i^-) = \{z\}$.*

The algorithm will keep point z and set of points Q_+, Q_- in doubly-ended queues such that

- z is the rightmost point in $g(\mathbf{0}, s_i^+) \cap g(\mathbf{0}, s_i^-)$.
- Q_+, Q_- repsectively contain the points in $g(z, s_i^+)$, $g(z, s_i^-)$.

Initially $z = \mathbf{0}$, $Q_+ = Q_- = \{\mathbf{0}\}$. Then, looping through increasing values of $i \in [1, n]$, the procedures 1 and its symmetric (where + is swapped with - and concavity with convexity) are both ran to restore the invariant for each i. These procedures incrementally generate the output (line 12). Since z is always part of the solution $g(\mathbf{0}, s_n)$, it is possible to append z in output as it is moved toward right, listing all points in solution.