Artificial Intelligence and Machine Learning

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Bernoulli Random Variables

Definition: Bernoulli Random Variable

Let X be a random variable taking only two possible values $\{0, 1\}$. Let $p = P(X = 1)$. Then X is said to be a **Bernoulli** random variable with parameter p, or equivalently $X \sim \text{Ber}(p)$.

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Examples:

- Flipping a "fair" coin: $X \sim \text{Ber}(1/2)$
- Success starting Windows: $X \sim \text{Ber}(0.95)$

Let $X \sim \text{Ber}(p)$. Then

$$\mathbb{E}[X] = 0 \times P(X = 0) + 1 \times P(X = 1) = p$$

$$\mathbb{E}[X^2] = 0^2 \times P(X = 0) + 1^2 \times P(X = 1) = p$$

$$\mathrm{V}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p(1 - p)$$

Bernoulli Trials

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Consider the following examples:

- Flip a coin 10 times. Let X be the number of heads obtained.
- The packets received during a UDP connection arrive corrupted 5% of the times. Suppose you send 100 packets and let X be the number of packets received in error.
- In the next 30 births at an hospital, let X be the number of female babies.

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These are all examples of binomial random variables (provided you assume each of the Bernoulli trials are independent).

Repeated Bernoulli Trials

Example: Suppose we send 3 packets through a communication channel. Assume

- Each packet is received with probability 0.9.
- The events {packet 1 is received}, {packet 2 is received}, and {packet 3 is received} are all **independent**.
- Let X be the total number of received packets.

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- Let X be the total number of received packets.

$$X_i \sim \text{Ber}(p) \rightarrow X_i = 1 \text{ iif } \{\text{packet i was received}\}, i = 1, 2, 3$$

$$X = X_1 + X_2 + X_3$$

x_1	x_2	x_3	$P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$	$x = x_1 + x_2 + x_3$
0	0	0	0.001	0
0	0	1	0.009	1
0	1	0	0.009	1
0	1	1	0.081	2
1	0	0	0.009	1
1	0	1	0.081	2
1	1	0	0.081	2
1	1	1	0.729	3

Binomial Distribution

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Therefore

$$P(X = x) = \begin{cases} 0.001 & \text{if } x = 0 \\ 0.027 & \text{if } x = 1 \\ 0.243 & \text{if } x = 2 \\ 0.729 & \text{if } x = 3 \end{cases}$$

This is what is called a Binomial Random Variable!!!

Binomial Distribution

Definition: Binomial Random Variable

Consider a random experiment consisting of $n \in \mathbb{N}$ **independent** Bernoulli trials, where each trial can result in either success or failure.

Assume the probability of success for each trial is equal to $0 \leq p \leq 1$, and remains the same throughout the experiment.

The random variable X that equals the number of trials that result in success is a **Binomial Random Variable** with parameters n and p. The probability mass function of X is given by

$$P(X = k) = f(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n \ .$$

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We have seen $\binom{n}{k} = C_k^n$ before, the number of possible combinations of k out of n elements.

Binomial Distribution

The name of this distribution comes from the similarity with the *binomial expansion*: For any $a, b \in \mathbb{R}$

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This immediately shows that the distribution is indeed valid, as

$$\sum_{k=0}^{n} f(k) = 1 \ .$$

The computation of the mean and variance of X can also be done directly from the p.m.f., but it is rather tedious and complicated... But there is a much easier way...

Binomial Distribution

Note that

$$X = X_1 + X_2 + \cdots + X_n \ ,$$

where $X_i \sim \text{Ber}(p)$ are independent Bernoulli trials.

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$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \ ,$$

and

$$\mathsf{V}(X_1 + X_2 + \cdots + X_n) = \mathsf{V}(X_1) + \mathsf{V}(X_2) + \cdots + \mathsf{V}(X_n) \ .$$

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Let $X \sim \mathsf{Bin}(n, p)$ be a binomial random variable with parameters n and p. Then

$$\mathbb{E}[X] = np, \qquad \text{and} \qquad \mathsf{V}(X) = np(1-p)$$

Binomial Distribution Example

Samples from a certain water supply have a 10% chance of containing an organic pollutant. Suppose you collect 20 samples over time, and these are taking in such a way that you can assume these are independent. Let X be the number of contaminated samples.

$$X \sim \text{Bin}(20, 0.1)$$

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$$X \sim \text{Bin}(20, 0.1)$$

$$P(X = 2) = \binom{20}{2} 0.1^2 (1 - 0.1)^{20-2} \approx 0.285$$

$$\begin{aligned} P(X \geq 2) = 1 - P(X < 2) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - 0.1^0\ 0.9^{20} - 200.10.9^{20-1} = 0.608 \end{aligned}$$

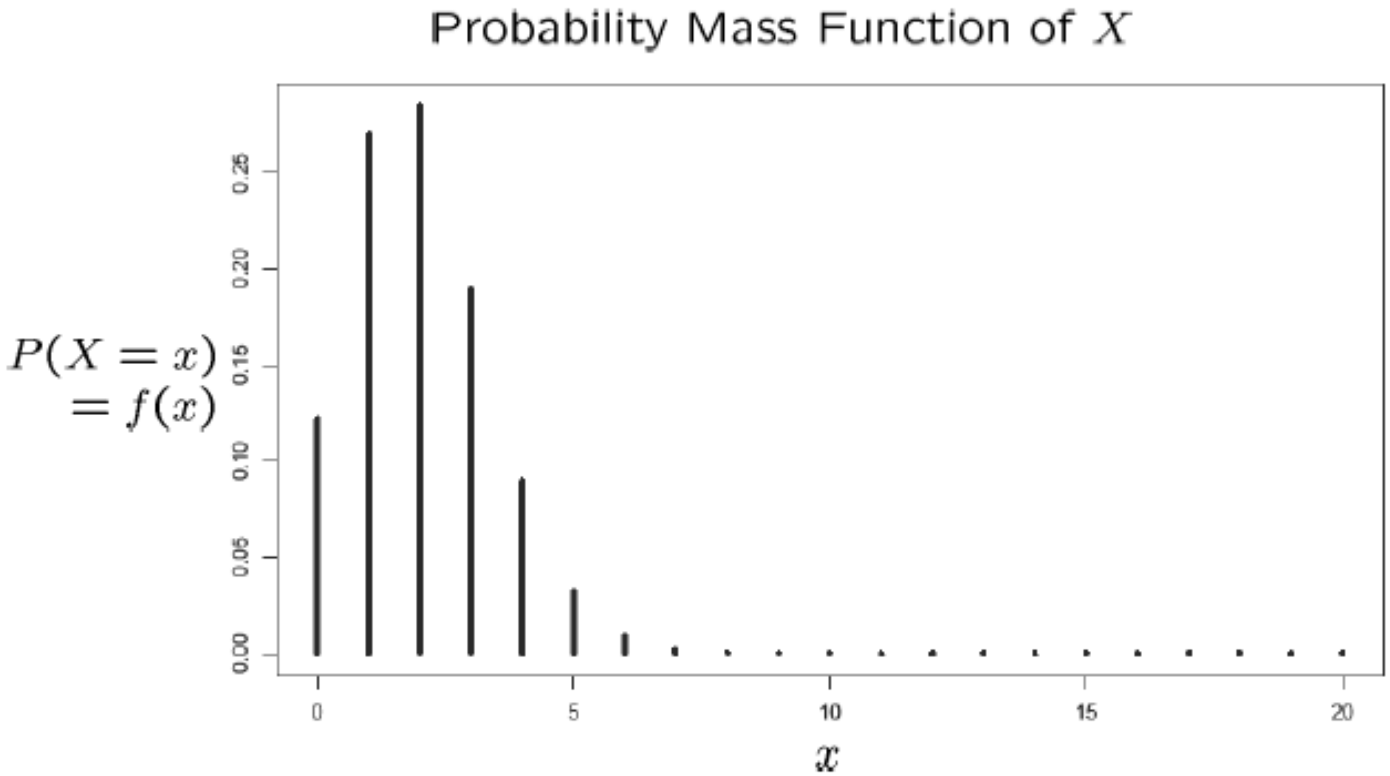
$$P(X \leq 7) = (\text{use table or computer}) = 0.9996$$

$$\begin{aligned} P(3 \leq X \leq 6) = P(X \leq 6) - P(X < 3) &= P(X \leq 6) - P(X \leq 2) \\ &= 0.3207 \end{aligned}$$

$$P(3 < X < 7) = 0.9976 - 0.8670 = 0.1306$$

$$\mathbb{E}[X] = 20 \times 0.1 = 2 \qquad \text{V}(X) = 20 \times 0.1(1 - 0.1) = 1.8$$

Binomial Distribution Example



I. Emily hits 60% of her free throws in basketball games. She had 25 free throws in last week's game. Use this information to answer the next **two** questions.

1. What is the average number of hits?
   (a) 10
   (b) 15
   (c) 20
   (d) 25

ANSWER: $\boxed{\text{(b)}}$

REASON:

The average (or the expected value) is $n \times p = 25 \times 0.60 = 15.$

2. What is the standard deviation of Emily's hit ?

   (a) 6
   (b) 3
   (c) 3.2
   (d) $\sqrt{6}$

ANSWER: $\boxed{\text{(d)}}$

REASON:

The standard deviation is $\sqrt{np(1-p)} = \sqrt{25 \times 0.60 \times (1 - 0.60)} = \sqrt{6} = 2.45$.

In the previous question, suppose Emily had 7 free throws in yesterday's game.

1. What is the probability that she made at least 5 hits?
   (a) 0.2613
   (b) 0.1306
   (c) 0.0280
   (d) 0.1586
   (e) 0.4199

ANSWER: $\boxed{\text{(e)}}$

REASON:

Denote Y the hits in Emily's 7 free throws. The event that she made at least 5 hits is then

$$(Y \geq 5) = (Y = 5 \text{ or } 6 \text{ or } 7).$$

So,

$$P(Y \geq 5) = P(Y = 5) + P(Y = 6) + P(Y = 7)$$

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$$\begin{aligned}
P(Y = 5) &= \frac{7!}{5!(7-5)!}(.6)^5(.4)^{7-5} = \frac{7 \times 6 \times 5!}{5! \times 2!}(.6)^5(.4)^2 \\
&= \frac{7 \times 6}{2 \times 1}(.6)^5(.4)^2 \ \ \text{(note that 5! was factored out)} \\
&= 21 \times (.6)^5(.4)^2 = 0.2613 \\
P(Y = 6) &= \frac{7!}{6!(7-6)!}(.6)^6(.4)^{7-6} = \frac{7 \times 6!}{6! \times 1!}(.6)^6(.4)^1 \\
&= \frac{7}{1}(.6)^6(.4)^1 \ \ \text{(note that 6! was factored out)} \\
&= 7 \times (.6)^6 \times .4 = 0.1306 \\
P(Y = 7) &= \frac{7!}{7!(7-7)!}(.6)^7(.4)^{7-7} = \frac{7!}{7! \times 0!}(.6)^7(.4)^0 \\
&= (.6)^7 \times 1 \ \ \text{(note that 7! was factored out)} \\
&= 0.0280.
\end{aligned}$$

Note that, $0! = 1$ and $(.4)^0 = 1$ above.
It follows that the probability of interest is $0.2613 + 0.1306 + 0.0280 = 0.4199$.