Analytical computations of an effective lattice theory for heavy quarks

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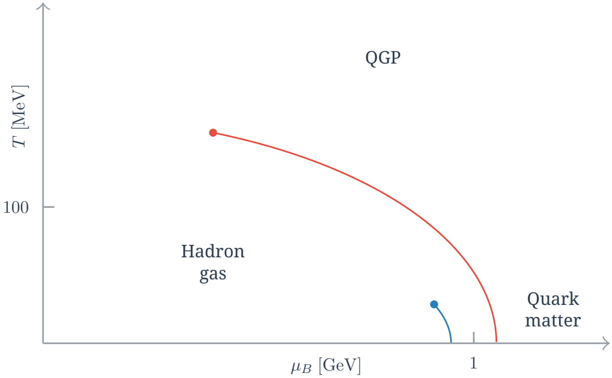
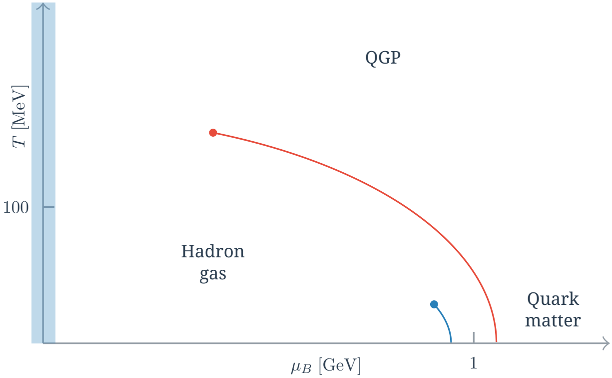
Introduction

Putative QCD phase diagram



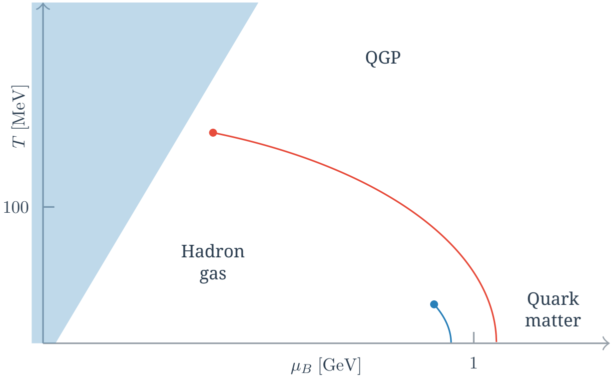
- Different approaches can access different regions

Putative QCD phase diagram



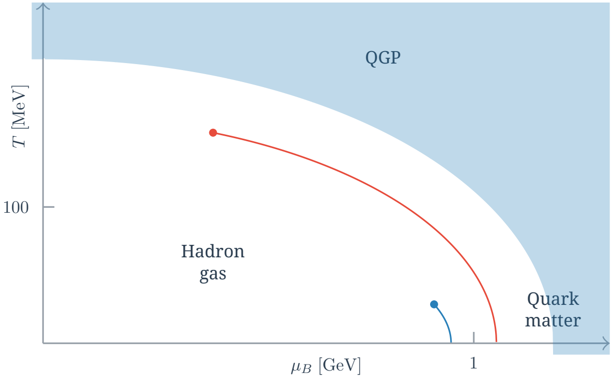
- Naive reach of Lattice QCD

Putative QCD phase diagram



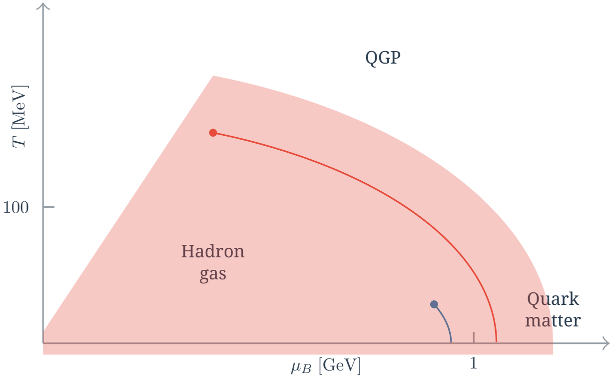
- Lattice QCD with additional methods (analytic continuation, taylor expansions, reweighting,...)

Putative QCD phase diagram



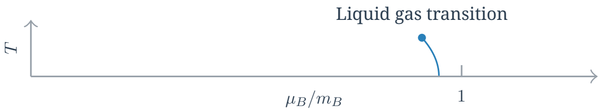
- Perturbation theory of QCD

Putative QCD phase diagram



- Region currently not accessible from first principles with traditional methods

The cold and dense



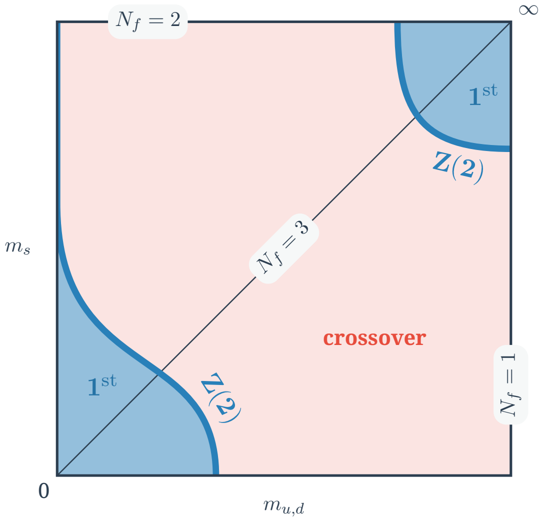
Lattice requirements

- $a \ll m_B^{-1} \quad \Rightarrow \quad a \ll 0.2\text{fm}$
- $T < 10\text{MeV} \quad \Rightarrow \quad N_t \gtrsim 200$

The cold and dense is very numerically demanding

... and there is the sign problem

Columbia plot



Crash course in lattice theory

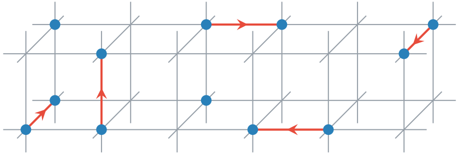
Fermions

Quarks

$$\mathcal{L}_F = \bar{\psi}(x)\left(i\gamma_\mu\partial^\mu + \gamma_\mu A^\mu\right)\psi(x) + m_q\bar{\psi}(x)\psi(x)$$

$$\Big\downarrow \; x \to an$$

$$L_F \sim \sum_\mu \bar{\psi}(n)U_\mu(n)\psi(n+\mu) + m_q\bar{\psi}(n)\psi(n)$$



Crash course in lattice theory

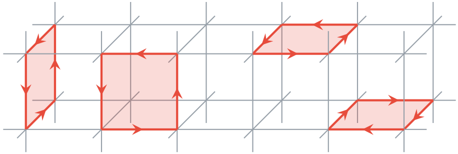
Gauge fields

Gluons

$$\mathcal{L}_G = \frac{1}{4} \operatorname{tr} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

$$\Big\downarrow \; x \to an$$

$$L_G \sim \beta \sum_{\mu,\nu} \operatorname{tr} U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)$$



The sign problem

QCD Integration Measure

$$\det Q_{\text{uark}}(\mu_B)\ \exp\left\{-S_g\right\}$$

For $\operatorname{Re}\{\,\mu_B\} \neq 0$: $\det Q_{\text{uark}} \in \mathbb{C}$

Workarounds:

- Reweighting
- Analytic continuation
- Complex Langevin

but still suffer from the need for exponential cancellations

The Effective 3D Theory

Guiding equation

Our goal

- Integrate out all spatial gauge links

$$\mathcal{Z} = \int DU_\mu \exp\left\{-S_{\text{action}}\right\}$$
$$= \int DU_0 \exp\left\{-S_{\text{effective action}}\right\}$$

Lattice expansions
Strong coupling expansion

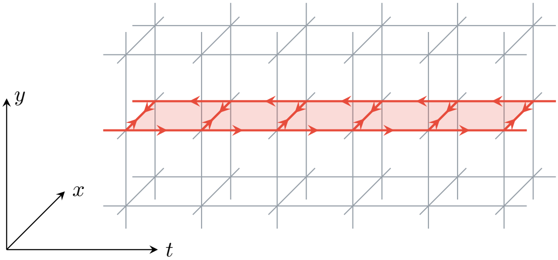
Expansion around $\beta = \frac{2N_c}{g^2} = 0$

Recap

$$L_G \sim \beta \sum_{\mu,\nu} \mathrm{tr}\, U_\mu(n) U_\nu(n+\mu) U_\mu^\dagger(n+\nu) U_\nu^\dagger(n)$$

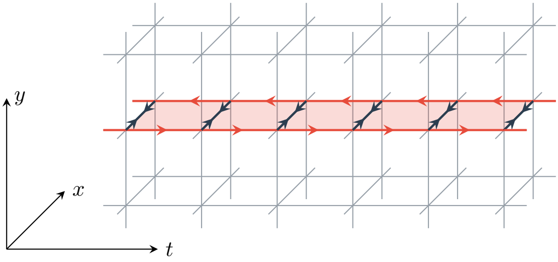
This is an expansion in the number of plaquettes on the lattice

Effective pure gauge theory



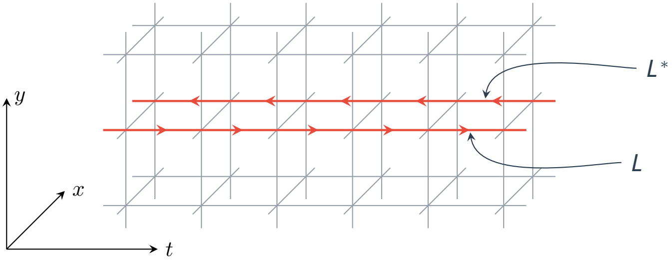
Put a strip of plaquettes in the time direction

Effective pure gauge theory



Integrate over all spatial gauge links

Effective pure gauge theory

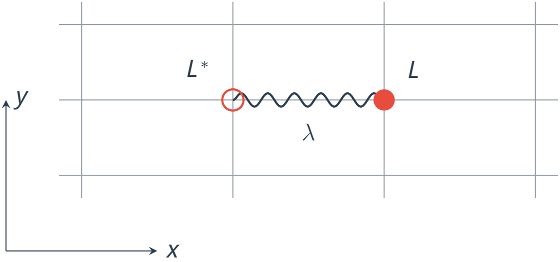


What remains is an interaction between Polyakov Loops

Effective pure gauge theory

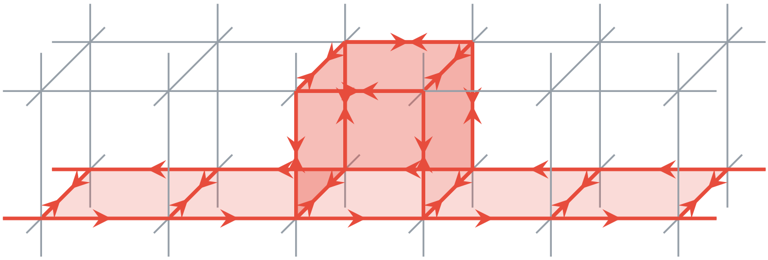
Effective Gluon Interactions

$$S_{\text{eff gluon}} \sim \lambda(\beta, N_t) \sum_{\langle x,y \rangle} L(x) L^*(y)$$



Effective pure gauge theory

Higher order β corrections



Rescales λ:

$$\lambda \to \lambda\Big(1 + 4N_t u(\beta)^4\Big)$$

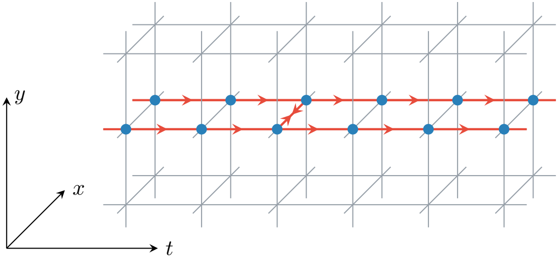
Lattice expansions
Hopping parameter expansion

One can rewrite the fermion matrix $Q_{\rm uark}$ as

$$\det Q_{\rm uark} = \exp\left\{ -\sum_{n=1}^{\infty} \frac{1}{n} \kappa^{n} \operatorname{tr} H_{\rm op}^{n} \right\}$$

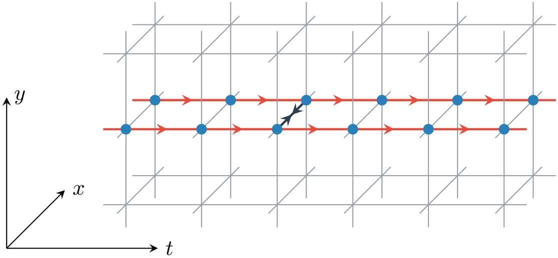
where $H_{\rm op}$ translates the quark one lattice spacing and $\kappa \sim 1/m_q$

Spatial hopping expansion



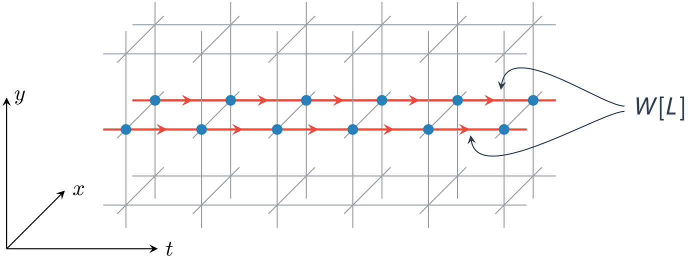
Can produce a closed quark loop with multiple temporal windings

Spatial hopping expansion



Once again integrate out spatial links

Spatial hopping expansion

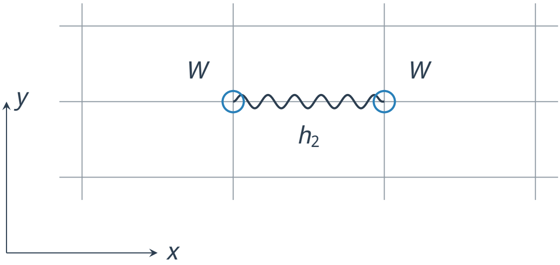


Producing an interaction between the W objects

Spatial hopping expansion

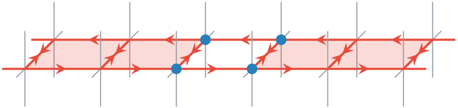
Effective Quark Interactions

$$S_{\text{eff quarks}} \sim h_2(\kappa, N_t) \sum_{\langle x,y \rangle} W(x)W(y)$$



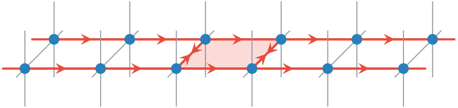
Mixed contributions

Correction to λ



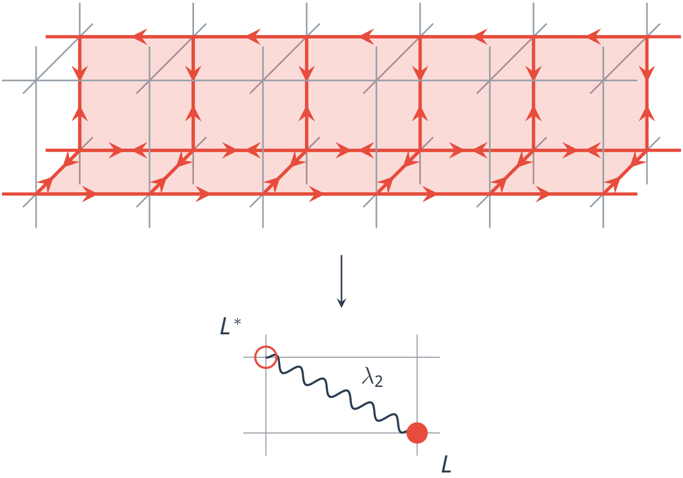
- Rescales λ
- $\lambda \to \lambda(\kappa)$

Correction to h_2

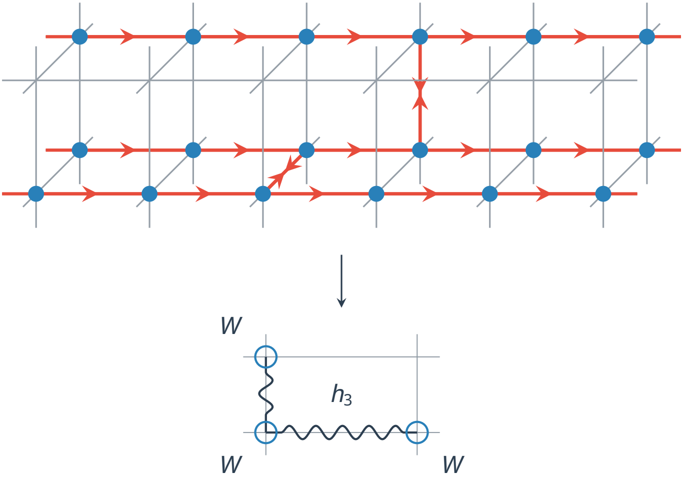


- Rescales h_2
- $h_2 \to h_2(\beta)$

Spatially extended contributions



Spatially extended contributions



The effective lattice theory

The Effective Action

$$\mathcal{Z} = \int \prod_x \mathrm{d}L(x)\ \exp\left\{-S_{\text{eff action}}\right\}$$

$$S_{\text{eff action}} \sim \lambda \sum_{\langle x,y\rangle} L(x)L^*(y) + h_2 \sum_{\langle x,y\rangle} W(x)W(y)$$

The theory is contained in the set of coupling constants

$$\lambda_1(\beta, N_t, \kappa), \lambda_2, \ldots \quad h_1(\beta, N_t, \kappa), h_2, \ldots$$

Simulating the effective theory

The effective theory is numerically cheap to simulate

- No fermion determinant to calculate
- One dimension less
- Polyakov loop as only degree of freedom per site
- N_t is only a parameter

Sign problem is mild $\Rightarrow$ Reweighting works well

Analytic Calculations

Motivation

SCALING LAWS AND TRIVIALITY BOUNDS IN THE LATTICE ϕ^4 THEORY
(I). One-component model in the symmetric phase

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The lattice ϕ^4 theory in four space-time dimensions is most likely "trivial", i.e. its continuum limit is a free field theory. However, for small but positive lattice spacing a and at energies well below the cutoff mass $\Lambda = 1/a$, the theory effectively behaves like a continuum theory with particle interactions, which may be appreciable. By a combination of known analytical methods, we here determine the maximal value of the renormalized coupling at zero momentum as a function of Λ/m, where m denotes the mass of the scalar particle in the theory. Moreover, a complete solution of the model is obtained in the sense that all low energy amplitudes can be computed with reasonable estimated accuracy for arbitrarily chosen bare coupling and mass in the symmetric phase region.

Similarity to the LCE

$\lambda\phi^4$ theory

$$\int \prod \mathrm{d}\phi(x)\ \exp\left\{ -S_0[\phi] - \kappa \sum_{\langle x,y\rangle} \phi(x)\phi(y) \right\}$$

Effective Theory

$$\int \prod \mathrm{d}L(x)\ \det Q^{\mathrm{stat}}\ \exp\left\{ -\lambda \sum_{\langle x,y\rangle} L(x)L^*(y) + \ldots \right\}$$

Effective action as graphs

The theory contains interactions at all distances

$$S_I[L] = \sum_{\text{terms}} \sum_{\text{dof}} v_i(1, 2, ..., n_i)\phi_1[L]\phi_2[L] \cdots \phi_{n_i}[L]$$

In our theory:

- $v_i(1, 2, ...n_i) \to \{\lambda_i, h_i\} \times \text{geometry}$
- $\phi_i \to \{L_i, L_i^*, W_i\}$

Graphs and embeddings
The N-point Linked Cluster Expansion

Classical Linked Cluster Expansion

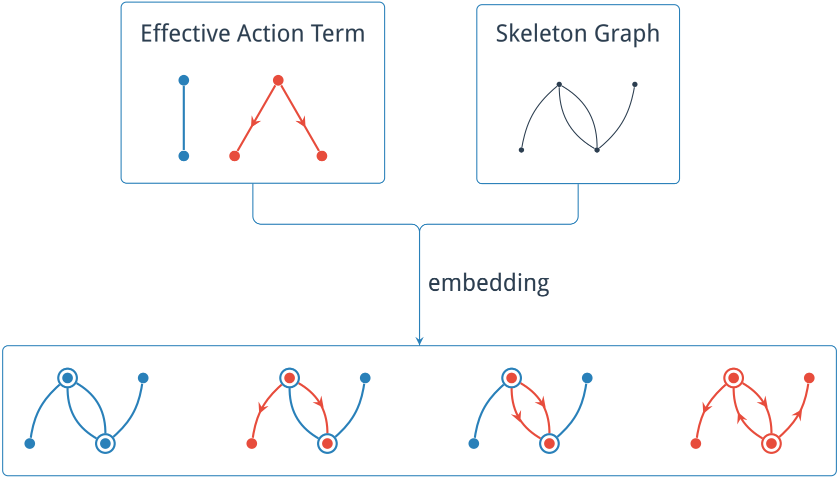
The action consists of two-point interactions which can be expanded in a set of connected graphs.

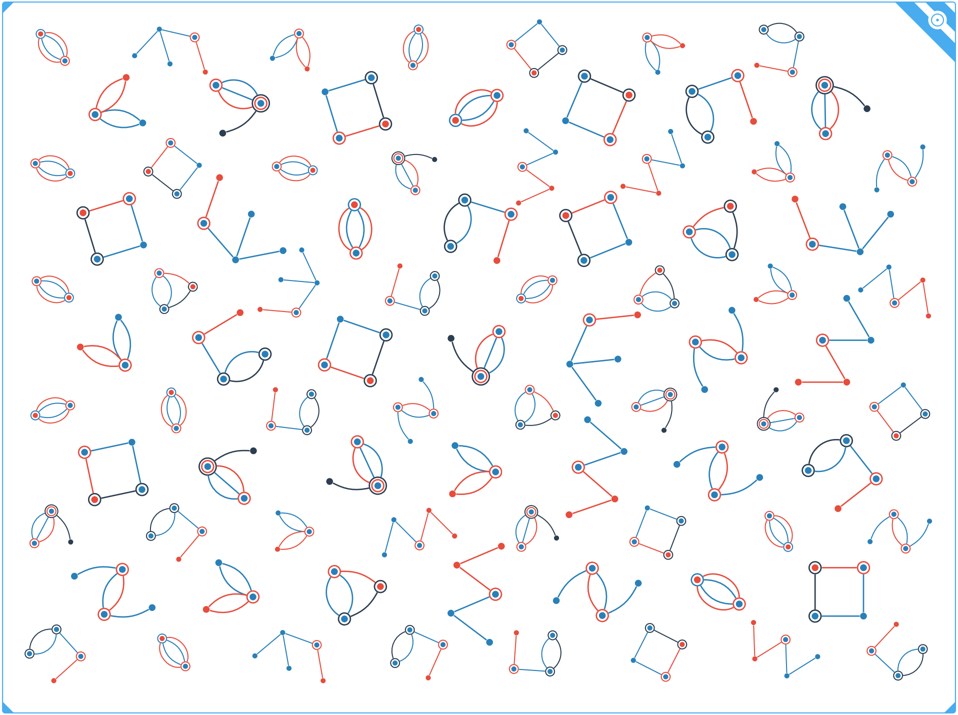
Our Problem

The action contains n-point interactions that we can embed on a set of connected graphs.

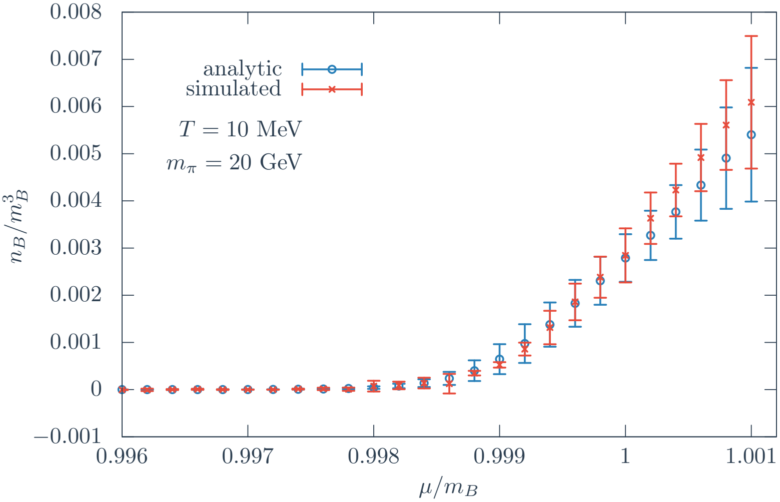
$\hookrightarrow$ Two step embedding

Graphs and embeddings
Two step embedding process





Continuum comparison



Two-step expansion

First expansion

Strong coupling expansion

+

Hopping parameter expansion

↓

$$\mathcal{Z} \approx \int \prod_x \mathrm{d}L(x)\ \exp\left\{-S_{\text{eff action}}\right\}$$

$$S_{\text{eff action}} = \sum_{n=0}^{N} \sum_{m=0}^{M} \kappa^n \beta^m S_{n,m}$$

Two-step expansion

Second expansion

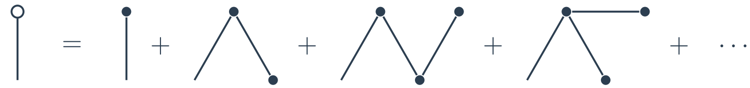
Linked cluster expansion

$$
\begin{aligned}
\mathcal{W} &= -\frac{1}{\Omega} \log \mathcal{Z} \\
&\approx \sum_{i,j} \sum_{n=0}^{N} \sum_{m=0}^{M} h_i^n \lambda_j^m W_{i,j,n,m} \\
&\equiv \sum_{n=0}^{N_\kappa} \sum_{m=0}^{M_\beta} \kappa^n \beta^m W_{n,m}
\end{aligned}
$$

Analytic resummations

Extending the theory

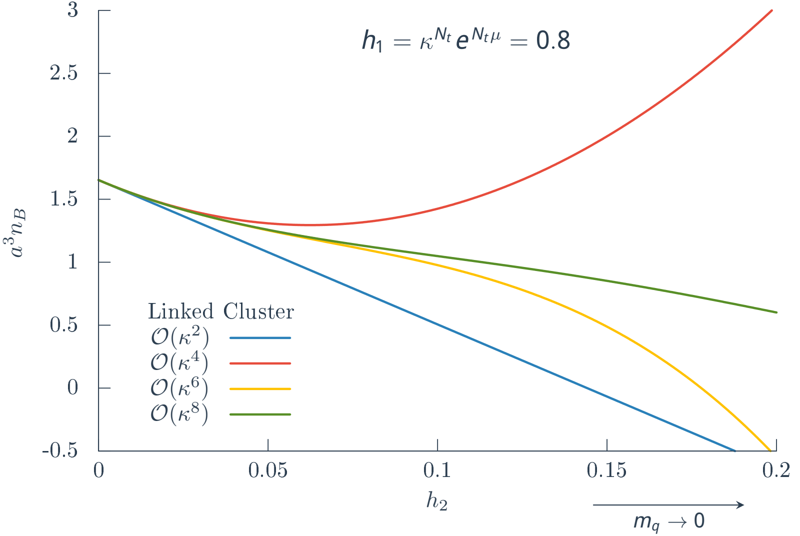
Using the resummed Linked Cluster Expansion as motivation



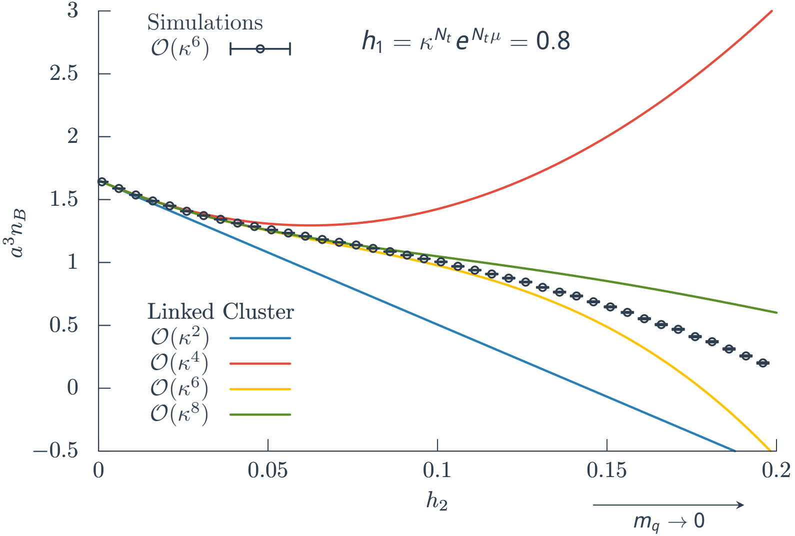
We can do the same resummation for the effective action itself, incorporating long-range effects

Results

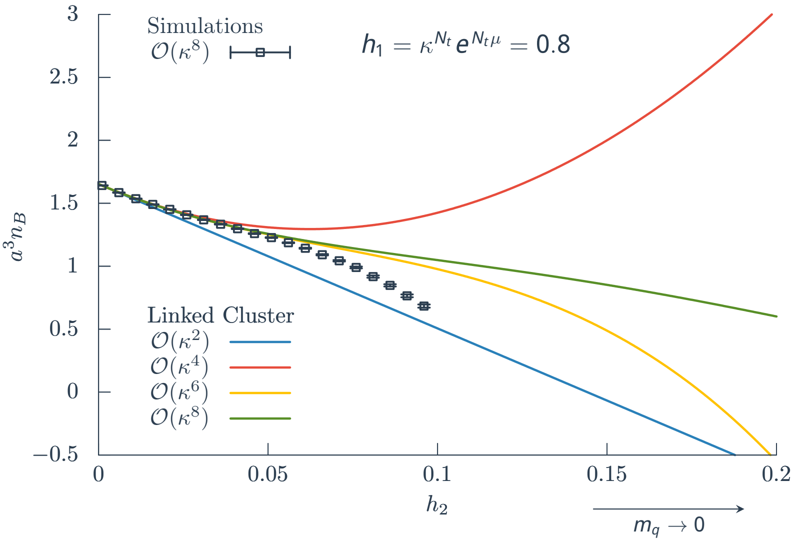
Convergence



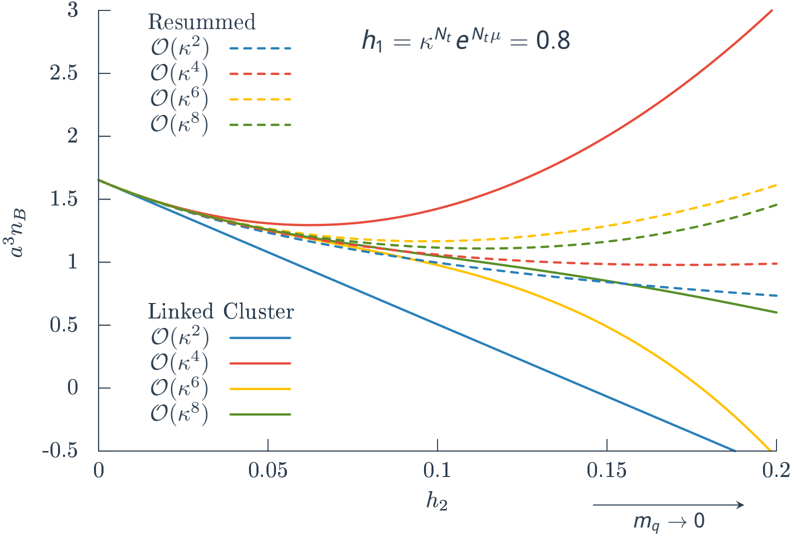
Convergence



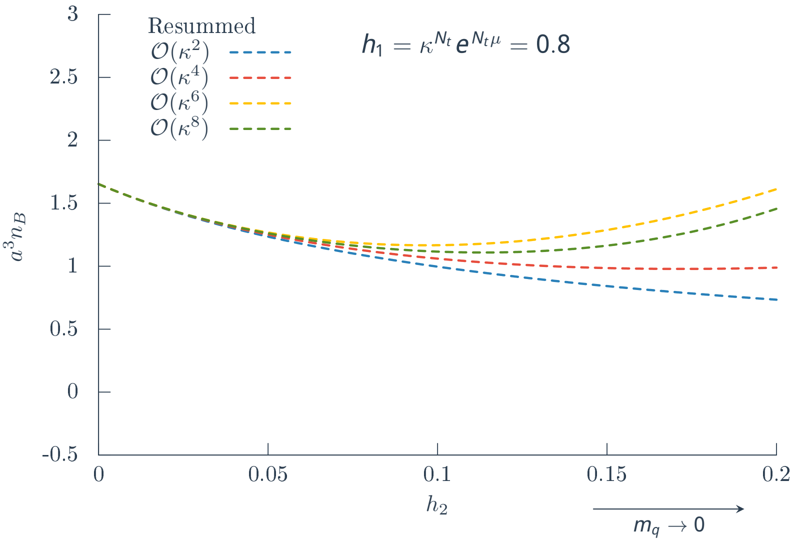
Convergence



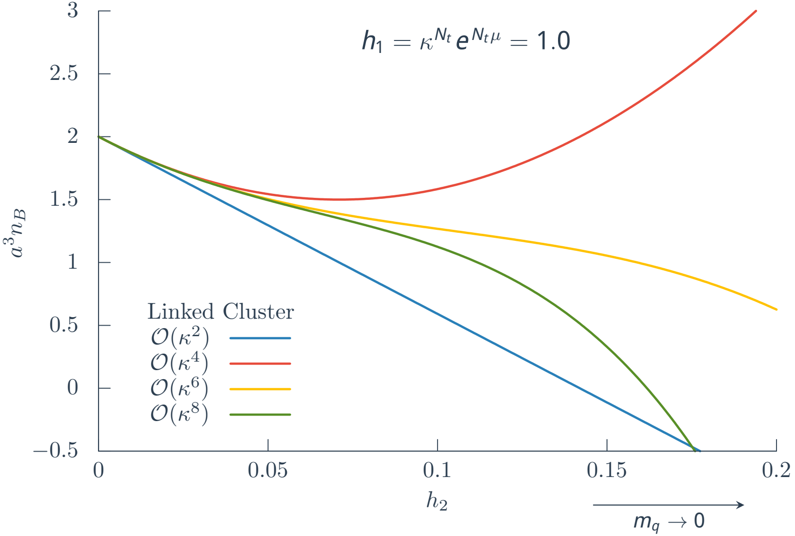
Effect of the resummations



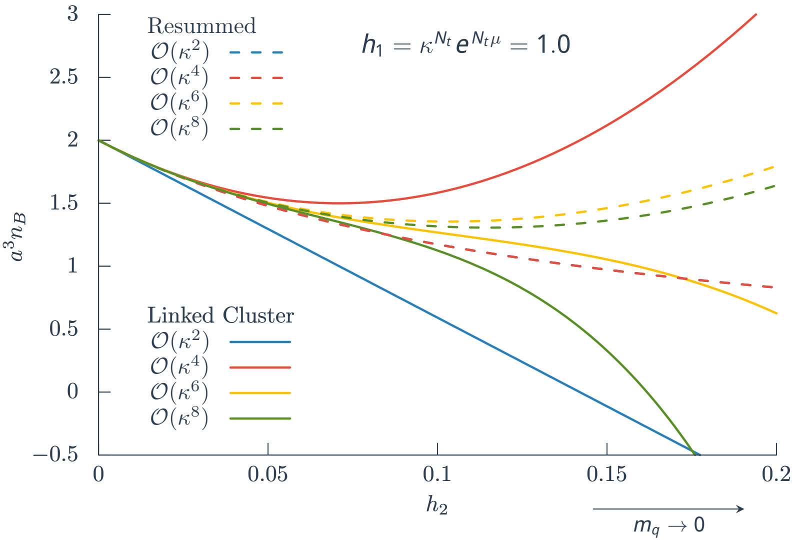
Effect of the resummations



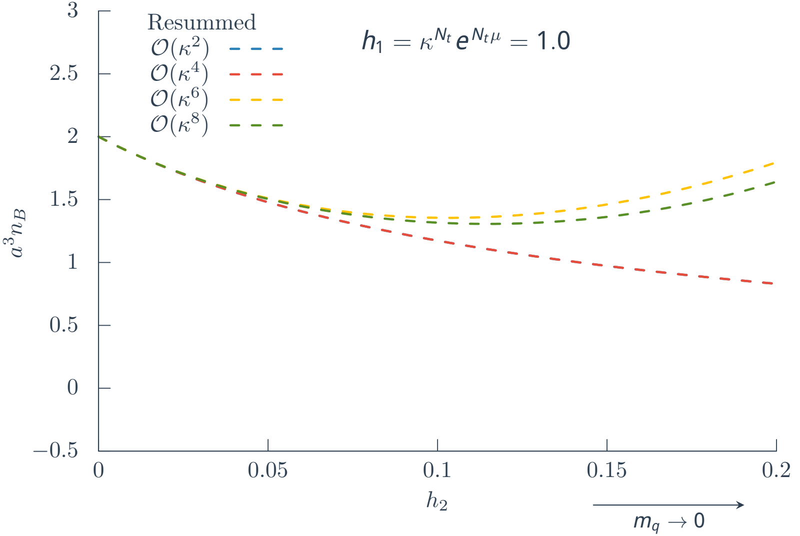
Effect of the resummations



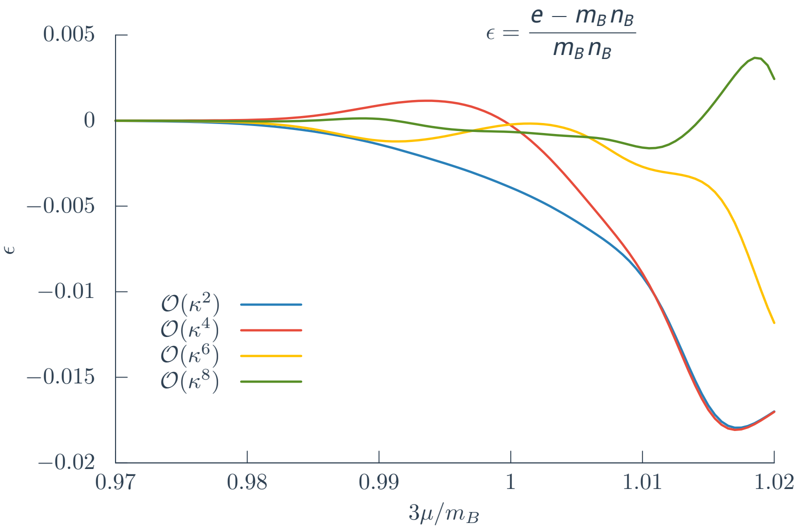
Effect of the resummations



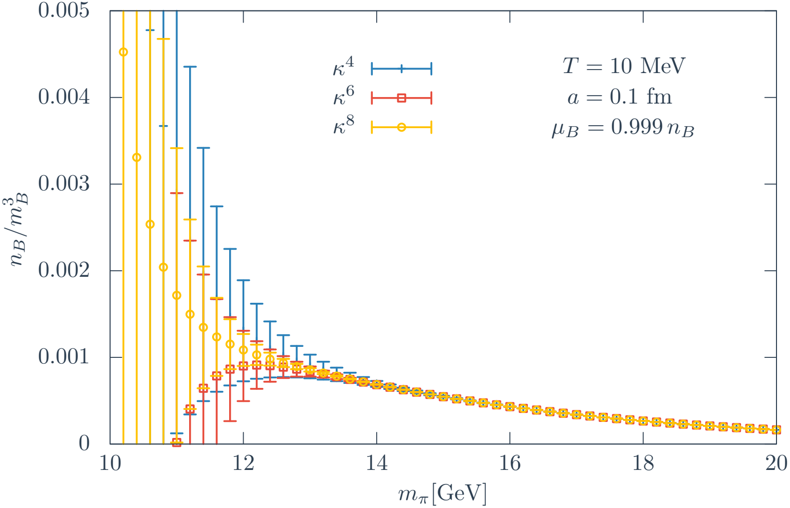
Effect of the resummations



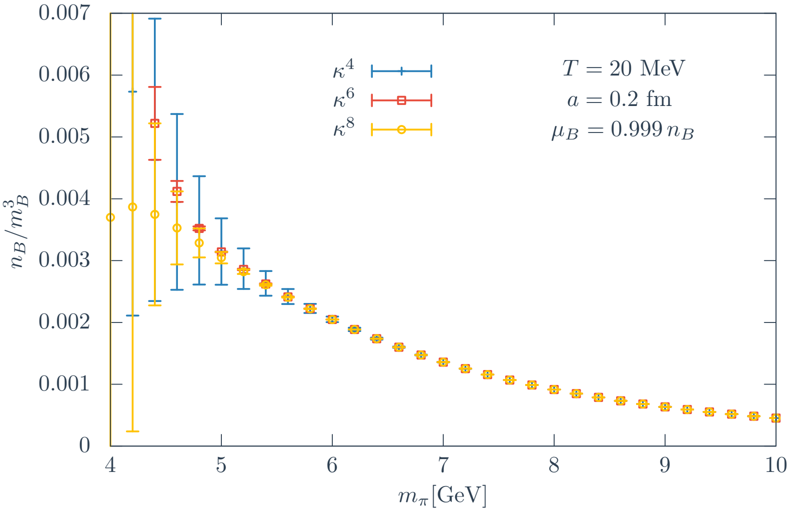
Attempts at the binding energy



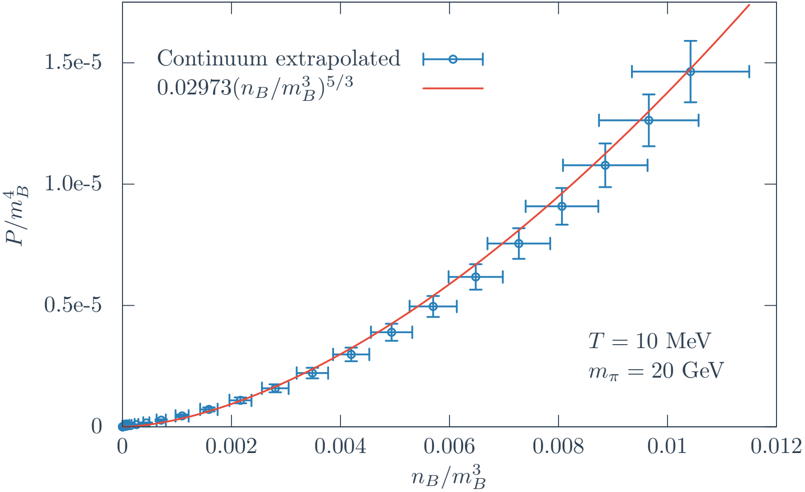
Towards lighter quarks



Towards lighter quarks



Equation of state



Conclusion

Summary

Summary

- Introduced the effective dimensionally reduced lattice theory
- Looked at how a consistent analytic calculation could be carried out
- Demonstrated convergence and comparisons with numerics

Outlook

Outlook

- Use the analytic results as a tool to study the characteristics of the effective theory
- Find analytic resummation schemes to incorporate long-range effects