DD2434 – Advanced Machine Learning

Lecture 3: **Linear regression, model comparison**

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Short outline for today

1. Recap from Lecture 2
2. Regression models
   a) maximum likelihood, regularization
   b) Bayesian approach
3. Model selection.

Main points in lecture 2

- Motivation for a probabilistic perspective of machine learning
- Other flavours of ML theory (e.g. PAC)
- General philosophy of a Bayesian approach

$$p(\mathbf{w} \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \mathbf{w})\, p(\mathbf{w})}{p(\mathcal{D})}$$

$$\text{posterior} \propto \text{likelihood} \times \text{prior}$$

Main points in lecture 2

- Motivation for a probabilistic perspective of machine learning – scientific discipline
- General philosophy of a Bayesian approach
- Bayesian vs frequentist viewpoint

Main points in lecture 2

- Motivation for a probabilistic perspective of machine learning – scientific discipline
- General philosophy of a Bayesian approach
- Bayesian vs frequentist viewpoint
- Learning as inference, regression example

Learning as inference – recap

1) **Maximise the likelihood**

$$\mathcal{D} \rightarrow \boldsymbol{w}_{\mathrm{ML}} \qquad \ln p(\boldsymbol{t} \mid \mathbf{X}, \boldsymbol{w}, \beta) = -\frac{\beta}{2}\sum_{n=1}^{N}\{y(x_n, \boldsymbol{w}) - t_n\}^2 + \frac{N}{2}\ln\beta - \frac{N}{2}\ln(2\pi)$$

2) Parameters of the model as random variables with a prior distribution

$$\mathcal{D}, p(\boldsymbol{w}) \rightarrow \boldsymbol{w}_{\mathrm{MAP}} \qquad p(\boldsymbol{w} \mid \alpha) = \mathcal{N}(\boldsymbol{w} \mid \mathbf{0}, \alpha^{-1}\mathbf{I})$$

Maximise the posterior: $\max\, p(\boldsymbol{w} \mid \mathbf{X}, \boldsymbol{t}, \alpha, \beta) \propto p(\boldsymbol{t} \mid \mathbf{X}, \boldsymbol{w}, \beta)\; p(\boldsymbol{w} \mid \alpha)$

3) **Marginalise** over parameters

Bayes: $\mathcal{D}, p(\boldsymbol{w}) \rightarrow p(\boldsymbol{w} \mid \mathcal{D})$

$$p(t \mid x, \mathbf{X}, \boldsymbol{t}) = \int p(t \mid x, \boldsymbol{w}, \beta)\; p(\boldsymbol{w} \mid \mathbf{X}, \boldsymbol{t}, \alpha, \beta)\, \mathrm{d}\boldsymbol{w}$$

Learning as inference – recap

1) $\mathcal{D} \rightarrow \boldsymbol{w}_{\mathrm{ML}}$

2) $\mathcal{D}, p(\boldsymbol{w}) \rightarrow \boldsymbol{w}_{\mathrm{MAP}}$

3) Bayes: $\mathcal{D}, p(\boldsymbol{w}) \rightarrow p(\boldsymbol{w} \mid \mathcal{D})$

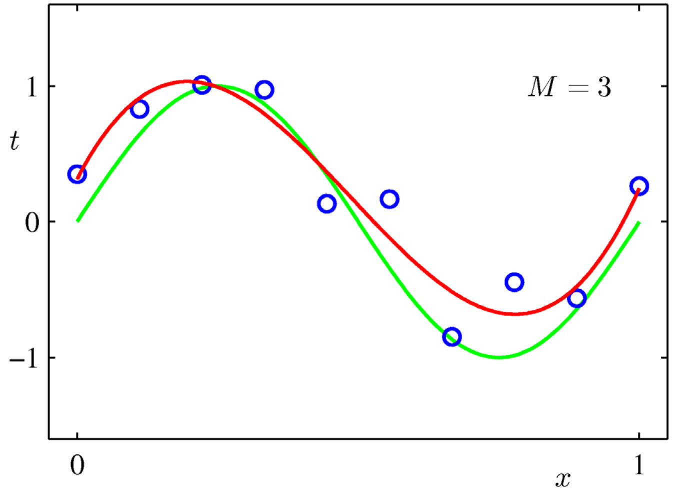
We will elaborate more particularly on the Bayesian regression in this lecture.

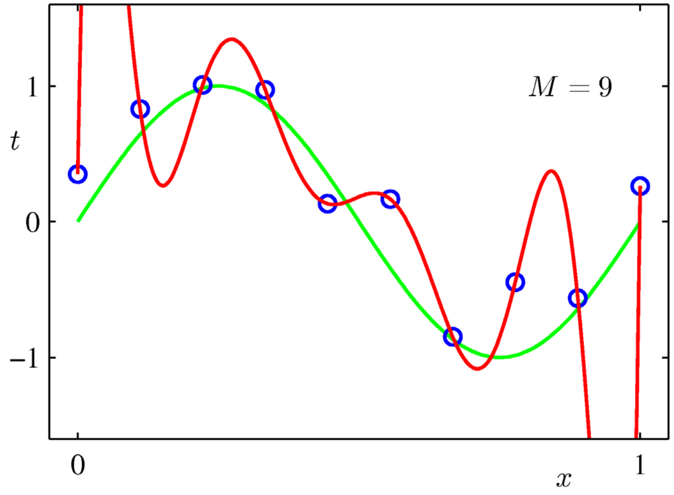
Main points in lecture 2

- Motivation for a probabilistic perspective of machine learning – scientific discipline
- General philosophy of a Bayesian approach
- Bayesian vs frequentist viewpoint
- Learning as inference, regression example
- Generative vs discriminative models
- Model complexity, overfitting, model selection

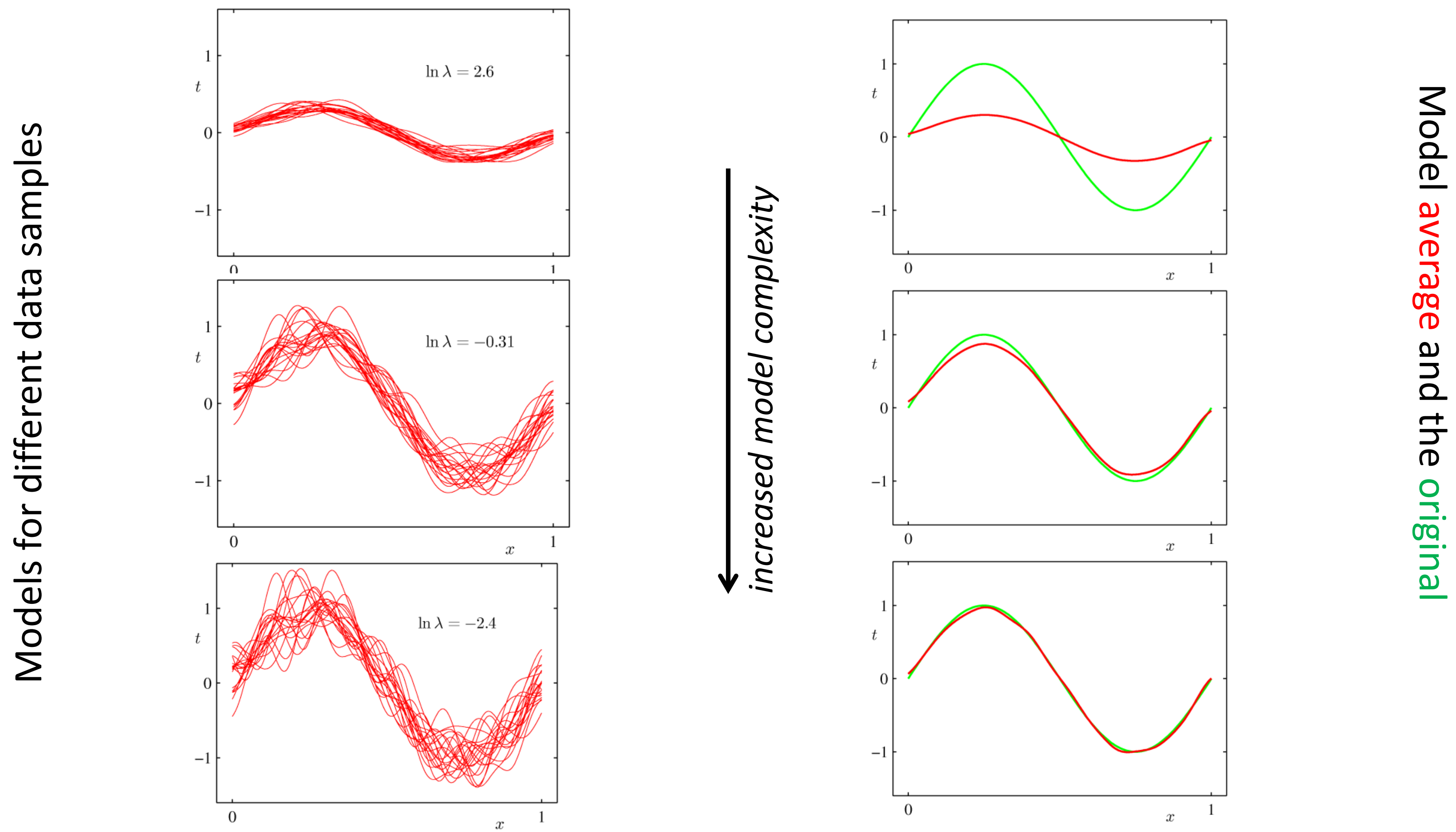
Model complexity

- Overfitting of maximum likelihood models



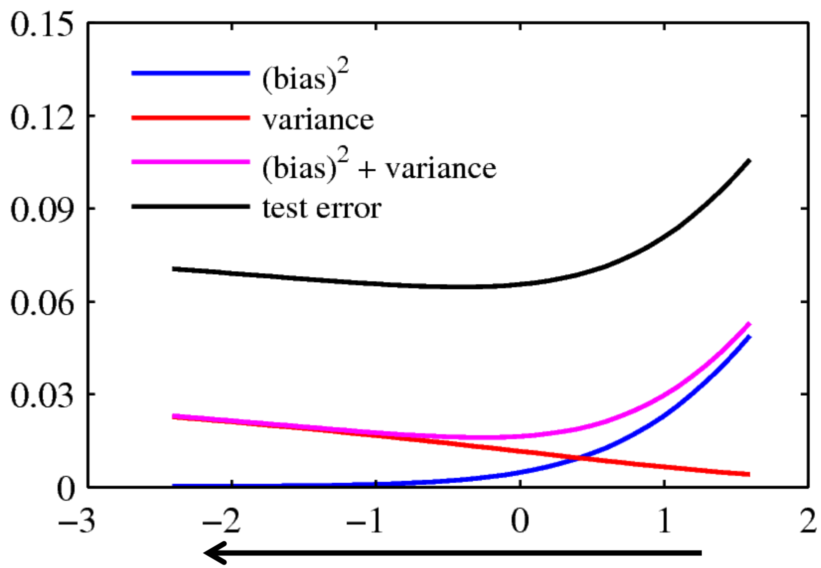


Bias-variance dilemma: model complexity



Bias-variance dilemma: model complexity

$$E[L] = (\text{bias})^2 + \text{variance} + \text{noise}$$



increased model complexity

Model selection

- Occam's razor – *"Accept the simplest explanation that fits the data."*
- Frequentist approach with maximum likelihood
  - bias-variance dilemma
  - need to control the model's complexity
    - regularisation
    - correction for the bias of ML estimates (AIC, BIC)
    - empirical estimate of generalisation error on a hold-out set (validation, resampling)
    - structural risk minimization (SRM) (minimise upper bound on the true risk), see also VC dimension (statistical learning theory)

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Linear regression – the "*work horse*" of ML

- Linear basis function models

$$y(\mathbf{x},\mathbf{w}) = \sum_{j=0}^{M-1} w_j \phi_j(\mathbf{x}) = \mathbf{w}^{\mathrm{T}} \phi(\mathbf{x})$$

$$\mathbf{w} = \left(w_0, \ldots, w_{M-1}\right)^{\mathrm{T}} \qquad \boldsymbol{\phi} = \left(\phi_0, \ldots, \phi_{M-1}\right)^{\mathrm{T}}$$

e.g., for polynomial one-dimensional regression basis functions are $\phi_j(x) = x^j$

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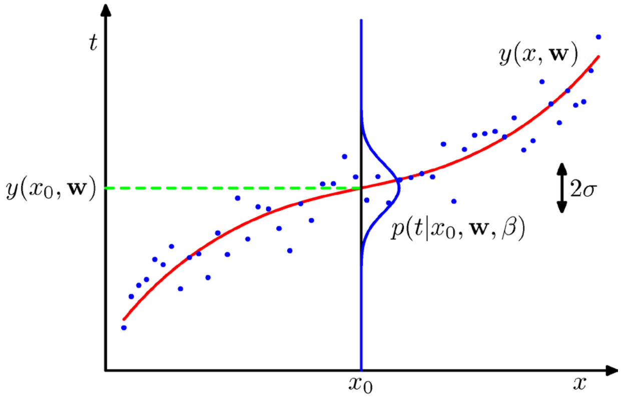
e.g., for polynomial one-dimensional regression basis functions are $\phi_j(x) = x^j$

$$t = y(\mathbf{x},\mathbf{w}) + \varepsilon = \mathbf{w}^{\mathrm{T}} \phi(\mathbf{x}) + \varepsilon$$

$$p(t \mid \mathbf{x},\mathbf{w},\beta) = \mathcal{N}(t \mid y(\mathbf{x},\mathbf{w}), \beta^{-1})$$



Linear regression

- Maximum likelihood estimation

$$y(\mathbf{x},\mathbf{w}) = \sum_{j=0}^{M-1} w_j \phi_j(\mathbf{x}) = \mathbf{w}^{\mathrm{T}} \phi(\mathbf{x})$$

Likelihood:

$$p(\mathbf{t}_{\mathcal{D}_{trn}} \mid \mathbf{X}_{\mathcal{D}_{trn}}, \mathbf{w}, \beta) = \prod_{n=1}^{N} \mathcal{N}(t_n \mid \mathbf{w}^{\mathrm{T}} \phi(\mathbf{x}_n), \beta^{-1})$$

$$\mathbf{X}_{\mathcal{D}_{trn}} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$$

$$\mathbf{t}_{\mathcal{D}_{trn}} = (t_1, \ldots, t_N)$$

$$\Downarrow$$

$$\ln p(\mathbf{t}_{\mathcal{D}_{trn}} \mid \mathbf{X}_{\mathcal{D}_{trn}}, \mathbf{w}, \beta) = \underbrace{-\frac{\beta}{2} \sum_{n=1}^{N} \left\{ t_n - \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n) \right\}^2}_{E_D(\mathbf{w})} + \frac{N}{2} \ln \beta - \frac{N}{2} \ln(2\pi)$$

Linear regression

- Maximum likelihood

$$0 = \sum_{n=1}^{N} t_n \boldsymbol{\phi}(\mathbf{x}_n) - \mathbf{w}^{\mathrm{T}} \left(\sum_{n=1}^{N} \boldsymbol{\phi}(\mathbf{x}_n) \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} \right)$$

Linear regression

- Maximum likelihood

$$0 = \sum_{n=1}^{N} t_n \boldsymbol{\phi}(\mathbf{x}_n) - \mathbf{w}^{\mathrm{T}} \left(\sum_{n=1}^{N} \boldsymbol{\phi}(\mathbf{x}_n) \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} \right)$$

Least-square solution:
(normal equations)

$$\mathbf{w}_{\mathrm{ML}} = \left(\boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\Phi} \right)^{-1} \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{t}$$

pseudo-inverse of the design matrix $\boldsymbol{\Phi}$

$$\beta_{\mathrm{ML}}^{-1} = \frac{1}{N} \sum_{n=1}^{N} \left\{ t_n - \mathbf{w}_{\mathrm{ML}}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n) \right\}^2$$

Linear regression

- Maximum likelihood

$$0 = \sum_{n=1}^{N} t_n \boldsymbol{\phi}(\mathbf{x}_n) - \mathbf{w}^{\mathrm{T}} \left(\sum_{n=1}^{N} \boldsymbol{\phi}(\mathbf{x}_n) \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} \right)$$

Least-square solution:
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$$\mathbf{w}_{\mathrm{ML}} = \left(\boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\Phi} \right)^{-1} \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{t}$$

pseudo-inverse of
the design matrix $\boldsymbol{\Phi}$

$$\boldsymbol{\Phi} = \begin{pmatrix} \phi_0(\mathbf{x}_1) & \cdots & \phi_{M-1}(\mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ \phi_0(\mathbf{x}_N) & \cdots & \phi_{M-1}(\mathbf{x}_N) \end{pmatrix}$$

Linear regression

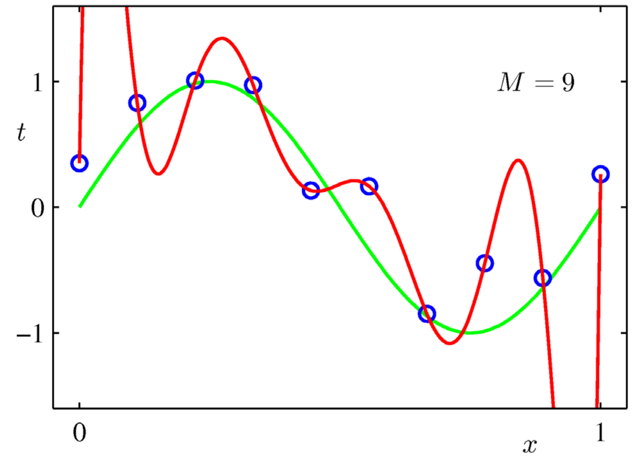
- Maximum likelihood

$$0 = \sum_{n=1}^{N} t_n \boldsymbol{\phi}(\mathbf{x}_n) - \mathbf{w}^{\mathrm{T}} \left(\sum_{n=1}^{N} \boldsymbol{\phi}(\mathbf{x}_n) \boldsymbol{\phi}(\mathbf{x}_n)^{\mathrm{T}} \right)$$

$$\begin{cases} \mathbf{w}_{\mathrm{ML}} = \left(\boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\Phi} \right)^{-1} \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{t} \\ \beta_{\mathrm{ML}}^{-1} = \dfrac{1}{N} \displaystyle\sum_{n=1}^{N} \left\{ t_n - \mathbf{w}_{\mathrm{ML}}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n) \right\}^2 \end{cases}$$

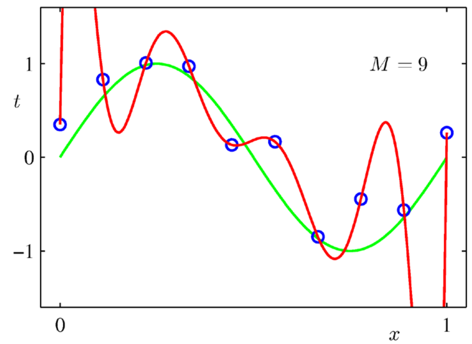
Linear regression - regularisation

- Problems with maximum likelihood estimate



Linear regression - regularisation

- Problems with maximum likelihood estimate



- We can address it by regularisation (parameter shrinkage)

$$\arg\min_{\mathbf{w}} \left\{ \frac{1}{2} \sum_{n=1}^{N} \{ t_n - \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n) \}^2 + \frac{\lambda}{2} \mathbf{w}^{\mathrm{T}} \mathbf{w} \right\}$$ (weight decay)

$$\arg\min_{\mathbf{w}} \left\{ \frac{1}{2} \sum_{n=1}^{N} \{ t_n - \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n) \}^2 + \frac{\lambda}{2} \sum_{j=1}^{M} |w_j|^q \right\}$$ (lasso for q=1)

Bayesian linear regression

- Bayesian treatment

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} \mid \mathbf{m}_0, \mathbf{S}_0)$$

Bayesian linear regression

- Bayesian treatment

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} \mid \mathbf{m}_0, \mathbf{S}_0)$$

For this **conjugate Gaussian prior** corresponding to the likelihood, the posterior has also Gaussian distribution:

$$p(\mathbf{w} \mid \mathbf{t}_{D_{trn}}) = \mathcal{N}(\mathbf{w} \mid \mathbf{m}_N, \mathbf{S}_N)$$

$$\mathbf{m}_N = \mathbf{S}_N \left(\mathbf{S}_0^{-1} \mathbf{m}_0 + \beta \mathbf{\Phi}^{\mathrm{T}} \mathbf{t}_{\mathcal{D}_{trn}} \right)$$

$$\mathbf{S}_N^{-1} = \mathbf{S}_0^{-1} + \beta \mathbf{\Phi}^{\mathrm{T}} \mathbf{\Phi}$$

$$\mathbf{\Phi}(\mathbf{X}_{\mathcal{D}_{trn}}) \to \mathbf{\Phi}$$

Bayesian linear regression

- Bayesian treatment

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} \mid \mathbf{0}, \alpha^{-1}\mathbf{I})$$

$$\Downarrow$$

$$p(\mathbf{w} \mid \mathbf{t}_{Dtrn}, \mathbf{X}_{\mathcal{D}_{trn}}, \alpha, \beta) = \mathcal{N}(\boldsymbol{w} \mid \boldsymbol{m}_N, \mathbf{S}_N)$$

$$\mathbf{m}_N = \beta \mathbf{S}_N \mathbf{\Phi}^{\mathrm{T}} \mathbf{t}_{\mathcal{D}_{trn}}$$

$$\mathbf{S}_N^{-1} = \alpha \mathbf{I} + \beta \mathbf{\Phi}^{\mathrm{T}} \mathbf{\Phi}$$

Bayesian linear regression

- Bayesian treatment

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{w} \mid \mathbf{0}, \alpha^{-1}\mathbf{I})$$

$$\Downarrow$$

the **likelihood part** of the log-posterior dependent on $\mathbf{w}$

the **prior part** of the log-posterior dependent on $\mathbf{w}$

$$\underbrace{\ln p(\mathbf{w} \mid \mathbf{t}_{\mathcal{D}_{trn}}, \mathbf{X}_{\mathcal{D}_{trn}}, \alpha, \beta)}_{\text{log-posterior}} = \boxed{-\frac{\beta}{2}\sum_{n=1}^{N}\left\{t_n - \mathbf{w}^{\mathrm{T}}\boldsymbol{\phi}(\mathbf{x}_n)\right\}^2} \boxed{-\frac{\alpha}{2}\mathbf{w}^{\mathrm{T}}\mathbf{w}} + \frac{N}{2}\ln\beta - \frac{N}{2}\ln(2\pi)$$

Bayesian linear regression

- Predictive distribution

$$\underbrace{p(t \mid \mathbf{x}, \mathbf{t}_{\mathcal{D}_{trn}}, \alpha, \beta)}_{\text{predictive}} = \int \underbrace{p(t \mid \mathbf{x}, \mathbf{w}, \beta)}_{\text{“noise” model}} \; \underbrace{p(\mathbf{w} \mid \mathbf{t}_{\mathcal{D}_{trn}}, \mathbf{X}_{\mathcal{D}_{trn}}, \alpha, \beta)}_{\text{posterior}} \, \mathrm{d}\mathbf{w}$$

Bayesian linear regression

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$$p(t \mid \mathbf{x}, \mathbf{t}_{Dtrn}, \alpha, \beta) = \mathcal{N}(t \mid \mathbf{m}_N^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}), \sigma_N^2(\mathbf{x}))$$

$$\mathbf{m}_N = \beta \mathbf{S}_N \boldsymbol{\Phi}^{\mathrm{T}} \mathbf{t}_{\mathcal{D}_{trn}}$$

$$\mathbf{S}_N^{-1} = \alpha \mathbf{I} + \beta \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\Phi}$$

$$\sigma_N^2(\mathbf{x}) = \frac{1}{\beta} + \boldsymbol{\phi}(\mathbf{x})^{\mathrm{T}} \mathbf{S}_N \boldsymbol{\phi}(\mathbf{x})$$

Bayesian linear regression

- Bayesian philosophy – overview
  - in order to make predictions we need posterior

$$p(\mathbf{w} \,|\, \mathbf{t}, \mathbf{x})$$

  - it can be updated (estimated) after the relevant information has been taken into account

Adapted from Carl Henrik Ek's lecture (DD2434, 2015)

Bayesian linear regression

- Bayesian philosophy
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Bayesian linear regression

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  - before we see the data, we express our belief about the regression params

$$p(\mathbf{w}) = \mathcal{N}(\mathbf{0}, \alpha^{-1}\mathbf{I})$$

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Bayesian linear regression

- Bayesian philosophy
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$$p(\mathbf{w}) = \mathcal{N}(\mathbf{0}, \alpha^{-1}\mathbf{I})$$

very often we do not really know too much in advance

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Bayesian linear regression

- Bayesian philosophy (cont.)
  - how well does my model predicts (explains) data?
    - likelihood (like an error function) for a single sample n

$$p(t_n \mid \mathbf{w}, \mathbf{x}_n) = \mathcal{N}(t_n \mid \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1}\mathbf{I})$$

    - if there is independence between samples, likelihood for all data $\mathbf{t}$ amounts to

$$p(\mathbf{t} \mid \mathbf{w}, \mathbf{X}) = \prod_{n=1}^{N} \mathcal{N}(t_n \mid \mathbf{w}^{\mathrm{T}} \boldsymbol{\phi}(\mathbf{x}_n), \beta^{-1}\mathbf{I})$$

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Bayesian linear regression

- Bayesian philosophy – summary
  - The goal is to reach posterior distribution after <u>all relevant</u> information has been taken into account.
  - Prediction should reflect my beliefs in the model and the information in the observations.

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Bayesian linear regression

- Bayesian philosophy – summary
  - The goal is to reach posterior distribution after <u>all relevant</u> information has been taken into account.
  - Prediction should reflect my beliefs in the model and the information in the observations.
  - So:
    1. Choose a model
    2. Formulate prediction error by likelihood
    3. Formulate belief of a model in prior
    4. Marginalise irrelevant variables (parameters)

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Bayesian linear regression

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Bayesian linear regression

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  - Prediction should reflect my beliefs in the model and the information in the observations.
  - So:

Gaussian distributions are *self-conjugate*, i.e.:

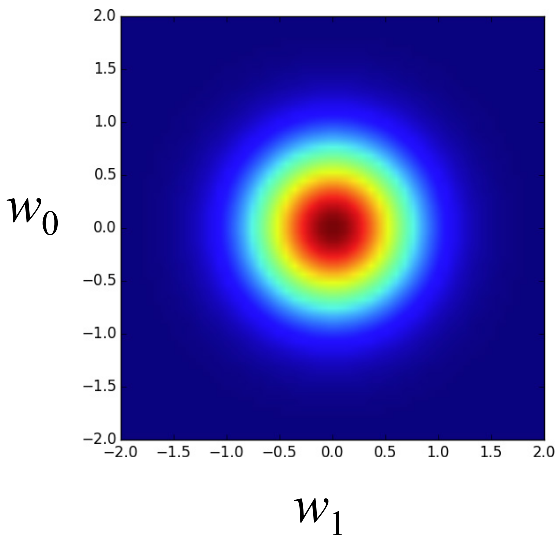
Gaussian prior + Gaussian likelihood → **Gaussian posterior**

Adapted from Carl Henrik Ek's lecture (DD2434, 2015)

Sequential Bayesian learning

$$y(x,\mathbf{w}) = -w_0 + w_1 x$$

$$p(\mathbf{w})$$



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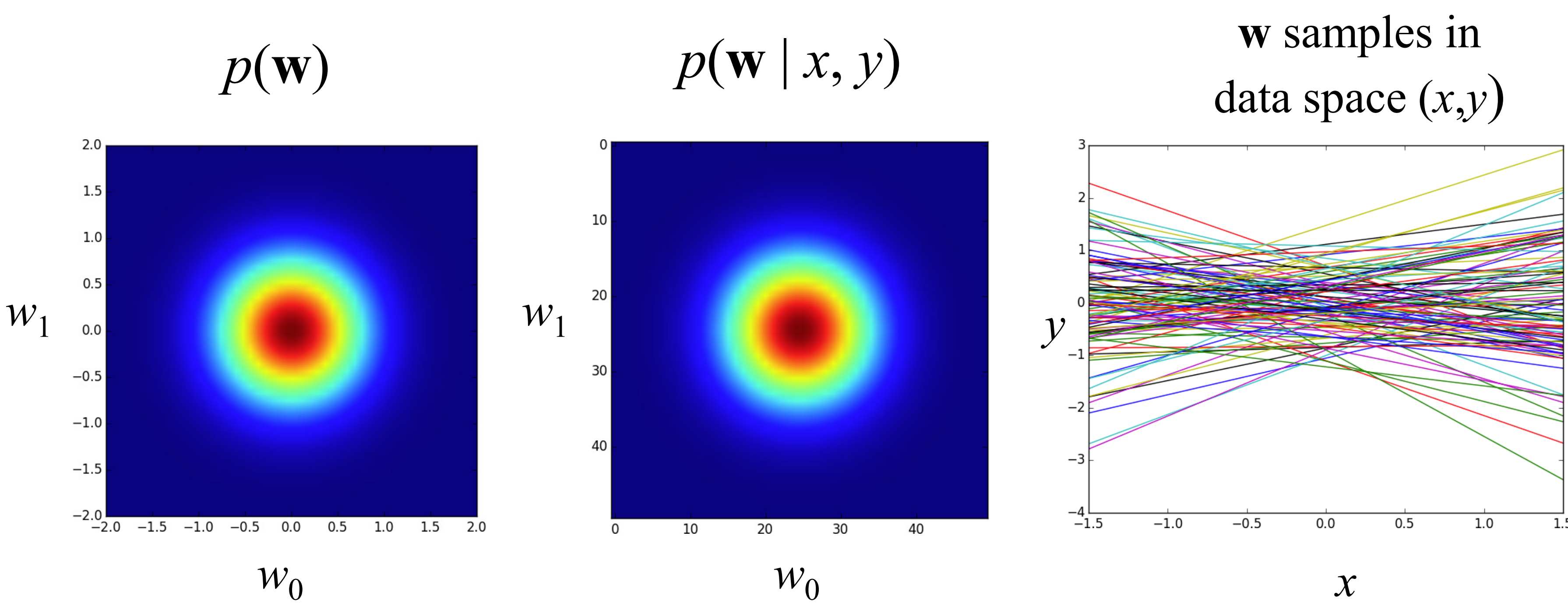
Sequential Bayesian learning



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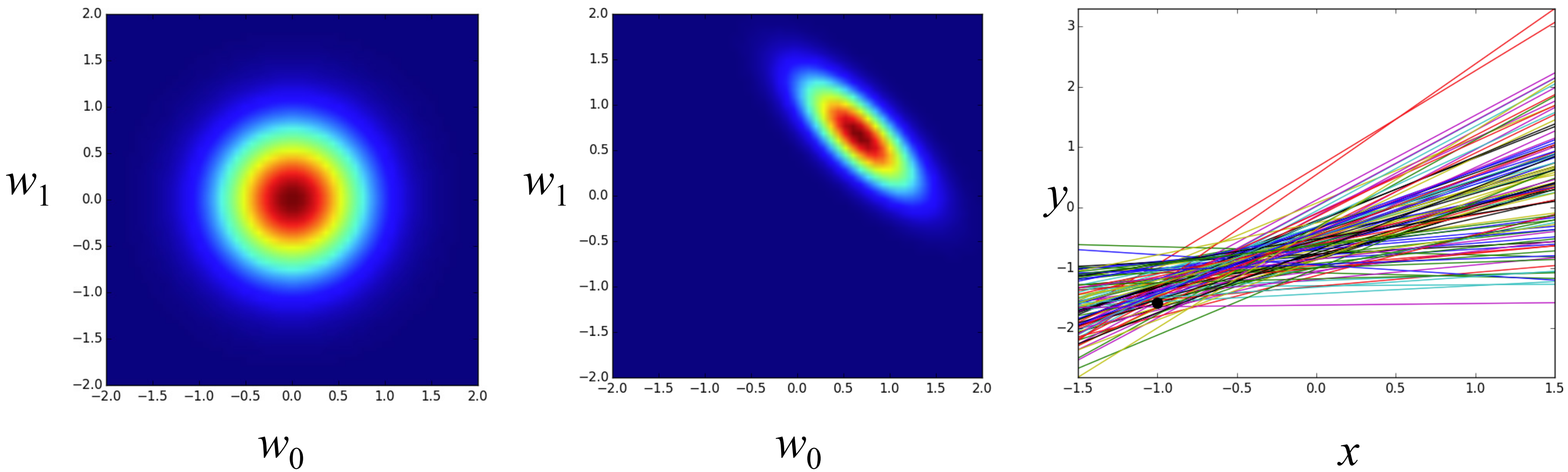
Sequential Bayesian learning

$p(\mathbf{w})$ $\quad$ $p(\mathbf{w} \mid x, y)$ $\quad$ $\mathbf{w}$ samples in data space (x,y)



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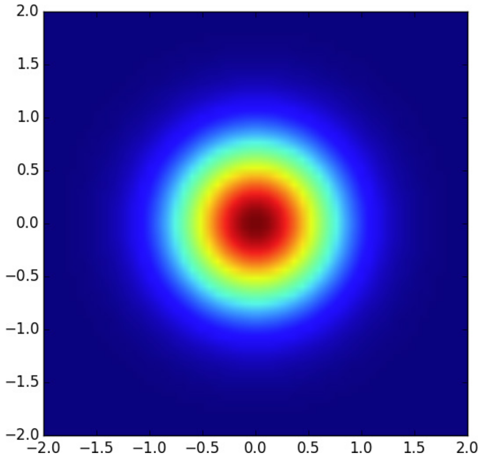
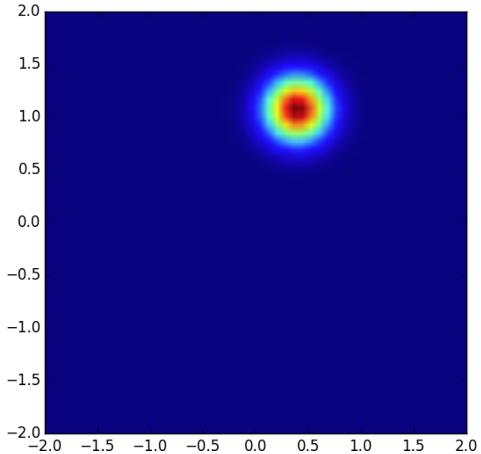
Sequential Bayesian learning

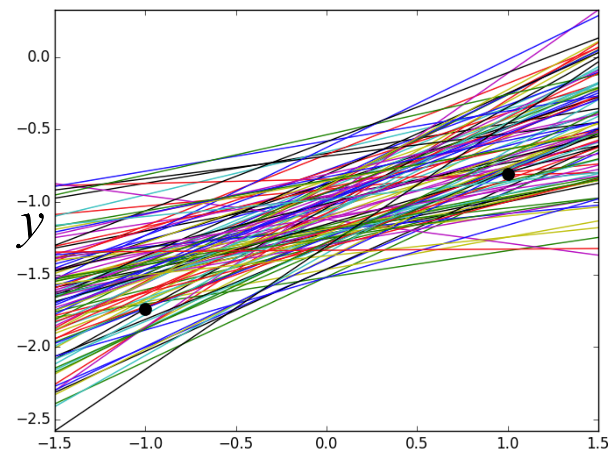
$p(\mathbf{w})$



$p(\mathbf{w} \mid x, y)$



$\mathbf{w}$ samples in data space (x,y)



w_1 w_0 w_1 w_0 y x

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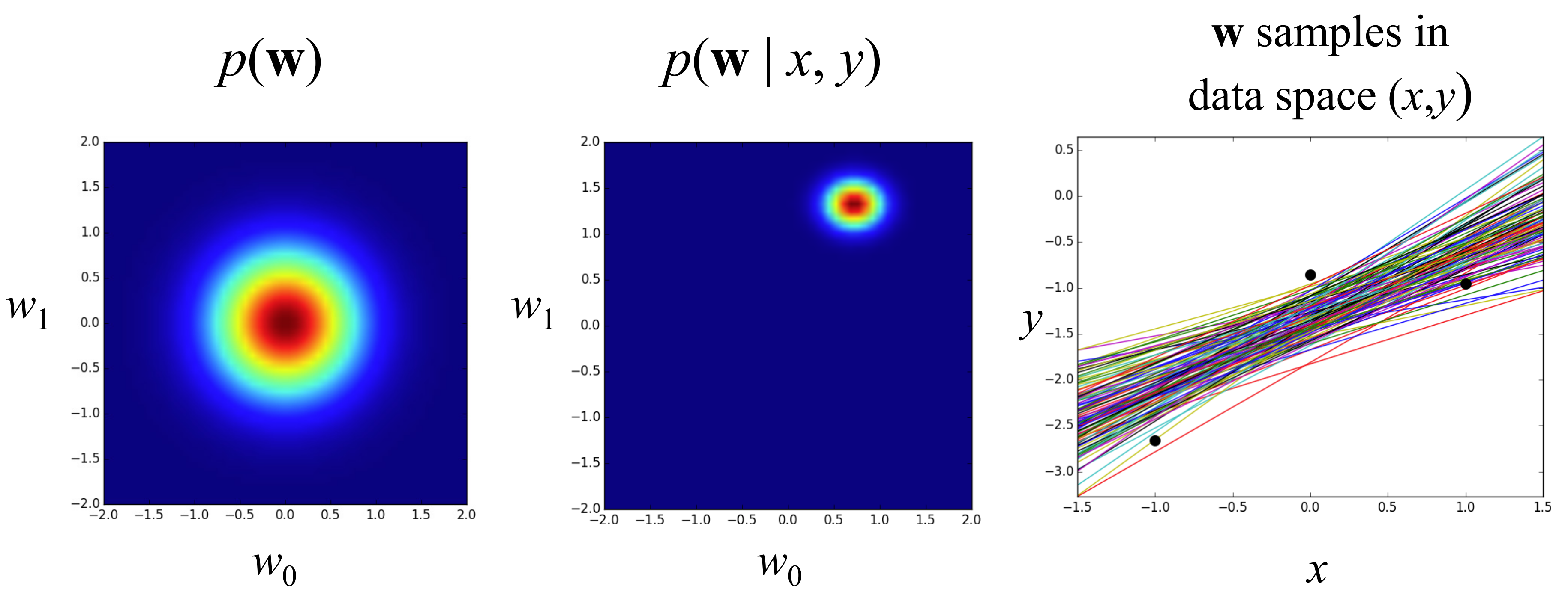
Sequential Bayesian learning



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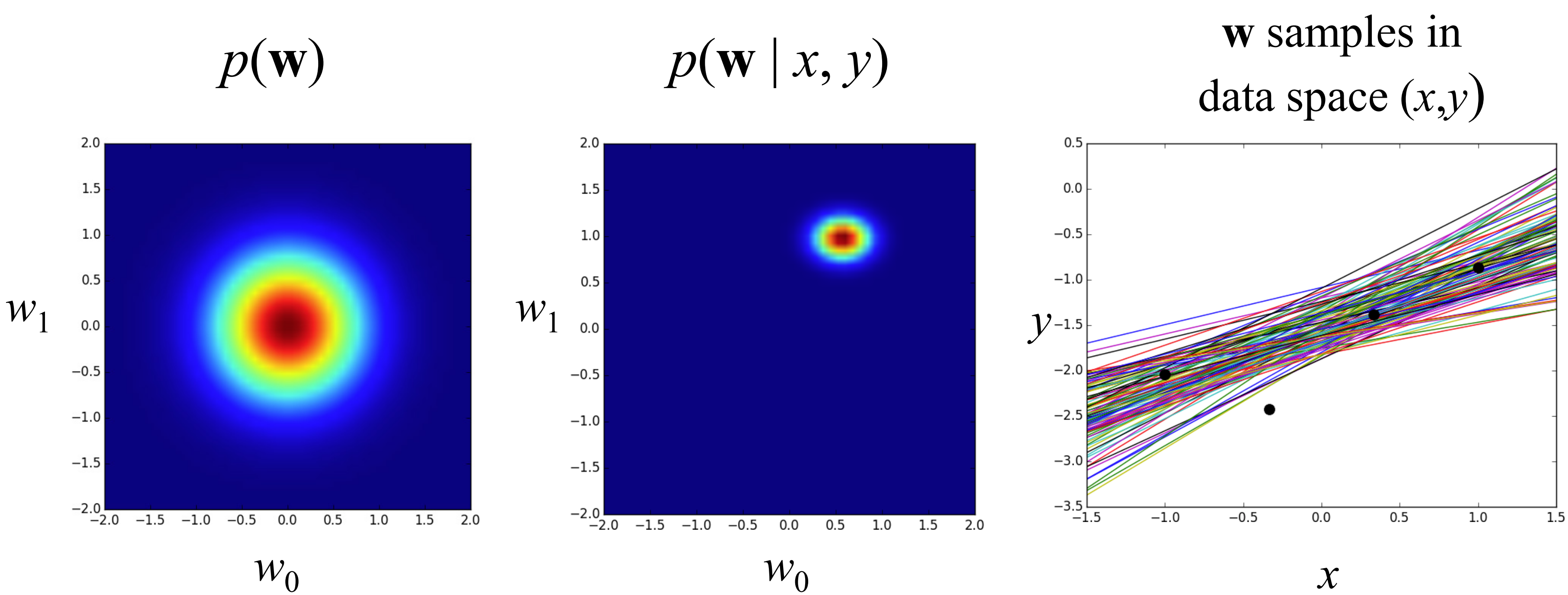

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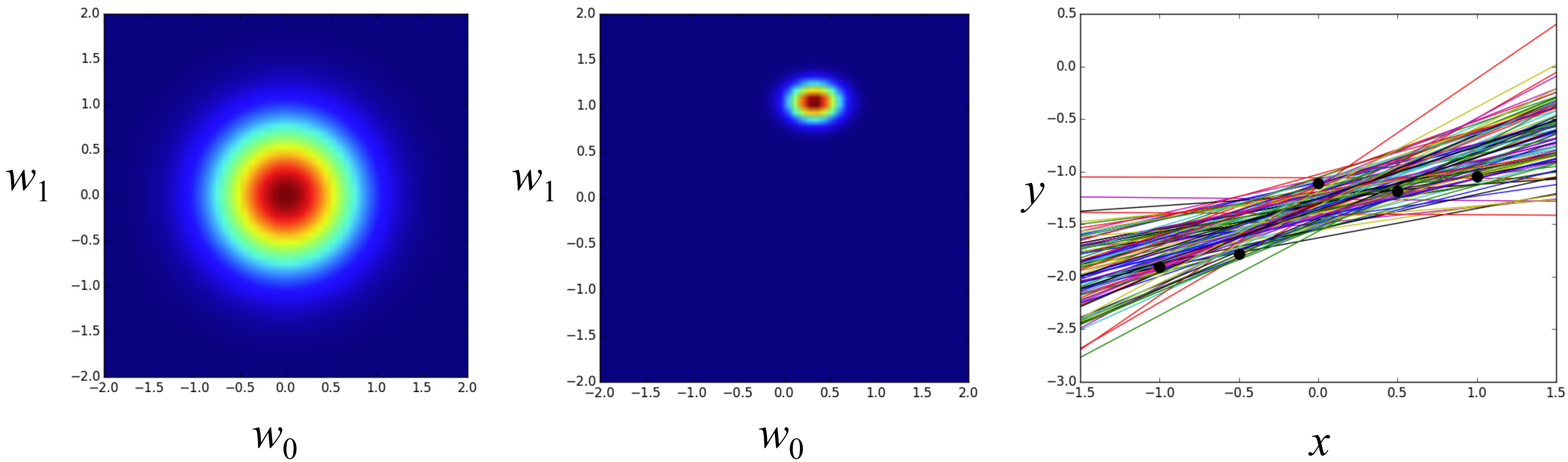

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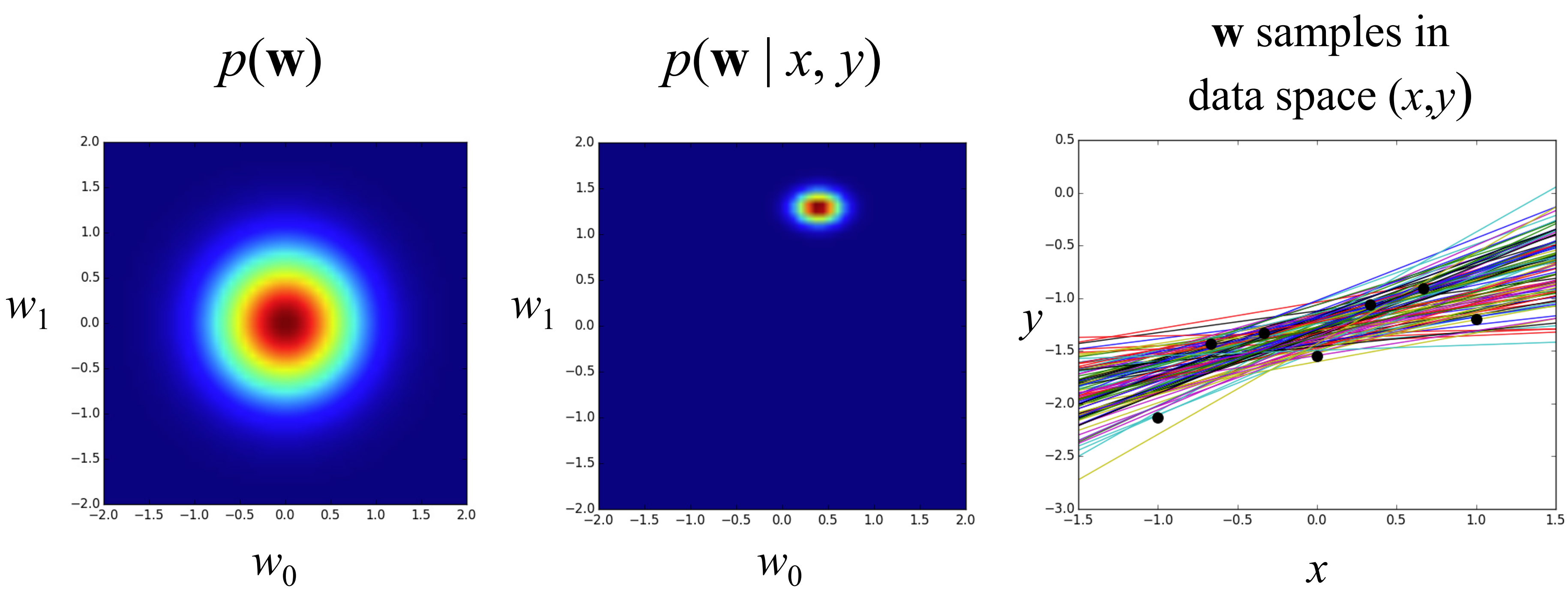

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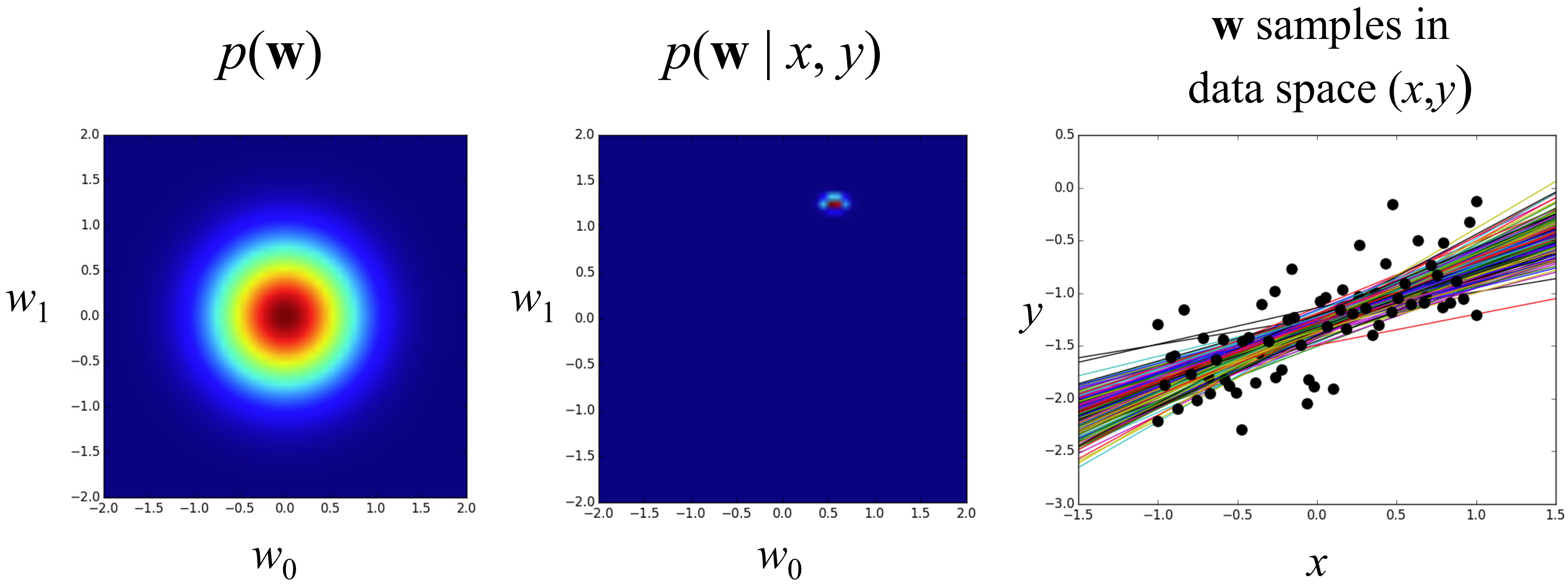

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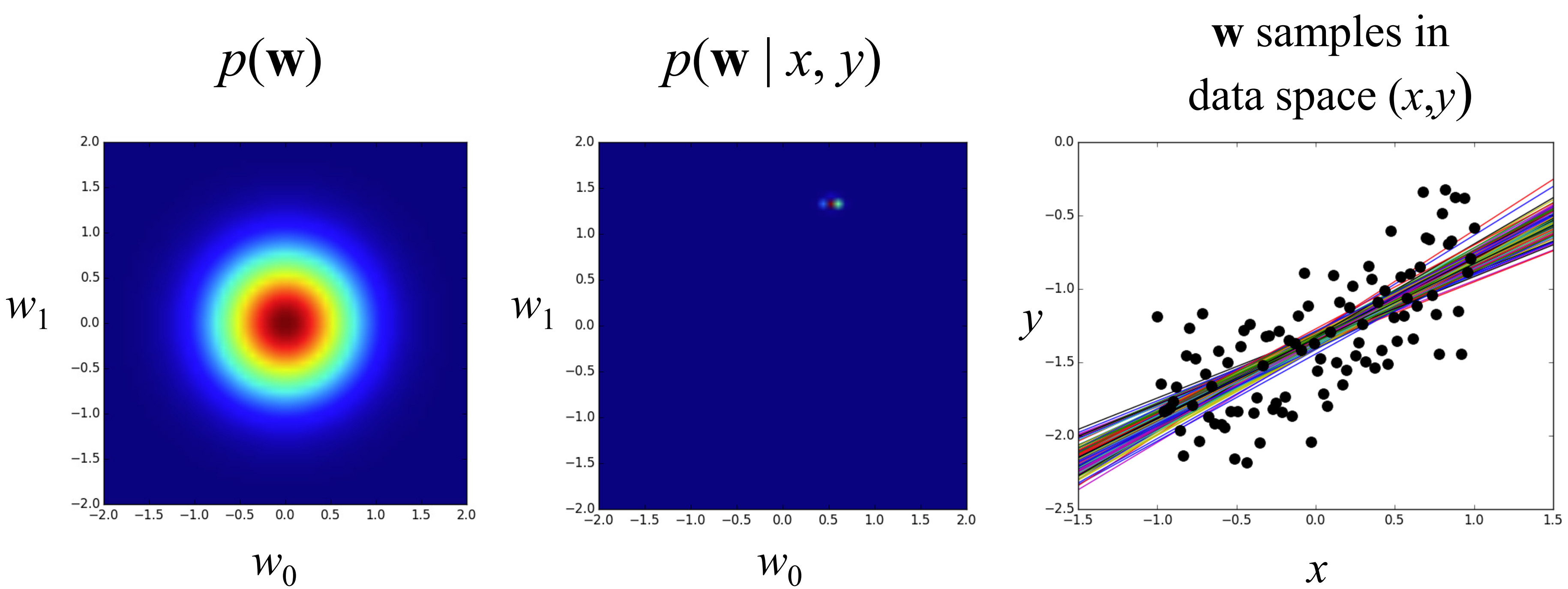

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Bayesian model comparison

- Evidence framework

$$p(\mathbf{w} \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \mathbf{w})\, p(\mathbf{w})}{p(\mathcal{D})}$$

  - The denominator does not change with $\mathbf{w}$

$$p(\mathbf{w} \mid \mathbf{t}_{\mathcal{D}}, \mathbf{X}_{\mathcal{D}}) \propto p(\mathbf{t}_{\mathcal{D}} \mid \mathbf{w}, \mathbf{X}_{\mathcal{D}})\, p(\mathbf{w})$$

Bayesian model comparison

- Evidence framework

$$p(\mathbf{w} \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \mathbf{w})\, p(\mathbf{w})}{p(\mathcal{D})}$$

  - The denominator does not change with $\boldsymbol{w}$

$$p(\mathbf{w} \mid \mathbf{t}_{\mathcal{D}}, \mathbf{X}_{\mathcal{D}}) \propto p(\mathbf{t}_{\mathcal{D}} \mid \mathbf{w}, \mathbf{X}_{\mathcal{D}})\, p(\mathbf{w})$$

  - $p(\mathcal{D})$ shows where the model spreads its probability mass over the data space (evidence of the model)

$$p(\mathbf{w} \mid \mathcal{D}, \mathcal{M}_i) = \frac{p(\mathcal{D} \mid \mathbf{w}, \mathcal{M}_i)\, p(\mathbf{w} \mid \mathcal{M}_i)}{p(\mathcal{D} \mid \mathcal{M}_i)}$$

Bayesian model comparison

- What can we do with model evidence?

posterior: $p(\mathcal{M}_i \mid \mathcal{D}) \propto p(\mathcal{M}_i)\, p(\mathcal{D} \mid \mathcal{M}_i)$

marginal likelihood

Bayesian model comparison

- What can we do with model evidence?

posterior: $p(\mathcal{M}_i \mid \mathcal{D}) \propto p(\mathcal{M}_i)\, p(\mathcal{D} \mid \mathcal{M}_i)$

  - for the same priors, the posterior ratio between $\mathcal{M}_1$ and $\mathcal{M}_2$

$$\frac{p(\mathcal{D} \mid \mathcal{M}_1)}{p(\mathcal{D} \mid \mathcal{M}_2)}$$ Bayes factor

Bayesian model comparison

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"The evidence can be seen as the probability of generating the data set from a model whose parameters are sampled at random from the prior" *Bishop, sec.3.4*

Bayesian model comparison

- What can we do with model evidence?

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  - prediction can be made using the “best” model (single most probable model) or can be averaged from the mixture of K models

Model mixture: $$p(t \mid \mathbf{x}, \mathcal{D}) = \sum_{i=1}^{K} p(t \mid \mathbf{x}, \mathcal{M}_i, \mathcal{D})\, p(\mathcal{M}_i \mid \mathcal{D})$$

Bayesian model comparison

- Simple approximation for a single parameter w

$$p(\mathcal{D}|\mathcal{M}_i) = \int p(\mathcal{D}|w,\mathcal{M}_i)\,p(w|\mathcal{M}_i)\,dw$$

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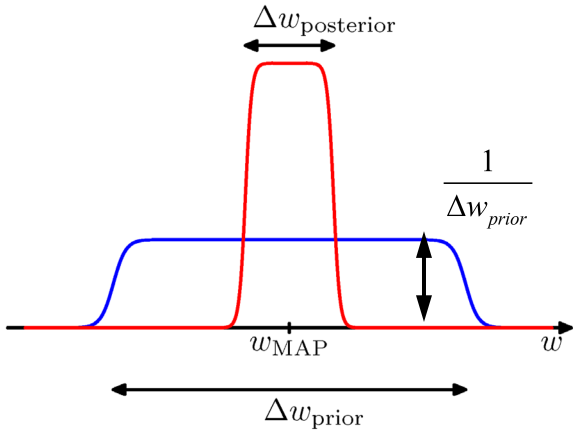
  - If the *posterior* is sharply peaked around some w_{MAP}, with the width $\Delta w_{\mathrm{posterior}}$, and the *prior* is flat with the width $\Delta w_{\mathrm{prior}}$, then

$$p(\mathcal{D}|\mathcal{M}_i) \approx p(\mathcal{D}|w_{\mathrm{MAP}},\mathcal{M}_i)\frac{\Delta w_{posterior}}{\Delta w_{prior}}$$



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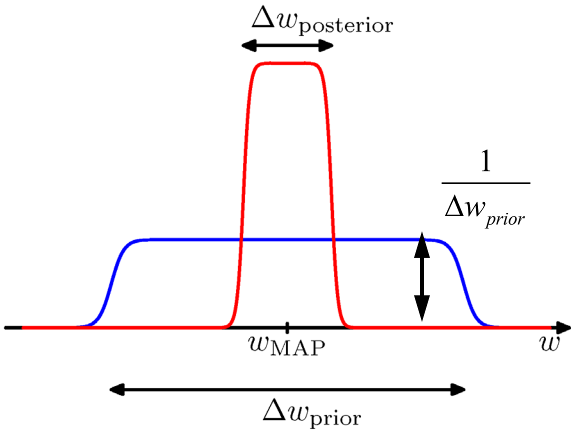
$$\Downarrow$$

$$\ln p(\mathcal{D}|\mathcal{M}_i) \approx \ln p(\mathcal{D}|w_{\mathrm{MAP}},\mathcal{M}_i) + \ln\left(\frac{\Delta w_{posterior}}{\Delta w_{prior}}\right)$$



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$\Bigg\{$ Similar estimate can be made for M parameters (assuming their distributions behave similarly)

Bayesian model comparison

- Simple approximation for a single parameter w

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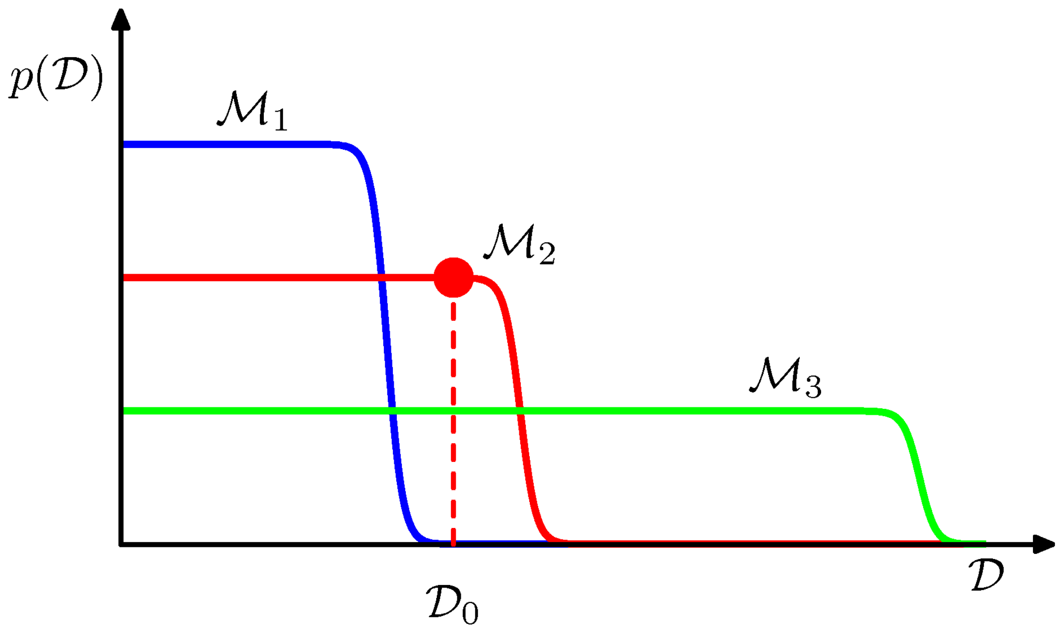
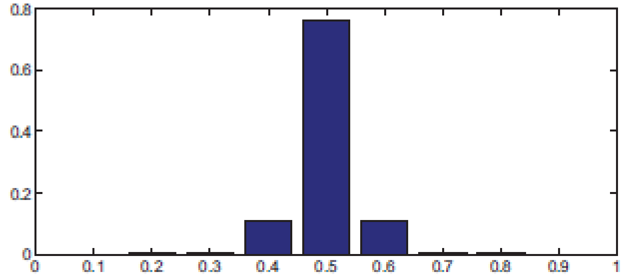
It can be seen as *Occam factor.*

Bayesian model comparison

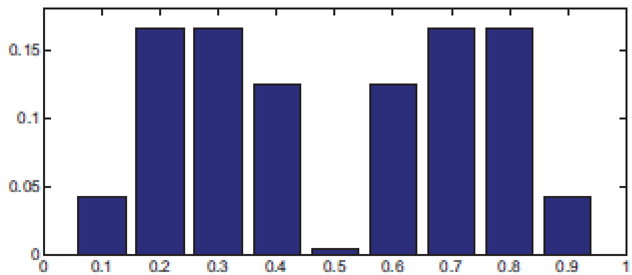


Example – a discrete parameter space

We have two discrete models: "fair" coin (M_1) and "biased" coin (M_2) model



prior for θ for "fair" coin (M_1)



prior for θ for "biased" coin (M_2)

The evidence for each model is:

$$p(\mathcal{D}|\,\mathcal{M}_i) = \sum_{\theta} p(\mathcal{D}|\,\theta, \mathcal{M}_i)\, p(\theta\,|\,\mathcal{M}_i)$$

Example – a discrete parameter space

Evidence for M_1 and M_2

$$p(\mathcal{D}|\,\mathcal{M}_i) = \sum_{\theta} p(\mathcal{D}|\,\theta, \mathcal{M}_i)\, p(\theta\,|\,\mathcal{M}_i)$$

For N_H heads and N_T tails, we obtain the following evidence for model i:

$$p(\mathcal{D}|\,\mathcal{M}_i) = \sum_{\theta} \theta^{N_H} (1-\theta)^{N_T} \;\; p(\theta|\,\mathcal{M}_i)$$

Example – a discrete parameter space

Evidence for M_1 and M_2

$$p(\mathcal{D}|\mathcal{M}_i) = \sum_{\theta} p(\mathcal{D}|\theta,\mathcal{M}_i)\,p(\theta\,|\,\mathcal{M}_i)$$

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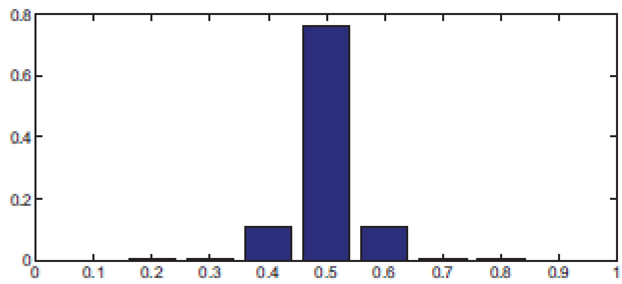
If we assume that $p(M_1)$=$p(M_2)$, i.e. both “fair” &“biased” coins are equally probable, then the Bayes’ factor is decisive:

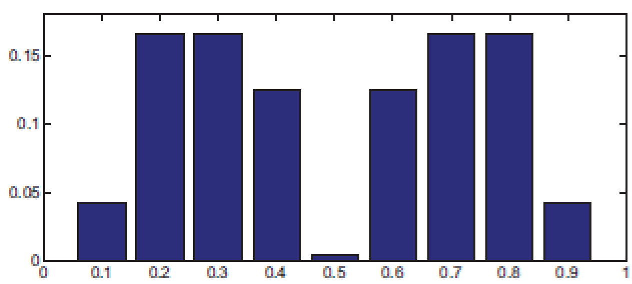
$$\frac{p(\mathcal{M}_{fair}\,|\,\mathcal{D})}{p(\mathcal{M}_{biased}\,|\,\mathcal{D})} = \frac{p(\mathcal{D}|\mathcal{M}_{fair})}{p(\mathcal{D}|\mathcal{M}_{biased})}$$

Example – a discrete parameter space

$$\theta \in \{0.1, 0.2, \ldots, 0.9\}$$



prior for θ for "fair" coin (M_1)



prior for θ for "biased" coin (M_2)

$$p(\mathcal{D} | \theta) = \theta^{N_H} (1-\theta)^{N_T}$$

Example – a discrete parameter space

$$\theta \in \{0.1, 0.2, \ldots, 0.9\}$$

Bayes' factor estimates:

N_H = 5 and N_T = 2

$$\frac{p(\mathcal{D} \mid \mathcal{M}_{fair})}{p(\mathcal{D} \mid \mathcal{M}_{biased})} = 1.09$$

N_H = 50 and N_T = 20

$$\frac{p(\mathcal{D} \mid \mathcal{M}_{fair})}{p(\mathcal{D} \mid \mathcal{M}_{biased})} = 0.109$$

The evidence approximation

If hyperpriors over α and β are introduced, we obtain the predictive distribution by marginalizing out $\mathbf{w}$, α and β

$$p(t \mid \mathbf{t}, \mathbf{x}) = \iiint p(t \mid \mathbf{w}, \mathbf{x}, \beta)\, p(\mathbf{w} \mid \mathbf{t}, \alpha, \beta)\, p(\alpha, \beta \mid \mathbf{t})\, dw\, d\alpha\, d\beta$$

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From Bayes theorem:

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For flat prior over α and β we can resort to maximising the ***marginal likelihood***.

(*type II maximum likelihood*)