Eigenvalue and Eigenvector

Exercise Find the eigenvalues and eigenvectors. Then find diagonalization if possible.

1) $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$

Answer

$$\det\begin{bmatrix} 3 - \lambda & 1 \\ 2 & 4 - \lambda \end{bmatrix} = (3 - \lambda)(4 - \lambda) - 2 = (2 - \lambda)(5 - \lambda).$$

For $\lambda_1 = 2$, we have

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix},$$

and $v_1 = (1, -1)$ is a basis of the eigenspace nul(A - 2I). For $\lambda_2 = 5$, we have

$$A - 5I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix},$$

and $v_2 = (1, 2)$ is a basis of the eigenspace nul(A - 5I). A diagonalization of A is

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}.$$

2) $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

Answer

$$\det\begin{bmatrix} 2 - \lambda & 1 \\ -1 & 4 - \lambda \end{bmatrix} = (2 - \lambda)(4 - \lambda) + 1 = (3 - \lambda)^2.$$

The only eigenvalue is $\lambda = 3$. We have

$$A - 3I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix},$$

and $v = (1, 1)$ is a basis of the eigenspace nul(A - 3I). A is not diagonalizable.

3) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

Answer

$$\det\begin{bmatrix} 2 - \lambda & 1 \\ 0 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2.$$

The only eigenvalue is $\lambda = 2$. We have

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and $v = (1, 0)$ is a basis of $\mathrm{nul}(A - 2I)$. A is not diagonalizable.

4) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Answer

$$\det\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix} = \lambda^2.$$

The only eigenvalue is $\lambda = 0$, and $v = (0, 1)$ is a basis of $\mathrm{nul}(A - 0I) = \mathrm{nul}A$. A is not diagonalizable.

5) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Answer One may follow the steps and carry out the computation. On the other hand, the matrix represents multiplying 1 to x_1-direction and multiplying 2 to x_2-direction. So the standard basis vectors are eigenvectors, with 1 and 2 as eigenvalues.

A is already diagonal, and is diagonalizable by tautology: $A = IAI^{-1}$.

Remark Any diagonal matrix has the standard basis as eigenvectors. The process of diagonalization is trying to find a basis so that, by setting the new basis to be the standard one, the matrix becomes a diagonal one.

6) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Answer The situation is the same as the previous one. A has the standard basis e_1, e_2, e_3 as eigenvectors, with eigenvalues 1, 2, 3. A is diagonalizable by tautology.

7) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Answer

$$\det\begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^3.$$

The only eigenvalue is $\lambda = 2$. We have

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $v = (1, 0, 0)$ is a basis of $\mathrm{nul}(A - 2I)$. A is not diagonalizable.

Remark The matrices in problems 3 and 7 are typical non-diagonalizable matrices. The general form is the Jordan block

$$J = \begin{bmatrix} a & 1 & 0 & .. & 0 \\ 0 & a & 1 & .. & 0 \\ 0 & 0 & a & .. & 0 \\ : & : & : & & : \\ 0 & 0 & 0 & .. & a \end{bmatrix}.$$

8) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

Answer

$$\det\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda)(3-\lambda).$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. We have

$$A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A - 2I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 2, 2)$ span the respective eigenspaces. A diagonalization is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}^{-1}.$$

9) $\begin{bmatrix} 1 & 10 & 5 \\ 0 & -4 & 0 \\ 5 & 10 & 1 \end{bmatrix}$

Answer By cofactor expansion along [row 2], we have

$$\det\begin{bmatrix} 1-\lambda & 10 & 5 \\ 0 & -4-\lambda & 0 \\ 5 & 10 & 1-\lambda \end{bmatrix} = (-4-\lambda)\det\begin{bmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{bmatrix} = (-4-\lambda)^2(6-\lambda).$$

The eigenvalues are $\lambda_1 = -4, \lambda_2 = 6$, and

$$A + 4I = \begin{bmatrix} 5 & 10 & 5 \\ 0 & 0 & 0 \\ 5 & 10 & 5 \end{bmatrix}, \quad A - 6I = \begin{bmatrix} -5 & 10 & 5 \\ 0 & -10 & 0 \\ 5 & 10 & -5 \end{bmatrix}.$$

The vectors $v_1 = (1, 0, -1)$, $v_2 = (0, 1, -2)$ form a basis of nul$(A + 4I)$. The vector $v_3 = (1, 0, 1)$ forms a basis of nul$(A - 6I)$. A diagonalization is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1}.$$

$$\begin{matrix} -1 & -2 & 1 \end{matrix} \quad \begin{matrix} 0 & 0 & 6 \end{matrix} \quad \begin{matrix} -1 & -2 & 1 \end{matrix}$$

10) $\begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix}$

Answer By an earlier exercise, the characterisitc equation is $-\lambda(6 + \lambda)^2 = 0$. The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -6$, and

$$A - 0I = \begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix}, \quad A + 6I = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}.$$

The vector $v_1 = (-1, -2, 1)$ forms a basis of nulA. The vectors $v_2 = (1, 0, 1)$, $v_3 = (-1, 1, 1)$ form a basis of nul(A + 6I). A diagonalization is

$$A = \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1}.$$

11) $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ 1 & 4 & -2 \end{bmatrix}$

Answer By an earlier exercise, the charcaterisitc equation is $(1 - \lambda)(3 - \lambda)(-1 - \lambda) = 0$. The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, $\lambda_3 = -1$, and

$$A - I = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 1 & 4 & -3 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 4 & -5 \end{bmatrix}, \quad A + I = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 4 & -1 \\ 1 & 4 & -1 \end{bmatrix}.$$

The vectors $v_1 = (-1, 1, 1)$, $v_2 = (1, 1, 1)$, $v_3 = (1, 1, 5)$ span the respective eigenspaces. A diagonalization is

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 5 \end{bmatrix}^{-1}.$$

12) $\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

Answer In the first equality, we use [row 2] + [row 1] and [row 3] + [row 1]. In the second equality, we use (-1)[col 1] + [col 2] and (-1)[col 1] + [col 3].

$$\det\begin{bmatrix} 4-\lambda & 1 & 1 \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{bmatrix} = \det\begin{bmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 1 & 4-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{bmatrix} = \det\begin{bmatrix} 6-\lambda & 0 & 0 \\ 1 & 3-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{bmatrix}.$$

Thus the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = 3$, and

$$A - 6I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

1 1 -2 1 1 1

The vector $v_1 = (1, 1, 1)$ forms a basis of nul(A - 6I). The vectors $v_2 = (1, 0, -1)$, $v_3 = (0, 1, -1)$ form a basis of nul(A - 3I). A diagonalization is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}^{-1}.$$

13) $\begin{bmatrix} 5 & -4 & -2 & 4 \\ 3 & -2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

Answer By this property, we have

$$\det\begin{bmatrix} 5-\lambda & -4 & -2 & 4 \\ 3 & -2-\lambda & 0 & 2 \\ 0 & 0 & 1-\lambda & 1 \\ 0 & 0 & 1 & 1-\lambda \end{bmatrix}$$

$$= \det\begin{bmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{bmatrix} \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = -\lambda(1-\lambda)(2-\lambda)^2.$$

The eigenvalues are 0, 1, 2, and

$$A - 0I = \begin{bmatrix} 5 & -4 & -2 & 4 \\ 3 & -2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A - 1I = \begin{bmatrix} 4 & -4 & -2 & 4 \\ 3 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A - 2I = \begin{bmatrix} 3 & -4 & -2 & 4 \\ 3 & -4 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

We then have the bases {(0, 0, 1, -1)}, {(1, 1, 0, 0)}, {(4, 3, 0, 0). (0, 0, 1, 1)} for the respective eigenspaces. A diagonalization is

$$A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}^{-1}.$$

14) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

Answer In the first equality, we use [row 2] + [row 1], [row 3] + [row 1], [row 4] + [row 1], (-1)[col 1] + [col 2], (-1)[col 1] + [col 3], (-1)[col 1] + [col 4]. In the second equality, we use the cofactor expansion along [row 1] and the direct computation of the 3 by 3 determinant.

$$\det\begin{bmatrix} -\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 0 \\ 1 & 1 & 0 & -\lambda \end{bmatrix} = \det\begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ 1 & -1 & -\lambda & -1 \\ 1 & 0 & -1 & -1-\lambda \end{bmatrix}$$

$$= (2 - \lambda)[(1 - \lambda)(- \lambda)(-1 - \lambda) + 1 - (1 - \lambda)] = - \lambda(2 - \lambda)(2 - \lambda^2).$$

The eigenvalues are $0, 2, \sqrt{2}, -\sqrt{2}$, and the vectors $(1, -1, -1, 1)$, $(1, 1, 1, 1)$, $(\sqrt{2} - 1, -1, 1, -\sqrt{2} + 1)$, $(-\sqrt{2} - 1, -1, 1, \sqrt{2} + 1)$ span the respective eigenspaces. A diagonalization is

$$A = \begin{bmatrix} 1 & 1 & \sqrt{2} - 1 & -\sqrt{2} - 1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -\sqrt{2} + 1 & \sqrt{2} + 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & \sqrt{2} - 1 & -\sqrt{2} - 1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -\sqrt{2} + 1 & \sqrt{2} + 1 \end{bmatrix}^{-1}.$$

15) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 1 & 2 & 1 \end{bmatrix}$

Answer The eigenvalues are 2 and 1, and

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & -1 \end{bmatrix}, \quad A - 1I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

We then have the bases $\{(0, 0, 1, 2)\}$, $\{(0, 0, 0, 1)\}$ for the respective eigenspaces. A is not diagonalizable.

16) $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer The eigenvalues are 2 and 1, and

$$A - 2I = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A - 1I = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We then have the bases $\{(1, 0, 0, 0)\}$, $\{(2, -1, 0, 0)\}$ for the respective eigenspaces. A is not diagonalizable.