Curl $\nabla \times \mathbf{F}$ in non-Cartesian coordinates

This is an optional extra for those who like to see where things come from.

Coordinate-free definition of curl

When trying to find an interpretation of $\nabla \times \mathbf{F}$ in Cartesian coordinates, we effectively evaluated a tangential line integral around a small loop. We were actually making use of the definition of the curl in terms of a line integral:

$$\boxed{\hat{\mathbf{n}} \cdot \mathrm{curl}\mathbf{F} = \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{F}(\mathbf{r})) = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r}}$$

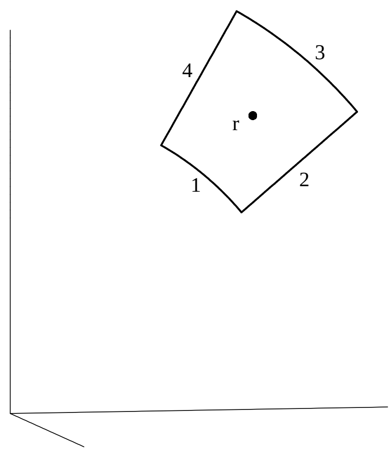
The path C encloses area A. $\hat{\mathbf{n}}$ is the normal to the area. As $A \to 0$, the path encloses point $\mathbf{r}$. The shape of A is arbitrary.

Curl in a general orthogonal coordinate system.

We know how to evaluate curl$\mathbf{F}$ in Cartesian coordinates. How do we evaluate $\nabla \times \mathbf{F}$ when $\mathbf{F}$ is a vector field expressed in a different orthogonal coordinate system? To derive the answer, we perform line integrals around chosen paths, using what we now know about orthogonal coordinate systems. We determine the component of curl in the direction $\hat{\mathbf{n}} = \hat{\mathbf{e}}_w$ by integrating around a curve on the surface $w = w_0$ enclosing the point $\mathbf{r} = (u_0, v_0, w_0)$ and which has corners $(u_0 \pm \delta u, v_0 \pm \delta v, w_0)$.

The curve has 4 sections, along each of which just one of the coordinates varies. We therefore calculate the line integral from a sum of four contributions. For each, the curve can be parameterised in terms of the one coordinate which varies:

1. $u_0 - \delta u \leq u \leq u_0 + \delta u \qquad v = v_0 - \delta v \qquad w = w_0$
2. $u = u_0 + \delta u \qquad v_0 - \delta v \leq v \leq v_0 + \delta v \qquad w = w_0$
3. $u_0 - \delta u \leq u \leq u_0 + \delta u \qquad v = v_0 + \delta v \qquad w = w_0$
4. $u = u_0 - \delta u \qquad v_0 - \delta v \leq v \leq v_0 + \delta v \qquad w = w_0$



We write the component of the vector $\mathbf{F}$ in the direction $\hat{\mathbf{e}}_u$ as F_u, so

$$\mathbf{F} = F_u \hat{\mathbf{e}}_u + F_v \hat{\mathbf{e}}_v + F_w \hat{\mathbf{e}}_w,$$

and the line element (see earlier notes) is

$$d\mathbf{r} = h_u \hat{\mathbf{e}}_u du + h_v \hat{\mathbf{e}}_v dv + h_w \hat{\mathbf{e}}_w dw.$$

Then

$$\mathbf{F} \cdot d\mathbf{r} = F_u(u,v,w) h_u(u,v,w) du + F_v(u,v,w) h_v(u,v,w) dv + F_w(u,v,w) h_w(u,v,w) dw$$

making the dependence on the coordinates explicit.

At this point, we could make life easy (as we have in the past) by assuming that $\mathbf{F}$ does not vary along a path, but may be different on each path. This time, we will take a bit more care, so you can see once how it is done, and that you haven't been cheated. We will soon pick up the answer we would have got from the easier method.

Along line segment 1, v and w are constant, and so $dv = dw = 0$. Therefore the line integral is

$$\int_1 \mathbf{F}\cdot d\mathbf{r} = \int_{u=u_0-\delta u}^{u_0+\delta u} F_u(u, v_0-\delta v, w_0) h_u(u, v_0-\delta v, w_0) du = \int_{u=u_0-\delta u}^{u_0+\delta u} H_u(u, v_0-\delta v, w_0) du$$

where $H_u = F_u h_u$ is introduced for convenience. [This is the key point – we have to remember that the h-factors depend on coordinates.]

Since u only varies a small amount over the range of the integral, we make a Taylor expansion of H_u in our integrand:

$$H_u(u, v_0-\delta v, w_0) = H_u(u_0, v_0-\delta v, w_0) + (u-u_0)\frac{\partial}{\partial u}H_u(u_0, v_0-\delta v, w_0) + \mathcal{O}(u-u_0)^2.$$

Performing the integral then gives

$$\int_1 \mathbf{F}\cdot d\mathbf{r} = 2\delta u H_u(u_0, v_0-\delta v, w_0) + \mathcal{O}(\delta u)^3$$

By the same methods, we obtain the following along the other line sections:

$$\begin{array}{ll} 2. & +2\delta v H_v(u_0+\delta u, v_0, w_0) + \mathcal{O}(\delta v)^3 \\ 3. & -2\delta u H_u(u_0, v_0+\delta v, w_0) + \mathcal{O}(\delta u)^3 \\ 4. & -2\delta v H_v(u_0-\delta u, v_0, w_0) + \mathcal{O}(\delta v)^3 \end{array}$$

That's the end of the extra bit – not so bad, was it? We can combine the results from paths 1 and 3 by making another Taylor expansion, this time in the variable v;

$$H_u(u_0, v_0 \pm \delta v, w_0) = H_u(u_0, v_0, w_0) \pm \delta v\frac{\partial}{\partial v}H_u(u_0, v_0, w_0) + \mathcal{O}(\delta v)^2,$$

and likewise combine the contributions from paths 2 and 4 by making a Taylor expansion in the variable u. Adding all four contributions we get

$$\oint \mathbf{F}\cdot d\mathbf{r} = 4\delta u \delta v\left(\frac{\partial H_v}{\partial u} - \frac{\partial H_u}{\partial v}\right).$$

The area A enclosed by the curve is $2h_u\delta u \times 2h_v \delta v = 4h_u h_v \delta u \delta v$ (the factors of 2 are because the ranges are $\pm\delta u, \pm\delta v$), so finally

$$\hat{\mathbf{e}}_w \cdot \text{curl } \mathbf{F} = \lim_{A\to 0}\frac{1}{4h_u h_v \delta u \delta v}\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{h_u h_v}\left(\frac{\partial h_v F_v}{\partial u} - \frac{\partial h_u F_u}{\partial v}\right).$$

We can obtain the remaining components by inspection, substituting $u \to v$, $v \to w$, $w \to u$ etc: Then

$$\begin{aligned} \text{curl } \mathbf{F} &= \frac{1}{h_v h_w}\left(\frac{\partial h_w F_w}{\partial v} - \frac{\partial h_v F_v}{\partial w}\right)\hat{\mathbf{e}}_u \\ &+ \frac{1}{h_w h_u}\left(\frac{\partial h_u F_u}{\partial w} - \frac{\partial h_w F_w}{\partial u}\right)\hat{\mathbf{e}}_v \\ &+ \frac{1}{h_u h_v}\left(\frac{\partial h_v F_v}{\partial u} - \frac{\partial h_u F_u}{\partial v}\right)\hat{\mathbf{e}}_w \end{aligned}$$

which can be conveniently expressed as a determinant

$$\boxed{\text{curl } \mathbf{F} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u\hat{\mathbf{e}}_u & h_v\hat{\mathbf{e}}_v & h_w\hat{\mathbf{e}}_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}}$$