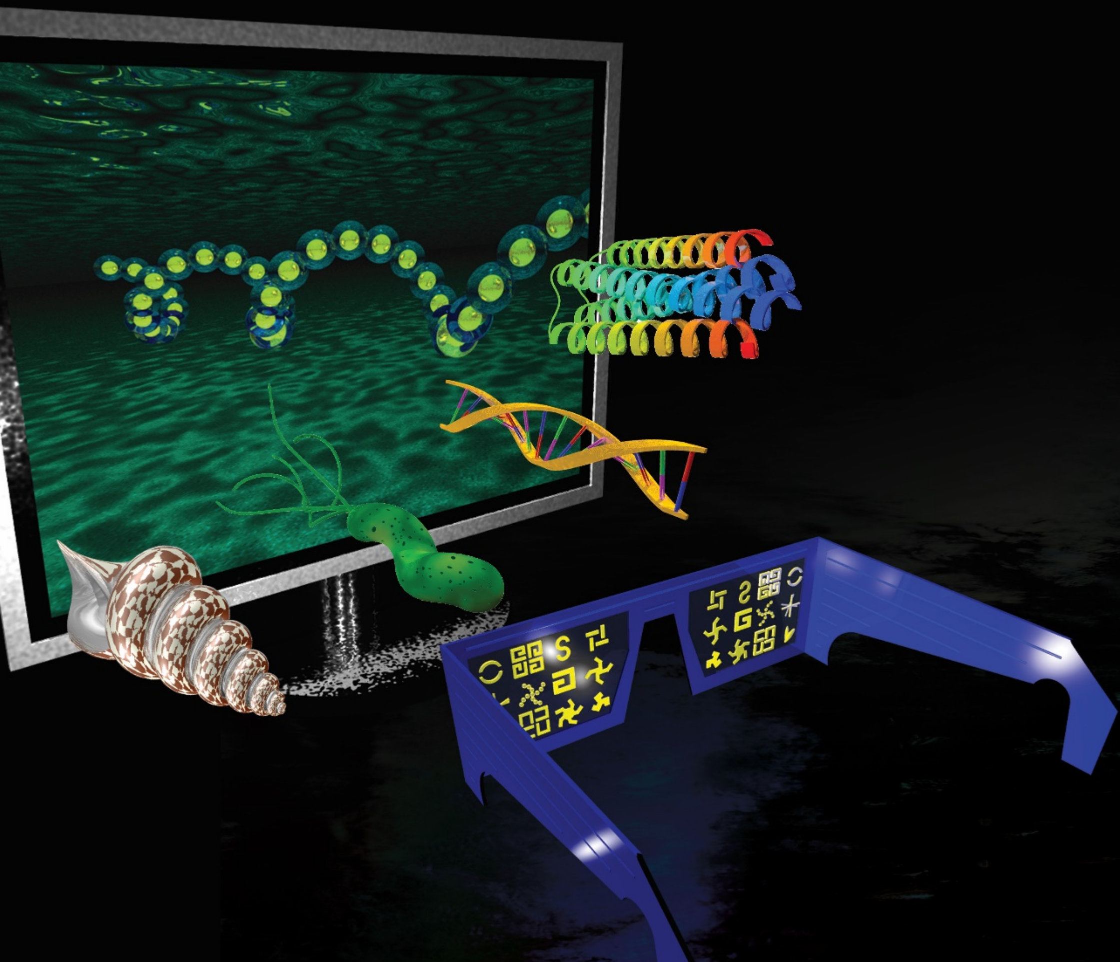
Lecture 15

Electromagnetic waves in linear, isotropic and homogeneous materials



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Summary

Electric current is a source of magnetic fields.

The magnetic dipole moment is current times area: $m = I \times (\text{area})$

The magnetisation is the magnetic dipole moment per unit volume: $\vec{M} = \dfrac{\sum \vec{m}}{V}$

When an external magnetic field is applied to a magnetic dipole moment, the torque acts to bring $\vec{m}$ parallel to $\vec{B}$.

The magnetisation induces a surface current density $\vec{k}_b = \vec{M} \times \hat{n}$ and bound current density $\vec{j}_b = \nabla \times \vec{M}$.

The magnetic flux density results from adding up the magnetic field strength and the magnetisation: $\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right)$

The magnetic field strength is defined as $\vec{H} = \dfrac{1}{\mu}\vec{B}$.

In LIH materials: $\vec{M} = \chi_m \vec{H}$

Examples of magnetic order in magnetic materials include: diamagnetic, paramagnetic and ferromagnetic.

Summary

In diamagnetic materials, the magnetic dipole moment $\vec{m}$ is opposite to the direction of $\vec{B}$.

The energy stored in a magnetic material is $w = \frac{1}{2}\vec{H} \cdot \vec{B}$.

Units: *B* is in Tesla; *M* is in A/m; *H* is in A/m.

Overview

In this Lecture we will look at:

- Maxwell's equations in vacuum and in LIH materials
- The wave equation in an ideal LIH dielectric
- Phase velocity in an ideal LIH dielectric
- Intrinsic impedance in an ideal LIH dielectric
- Conducting materials: Ohm's law and the current density
- The modified wave equation
- Solving the modified wave equation
- The dielectric function
- Intrinsic impedance in a conducting medium
- The complex refractive index

EM waves in LIH materials

- How do EM waves propagate through materials?
- How can we understand an EM wave propagating through air (a mostly lossless dielectric), or thick glass (a more lossy dielectric), or a thin layer of gold (a good conductor)?
- What is the difference between dielectric and conducting materials for EM propagation?

Let’s look at Maxwell’s equations!

Maxwell's equations in vacuum and in LIH materials

Maxwell's equations in vacuum

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Maxwell's equations in LIH materials :

$$\boxed{\nabla \cdot \vec{D} = \rho_f}$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$\boxed{\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$$

We know more equations.

Maxwell's equations in vacuum and in LIH materials

Further equations in LIH materials:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

$$\vec{H} = \frac{1}{\mu_0 \mu_r} \vec{B} = \frac{1}{\mu} \vec{B}$$

$$\vec{M} = \chi_m \vec{H}$$

Maxwell's equations in LIH materials :

$$\boxed{\nabla \cdot \vec{D} = \rho_f}$$

$$\boxed{\nabla \cdot \vec{B} = 0}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

$$\boxed{\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}}$$

How about the wave equation?

The wave equation in an ideal LIH dielectric

Taking the curl of:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

we obtain:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

On the left hand side:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

But without free charges $\rho_f = 0$

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 0 \qquad \nabla \cdot \vec{D} = \rho_f$$

It follows that

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{E}$$

On the right hand side $\quad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \nabla \times \left(-\frac{\partial\left(\mu \vec{H}\right)}{\partial t}\right) =$$

$$= -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) = -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right)$$

But without free current: $\vec{J}_f = 0$ Ideal dielectric

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu \frac{\partial^2 \vec{D}}{\partial t^2} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Combining left and right hand sides:

$$\boxed{\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$$

How about the phase velocity?

Phase velocity in an ideal LIH dielectric

The **phase velocity** is:

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$$

Therefore:

$$\boxed{v_p = \frac{c}{\sqrt{\mu_r\varepsilon_r}}} \qquad c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

This means that in an ideal LIH dielectric, the electromagnetic wave propagates similarly to the way it does in free space – it remains a plane wave but its phase velocity is lowered.

We now introduce the refractive index:

$$n = \frac{c}{v_p} = \sqrt{\mu_r\varepsilon_r}$$

So the phase velocity is:

$$v_p = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}} = \frac{1}{\sqrt{\mu\varepsilon}} = c/n$$

Note: **The Maxwell relation is**

$n = \sqrt{\mu_r\varepsilon_r} \approx \sqrt{\varepsilon_r}$, since for most materials we have $\mu_r \approx 1$. non magnetic material $\mu_r = 1$

The impedance then?

Intrinsic impedance in an ideal LIH dielectric

Consider a plane wave:

$$\vec{E} = E_0 e^{i(kx-\omega t)}\hat{y}$$

Take the curl:

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & E_0 e^{i(kx-\omega t)} & 0 \end{vmatrix} =$$

$$= \frac{\partial}{\partial x}\left[E_0 e^{i(kx-\omega t)}\right]\hat{z} = ikE_0 e^{i(kx-\omega t)}\hat{z}$$

We shorten this.

Intrinsic impedance in an ideal LIH dielectric

Consider a plane wave:

$$\vec{E} = E_0 e^{i(kx-\omega t)}\hat{y}$$

Take the curl:

$$\nabla \times \vec{E} = ikE_0 e^{i(kx-\omega t)}\hat{z}$$

Then we use Maxwell's equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and write:

$$ikE_0 e^{i(kx-\omega t)}\hat{z} = -\frac{\partial \vec{B}}{\partial t}$$

We shorten this.

Intrinsic impedance in an ideal LIH dielectric

Consider a plane wave:

$$\vec{E} = E_0 e^{i(kx-\omega t)}\hat{y}$$

Take the curl:

$$\nabla \times \vec{E} = ikE_0 e^{i(kx-\omega t)}\hat{z}$$

Then we use Maxwell's equation:

$$ikE_0 e^{i(kx-\omega t)}\hat{z} = -\frac{\partial \vec{B}}{\partial t}$$

We can integrate with respect to time:

$$\int\left(-\frac{\partial \vec{B}}{\partial t}\right)dt = -\vec{B} = \int\left(ikE_0 e^{i(kx-\omega t)}\hat{z}\right)dt = \frac{ik}{-i\omega}E_0 e^{i(kx-\omega t)}\hat{z} + C$$

The constant is 0 because we are only interested in the oscillating field.

We shorten this.

Intrinsic impedance in an ideal LIH dielectric

Consider a plane wave:

$$\vec{E} = E_0 e^{i(kx-\omega t)}\hat{y}$$

Take the curl:

$$\nabla \times \vec{E} = ikE_0 e^{i(kx-\omega t)}\hat{z}$$

Then we use Maxwell's equation:

$$ikE_0 e^{i(kx-\omega t)}\hat{z} = -\frac{\partial \vec{B}}{\partial t}$$

We can integrate with respect to time:

$$-\vec{B} = \frac{ik}{-i\omega} E_0 e^{i(kx-\omega t)}\hat{z}$$

Therefore:

$$\vec{B} = \frac{k}{\omega} E_0 e^{i(kx-\omega t)}\hat{z}$$

We also have $\vec{B} = \mu \vec{H}$ so

$$\vec{H} = \frac{k}{\mu\omega} E_0 e^{i(kx-\omega t)}\hat{z} = H_0 e^{i(kx-\omega t)}\hat{z}$$

We then calculate the intrinsic impedance:

$$Z = \frac{E_0}{H_0} = \frac{E_0}{\frac{k}{\mu\omega}E_0} = \frac{\mu\omega}{k} = \mu v_p = \mu\frac{1}{\sqrt{\mu\varepsilon}} =$$

$$= \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{\mu_r}{\varepsilon_r}}Z_0$$

We shorten this.

Intrinsic impedance in an ideal LIH dielectric

Consider a plane wave:

$$\vec{E} = E_0 e^{i(kx-\omega t)}\hat{y}$$

Take the curl:

$$\nabla \times \vec{E} = ikE_0 e^{i(kx-\omega t)}\hat{z}$$

Then we use Maxwell's equation:

$$ikE_0 e^{i(kx-\omega t)}\hat{z} = -\frac{\partial \vec{B}}{\partial t}$$

We can integrate with respect to time:

$$-\vec{B} = \frac{ik}{-i\omega} E_0 e^{i(kx-\omega t)}\hat{z}$$

Therefore:

$$\vec{B} = \frac{k}{\omega} E_0 e^{i(kx-\omega t)}\hat{z}$$

We also have $\vec{B} = \mu \vec{H}$ so

$$\vec{H} = \frac{k}{\mu\omega} E_0 e^{i(kx-\omega t)}\hat{z} = H_0 e^{i(kx-\omega t)}\hat{z}$$

We then calculate the **intrinsic impedance**:

$$Z = \sqrt{\frac{\mu_r}{\varepsilon_r}} Z_0$$

Where the wave impedance in vacuum is :

$$Z_0 \approx 377\ \Omega$$

In a perfect dielectric, an EM wave propagates in the same way as in vacuum but with scaled impedance:

$$Z = \sqrt{\frac{\mu_r}{\varepsilon_r}} Z_0$$

intrinsic materials ↑

How about conducting materials?

conducting materials ↓

Conducting materials: Ohm's law and the current density

We know that Ohm's law is:

$$V = R \cdot I$$

which we can write as:

$$I = V/R$$

So we need the voltage and the resistance.

> Note: **Pouillet's law** is
>
> $R = \rho \frac{L}{A}$, where L: length, A: area
>
> and ρ is the **electric resistivity**.

$\rho =$

$=$ density of charge

$= \cdots$

We can also use $E = V/L$

$$V = L \cdot E$$

Now we substitute:

$$I = \frac{V}{R} = \frac{L \cdot E}{\rho \frac{L}{A}} = \frac{1}{\rho} A \cdot E$$

Which leads to:

$$\frac{I}{A} = \frac{1}{\rho} E = J = \sigma E$$

And generally: $\boxed{\vec{J} = \sigma \vec{E}}$

The vector J is the **current density** (the amount of charge per unit time that flows through a unit area of a chosen cross section) and σ is the **conductivity** (the inverse of the resistivity).

σ = conductivity

= surface density of charge

How about the phase velocity?

The modified wave equation

We start with Maxwell's equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and we take the curl.

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

We can apply a maths formula:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

Without free charges: $\rho_f = 0$

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 0$$

It follows that:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{E}$$

what we saw before

On the right hand side:

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \nabla \times \left(-\frac{\partial\left(\mu \vec{H}\right)}{\partial t}\right) =$$

$$= -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) = -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right)$$

We also have:

$\vec{J}_f = \sigma \vec{E}$ and $\vec{D} = \varepsilon \vec{E}$

It follows that:

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial^2 \vec{D}}{\partial t^2} =$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

We shorten this.

The modified wave equation

We start with Maxwell's equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

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$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

Without free charges: $\rho_f = 0$

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 0$$

$\nabla \cdot \vec{D} = \rho_f$

It follows that:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{E}$$

what we saw before

On the right hand side:

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right)$$

We also have:

$$\vec{J}_f = \sigma \vec{E} \quad \text{and} \quad \vec{D} = \varepsilon \vec{E}$$

It follows that:

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Combining both sides we obtain the modified wave equation:

$$\boxed{\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}}$$

Now we need to solve it...

Solving the modified wave equation

We start with the usual form:

$$\boxed{\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}} \qquad \gamma = ik$$

On the left side of the equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$

$$\nabla^2 \vec{E} \rightarrow \frac{\partial^2 \vec{E}}{\partial x^2} = \frac{\partial^2 \left(\vec{E}_0 e^{(\gamma x - i\omega t)} \right)}{\partial x^2} = \gamma^2 \vec{E}$$

On the right side of the equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$

$$\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu\varepsilon (-i\omega)^2 \vec{E} + \mu\sigma (-i\omega) \vec{E}$$

Combining both sides:

$$\gamma^2 \vec{E} = \mu\varepsilon (-i\omega)^2 \vec{E} + \mu\sigma (-i\omega) \vec{E} =$$

$$= -\mu\varepsilon\omega^2 \vec{E} - \mu\sigma\omega \vec{E}$$

We take gamma to be complex so,

$$\gamma = -\alpha + i\beta$$

We clear up.

Note: the minus sign in front of α is just a convenience, it will become clear later why this is convenient..

Solving the modified wave equation

We start with the usual form:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)} \qquad \gamma = ik$$

On the left side of the equation

$$\nabla^2 \vec{E} \rightarrow \frac{\partial^2 \vec{E}}{\partial x^2} = \frac{\partial^2 \left(\vec{E}_0 e^{(\gamma x - i\omega t)}\right)}{\partial x^2} = \gamma^2 \vec{E}$$

On the right side of the equation

$$\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu\varepsilon (-i\omega)^2 \vec{E} + \mu\sigma (-i\omega) \vec{E}$$

Combining both sides:

$$\gamma^2 \vec{E} = -\mu\varepsilon\omega^2 \vec{E} - \mu\sigma\omega \vec{E}$$

We take gamma to be complex so,

$$\boxed{\gamma = -\alpha + i\beta}$$

We then obtain:

$$\gamma^2 = (-\alpha + i\beta)^2 = \alpha^2 - \beta^2 - i2\alpha\beta =$$

$$= -\mu\varepsilon\omega^2 - i\mu\sigma\omega$$

Where we can ID the real and imaginary parts:

$$\left.\begin{array}{l} \alpha^2 - \beta^2 = -\mu\varepsilon\omega^2 \\ -2\alpha\beta = -\mu\sigma\omega \end{array}\right\} \rightarrow \frac{-2\alpha\beta}{\alpha^2 - \beta^2} =$$

$$= \frac{2\alpha\beta}{\beta^2 - \alpha^2} = \frac{\cancel{\mu}\sigma\cancel{\omega}}{\cancel{\mu}\varepsilon\omega^{\cancel{2}}} = \frac{\sigma}{\varepsilon\omega}$$

This ratio is dimensionless:

$$\frac{\sigma}{\varepsilon\omega} = \frac{\left[s \cdot \cancel{m}^{-1}\right]}{\left[F \cdot \cancel{m}^{-1}\right]\left[s^{-1}\right]} = \frac{\left[\dfrac{s^3 \cdot A^2}{m^2 \cdot kg}\right]}{\left[\dfrac{s^4 \cdot A^2}{m^2 \cdot kg}\right]\left[s^{-1}\right]}$$

Our new ratio appears in the dielectric function

Solving the modified wave equation

Same wave but different Maxwell equation:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)} \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Here, we can use:

$$\frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E} \quad \text{so} \quad \vec{E} = -\frac{1}{i\omega}\frac{\partial \vec{E}}{\partial t}$$

We can substitute into Maxwell's equation, using $\vec{J}_f = \sigma \vec{E}$ and $\vec{D} = \varepsilon \vec{E}$

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma\left(-\frac{1}{i\omega}\frac{\partial \vec{E}}{\partial t}\right) + \varepsilon \frac{\partial \vec{E}}{\partial t} = \left(i\frac{\sigma}{\omega} + \varepsilon\right)\frac{\partial \vec{E}}{\partial t}$$

Therefore:

$$\nabla \times \vec{H} = \left(i\frac{\sigma}{\omega} + \varepsilon_0\varepsilon_r\right)\frac{\partial \vec{E}}{\partial t} = \varepsilon_0\left(\varepsilon_r + i\frac{\sigma}{\varepsilon_0\omega}\right)\frac{\partial \vec{E}}{\partial t} = \varepsilon_0\varepsilon(\omega)\frac{\partial \vec{E}}{\partial t}$$

We clear up.

The dielectric function

Same wave but different Maxwell equation:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)} \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Here, we can use:

$$\frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E} \quad \text{so} \quad \vec{E} = -\frac{1}{i\omega}\frac{\partial \vec{E}}{\partial t}$$

We can substitute

$$\nabla \times \vec{H} = \left(i\frac{\sigma}{\omega} + \varepsilon \right) \frac{\partial \vec{E}}{\partial t}$$

Therefore:

$$\nabla \times \vec{H} = \varepsilon_0 \varepsilon(\omega) \frac{\partial \vec{E}}{\partial t}$$

The last equation has a familiar mathematical shape and we have introduced a new quantity – **the dielectric** function:

$$\boxed{\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}}$$

Sometimes written as:

$$\varepsilon(\omega) = \varepsilon' + i\varepsilon''$$

The dielectric function describes the response of a material to E-fields oscillating with angular frequency ω. It captures both the 'insulating aspect', expressed by the dielectric "constant" , and the conduction of a material, through the appearance of conductivity σ in the dimensionless ratio $\sigma/\varepsilon\omega$.

Intrinsic impedance in a conducting medium

For the intrinsic impedance:

$$Z = \frac{\mu\omega}{k} = \frac{\mu\omega}{\text{wave vector}}$$

Our field here is:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)} \quad \text{with} \quad \gamma = ik$$

Therefore:

$$k = \frac{\gamma}{i} = -i\gamma$$

Which leads to:

$$Z = \frac{\mu\omega}{-i\gamma}$$

And we also have

$$\gamma^2 = (-\alpha + i\beta)^2 = \alpha^2 - \beta^2 - i2\alpha\beta =$$
$$= -\mu\varepsilon\omega^2 - i\mu\sigma\omega = \mu\omega(-\varepsilon\omega - i\sigma)$$

So we can write:

$$k = -i\sqrt{\mu\omega(-\varepsilon\omega - i\sigma)} = -i\sqrt{-(\mu\varepsilon\omega^2 + i\mu\sigma\omega)}$$

Hence, we can calculate Z:

$$Z = \frac{\mu\omega}{-i\sqrt{-(\mu\varepsilon\omega^2 + i\mu\sigma\omega)}} = \frac{\mu\omega}{-i\sqrt{i^2(\mu\varepsilon\omega^2 + i\mu\sigma\omega)}}$$

$$= \frac{\sqrt{\mu^2\omega^2}}{-ii\sqrt{\mu\varepsilon\omega^2 + i\mu\sigma\omega}} = \sqrt{\frac{\mu^2\omega^2}{\mu\varepsilon\omega^2\left(1 + i\frac{\mu\sigma\omega}{\mu\varepsilon\omega^2}\right)}} =$$

$$= \sqrt{\frac{\mu}{\varepsilon\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)}}$$

We can clear up the notations.

Intrinsic impedance in a conducting medium

For the intrinsic impedance:

$$Z = \frac{\mu\omega}{k} = \frac{\mu\omega}{\text{wave vector}}$$

Our field here is:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)} \quad \text{with} \quad \gamma = ik$$

Therefore:

$$k = \frac{\gamma}{i} = -i\gamma$$

Which leads to:

$$Z = \frac{\mu\omega}{-i\gamma}$$

And we also have

$$\gamma^2 = \mu\omega(-\varepsilon\omega - i\sigma)$$

So we can write:

$$k = -i\sqrt{-(\mu\varepsilon\omega^2 + i\mu\sigma\omega)}$$

Hence, we can calculate Z:

$$Z = \sqrt{\frac{\mu}{\varepsilon\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)}}$$

The intrinsic impedance is now a complex number, where the imaginary part is present because of the dimensionless ratio: $\sigma/\varepsilon\omega$

If $\frac{\sigma}{\varepsilon\omega} << 1$ then $Z \approx \sqrt{\frac{\mu}{\varepsilon}}$

which is the expression for a lossless dielectric material. ideal dielectrics

Intrinsic impedance in a conducting medium

At a given frequency, the ratio $\frac{\sigma}{\varepsilon\omega}$ quantifies how good a conductor a material is.

ratio = 0 not conductor

The complex refractive index

The wave vector is complex in:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$$

So, we can rewrite as:

$$\vec{E} = \vec{E}_0 e^{(-\alpha x + i\beta x - i\omega t)}$$

$$\boxed{\vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)}}$$

In this form, the term $e^{-\alpha x}$ is an exponential decay with distance *x* and decay constant α.

Also, for the wave vector:

$$\vec{E} = \vec{E}_0 e^{(-\alpha x + i\beta x - i\omega t)} =$$

$$= \vec{E}_0 e^{i\left(-\frac{\alpha x}{i} + \beta x - \omega t\right)} = \vec{E}_0 e^{i(i\alpha x + \beta x - \omega t)} =$$

$$= \vec{E}_0 e^{i\left[(\beta + i\alpha)x - \omega t\right]}$$

where the wave vector is:

$$k \to k + i\kappa = \beta + i\alpha$$

And because we have a complex wave vector, we also have a complex refractive index:

$$\boxed{\tilde{n} = n + i\eta = \sqrt{\varepsilon(\omega)} = \sqrt{\varepsilon' + i\varepsilon''}}$$

where n is the refractive index (describing the change in phase velocity) and η is the **extinction coefficient** (describing the attenuation/absorption of the EM wave).

Our new ratio appears in the dielectric function

Example question

[from Sadiku] An EM wave propagating in free space is described by:

$$\vec{H} = 0.1\cos\left(2\times10^{8}t + kx\right)\vec{a}_y \ \text{Am}^{-1}$$

Determine:

(a) The direction of propagation.

(b) The period.

(c) The wavelength (λ).

(d) The wave vector magnitude.

Example question

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$$\vec{H} = 0.1\cos\left(2\times 10^{8}t + kx\right)\vec{a}_y \ \text{Am}^{-1}$$

Determine:

(a) The direction of propagation.

(b) The period.

(c) The wavelength (λ).

(d) The wave vector magnitude.

(a) The wave propagates along the negative x direction.

(b) $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2\pi \times 10^{8}} = \underline{\underline{31.42}}$ ns

(c) $\lambda = uT = 3\times 10^{8} \times 31.42 \times 10^{-9} = \underline{\underline{9.425}}$ m

(b) $k = \beta = \dfrac{2\pi}{\lambda} = \underline{\underline{0.67}}$ $\text{rad}\cdot\text{m}^{-1}$

Example question

[from Sadiku] An EM wave propagating in a certain medium is described by:

$$\vec{E} = 25\sin\left(2\pi \times 10^{6} t - 6x\right)\vec{a}_z \ \text{Vm}^{-1}$$

(a) Determine the direction of propagation.
(b) Compute the period (T).
(c) Calculate the wavelength (λ).
(d) Establish the velocity (u). [hint $u = \omega/\beta$]

Example question

[from Sadiku] An EM wave propagating in a certain medium is described by:

$$\vec{E} = 25\sin\left(2\pi \times 10^6 t - 6x\right)\vec{a}_z \ \text{Vm}^{-1}$$

(a) Determine the direction of propagation.
(b) Compute the period (T).
(c) Calculate the wavelength (λ).
(d) Establish the velocity (u). [hint $u = \omega/\beta$]

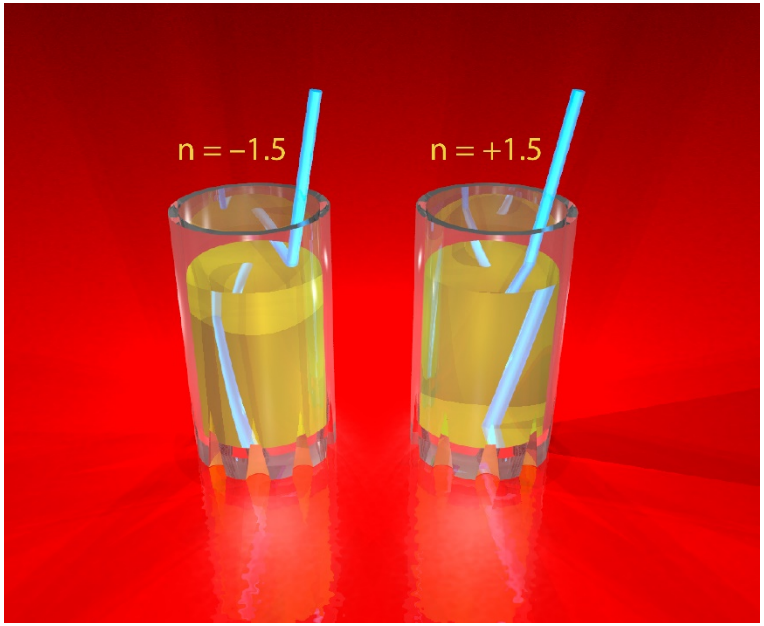
(a) The wave propagates along the positive x direction.

(b) $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2\pi \times 10^6} = \underline{\underline{1\ \mu\text{s}}}$

(c) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6} = \underline{\underline{1.047}}\ \text{m}$

(b) $u = \dfrac{\omega}{\beta} = \dfrac{2\pi \times 10^6}{6} = \underline{\underline{1.047 \times 10^6}}\ \text{ms}^{-1}$

Negative refractive index in chiral metamaterials



Constitutive relations in chiral metamaterials :

$$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E} + i\xi\sqrt{\mu_0 \varepsilon_0}\,\mathbf{H}$$

$$\mathbf{B} = \mu_r \mu_0 \mathbf{H} - i\xi\sqrt{\mu_0 \varepsilon_0}\,\mathbf{E}$$

D: electric displacement field
B: the magnetic field
E: the electric field
H: the magnetic field

ε_r: the relative permittivity
ε_0: the permittivity of vacuum
μ_r: the relative permeability
μ_0: the permeability of vacuum
ζ: the chirality parameter

The refractive index of circularly polarized light is: $n^{\pm} = n \pm \xi$, where $n = \sqrt{\varepsilon_r \mu_r}$.

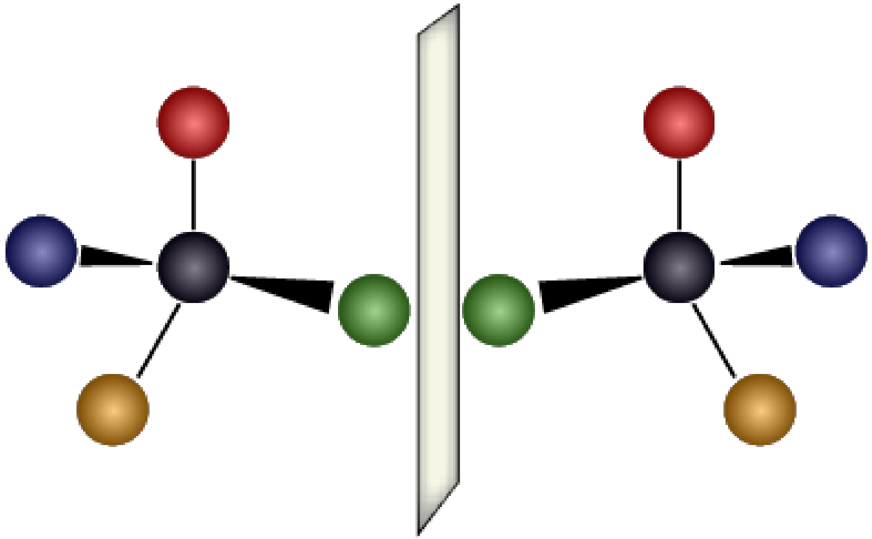
Consequently, a large $|\xi|$ leads directly to negative refractive index for one of the circularly-polarized electromagnetic waves in chiral metamaterials.

But what is chirality?

J.B. Pendry, "A chiral route to negative refraction," Science **306**, 1353 (2004).

Chirality and the mirror

Enantiomers



In his Baltimore Lectures on Molecular Dynamics and the wave theory of light, Lord Kelvin defined chirality as follows: "I call any geometric figure, or group of points, chiral, and say it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself".

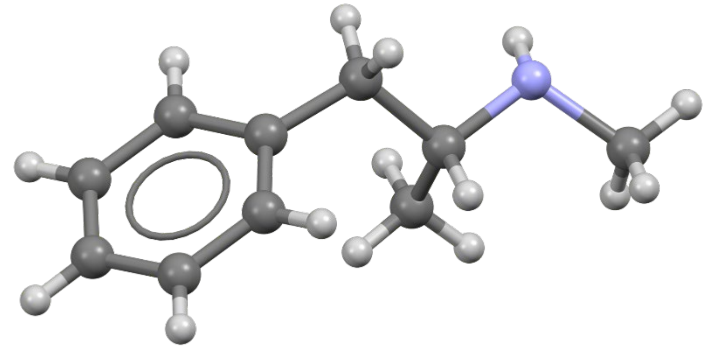
Lord Kelvin, in Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light, Clay and Sons: London, 1904, p.449.

How important is chirality?

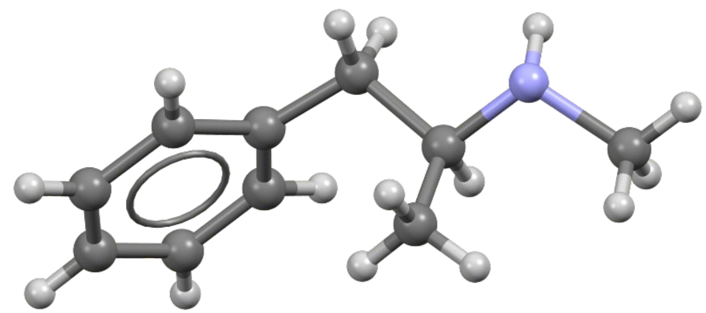
The scientist's evil image in the mirror

In the episode "Cat's in the Bag…" (S1E02) of the television show Breaking Bad, there is a good introduction to chirality.

R-methamphetamine



S-methamphetamine



Chirality is used as a metaphor for the transformation that the main character Walter White undergoes.

In summary…

Summary

In LIH materials, Maxwell's equations become: $\nabla \cdot \vec{D} = \rho_f \qquad \nabla \cdot \vec{B} = 0$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

In an ideal LIH dielectric, the electromagnetic wave propagates similarly to the way it does in free space – it remains a plane wave but its phase velocity is lowered.

$$v_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

The wave equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ becomes $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$, which is the modified wave equation.

A solution to the modified wave equation is $\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$ with $\gamma = -\alpha + i\beta$

So: $\vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)}$ Upon solving, an important ratio appears: $\sigma/\varepsilon\omega$

This ratio is part of the dielectric function of the material: $\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}$

and of its complex refractive index: $\tilde{n} = n + i\eta = \sqrt{\varepsilon(\omega)} = \sqrt{\varepsilon' + i\varepsilon''}$