6. Differentiation in non-Cartesian coordinates

Spatial derivatives like

$$\nabla\psi, \quad \nabla\cdot\mathbf{F}, \quad \nabla^2\psi, \quad \nabla\times\mathbf{F}$$

have the same value at a given point $\mathbf{r}$ whatever coordinates are used BUT we must take care to use the correct formula to calculate them.

General points

Don't assume Cartesians are always best

Choose coordinates that suit the problem

Derivation of formulae usually from "coordinate-free" expressions for each derivative:

- just "book-keeping" to keep track of scale-factors

- recall

Change u by $du \quad \Rightarrow$ move distance $h_u du$.

This unit – care more about accurate calculation of div, grad, curl, than derivation of formulae

6.1 Gradient of a scalar field, $\nabla\psi$

The spatial derivative of a scalar field (temperature, concentration etc) depends on direction.

If we move $d\mathbf{r}\ (=\lim_{\delta s\to 0}\delta\mathbf{r})$ from $\mathbf{r}$ then (§2.1)

$$d\psi = \operatorname{grad}\psi\cdot d\mathbf{r} = \nabla\psi\cdot d\mathbf{r}$$

We use this (coordinate-free) expression to derive

$\nabla\psi$ when $\psi=\psi(u,v,w)$

First write

$$\nabla\psi = f_u\hat{\mathbf{e}}_u + f_v\hat{\mathbf{e}}_v + f_w\hat{\mathbf{e}}_w.$$

Next recall that as $\mathbf{r}=\mathbf{r}(u,v,w)$, the "total differential" of $\mathbf{r}$ is

$$d\mathbf{r} = \frac{\partial\mathbf{r}}{\partial u}du + \frac{\partial\mathbf{r}}{\partial v}dv + \frac{\partial\mathbf{r}}{\partial w}dw,$$

which we can re-write as

$$d\mathbf{r} = h_u du\,\hat{\mathbf{e}}_u + h_v dv\,\hat{\mathbf{e}}_v + h_w dw\,\hat{\mathbf{e}}_w$$

by using the definition of a scale factor, $h_u\hat{\mathbf{e}}_u = \dfrac{\partial\mathbf{r}}{\partial u}$.

So

$$d\psi = \nabla\psi \cdot d\mathbf{r} = f_u h_u du + f_v h_v dv + f_w h_w dw\,. \qquad (1)$$

But as $\psi = \psi(u,v,w)$ we can also write $d\psi$ as

$$d\psi = \frac{\partial\psi}{\partial u}du + \frac{\partial\psi}{\partial v}dv + \frac{\partial\psi}{\partial w}dw. \qquad (2)$$

Eqns (1) and (2) must agree for arbitrary du, dv, dw.

$$\Rightarrow \quad f_u h_u = \frac{\partial\psi}{\partial u} \quad \Rightarrow \quad f_u = \frac{1}{h_u}\frac{\partial\psi}{\partial u}$$ etc, so

$$\nabla\psi = \frac{1}{h_u}\frac{\partial\psi}{\partial u}\hat{\mathbf{e}}_u + \frac{1}{h_v}\frac{\partial\psi}{\partial v}\hat{\mathbf{e}}_v + \frac{1}{h_w}\frac{\partial\psi}{\partial w}\hat{\mathbf{e}}_w \qquad (3)$$

Check – in Cartesians, $(u,v,w) \rightarrow (x,y,z)$;
$h_x = h_y = h_z = 1$ and

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial\psi}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial\psi}{\partial z}\hat{\mathbf{e}}_z$$ as before [$\hat{\mathbf{e}}_x = \mathbf{i}$ etc]

Explicit expressions for $\nabla\psi$

coordinate system	$\nabla\psi$
Cartesian	$\frac{\partial\psi}{\partial x}\mathbf{i}+\frac{\partial\psi}{\partial y}\mathbf{j}+\frac{\partial\psi}{\partial z}\mathbf{k}$
Cylindrical polar	$\frac{\partial\psi}{\partial\rho}\hat{\mathbf{e}}_\rho+\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\mathbf{e}}_\phi+\frac{\partial\psi}{\partial z}\hat{\mathbf{e}}_z$
Spherical polar	$\frac{\partial\psi}{\partial r}\hat{\mathbf{e}}_r+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\mathbf{e}}_\theta+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\mathbf{e}}_\phi$

You should remember that these formulae exist.

All on p19 of “Basic Formulae and Statistical Tables”

Example: Find $\nabla\Phi$ when $\Phi=\frac{1}{r}$.

Always use the most appropriate coordinate system.

Easiest here in spherical polars, as $\frac{\partial\Phi}{\partial\theta}=\frac{\partial\Phi}{\partial\phi}=0$.

Then

$$\nabla\Phi=\frac{1}{h_r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{e}}_r=\frac{1}{1}\left(\frac{-1}{r^2}\right)\hat{\mathbf{e}}_r=-\frac{1}{r^2}\hat{\mathbf{e}}_r.$$

In Cartesians, $\Phi = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$. Then need to find

eg

$$\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}}.2x = \frac{-x}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}$$

and similarly for the other 2 components, to give

$$\nabla \Phi = \frac{-x\mathbf{i} - y\mathbf{j} - z\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}.$$

[Note the 2 answers are the same, as

$$\hat{\mathbf{e}}_r = \frac{\mathbf{r}}{r} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}.]$$

Feel free to try the same in Cylindrical polars…

6.2 Divergence of a vector field, $\nabla\cdot\mathbf{F}$

Recall our interpretation of $\nabla\cdot\mathbf{F}$ in Cartesian coordinates. We effectively took a flux integral over volume $\delta V = \delta x\,\delta y\,\delta z$, and showed that divergence measures local sources of flux. In fact we were using the (coordinate-free) definition of divergence in terms of a flux integral:

$$\operatorname{div}\mathbf{F}(\mathbf{r}) = \nabla\cdot\mathbf{F}(\mathbf{r}) = \lim_{V\to 0}\frac{1}{V}\oint_S \mathbf{F}\cdot d\mathbf{S}$$

Surface S encloses volume V, which as $V\to 0$ converges around the point $\mathbf{r}$. The integral $\oint_S \mathbf{F}\cdot d\mathbf{S}$ gives net flux from V (remember, direction of $d\mathbf{S}$ is the *outward* normal).

This definition can be used to derive a formula for $\nabla\cdot\mathbf{F}$ in other orthogonal coordinate systems – see handout on moodle if you wish.

Result…

$$\nabla\cdot\mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(F_u\, h_v\, h_w\right) + \frac{\partial}{\partial v}\left(F_v\, h_w\, h_u\right) + \frac{\partial}{\partial w}\left(F_w\, h_u\, h_v\right)\right]$$

Explicit expressions for $\nabla \cdot \mathbf{F}$

system	$\nabla \cdot \mathbf{F}$
Cartesian	$\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}$
Cylindrical polar	$\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(F_\rho \rho\right)+\frac{\partial F_\phi}{\partial \phi}\right]+\frac{\partial F_z}{\partial z}$
Spherical polar	$\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(F_r r^2 \sin\theta\right)+\frac{\partial}{\partial \theta}\left(F_\theta r \sin\theta\right)+\frac{\partial}{\partial \phi}\left(F_\phi r\right)\right]$

You should remember that these formulae exist.

All on p19 of “Basic Formulae and Statistical Tables”

Example:

Find the divergence of the vector field

$$\mathbf{F} = \frac{5\sin\theta}{r^2}\hat{\mathbf{e}}_r + r\cot\theta\,\hat{\mathbf{e}}_\theta + \frac{1}{2}r\sin 2\theta\,\hat{\mathbf{e}}_\phi.$$

The field is expressed in spherical polars & is not obviously more straightforward in another coordinate system, so use $\nabla\cdot\mathbf{F} =$

$$\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(F_r\, r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(F_\theta\, r\sin\theta\right) + \frac{\partial}{\partial\phi}\left(F_\phi\, r\right)\right]$$

Here

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2\sin\theta}\left[\begin{array}{l}\dfrac{\partial}{\partial r}\left(\dfrac{5\sin\theta}{r^2}r^2\sin\theta\right)\\ +\dfrac{\partial}{\partial\theta}\left(r\cot\theta\, r\sin\theta\right)\\ +\dfrac{\partial}{\partial\phi}\left(\dfrac{1}{2}r\sin 2\theta\, r\right)\end{array}\right]$$

$$= \frac{1}{r^2\sin\theta}\left[0 - r^2\sin\theta + 0\right] = -1$$