# VECTOR CALCULUS

Many physical quantities are **vectors**, which change with time or with position. How do we describe this mathematically?

In this part of the unit, we will deal with differentiation & integration first in Cartesian coordinates $(x, y, z)$, then in other “coordinate systems”.

**Contents:**



Sections are not equal in length.
Each has a “Problem Sheet” and an “Extra Sheet”.

## 1. Differentiation of vectors & space curves

Consider a vector quantity $\mathbf{a}$ which changes according to a scalar $t$.

i.e. $\mathbf{a}$ is really $\mathbf{a}(t)$

[eg: If the position of a particle $\mathbf{r}$ moves with time $\Rightarrow \mathbf{r} = \mathbf{r}(t)$].

The **components** of **a** also change with $t$ so write

$$\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k}$$

Note that the Cartesian basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are FIXED.

The derivative of $\mathbf{a}$ with respect to $t$ is defined by

$$\frac{d\mathbf{a}}{dt} = \lim_{\delta t \to 0}\left[\frac{\mathbf{a}(t+\delta t) - \mathbf{a}(t)}{\delta t}\right] = \lim_{\delta t \to 0}\left[\frac{\delta \mathbf{a}}{\delta t}\right],$$

or $$\boxed{\frac{d\mathbf{a}}{dt} = \frac{da_x}{dt}\mathbf{i} + \frac{da_y}{dt}\mathbf{j} + \frac{da_z}{dt}\mathbf{k}} \qquad (1)$$

A good example to keep in mind is

**position**, **velocity**, **acceleration**.

Position: $\mathbf{r}(t)$

NB: $\mathbf{r}$ is ALWAYS position vector on this course.

Velocity: $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$

Acceleration: $\mathbf{a}(t) = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ then

$$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j} + v_z(t)\mathbf{k},$$

with

$$v_x(t) = \frac{dx(t)}{dt} \qquad v_y(t) = \frac{dy(t)}{dt} \qquad v_z(t) = \frac{dz(t)}{dt}$$

etc

*Example*

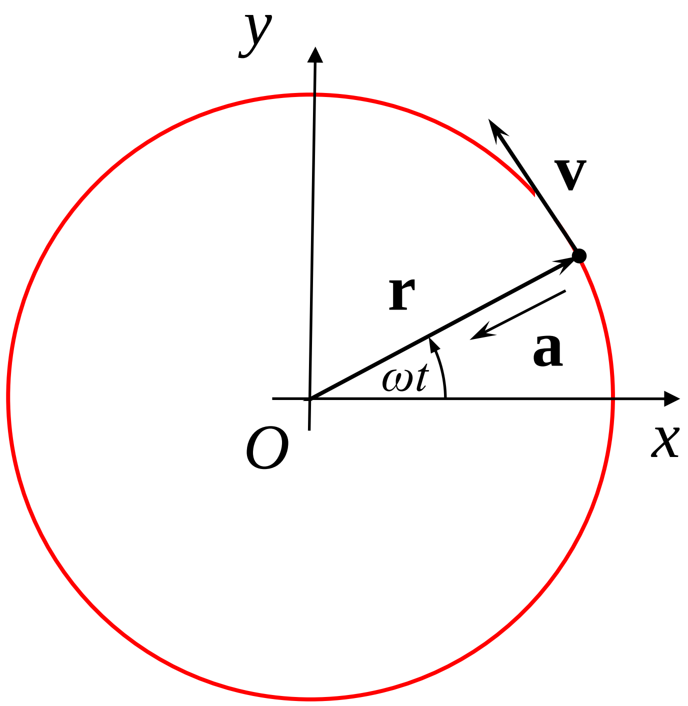
Let $$\mathbf{r} = \underbrace{\cos(\omega t)}_{x(t)}\,\mathbf{i} + \underbrace{\sin(\omega t)}_{y(t)}\,\mathbf{j}.$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \underbrace{-\omega\sin(\omega t)}_{v_x(t)}\,\mathbf{i} + \underbrace{\omega\cos(\omega t)}_{v_y(t)}\,\mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \underbrace{-\omega^2\cos(\omega t)}_{a_x(t)}\,\mathbf{i} \underbrace{-\omega^2\sin(\omega t)}_{a_y(t)}\,\mathbf{j}.$$

So $$\mathbf{a}(t) = -\omega^2\mathbf{r}(t)$$

This is just a description of **circular motion** with angular frequency $\omega$.



## 1.1 Rules for differentiation of vectors

(i) $$\frac{d}{dt}(c\mathbf{a}) = c\frac{d\mathbf{a}}{dt}$$ ($c$ is a constant)

(ii) $$\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$$

(iii) $$\frac{d}{dt}(\phi(t)\mathbf{a}(t)) = \phi\frac{d\mathbf{a}}{dt} + \frac{d\phi}{dt}\mathbf{a}$$ ($\phi$ is a scalar fn of $t$)

(iv) $$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt}\cdot\mathbf{b}$$

(v) $$\frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt}\times\mathbf{b}$$ (care with order)

(vi) $$\frac{d}{dt}\mathbf{a}(s) = \frac{d\mathbf{a}}{ds}\frac{ds}{dt}$$ ($\mathbf{a}$ is a function of $s$, which is a scalar function of $t$)

All these can be proved by looking at components

*Example of the use of these rules:*

Consider a vector $\mathbf{a}(t)$ with constant **magnitude** $a$.

ie $\mathbf{a}(t)\cdot\mathbf{a}(t)=a^2$.

Though the magnitude is constant, the direction need not be, so the derivative of $\mathbf{a}(t)$ need not be zero. Differentiating,

LHS: $$\frac{d}{dt}\left[\mathbf{a}(t)\cdot\mathbf{a}(t)\right]=\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}+\frac{d\mathbf{a}}{dt}\cdot\mathbf{a}=2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt} \qquad \text{(iv)}$$

RHS: $\frac{d}{dt}(a^2)=0$, as $a^2$ is a **constant.**

$$\therefore \quad \mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0$$

So either $\frac{d\mathbf{a}}{dt}=0$ [so $\mathbf{a}$ **is** a constant vector],

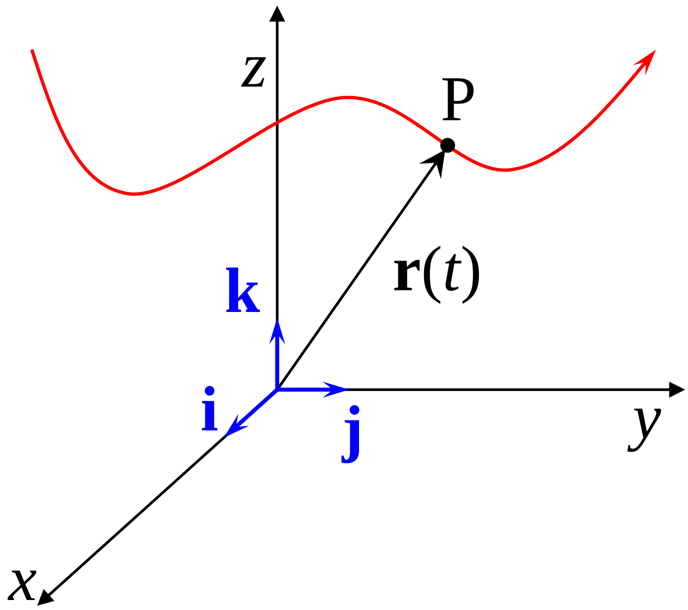
or $\frac{d\mathbf{a}}{dt}$ is perpendicular to $\mathbf{a}$

The latter case is what we have for circular motion, where the position vector $\mathbf{r}$ is a vector of constant magnitude; we found that the velocity vector $\mathbf{v}$ is always perpendicular to $\mathbf{r}$.

## 1.2 Space Curves

How do we describe *trajectories* of particles moving through space?

Think about point P whose coordinates $x(t)$, $y(t)$, and $z(t)$ are continuous functions of time $t$. As $t$ increases, P traces out a **curve in space**.




eg $\mathrm{P} \equiv \mathbf{r}(t) = \underbrace{\cos(\omega t)}_{x(t)}\,\mathbf{i} + \underbrace{\sin(\omega t)}_{y(t)}\,\mathbf{j} + \underset{z(t)}{\alpha t}\,\mathbf{k}$

This is a space curve with circular motion in 2d (the $xy$ plane in this case), with angular frequency $\omega$, and constant motion, with speed $\alpha$, in $z$. This is a HELIX.

Q: How do we write a known trajectory or curve in the form $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$?

## 1.3 Parameterisation of curves

This is a very useful technique.

Key point is that all lines (curves) are 1-dimensional.

We can $\therefore$ follow any line by increasing or decreasing a parameter that takes us along the line.

Let such a parameter be $u$

We parameterise a curve by finding expressions for $x(u)$, $y(u)$, and $z(u)$ for that curve.

Eg (2d): take the straight line $y = 3x + 4$.

Let $u = x$. Then the line is followed as $u$ is varied, provided we set $y = 3u + 4$ and $z = 0$.

The position vector varies with parameter $u$ as

$$\begin{aligned}\mathbf{r}(u) &= x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k} \\ &= u\mathbf{i} + (3u + 4)\mathbf{j}\end{aligned}$$

Note that any curve can be parameterised an infinite number of ways – pick the one that suits you best!

Eg let $v = 3x$. Then the same line $y = 3x + 4$ can be written

$$\mathbf{r}(v) = \frac{v}{3}\mathbf{i} + (v+4)\mathbf{j}.$$

For space curves, two useful parameters are time $t$ and distance along the curve $s$.

*More 2d examples of parameterisation*

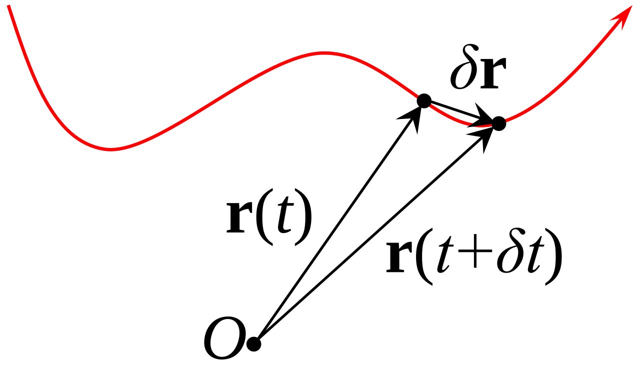
- Express $y = 3x^2$ as $\mathbf{r}(t)$.

- Express $x^2 + y^2 = 9$ as $\mathbf{r}(u)$.

- Express $\mathbf{r}(u) = e^u\mathbf{i} + e^{-u}\mathbf{j}$ as $y(x)$.

## 1.4 Tangent to a space curve

Consider 2 adjacent points $\mathbf{r}(t)$ and $\mathbf{r}(t+\delta t)$on a space curve.




The separation of the 2 points is

$$\delta\mathbf{r} = \mathbf{r}(t+\delta t) - \mathbf{r}(t) \approx \frac{d\mathbf{r}}{dt}\delta t$$ [from defn of $\frac{d\mathbf{r}}{dt}$]

In the limit that $\delta t \to 0$ this approximation becomes exact – then $\delta\mathbf{r}$ becomes **parallel to the local direction of the curve**, which is the **tangent direction** at that point.

$\therefore$ tangent is in direction of $d\mathbf{r} = \lim_{\delta t \to 0} \delta\mathbf{r}$

$\therefore$ tangent is in direction of $\frac{d\mathbf{r}}{dt}$ [from above]

So UNIT TANGENT VECTOR $\hat{\mathbf{T}} = \dfrac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|}$.

Also, $\dfrac{d\mathbf{r}}{dt} = \mathbf{v}$ (velocity), and $\left|\dfrac{d\mathbf{r}}{dt}\right| = |\mathbf{v}|$ (speed).

$\therefore$ tangent is in direction of velocity, with

$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \hat{\mathbf{v}}.$$

Space curves can also be expressed in terms of a position vector as a function of **distance travelled along the curve**, $s$.

ie consider $\mathbf{r}(s)$ instead of $\mathbf{r}(t)$

From above, tangent vector is in direction of $d\mathbf{r}$, and the magnitude of $d\mathbf{r}$ must be $ds$, the distance travelled in $dt$.

$$\therefore \quad \hat{\mathbf{T}} = \frac{d\mathbf{r}}{ds}$$

But $\dfrac{d\mathbf{r}(s)}{dt} = \dfrac{d\mathbf{r}(s)}{ds}\dfrac{ds}{dt} = \hat{\mathbf{T}}\dfrac{ds}{dt}$ which means we have 3 useful ways of writing the speed of a particle:

Speed = $\quad |\mathbf{v}| = \left|\dfrac{d\mathbf{r}}{dt}\right| = \dfrac{ds}{dt}$

NOTE: if we know $\mathbf{r}$ as a function of $t$, then use $\left|\frac{d\mathbf{r}}{dt}\right| = \frac{ds}{dt}$ to find the distance travelled as a function of $t$, $s(t)$. Simply differentiate $\mathbf{r}$ with respect to $t$, find the magnitude of the resulting vector, then integrate with respect to $t$.

*Example*

Consider circular motion again…

If $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$ [$a$ is the radius]

then $$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -a\omega\sin(\omega t)\mathbf{i} + a\omega\cos(\omega t)\mathbf{j},$$

$$\left|\frac{d\mathbf{r}}{dt}\right| = [a^2\omega^2\sin^2(\omega t) + a^2\omega^2\cos^2(\omega t)]^{\frac{1}{2}} = a\omega$$

so $\hat{\mathbf{T}} = \dfrac{\frac{d\mathbf{r}}{dt}}{\left|\frac{d\mathbf{r}}{dt}\right|} = -\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}$ for any $t$.

For example, to find $\hat{\mathbf{T}}$ at the point $(0, a)$ on the curve,

(i) find $t$. Here $x(t) = 0 = a\cos(\omega t)$ and $y(t) = a = a\sin(\omega t)$. This is true for $\omega t = \dfrac{\pi}{2}$ (and other values…)

(ii) find $\hat{\mathbf{T}}$. Here $\hat{\mathbf{T}} = -\sin(\frac{\pi}{2})\mathbf{i} + \cos(\frac{\pi}{2})\mathbf{j} = -\mathbf{i}$.

Also, $\left|\dfrac{d\mathbf{r}}{dt}\right| = \dfrac{ds}{dt} = a\omega$, so $s = a\omega t + c$

[$c$ is integration constant]

## 1.5 Newton's Law in vector form

Newton's Law "Force = mass x acceleration" is really a vector differential equation

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2}.$$

So, to find the trajectory $\mathbf{r}(t)$ of an object subjected to a force $\mathbf{F}$ we solve equations like this one.

For example, the electromagnetic force on a particle of charge $q$ in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

so the full "equations of motion" for the particle are

$$m\frac{d^2\mathbf{r}}{dt^2} = q\mathbf{E} + q\frac{d\mathbf{r}}{dt} \times \mathbf{B}.$$

In 3d, we solve 3 ODEs, one for each component:

$$F_x = m\frac{d^2x}{dt^2} \qquad F_y = m\frac{d^2y}{dt^2} \qquad F_z = m\frac{d^2z}{dt^2}$$