Artificial Intelligence and Machine Learning

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Bayes Decision Rule

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Therefore:

whenever we observe a particular x, the probability of error is :

$$P(error \mid x) = P(\omega_1 \mid x) \text{ if we decide } \omega_2$$

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Therefore:

$$P(error \mid x) = min\ [\ P(\omega_1 \mid x),\ \ P(\omega_2 \mid x)\ ]$$

(Bayes decision)

Bayes Decision Theory – Continuous Features

- Generalization of the preceding ideas
  - Use of more than one feature
  - Use more than two states of nature
  - Allowing actions and not only decide on the state of nature
  - Introduce a loss of function which is more general than the probability of error

Loss Function

- Allowing actions other than classification primarily allows the possibility of rejection
- Refusing to make a decision in close or bad cases!
- The loss function states how costly each action taken is

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Let $\{\alpha_1, \alpha_2, \ldots, \alpha_a\}$ be the set of possible actions

Let $\lambda(\alpha_i \mid \omega_j)$ be the loss incurred for taking

action α_i when the state of nature is ω_j

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$$R(\alpha_i \mid x) = \sum_{j=1}^{j=c} \lambda(\alpha_i \mid \omega_j) P(\omega_j \mid x)$$

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Select the action α_i for which $R(\alpha_i \mid x)$ is minimum

R is minimum and R in this case is called the Risk

Bayes risk = best performance that can be achieved

Two-category classification

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α_1 : *deciding* ω_1

α_2 : *deciding* ω_2

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Conditional risk:

$$R(\alpha_1 \mid x) = \lambda_{11}P(\omega_1 \mid x) + \lambda_{12}P(\omega_2 \mid x)$$
$$R(\alpha_2 \mid x) = \lambda_{21}P(\omega_1 \mid x) + \lambda_{22}P(\omega_2 \mid x)$$

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decide ω_1 if:

$$(\lambda_{21} - \lambda_{11})\, P(x \mid \omega_1)\, P(\omega_1) > (\lambda_{12} - \lambda_{22})\, P(x \mid \omega_2)\, P(\omega_2)$$

and decide ω_2 otherwise

Minimum-Error-Rate Classification

- Actions are decisions on classes

  If action α_i is taken and the true state of nature is ω_j then: decision is correct if $i = j$ and in error if $i \neq j$

  Seek a decision rule that minimizes the *probability of error* which is the *error rate*

Let’s take a break!

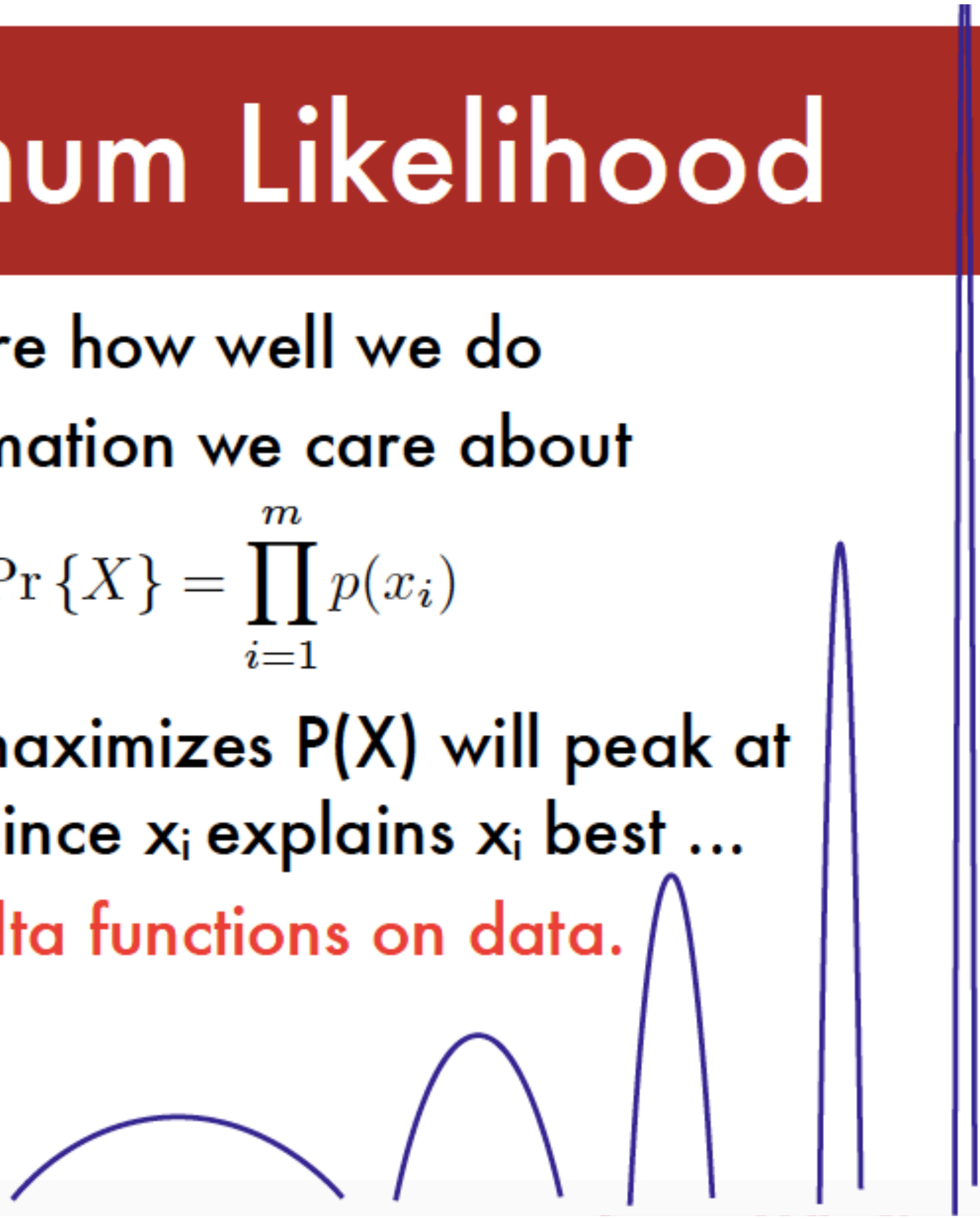
Model Selection

Maximum Likelihood

- Need to measure how well we do
- For density estimation we care about

$$\Pr\{X\} = \prod_{i=1}^{m} p(x_i)$$

- Finding a that maximizes P(X) will peak at all data points since x_i explains x_i best ...
- Maxima are delta functions on data.
- Overfitting!



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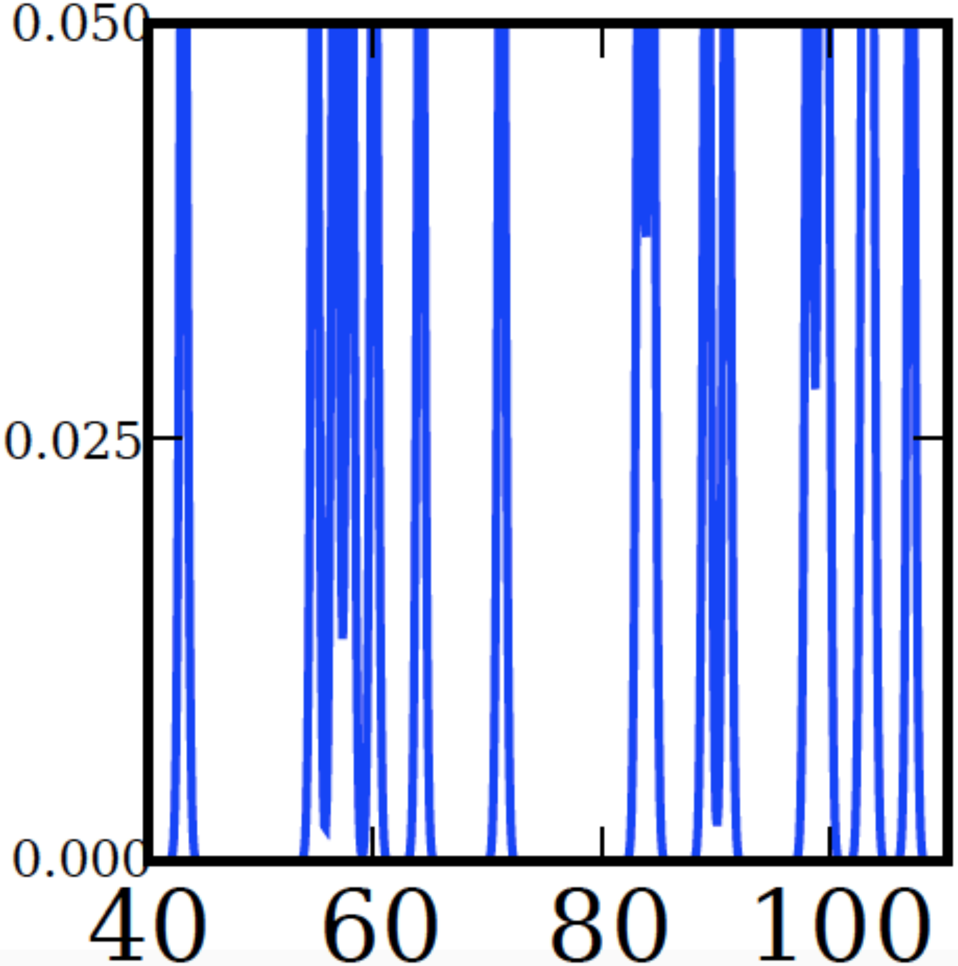

Likelihood on training set is much higher than typical.

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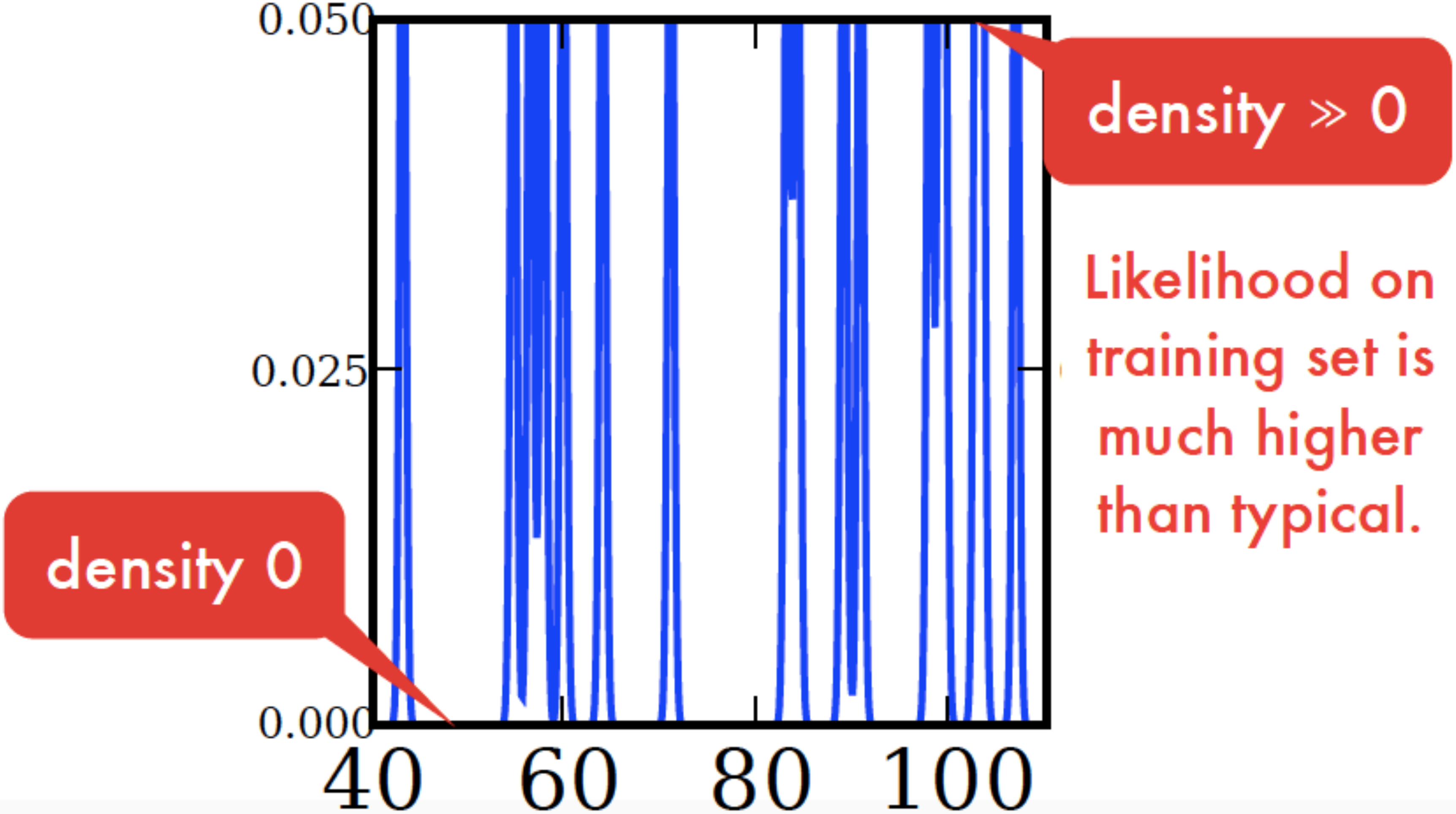

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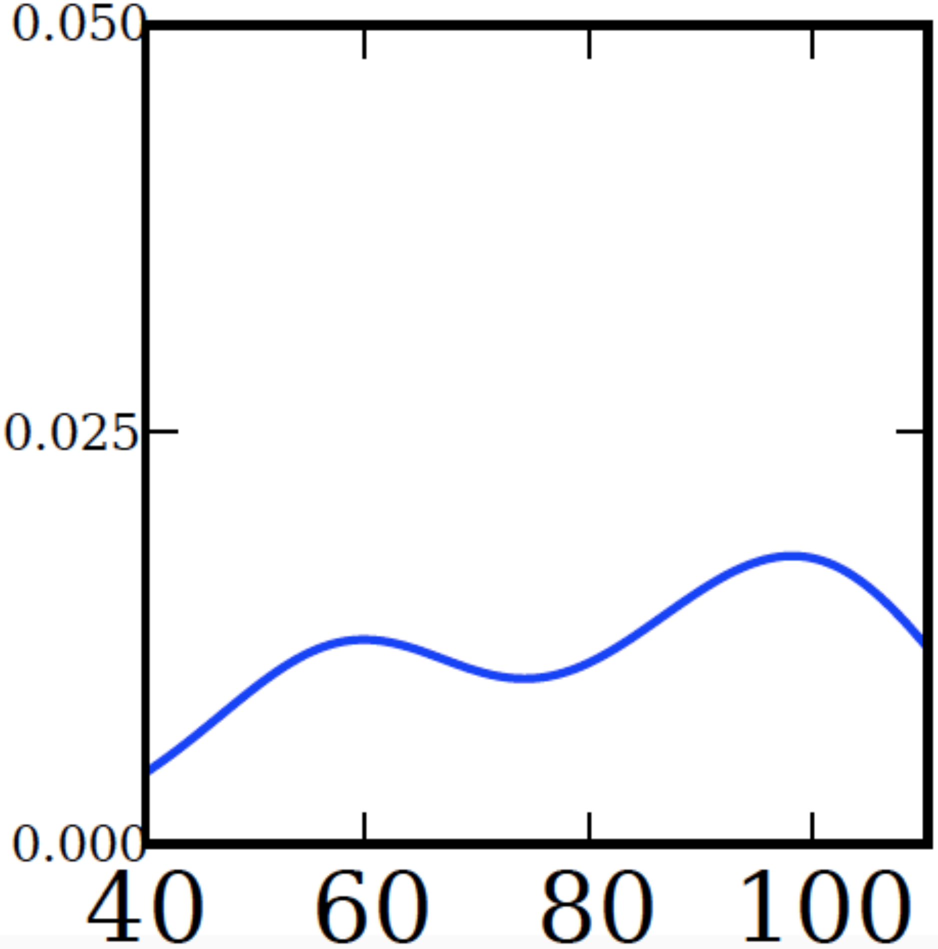

Likelihood on training set is very similar to typical one.

Too simple.

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- Validation
  - Use some of the data to estimate density.
  - Use other part to evaluate how well it works
  - Pick the parameter that works best

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$$\mathcal{L}(X'|X) := \frac{1}{n'} \sum_{i=1}^{n'} \log \hat{p}(x_i')$$

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- Simple implementation
$$\frac{1}{n} \sum_{i=1}^{n} \log \left[\frac{n}{n-1} p(x_i) - \frac{1}{n-1} k(x_i, x_i) \right] \text{ where } p(x) = \frac{1}{n} \sum_{i=1}^{n} k(x_i, x)$$

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That’s all