# Generating the Strong Coupling Hopping Expansion

$\hookrightarrow$ and the SU($N$) Gauge Integrals

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# Introduction

## Goal

Hopping expand the fermion determinant and calculate the spatial gauge links analytically

## Motivation

- The fermion determinant is computationally expensive
- Reduce the DOF to traces over Polyakov loop constructs
- Makes complex Langevin a viable simulation algorithm
- The series is convergent at some order in $\kappa$

# Preliminaries: The hopping expansion

Start off with the standard Wilson QCD action:

$$Z = \int \mathcal{D}U_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S_F - S_G}, \text{ with}$$
$$S_F = a^4 \sum_{n,m\in\Omega} \bar{\psi}(n) D(n|m) \psi(m),$$

where the Fermion determinant is

$$D(n|m) = (m + \tfrac{4}{a})\delta_{n,m} - \tfrac{1}{2a}\sum_{\mu=\pm 0}^{\pm 3}(1-\gamma_\mu)\, U_\mu(n)\, \delta(n+\hat{\mu}, m)$$
$$= D_0\big(\delta_{n,m} - \kappa H(n|m)\big), \;\; \kappa = \frac{1}{2(am+4)}$$

# Preliminaries: The hopping expansion

Carrying out the fermion integrals in the partition function

$$Z = \int \mathcal{D}U_\mu \, e^{-S_G} \det\left[D\right],$$

and using the standard trace-log identity:

**Determinant expansion**

$$\begin{aligned}\det\left[D\right] &= \exp\left\{ -\sum_{n=1}^{\infty} \frac{\kappa^n}{n} \mathrm{tr}\left[H^n\right] \right\} \\ &= 1 - \kappa\,\mathrm{tr}\left[H\right] + \frac{1}{2}\kappa^2\Big(\mathrm{tr}\left[H\right]\Big)^2 - \frac{1}{2}\kappa^2 \mathrm{tr}\left[H^2\right] + \mathcal{O}(\kappa^3)\end{aligned}$$

# The effective 3D theory

Start by separating the temporal and spatial hops:

$$
\begin{aligned}
\det\left[D\right] &= \det\left[1 - T - S^+ - S^-\right] \\
&= \det\left[1 - T\right]\det\left[1 - (1-T)^{-1}(S^+ + S^-)\right] \\
&\equiv \det\left[Q_{\mathrm{stat}}\right]\det\left[1 - P - M\right]
\end{aligned}
$$

$\det Q_{\mathrm{det}}$ is known so the terms of interest come from

$$
\det\left[1 - P - M\right] = \exp\left\{ -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\left[(P+M)^n\right]\right\},
$$

where the trace is over all free indices (space, time, Dirac, colour).

# Index overload

## Ingredients

$$P(\vec{x}, t \,|\, \vec{y}, \tau) = \kappa Q_{\text{stat}}^{-1}(t, \tau) \sum_{i=1}^{3} (1 - \gamma_i)\, U_i(\vec{x}) \delta_{\vec{x}, \vec{y} - \hat{\imath}}$$

$$M(\vec{x}, t \,|\, \vec{y}, \tau) = \kappa Q_{\text{stat}}^{-1}(t, \tau) \sum_{i=1}^{3} (1 + \gamma_i)\, U_i^{\dagger}(\vec{y}) \delta_{\vec{x} - \hat{\imath}, \vec{y}}$$

Every factor of $P$ or $M$ bring a full set of free indices which needs to be reduced before we can carry out the spatial gauge integrals.

# The trace is our friend

**Initial expression**

$$\det D = \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\big[(P+M)^n\big] \right\}$$

Both $P$ and $M$ has temporal movement because of $Q_{\mathrm{stat}}^{-1}$, but primarily

- $P$ does one forwards spatial jump
- $M$ does one backwards spatial jump

**Rule #1**

$N_P = N_M$, to satisfy the spatial trace

# The trace is our friend

## Initial expression

$$\det D = \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\left[(P+M)^n\right] \right\}$$

## Rule #1

$N_P = N_M$, to satisfy the spatial trace

$$\begin{aligned}\det D = \exp\Big\{ &- \cancel{\mathrm{tr}\left[P\right]} - \cancel{\mathrm{tr}\left[M\right]} - \frac{1}{2}\cancel{\mathrm{tr}\left[P^2\right]} \\ &- \frac{1}{2}\mathrm{tr}\left[PM\right] - \frac{1}{2}\mathrm{tr}\left[MP\right] - \frac{1}{2}\cancel{\mathrm{tr}\left[M^2\right]} + \mathcal{O}(\kappa^4)\Big\} \\ = \exp\Big\{ &- \mathrm{tr}\left[PM\right] + \mathcal{O}(\kappa^4)\Big\}\end{aligned}$$

# Limiting spatial indices

Every term includes the direction the jump is to be taken, we want to limit this as much as possible.

$$P(\vec{x},t\,|\,\vec{y},\tau) = \kappa Q_{\mathrm{stat}}^{-1}(t,\tau)\sum_{i=1}^{3}(1-\gamma_i)\,U_i(\vec{x})\delta_{\vec{x},\vec{y}-\hat{i}}$$

Tracing a contribution and carrying out the intermediate sums leaves us with:

Rule #2

$$\sum_{i_1,i_2,\ldots,i_n}\delta\big(s_1\hat{i_1}+s_2\hat{i_2}+\cdots+s_n\hat{i_n}\big),\quad s_j = \begin{matrix} 1,\ i_j \text{ from } P \\ -1,\ i_j \text{ from } M \end{matrix}$$

# Example: PPMM

Assume we pull the jump-indices out of $P$ and $M$:

$$\mathrm{tr}\big[PPMM\big] = \mathrm{tr} \sum_{i,j,k,l \in \{\hat{x},\hat{y},\hat{z}\}} P_i P_j M_k M_l$$

This gives the following Delta:

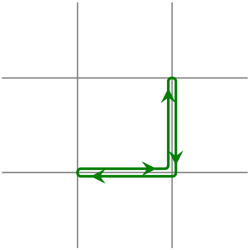
$$\sum_{i,j,k,l \in \{\hat{x},\hat{y},\hat{z}\}} \delta\big(i + j - k - l\big)$$

which has two solutions

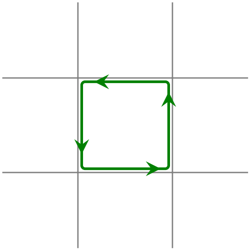
$$\begin{aligned} i = l \ \text{ and } \ j = k, \quad \text{or} \\ i = k \ \text{ and } \ j = l \end{aligned}$$

# Example: PPMM cont.

These two solutions can graphically be represented as



$i = l$ and $j = k$



$i = k$ and $j = l$

# A look into the future

These diagrams yield the following spatial gauge integrals

**$i = l$ and $j = k$**

$$\int \mathcal{D}U_\mu \, U_i(\vec{x})\, U_j(\vec{x}+\hat{i})\, U_j^\dagger(\vec{x}+\hat{i})\, U_i^\dagger(\vec{x})$$

**$i = k$ and $j = l$**

$$\int \mathcal{D}U_\mu \, U_i(\vec{x})\, U_j(\vec{x}+\hat{i})\, U_i^\dagger(\vec{x}+\hat{j})\, U_j^\dagger(\vec{x})$$

# A look into the future

These diagrams yield the following spatial gauge integrals

$i = l$ and $j = k$

$$\left( \int \mathrm{d}U \; U U^\dagger \right)^2$$

$i = k$ and $j = l$

$$\left( \int \mathrm{d}U \; U \right)^2 \left( \int \mathrm{d}U \; U^\dagger \right)^2$$

# Vanishing gauge integrals

A quick result from a paper by Creutz[1]

**Vanishing integrals**

$$\int_{U \in \mathrm{SU}(3)} \mathrm{d}U \; U^n \big(U^\dagger\big)^m = 0, \quad \text{if} \quad \begin{cases} n + 2m \neq 0 \pmod 3, \text{ or} \\ 2n + m \neq 0 \pmod 3 \end{cases}$$

[1] M. Creutz, "On Invariant Integration Over SU($N$)", *J.Math.Ph, 1978*

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### Examples

$$\int \mathrm{d}U\; U^\dagger = \int \mathrm{d}U\; UU = \int \mathrm{d}U\; UUUU^\dagger = 0$$

[1] M. Creutz, "On Invariant Integration Over SU($N$)", *J.Math.Ph, 1978*

# Different contributions

## Mesonic contributions

$$\int \mathrm{d}U \left(UU^\dagger\right)^{n_m}$$

## Baryonic contributions

$$\int \mathrm{d}U \left(UUU\right)^{n_b} \quad \text{or} \quad \int \mathrm{d}U \left(U^\dagger U^\dagger U^\dagger\right)^{\bar{n}_b}$$

## Mixed contributions

$$\int \mathrm{d}U \left(UU^\dagger\right)^{n_m} \left(UUU\right)^{n_b} \left(U^\dagger U^\dagger U^\dagger\right)^{\bar{n}_b}$$

# Mesonic contributions

One can get these contributions by connecting pairs of $P$'s and $M$'s with the following rule:

**Rule #3**

Every connected $P$ and $M$ must have a equal number of $P$'s and $M$'s in between, to allow for complete backtracking.

A connected pair must be at the same space-time position, and must jump in the same spatial direction.

# Diagrammic notation

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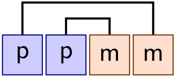
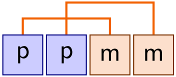
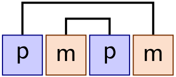
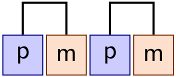


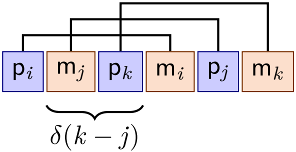



# Diagrammic notation


Sometimes also impose additional index restrictions



# Multi-trace contributions

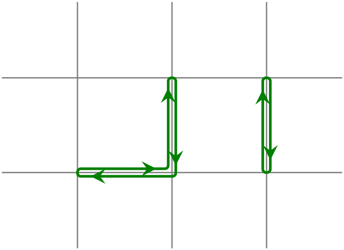
**Determinant expansion**

$$\det\left[D\right] = \exp\left\{-\sum_{n=1}^{\infty}\frac{\kappa^n}{n}\mathrm{tr}\left[H^n\right]\right\}$$
$$= 1 - \kappa\,\mathrm{tr}\left[H\right] + \frac{1}{2}\kappa^2\left(\mathrm{tr}\left[H\right]\right)^2 - \frac{1}{2}\kappa^2\mathrm{tr}\left[H^2\right] + \mathcal{O}(\kappa^3)$$

Multi-trace contributions are separate diagrams which can be located anywhere on the lattice.

# Multi-trace contributions: Example

So the $\boxed{\text{p}}\boxed{\text{p}}\boxed{\text{m}}\boxed{\text{m}}\bullet\boxed{\text{p}}\boxed{\text{m}}$ would graphically be represented as:



However, it doesn't get interesting until they somehow overlap.

# Resummation

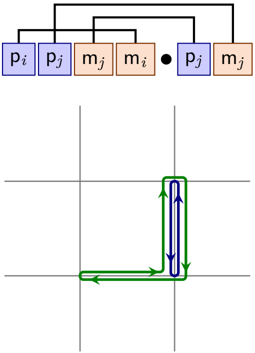
$$Z = 1 + \sum_{\vec{x},t} S_{PM} + S_{PPMM} + S_{PMPM} + S_{PM.PM} + \mathcal{O}(\kappa^6)$$
$$= \exp\left\{ \sum_{\vec{x},t} S_{PM} + S_{PPMM} + S_{PMPM} + S_{PM.PM} \right.$$
$$\left. - \text{counter terms} + \mathcal{O}(\kappa^6) \right\}$$

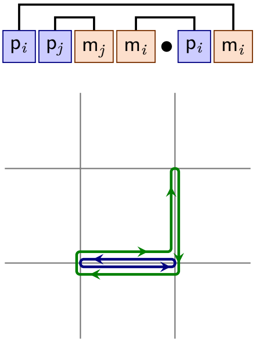
The counter terms are the disconnected multi-trace terms.

$$\sum_{\vec{x},\vec{y},t,\tau} S_{PM}(\vec{x},t) S_{PM}(\vec{y},\tau) = \sum_{\vec{x},t} S^{\mathrm{d}}_{PM.PM}(\vec{x},t)$$

# Contributing multi-trace diagrams

Only new contributions from $\mathrm{tr}\big[PPMM\big]\mathrm{tr}\big[PM\big]$ are:



# The large $N_T$ limit

The last two steps of the analytic part of the calculation are:

- Further restricting temporal variables
- Inserting the proper gauge integrals

$P_i M_i P_j M_j$

$$\sum_{i,j} \sum_{t,\tau} \int \mathrm{d}U_i(t)\mathrm{d}U_j(\tau)\; U_i(t)\, U_i^\dagger(t)\, U_j(\tau)\, U_j^\dagger(\tau)$$

## The large $N_T$ limit

**$P_i M_i P_j M_j$**

$$\sum_{i,j} \sum_{t,\tau} \int \mathrm{d}U_i(t) \mathrm{d}U_j(\tau)\; U_i(t)\, U_i^\dagger(t)\, U_j(\tau)\, U_j^\dagger(\tau)$$

The sums over $t$ and $\tau$ can be divided into two different regions:

**$i \neq j$ or $t \neq \tau$**

$$\sim N_T(N_T - 1)\left( \int \mathrm{d}U \; UU^\dagger \right)^2$$

**$i = j$ and $t = \tau$**

$$\sim N_T \left( \int \mathrm{d}U \; UU^\dagger\, UU^\dagger \right)$$

## The large $N_T$ limit

The sums over $t$ and $\tau$ can be divided into two different regions:

**$i \neq j$ or $t \neq \tau$**

$$\sim N_T(N_T - 1)\left(\int \mathrm{d}U\; UU^\dagger\right)^2$$

**$i = j$ and $t = \tau$**

$$\sim N_T\left(\int \mathrm{d}U\; UU^\dagger UU^\dagger\right)$$

Thus in the limit $N_T \longrightarrow \infty$, only the simple mesonic integrals contribute.

## Intermediate summary

Rules of the game

1. Only terms with equally many $P$'s and $M$'s contribute
2. One have to link every $P$ with an $M$ (or sets of three $P$'s and $M$'s)
3. Between every single-trace link there must be a valid term
4. (Multi-trace links are a bit more difficult. . . )

# Intermediate summary

Rules of the game

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And Now for Something Completely Different™

# SU($N$) gauge integrals

**Goal**

Calculate integrals of the form:

$$I = \int\limits_{U \in \mathrm{SU}(N)} \mathrm{d}U \; U_{i_1,j_1} \dots U_{i_n,j_n} U^\dagger_{k_1,l_1} \dots U^\dagger_{k_m,l_m}$$

in the fundamental representation.

This presentation will closely follow the previously cited paper by M. Creutz[2]

[2]M. Creutz, "On Invariant Integration Over SU($N$)", *J.Math.Ph, 1978*

## Generating function

Introduce the generating function $W$ in the standard way:

**Generating function**

$$W(J,K) = \int \mathrm{d}U \, \exp\left\{ \mathrm{tr}\left[ JU + KU^{\dagger} \right] \right\}$$

where $J$ and $K$ are arbitray $N \times N$ matrices. Thus:

$$I = \left( \frac{\delta}{\delta J_{j_1,i_1}} \cdots \frac{\delta}{\delta J_{j_n,i_n}} \frac{\delta}{\delta K_{l_1,k_1}} \cdots \frac{\delta}{\delta K_{l_m,k_m}} W(J,K) \right) \Bigg|_{J=K=0}$$

# Rewriting $U^\dagger$

We can rewrite all $U^\dagger$ $(= U^{-1})$ matrices in terms of the cofactor of $U$ using the identity $U^{-1} = \left(\mathrm{cof}\, U\right)^T$

$$U^\dagger_{i,j} = \frac{1}{(N-1)!} \varepsilon_{j,k_1,\ldots,k_{N-1}} \varepsilon_{i,l_1,\ldots,l_{N-1}} U_{k_1,l_1} \cdots U_{k_{N-1},l_{N-1}}$$

We can thus eliminate $K$ from the generating funtional, and from now on, $W = W(J)$.

## An analytic expression for $W(J)$

Because of the properties of the gauge group elements of SU($N$), $W(J)$ can only be a function of $J$'s determinant:

$$W(J) = \sum_{n=0}^{\infty} a_n \big( \det J \big)^n$$

Using the fact that, $\det U = 0 \;\; \forall \;\; U \in \mathrm{SU}(N)$, and a bit of algebra, it follows that

$$W(J) = \sum_{n=0}^{\infty} \frac{2! \cdots (N-1)!}{n! \cdots (n+N-1)!} \big( \det J \big)^n$$

## The final piece

Finally, inserting the analytic expression for a matrix determinant, one is ready to go

$$\det J = \frac{1}{N!}\varepsilon_{i_1,\dots,i_N}\varepsilon_{j_1,\dots,j_N}J_{i_1,j_1}\cdots J_{i_N,j_N}$$

From this we can see that the equation for the vanishing integrals presented earlier is correct as $W \sim 1 + J^N + J^{2N} + \dots$, and an integral of type

$$\int \mathrm{d}U\, U^n \left(U^\dagger\right)^m$$

requires $n + (N-1)m$ derivatives w.r.t. $J$.

## The Levi-Civita symbol

To finalise the computation we need a way to sum over repeated indices in Levi-Civita symbols:

$$\varepsilon_{i_1,i_2,i_3,\dots,i_N}\varepsilon_{j_1,j_2,\dots,j_N} = \begin{vmatrix} \delta_{i_1,j_1} & \delta_{i_1,j_2} & \cdots & \delta_{i_1,j_N} \\ \delta_{i_2,j_1} & \delta_{i_2,j_2} & \cdots & \delta_{i_2,j_N} \\ \delta_{i_3,j_1} & \delta_{i_3,j_2} & \cdots & \delta_{i_3,j_N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_N,j_1} & \delta_{i_N,j_2} & \cdots & \delta_{i_N,j_N} \end{vmatrix}$$

# The Levi-Civita symbol

To finalise the computation we need a way to sum over repeated indices in Levi-Civita symbols:

$$\sum_i \varepsilon_{i_1,i_2,i,\ldots,i_N}\varepsilon_{j_1,i,\ldots,j_N} = \begin{vmatrix} \delta_{i_1,j_1} & \delta_{i_1,j_2} & \cdots & \delta_{i_1,j_N} \\ \delta_{i_2,j_1} & \delta_{i_2,j_2} & \cdots & \delta_{i_2,j_N} \\ \delta_{i_3,j_1} & \delta_{i_3,j_2} & \cdots & \delta_{i_3,j_N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_N,j_1} & \delta_{i_N,j_2} & \cdots & \delta_{i_N,j_N} \end{vmatrix}$$

# An example

Finally, a short example

$$I = \int\limits_{U \in \mathrm{SU}(N)} \mathrm{d}U \; U_{i,j} U^{\dagger}_{k,l}$$

$$\begin{aligned}
I &= \frac{1}{(N-1)!} \varepsilon_{l,i_1,\ldots,i_{N-1}} \varepsilon_{k,j_1,\ldots,j_{N-1}} \int \mathrm{d}U \; U_{i,j} U_{i_1,j_1} \cdots U_{i_{N-1},j_{N-1}} \\
&= \frac{1}{(N-1)!} \varepsilon_{l,i_1,\ldots,i_{N-1}} \varepsilon_{k,j_1,\ldots,j_{N-1}} \left( \frac{\delta}{\delta J_{i,j}} \frac{\delta}{\delta J_{i_1,j_1}} \cdots \frac{\delta}{\delta J_{i_{N-1},j_{N-1}}} W(J) \right) \Bigg|_{J=0} \\
&= \frac{1}{(N-1)!N!} \varepsilon_{l,i_1,\ldots,i_{N-1}} \varepsilon_{k,j_1,\ldots,j_{N-1}} \varepsilon_{i,i_1,\ldots,i_{N-1}} \varepsilon_{j,j_1,\ldots,j_{N-1}} \\
&= \frac{1}{(N-1)!N!} \,(N-1)!\, \delta_{i,l} \,(N-1)!\, \delta_{j,k} = \frac{1}{N} \delta_{i,l} \delta_{j,k}
\end{aligned}$$

# Final summary

- Used a generating function to simplify the calculation of SU($N$) group integrals
- Rewrote the generating function as the determinant of its sources
- Showed how gauge integrals can be written as products of Levi-Civita symbols