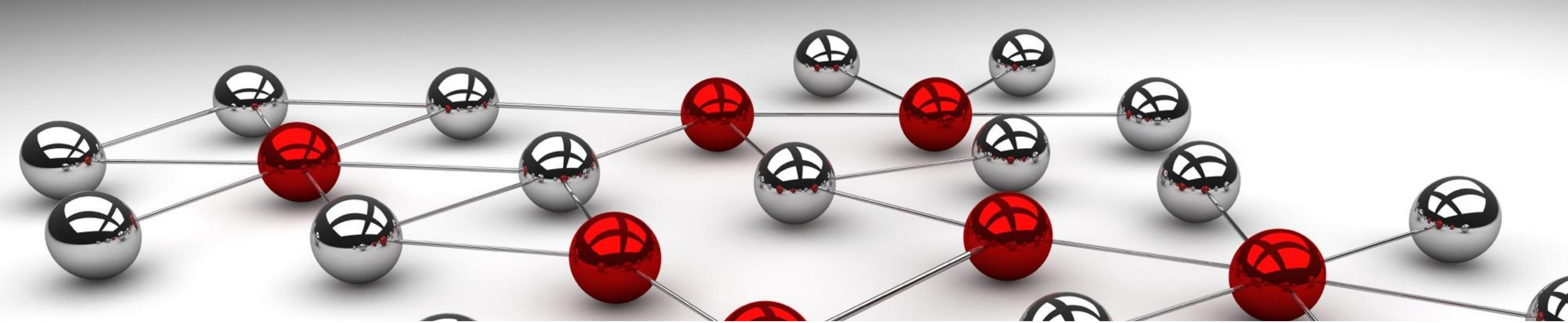
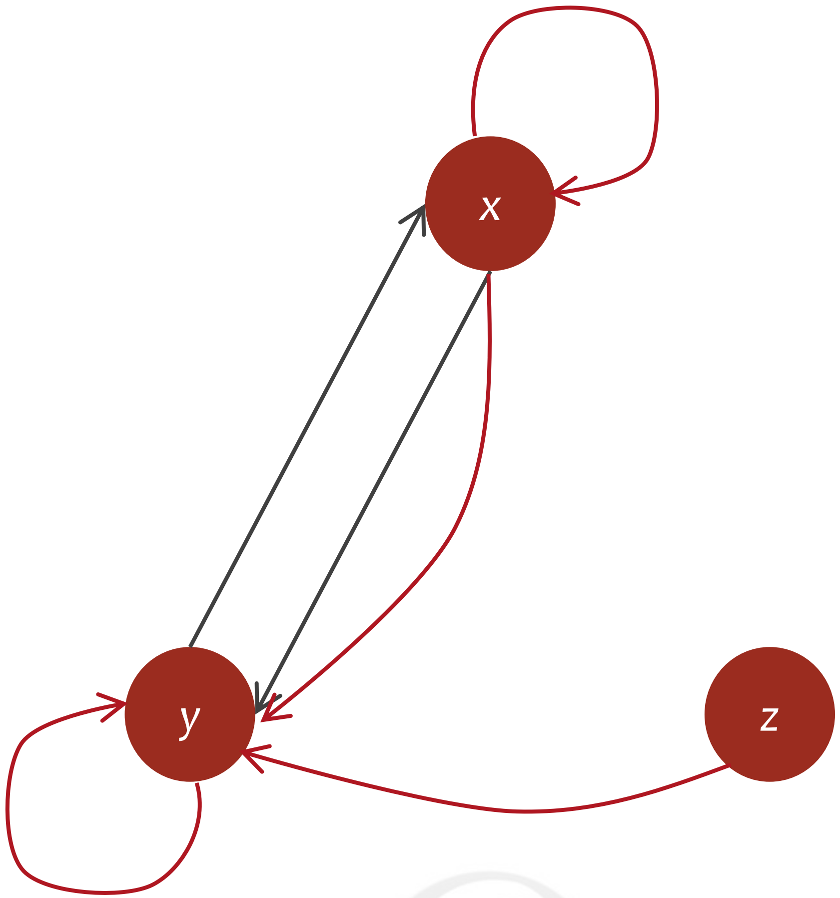


05 Introduction to Static Complex Networks (Part I)

By NTU Complexity Institute

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

The ‘Logic’ of Complex Systems



Lorenz Equation:

$$\frac{dx}{dt} = \sigma(y - x)$$

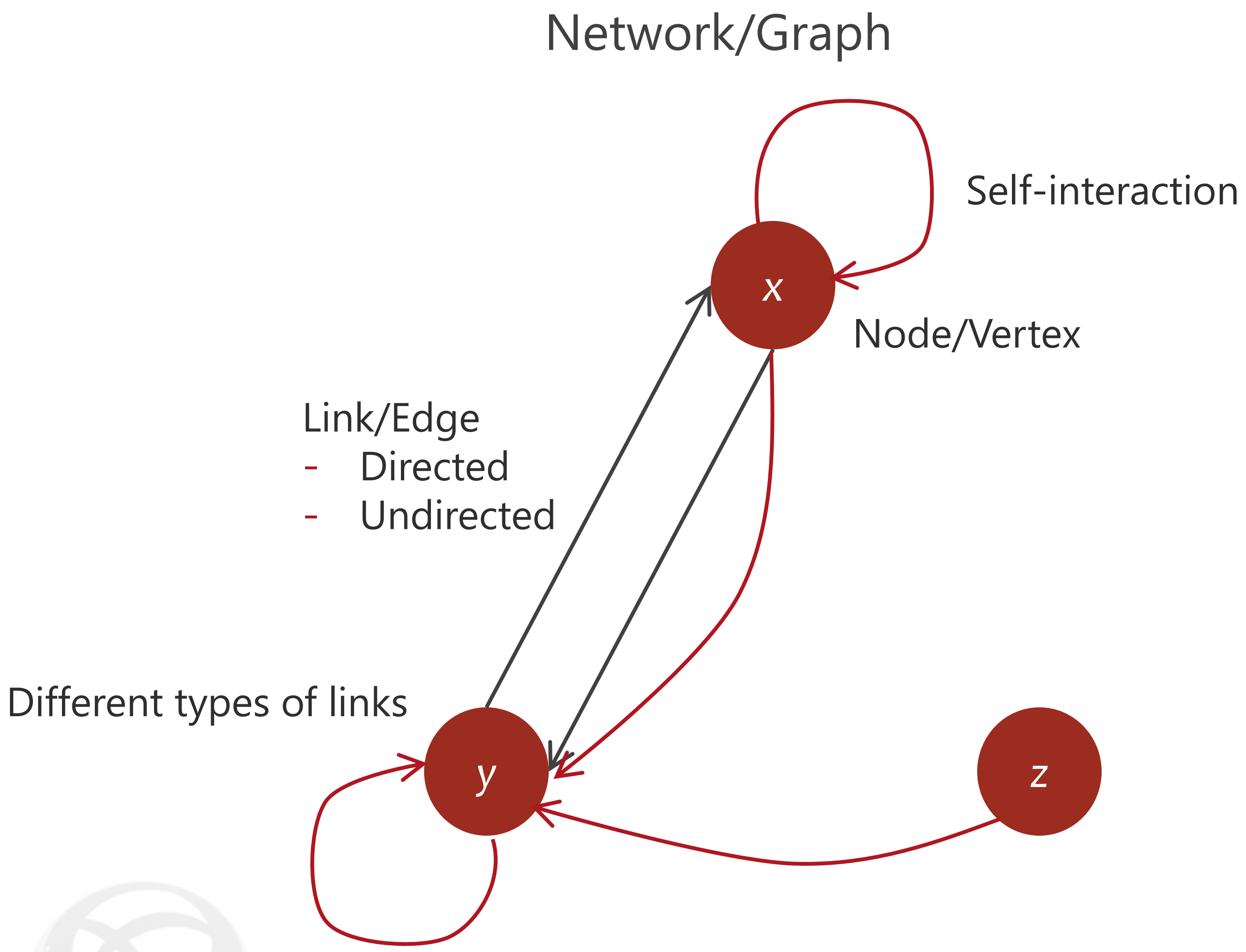
$$\frac{dy}{dt} = x(p - z) - y$$

$$\frac{dz}{dt} = xy = \beta z$$

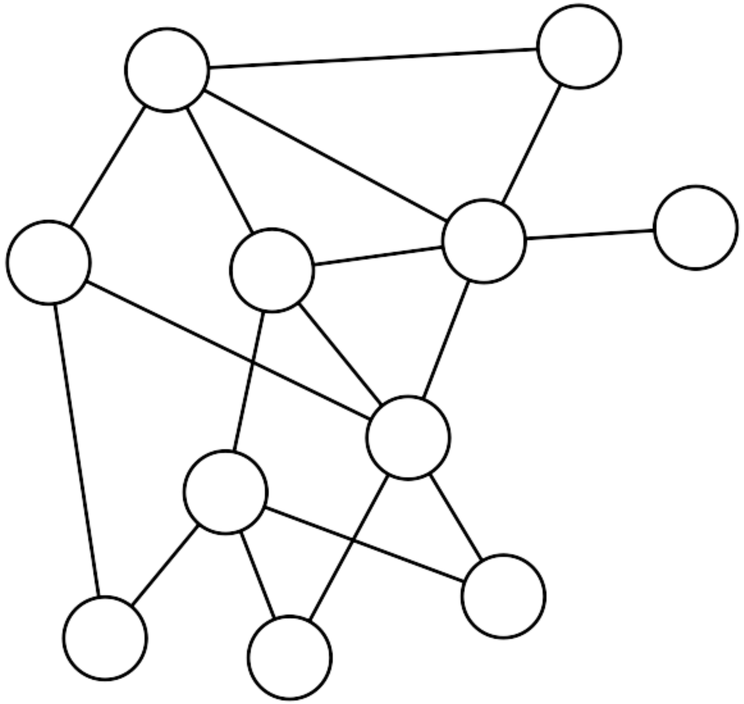
Equation is unknown.

Can still understand a lot!

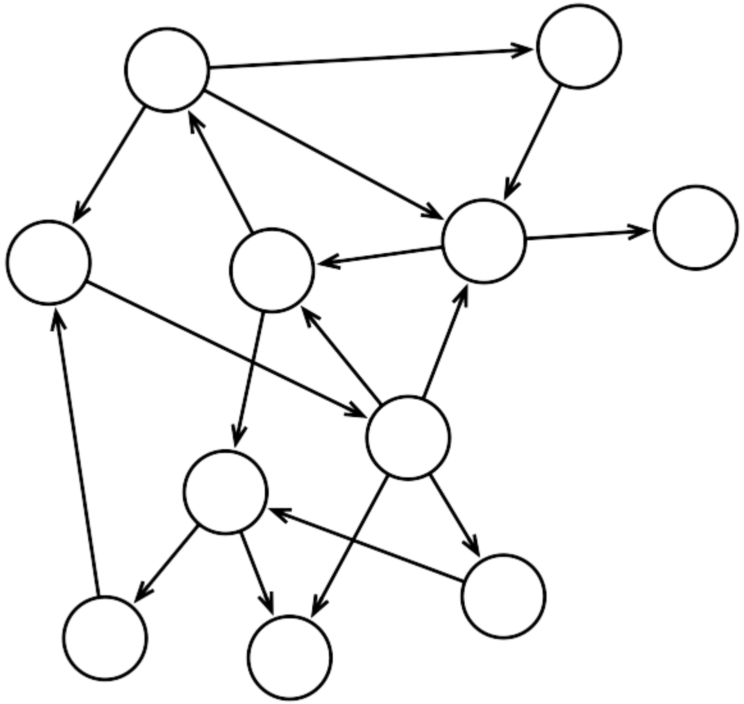
Definitions



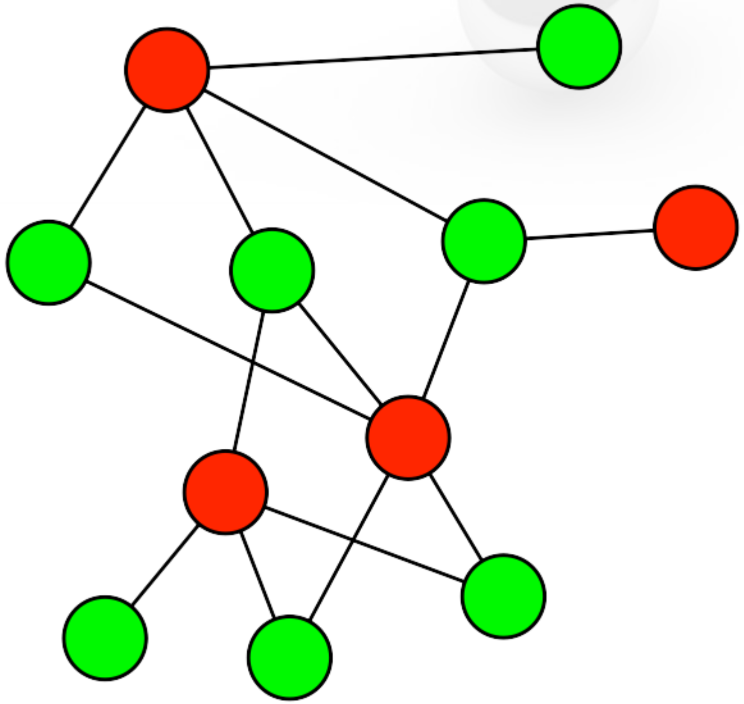
Types of Networks I



Undirected

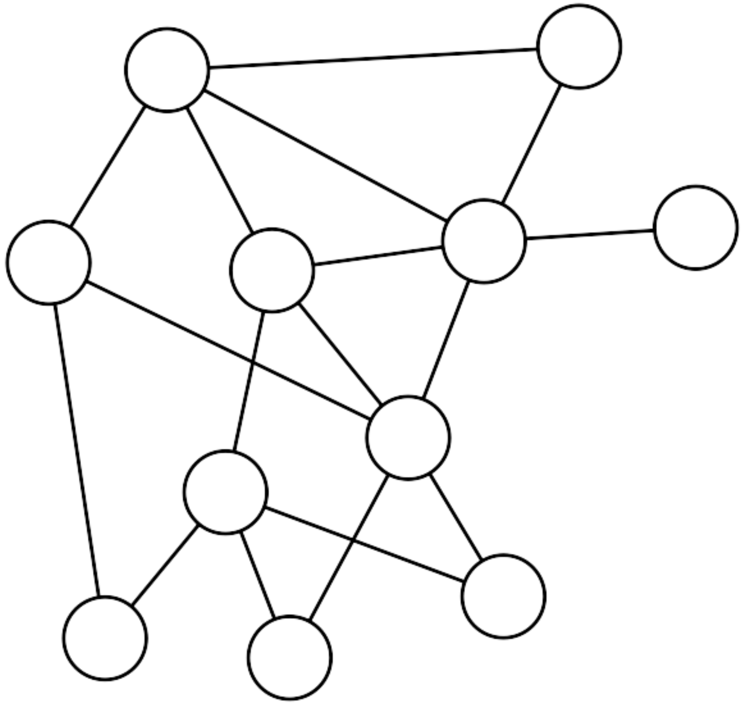


Directed

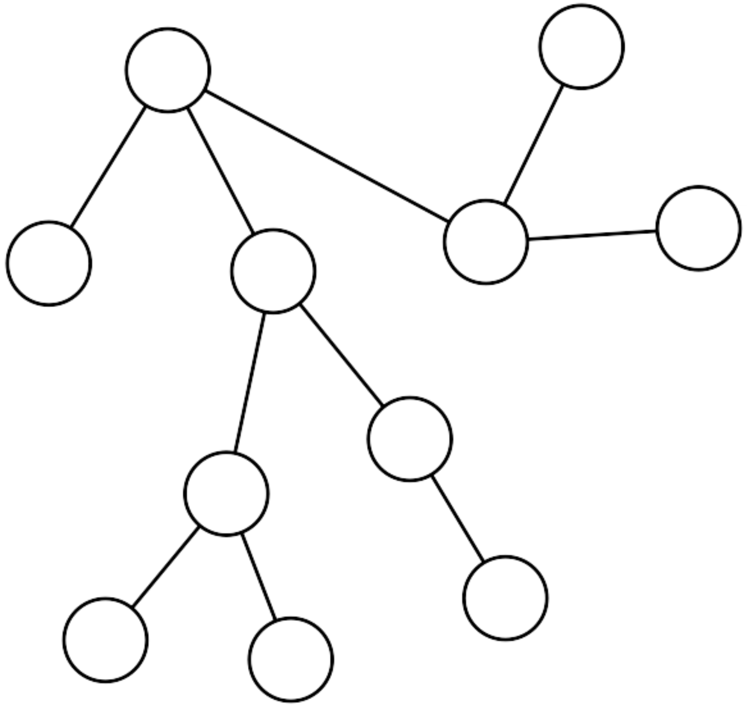


Bipartite

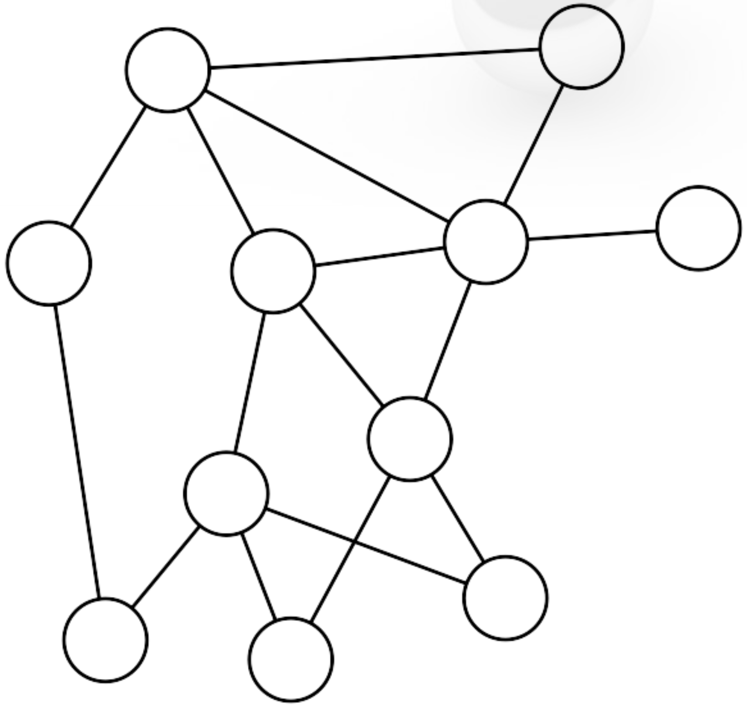
Types of Networks II



Undirected

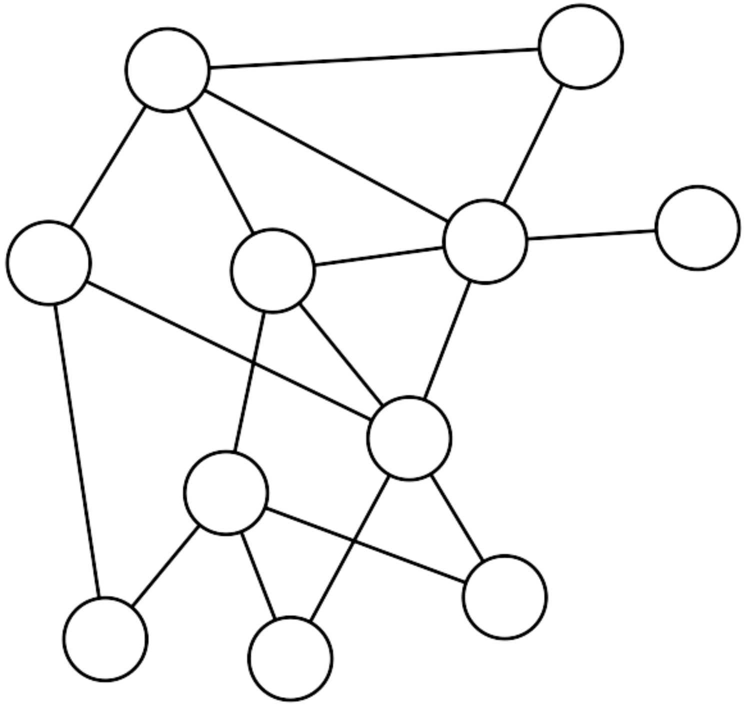


Tree

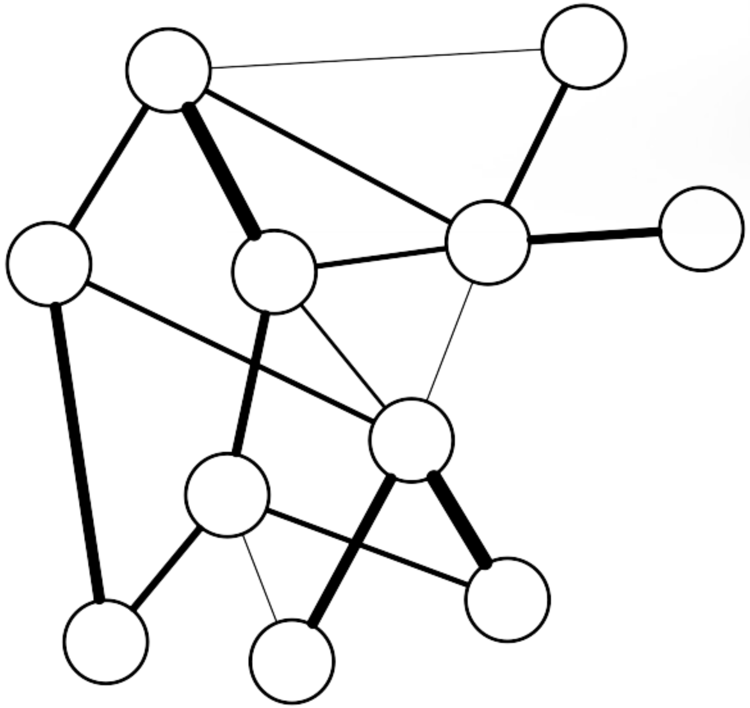


Planar

Types of Networks III



Unweighted

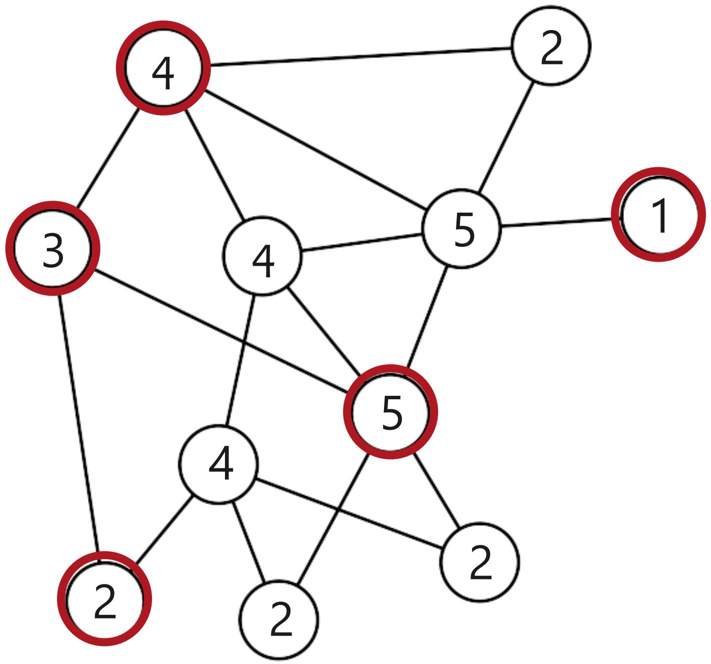


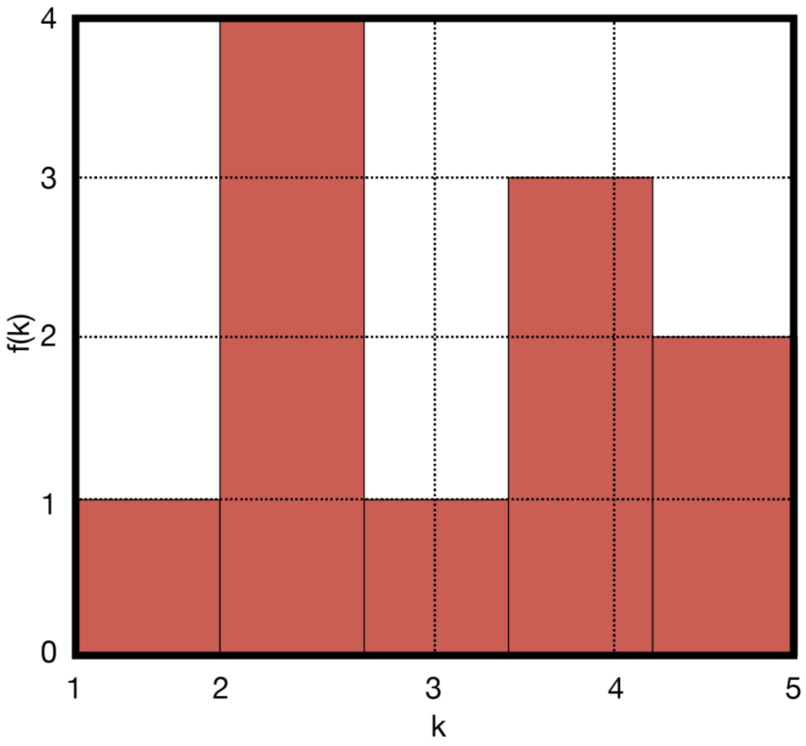
Weighted

Measures on Networks

- Nodes
    - Degree
    - Clustering
    - Assortativity
- Paths
    - Path length
    - Diameter
    - Betweenness

Degree of a Node I





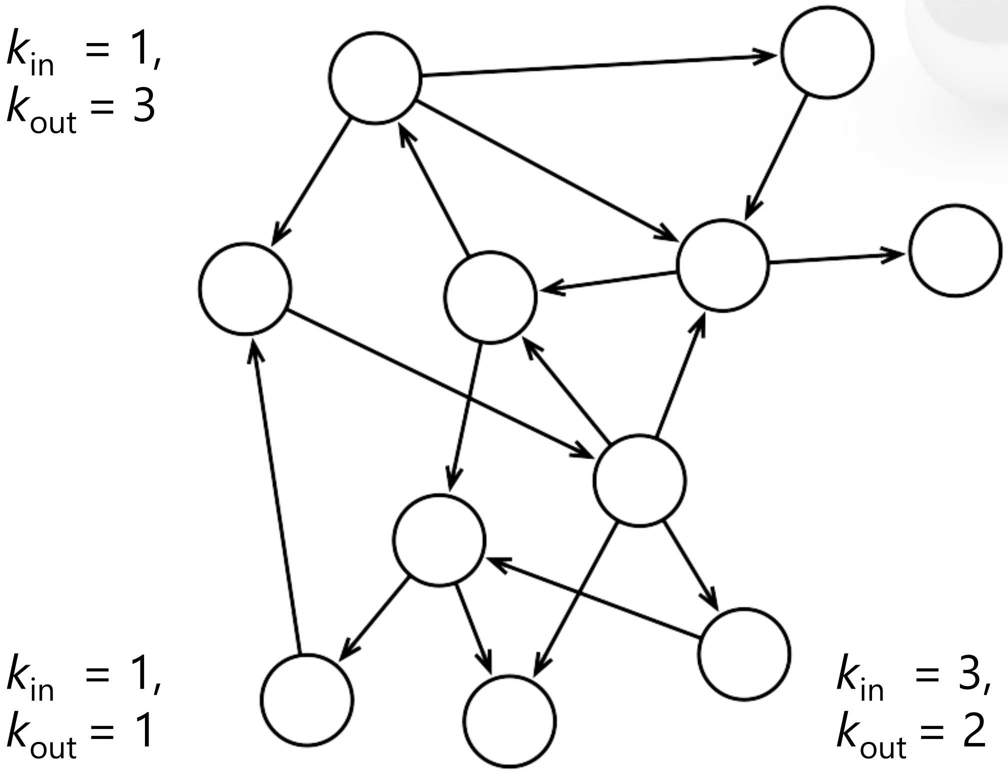
Degree of node, *k* = number of neighbours (nodes it is linked)

k_{min} = 1
k_{max} = 5

$$\bar{k} = \frac{1}{N}\sum_{i=1}^{N} k_i = 3.1$$

Degree of a Node II

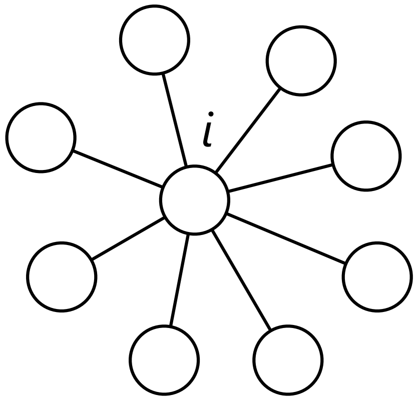
in-degree of node, k_{in} + out-degree of node, k_{out}

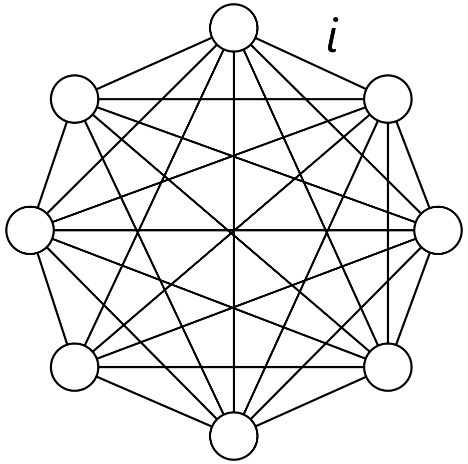


Clustering Coefficient I

$$c_i = \frac{\text{Number of neighbours of } i \text{ that are also neighbours of each other}}{\text{Maximum number of mutual neighbours } i \text{ can have}}$$

8 neighbours



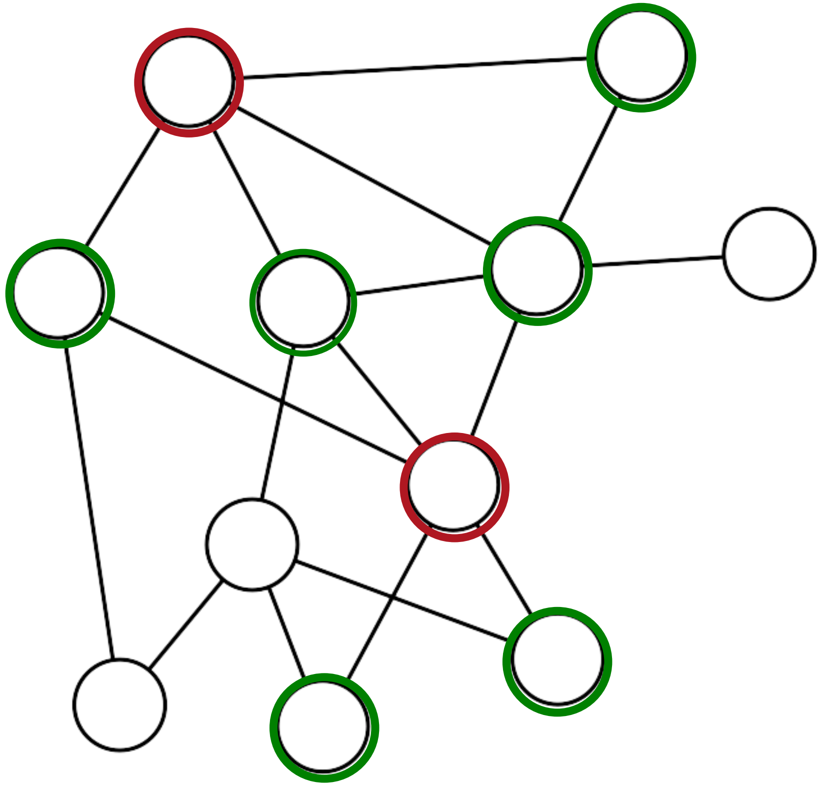


$$\text{Max mutual neighbours} = \frac{8(8-1)}{2}$$

$$= 28 \text{ pairs}$$

Mutual neighbours = 0 $c_{ij} = 0$ $c_{ij} = 1$

Clustering Coefficient II



4 neighbours

2 pairs of mutual neighbours

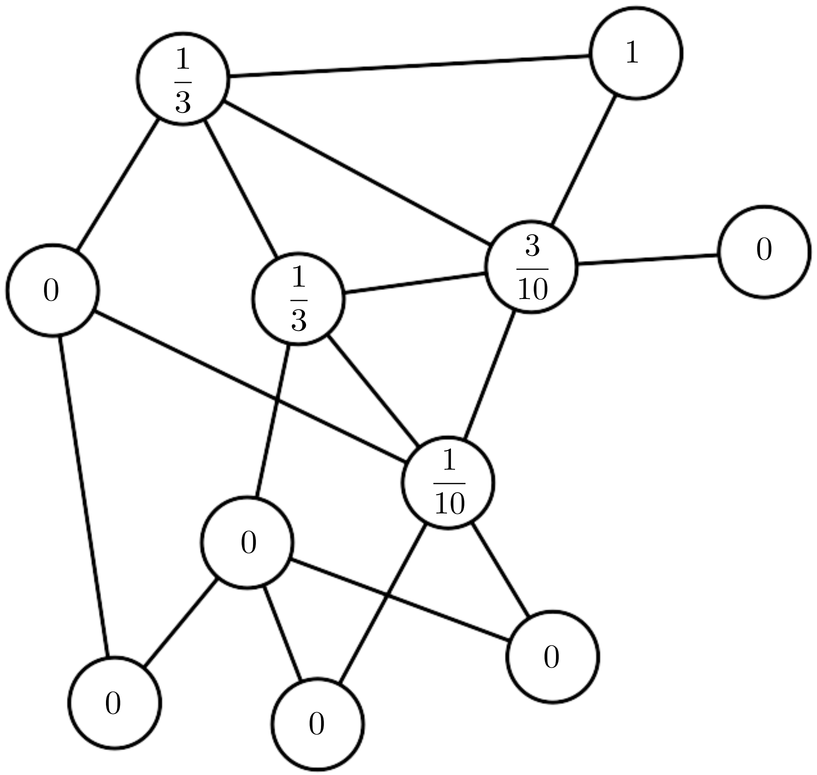
$$c = \frac{2}{6} = \frac{1}{3}$$

5 neighbours

1 pair of mutual neighbours

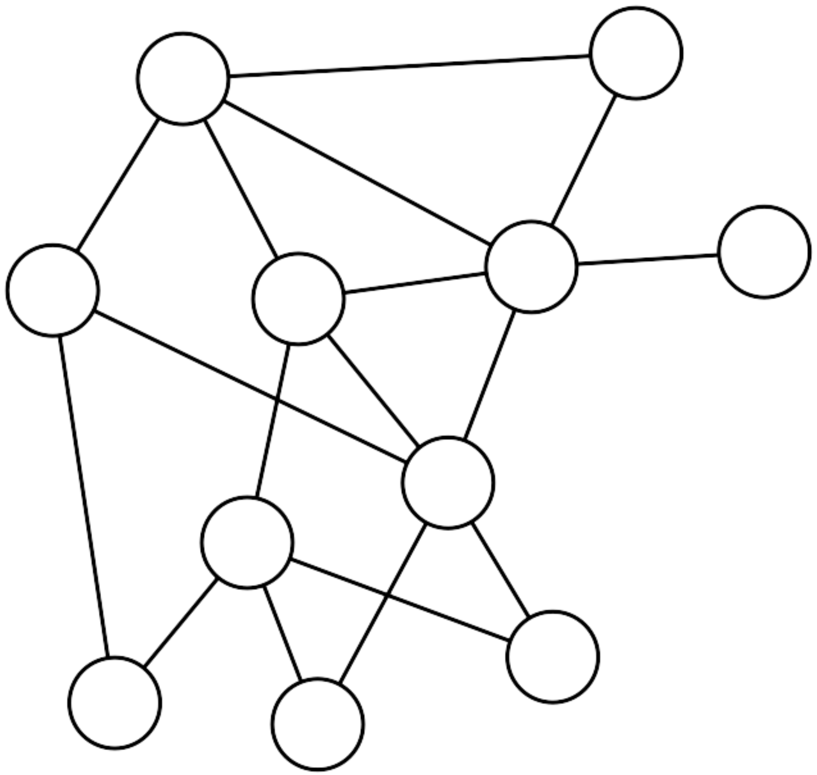
$$c = \frac{1}{10}$$

Clustering Coefficient II



$$\bar{c} = 0.188$$

Assortativity



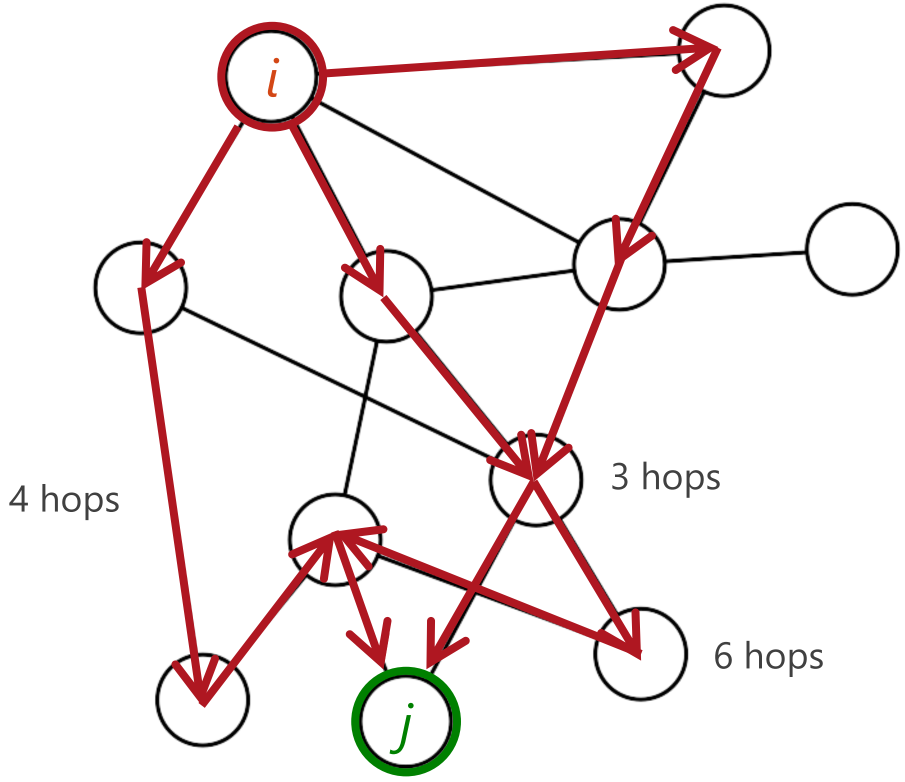
$\bar{k} = 3.1$

$\sigma^2 = 1.89$

1-5: 1	-2.1-1.9: 1
2-3: 1, 2-4: 3, 2-5: 3	-1.1--0.1: 1, -1.1-0.9: 3, -1.1-1.9: 2
3-4: 1, 3-5: 1	-0.1-0.9: 1, -0.1-1.9: 1
4-4: 2, 4-5: 3	0.9-0.9: 2, 0.9-1.9: 3
5-5: 1	1.9-1.9: 1

$$r = \frac{1}{17}\left[\frac{(-2.1)(1.9) + (-1.1)(-0.1) + \cdots + 3(0.9)(1.9) + (1.9)(1.9)}{1.89}\right]$$
$$= -0.095$$

Path Length

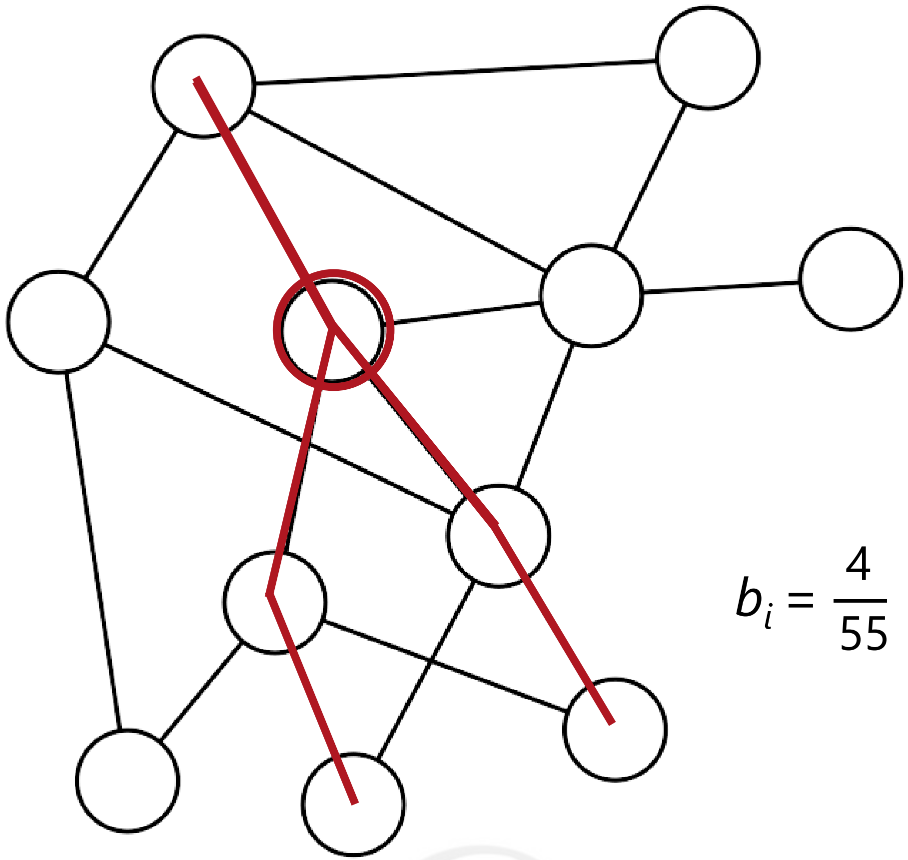


$d(i\,,j) = 3$

$\bar{d}$ = average path length (over all pairs of nodes)

$D = \max_{(i,j)} d(i\,,j)$ = diameter of network

Betweenness

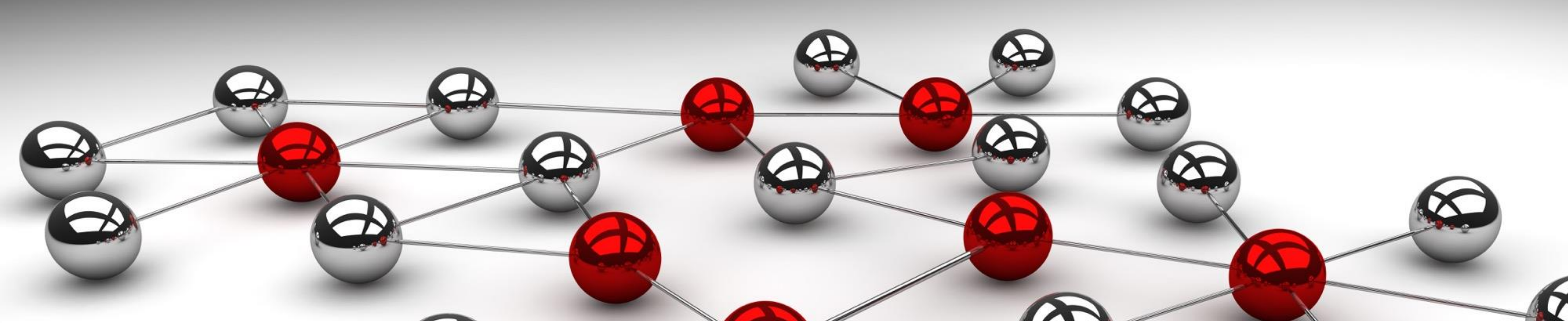


$$b_i = \frac{4}{55}$$

4 shortest paths through node

11 nodes

11(10)/2 = 55 shortest paths



05 Introduction to Static Complex Networks (Part II)

By NTU Complexity Institute

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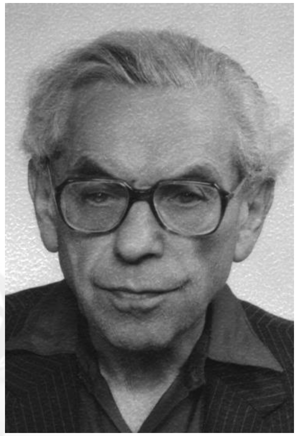
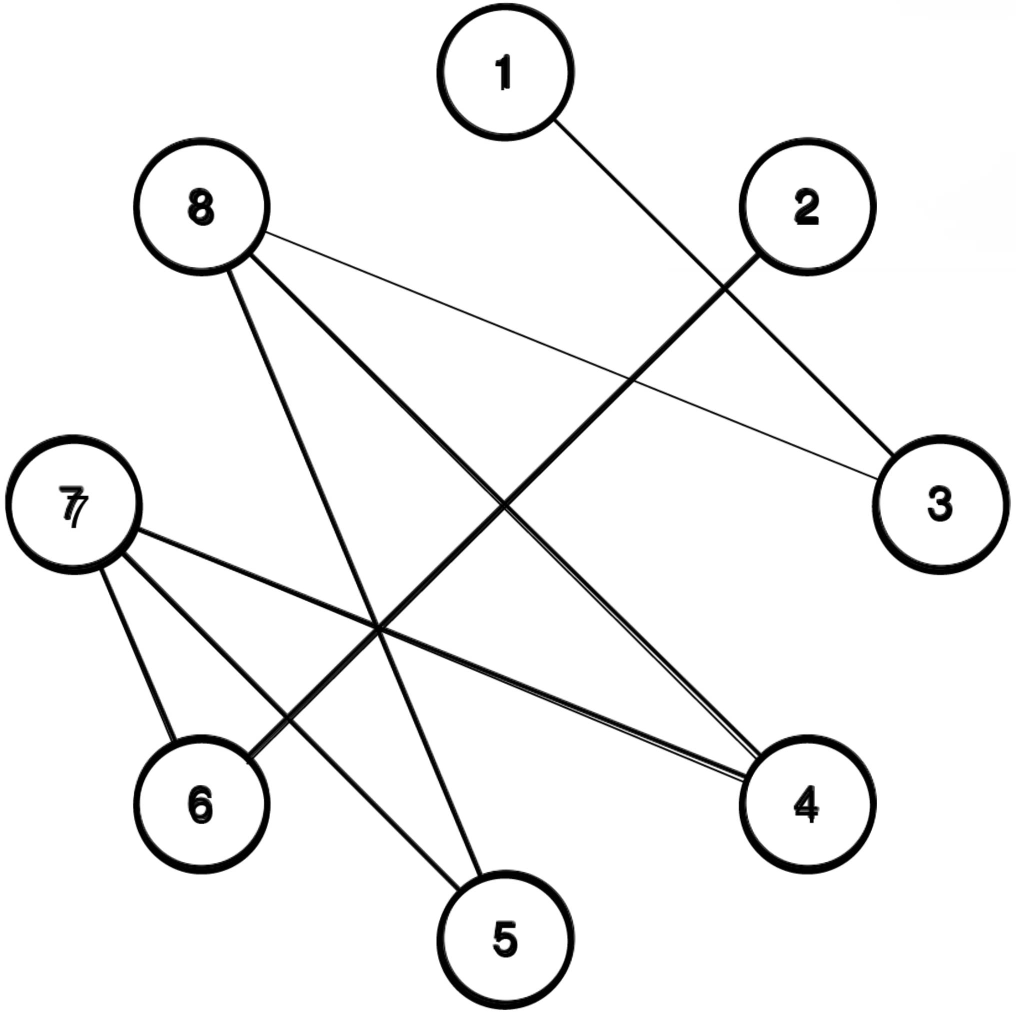
Random Networks I

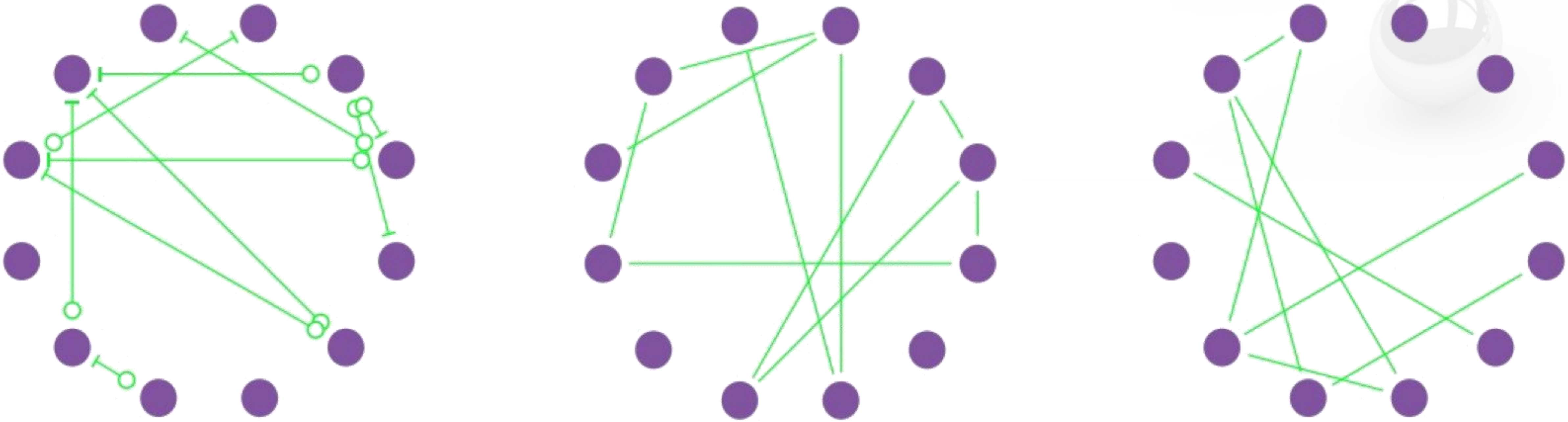
Alfred Rényi
(1921–1970)



Paul Erdös
(1913–1996)



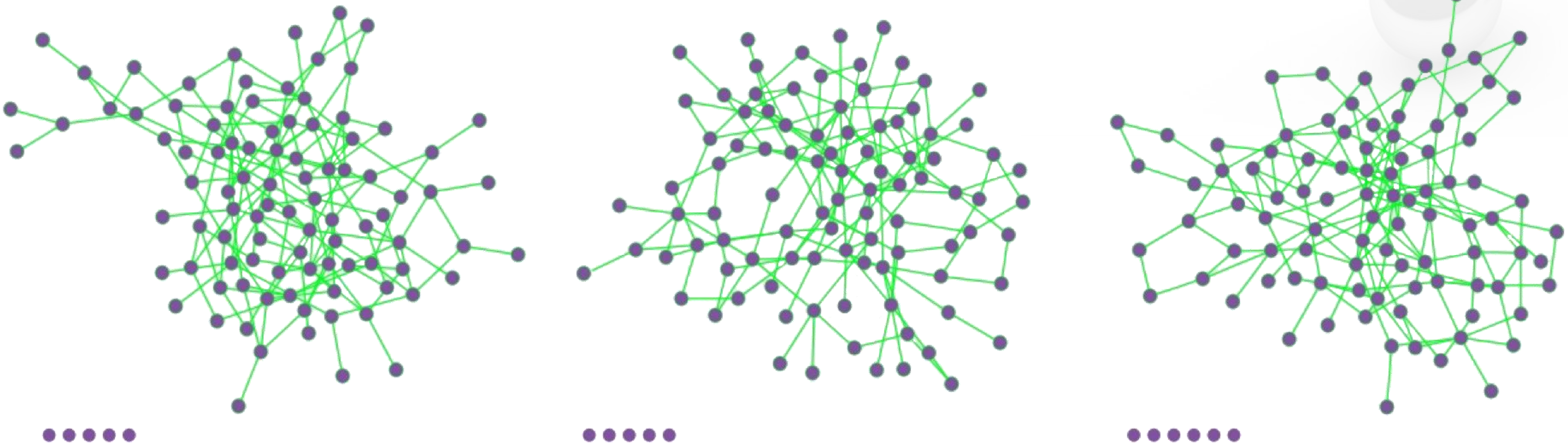
Random Networks II



$$p = \frac{1}{6}$$

$$N = 12$$

Random Networks III



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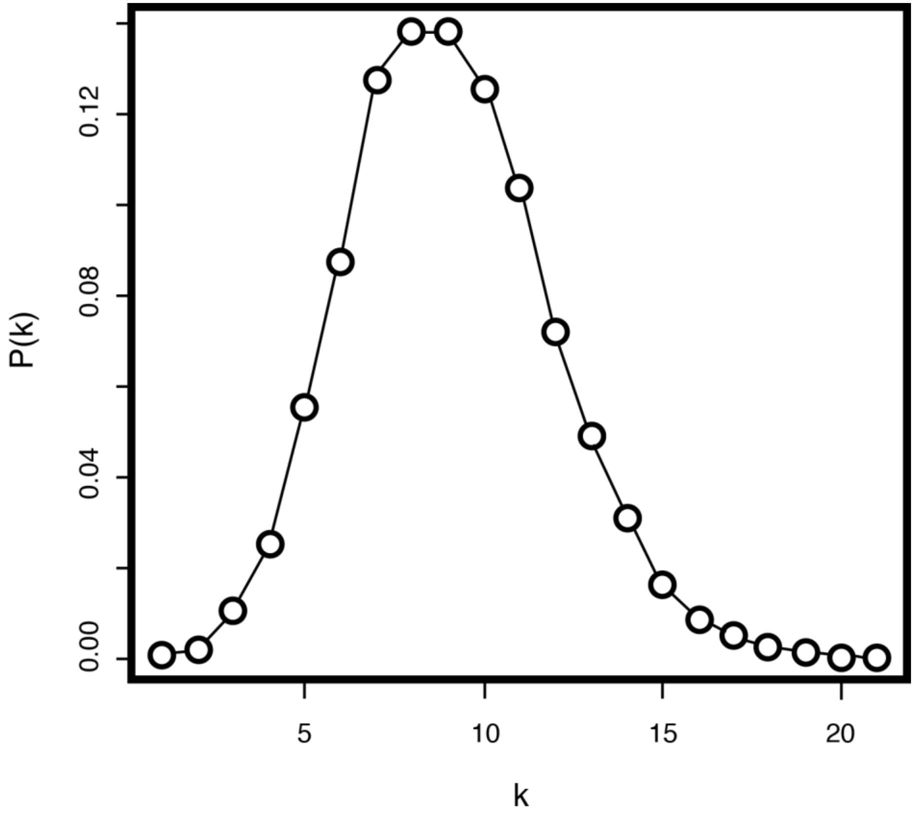
p = 0.03

N = 100

Random Networks IV



Tyler, Asselbergs, Williams and Moore. (2009). *Bioessays*, *31*, 220-7.

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$

Select *k* nodes from *N* – 1

Probability of having *k* edges

Probability of missing (*N* – 1)–*k* edges

$$\langle k \rangle = p(N-1)$$

$$\sigma_k^2 = p(1-p)(N-1)$$

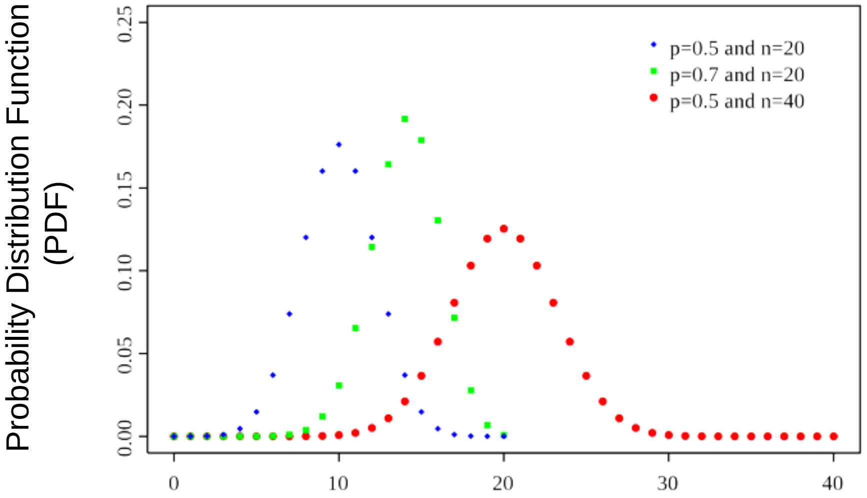
$$\frac{\sigma_k}{\langle k \rangle} = \left[\frac{1-p}{p} \frac{1}{(N-1)} \right]^{1/2} \approx \frac{1}{(N-1)^{1/2}}$$

Random Networks V

Exact result

(Binomial distribution)

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$



$<k> = (N-1)p$

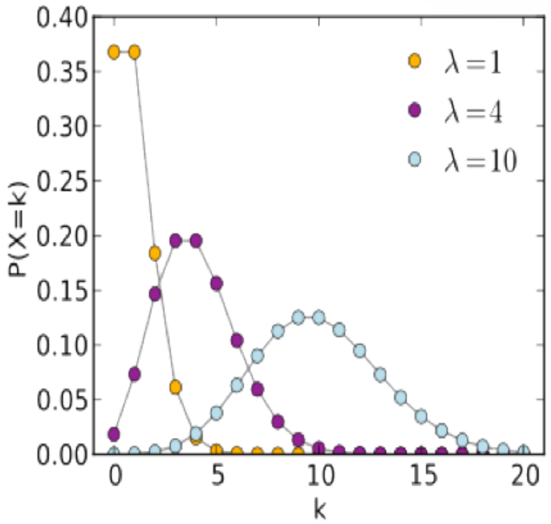
$<k^2> = p(1-p)(N-1) + p^2(N-1)^2$

$\sigma = (<k^2> - <k>^2)^{1/2} = [p(1-p)(N-1)]^{1/2}$

Large *N* limit

(Poisson distribution)

$$P(k) = e^{-<k>} \frac{<k>^k}{k!}$$

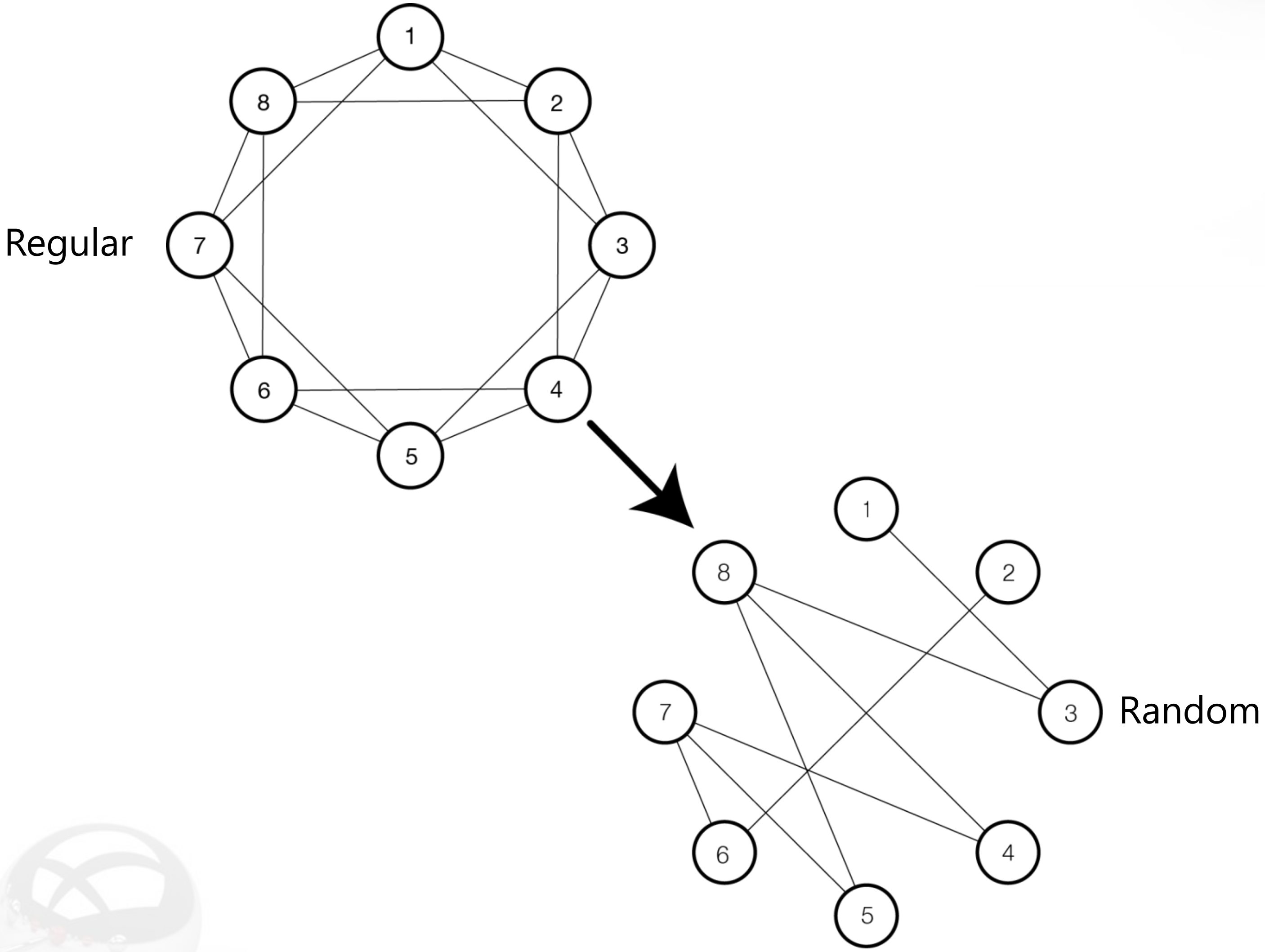


$<k> = <k>$

$<k^2> = <k>(1 + <k>)$

$\sigma_k = (<k^2> - <k>^2)^{1/2} = <k>^{1/2}$

Small-World Networks I

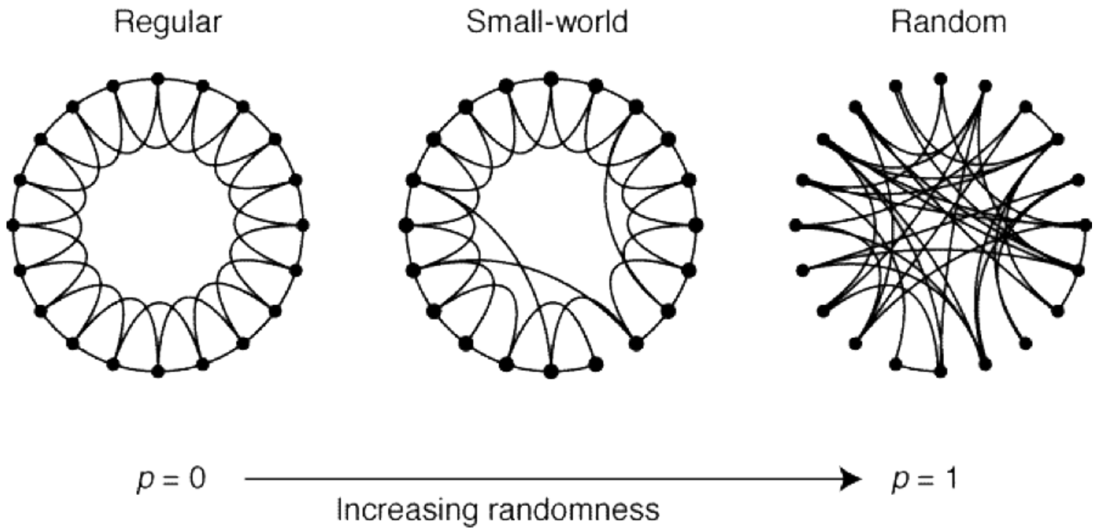


Duncan Watts
(1971–)

Steve Strogatz

(1959–)

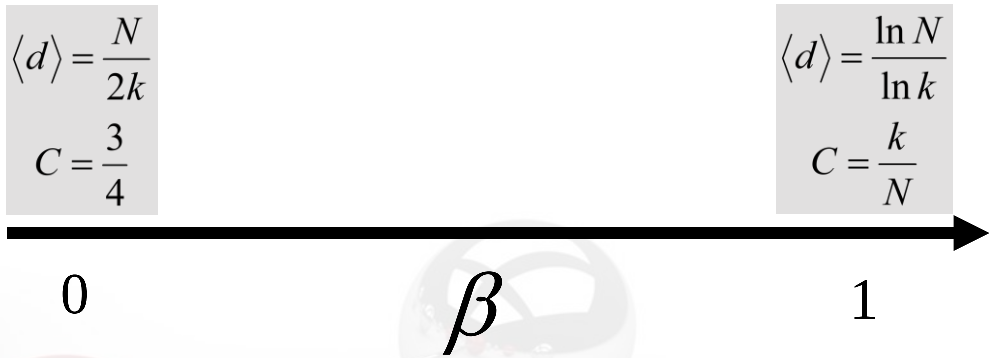
Small-World Networks II



Reprinted by permission from Macmillan Publishers Ltd: Nature Publishing Group (http://www.nature.com/nature/index.html), 1998.

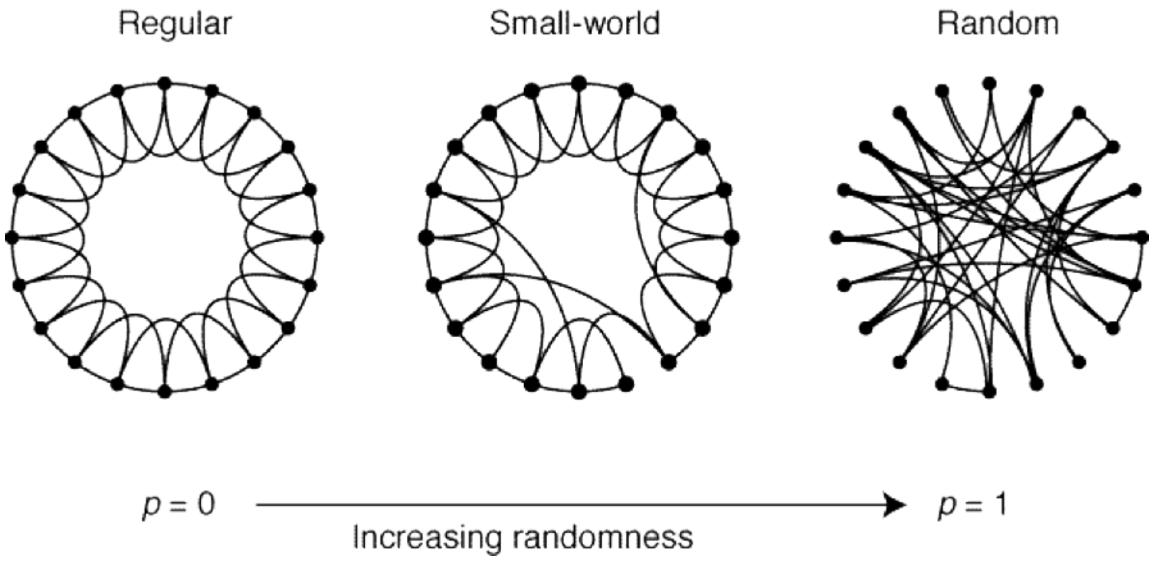
The Watts-Strogatz Model:

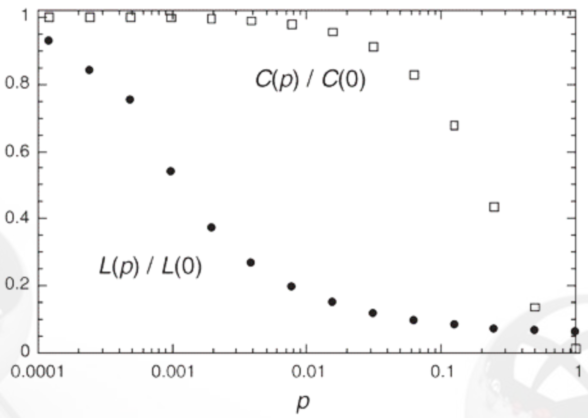
- Start with a lattice network
- For every edge, rewire with a probability β



Watts and Strogatz. (1998). *Nature*, *393*, 440.

Small-World Networks III

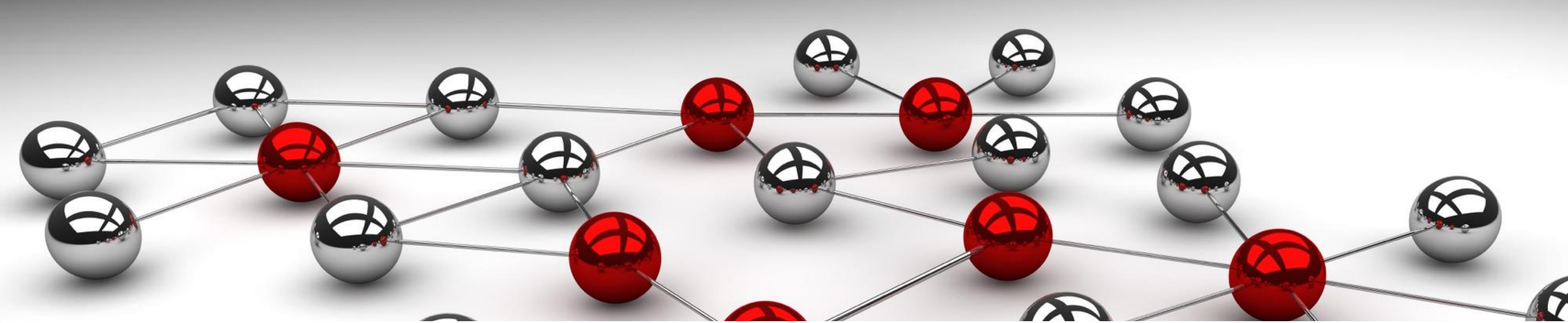




$$C(\beta) \approx C(0)(1-\beta)$$
$$\langle d(\beta) \rangle \sim e^{-\beta N k}$$

The Watts-Strogatz Model:

- It takes a lot of randomness to ruin the clustering, but a very small amount to overcome locality



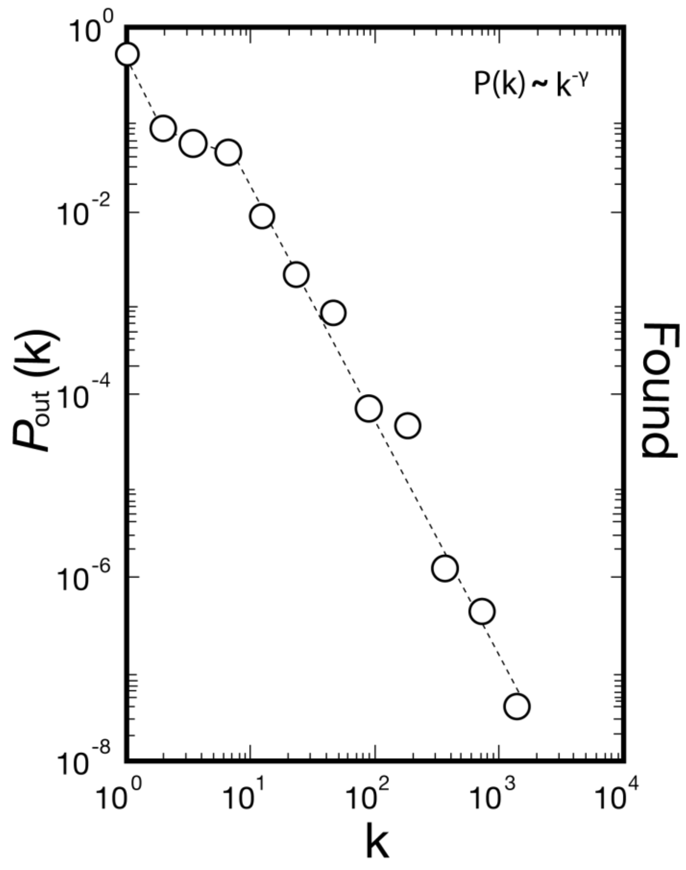
05 Introduction to Static Complex Networks (Part III)

By NTU Complexity Institute

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

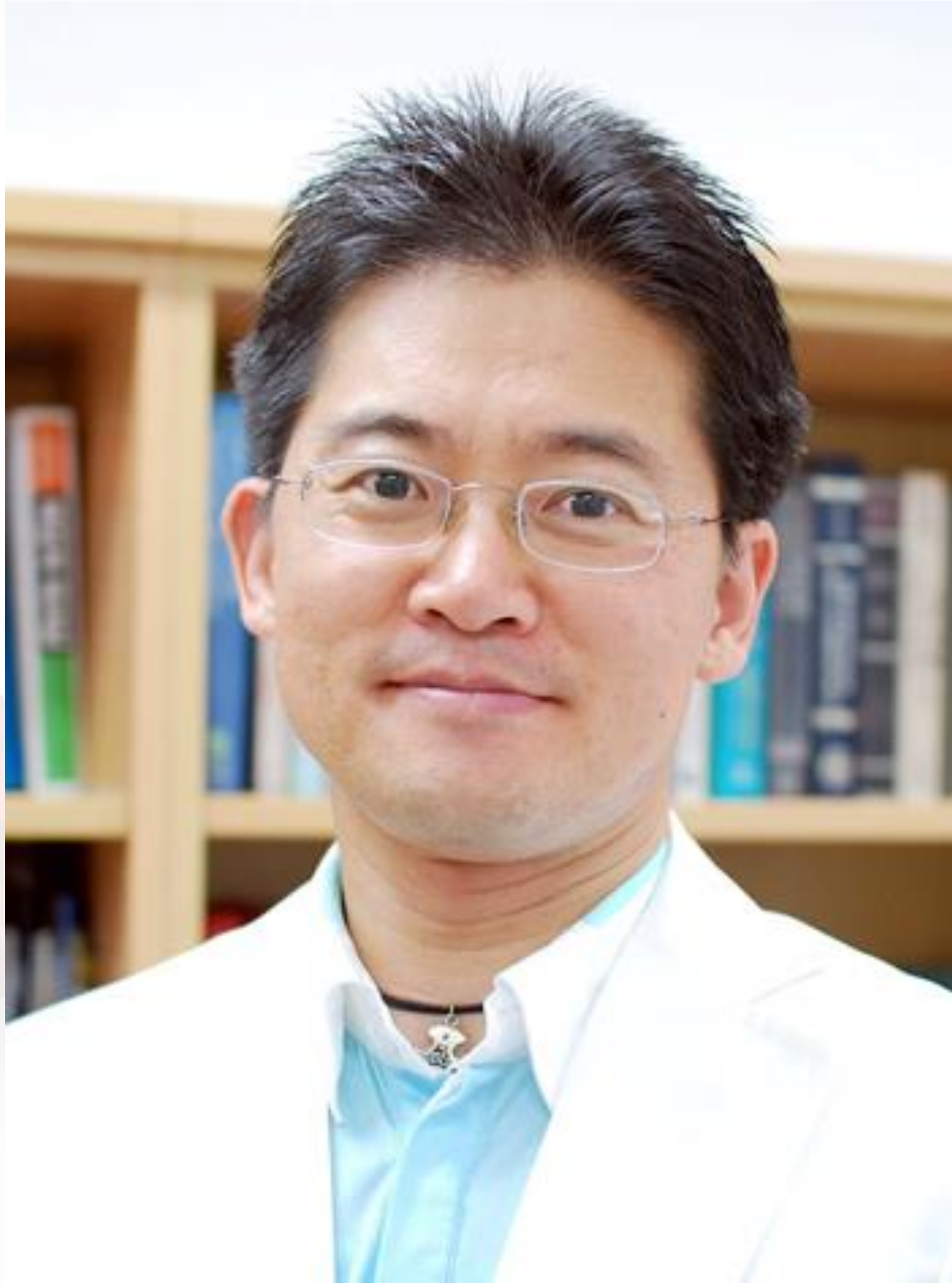
Albert-László Barabási

Scale-Free Networks I



Réka Albert

Hawoong Jeong



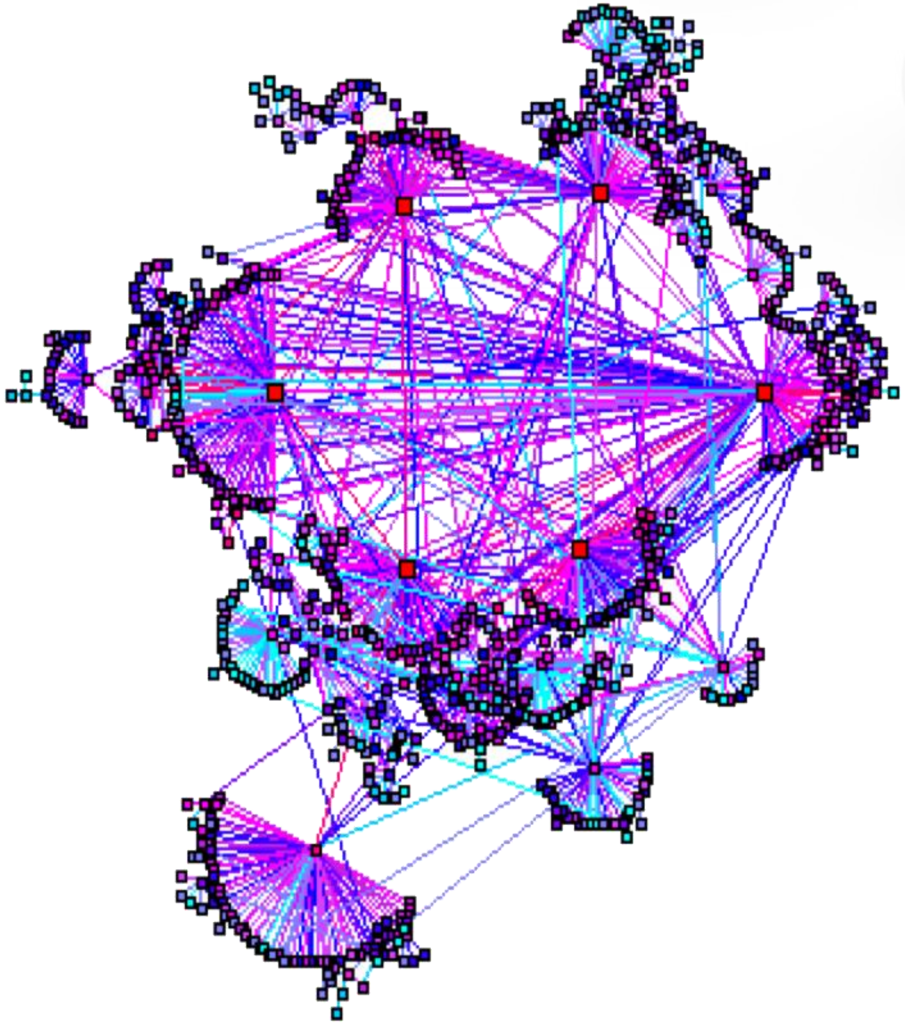
Scale-Free Networks I

Nodes: WWW documents

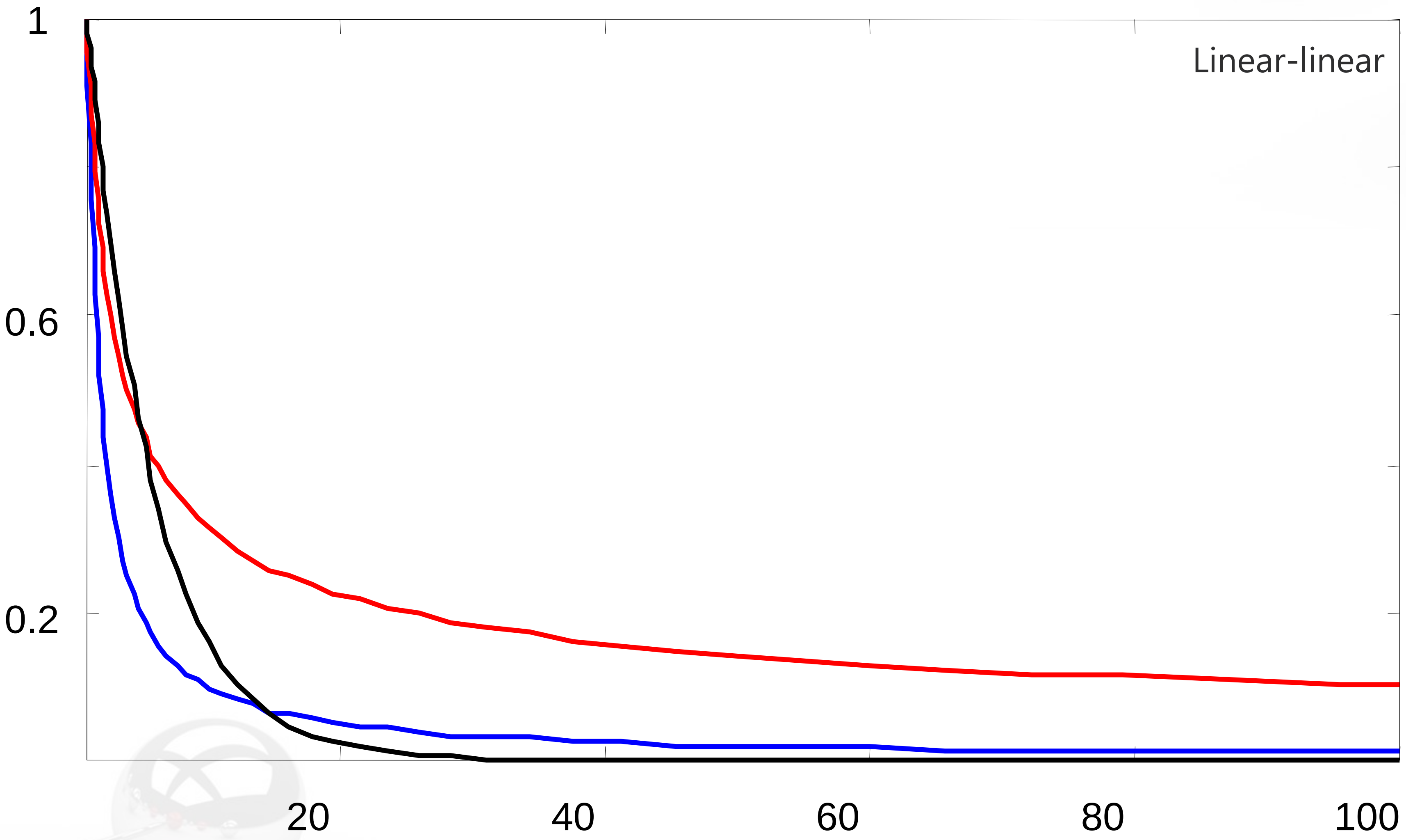
Links: URL links

Over three billion documents

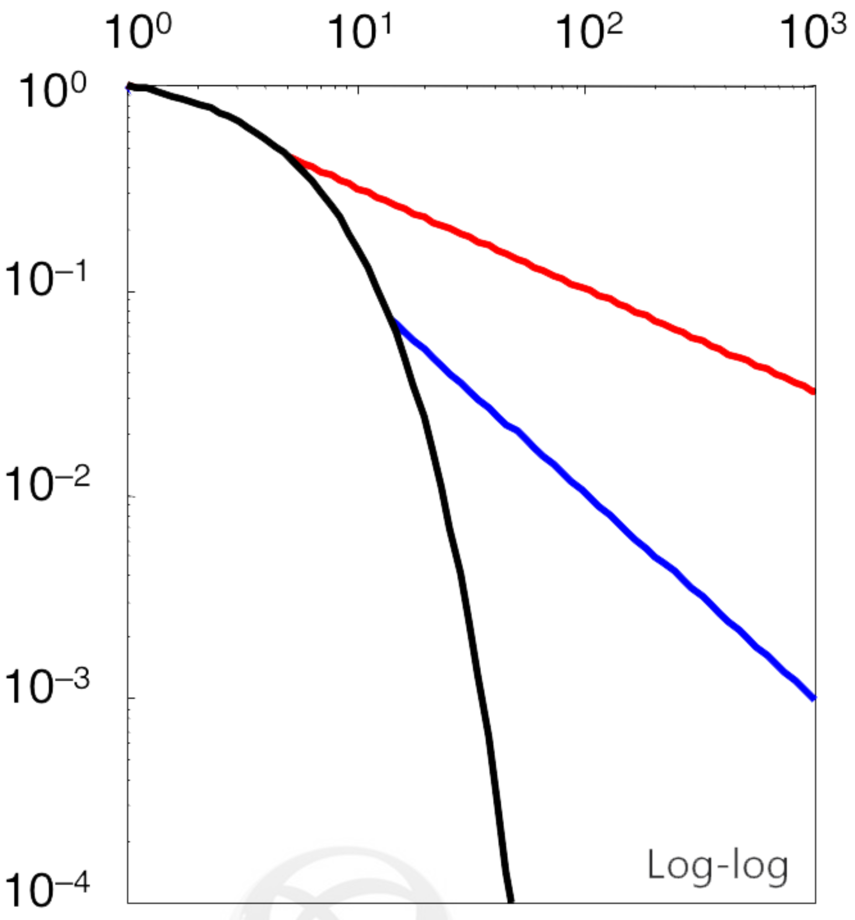
ROBOT: Collects all URLs found in a document and follows them recursively

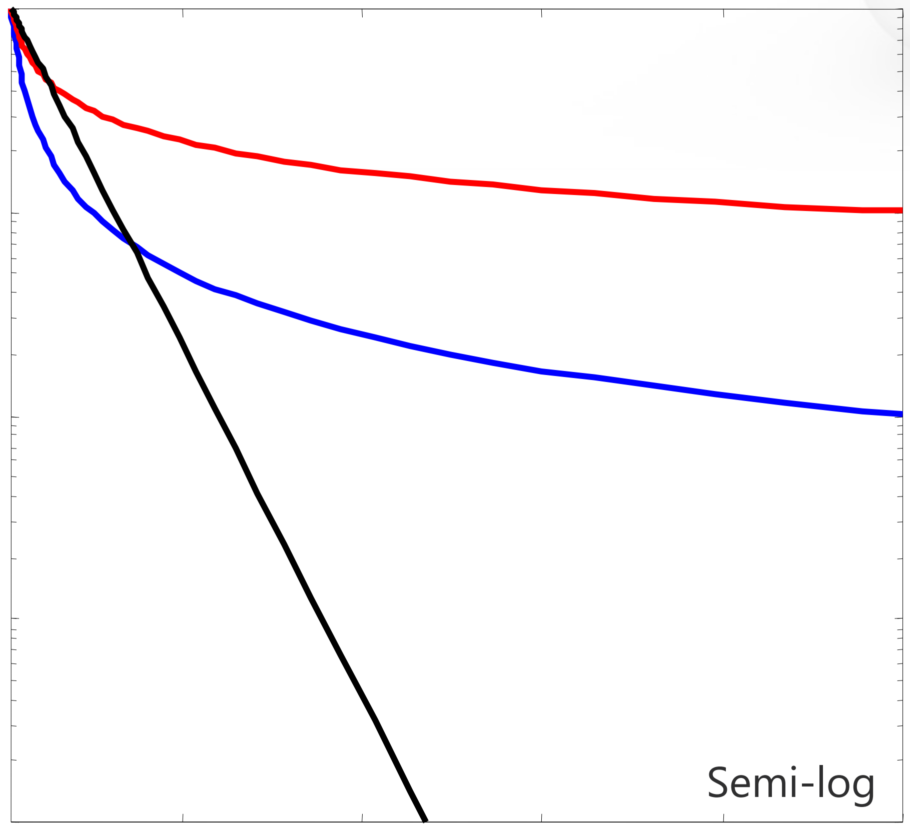


Scale-Free Networks II

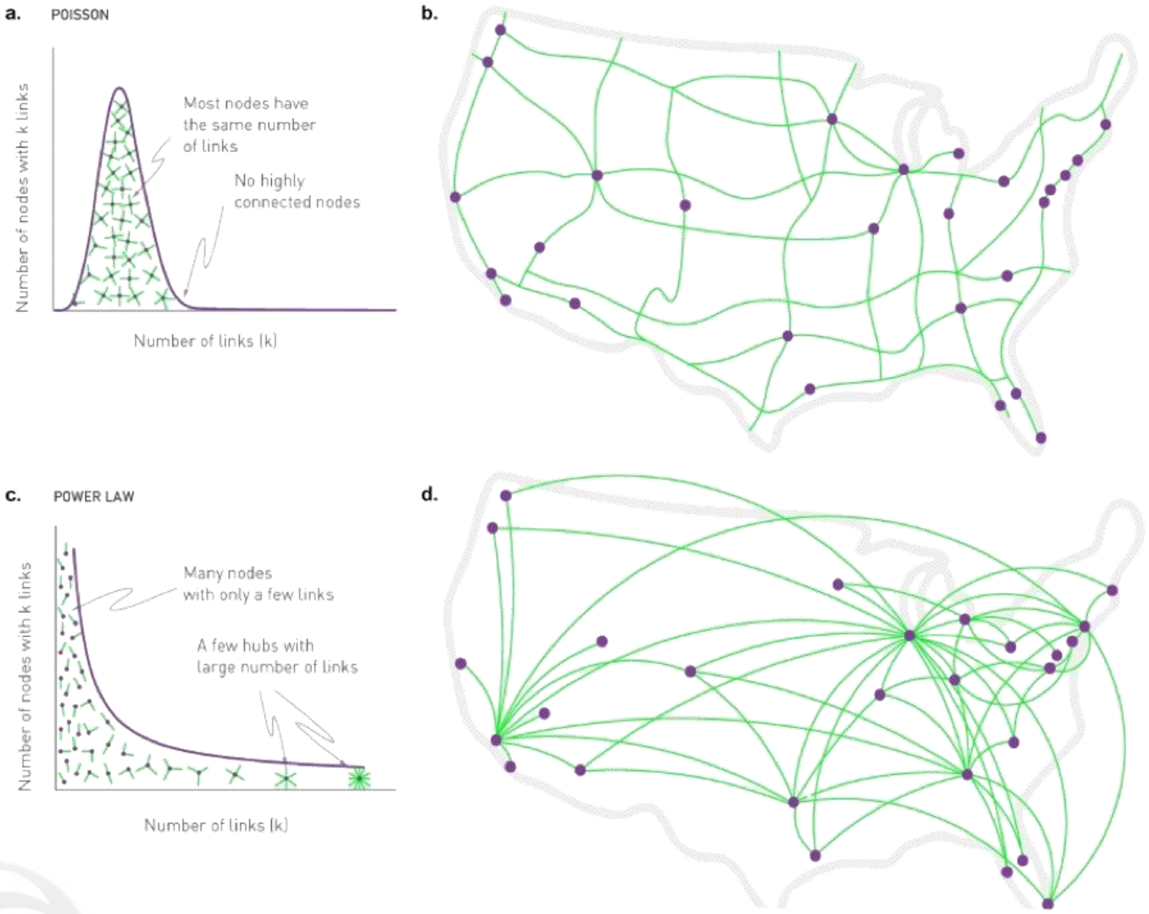


Scale-Free Networks II





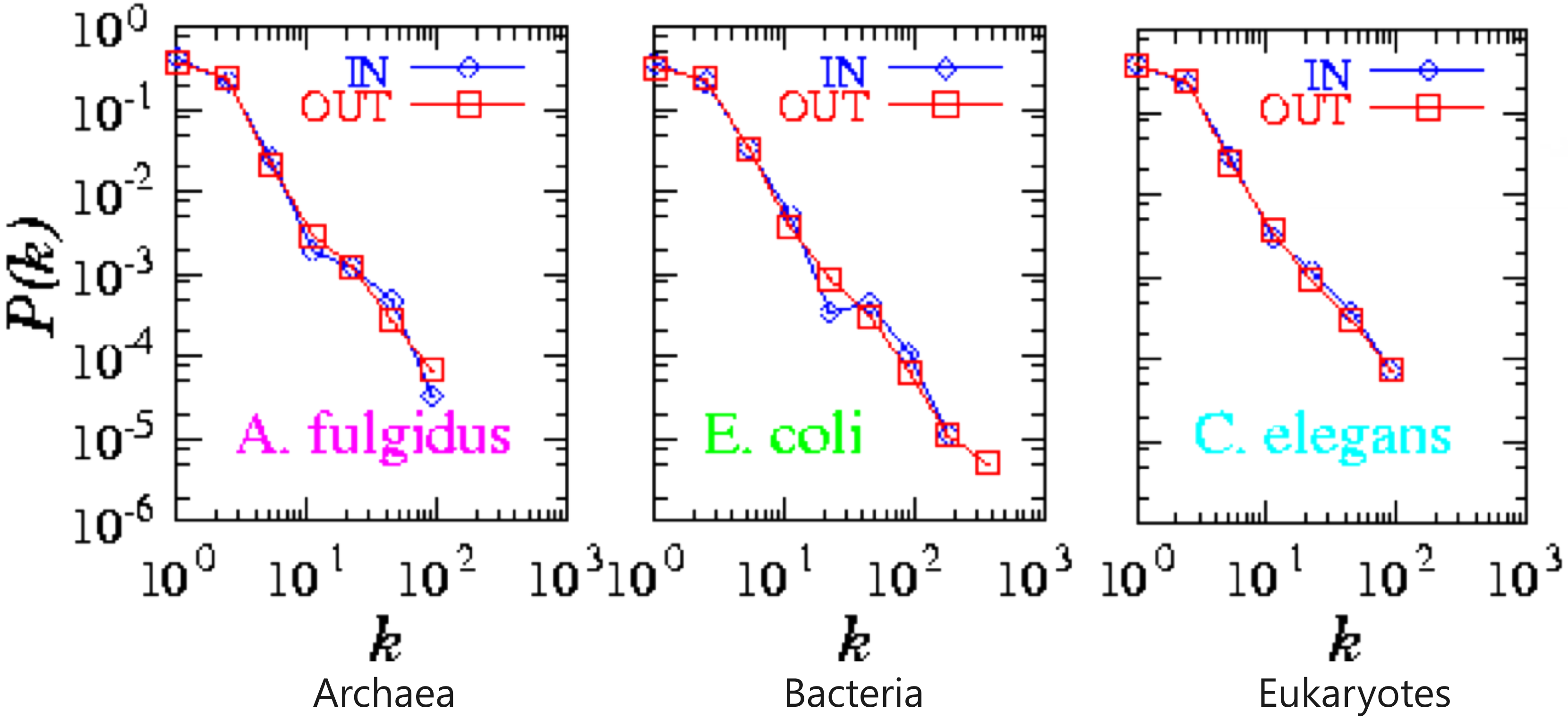
Scale-Free Networks III



Road network

Air travel network

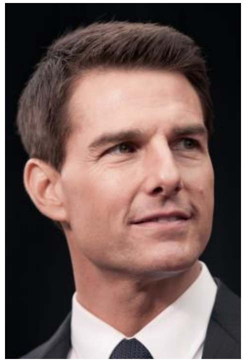
Scale-Free Networks IV



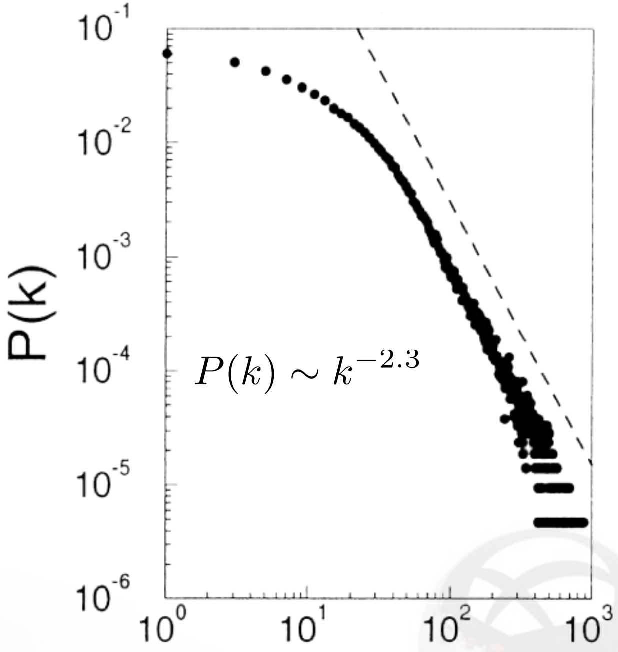
$$P_{in}(k) \approx k^{-2.2}$$

$$P_{out}(k) \approx k^{-2.2}$$

Scale-Free Networks IV



Days of Thunder *(1990)*
Far and Away *(1992)*
Eyes Wide Shut *(1999)*

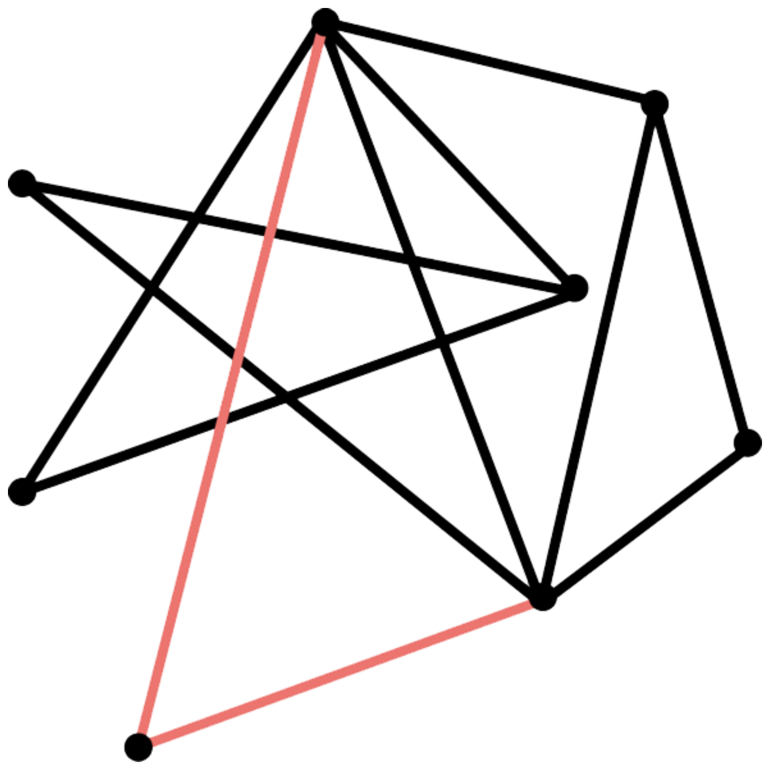


From Barabási, A.-L. and Albert, R. (1999). Emergence of Scaling in Random Networks. *Science*, *286*(5439), 509-512. doi: 110.1126/science.286.5439.509. Reprinted with permission from AAAS.

IMDb Internet Movie Database

- Nodes: Actors
- Links: Cast
- N = 212,250 actors
- $\langle k \rangle$ = 28.78

Barabási-Albert Model I

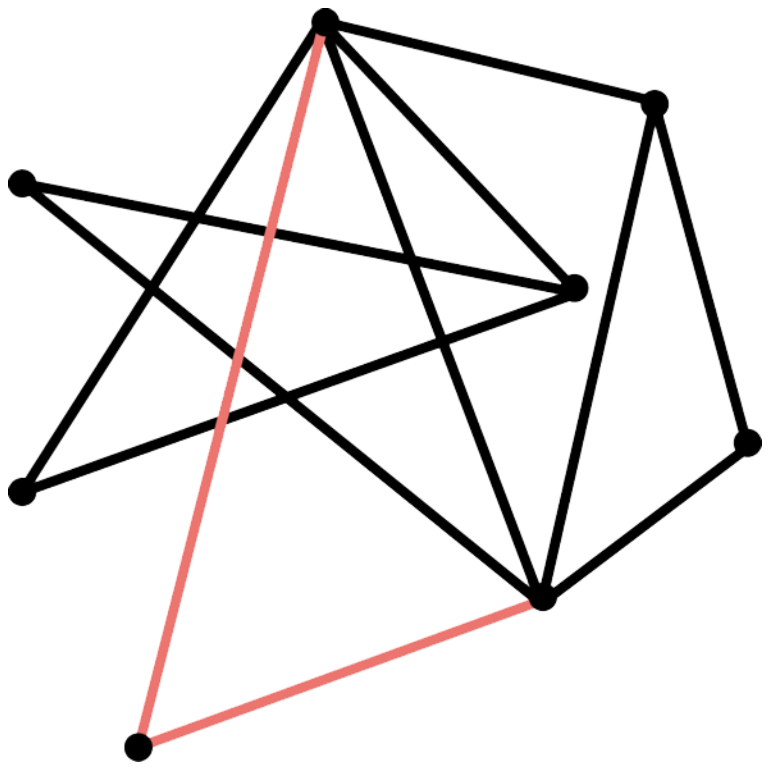


Networks continuously expand with the addition of new nodes:

- Add a new node with *m* links.

Barabási and Albert. (1999). *Science*, *286*, 509.

Barabási-Albert Model II



Barabási and Albert. (1999). *Science*, *286*, 509.

Where will the new node link to?

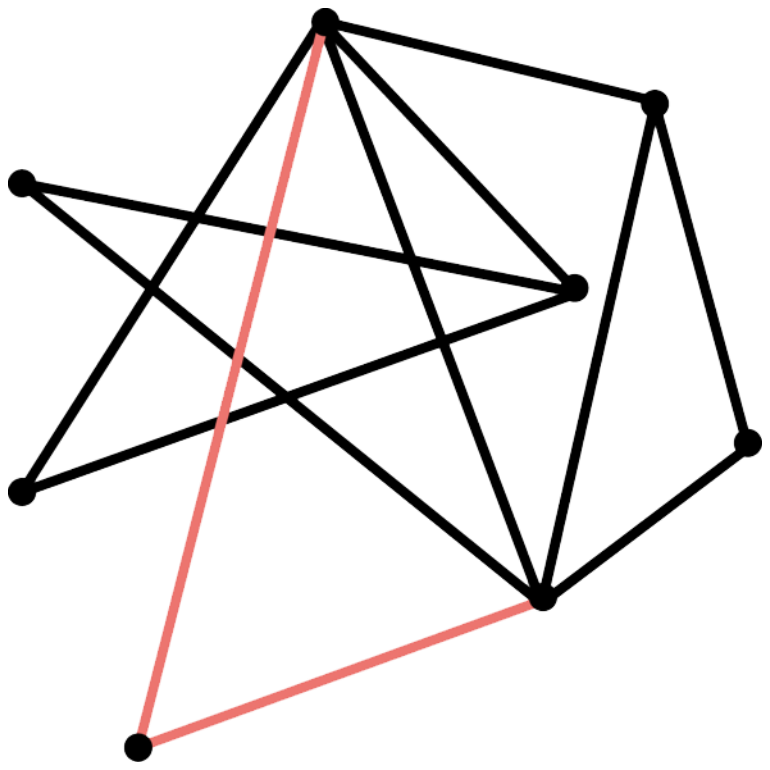
- According to the Erdös-Rényi and Watts-Strogatz models, choose randomly.
- New nodes prefer to link to highly connected nodes (E.g., www, citations, IMDB)

Preferential attachment:

- The probability that a node connects to a node with *k* links is proportional to *k*.

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$

Barabási-Albert Model III



Barabási and Albert. (1999). *Science*, *286*, 509.

- Networks continuously expand by the addition of new nodes
- WWW: Addition of new documents

Growth:

- Add a new node with *m* links.
- New nodes prefer to link to highly connected nodes
- WWW: Linking to well-known sites

Preferential attachment:

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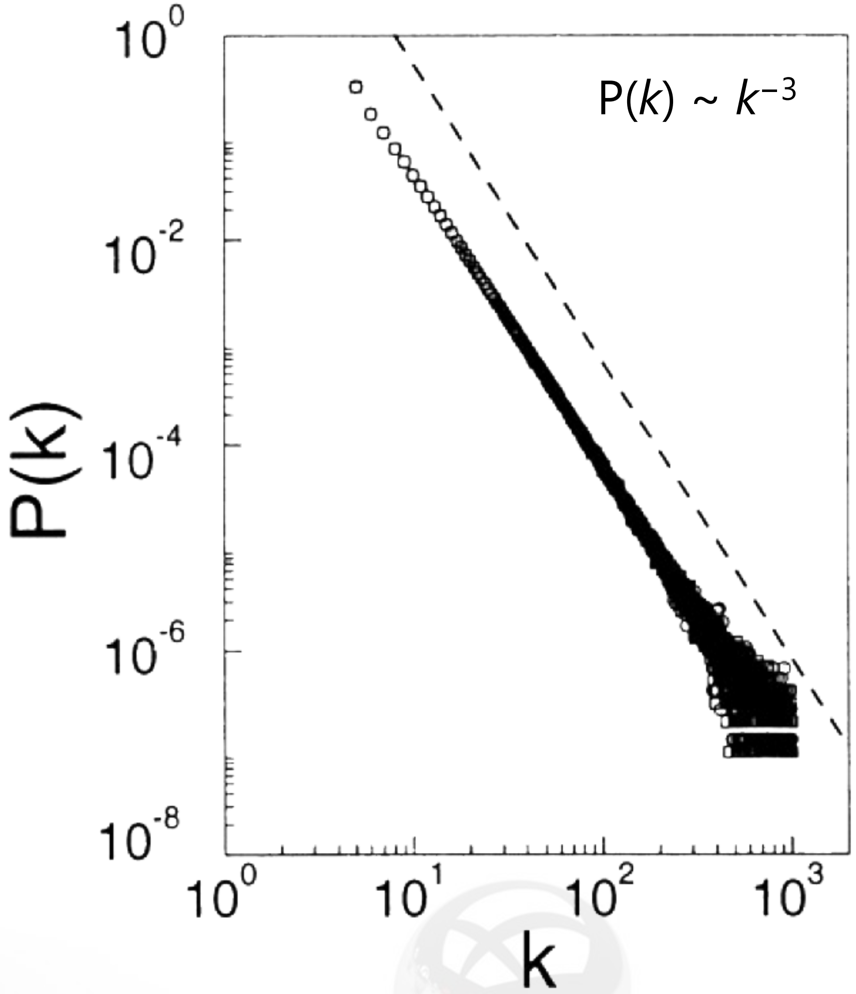
Barabási-Albert Model III



From Barabási, A.-L. and Albert, R. (1999). Emergence of Scaling in Random Networks. Science, 286(5439), 509-512. doi: 110.1126/science.286.5439.509. Reprinted with permission from AAAS.

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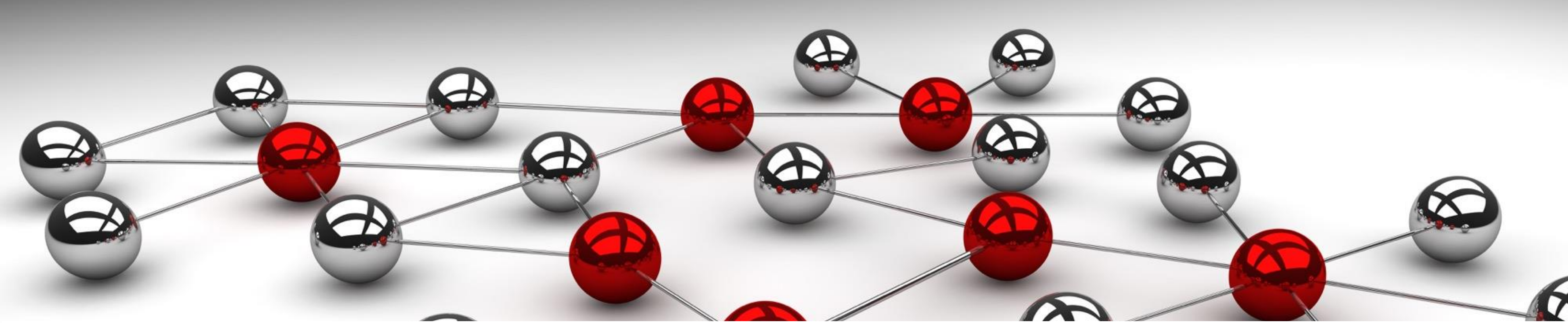

05 Introduction to Static Complex Networks (Part IV)

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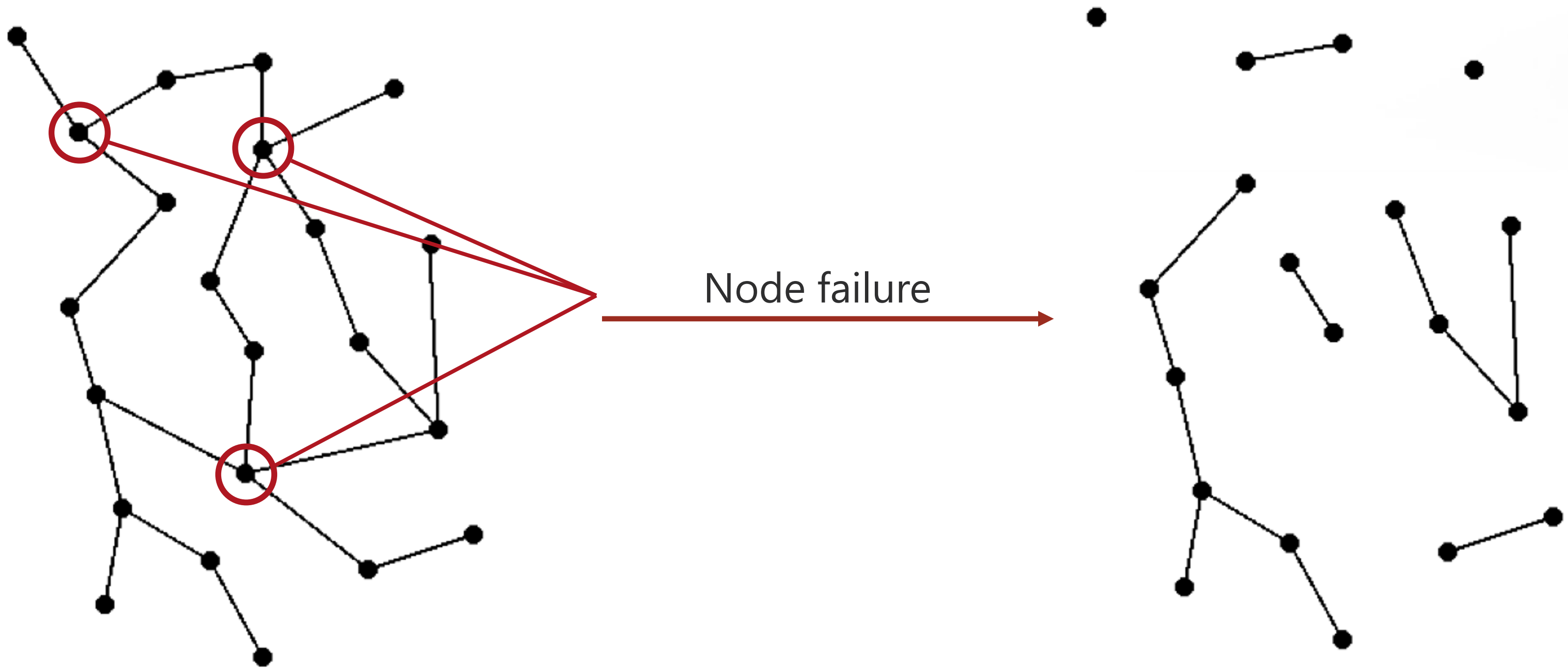
Robustness of a Network

Could the network structure contribute to robustness?

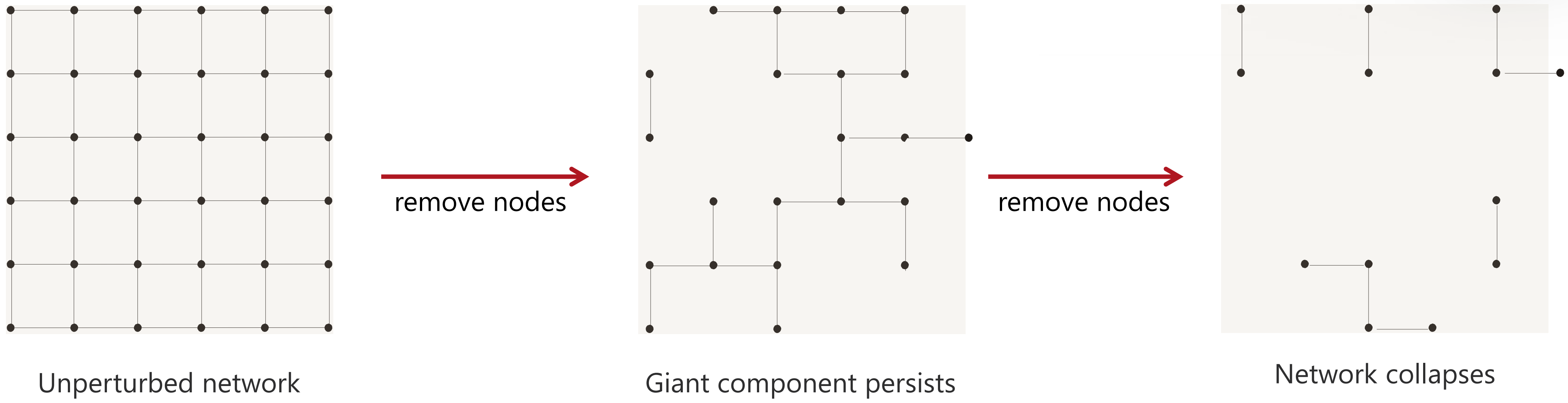


Retrieved from http://slideplayer.com/slide/9514512/. Copyright 2011 by Barabási A.L., Barzel, B. and Martino, M. Reproduced with permission.

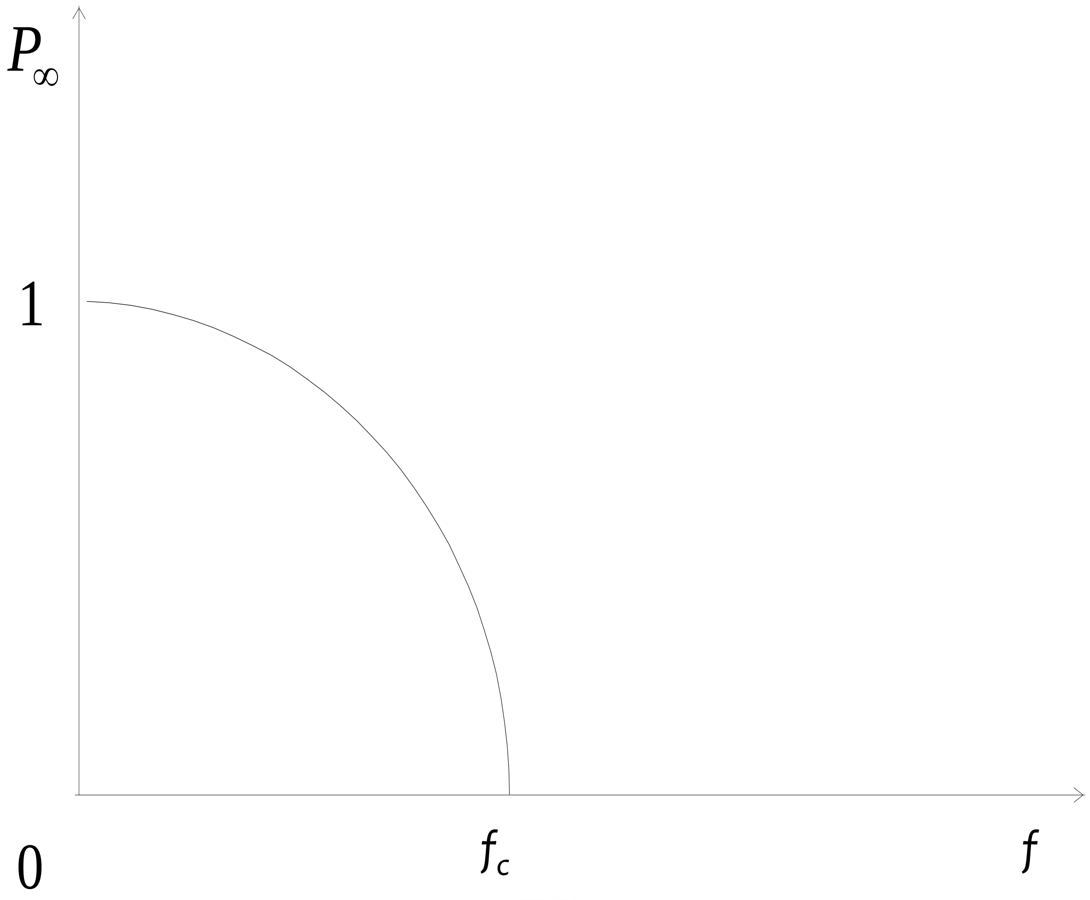
How do we describe the breakdown of a network under node removal in quantitative terms?

Percolation theory

Percolation Transition



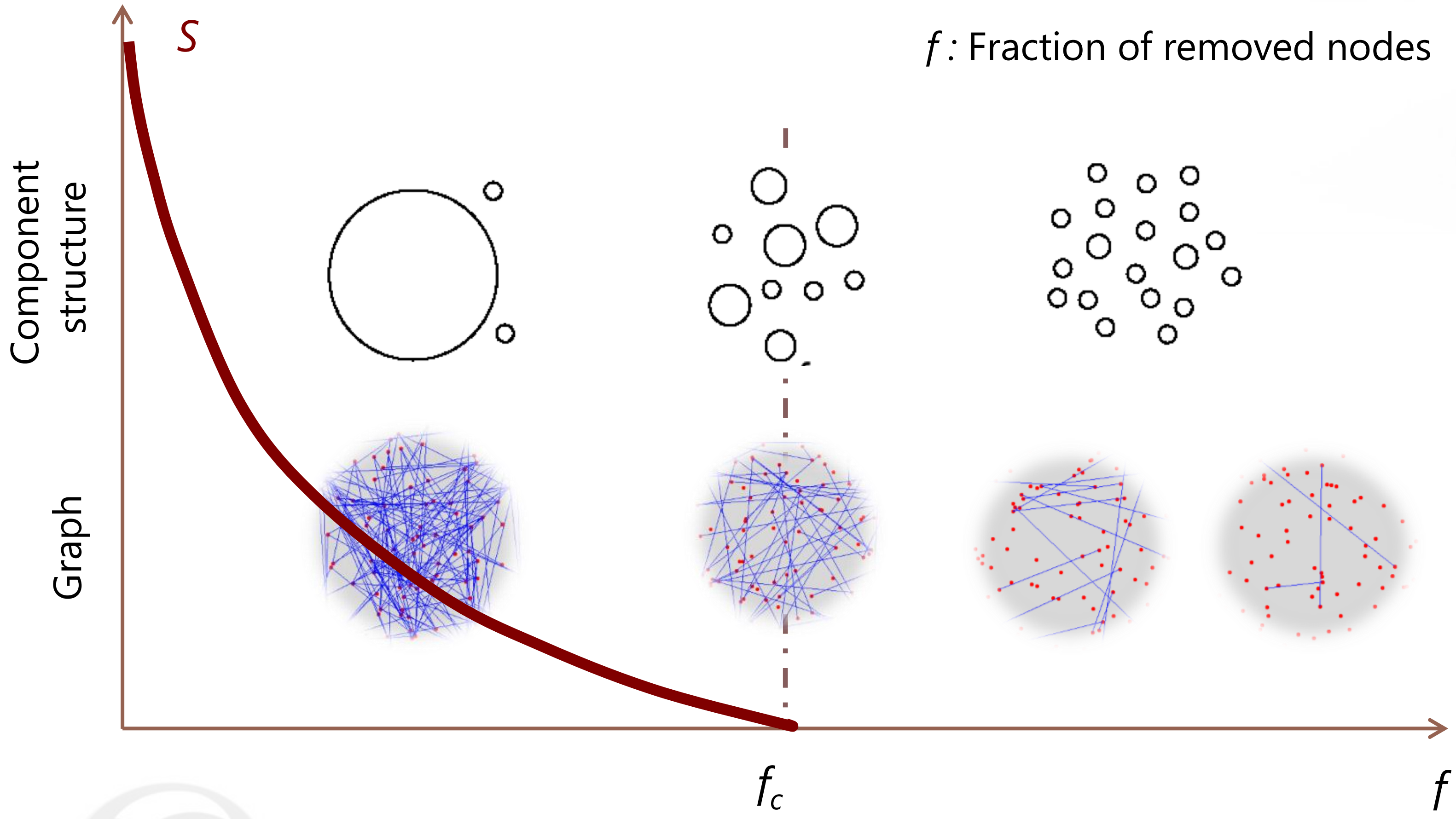
Percolation Transition



P_∞ : Probability that a node belongs to the giant component

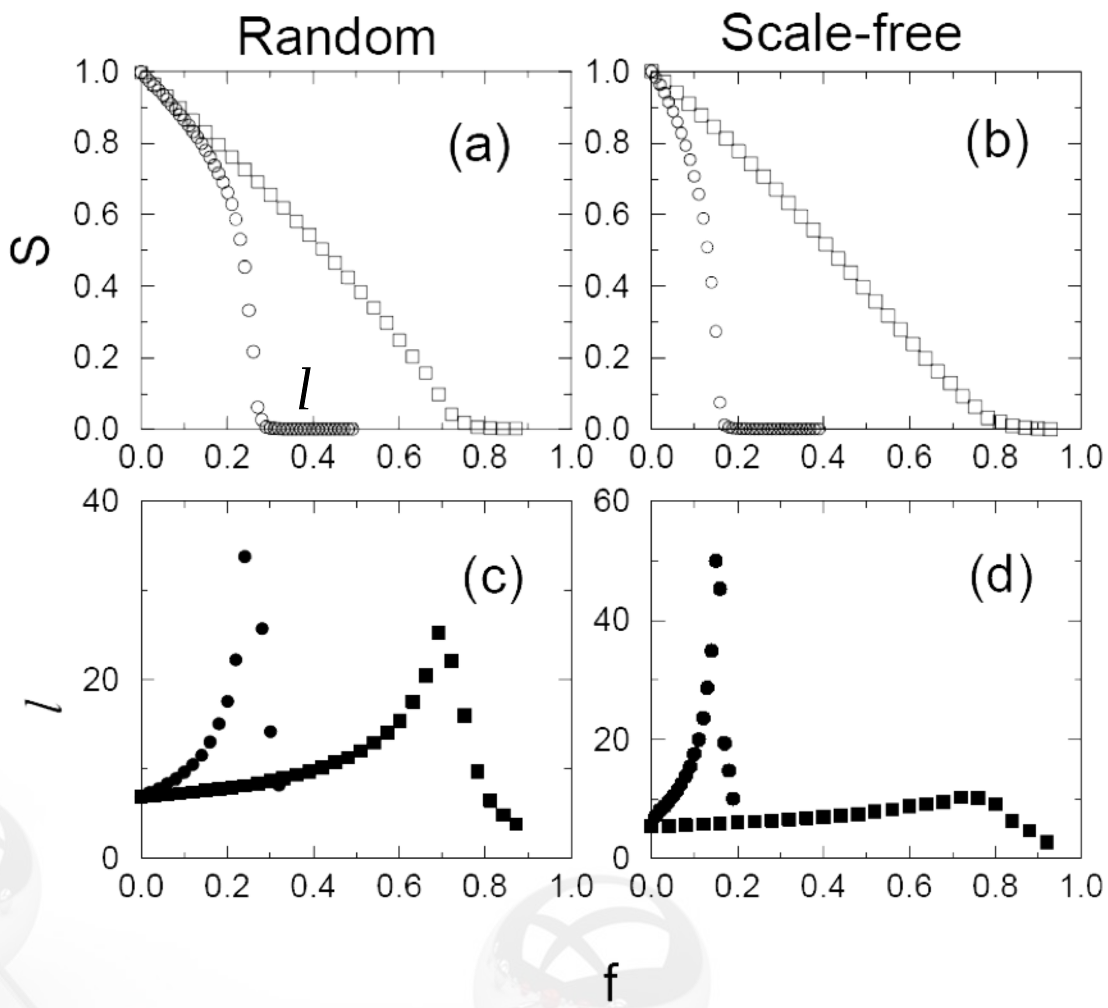
f : Fraction of removed nodes

Damage to Network as Percolation



(Inverse percolation phase transition)

Failures vs Attacks



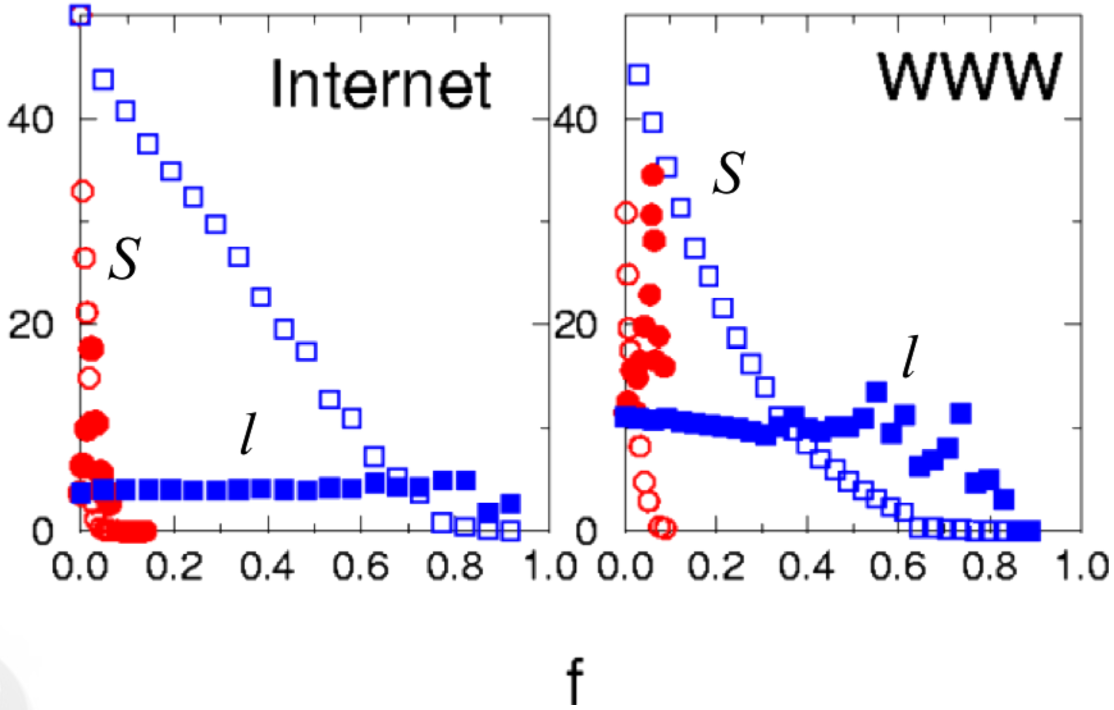
Squares: Random failure

Circles: Targeted attack

Failures: Little effect on the integrity of the network

Attacks: Fast breakdown

Real-World Examples



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Blue squares: Random failure

Red circles: Targeted attack

Open symbols: *S*

Filled symbols: *l*

- Breakdown if 5% of the nodes are eliminated selectively (always the node with the highest degree)
- Resilient to the random failure of 50% of the nodes

Similar results have been obtained for metabolic networks and food webs.

Acknowledgements

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