

DD2434 Machine Learning, Advanced Course

Assignment 1

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1 Knowing the Rules

Question 2.1.1: *It is mandatory to read the above text. Have you read it?*

Yes.

Question 2.1.2: *List all your collaborations concerning the problem formulations in this assignment.*

Question 2.1.3: *Have you discussed solutions with anybody?*

No.

2 Dependencies in a Directed Graphical Model

Question 2.2.4: *In the graphical model of Figure 9, is $\mu_k \perp \tau_k$ (not conditioned by anything)?*

Yes, $\mu_k \perp \tau_k$ is true.

$$p(\mu_k, \tau_k) = p(\mu_k)p(\tau_k)p(X^1, \dots, X^N \mid \mu_k, \tau_k).$$

Since none of the variables are observed, after “marginalizing both sides of” the above equation over X^1, \dots, X^N we obtain [1]:

$$p(\mu_k, \tau_k) = p(\mu_k)p(\tau_k).$$

So, $\mu_k \perp \tau_k$.

Question 2.2.5: *In the graphical model of Figure 9, is $\mu_k \perp \tau_k \mid X^1, \dots, X^N$?*

No, $\mu_k \perp \tau_k \mid X^1, \dots, X^N$ is false.

$$\begin{aligned} p(\mu_k, \tau_k \mid X^1, \dots, X^N) &= \frac{p(\mu_k, \tau_k, X^1, \dots, X^N)}{p(X^1, \dots, X^N)} \\ &= \frac{p(\mu_k)p(\tau_k)p(X^1, \dots, X^N \mid \mu_k, \tau_k)}{p(X^1, \dots, X^N)}, \end{aligned}$$

which isn't factorized into the product $p(\mu_k)p(\tau_k)$, thus $\mu_k \perp \tau_k \mid X^1, \dots, X^N$ is false.

Question 2.2.6: *In the graphical model of Figure 12, is $\mu \perp b'$ (not conditioned by anything)?*

No, $\mu \perp b'$ is false.

$$p(\mu, b', \tau) = p(b')p(\tau \mid b')p(\mu \mid \tau).$$

By marginalizing over τ , we can obtain

$$p(\mu, b') = p(b') \sum_{\tau} p(\tau \mid b')p(\mu \mid \tau) = p(\mu \mid b')p(b'),$$

which isn't factorized into the product $p(\mu)p(b')$, thus $\mu \perp b'$ is false.

Question 2.2.7: In the graphical model of Figure 12, is $\mu \perp b' \mid X^1, \dots, X^N$?

No.

Question 2.2.8: In the graphical model of Figure 12, is $X^n \perp S^n$ (not conditioned by anything)?

No.

Question 2.2.9: In the graphical model of Figure 12, is $X^n \perp S^n \mid \mu_k, \tau_k$?

No.

3 Likelihood of a tree GM only for E level

4 Simple VI

Question 2.4.12: Implement the VI algorithm for the variational distribution in Equation (10.24) in Bishop.

From Bishop [1], the variational distribution is given by

$$\begin{aligned} q(\mu, \tau) &= q_\mu(\mu)q_\tau(\tau) \\ q_\mu(\mu) &= \mathcal{N}(\mu \mid \mu_N, \lambda_N^{-1}) \\ q_\tau(\tau) &= \text{Gam}(\tau \mid a_N, b_N), \end{aligned}$$

where

$$\begin{aligned} \mu_N &= \frac{\lambda_0 \mu_0 + N \bar{x}}{\lambda_0 + N} \\ \lambda_N &= (\lambda_0 + N) \mathbb{E}[\tau] \\ a_N &= a_0 + \frac{N}{2} \\ b_N &= b_0 + \frac{1}{2} \mathbb{E}_\mu \left[\sum_{n=1}^N (x_n - \mu)^2 + \lambda_0 (\mu - \mu_0)^2 \right] \\ &= b_0 - \left(\sum_{n=1}^N x_n + \lambda_0 \mu_0 \right) \mathbb{E}_\mu[\mu] + \frac{1}{2} \left(\sum_{n=1}^N x_n^2 + \lambda_0 \mu_0^2 + (\lambda_0 + N) \mathbb{E}_\mu[\mu^2] \right). \end{aligned}$$

Where

$$\begin{aligned} \mathbb{E}_\mu[\mu] &= \mu_N \\ \mathbb{E}_\mu[\mu^2] &= \frac{1}{\lambda_N} + \mu_N^2 \\ \mathbb{E}_\tau[\tau] &= \frac{a_N}{b_N}. \end{aligned}$$

Due to non-informative condition, we set $\mu_0 = \lambda_0 = a_0 = b_0 = 0$. Then the VI algorithm would be an iterative algorithm that start with a set of initial values of μ_N , λ_N , a_N , and b_N . Then the values of μ_N , λ_N , a_N , and b_N should be updated to obtain a converging result of approximated posterior distribution. The algorithm is shown in the following code:

```

1 from math import exp, pi, sqrt
  from scipy.special import gamma

  import matplotlib.pyplot as plt
  import numpy as np

6 np.random.seed(1000)
  MAXITER = 1000
  THRES = 0.001

11 def _q(muN, lambdaN, aN, bN, mu, tau):
    q_mu = sqrt(lambdaN/(2*pi)) * np.exp(-0.5 * np.dot(lambdaN, np.transpose((mu-muN)
    )**2)))
    q_tau = (1.0/gamma(aN)) * bN**aN * tau**(aN-1) * np.exp(-bN*tau)
    q = q_tau * q_mu
    return q

16 def _update(D, N, mu0, lambda0, a0, b0, muN, lambdaN, aN, bN):
    E_mu = muN
    E_mu2 = 1.0 / lambdaN + muN**2
    E_tau = aN / bN
    lambdaN = (lambda0 + N) * E_tau
    bN = b0 - (sum(D) + lambda0*mu0)*E_mu \
    + 0.5*(sum(D**2) + lambda0*mu0**2 + (lambda0+N)*E_mu2)
    return lambdaN, bN

26 def SimpleVI():
    # Generate data set.
    mu_D = 0.0
    sigma_D = 1.0
    N = 100
    D = np.random.normal(loc=mu_D, scale=sigma_D, size=N)

    # Initial values.
    x_bar = D.mean()
    mu0 = 0
    lambda0 = 0
    a0 = 0
    b0 = 0

    muN = (lambda0*mu0 + N*x_bar) / (lambda0 + N)
    lambdaN = 10
    aN = a0 + N / 2
    bN = 5

    lambdaOld = lambdaN
    bOld = bN

    for _ in range(MAXITER):
        lambdaN, bN = _update(D, N, mu0, lambda0, a0, b0, muN, lambdaN, aN, bN)
        if (abs(lambdaN - lambdaOld) < THRES) and (abs(bN - bOld) < THRES):
            break
        lambdaOld = lambdaN

```

56

```

        bOld = bN

    return muN, lambdaN, aN, bN

if __name__ == "__main__":
    muN, lambdaN, aN, bN = SimpleVI()

```

Listing 1: Simple VI.

Question 2.4.13: *What is the exact posterior?*

Since gamma distribution is a conjugate prior of the normal distribution, the joint distribution of the Normal-Gamma distribution is

$$p(\mu, \tau \mid \mu_0, \lambda_0, a_0, b_0) = \frac{b_0^{a_0} \sqrt{\lambda_0}}{\Gamma(a_0) \sqrt{2\pi}} \tau^{a_0 - \frac{1}{2}} \exp[-b_0 \tau] \exp\left[-\frac{\lambda \tau (\mu - \mu_0)^2}{2}\right]$$

$$p(\mu, \tau) \propto \tau^{a_0 - \frac{1}{2}} \exp[-b_0 \tau] \exp\left[-\frac{\lambda \tau (\mu - \mu_0)^2}{2}\right]$$

According to Bayesian theorem, we have

$$p(\mu, \tau \mid \mathcal{D}) \propto p(\mathcal{D} \mid \mu, \tau) p(\mu, \tau),$$

where $p(\mathcal{D} \mid \mu, \tau)$ is the likelihood of the data points which have such relationship:

$$\begin{aligned}
 p(\mathcal{D} \mid \mu, \tau) &= \prod_{i=1}^N p(x_i \mid \mu, \tau) \\
 &\propto \prod_{i=1}^N \tau^{1/2} \exp\left[-\frac{\tau}{2}(x_i - \mu)^2\right] \\
 &\propto \tau^{N/2} \exp\left[-\frac{\tau}{2} \sum_{i=1}^N (x_i - \mu)^2\right] \\
 &\propto \tau^{N/2} \exp\left[-\frac{\tau}{2} \sum_{i=1}^N (x_i - \bar{x} + \bar{x} - \mu)^2\right] \\
 &\propto \tau^{N/2} \exp\left[-\frac{\tau}{2} \sum_{i=1}^N ((x_i - \bar{x})^2 + (\bar{x} - \mu)^2)\right] \\
 &\propto \tau^{N/2} \exp\left[-\frac{\tau}{2} (Ns + N(\bar{x} - \mu)^2)\right],
 \end{aligned}$$

where $\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$ and $s = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2$. Thus the posterior can be written into [2]

$$\begin{aligned}
 p(\mu, \tau \mid \mathcal{D}) &\propto p(\mathcal{D} \mid \mu, \tau) p(\mu, \tau) \\
 &\propto \tau^{N/2} \exp\left[-\frac{\tau}{2} (Ns + N(\bar{x} - \mu)^2)\right] \tau^{a_0 - \frac{1}{2}} \exp[-b_0 \tau] \exp\left[-\frac{\lambda \tau (\mu - \mu_0)^2}{2}\right] \\
 &\propto \tau^{\frac{N}{2} + a_0 - \frac{1}{2}} \exp\left[-\tau \left(\frac{1}{2} Ns + b_0\right)\right] \exp\left[-\frac{\tau}{2} (\lambda_0 (\mu - \mu_0)^2 + N(\bar{x} - \mu)^2)\right].
 \end{aligned}$$

The exponent (except the $-\frac{\tau}{2}$) in the last exponential term can be simplified into

$$\begin{aligned}
\lambda_0(\mu - \mu_0)^2 + N(\bar{x} - \mu)^2 &= \lambda_0\mu^2 - 2\lambda_0\mu\mu_0 + \lambda_0\mu_0^2 + N\mu^2 - 2N\bar{x}\mu + N\bar{x}^2 \\
&= (\lambda_0 + N)\mu^2 - 2(\lambda_0\mu_0 + N\bar{x})\mu + \lambda_0\mu_0^2 + N\bar{x}^2 \\
&= (\lambda_0 + N) \left(\mu^2 - 2\frac{\lambda_0\mu_0 + N\bar{x}}{\lambda_0 + N}\mu \right) + \lambda_0\mu_0^2 + N\bar{x}^2 \\
&= (\lambda_0 + N) \left(\mu - \frac{\lambda_0\mu_0 + N\bar{x}}{\lambda_0 + N} \right)^2 + \lambda_0\mu_0^2 + N\bar{x}^2 - \frac{(\lambda_0\mu_0 + N\bar{x})^2}{\lambda_0 + N} \\
&= (\lambda_0 + N) \left(\mu - \frac{\lambda_0\mu_0 + N\bar{x}}{\lambda_0 + N} \right)^2 + \frac{\lambda_0N(\bar{x} - \mu_0)^2}{\lambda_0 + N}.
\end{aligned}$$

Insert the above equation into the posterior distribution, we have

$$\begin{aligned}
p(\mu, \tau \mid \mathcal{D}) &\propto \tau^{\frac{N}{2} + a_0 - \frac{1}{2}} \exp \left[-\tau \left(\frac{1}{2}Ns + b_0 \right) \right] \exp \left[-\frac{\tau}{2} \left((\lambda_0 + N) \left(\mu - \frac{\lambda_0\mu_0 + N\bar{x}}{\lambda_0 + N} \right)^2 + \frac{\lambda_0N(\bar{x} - \mu_0)^2}{\lambda_0 + N} \right) \right] \\
&\propto \tau^{\frac{N}{2} + a_0 - \frac{1}{2}} \exp \left[-\tau \left(\frac{1}{2}Ns + b_0 + \frac{\lambda_0N(\bar{x} - \mu_0)^2}{2(\lambda_0 + N)} \right) \right] \exp \left[-\frac{\tau}{2}(\lambda_0 + N) \left(\mu - \frac{\lambda_0\mu_0 + N\bar{x}}{\lambda_0 + N} \right)^2 \right],
\end{aligned}$$

which is in the form of a Normal-Gamma distribution, thus the exact posterior is

$$\begin{aligned}
p(\mu, \tau \mid \mathcal{D}) &= \text{NormalGamma}(\mu', \lambda', a', b') \\
\mu' &= \frac{\lambda_0\mu_0 + N\bar{x}}{\lambda_0 + N} \\
\lambda' &= \lambda_0 + N \\
a' &= a_0 + \frac{N}{2} \\
b' &= b_0 + \frac{1}{2} \left(Ns + \frac{\lambda_0N(\bar{x} - \mu_0)^2}{\lambda_0 + N} \right).
\end{aligned}$$

Question 2.4.14: Compare the inferred variational distribution with the exact posterior. Run the inference on data points drawn from iid Gaussians. Do this for three interesting cases and visualize the results. Describe the differences.

1. In this case, the number of data set is $N = 10$ and the initial values are $b_N = 5$ and $\lambda_N = 5$. The result is shown in fig 1, where the true posterior distribution is shown by the colorful contour lines, while the approximated distribution given by the VI algorithm is shown in blue contour lines and red contour lines when reached convergence.

Although we are using zero for parameters in the prior distributions, so that $\mu_0 = \lambda_0 = a_0 = b_0 = 0$, which implies that we have no information about the distribution, the convergence is really quick with VI algorithm.

And not only from this case, but also from the following cases, we can see that the mean of the μ is always correct, which is true since we get value μ_N at the beginning and don't update it during the iterations.

2. In this case, the number of data set is $N = 10$ and the initial values are $b_N = 0.1$ and $\lambda_N = 0.1$. The result is illustrated in fig 2.

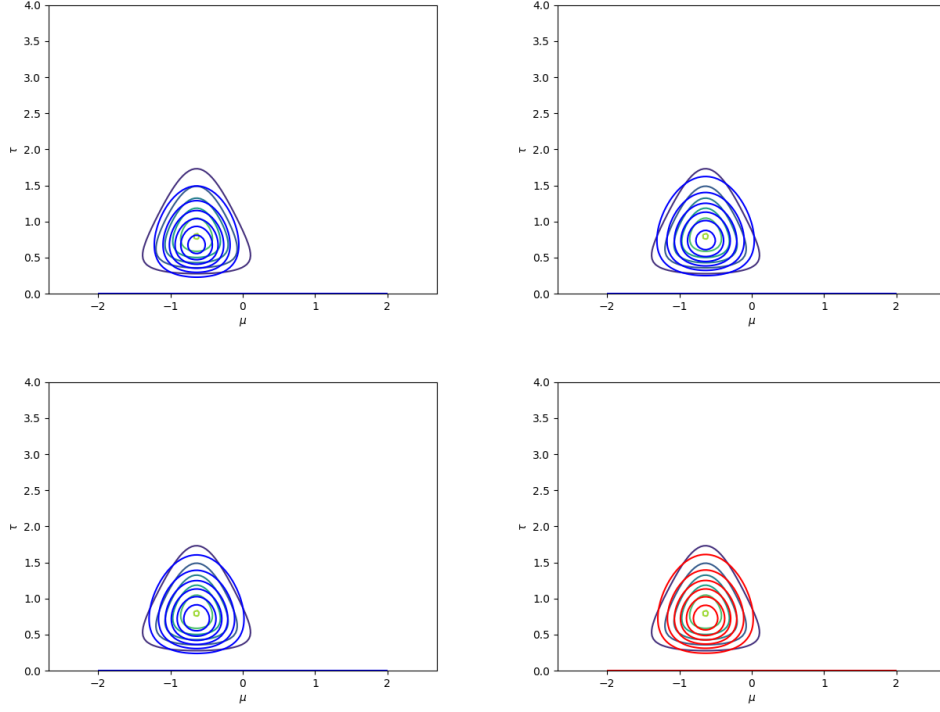


Figure 1: Simple VI with $\mu_0 = \lambda_0 = a_0 = b_0 = 0$, $b_N = 5$, and $\lambda_N = 5$.

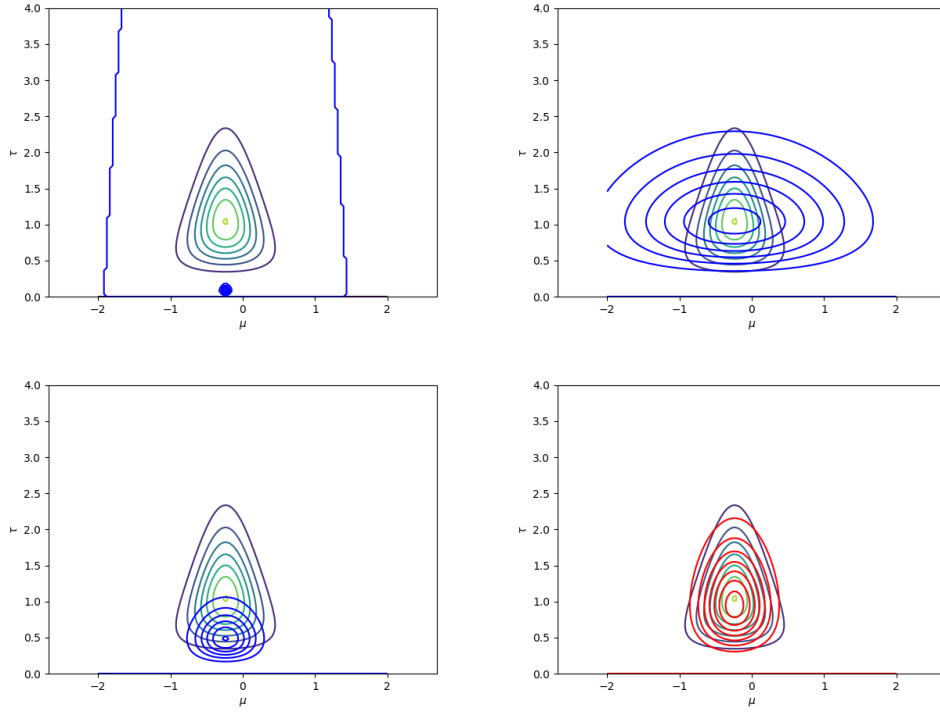


Figure 2: Simple VI with $\mu_0 = \lambda_0 = a_0 = b_0 = 0$, $b_N = 0.1$, and $\lambda_N = 0.1$.

By setting the value of the initial values b_N and λ_N to smaller values which are much different to the true values in the true posterior distribution, the converging procedure is slower and requires more iterations to reach the convergence.

3. In this case, the number of data set is $N = 100$ and the initial values are $b_N = 5$ and $\lambda_N = 5$. The result is illustrated in fig 3.

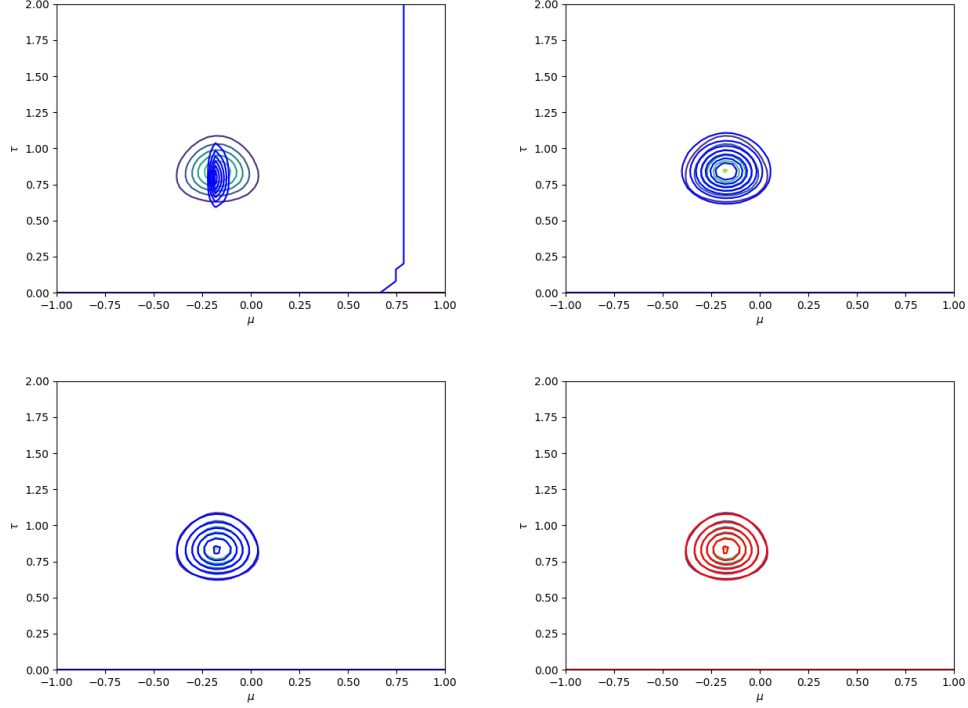


Figure 3: Simple VI with $\mu_0 = \lambda_0 = a_0 = b_0 = 0$, $b_N = 5$, and $\lambda_N = 5$.

In this case, we use 100 data sets generated from a normal distribution. The true posterior distribution for μ and τ looks more like a circular shape. And since we have more data sets, we would be confident to the result and the convergence is quick at the same time.

5 Mixture of trees with observable variables

Question 2.5.15: *Implement this EM algorithm.*

For each n, k the responsibilities is

$$\begin{aligned} r_{n,k} &= \frac{\pi_k p(x^n | T_k, \Theta_k)}{p(x^n)} \\ &= \frac{\pi_k p(x^n | T_k, \Theta_k)}{\sum_{k=1}^K \pi_k p(x^n | T_k, \Theta_k)}. \end{aligned}$$

The algorithm is implemented below, where the function in *Kruskal_v1.py* is slightly modified so that the result of the maximum spanning tree can be returned:


```

from Kruskal_v1 import Graph
2 from Tree import TreeMixture, Tree, Node

import numpy as np
import sys

7 epsilon = sys.float_info.epsilon

def em_algorithm(seed_val, samples, num_clusters, max_num_iter=100):
    print("Running EM algorithm...")

12     # Set threshold for convergence
    THRES = 1e-4

    # Set rounds for sieving
    num_sieving = 10

17     # Get the dimension of the data
    num_samples = np.size(samples, 0)
    num_nodes = np.size(samples, 1)

22     # Sieving
    np.random.seed(seed_val)
    seeds = np.random.randint(0, 100000000, num_sieving)
    last_loglikelihoods = []
    tms = []
27     for seed in seeds:
        np.random.seed(seed)
        tm = TreeMixture(num_clusters=num_clusters, num_nodes=num_nodes)
        tm.simulate_pi(seed_val=seed)
        tm.simulate_trees(seed_val=seed)
32         tm_loglikelihood, tm = em_helper(tm, samples, num_clusters, max_num_iter=10)
        last_loglikelihoods.append(tm_loglikelihood[-1])
        tms.append(tm)

    # Main procedure for EM algorithm
37     print("=> Sieving finished")
    seed = seeds[last_loglikelihoods.index(max(last_loglikelihoods))]
    tm = TreeMixture(num_clusters=num_clusters, num_nodes=num_nodes)
    tm.simulate_pi(seed_val=seed)
    tm.simulate_trees(seed_val=seed)
42     loglikelihood, tm = em_helper(tm, samples, num_clusters, max_num_iter=
max_num_iter)

    print("=> EM finished")
    topology_list = []
    theta_list = []
47     for t in tm.clusters:
        topology_list.append(t.get_topology_array())
        theta_list.append(t.get_theta_array())
        loglikelihood = np.array(loglikelihood)
        topology_list = np.array(topology_list)
52     theta_list = np.array(theta_list)

    return loglikelihood, topology_list, theta_list, tm

def em_helper(tm, samples, num_clusters, max_num_iter=10):
57     num_samples = np.size(samples, 0)

```

```

num_nodes = np.size(samples, 1)

loglikelihood = []
for iter in range(max_num_iter):
62     print("===== "+str(iter)+"-th iteration =====")
)

# Step 1: Compute the responsibilities
r = np.ones((num_samples, num_clusters))

67     for n, x in enumerate(samples):
        for k, t in enumerate(tm.clusters):
            r[n,k] *= tm.pi[k]
            visit_list = [t.root]
            while len(visit_list) is not 0:
                cur_node = visit_list[0]
72                 visit_list = visit_list[1:]
                visit_list = visit_list + cur_node.descendants
                if cur_node.ancestor is None:
                    r[n,k] *= cur_node.cat[x[int(cur_node.name)]]
                else:
77                     r[n,k] *= cur_node.cat[x[int(cur_node.ancestor.name)]] [x[int
(cur_node.name)]]

            r += epsilon
            marginal = np.reshape(np.sum(r, axis=1), (num_samples, 1))
            loglikelihood.append(np.sum(np.log(marginal)))
82            marginal_expand = np.repeat(marginal, num_clusters, axis=1)
            r /= marginal_expand

# Step 2: Update categorical distribution
tm.pi = np.mean(r, axis=0)

87

# Step 3: Construct directed graphs
denom = np.sum(r, axis=0)
q = np.zeros((num_nodes, num_nodes, 2, 2, num_clusters)) # (s, t, a, b, k)
92     for s in range(num_nodes):
        for t in range(num_nodes):
            for a in range(2):
                for b in range(2):
                    index = np.where((samples[:,(s,t)]==[a,b]).all(1))[0]
                    number = np.sum(r[index], axis=0)
97                     q[s, t, a, b] = number / denom

q += epsilon

q_s = np.zeros((num_nodes, 2, num_clusters))
102     for s in range(num_nodes):
        for a in range(2):
            index = np.where(samples[:, s]==a)
            number = np.sum(r[index], axis=0)
            q_s[s,a] = number / denom
q_s += epsilon

107

I = np.zeros((num_nodes, num_nodes, num_clusters)) # (s, t, k)
for s in range(num_nodes):
    for t in range(num_nodes):
        for a in range(2):
            for b in range(2):
112                 I[s,t] += q[s,t,a,b] * np.log(q[s,t,a,b] / q_s[s,a] / q_s[t,
b])

```

```

clusters = []
for k in range(num_clusters):
    g = Graph(num_nodes)
    for s in range(num_nodes):
        for t in range(s+1, num_nodes):
            g.addEdge(s, t, I[s, t, k])

# Step 4: Construct maximum spanning trees
edges = np.array(g.maximum_spanning_tree())[ :, 0:2]
topology_array = np.zeros(num_nodes)
topology_array[0] = np.nan
visit_list = [0]
while len(visit_list) != 0:
    cur_node = visit_list[0]
    index = np.where(edges==cur_node)
    index = np.transpose(np.stack(index))
    visit_list = visit_list[1:]
    for id in index:
        child = edges[id[0], 1-id[1]]
        topology_array[int(child)] = cur_node
        visit_list.append(int(child))
    if np.size(index) is not 0:
        edges = np.delete(edges, index[:,0], axis=0)

tree = Tree()
tree.load_tree_from_direct_arrays(topology_array)
tree.k = 2
tree.alpha = [1.0] * 2

# Step 5: Update CPDs
visit_list = [tree.root]
while len(visit_list) != 0:
    cur_node = visit_list[0]
    visit_list = visit_list[1:]
    visit_list = visit_list + cur_node.descendants
    if cur_node.ancestor is None:
        cur_node.cat = q_s[int(cur_node.name), :, k].tolist()
    else:
        cat = q[int(cur_node.ancestor.name), int(cur_node.name), :, k]
        cur_node.cat = [cat[0].tolist(), cat[1].tolist()]

    clusters.append(tree)
tm.clusters = clusters

return loglikelihood, tm

```

Listing 2: EM algorithm for mixture of trees with observable variables.

Question 2.5.16: *Apply your algorithm to the provided data and show how well you reconstruct the mixtures. First, compare the real and inferred trees with the unweighted Robinson-Foulds (aka symmetric difference) metric. Do the trees have similar structure (don't worry if the inferred trees don't match with the real trees)? Then, compare the likelihoods of real and inferred mixtures. Finally, simulate more data and analyze the results (try to find some interesting and more challenging cases).*

- **q_2_5_tm_10node_20sample_4clusters:** The result of the EM algorithm is illustrated in

fig 4.

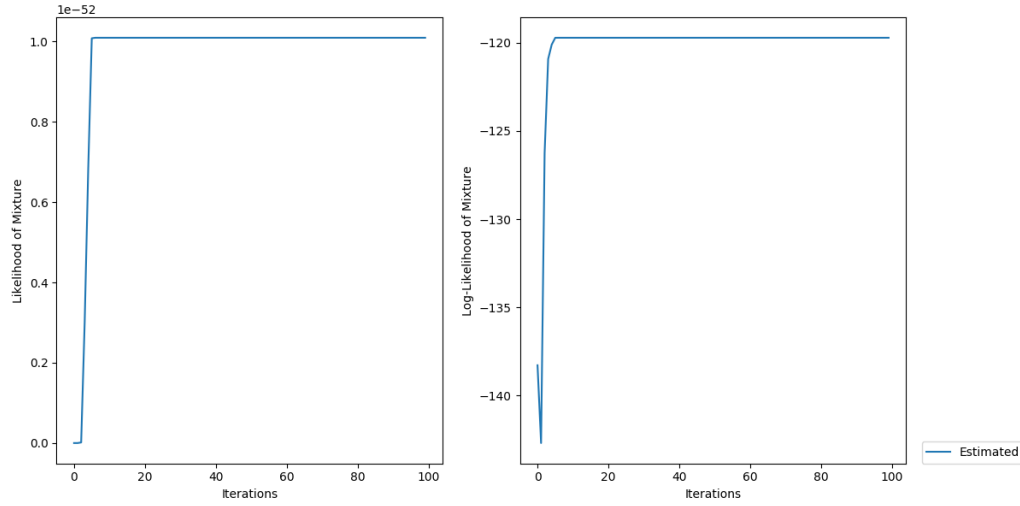


Figure 4: The likelihood and log-likelihood at each iteration of the EM algorithm for data *q_2_5-tm_10node_20sample_4clusters*.

The comparison of the real and inferred trees with unweighted RF metric is show in tab 1, from which we can see that the trees don't have similar structures which can caused by small number of samples.

Table 1: RF distance between the true tree mixture and the result obtained from the EM algorithm for data *q_2_5-tm_10node_20sample_4clusters*, where the rows represents the obtained tree mixture and the columns represents the true tree mixture.

	0	1	2	3
0	9	8	10	10
1	9	12	10	10
2	8	7	9	9
3	8	11	11	9

The log-likelihood of the result tree mixture obtained by the EM algorithm is -119.724756 , while the log-likelihood of the ground truth is -113.143100 .

Also, in the next question, 1000 data points would be sampled from the tree mixture to see what would happen with the result of the EM algorithm.

- ***q_2_5-tm_10node_50sample_4clusters***: The result of the EM algorithm is illustrated in fig 5.

The comparison of the real and inferred trees with unweighted RF metric is show in tab 2. Still the trees don't have similar structures but the distance looks more average than the one with only 20 samples.

The log-likelihood of the result tree mixture obtained by the EM algorithm is -416.226204 , while the log-likelihood of the ground truth is -280.856624 .

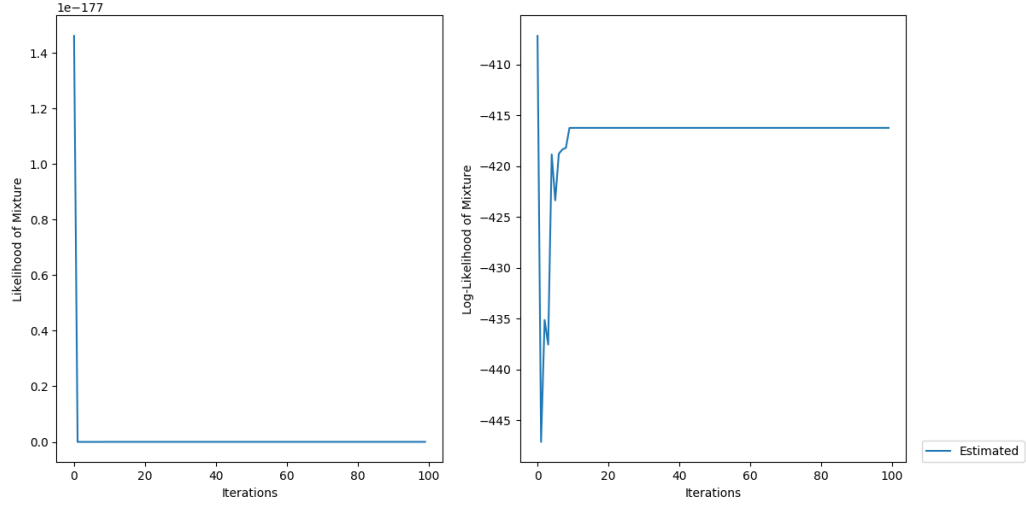


Figure 5: The likelihood and log-likelihood at each iteration of the EM algorithm for data *q_2_5_tm_10node_50sample_4clusters*.

Table 2: RF distance between the true tree mixture and the result obtained from the EM algorithm for data *q_2_5_tm_10node_50sample_4clusters*, where the rows represents the obtained tree mixture and the columns represents the true tree mixture.

	0	1	2	3
0	10	9	11	9
1	11	8	10	10
2	7	10	10	8
3	10	9	9	11

- ***q_2_5_tm_20node_20sample_4clusters***: The result of the EM algorithm is illustrated in fig 6.

The comparison of the real and inferred trees with unweighted RF metric is show in tab 3, from which we can see that the trees don't have similar structures which can caused by small number of samples and the result is even worse than the one with 10 nodes due to the number of nodes is bigger, thus making the inference much harder and making it more difficult to have similar structures.

Table 3: RF distance between the true tree mixture and the result obtained from the EM algorithm for data *q_2_5_tm_20node_20sample_4clusters*, where the rows represents the obtained tree mixture and the columns represents the true tree mixture.

	0	1	2	3
0	19	19	19	20
1	19	19	19	22
2	19	15	17	18
3	18	20	24	21

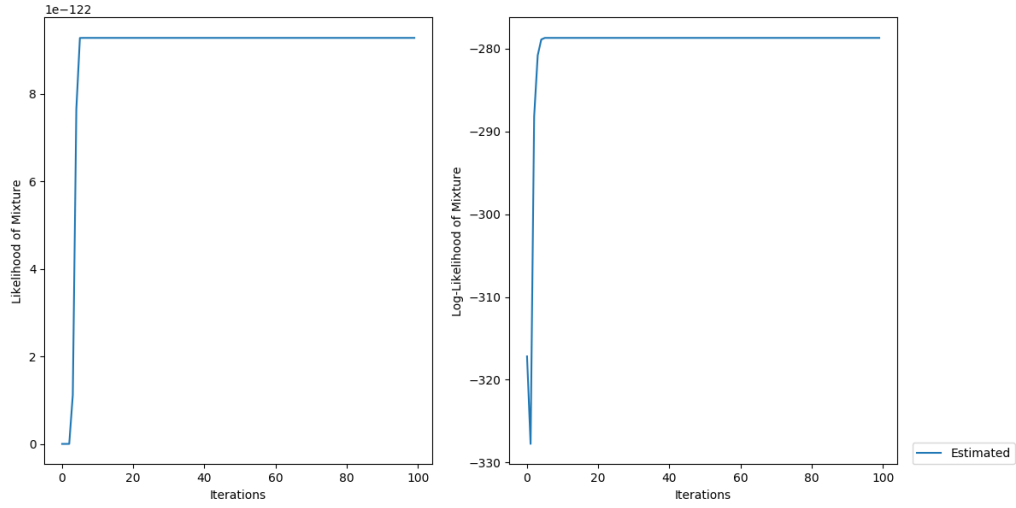


Figure 6: The likelihood and log-likelihood at each iteration of the EM algorithm for data *q_2_5-tm_20node_20sample_4clusters*.

The log-likelihood of the result tree mixture obtained by the EM algorithm is -278.687237 , while the log-likelihood of the ground truth is -217.005755 .

Question 2.5.17: *Simulate new tree mixtures with different number of nodes, samples and clusters. Try to find some interesting cases. Analyze your results as in the previous question.*

By simulating a tree mixture with only 3 nodes and 10 clusters and sample 1000 data points from the tree mixture, the result is shown in fig 7.

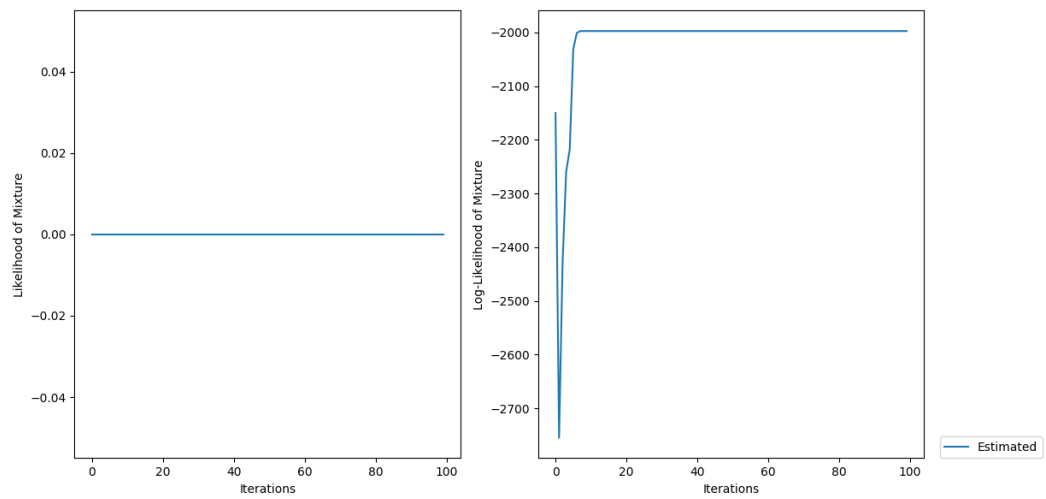


Figure 7: The likelihood and log-likelihood at each iteration of the EM algorithm for tree mixture with 3 nodes and 10 clusters.

The comparison of the real and inferred trees with unweighted RF metric is show in tab 4, from which we can see that the trees have similar structures because the number of nodes is only three which can be easily inferred.

Table 4: RF distance between the true tree mixture and the result obtained from the EM algorithm for tree mixture with 3 nodes and 10 clusters, where the rows represents the obtained tree mixture and the columns represents the true tree mixture.

	0	1	2	3	4	5	6	7	8	9
0	2	2	0	0	2	0	2	2	0	2
1	2	2	0	0	2	0	2	2	0	2
2	2	2	0	0	2	0	2	2	0	2
3	0	0	2	2	0	2	0	0	2	0
4	2	2	0	0	2	0	2	2	0	2
5	0	0	2	2	0	2	0	0	2	0
6	2	2	0	0	2	0	2	2	0	2
7	0	0	2	2	0	2	0	0	2	0
8	0	0	2	2	0	2	0	0	2	0
9	2	2	0	0	2	0	2	2	0	2

The log-likelihood of the result tree mixture obtained by the EM algorithm is -1997.297679 , while the log-likelihood of the ground truth is -2003.252791 .

Another example simulating a tree mixture with 10 nodes and 10 clusters, and sample 50 data points from the tree mixture (notice that it is similar to *q_2_5_tm_10node_50sample_4clusters* while the only difference is the number of clusters), the result of the EM algorithm is illustrated in fig 8.

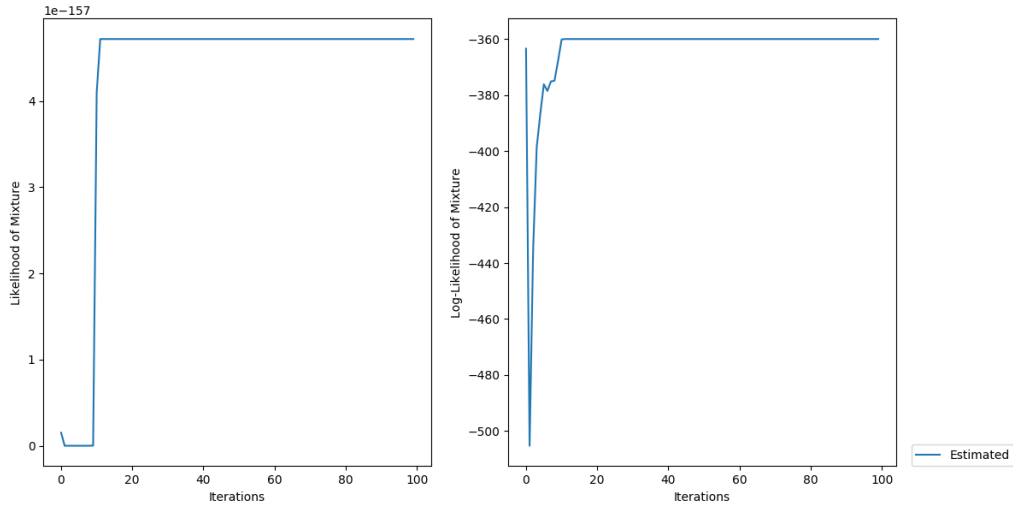


Figure 8: The likelihood and log-likelihood at each iteration of the EM algorithm for tree mixture with 10 nodes and 10 clusters.

The comparison of the real and inferred trees with unweighted RF metric is show in tab 5, from which we can see that the trees have more similar structures when compared to the case of

q_2_5_tm_10node_50sample_4clusters.

Table 5: RF distance between the true tree mixture and the result obtained from the EM algorithm for tree mixture with 10 nodes and 10 clusters, where the rows represents the obtained tree mixture and the columns represents the true tree mixture.

	0	1	2	3	4	5	6	7	8	9
0	9	10	8	7	7	9	5	11	9	6
1	10	9	7	12	10	6	8	10	8	9
2	11	8	8	11	9	7	7	11	9	8
3	10	9	7	10	8	10	6	12	8	7
4	9	8	8	9	9	7	9	9	9	8
5	10	9	9	10	12	10	10	10	10	11
6	9	10	8	7	9	9	7	11	9	8
7	10	7	7	8	8	8	8	10	8	7
8	9	8	10	9	11	11	9	9	11	10
9	11	8	8	9	9	7	9	11	9	8

The log-likelihood of the result tree mixture obtained by the EM algorithm is -359.954244 , while the log-likelihood of the ground truth is -323.847884 . Which are relatively similar when compare to the case of *q_2_5_tm_10node_50sample_4clusters*.

6 Super epicentra - EM

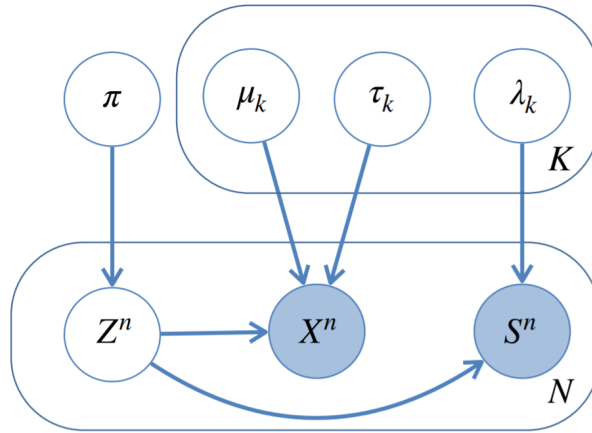


Figure 9: Mixture of components modeling location and strengths of earthquakes associated with a super-epicentra. In the figure, $\mu_k = (\mu_{k,1}, \mu_{k,2})$ and $\tau_k = (\tau_{k,1}, \tau_{k,2})$.

Question 2.6.18: *Derive an EM algorithm for the model.*

The conditional distribution of X^n, S^n given a particular value for Z^n is

$$\begin{aligned} p(X^n, S^n | Z_k^n = 1) &= \mathcal{N}(X^n | \mu_k, \tau_k) \text{Poisson}(S^n | \lambda_k) \\ &= \frac{\sqrt{\tau_{k,1}\tau_{k,2}}}{2\pi} \exp \left[-\frac{\tau_{k,1}}{2}(X_1^n - \mu_{k,1})^2 - \frac{\tau_{k,2}}{2}(X_2^n - \mu_{k,2})^2 \right] \frac{\lambda_k^{S^n}}{S^n!} \exp[-\lambda_k] \\ &= \frac{\sqrt{\tau_{k,1}\tau_{k,2}}}{2\pi} \frac{\lambda_k^{S^n}}{S^n!} \exp \left[-\frac{\tau_{k,1}}{2}(X_1^n - \mu_{k,1})^2 - \frac{\tau_{k,2}}{2}(X_2^n - \mu_{k,2})^2 - \lambda_k \right] \end{aligned}$$

which can also be written in the form

$$p(X^n, S^n | Z^n) = \prod_{k=1}^K p(X^n, S^n | Z_k^n = 1)^{Z_k^n}.$$

The joint distribution is given by

$$p(X^n, S^n, Z^n) = p(Z^n) p(X^n, S^n | Z^n),$$

and the marginal distribution of X^n, S^n is then obtained by summing the joint distribution over all possible states of Z^n to give

$$p(X^n, S^n) = \sum_{k=1}^K \pi_k p(X^n, S^n | Z_k^n = 1).$$

Another quantity that will play an important role is the conditional probability of Z^n given X^n, S^n . We shall use $\gamma(Z_k^n)$ to denote $p(Z_k^n = 1 | X^n, S^n)$, whose value can be found using Bayes' theorem

$$\begin{aligned} \gamma(Z_k^n) \equiv p(Z_k^n = 1 | X^n, S^n) &= \frac{p(Z_k^n = 1) p(X^n, S^n | Z_k^n = 1)}{\sum_{j=1}^K p(Z_j^n = 1) p(X^n, S^n | Z_j^n = 1)} \\ &= \frac{\pi_k p(X^n, S^n | Z_k^n = 1)}{\sum_{j=1}^K \pi_j p(X^n, S^n | Z_j^n = 1)}. \end{aligned}$$

The M-step in general EM algorithm is to evaluate θ^{new} given by

$$\theta^{\text{new}} = \arg_{\theta} \max Q(\theta, \theta^{\text{old}})$$

where

$$\begin{aligned} Q(\theta, \theta^{\text{old}}) &= \sum_{\mathbf{Z}} p(\mathbf{Z} | \mathbf{X}, \mathbf{S}, \theta^{\text{old}}) \ln p(\mathbf{X}, \mathbf{S}, \mathbf{Z} | \theta) \\ &= \sum_{\mathbf{Z}} p(\mathbf{Z} | \mathbf{X}, \mathbf{S}, \theta^{\text{old}}) \ln p(\mathbf{Z} | \theta) p(\mathbf{X}, \mathbf{S} | \mathbf{Z}, \theta) \end{aligned}$$

Setting the derivatives of $p(\mathbf{Z} | \mathbf{X}, \mathbf{S}, \theta^{\text{old}}) \ln p(\mathbf{X}, \mathbf{S}, \mathbf{Z} | \theta)$ with respect to θ to zero, we have the

updated parameters given by

$$\begin{aligned}\mu_k^{\text{new}} &= \frac{1}{N_k} \sum_{n=1}^N \gamma(Z_k^n) X^n \\ \tau_{k,1}^{\text{new}} &= \frac{N_k}{\sum_{n=1}^N \gamma(Z_k^n) (X_1^n - \mu_{k,1}^{\text{new}})^2} \\ \tau_{k,2}^{\text{new}} &= \frac{N_k}{\sum_{n=1}^N \gamma(Z_k^n) (X_2^n - \mu_{k,2}^{\text{new}})^2} \\ \lambda_k^{\text{new}} &= \frac{1}{N_k} \sum_{n=1}^N \gamma(Z_k^n) S^n \\ \pi_k^{\text{new}} &= \frac{N_k}{N}.\end{aligned}$$

EM algorithm:

1. Initialize the means μ_k , precision τ_k , Poisson parameter λ_k and mixing coefficients π_k , and evaluate the initial value of the log likelihood.
2. **E Step.** Evaluate the responsibilities using the current parameter values

$$\gamma(Z_k^n) = \frac{\pi_k p(X^n, S^n | Z_k^n = 1)}{\sum_{j=1}^K \pi_j p(X^n, S^n | Z_j^n = 1)}.$$

3. **M Step.** Re-estimate the parameters using the current responsibilities

$$\begin{aligned}\mu_k^{\text{new}} &= \frac{1}{N_k} \sum_{n=1}^N \gamma(Z_k^n) X^n \\ \tau_{k,1}^{\text{new}} &= \frac{N_k}{\sum_{n=1}^N \gamma(Z_k^n) (X_1^n - \mu_{k,1}^{\text{new}})^2} \\ \tau_{k,2}^{\text{new}} &= \frac{N_k}{\sum_{n=1}^N \gamma(Z_k^n) (X_2^n - \mu_{k,2}^{\text{new}})^2} \\ \lambda_k^{\text{new}} &= \frac{1}{N_k} \sum_{n=1}^N \gamma(Z_k^n) S^n \\ \pi_k^{\text{new}} &= \frac{N_k}{N}\end{aligned}$$

where

$$N_k = \sum_{n=1}^N \gamma(Z_k^n).$$

4. Evaluate the log likelihood

$$\ln p(\mathbf{X}, \mathbf{S} | \boldsymbol{\mu}, \boldsymbol{\tau}, \boldsymbol{\lambda}, \boldsymbol{\pi}) = \sum_{n=1}^N \ln \left\{ \sum_{k=1}^K \pi_k \mathcal{N}(X^n | \mu_k, \tau_k) \text{Poisson}(S^n | \lambda_k) \right\}$$

and check for convergence of either the parameters or the log likelihood. If the convergence criterion is not satisfied return to step 2.

Question 2.6.19: *Implement your EM algorithm.*

```

def _do_estep(self, X, S):
    """
    E-step
    """
    r = np.zeros((self.n_row, self.n_components)) # r[n,k]
    for n in range(self.n_row):
        for k in range(self.n_components):
            r[n,k] = self.weights[k] \
                * multivariate_normal(self.means[k], self.covs[k]).pdf(X[n]) \
                * poisson(self.rates[k]).pmf(S[n])

    r += epsilon
    marginal = np.sum(r, axis=1).reshape((self.n_row, 1))
    marginal = np.repeat(marginal, self.n_components, axis=1)
    r /= marginal

    self.r = r
    return self

def _do_mstep(self, X, S):
    """M-step, update parameters"""
    N = np.sum(self.r, axis=0) # N[k] = Nk
    N_expand = np.repeat(N.reshape((self.n_components, 1)), self.n_col, axis=1)
    self.means = np.dot(np.transpose(self.r), X) / N_expand

    for k in range(self.n_components):
        mean_expand = np.repeat(np.atleast_2d(self.means[k]), self.n_row, axis=0)
        r_expand = np.repeat(self.r[:,k].reshape((self.n_row, 1)), self.n_col, axis=1)
        variance = np.sum(r_expand * (X - mean_expand)**2, axis=0) / N[k]
        self.covs[k] = np.diag(variance)

    self.rates = np.dot(S, self.r) / N

    self.weights = N / self.n_row

    return self

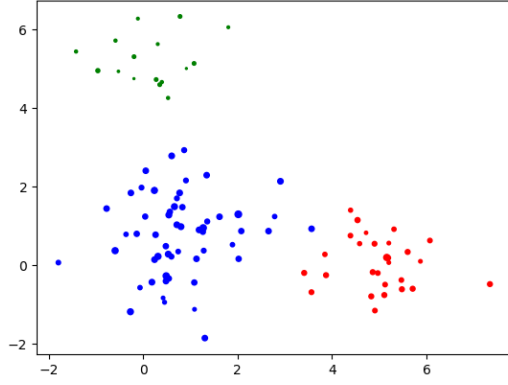
def _compute_log_likelihood(self, X, S):
    """compute the log likelihood of the current parameter"""
    log_likelihood = 0
    for n in range(self.n_col):
        likelihood = 1
        for k in range(self.n_components):
            likelihood *= self.weights[k] \
                * multivariate_normal(self.means[k], self.covs[k]).pdf(X[n]) \
                * poisson(self.rates[k]).pmf(S[n])
        log_likelihood += np.log(likelihood)

    return log_likelihood

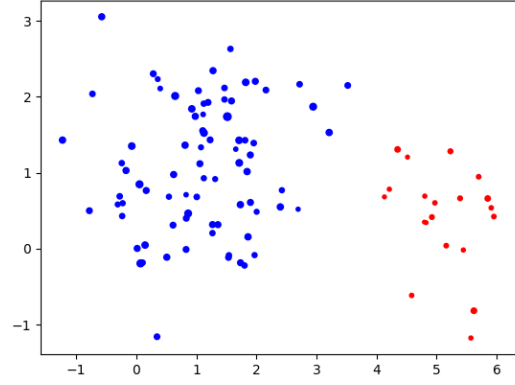
```

Listing 3: EM algorithm for super epicentra.

Question 2.6.20: *Apply it to the data provided separately, give an account of the success, and provide visualizations for a couple of examples.*



(a) Data set 1.



(b) Data set 2.

Figure 10: Ground truth for data sets

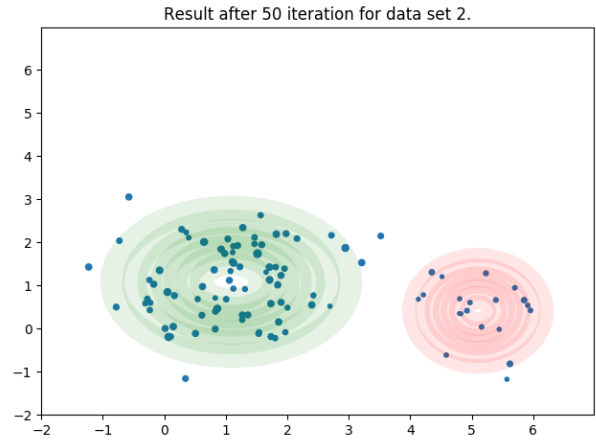
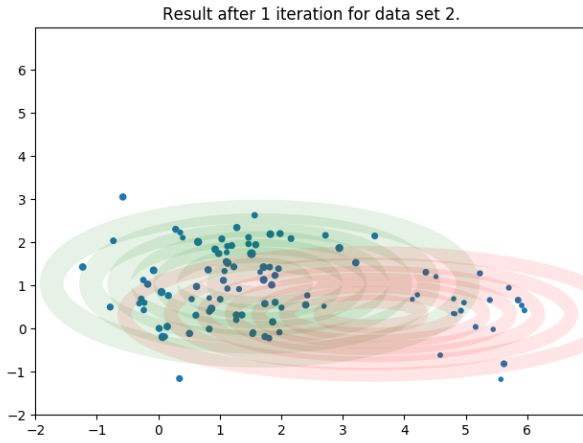
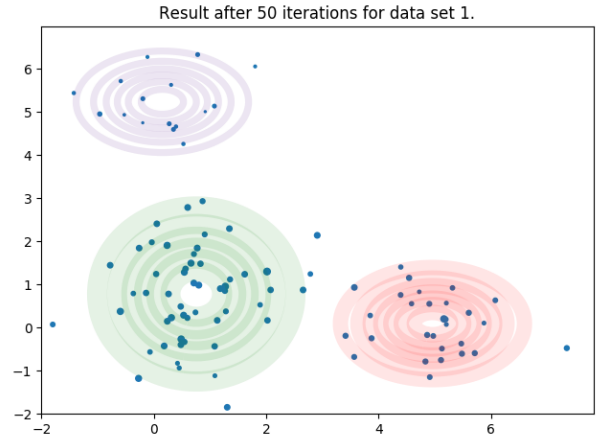
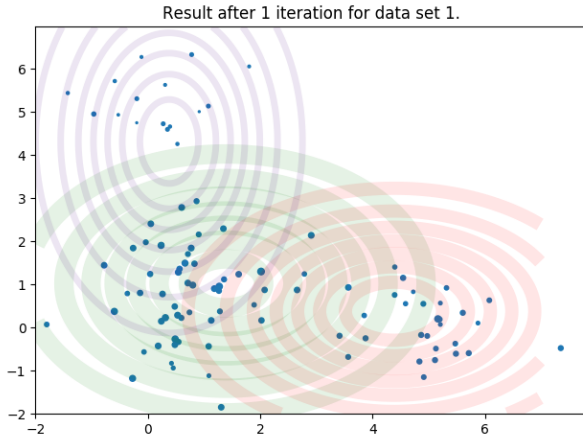


Figure 11: Result of the EM algorithm for different data sets.

The ground truth of the data is illustrated in fig 10 and the inference result from EM algorithm

is illustrated in fig 11. From those two figures, we can tell that the result of EM algorithm gives good inference of the means and covariances for each Gaussian model, and ratios for each Poisson model.

7 Super epicentra - VI

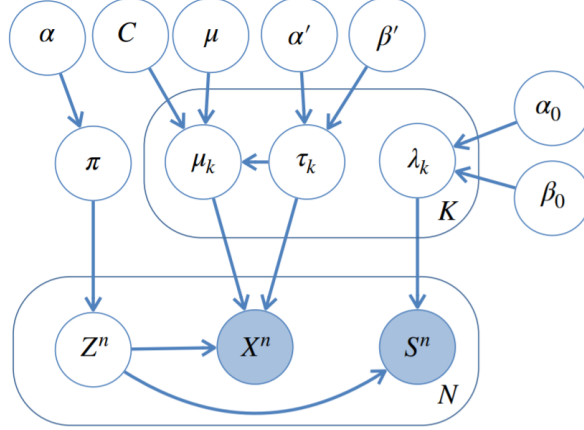


Figure 12: The K super epicentra model with priors.

Question 2.7.21: *Derive a VI algorithm that estimates the posterior distribution for this model.*

The VI algorithm is derived based on [1] and lecture notes.

Since $\pi \sim \text{Dir}(\alpha)$, we have

$$\begin{aligned} p(\pi|\alpha) &= p(\pi_1, \dots, \pi_K | \alpha_1, \dots, \alpha_K) \\ &= \frac{\Gamma\left(\sum_{k=1}^K \alpha_k\right)}{\prod_{k=1}^K \Gamma(\alpha_k)} \prod_{k=1}^K \pi_k^{\alpha_k - 1} \\ &= \frac{1}{B(\alpha)} \prod_{k=1}^K \pi_k^{\alpha_k - 1}. \end{aligned}$$

Also, for other latent variables, we have

$$\begin{aligned} p(\mu_k | \tau_k) &= \mathcal{N}(\mu_k | \mu, (C\tau_k)^{-1}) \\ p(\tau_k) &= \text{Gam}(\tau_k | \alpha', \beta') \\ p(\lambda_k) &= \text{Gam}(\lambda_k | \alpha_0, \beta_0). \end{aligned}$$

The joint distribution of all of the random variables is given by

$$p(\mathbf{X}, \mathbf{S}, \mathbf{Z}, \pi, \mu, \tau, \lambda) = p(\mathbf{X} | \mathbf{Z}, \mu, \tau) p(\mathbf{S} | \mathbf{Z}, \lambda) p(\mathbf{Z} | \pi) p(\pi) p(\mu | \tau) p(\tau) p(\lambda).$$

A variational distribution which factorizes between the latent variables and the parameters is given by

$$q(\mathbf{Z}, \pi, \mu, \tau, \lambda) = q(\mathbf{Z}) q(\pi, \mu, \tau, \lambda).$$

The log of the optimized factor is then given by

$$\begin{aligned}
\ln q^*(\mathbf{Z}) &= \mathbb{E}_{\pi, \mu, \tau, \lambda} [\ln p(\mathbf{X}, \mathbf{S}, \mathbf{Z}, \pi, \mu, \tau, \lambda)] + \text{const} \\
&= \mathbb{E}_{\pi} [\ln p(\mathbf{Z}|\pi)] + \mathbb{E}_{\mu, \tau} [\ln p(\mathbf{X}|\mathbf{Z}, \mu, \tau)] + \mathbb{E}_{\lambda} [\ln p(\mathbf{S}|\mathbf{Z}, \lambda)] + \text{const} \\
&= \sum_{n=1}^N \sum_{k=1}^K Z_k^n \ln \rho_{nk} + \text{const}
\end{aligned}$$

where

$$\begin{aligned}
\ln \rho_{nk} &= \mathbb{E} [\ln \pi_k] + \frac{1}{2} \mathbb{E} [\ln |\tau_k|] - \ln(2\pi) \\
&\quad - \frac{1}{2} \mathbb{E}_{\mu_k, \tau_k} [(X^n - \mu_k)^T \tau_k (X^n - \mu_k)] \\
&\quad + S^n \mathbb{E} [\ln \lambda_k] - \mathbb{E} [\lambda_k] - \ln(S^n!).
\end{aligned}$$

Taking the exponential of both sides of the equation and normalize it, we obtain

$$q^*(\mathbf{Z}) = \prod_{n=1}^N \prod_{k=1}^K r_{nk}^{Z_k^n}$$

where

$$r_{nk} = \frac{\rho_{nk}}{\sum_{j=1}^K \rho_{nj}}.$$

For the discrete distribution $q^*(\mathbf{Z})$ we have

$$\mathbb{E}[Z_k^n] = r_{nk}$$

and define

$$N_k = \sum_{n=1}^N r_{nk}.$$

Then consider the factor $q(\pi, \mu, \tau, \lambda)$ in the variational posterior distribution, we have

$$\begin{aligned}
\ln q^*(\pi, \mu, \tau, \lambda) &= \ln p(\pi) + \mathbb{E}_{\mathbf{Z}} [\ln p(\mathbf{Z}|\pi)] + \text{const} \\
&\quad + \sum_{k=1}^K \ln p(\mu_k, \tau_k) + \sum_{k=1}^K \sum_{n=1}^N \mathbb{E}[Z_k^n] \ln \mathcal{N}(X^n | \mu_k, \tau_k^{-1}) \\
&\quad + \sum_{k=1}^K \ln p(\lambda_k) + \sum_{k=1}^K \sum_{n=1}^N \mathbb{E}[Z_k^n] \ln \text{Poisson}(S^n | \lambda_k).
\end{aligned}$$

Since the right-hand side of this expression decomposes into a sum of terms involving only π , terms only involving μ and τ , and terms only involving λ , which implies that the variational posterior $q(\pi, \mu, \tau, \lambda)$ factorizes to give $q(\pi)q(\mu, \tau)q(\lambda)$ which gives

$$q(\pi, \mu, \tau, \lambda) = q(\pi) \prod_{k=1}^K q(\mu_k, \tau_k) q(\lambda_k).$$

- First, by identifying the terms that depend on π , we have

$$\ln q^*(\pi) = (\alpha - 1) \sum_{k=1}^K \ln \pi_k + \sum_{k=1}^K \sum_n^N r_{nk} \ln \pi_k + \text{const}$$

and taking the exponential of both sides, we have

$$q^*(\pi) = \text{Dir}(\pi | \alpha^*)$$

where α^* has components α_k^* given by

$$\alpha_k^* = \alpha_k + N_k.$$

- Similarly, by identifying the terms that depends on μ_k and τ_k , we have

$$\begin{aligned} \ln q^*(\mu_k, \tau_k) &= \sum_{n=1}^N r_{nk} \left(\frac{1}{2} \ln \tau_k - \frac{1}{2} \tau_k (X^n - \mu_k)^2 \right) \\ &\quad + \frac{1}{2} \ln \tau_k - \frac{1}{2} C (\mu_k - \mu)^2 \\ &\quad + (\alpha' - 1) \ln \tau_k - \beta' \tau_k. \end{aligned}$$

So, we have

$$\begin{aligned} q^*(\mu_k | \tau_k) &= \mathcal{N}(\mu_k | \mu^*, \tau^*) \\ \tau^* &= (C + N_k) \tau_k \\ \mu^* &= \frac{\left(\tau_k \sum_{n=1}^N r_{nk} X^n \right) + C \tau_k \mu}{\tau^*}, \end{aligned}$$

and

$$\begin{aligned} q^*(\tau_k) &= \text{Gam}(\tau_k | \alpha'^*, \beta'^*) \\ \alpha'^* &= \alpha' + N_k \\ \beta'^* &= \beta' + \frac{1}{2} C \mu^2 + \frac{1}{2} \sum_{n=1}^N r_{nk} X^{n2}. \end{aligned}$$

- Finally, by identifying the terms that depends on λ_k , we have

$$\begin{aligned} \ln q^*(\lambda_k) &= (\alpha_0 - 1) \ln \lambda_k - \beta_0 \lambda_k + \text{const} \\ &\quad + \sum_{n=1}^N (r_{nk} (S^n \ln \lambda_k - \lambda_k - \ln(S^n!))). \end{aligned}$$

So, we have

$$\begin{aligned} q^*(\lambda_k) &= \text{Gam}(\lambda_k | \alpha_0^*, \beta_0^*) \\ \alpha_0^* &= \alpha_0 + \sum_{n=1}^N r_{nk} S^n \\ \beta_0^* &= \beta_0 + N_k. \end{aligned}$$

8 Sampling from a tree GM

9 Failing components VI

Question 2.9.26: *Derive a VI algorithm that estimates the posterior distribution for this model.*

From the question, we have

$$p(Z_1) = \frac{1}{R}$$

$$p(Z_{c+1}|Z_c) = 1 - \left(\frac{2}{3}\right)^{N_{Z_c}(Z_{c+1})}$$

where $c \leq 1$, and

$$N_{Z_c}(Z_{c+1}) = \begin{cases} 0, & \text{if } Z_{c+1} \notin N(Z_c) \\ 1, & \text{if } Z_{c+1} \in N(Z_c) \end{cases}$$

and define $N_{Z_0}(Z_1) = 0$. Also, we have

$$\begin{aligned} p(X_{r,c}^n | Z_c, \Theta_{r,c}) &= \text{Bernoulli}(X_{r,c}^n | \theta_f)^{1-I_r(Z_c)} \text{Bernoulli}(X_{r,c}^n | \theta_{r,c})^{I_r(Z_c)} \\ &= \theta_f^{X_{r,c}^n(1-I_r(Z_c))} (1 - \theta_f)^{(1-X_{r,c}^n)(1-I_r(Z_c))} \\ &\quad \theta_{r,c}^{X_{r,c}^n I_r(Z_c)} (1 - \theta_{r,c})^{(1-X_{r,c}^n)I_r(Z_c)}, \end{aligned}$$

where

$$I_r(Z_c) = \begin{cases} 0, & \text{if } Z_c = r \\ 1, & \text{if } Z_c \neq r \end{cases}.$$

And also

$$\begin{aligned} p(\theta_f)p(X_{r,c}|Z_c = r, \Theta_{r,c}) &= \text{Bernoulli}(X_{r,c}|\theta_f) \text{Beta}(\theta_f|a_f, b_f) \\ &= \text{Beta}\left(\theta_f|a_f + \sum_{n=1}^N X_{r,c}^n, b_f + N - \sum_{n=1}^N X_{r,c}^n\right) \\ p(\theta_{r,c})p(X_{r,c}|Z_c \neq r, \Theta_{r,c}) &= \text{Bernoulli}(X_{r,c}|\theta_{r,c}) \text{Beta}(\theta_{r,c}|a, b) \\ &= \text{Beta}\left(\theta_{r,c}|a + \sum_{n=1}^N X_{r,c}^n, b + N - \sum_{n=1}^N X_{r,c}^n\right) \end{aligned}$$

from which we obtain

$$\begin{aligned} p(\Theta_{r,c})p(X_{r,c}|Z_c, \Theta_{r,c}) &= [\text{Beta}(\theta_{r,c}|a, b) \text{Beta}(\theta_f|a'_f, b'_f)]^{1-I_r(Z_c)} \\ &\quad [\text{Beta}(\theta_f|a_f, b_f) \text{Beta}(\theta_{r,c}|a', b')]^{I_r(Z_c)} \end{aligned}$$

where

$$\begin{aligned} a'_f &= a_f + \sum_{n=1}^N X_{r,c}^n & b'_f &= b_f + N - \sum_{n=1}^N X_{r,c}^n \\ a' &= a + \sum_{n=1}^N X_{r,c}^n & b' &= b + N - \sum_{n=1}^N X_{r,c}^n \end{aligned}$$

Also, since the model follows the same structure with Hidden Markov Model (HMM), then the joint distribution of all the random variables is given by

$$\begin{aligned}
p(X, Z, \Theta) &= p(Z_1)p(\Theta_1)P(X_1|Z_1, \Theta_1) \\
&\quad p(Z_2|Z_1)p(\Theta_2)p(X_2|Z_2, \Theta_2) \\
&\quad \dots \\
&\quad p(Z_C|Z_{C-1})p(\Theta_C)P(X_C|Z_C, \Theta_C) \\
&= p(Z_1) \prod_{r=1}^R p(\Theta_{r,1})P(X_{r,1}|Z_1, \Theta_{r,1}) \\
&\quad p(Z_2|Z_1) \prod_{r=1}^R p(\Theta_{r,2})p(X_{r,2}|Z_2, \Theta_{r,2}) \\
&\quad \dots \\
&\quad p(Z_C|Z_{C-1}) \prod_{r=1}^R p(\Theta_{r,C})P(X_{r,C}|Z_C, \Theta_{r,C}) \\
&= p(Z) \prod_{c=1}^C \prod_{r=1}^R p(\Theta_{r,c})P(X_{r,c}|Z_c, \Theta_{r,c})
\end{aligned}$$

A variational distribution is consider so that

$$q(Z, \Theta) = q(Z)q(\Theta).$$

The log of the optimized factor is given by

$$\begin{aligned}
\ln q^*(Z) &= \ln q^*(Z_1) + \ln q^*(Z_2|Z_1) + \dots + \ln q^*(Z_C|Z_{C-1}) \\
&= \mathbb{E}[\ln p(Z_1)] + \mathbb{E}[\ln p(Z_2|Z_1)] + \dots + \mathbb{E}[\ln p(Z_C|Z_{C-1})] + \mathbb{E}_\Theta[p(X|Z, \Theta)] + \text{const} \\
&= \frac{1}{R} + \sum_{c=1}^{C-1} \ln \left(1 - \left(\frac{2}{3} \right)^{N_{Z_c}(Z_{c+1})} \right) + \text{const} \\
&\quad + \sum_{n=1}^N \sum_{c=1}^C \sum_{r=1}^R \{ X_{r,c}^n (1 - I_r(Z_c)) \mathbb{E}[\ln \theta_f] + (1 - X_{r,c}^n) (1 - I_r(Z_c)) \mathbb{E}[\ln(1 - \theta_k)] \\
&\quad + x_{r,c}^n I_r(Z_c) \mathbb{E}[\ln \theta_{r,c}] + (1 - X_{r,c}^n) I_r(Z_c) \mathbb{E}[\ln(1 - \theta_{r,c})] \} \\
&= \sum_{n=1}^N \sum_{c=1}^C \sum_{r=1}^R \frac{1}{NR} \ln \left(1 - \left(\frac{2}{3} \right)^{N_{Z_c}(Z_{c+1})} \right) + \text{const} \\
&\quad + \sum_{n=1}^N \sum_{c=1}^C \sum_{r=1}^R \{ X_{r,c}^n (1 - I_r(Z_c)) \mathbb{E}[\ln \theta_f] + (1 - X_{r,c}^n) (1 - I_r(Z_c)) \mathbb{E}[\ln(1 - \theta_k)] \\
&\quad + x_{r,c}^n I_r(Z_c) \mathbb{E}[\ln \theta_{r,c}] + (1 - X_{r,c}^n) I_r(Z_c) \mathbb{E}[\ln(1 - \theta_{r,c})] \} \\
&= \sum_{n=1}^N \sum_{r=1}^R \sum_{c=1}^C \ln \rho_{n,r,c} + \text{const}
\end{aligned}$$

where could have $\ln(0)$ problem, which can be solved in practice by calculating $\ln(\text{sys.float_info.epsilon})$

instead. Then take the exponential of both side of the equation and normalize it, we obtain

$$q^*(Z) = \prod_{n=1}^N \prod_{r=1}^R \prod_{c=1}^C r_{n,r,c}$$

where

$$r_{n,r,c} = \frac{\rho_{n,r,c}}{\sum_{r=1}^R \sum_{c=1}^C \rho_{n,r,c}}.$$

Then for $q(\Theta)$, we have

$$\begin{aligned} \ln q^*(\Theta_{r,c}) = & \mathbb{E}[1 - I_r(Z_c)] \ln [\text{Beta}(\theta_{r,c}|a, b) \text{Beta}(\theta_f|a'_f, b'_f)] \\ & + \mathbb{E}[I_r(Z_c)] \ln [\text{Beta}(\theta_f|a_f, b_f) \text{Beta}(\theta_{r,c}|a', b')]. \end{aligned}$$

So, we have

$$\begin{aligned} q^*(\Theta_{r,c}) = & q^*(\theta_f) q^*(\theta_{r,c}) \\ = & \text{Beta}(\theta_f|a_f^*, b_f^*) \text{Beta}(\theta_{r,c}|a_{r,c}^*, b_{r,c}^*) \end{aligned}$$

where

$$\begin{aligned} a_f^* = & \mathbb{E}[I_r(Z_c)](a_f - 1) + \mathbb{E}[1 - I_r(Z_c)](a'_f - 1) + 1 \\ b_f^* = & \mathbb{E}[I_r(Z_c)](b_f - 1) + \mathbb{E}[1 - I_r(Z_c)](b'_f - 1) + 1 \\ a_{r,c}^* = & \mathbb{E}[I_r(Z_c)](a' - 1) + \mathbb{E}[1 - I_r(Z_c)](a - 1) + 1 \\ b_{r,c}^* = & \mathbb{E}[I_r(Z_c)](b' - 1) + \mathbb{E}[1 - I_r(Z_c)](b - 1) + 1 \end{aligned}$$

References

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