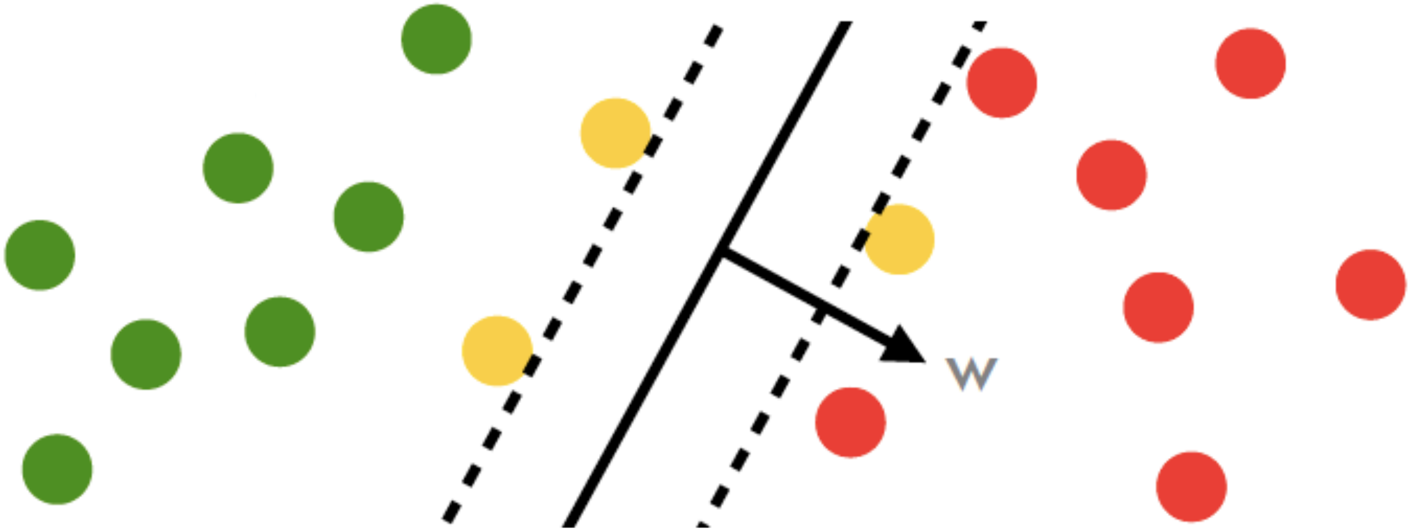
Artificial Intelligence and Machine Learning

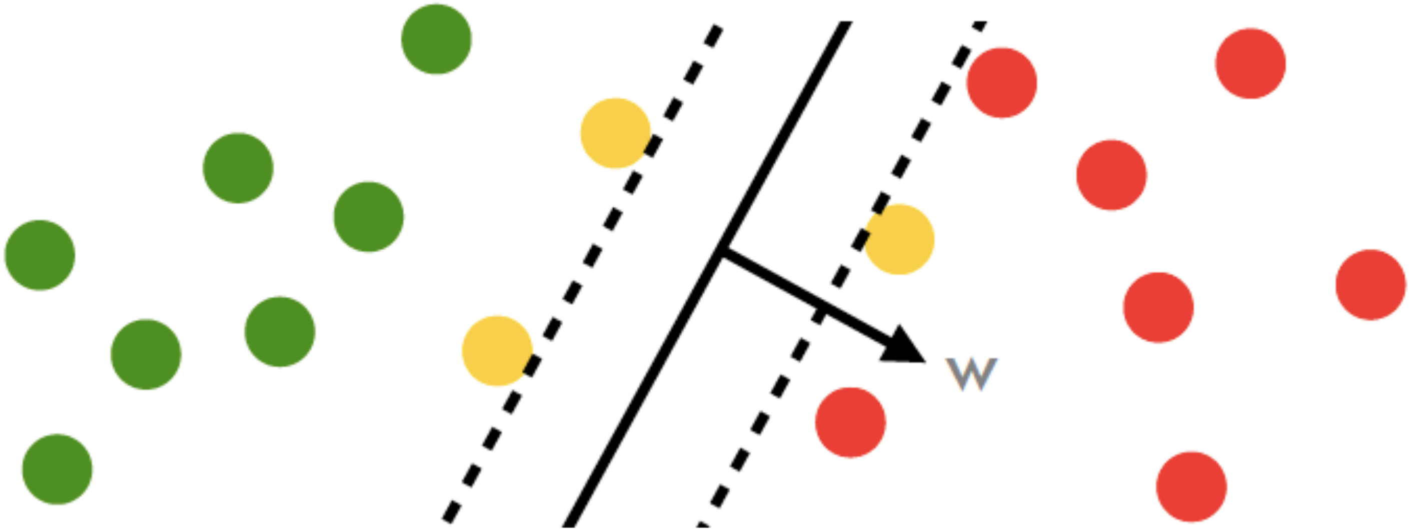
Barbara Caputo

Support Vector Machines



Support Vector Machines

$$\underset{w.b}{\text{minimize}} \frac{1}{2} \|w\|^2 \text{ subject to } y_i \left[\langle x_i, w\rangle + b\right] \geq 1$$



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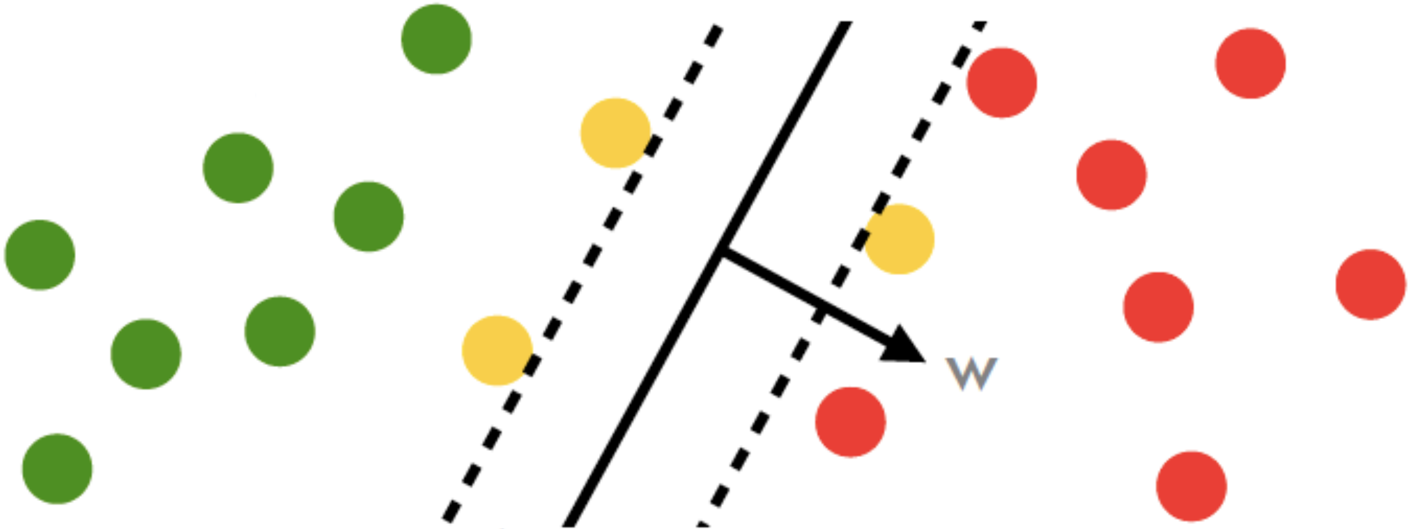


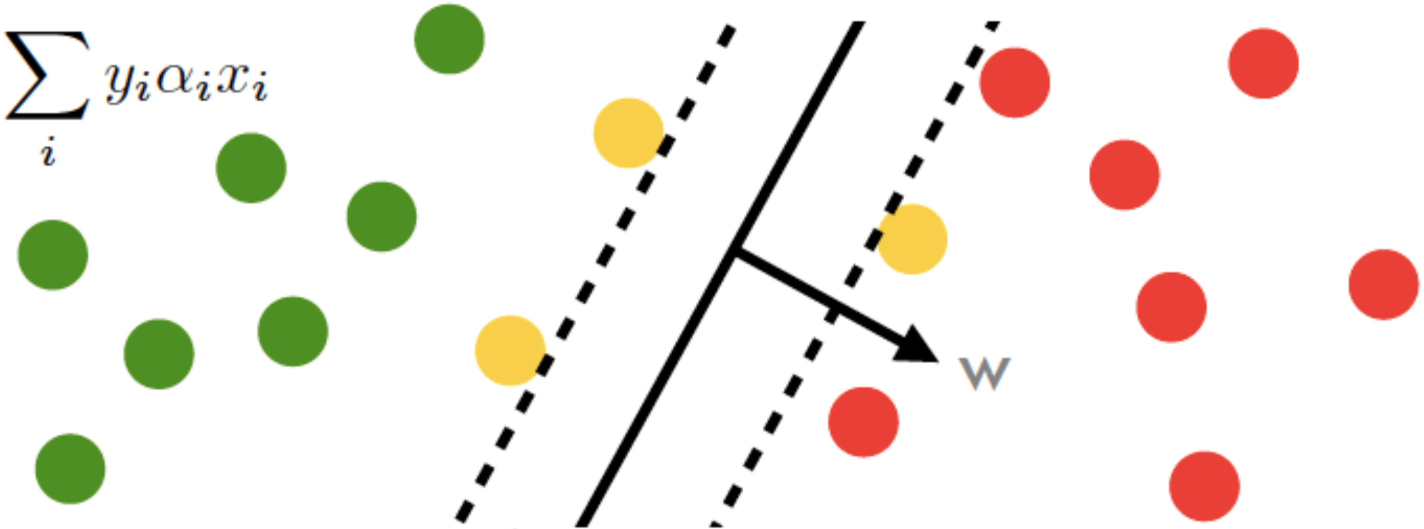
$$\underset{\alpha}{\text{maximize}} - \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j y_i y_j \langle x_i, x_j\rangle + \sum_i \alpha_i$$

$$\text{subject to}\ \sum \alpha_i y_i = 0 \ \text{and}\ \alpha_i \geq 0$$

Support Vector Machines

$$\underset{w,b}{\text{minimize}} \frac{1}{2} \|w\|^2 \text{ subject to } y_i \left[\langle x_i, w \rangle + b\right] \geq 1$$

$$w = \sum_i y_i \alpha_i x_i$$



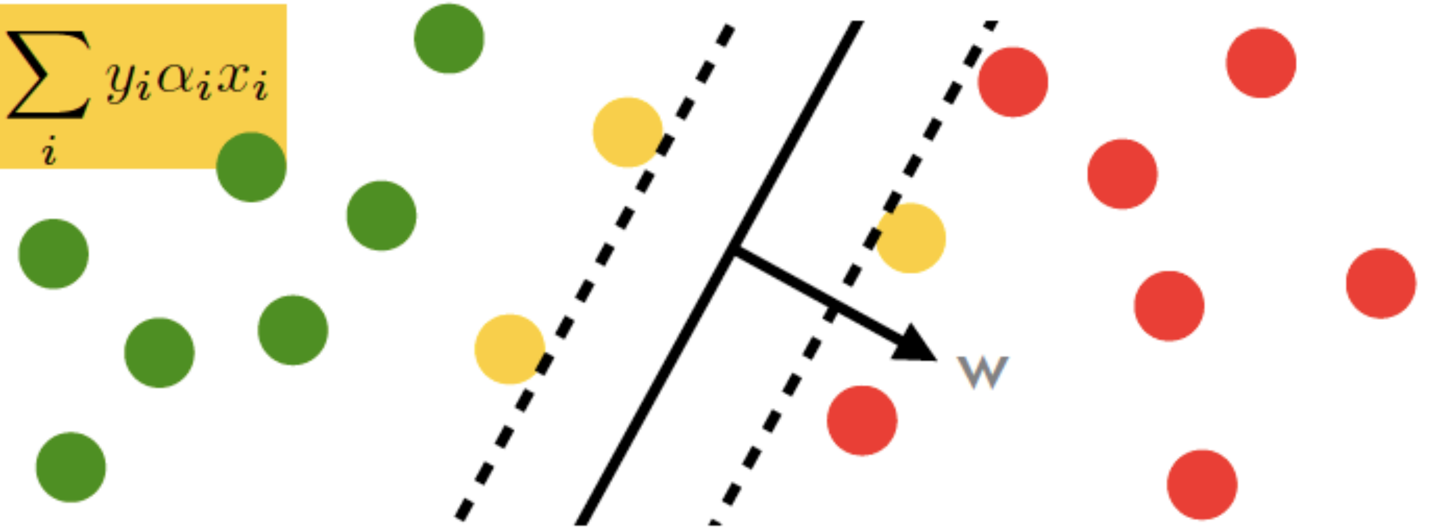
$$\underset{\alpha}{\text{maximize}} - \frac{1}{2} \sum_{i,j} \alpha_i \alpha_j y_i y_j \langle x_i, x_j \rangle + \sum_i \alpha_i$$

$$\text{subject to } \sum \alpha_i y_i = 0 \text{ and } \alpha_i \geq 0$$

Support Vectors

$$\underset{w,b}{\text{minimize}}\ \frac{1}{2}\|w\|^2 \text{ subject to } y_i\left[\langle x_i, w\rangle + b\right] \geq 1$$

$$w = \sum_i y_i \alpha_i x_i$$



Karush Kuhn Tucker

Optimality condition

$$\alpha_i\left[y_i\left[\langle w, x_i\rangle + b\right] - 1\right] = 0$$

$$\alpha_i = 0$$

$$\alpha_i > 0 \Longrightarrow y_i\left[\langle w, x_i\rangle + b\right] = 1$$

Soft Margin Classifiers

Adding slack variables

- Hard margin problem

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$$\underset{w,b}{\text{minimize}}\ \frac{1}{2}\|w\|^2 \text{ subject to } y_i\left[\langle w, x_i\rangle + b\right] \geq 1$$

Adding slack variables

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- With slack variables

Adding slack variables

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- With slack variables

$$\underset{w,b}{\text{minimize}}\;\frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

Adding slack variables

- Hard margin problem

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- With slack variables

$$\underset{w,b}{\text{minimize}}\ \frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

$$\text{subject to } y_i\left[\langle w, x_i\rangle + b\right] \geq 1 - \xi_i \text{ and } \xi_i \geq 0$$

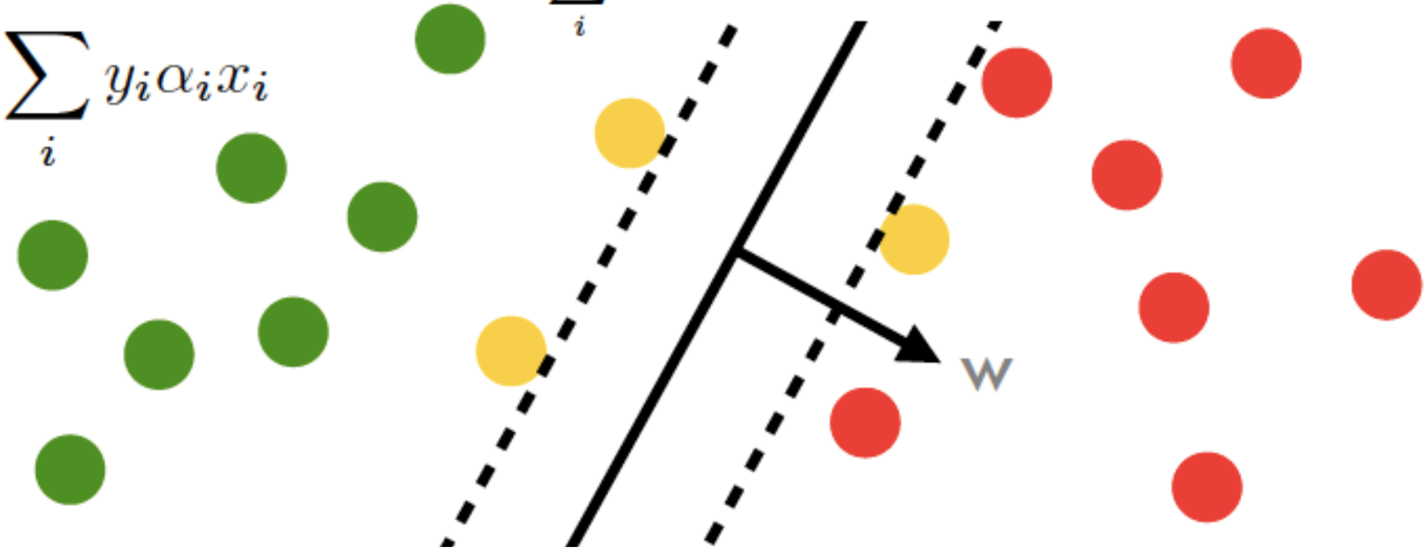
Problem is always feasible.

Karush Kuhn Tucker Conditions

$$\underset{\alpha}{\text{maximize}} - \frac{1}{2}\sum_{i,j} \alpha_i \alpha_j y_i y_j \langle x_i, x_j \rangle + \sum_i \alpha_i$$

$$\text{subject to } \sum_i \alpha_i y_i = 0 \text{ and } \alpha_i \in [0, C]$$

$$w = \sum_i y_i \alpha_i x_i$$



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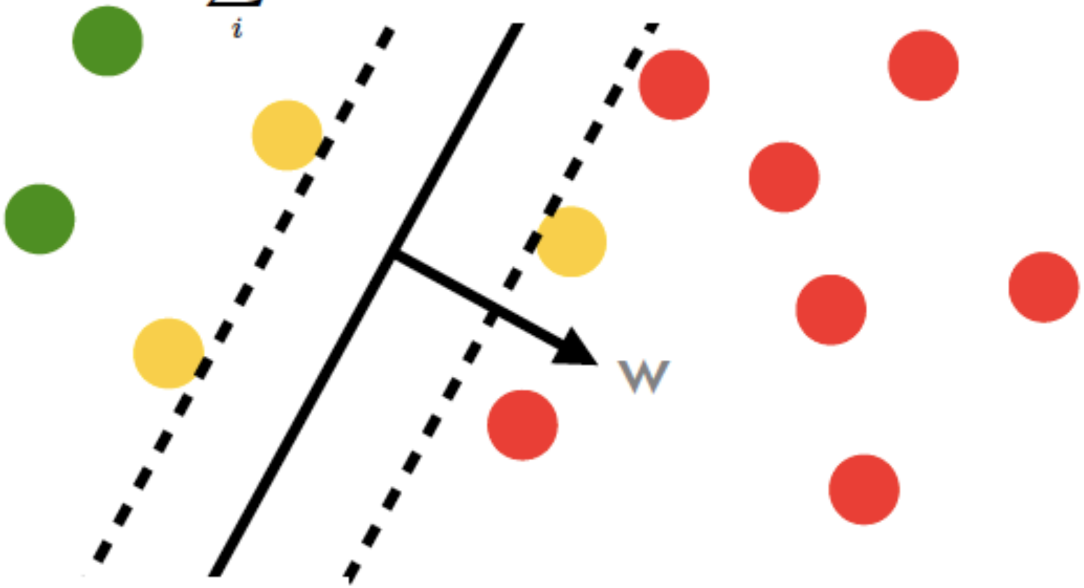

$$\alpha_i \left[y_i \left[\langle w, x_i\rangle + b\right] + \xi_i - 1\right] = 0$$

$$\eta_i \xi_i = 0$$

Karush Kuhn Tucker Conditions

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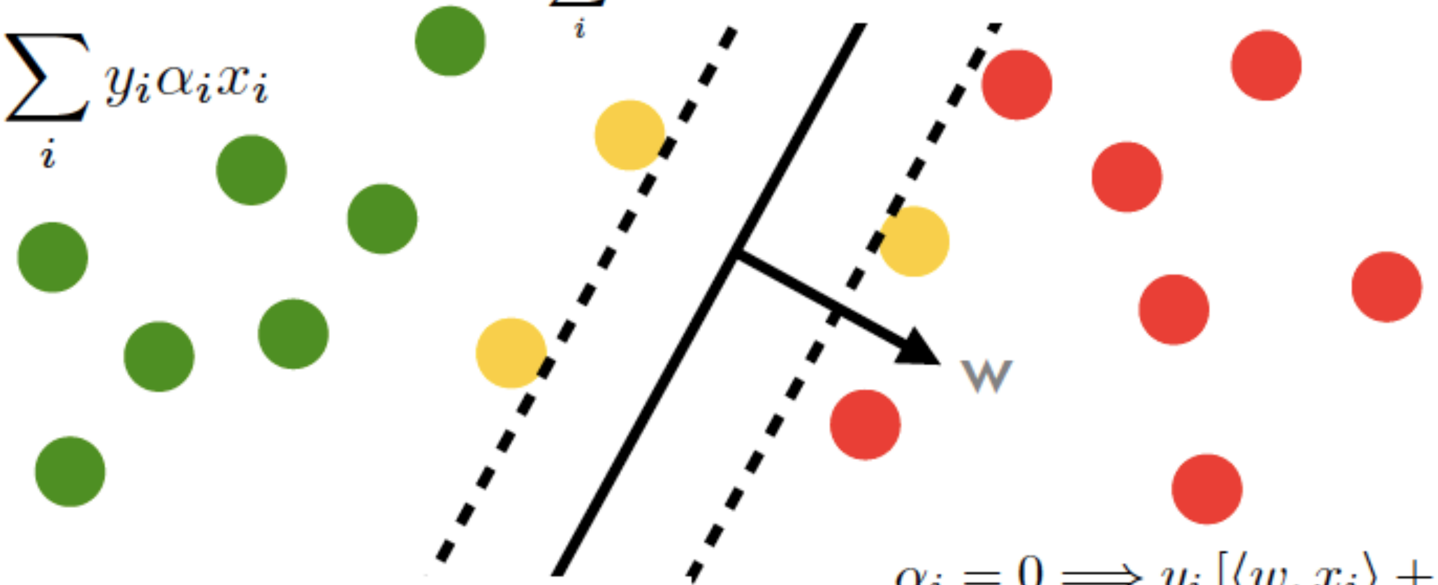

$$\alpha_i \left[y_i \left[\langle w, x_i\rangle + b\right] + \xi_i - 1\right] = 0$$

$$\eta_i \xi_i = 0$$

$$\alpha_i = 0 \Longrightarrow y_i \left[\langle w, x_i\rangle + b\right] \geq 1$$

$$0 < \alpha_i < C \Longrightarrow y_i \left[\langle w, x_i\rangle + b\right] = 1$$

$$\alpha_i = C \Longrightarrow y_i \left[\langle w, x_i\rangle + b\right] \leq 1$$

Nonlinear Separation

The Kernel Trick

The Kernel Trick

- Linear soft margin problem

$$\underset{w,b}{\text{minimize}}\ \frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

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$$\underset{w,b}{\text{minimize}}\ \frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

$$\text{subject to } y_i\left[\langle w, x_i\rangle + b\right] \geq 1 - \xi_i \text{ and } \xi_i \geq 0$$

- Dual problem

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$$\underset{w,b}{\text{minimize}} \; \frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

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The Kernel Trick

- **Linear soft margin problem**

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$$\text{subject to } y_i\left[\langle w, x_i\rangle + b\right] \geq 1 - \xi_i \text{ and } \xi_i \geq 0$$

- **Dual problem**

$$\underset{\alpha}{\text{maximize}} - \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j y_i y_j \langle x_i, x_j\rangle + \sum_i \alpha_i$$

$$\text{subject to } \sum_i \alpha_i y_i = 0 \text{ and } \alpha_i \in [0, C]$$

- **Support vector expansion**

$$f(x) = \sum_i \alpha_i y_i \langle x_i, x\rangle + b$$

The Kernel Trick

- Linear soft margin problem

$$\underset{w,b}{\text{minimize}} \ \frac{1}{2}\|w\|^2 + C\sum_i \xi_i$$

$$\text{subject to } y_i\left[\langle w, \phi(x_i)\rangle + b\right] \geq 1 - \xi_i \text{ and } \xi_i \geq 0$$

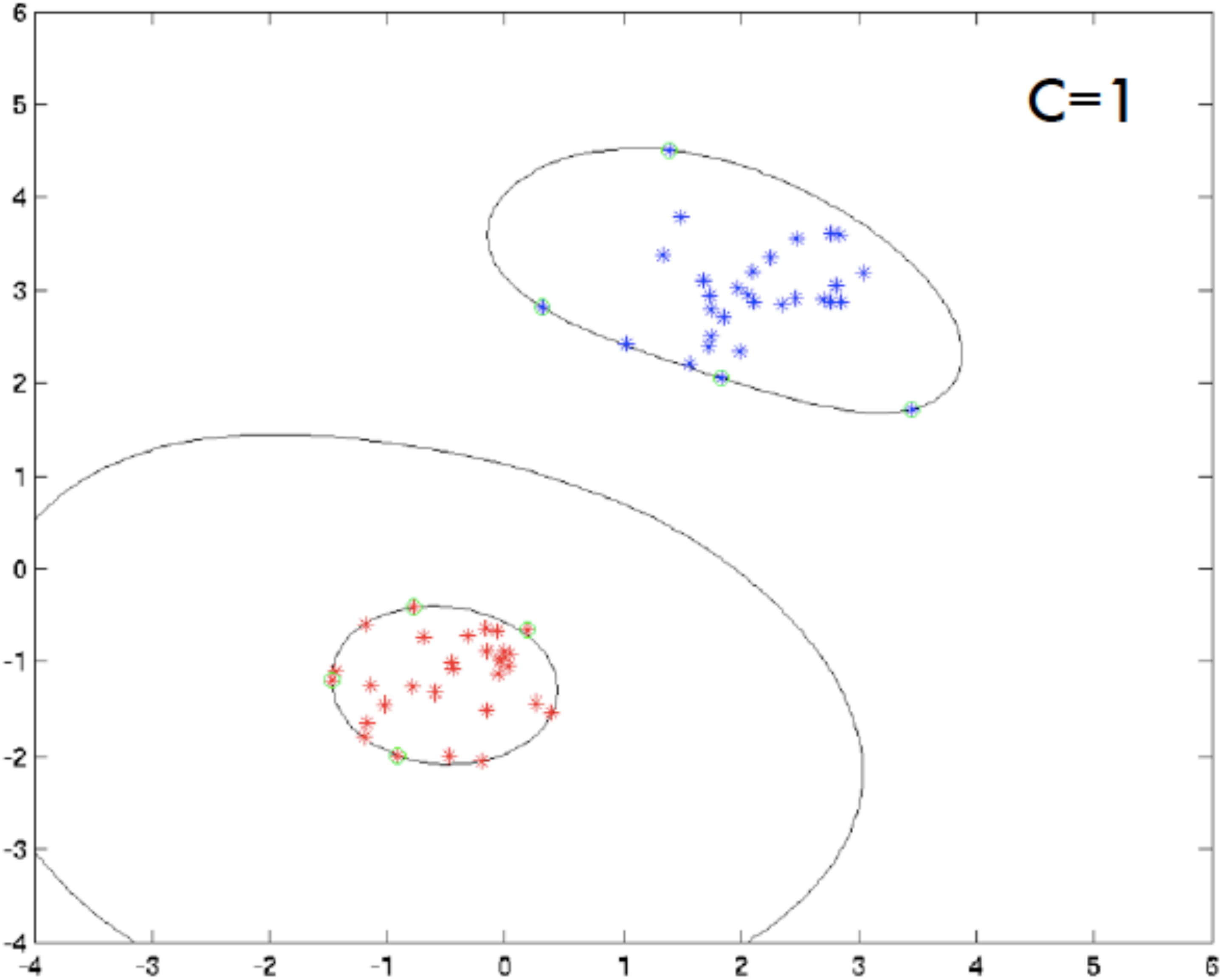
- Dual problem

$$\underset{\alpha}{\text{maximize}} - \frac{1}{2}\sum_{i,j}\alpha_i\alpha_j y_i y_j k(x_i, x_j) + \sum_i \alpha_i$$

$$\text{subject to } \sum_i \alpha_i y_i = 0 \text{ and } \alpha_i \in [0, C]$$

- Support vector expansion

$$f(x) = \sum_i \alpha_i y_i k(x_i, x) + b$$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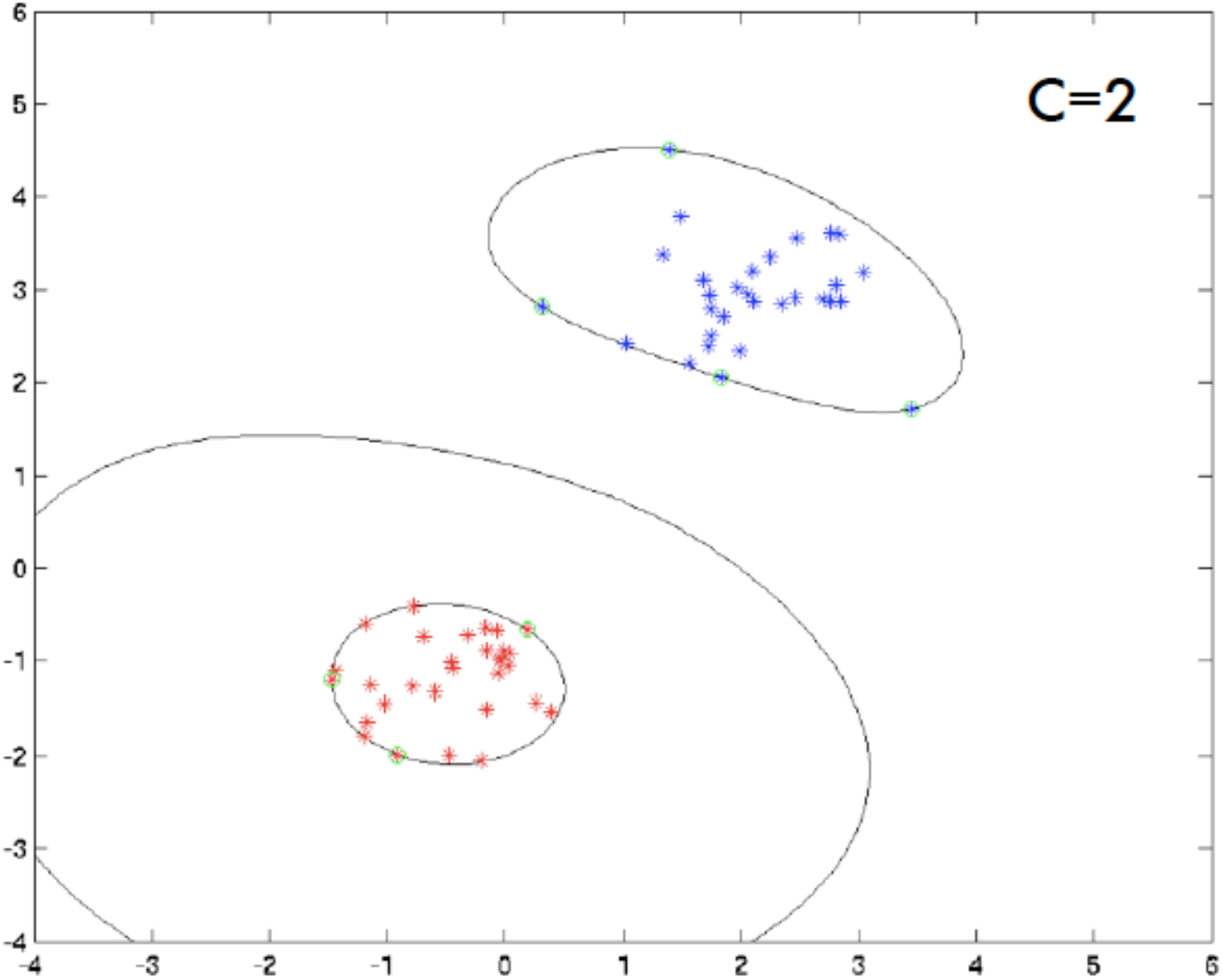
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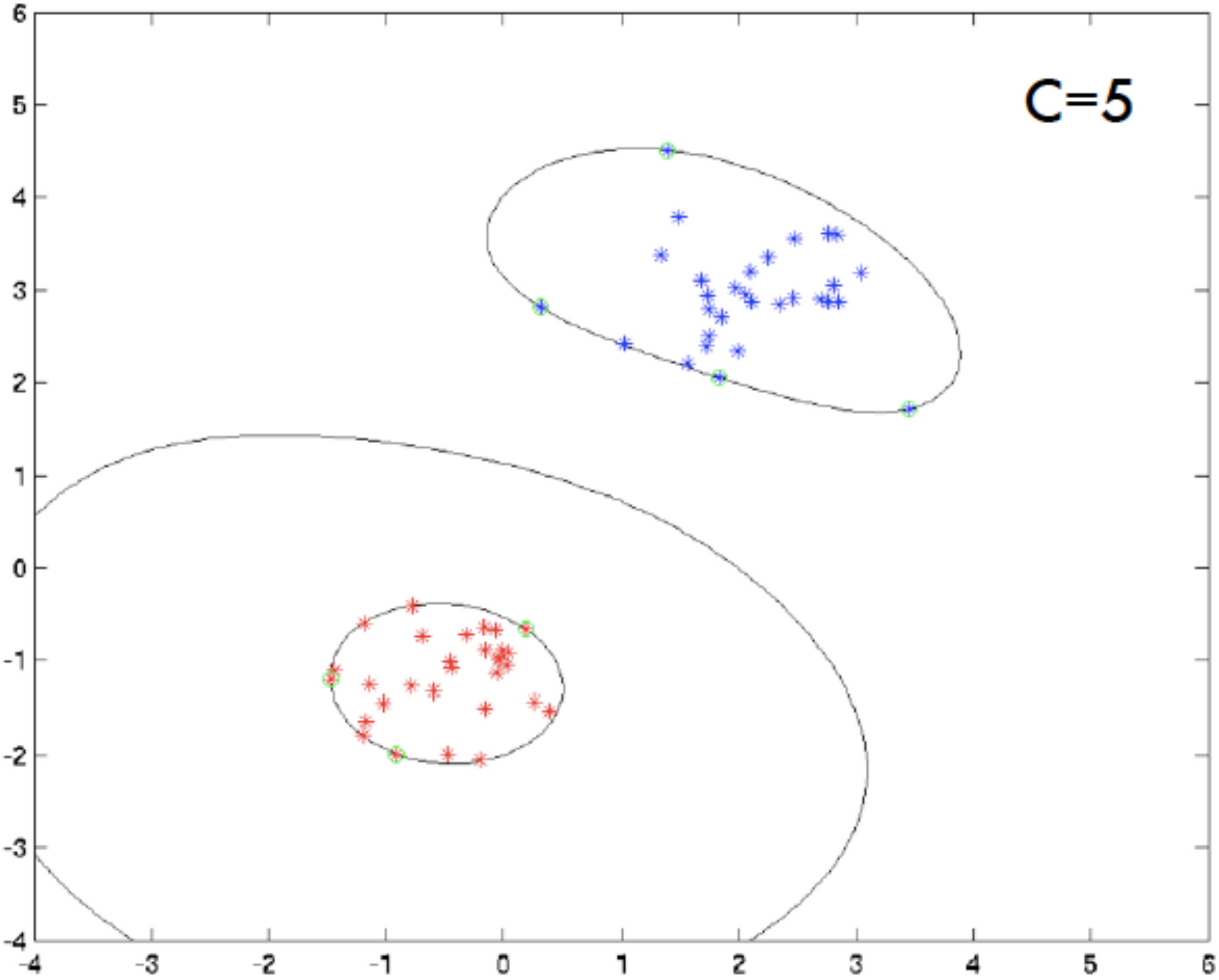
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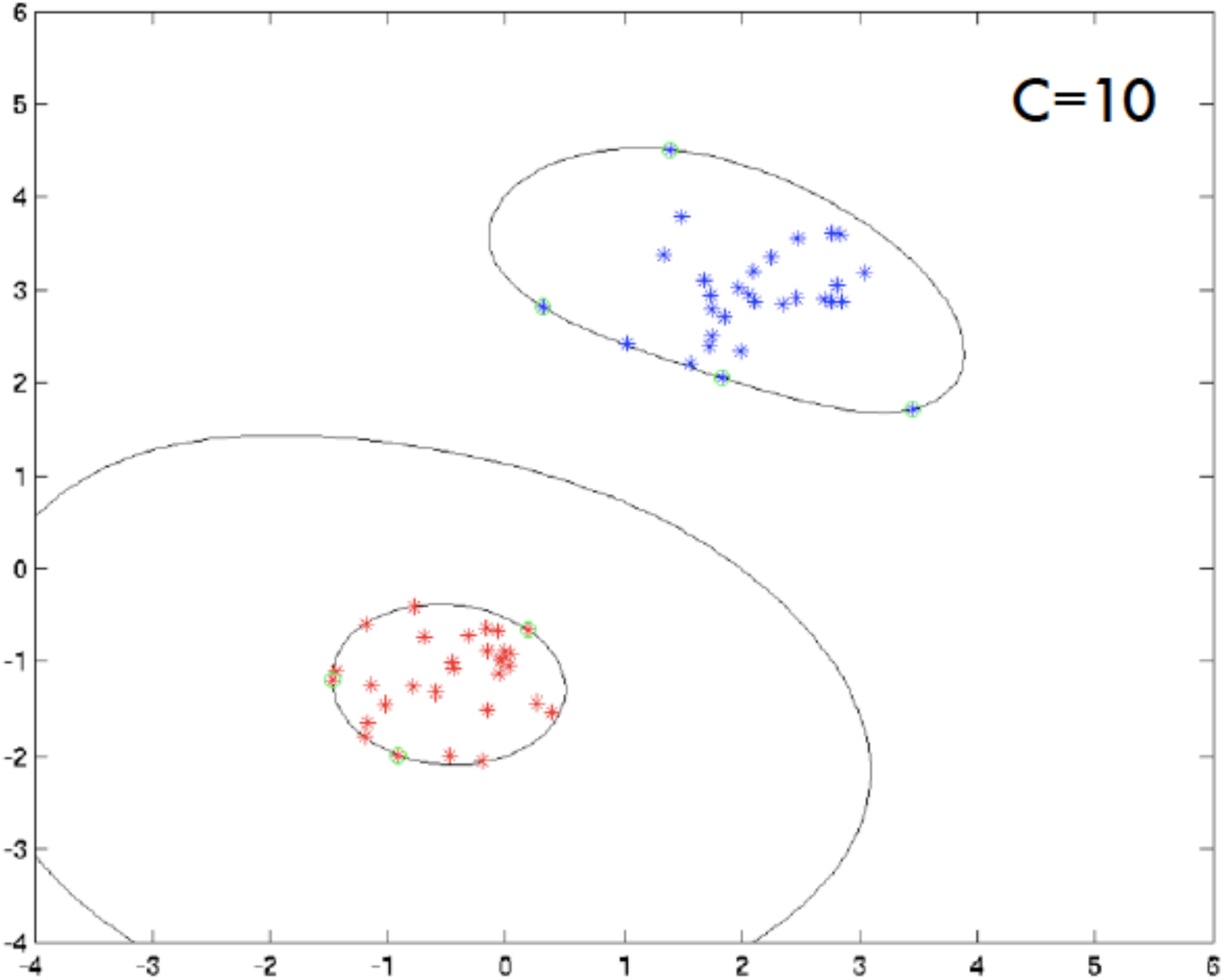
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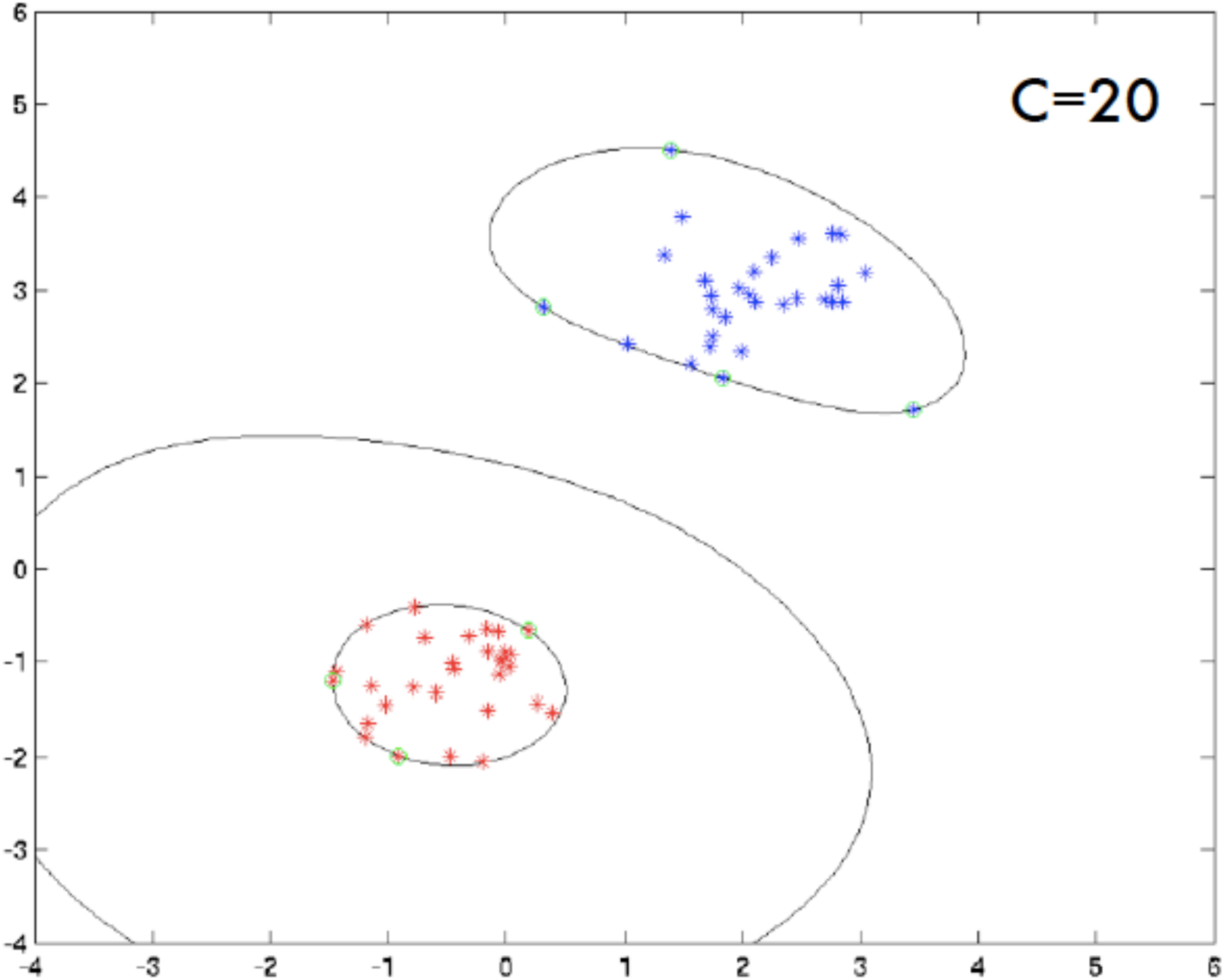
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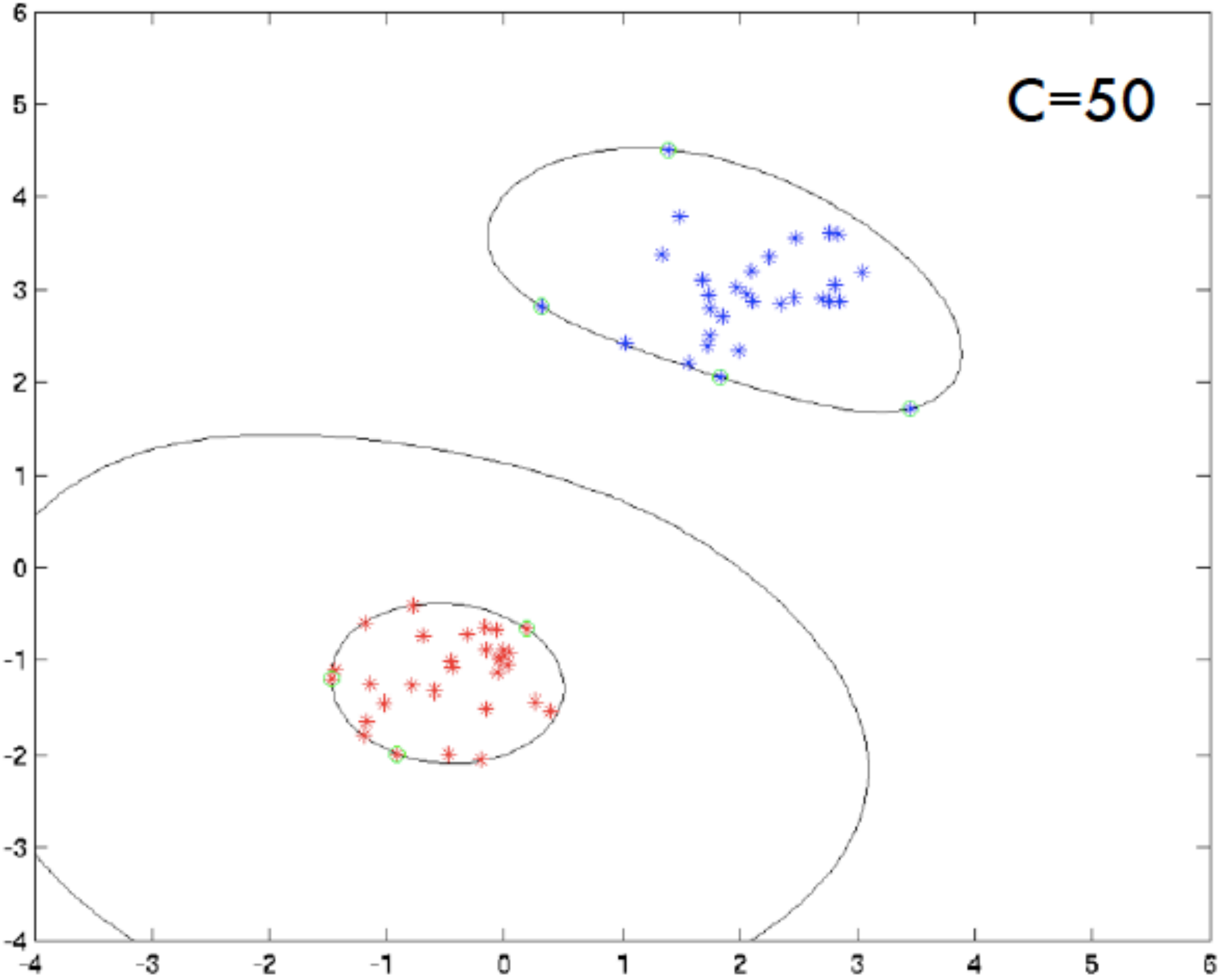
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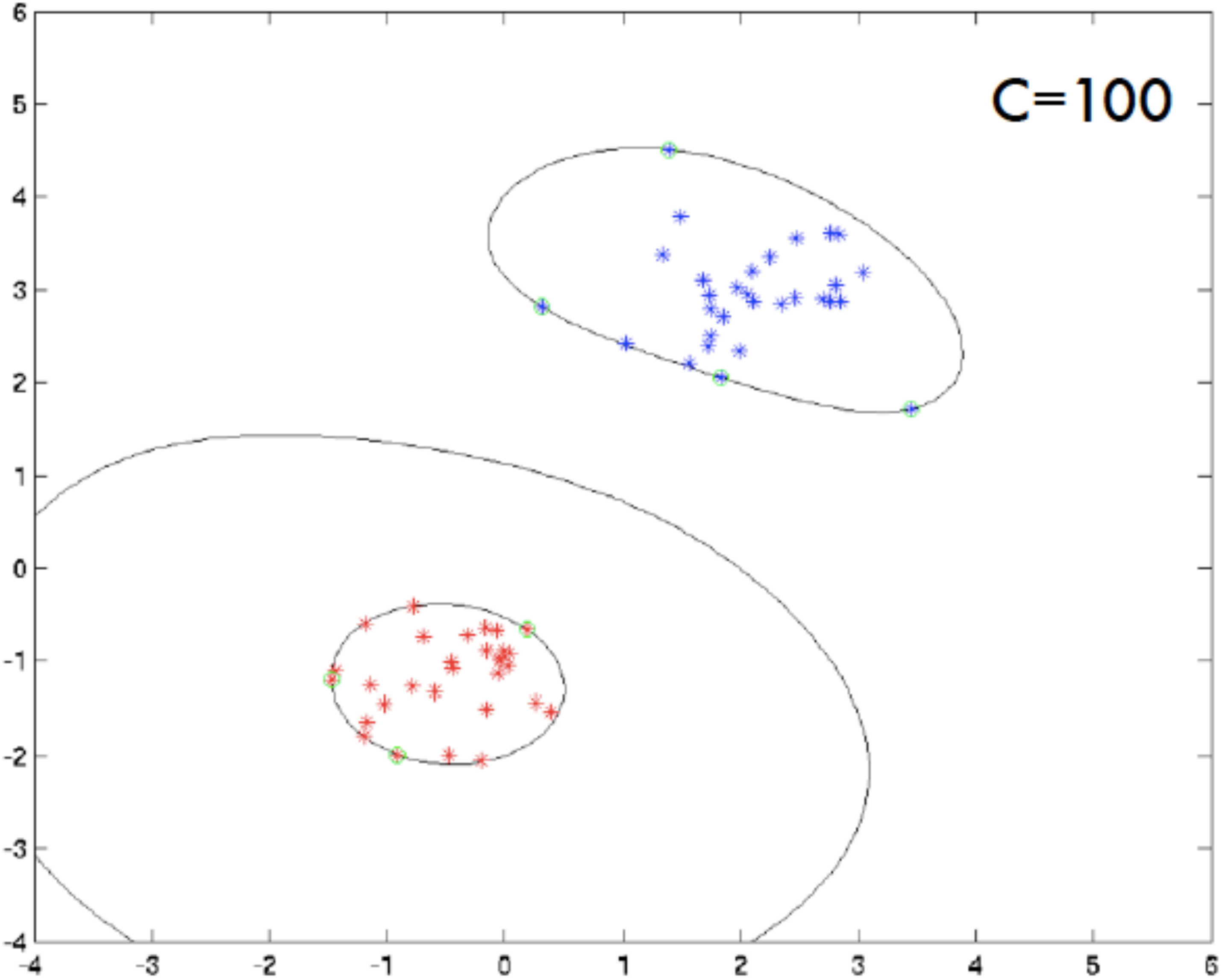
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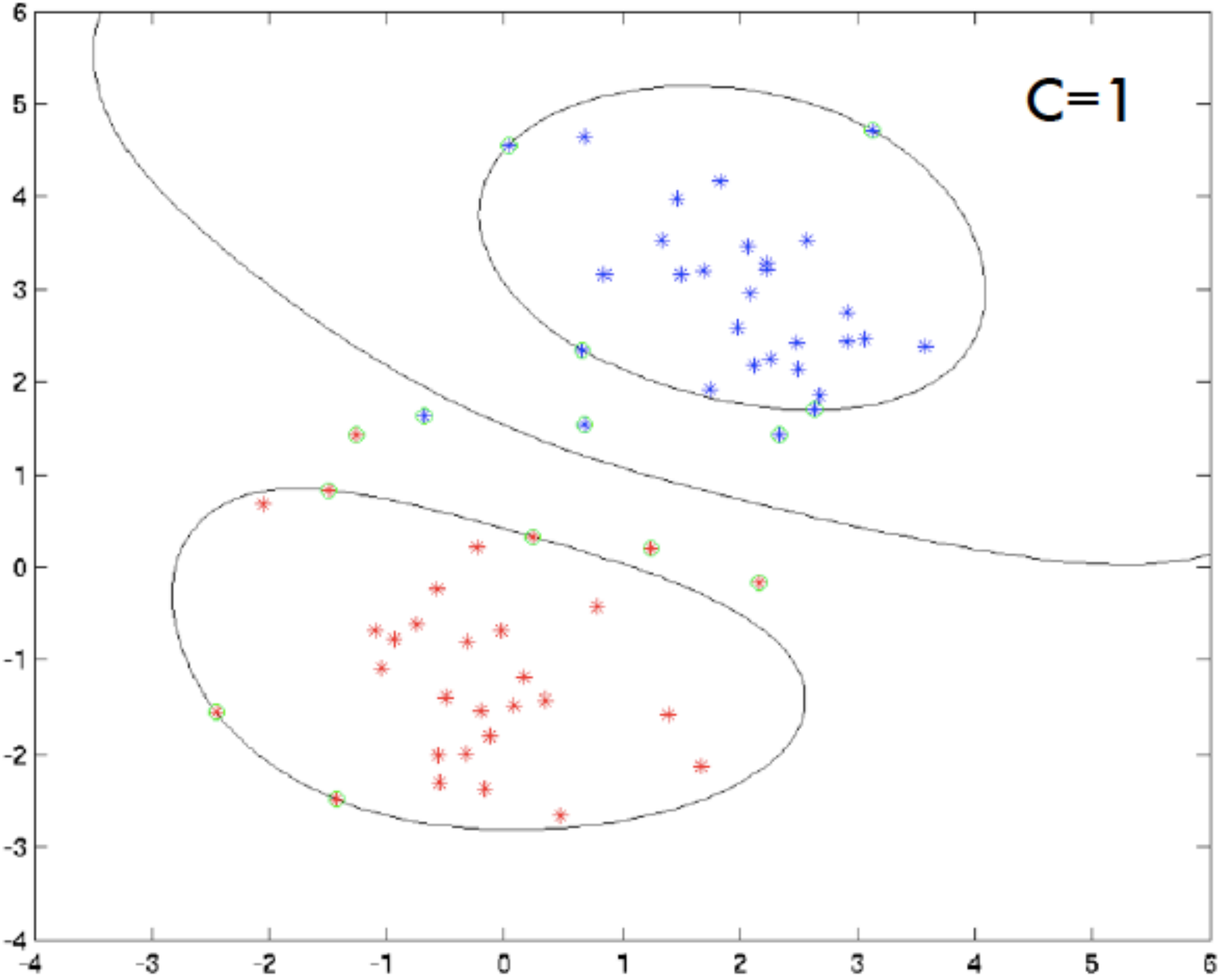
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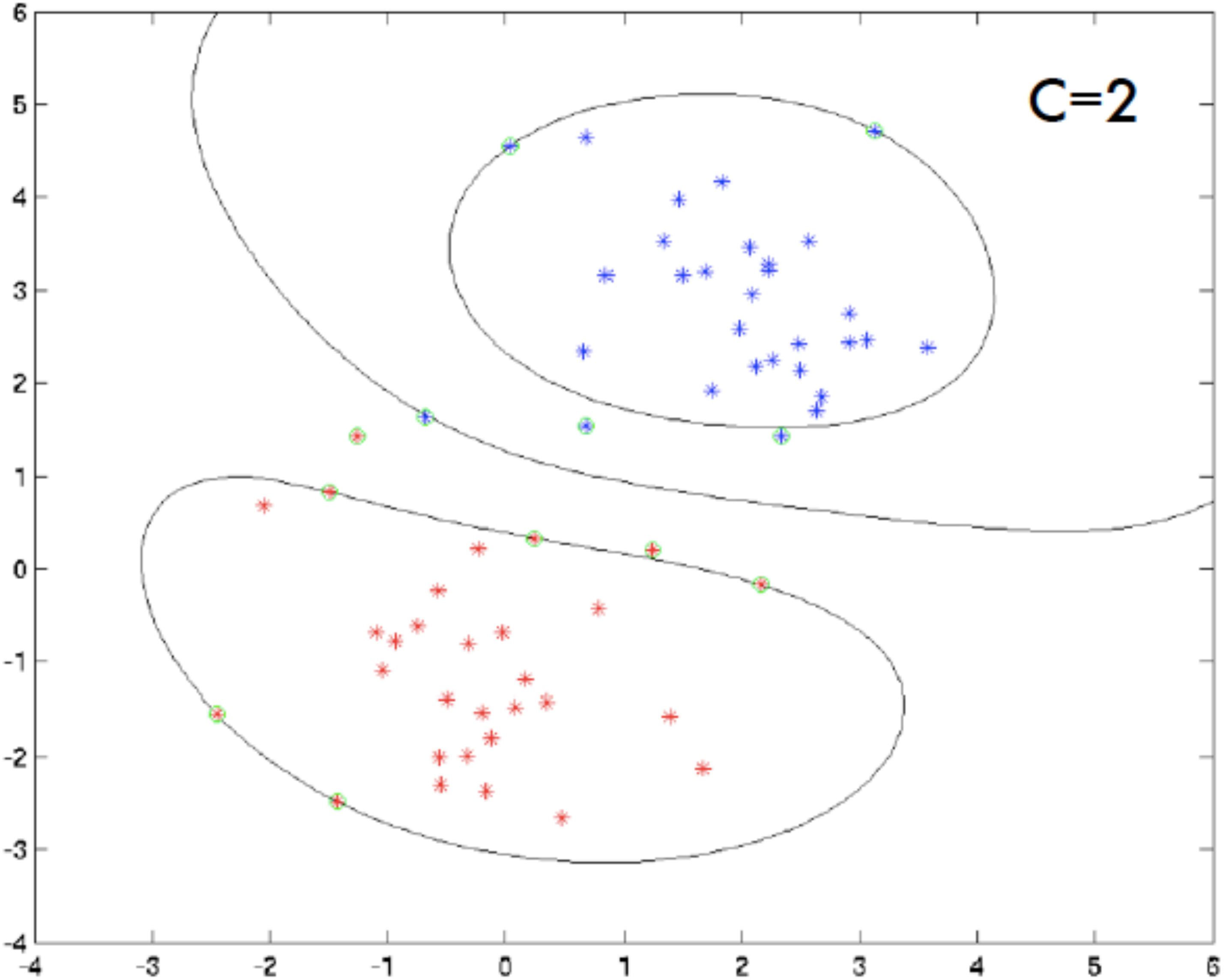

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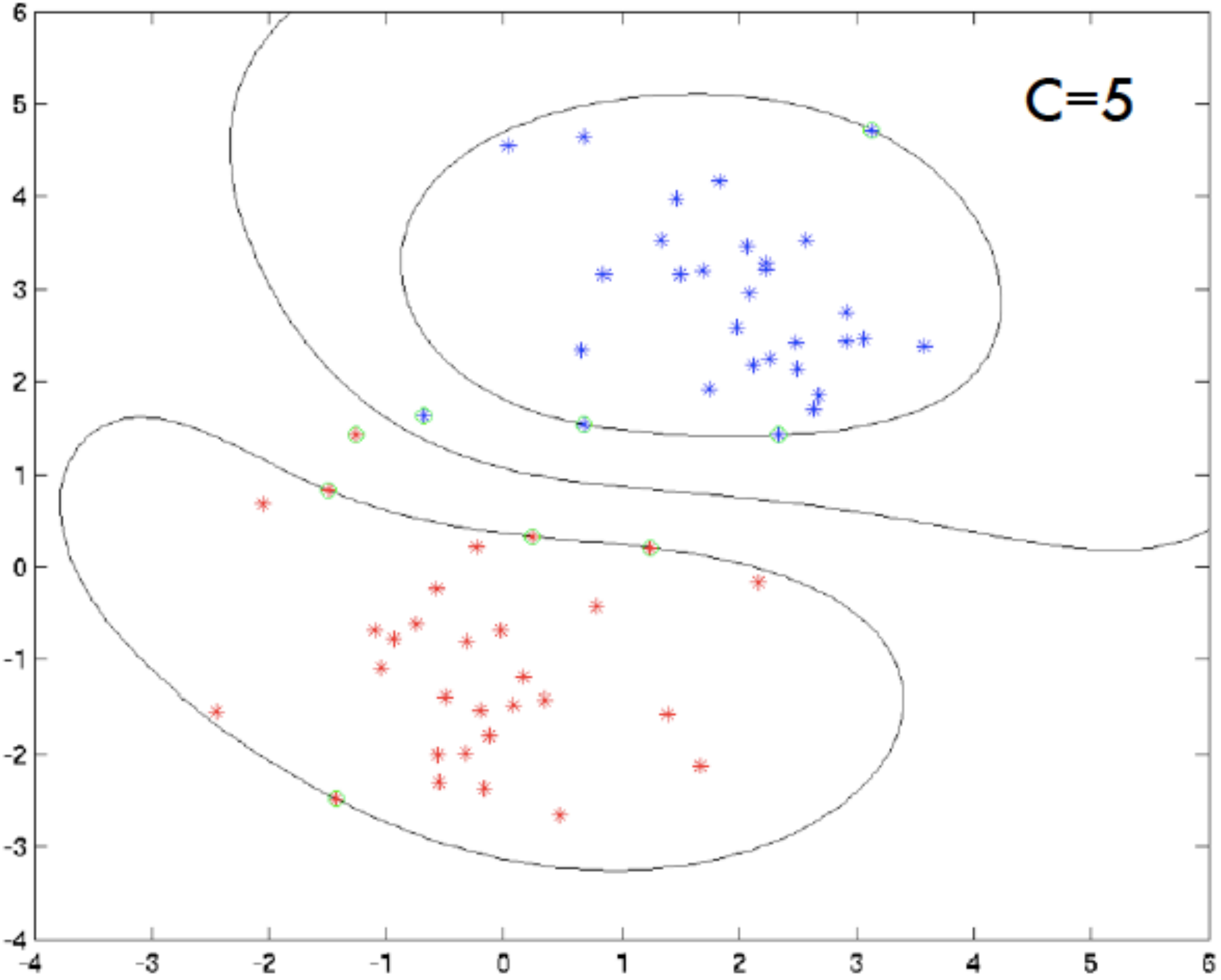
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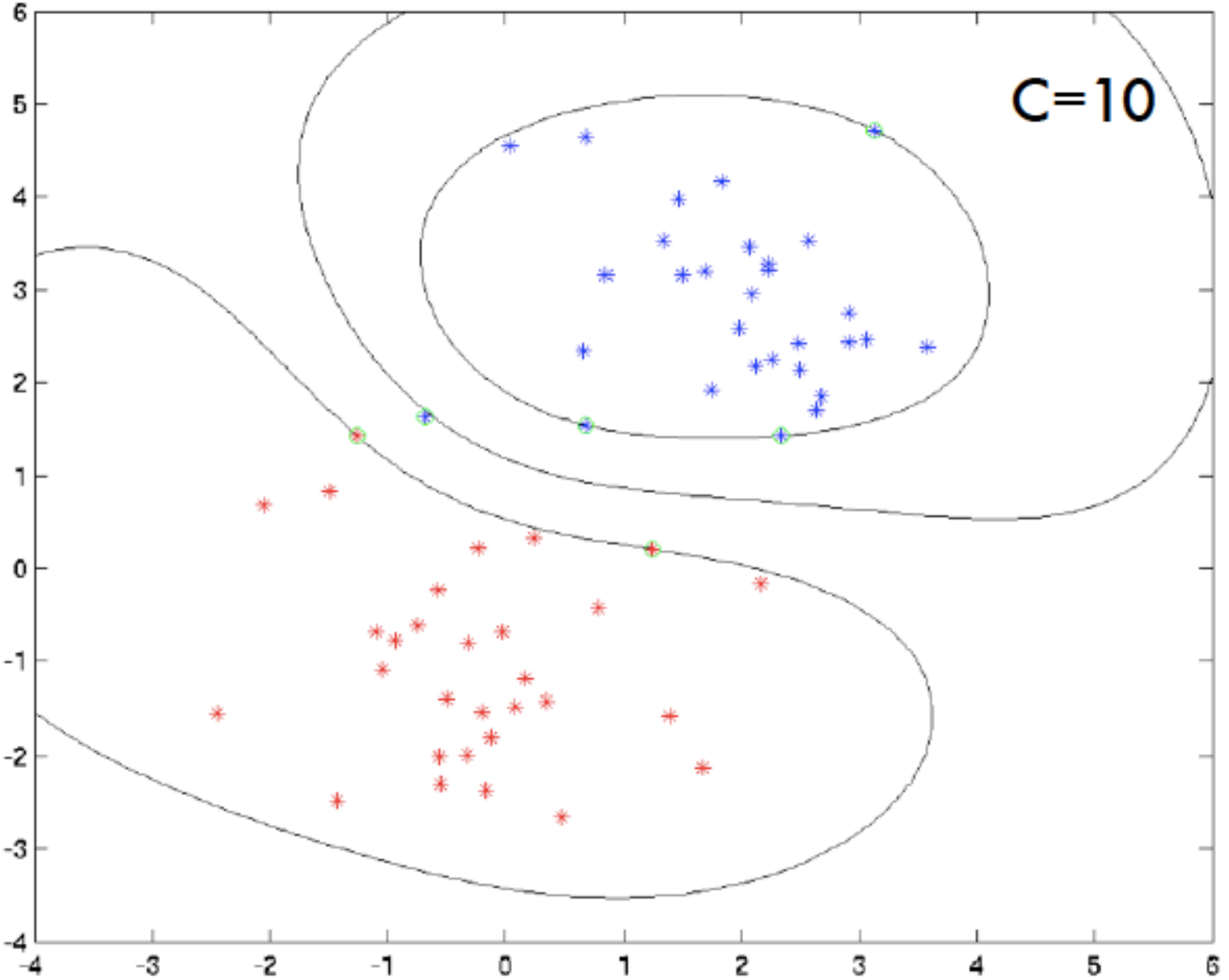
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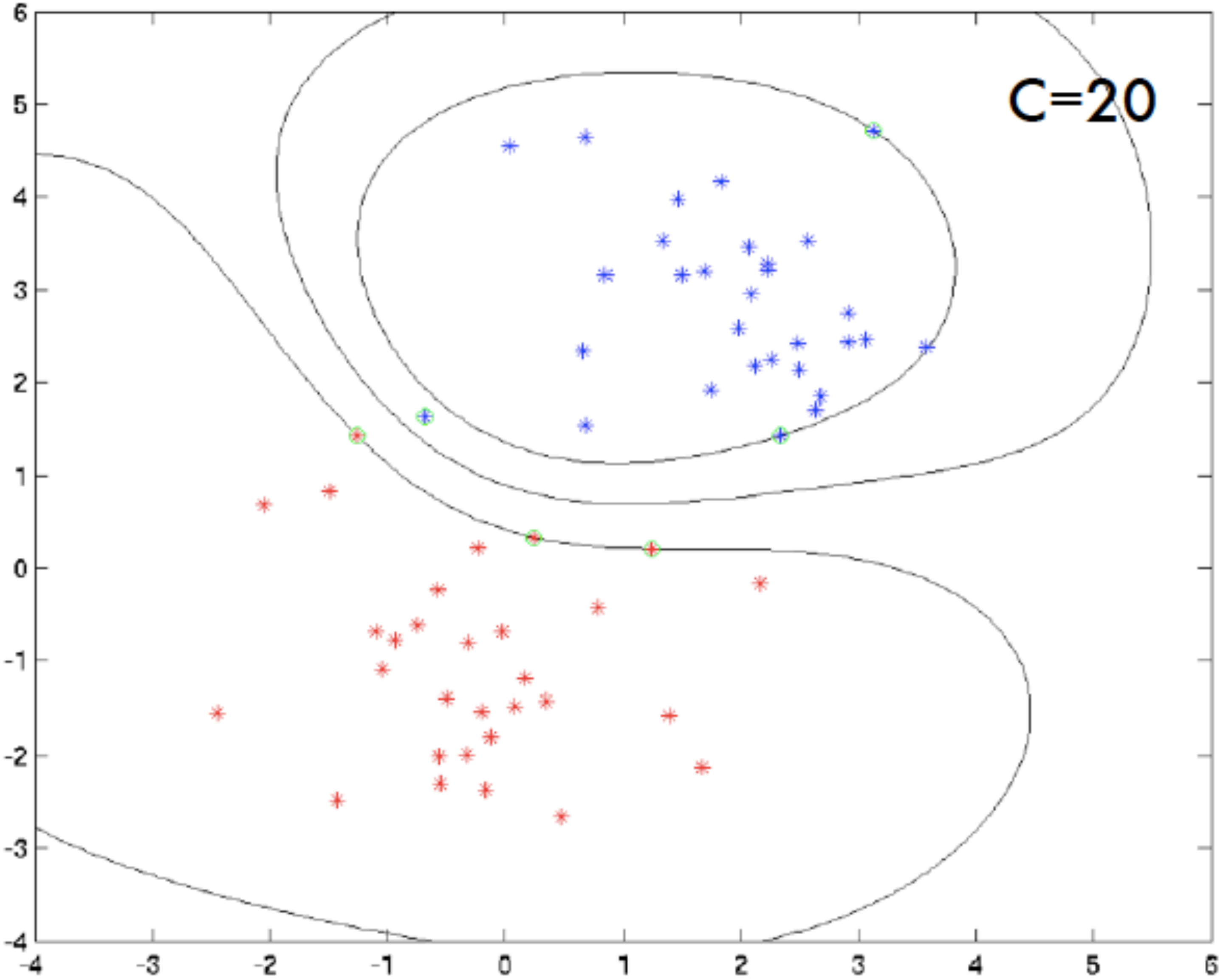
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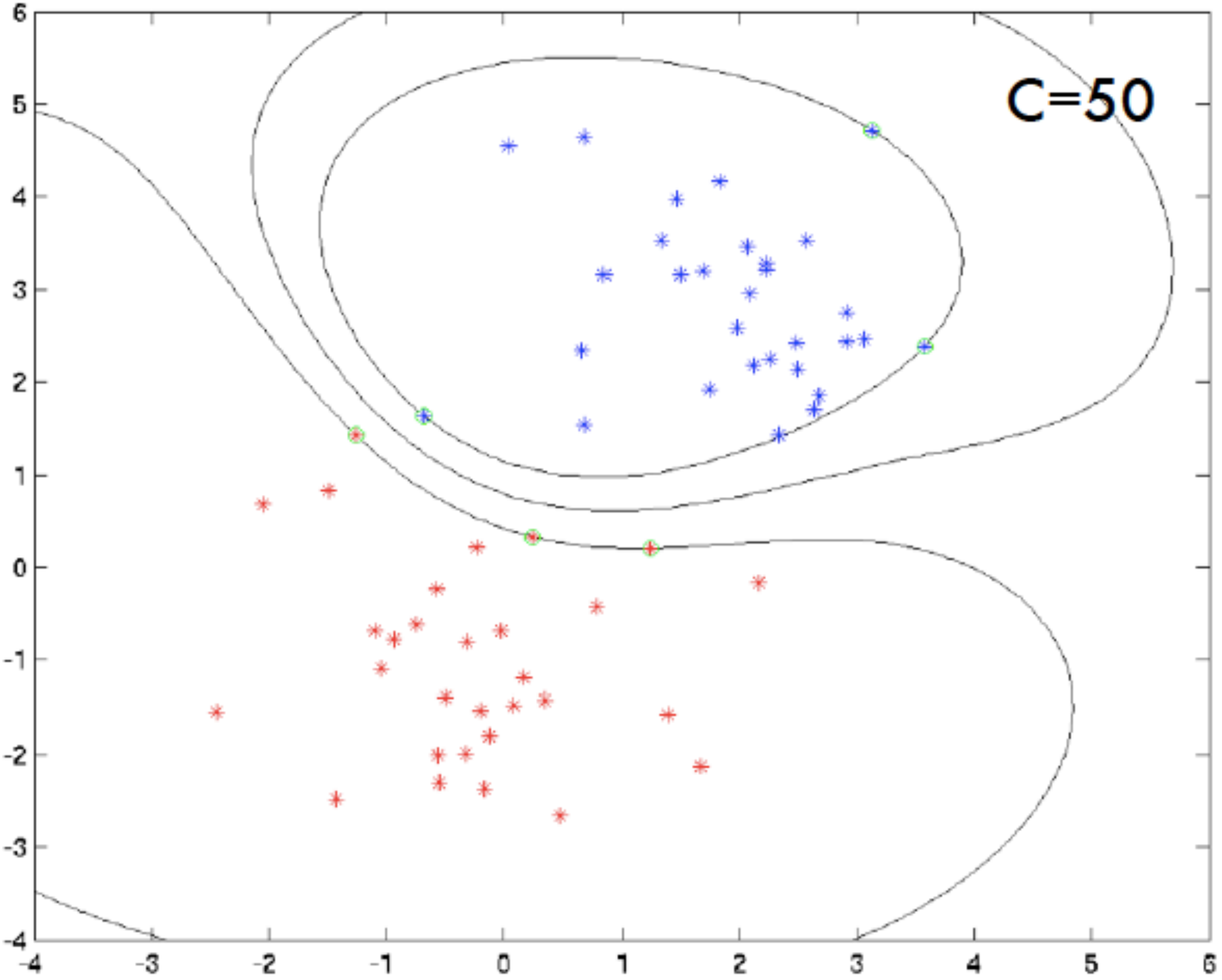
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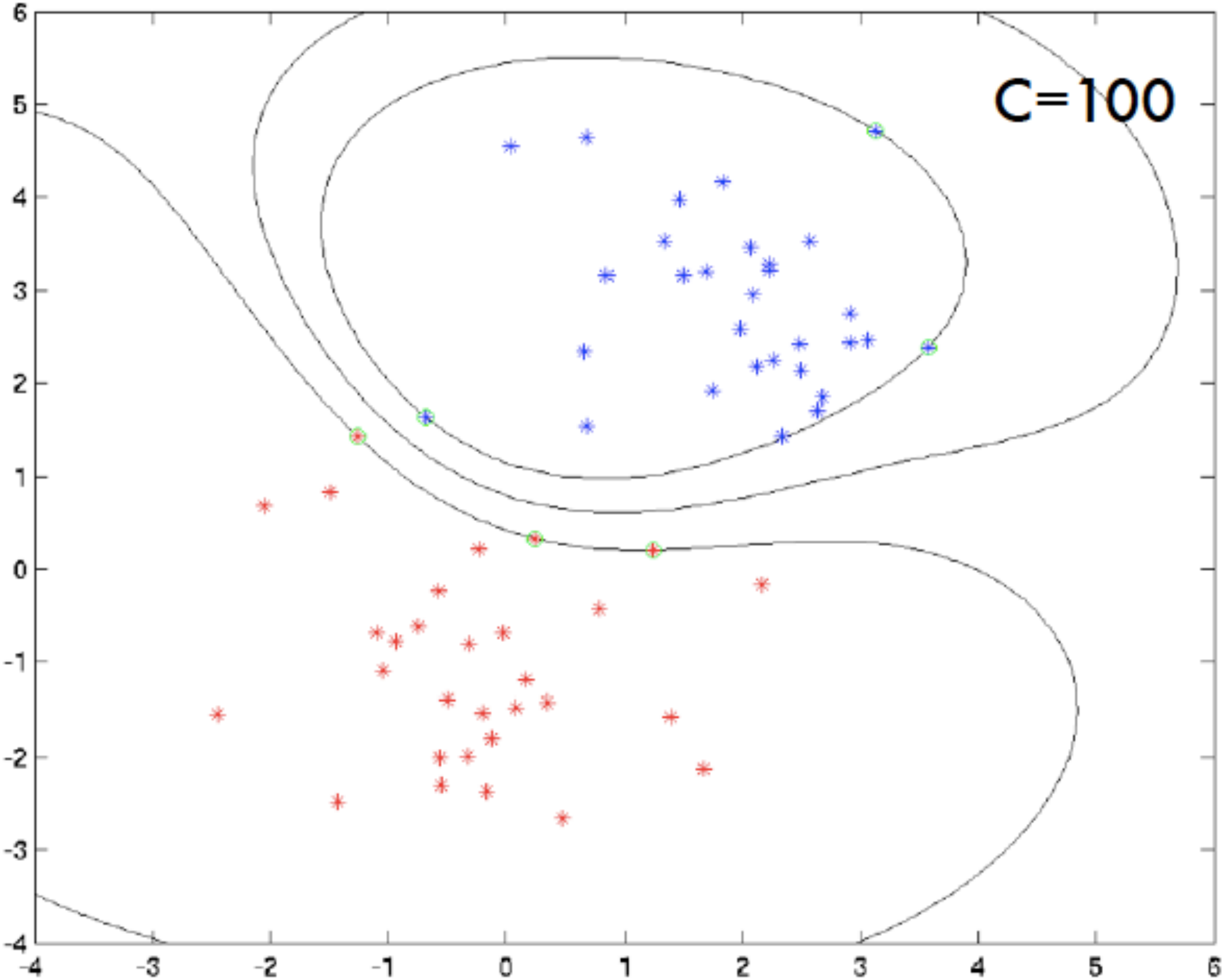
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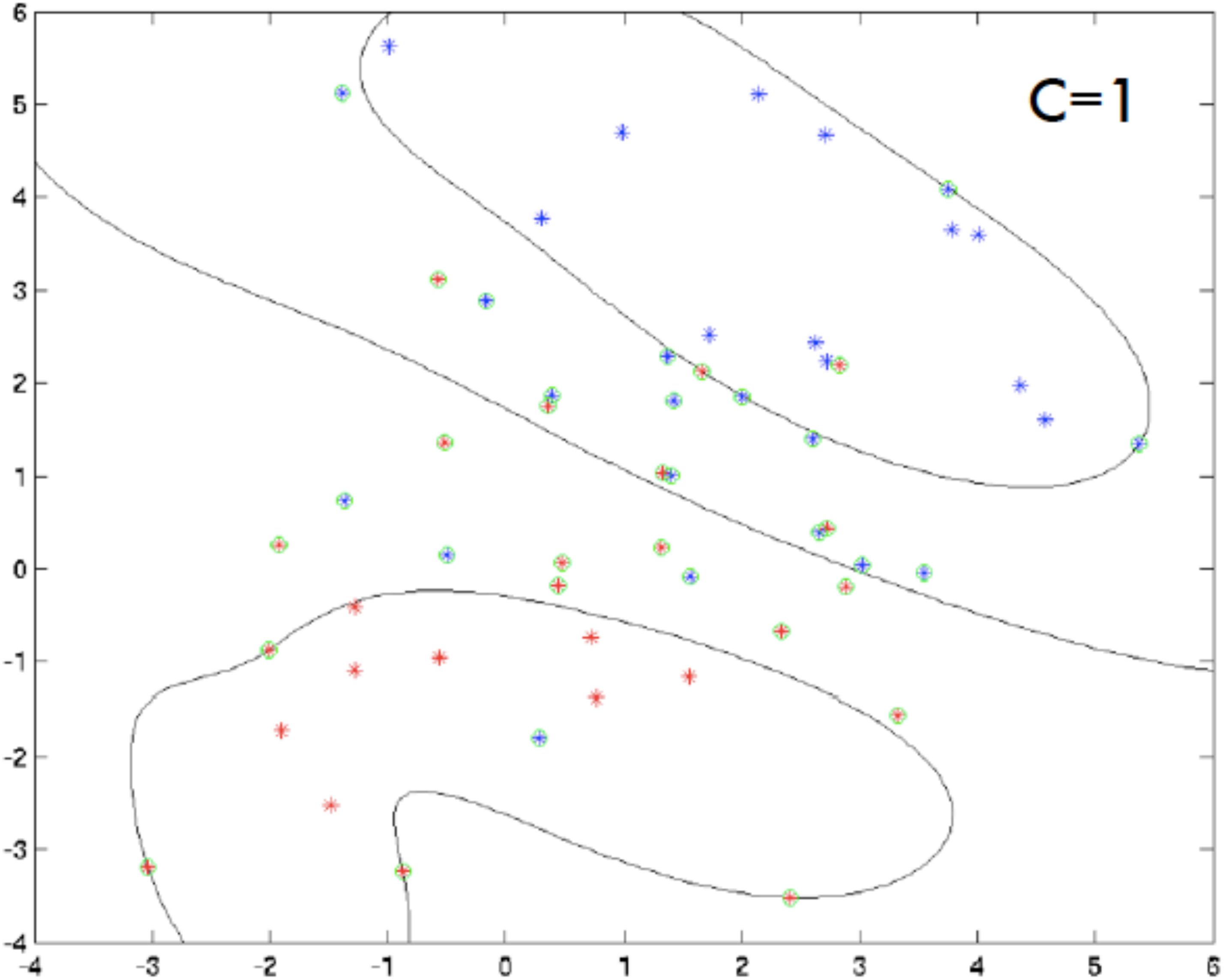
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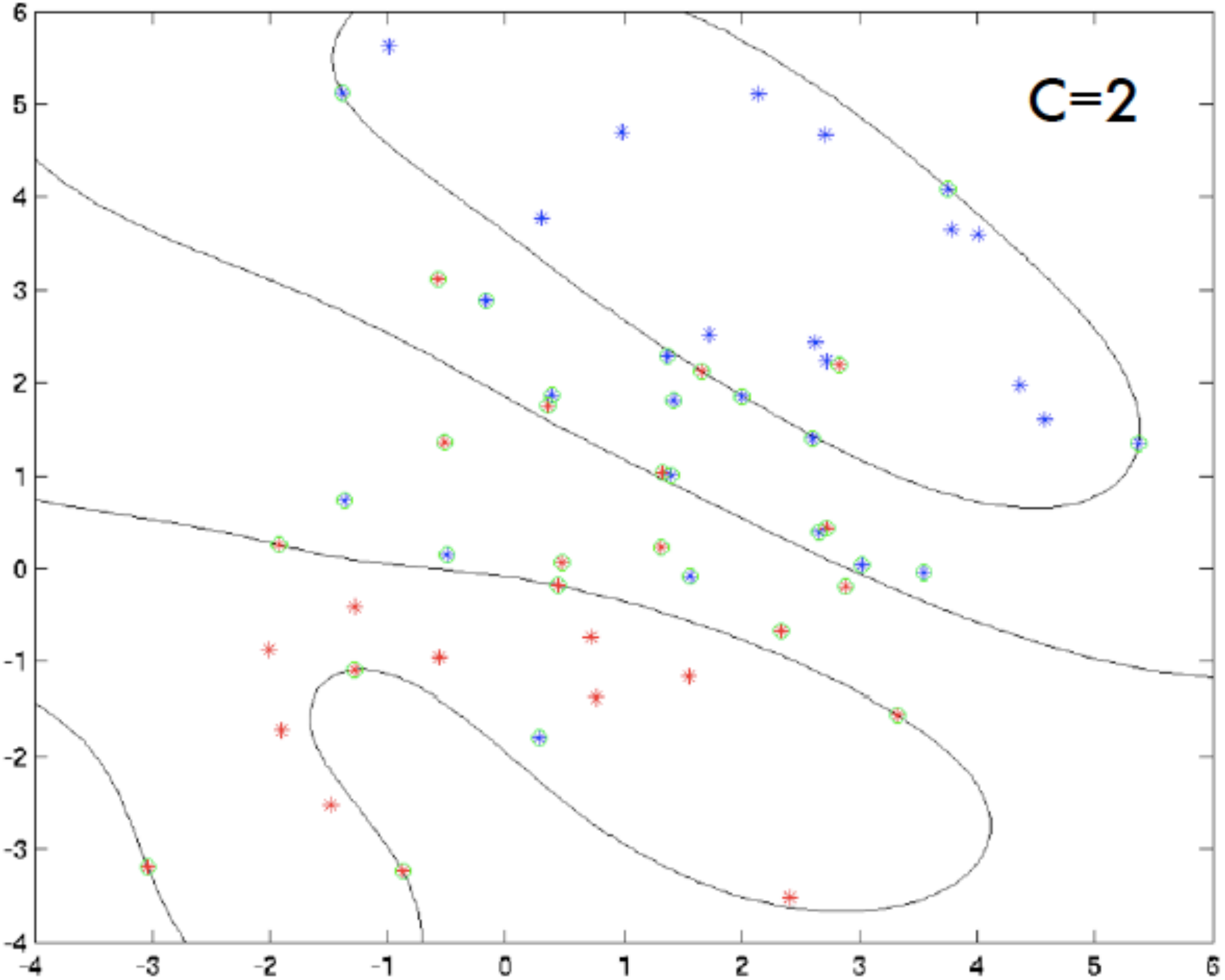
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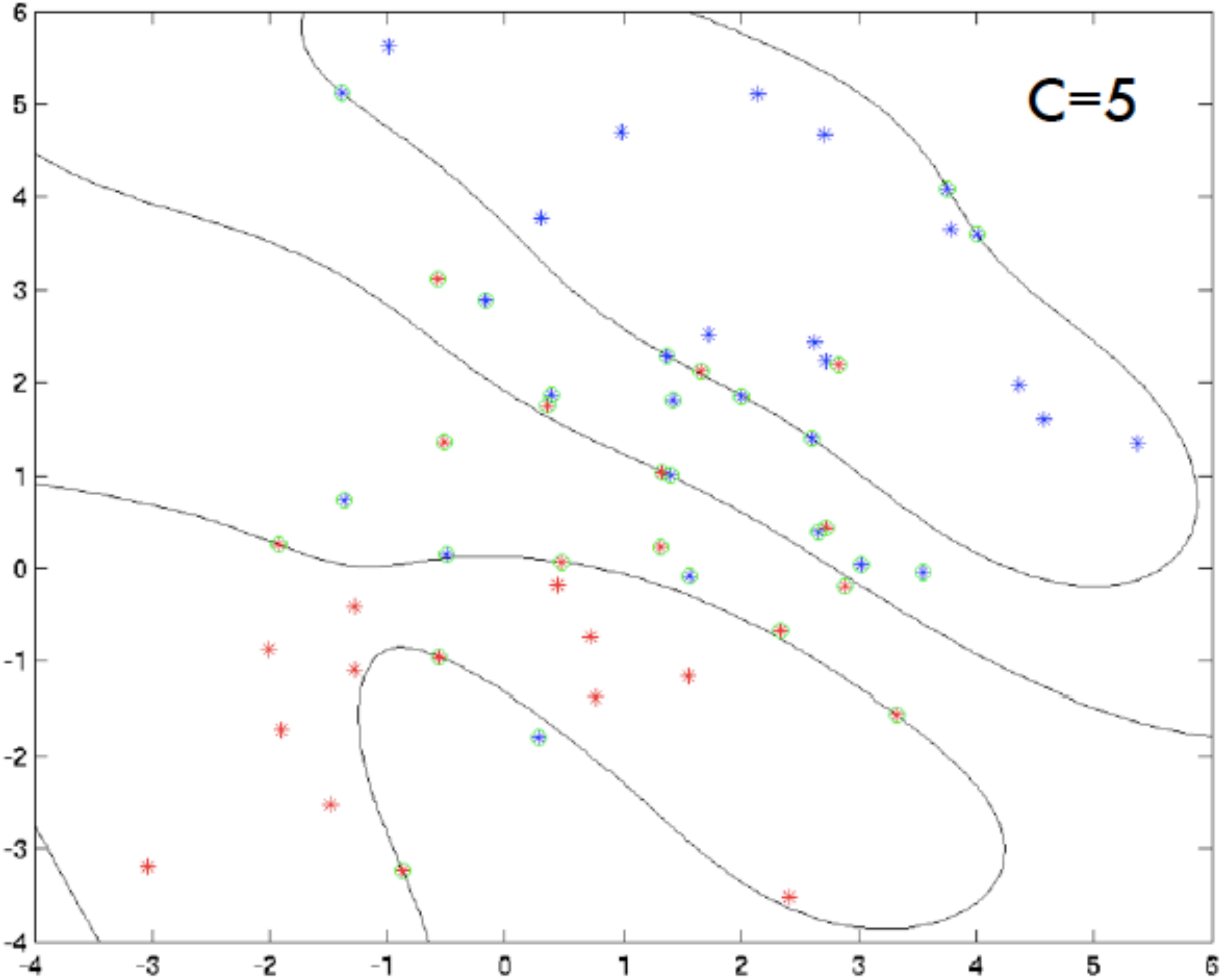
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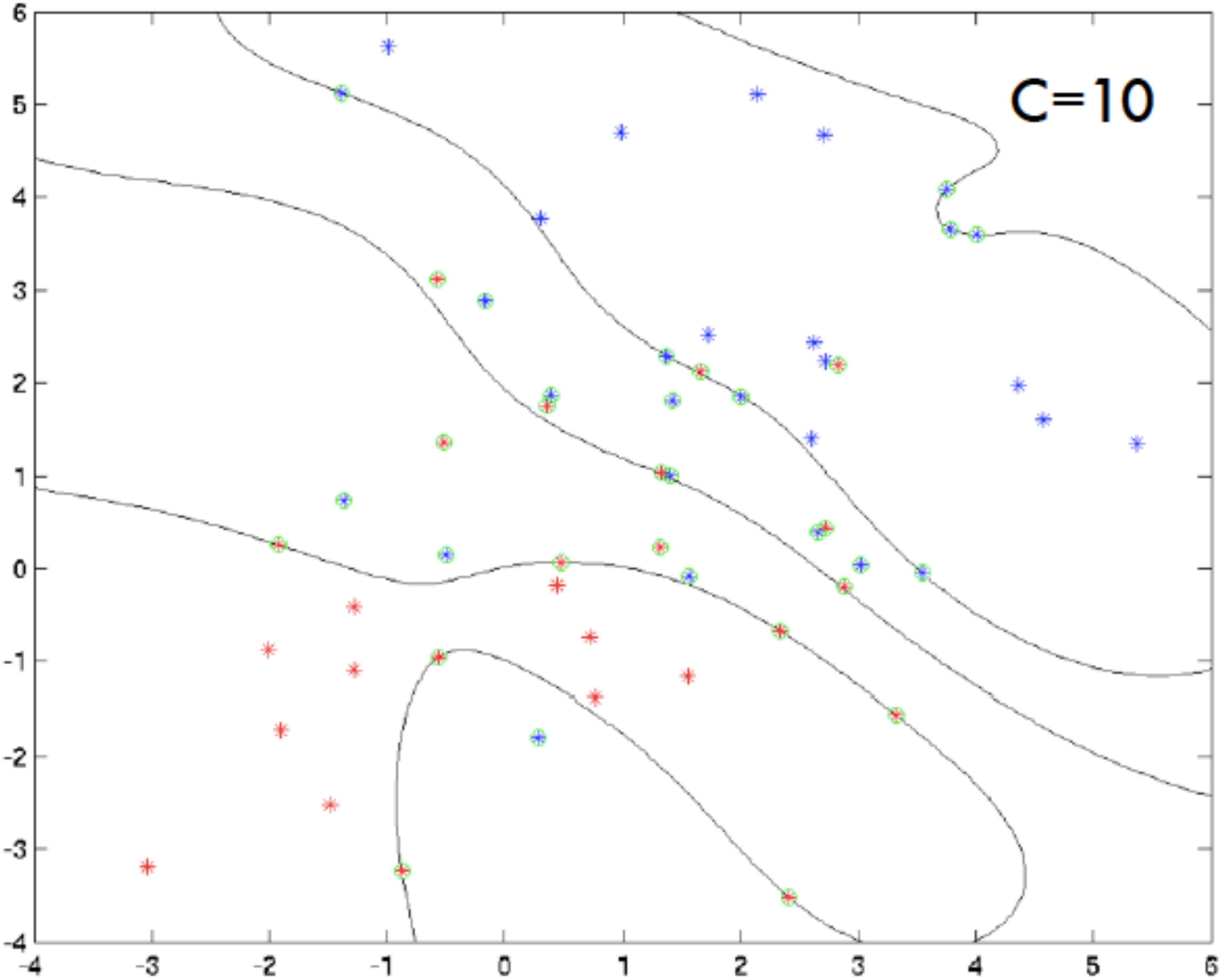
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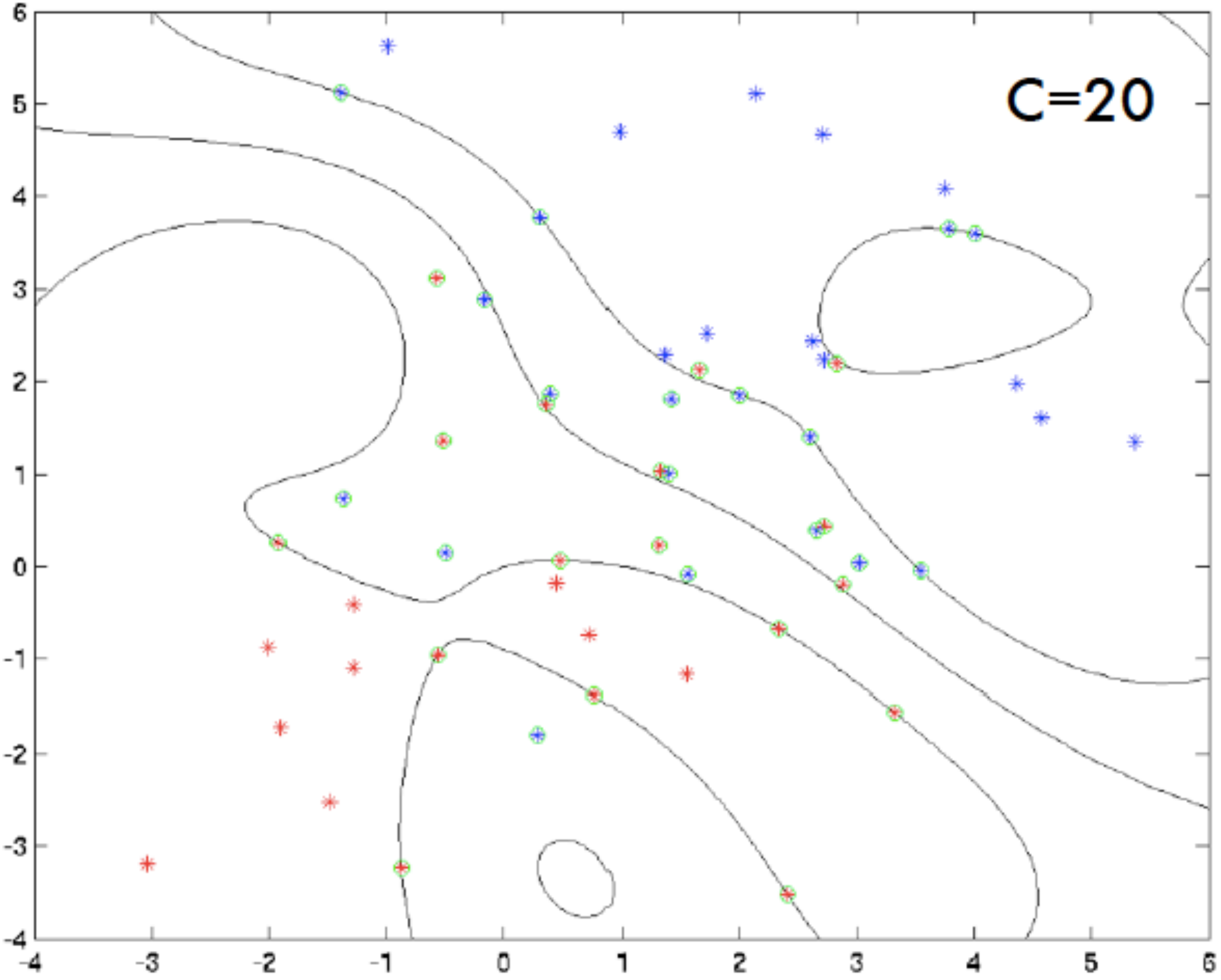
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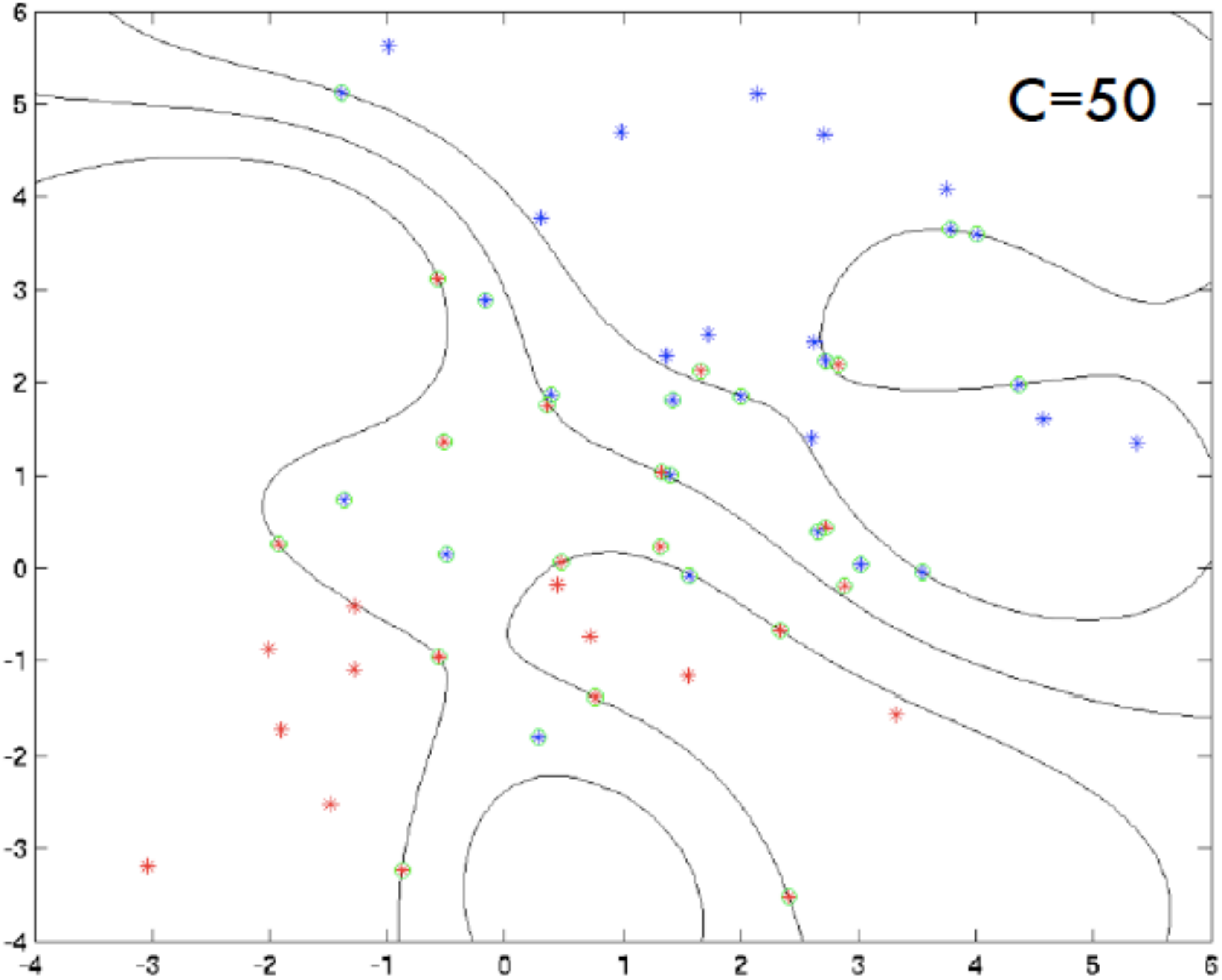
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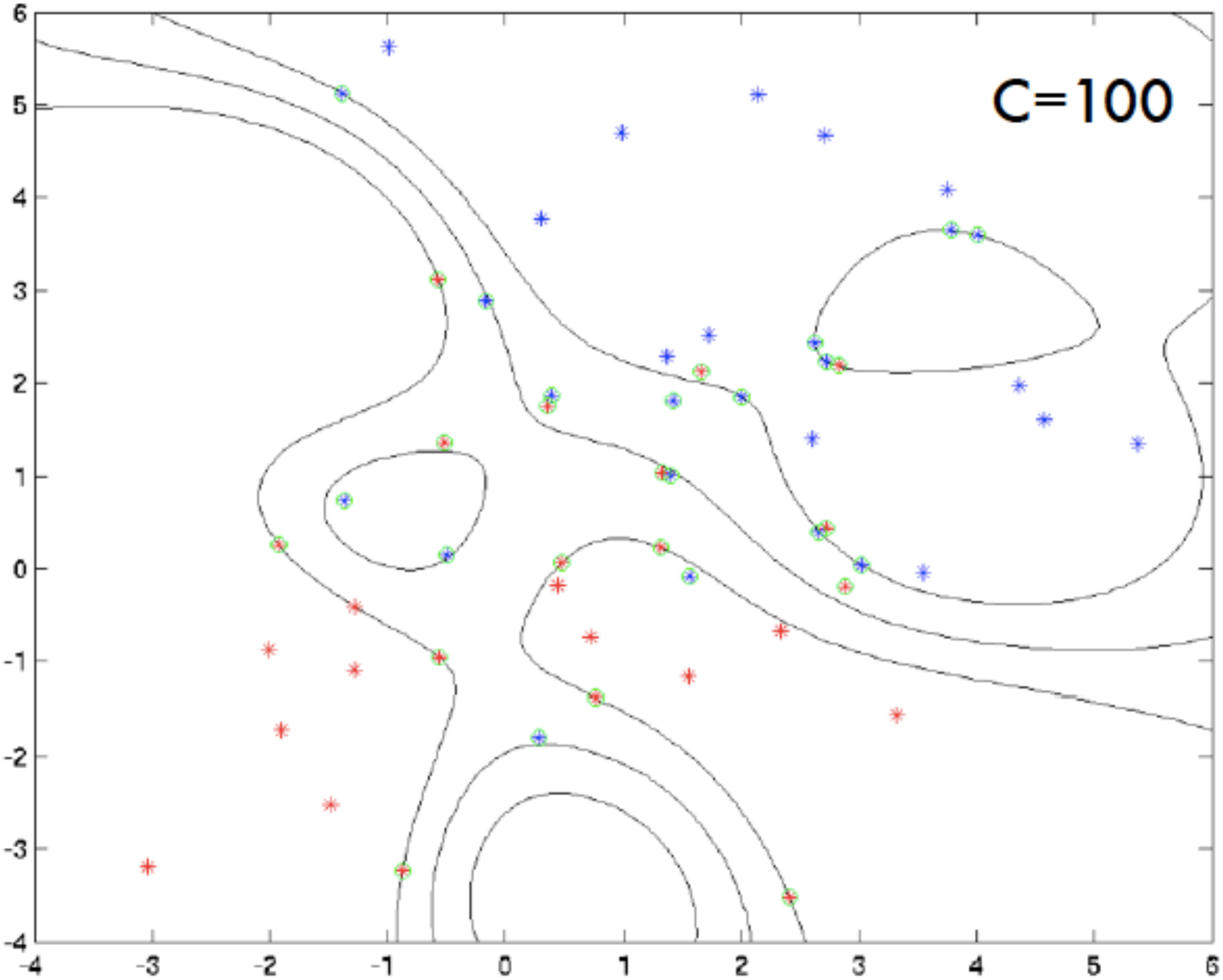
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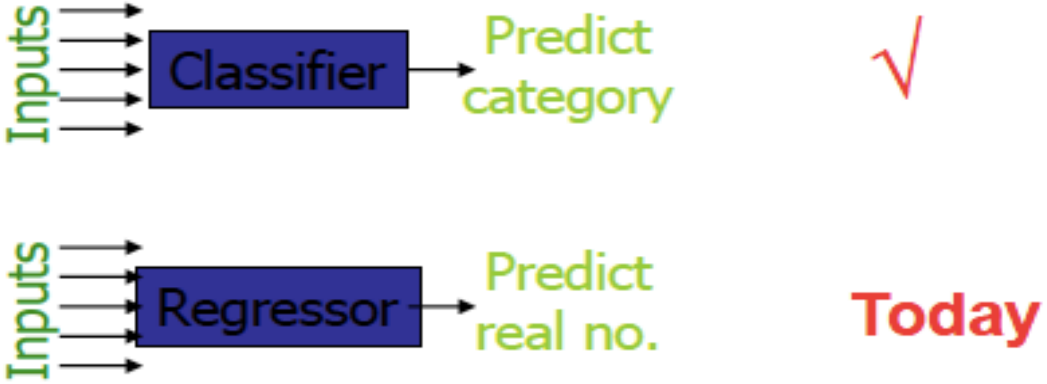
Regression

Where we are



Inputs

Classifier

Predict category

√

Inputs

Regressor

Predict real no.

Today

Choosing a restaurant

- In everyday life we need to make decisions by taking into account lots of factors
- The question is what weight we put on each of these factors (how important are they with respect to the others).

Choosing a restaurant

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- The question is what weight we put on each of these factors (how important are they with respect to the others).
- Assume we would like to build a recommender system for ***ranking*** potential restaurants based on an individuals' preferences

Reviews (out of 5 stars)	$	Distance	Cuisine (out of 10)
4	30	21	7
2	15	12	8
5	27	53	9
3	20	5	6

Choosing a restaurant

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- The question is what weight we put on each of these factors (how important are they with respect to the others).
- Assume we would like to build a recommender system for ***ranking*** potential restaurants based on an individuals' preferences
- If we have many observations we may be able to recover the weights

Reviews (out of 5 stars)	$	Distance	Cuisine (out of 10)
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5	27	53	9
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?

Linear regression

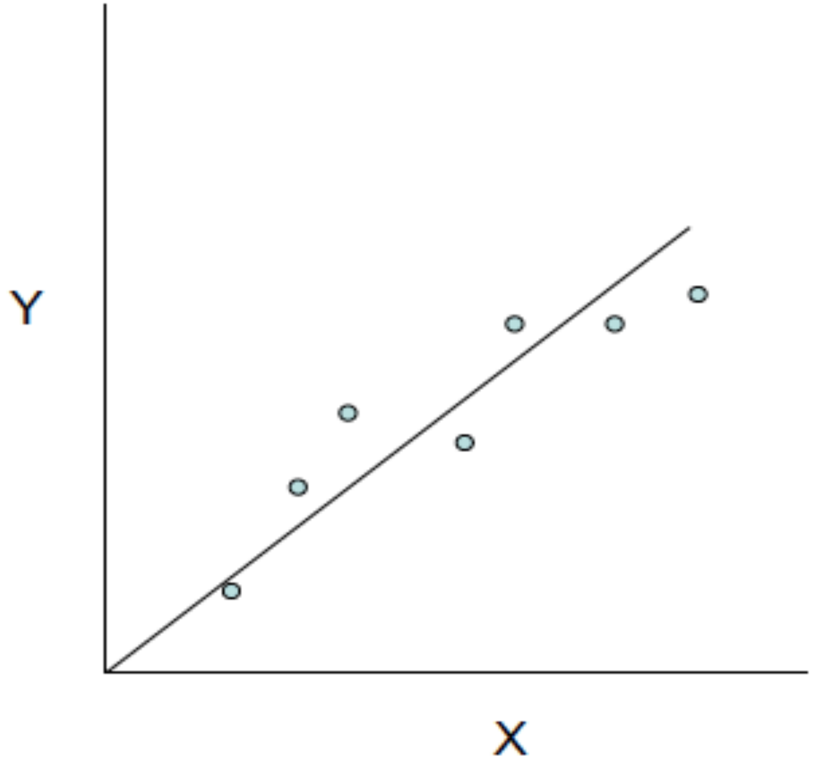
Linear regression

- Given an input x we would like to compute an output y
- For example:

  - Predict height from age

  - Predict Google's price from Yahoo's price

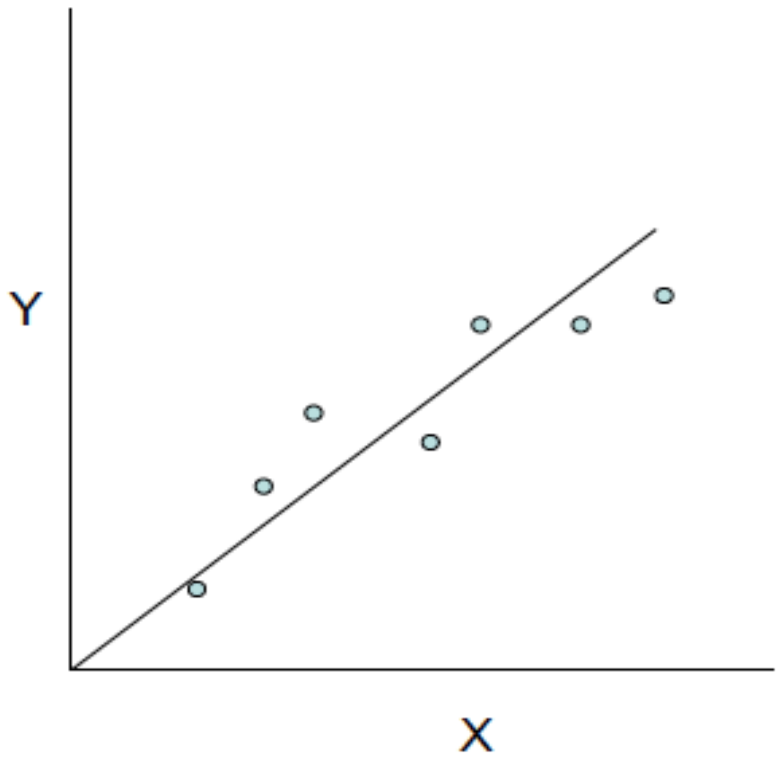
  - Predict distance from wall using sensor readings



Note that now Y can be **continuous**

Linear regression

- Given an input x we would like to compute an output y
- In linear regression we assume that y and x are related with the following equation:



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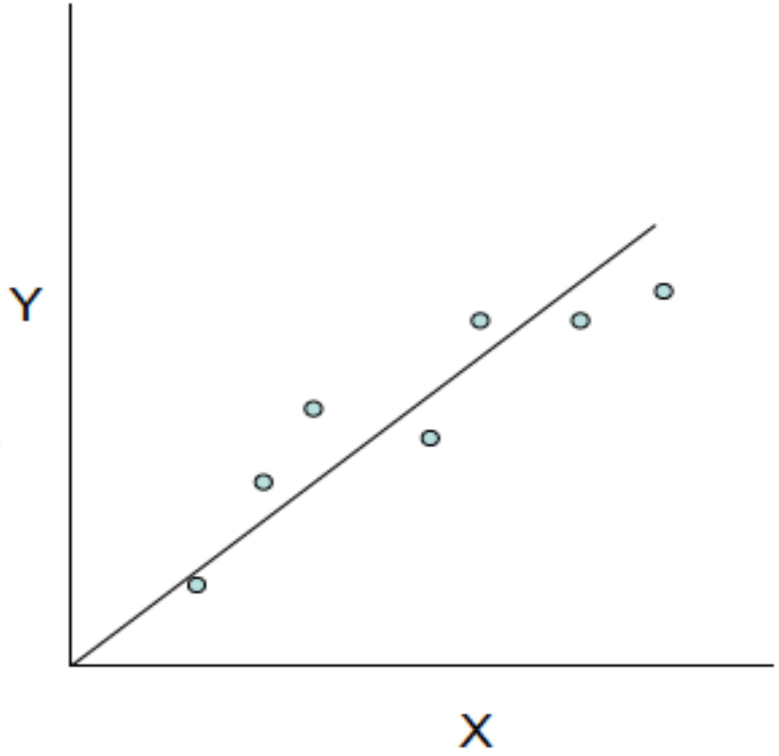
What we are trying to predict

Observed values

$$y = wx+\varepsilon$$



Linear regression

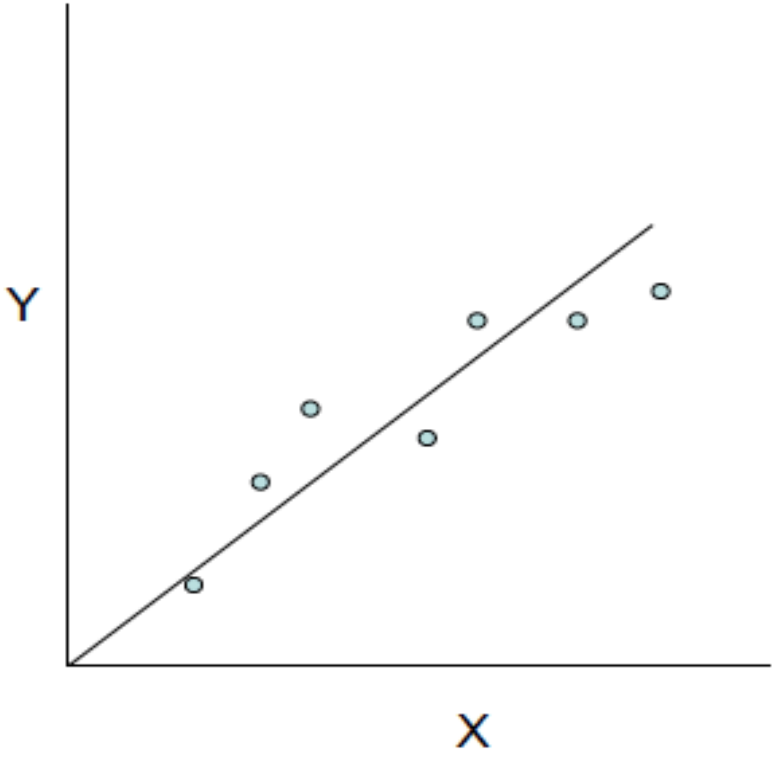
- Given an input x we would like to compute an output y
- In linear regression we assume that y and x are related with the following equation:

What we are trying to predict

Observed values

$$y = wx+\varepsilon$$

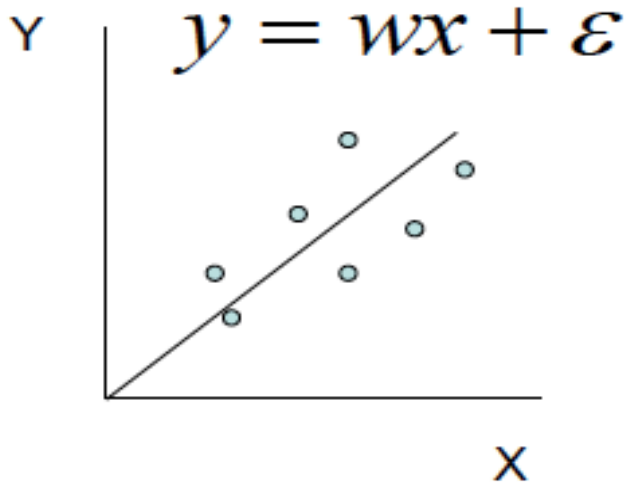
where w is a parameter and ε represents measurement or other noise



Linear regression

Linear regression

- Our goal is to estimate *w* from a training data of $<x_i,y_i>$ pairs

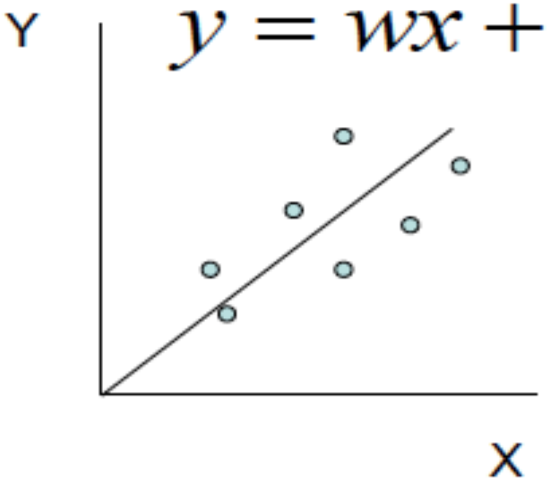


Linear regression

- Our goal is to estimate *w* from a training data of <x_i,y_i> pairs
- One way to find such relationship is to minimize the a least squares error:

$$\arg\min_w \sum_i (y_i - wx_i)^2$$

$$y = wx + \varepsilon$$



- Several other approaches can be used as well
- So why least squares?

  - minimizes squared distance between measurements and predicted line
  - has a nice probabilistic interpretation
  - easy to compute

If the noise is Gaussian with mean 0 then least squares is also the maximum likelihood estimate of w

5 minutes break

Solving linear regression using least squares minimization

- We just take the derivative w.r.t. to w and set to 0:

Solving linear regression using least squares minimization

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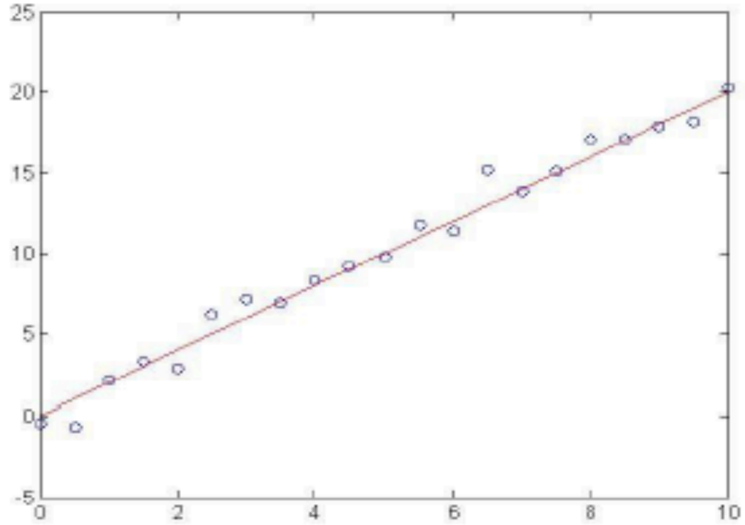
$$\sum_i x_i y_i = \sum_i wx_i^2 \Rightarrow$$

$$w = \frac{\sum_i x_i y_i}{\sum_i x_i^2}$$

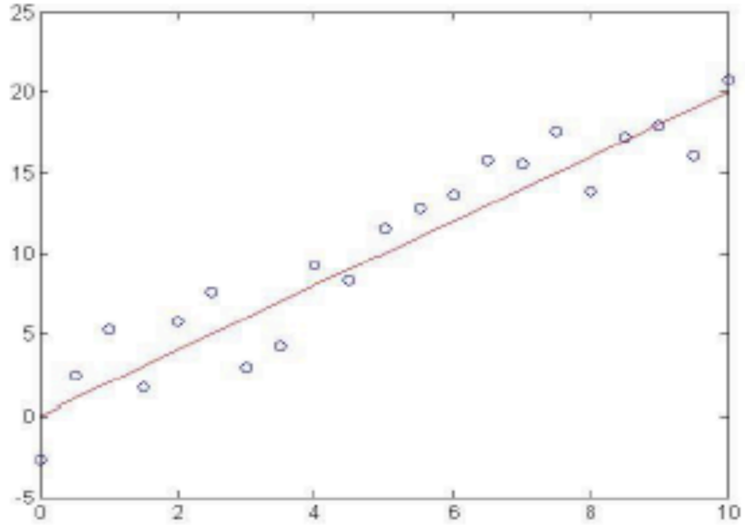
Regression example

- Generated: w=2
- Recovered: w=2.03
- Noise: std=1



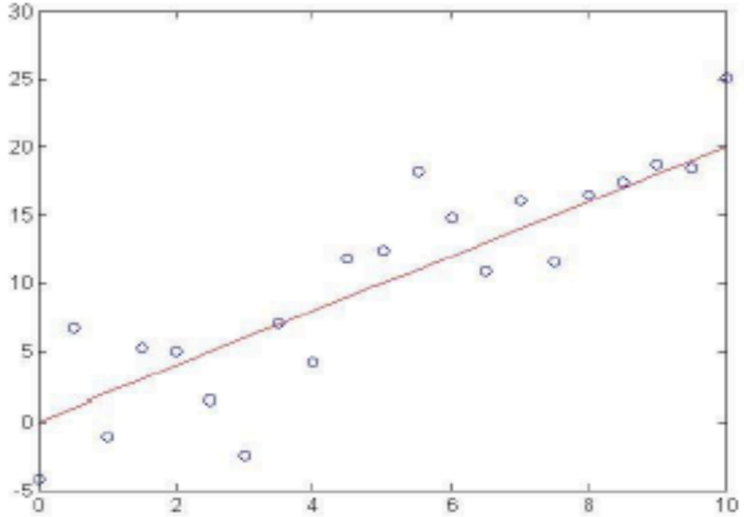
Regression example

- Generated: w=2
- Recovered: w=2.05
- Noise: std=2



Regression example

- Generated: w=2
- Recovered: w=2.08
- Noise: std=4

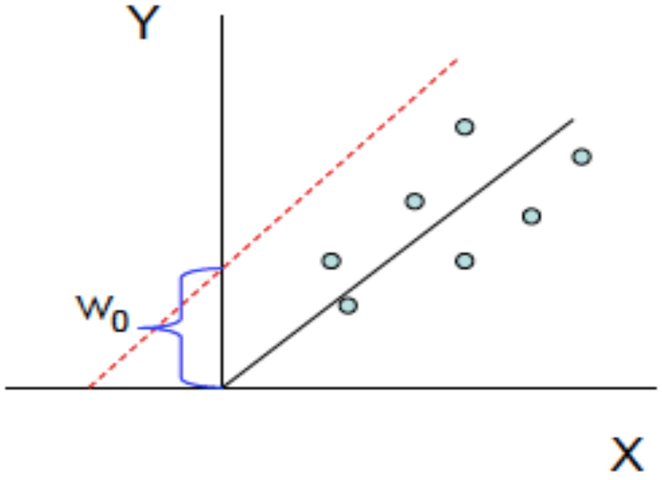


Bias term

Bias term

- So far we assumed that the line passes through the origin
- What if the line does not?
- No problem, simply change the model to

$$y = w_0 + w_1 x + \varepsilon$$



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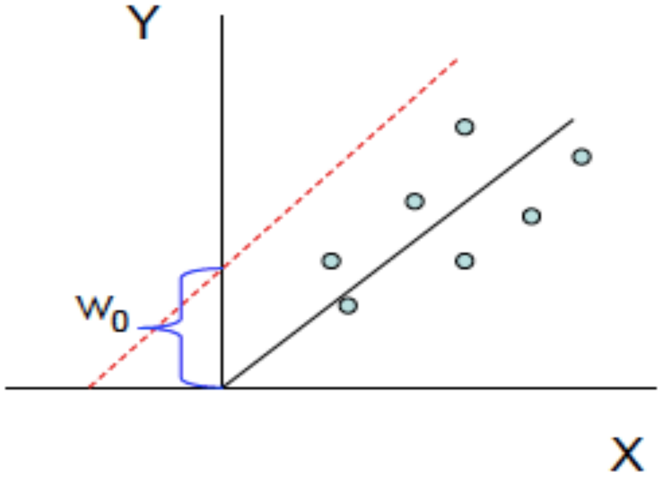

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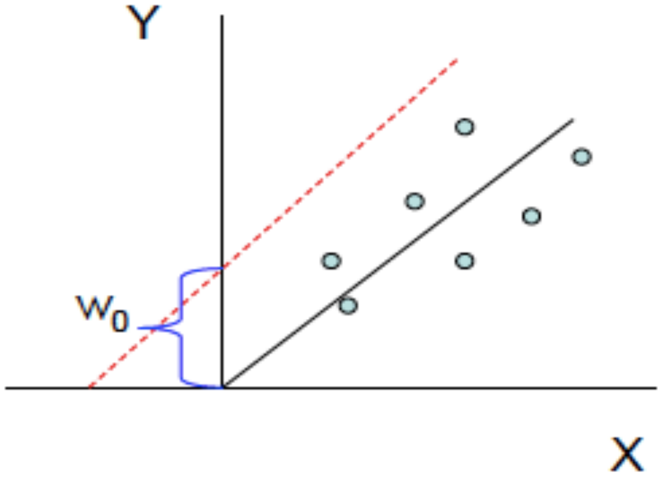

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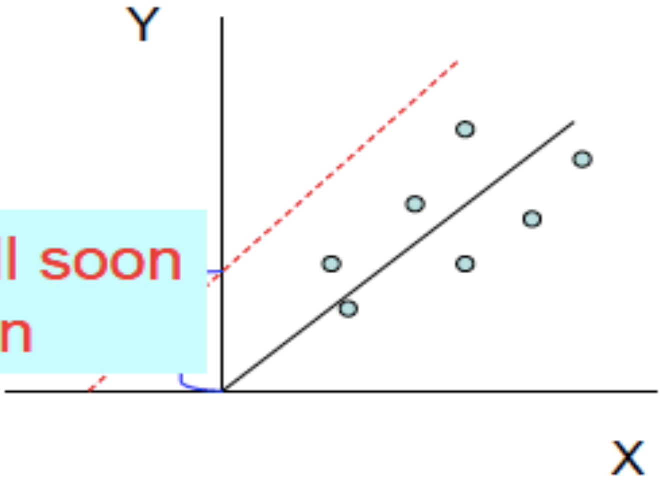
y = w

Just a second, we will soon give a simpler solution



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Multivariate regression

- What if we have several inputs?

  - Stock prices for Yahoo, Microsoft and Ebay for the Google prediction task

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Google's stock price

Microsoft's stock price

Yahoo's stock price

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Yes. As long as the coefficients are linear the equation is still a linear regression problem!

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- Gaussian: $\phi_j(x) = \frac{(x-\mu_j)}{2\sigma_j^2}$

- Sigmoid: $\phi_j(x) = \frac{1}{1+\exp(-s_j x)}$

Any function of the input values can be used. The solution for the parameters of the regression remains the same.

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- Once again we can use 'least squares' to find the optimal solution.

LMS for the general linear regression problem

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Our goal is to minimize the following loss function:

$$J(\mathrm{w}) = \sum_{i} (y^i - \sum_{j} w_j \phi_j(x^i))^2$$

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Define:

$$\Phi = \begin{pmatrix} \phi_0(x^1) & \phi_1(x^1) & \cdots & \phi_m(x^1) \\ \phi_0(x^2) & \phi_1(x^2) & \cdots & \phi_m(x^2) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_0(x^n) & \phi_1(x^n) & \cdots & \phi_m(x^n) \end{pmatrix}$$

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Then deriving w
we get:

$$\mathbf{w} = (\Phi^T \Phi)^{-1} \Phi^T \mathbf{y}$$

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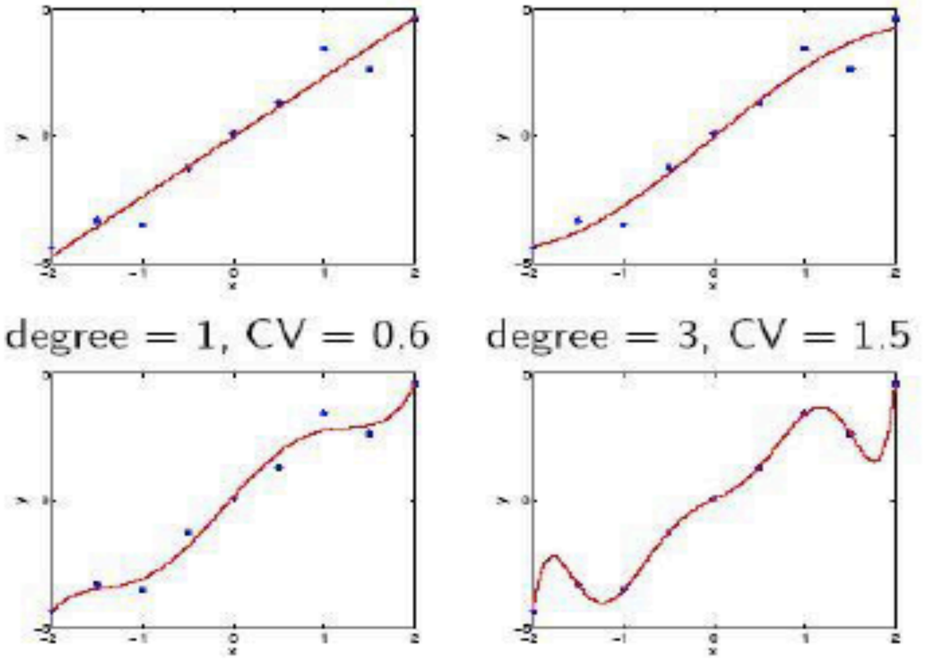
k+1 entries vector

n by k+1 matrix

n entries vector

This solution is also known as ‘psuedo inverse’

Example: Polynomial regression



degree = 1, CV = 0.6   degree = 3, CV = 1.5

degree = 5, CV = 6.0   degree = 7, CV = 15.6

A probabilistic interpretation

Our least squares minimization solution can also be motivated by a probabilistic in interpretation of the regression problem: $y = \mathbf{w}^{\mathrm{T}} \phi(x) + \varepsilon$

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The MLE for w in this model is the same as the solution we derived for least squares criteria:

$$\mathbf{w} = (\Phi^T\Phi)^{-1}\Phi^T\mathbf{y}$$

Other types of linear regression

- Linear regression is a useful model for many problems
- However, the parameters we learn for this model are **global**; they are the same regardless of the value of the input x
- Extension to linear regression adjust their parameters based on the region of the input we are dealing with

That’s all!