

# Generating the Strong Coupling Hopping Expansion ↪ and the $SU(N)$ Gauge Integrals

Aleksandra R. Glesaaen - Goethe-Universität Frankfurt am Main

18. June 2014

## Goal

Hopping expand the fermion determinant and calculate the spatial gauge links analytically

## Motivation

- The fermion determinant is computationally expensive
- Reduce the DOF to traces over Polyakov loop constructs
- Makes complex Langevin a viable simulation algorithm
- The series is convergent at some order in  $\kappa$

## Preliminaries: The hopping expansion

Start off with the standard Wilson QCD action:

$$Z = \int \mathcal{D}U_\mu \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_F - S_G}, \text{ with}$$
$$S_F = a^4 \sum_{n,m \in \Omega} \bar{\psi}(n) D(n|m) \psi(m),$$

where the Fermion determinant is

$$D(n|m) = \left(m + \frac{4}{a}\right) \delta_{n,m} - \frac{1}{2a} \sum_{\mu=\pm 0}^{\pm 3} (1 - \gamma_\mu) U_\mu(n) \delta(n + \hat{\mu}, m)$$
$$= D_0 \left( \delta_{n,m} - \kappa H(n|m) \right), \quad \kappa = \frac{1}{2(am + 4)}$$

## Preliminaries: The hopping expansion

Carrying out the fermion integrals in the partition function

$$Z = \int \mathcal{D}U_\mu e^{-S_G} \det [D],$$

and using the standard trace-log identity:

Determinant expansion

$$\begin{aligned} \det [D] &= \exp \left\{ - \sum_{n=1}^{\infty} \frac{\kappa^n}{n} \text{tr} [H^n] \right\} \\ &= 1 - \kappa \text{tr} [H] + \frac{1}{2} \kappa^2 \left( \text{tr} [H] \right)^2 - \frac{1}{2} \kappa^2 \text{tr} [H^2] + \mathcal{O}(\kappa^3) \end{aligned}$$

## The effective 3D theory

Start by separating the temporal and spatial hops:

$$\begin{aligned}\det [D] &= \det [1 - T - S^+ - S^-] \\ &= \det [1 - T] \det [1 - (1 - T)^{-1}(S^+ + S^-)] \\ &\equiv \det [Q_{\text{stat}}] \det [1 - P - M]\end{aligned}$$

$\det Q_{\text{det}}$  is known so the terms of interest come from

$$\det [1 - P - M] = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \text{tr} [(P + M)^n] \right\},$$

where the trace is over all free indices (space, time, Dirac, colour).

## Ingredients

$$P(\vec{x}, t | \vec{y}, \tau) = \kappa Q_{\text{stat}}^{-1}(t, \tau) \sum_{i=1}^3 (1 - \gamma_i) U_i(\vec{x}) \delta_{\vec{x}, \vec{y} - \hat{i}}$$
$$M(\vec{x}, t | \vec{y}, \tau) = \kappa Q_{\text{stat}}^{-1}(t, \tau) \sum_{i=1}^3 (1 + \gamma_i) U_i^{\dagger}(\vec{y}) \delta_{\vec{x} - \hat{i}, \vec{y}}$$

Every factor of  $P$  or  $M$  bring a full set of free indices which needs to be reduced before we can carry out the spatial gauge integrals.

# The trace is our friend

## Initial expression

$$\det D = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \text{tr}[(P + M)^n] \right\}$$

Both  $P$  and  $M$  has temporal movement because of  $Q_{\text{stat}}^{-1}$ , but primarily

- $P$  does one forwards spatial jump
- $M$  does one backwards spatial jump

## Rule #1

$N_P = N_M$ , to satisfy the spatial trace

# The trace is our friend

## Initial expression

$$\det D = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} [(P + M)^n] \right\}$$

## Rule #1

$N_P = N_M$ , to satisfy the spatial trace

$$\begin{aligned} \det D &= \exp \left\{ - \cancel{\operatorname{tr}[P]} - \cancel{\operatorname{tr}[M]} - \frac{1}{2} \cancel{\operatorname{tr}[P^2]} \right. \\ &\quad \left. - \frac{1}{2} \operatorname{tr}[PM] - \frac{1}{2} \operatorname{tr}[MP] - \frac{1}{2} \cancel{\operatorname{tr}[M^2]} + \mathcal{O}(\kappa^4) \right\} \\ &= \exp \left\{ - \operatorname{tr}[PM] + \mathcal{O}(\kappa^4) \right\} \end{aligned}$$



Every term includes the direction the jump is to be taken, we want to limit this as much as possible.

$$P(\vec{x}, t | \vec{y}, \tau) = \kappa Q_{\text{stat}}^{-1}(t, \tau) \sum_{i=1}^3 (1 - \gamma_i) U_i(\vec{x}) \delta_{\vec{x}, \vec{y} - \hat{i}}$$

Tracing a contribution and carrying out the intermediate sums leaves us with:

Rule #2

$$\sum_{i_1, i_2, \dots, i_n} \delta(s_1 \hat{i}_1 + s_2 \hat{i}_2 + \dots + s_n \hat{i}_n), \quad s_j = \begin{array}{l} 1, i_j \text{ from } P \\ -1, i_j \text{ from } M \end{array}$$

## Example: PPMM

Assume we pull the jump-indices out of  $P$  and  $M$ :

$$\text{tr}[PPMM] = \text{tr} \sum_{i,j,k,l \in \{\hat{x}, \hat{y}, \hat{z}\}} P_i P_j M_k M_l$$

This gives the following Delta:

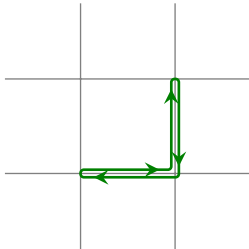
$$\sum_{i,j,k,l \in \{\hat{x}, \hat{y}, \hat{z}\}} \delta(i + j - k - l)$$

which has two solutions

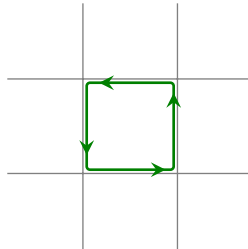
$$\begin{aligned} i = l \text{ and } j = k, \text{ or} \\ i = k \text{ and } j = l \end{aligned}$$

## Example: PPMM cont.

These two solutions can graphically be represented as



$$i = l \text{ and } j = k$$



$$i = k \text{ and } j = l$$

## A look into the future

These diagrams yield the following spatial gauge integrals

$i = l$  and  $j = k$

$$\int \mathcal{D}U_\mu U_i(\vec{x}) U_j(\vec{x} + \hat{i}) U_j^\dagger(\vec{x} + \hat{i}) U_i^\dagger(\vec{x})$$

$i = k$  and  $j = l$

$$\int \mathcal{D}U_\mu U_i(\vec{x}) U_j(\vec{x} + \hat{i}) U_i^\dagger(\vec{x} + \hat{j}) U_j^\dagger(\vec{x})$$

## A look into the future

These diagrams yield the following spatial gauge integrals

$i = l$  and  $j = k$

$$\left( \int dU U U^\dagger \right)^2$$

$i = k$  and  $j = l$

$$\left( \int dU U \right)^2 \left( \int dU U^\dagger \right)^2$$

# Vanishing gauge integrals

A quick result from a paper by Creutz<sup>1</sup>

## Vanishing integrals

$$\int_{U \in \text{SU}(3)} dU U^n (U^\dagger)^m = 0, \text{ if } \begin{cases} n + 2m \neq 0 \pmod{3}, \text{ or} \\ 2n + m \neq 0 \pmod{3} \end{cases}$$

---

<sup>1</sup>M. Creutz, "On Invariant Integration Over  $\text{SU}(N)$ ", *J.Math.Ph*, 1978

# Vanishing gauge integrals

A quick result from a paper by Creutz<sup>1</sup>

## Vanishing integrals

$$\int_{U \in \text{SU}(3)} dU U^n (U^\dagger)^m = 0, \text{ if } \begin{cases} n + 2m \neq 0 \pmod{3}, \text{ or} \\ 2n + m \neq 0 \pmod{3} \end{cases}$$

## Examples

$$\int dU U^\dagger = \int dU UU = \int dU UUUU^\dagger = 0$$

---

<sup>1</sup>M. Creutz, “On Invariant Integration Over  $\text{SU}(N)$ ”, *J.Math.Ph.*, 1978

# Different contributions

Mesonic contributions

$$\int dU (UU^\dagger)^{n_m}$$

Baryonic contributions

$$\int dU (UUU)^{n_b} \text{ or } \int dU (U^\dagger U^\dagger U^\dagger)^{\bar{n}_b}$$

Mixed contributions

$$\int dU (UU^\dagger)^{n_m} (UUU)^{n_b} (U^\dagger U^\dagger U^\dagger)^{\bar{n}_b}$$



## Mesonic contributions

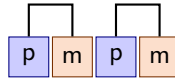
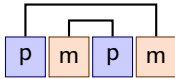
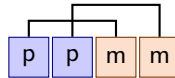
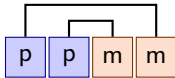
One can get these contributions by connecting pairs of  $P$ 's and  $M$ 's with the following rule:

### Rule #3

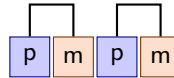
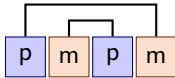
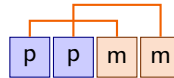
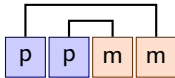
Every connected  $P$  and  $M$  must have a equal number of  $P$ 's and  $M$ 's in between, to allow for complete backtracking.

A connected pair must be at the same space-time position, and must jump in the same spatial direction.

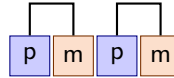
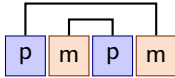
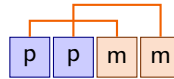
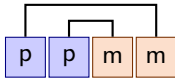
# Diagrammic notation



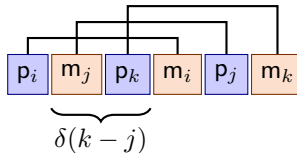
# Diagrammic notation



# Diagrammic notation



Sometimes also impose additional index restrictions



## Multi-trace contributions

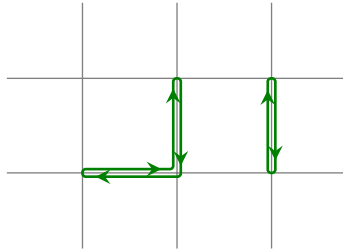
### Determinant expansion

$$\begin{aligned}\det [D] &= \exp \left\{ - \sum_{n=1}^{\infty} \frac{\kappa^n}{n} \text{tr} [H^n] \right\} \\ &= 1 - \kappa \text{tr} [H] + \frac{1}{2} \kappa^2 \left( \text{tr} [H] \right)^2 - \frac{1}{2} \kappa^2 \text{tr} [H^2] + \mathcal{O}(\kappa^3)\end{aligned}$$

Multi-trace contributions are separate diagrams which can be located anywhere on the lattice.

## Multi-trace contributions: Example

So the  $\begin{array}{|c|c|c|c|} \hline p & p & m & m \\ \hline \end{array} \bullet \begin{array}{|c|c|} \hline p & m \\ \hline \end{array}$  would graphically be represented as:



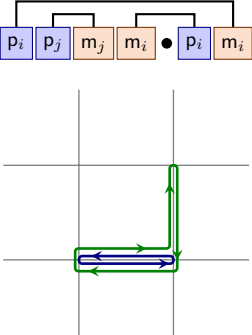
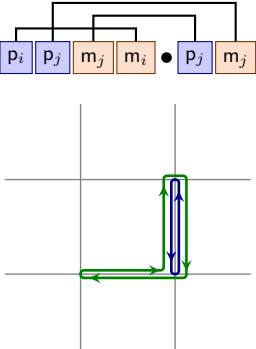
However, it doesn't get interesting until they somehow overlap.

$$\begin{aligned}
 Z &= 1 + \sum_{\vec{x},t} S_{PM} + S_{PPMM} + S_{PM\overline{PM}} + S_{PM.PM} + \mathcal{O}(\kappa^6) \\
 &= \exp \left\{ \sum_{\vec{x},t} S_{PM} + S_{PPMM} + S_{PM\overline{PM}} + S_{PM.PM} \right. \\
 &\quad \left. - \text{counter terms} + \mathcal{O}(\kappa^6) \right\}
 \end{aligned}$$

The counter terms are the disconnected multi-trace terms.

$$\sum_{\vec{x},\vec{y},t,\tau} S_{PM}(\vec{x},t) S_{PM}(\vec{y},\tau) = \sum_{\vec{x},t} S_{PM.PM}^d(\vec{x},t)$$

Only new contributions from  $\text{tr}[PPMM]\text{tr}[PM]$  are:





## The large $N_T$ limit

The last two steps of the analytic part of the calculation are:

- Further restricting temporal variables
- Inserting the proper gauge integrals

$$P_i M_i P_j M_j \sum_{i,j} \sum_{t,\tau} \int dU_i(t) dU_j(\tau) U_i(t) U_i^\dagger(t) U_j(\tau) U_j^\dagger(\tau)$$

## The large $N_T$ limit

$$P_i M_i P_j M_j \sum_{i,j} \sum_{t,\tau} \int dU_i(t) dU_j(\tau) U_i(t) U_i^\dagger(t) U_j(\tau) U_j^\dagger(\tau)$$

The sums over  $t$  and  $\tau$  can be divided into two different regions:

$i \neq j$  or  $t \neq \tau$

$$\sim N_T(N_T - 1) \left( \int dU U U^\dagger \right)^2$$

$i = j$  and  $t = \tau$

$$\sim N_T \left( \int dU U U^\dagger U U^\dagger \right)$$

## The large $N_T$ limit

The sums over  $t$  and  $\tau$  can be divided into two different regions:

$i \neq j$  or  $t \neq \tau$

$$\sim N_T(N_T - 1) \left( \int dU U U^\dagger \right)^2$$

$i = j$  and  $t = \tau$

$$\sim N_T \left( \int dU U U^\dagger U U^\dagger \right)$$

Thus in the limit  $N_T \rightarrow \infty$ , only the simple mesonic integrals contribute.

### Rules of the game

- 1 Only terms with equally many  $P$ 's and  $M$ 's contribute
- 2 One have to link every  $P$  with an  $M$  (or sets of three  $P$ 's and  $M$ 's)
- 3 Between every single-trace link there must be a valid term
- 4 (Multi-trace links are a bit more difficult. . . )

### Rules of the game

- 1 Only terms with equally many  $P$ 's and  $M$ 's contribute
- 2 One have to link every  $P$  with an  $M$  (or sets of three  $P$ 's and  $M$ 's)
- 3 Between every single-trace link there must be a valid term
- 4 (Multi-trace links are a bit more difficult. . . )

And Now for Something Completely Different™

## Goal

Calculate integrals of the form:

$$I = \int_{U \in SU(N)} dU U_{i_1, j_1} \dots U_{i_n, j_n} U_{k_1, l_1}^\dagger \dots U_{k_m, l_m}^\dagger$$

in the fundamental representation.

This presentation will closely follow the previously cited paper by M. Creutz<sup>2</sup>

---

<sup>2</sup>M. Creutz, “On Invariant Integration Over  $SU(N)$ ”, *J.Math.Ph.*, 1978

## Generating function

Introduce the generating function  $W$  in the standard way:

Generating function

$$W(J, K) = \int dU \exp \left\{ \text{tr} [JU + KU^\dagger] \right\}$$

where  $J$  and  $K$  are arbitray  $N \times N$  matrices. Thus:

$$I = \left( \frac{\delta}{\delta J_{j_1, i_1}} \cdots \frac{\delta}{\delta J_{j_n, i_n}} \frac{\delta}{\delta K_{l_1, k_1}} \cdots \frac{\delta}{\delta K_{l_m, k_m}} W(J, K) \right) \Big|_{J=K=0}$$

## Rewriting $U^\dagger$

We can rewrite all  $U^\dagger (= U^{-1})$  matrices in terms of the cofactor of  $U$  using the identity  $U^{-1} = (\text{cof} U)^T$

$$U_{i,j}^\dagger = \frac{1}{(N-1)!} \varepsilon_{j,k_1,\dots,k_{N-1}} \varepsilon_{i,l_1,\dots,l_{N-1}} U_{k_1,l_1} \cdots U_{k_{N-1},l_{N-1}}$$

We can thus eliminate  $K$  from the generating functional, and from now on,  $W = W(J)$ .



## An analytic expression for $W(J)$

Because of the properties of the gauge group elements of  $SU(N)$ ,  $W(J)$  can only be a function of  $J$ 's determinant:

$$W(J) = \sum_{n=0}^{\infty} a_n (\det J)^n$$

Using the fact that,  $\det U = 0 \quad \forall \quad U \in SU(N)$ , and a bit of algebra, it follows that

$$W(J) = \sum_{n=0}^{\infty} \frac{2! \cdots (N-1)!}{n! \cdots (n+N-1)!} (\det J)^n$$

## The final piece

Finally, inserting the analytic expression for a matrix determinant, one is ready to go

$$\det J = \frac{1}{N!} \varepsilon_{i_1, \dots, i_N} \varepsilon_{j_1, \dots, j_N} J_{i_1, j_1} \cdots J_{i_N, j_N}$$

From this we can see that the equation for the vanishing integrals presented earlier is correct as  $W \sim 1 + J^N + J^{2N} + \dots$ , and an integral of type

$$\int dU U^n (U^\dagger)^m$$

requires  $n + (N - 1)m$  derivatives w.r.t.  $J$ .

# The Levi-Civita symbol

To finalise the computation we need a way to sum over repeated indices in Levi-Civita symbols:

$$\varepsilon_{i_1, i_2, i_3, \dots, i_N} \varepsilon_{j_1, j_2, \dots, j_N} = \begin{vmatrix} \delta_{i_1, j_1} & \delta_{i_1, j_2} & \cdots & \delta_{i_1, j_N} \\ \delta_{i_2, j_1} & \delta_{i_2, j_2} & \cdots & \delta_{i_2, j_N} \\ \delta_{i_3, j_1} & \delta_{i_3, j_2} & \cdots & \delta_{i_3, j_N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_N, j_1} & \delta_{i_N, j_2} & \cdots & \delta_{i_N, j_N} \end{vmatrix}$$

# The Levi-Civita symbol

To finalise the computation we need a way to sum over repeated indices in Levi-Civita symbols:

$$\sum_i \varepsilon_{i_1, i_2, i, \dots, i_N} \varepsilon_{j_1, i, \dots, j_N} =$$

$\delta_{i_1, j_1}$	$\delta_{i_1, j_2}$	$\cdots$	$\delta_{i_1, j_N}$
$\delta_{i_2, j_1}$	$\delta_{i_2, j_2}$	$\cdots$	$\delta_{i_2, j_N}$
$\delta_{i_3, j_1}$	$\delta_{i_3, j_2}$	$\cdots$	$\delta_{i_3, j_N}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\delta_{i_N, j_1}$	$\delta_{i_N, j_2}$	$\cdots$	$\delta_{i_N, j_N}$

## An example

Finally, a short example

$$I = \int_{U \in \text{SU}(N)} dU U_{i,j} U_{k,l}^\dagger$$

$$\begin{aligned} I &= \frac{1}{(N-1)!} \varepsilon_{l,i_1,\dots,i_{N-1}} \varepsilon_{k,j_1,\dots,j_{N-1}} \int dU U_{i,j} U_{i_1,j_1} \cdots U_{i_{N-1},j_{N-1}} \\ &= \frac{1}{(N-1)!} \varepsilon_{l,i_1,\dots,i_{N-1}} \varepsilon_{k,j_1,\dots,j_{N-1}} \left( \frac{\delta}{\delta J_{i,j}} \frac{\delta}{\delta J_{i_1,j_1}} \cdots \frac{\delta}{\delta J_{i_{N-1},j_{N-1}}} W(J) \right) \Big|_{J=0} \\ &= \frac{1}{(N-1)!N!} \varepsilon_{l,i_1,\dots,i_{N-1}} \varepsilon_{k,j_1,\dots,j_{N-1}} \varepsilon_{i,i_1,\dots,i_{N-1}} \varepsilon_{j,j_1,\dots,j_{N-1}} \\ &= \frac{1}{(N-1)!N!} (N-1)! \delta_{i,l} (N-1)! \delta_{j,k} = \frac{1}{N} \delta_{i,l} \delta_{j,k} \end{aligned}$$

## Final summary

- Used a generating function to simplify the calculation of  $SU(N)$  group integrals
- Rewrote the generating function as the determinant of its sources
- Showed how gauge integrals can be written as products of Levi-Civita symbols