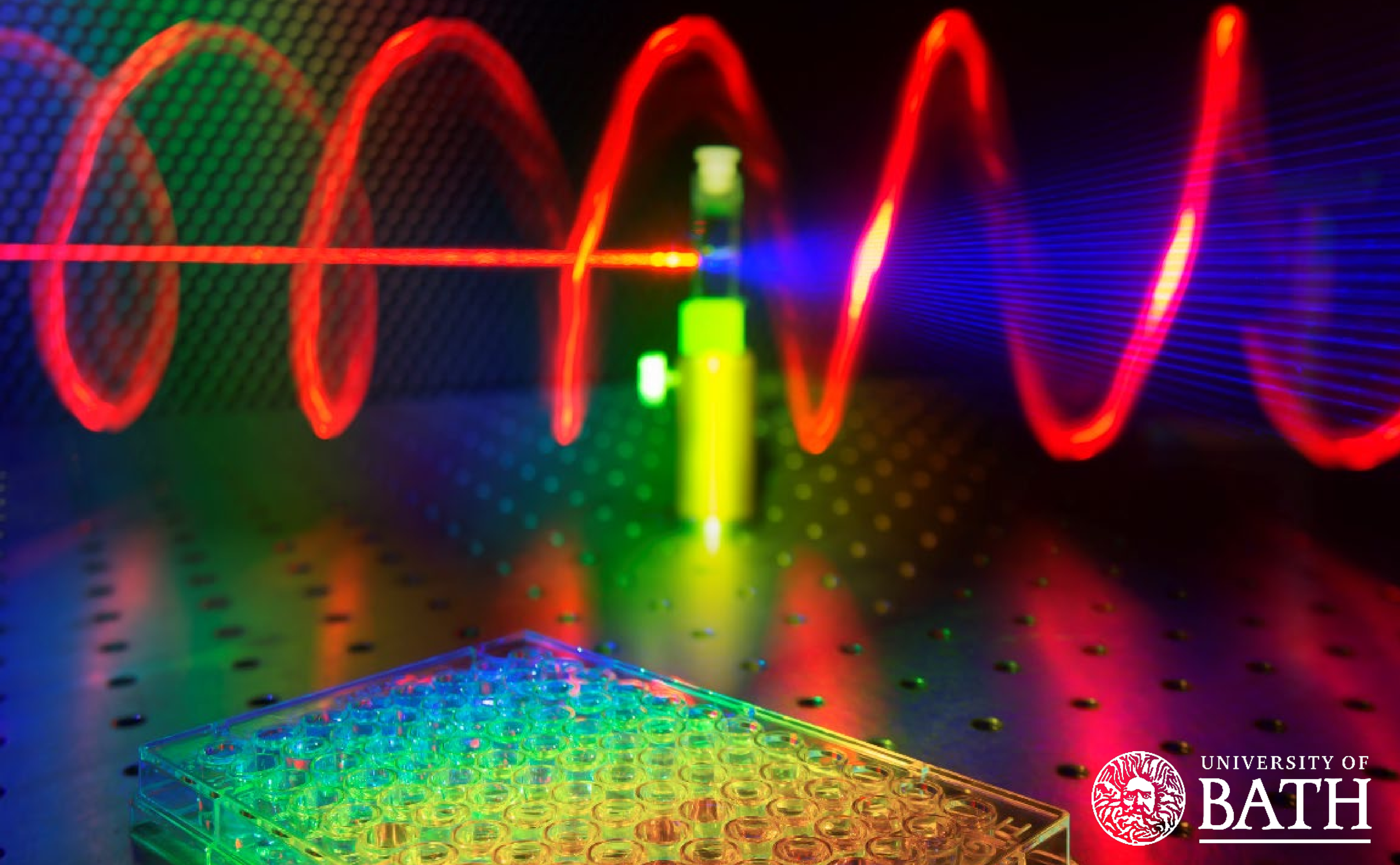
Lecture 12

Electric fields in materials



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Last time we saw that

In a dielectric, microscopic dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.

Macroscopically, the combined effect of these dipole moments produces a polarisation, defined as the induced dipole moment per unit volume: $\vec{P} = \frac{\sum \vec{p}}{V}$

In LIH dielectrics, the polarisation originates from the surface charge density $\sigma_p = P$.

In LIH dielectrics: $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ with χ_e being the electric susceptibility of the material.

The relative permeability of the material is given by: $\varepsilon_r = 1 + \chi_e$

We also talked about EM in cutting edge metamaterials.

The Blue Morpho – symbol of nanophotonics & matamaterials

Modern metamaterials use structure to manipulate EM waves.

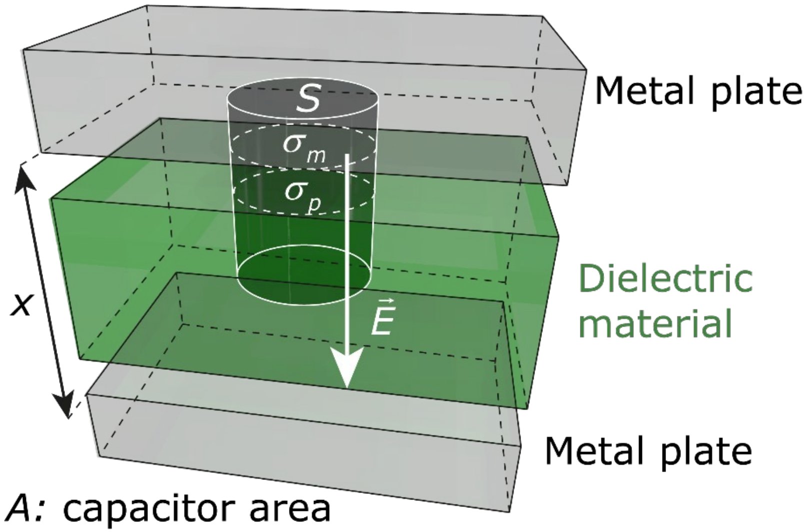
The Blue Morpho does this naturally. The colour of its wings is structural.

The colour changes depending on the angle of observation. This is called iridescence.

We also saw some typos…

Corrections – slide 13

We consider a plate capacitor:



A dielectric materials between two metal plates.

The E-field leaves the cylinder on three sides: side, top and bottom.

Therefore (side+top+bottom):

$$\Phi_e = 0 + 0 + \vec{E} \cdot \vec{S} = \vec{E} \cdot \vec{S}$$

The charge enclosed is (metal and dielectric):

$$\sigma_m S + \left(-\sigma_p\right) S = \left(\sigma_m - \sigma_p\right) S$$

Therefore, by Gauss' law:

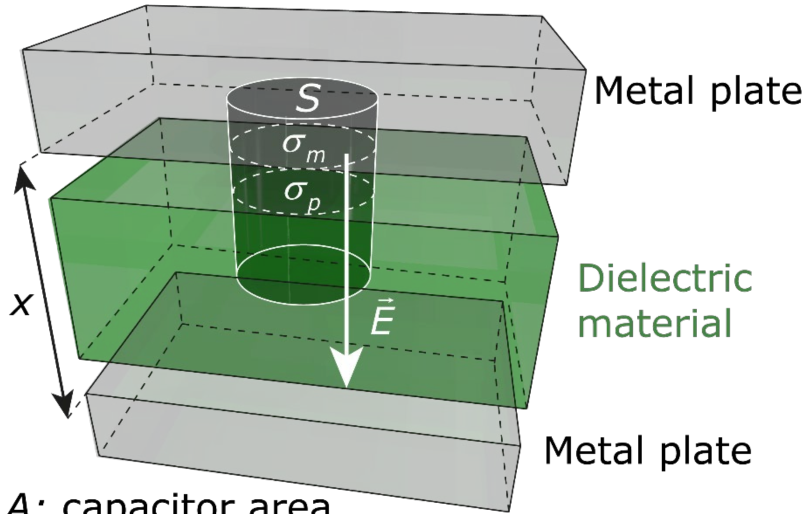
$$ES = \frac{\left(\sigma_m - \sigma_p\right) S}{\varepsilon_0} \quad \text{and} \quad E = \frac{\left(\sigma_m - \sigma_p\right)}{\varepsilon_0}$$

We can re-arrange using the magnitudes of the polarization vector $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

$$\varepsilon_0 E = \sigma_m - \sigma_p = \sigma_m - P = \sigma_m - \varepsilon_0 \chi_e E \quad \text{so that} \quad \boxed{E = \frac{\sigma_m}{\varepsilon_0 \left(1 + \chi_e\right)}}$$

Can we find another expression for the E-field?

We consider a plate capacitor:



A: capacitor area

A dielectric materials between two metal plates.

We just found: $E = \dfrac{\sigma_m}{\varepsilon_0 \left(1 + \chi_e\right)}$

From A-level physics: $E = \dfrac{V}{x}$, so:

$$\frac{V}{x} = \frac{\sigma_m}{\varepsilon_0 \left(1 + \chi_e\right)} = \frac{Q/A}{\varepsilon_0 \left(1 + \chi_e\right)}$$

Rearranging: $\dfrac{Q}{V} = \dfrac{\varepsilon_0 \left(1 + \chi_e\right) A}{x}$

From A-Level physics: $C = \dfrac{Q}{V}$; $C = \dfrac{A\varepsilon_0\varepsilon_r}{x}$

It follows that: $\dfrac{A\varepsilon_0\varepsilon_r}{x} = \dfrac{\varepsilon_0 \left(1 + \chi_e\right) A}{x}$

Where we see that $\boxed{\varepsilon_r = 1 + \chi_e}$. This is the **relative permittivity of the material**.

Consider an example question!

Overview

In this Lecture we will look at:

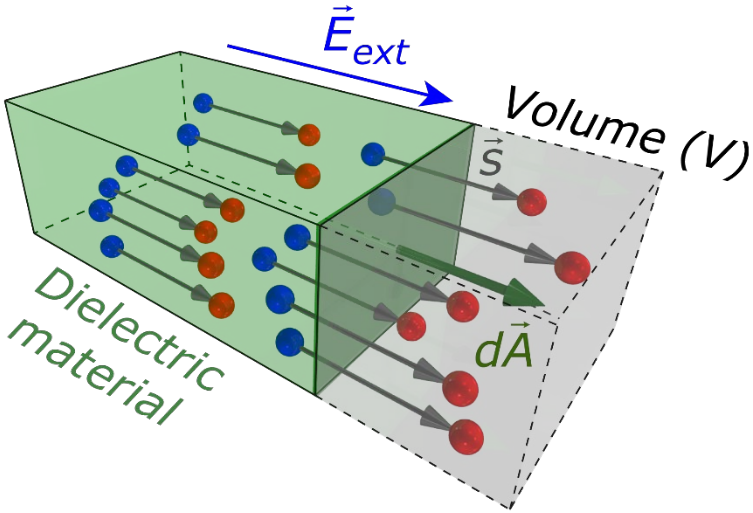
- ❑ Electric polarisation at the surface
- ❑ The sources of electric polarization
- ❑ Bound current density and the continuity equation
- ❑ Free charges and the electric flux density
- ❑ Free charges versus bound charges
- ❑ The energy stored in a dielectric

First, the electric polarisation at the surface!

Electric polarisation at the surface

We saw that: $\sigma_p = P$

Now, consider a unit volume V near a surface dA of a dielectric:



An external E-field splits the charges by a distance s, near an element of area dA, inside a volume $V=sdA$.

When E increases, $N+$ positive charges and $N-$ negative charges cross the small element of area dA. The net charge that crosses dA is then

$$dq_b = N^+ q_b - N^- (-q_b) = \left(N^+ + N^-\right) q_b$$

This is the sum of all charges crossing dA times the charge value. But, this is also the number of molecules within the unit volume: $V = \vec{s} \cdot d\vec{A}$

Then remembering that n is the number of molecules per unit volume and $\vec{s} = \vec{r}$:

$$dq_b = nVq_b = n\left(\vec{s} \cdot d\vec{A}\right) q_b =$$

$$= nq_b\vec{s} \cdot d\vec{A} = n\vec{p} \cdot d\vec{A} = \vec{P} \cdot d\vec{A}$$

And also, we can check that:

$$P = dq_b / dA = \sigma_p$$

How is this important?

The sources of electric polarisation

We just saw that:

$$dq_b = \vec{P} \cdot d\vec{A}$$

Inside the dielectric, the net charge that flows out of the closed surface A which encloses the volume V is

$$Q_{out} = \int_A \vec{P} \cdot d\vec{A}$$

but then, it is clear that the charge that remains within V must be $-Q_{out}$

We can also use the **volume density of charge** ρ_b to find the charge inside the dielectric:

$$\int_V \rho_b \cdot dv = -Q_{out}$$

If only we had something to link surface and volume integrals…

The sources of electric polarisation

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Inside the dielectric, the net charge that flows out of the closed surface *A* which encloses the volume *V* is

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We can also use the **volume density of charge** ρ_b to find the charge inside the dielectric:

$$\int_V \rho_b \cdot dv = -Q_{out}$$

So, using the divergence theorem:

$$\int_V \rho_b \cdot dv = -\int_A \vec{P} \cdot d\vec{A} = -\int_V \left(\nabla \cdot \vec{P}\right) dv$$

Therefore: $\boxed{\rho_b = -\nabla \cdot \vec{P}}$

This means that there are sources of polarisation.

The total bound polarisation charge within a region is obtained by integrating over a volume

$$Q_b = \int_V \rho_b \cdot dv = -\int_V \left(\nabla \cdot \vec{P}\right) dv = -\int_A \vec{P} \cdot d\vec{A}$$

How does this bound charge change with time?

Bound current density and the continuity equation

We just saw that:

$$Q_b = \int_V \rho_b \cdot dv = -\int_V \left(\nabla \cdot \vec{P}\right) dv$$

$$\boxed{\rho_b = -\nabla \cdot \vec{P}}$$

Under the influence of a time dependent electric field (such as the one in an EM wave), by definition, the current flowing through a surface A is related to the **current density** $\vec{J}$ by :

$$\frac{dQ_b}{dt} = I = \int_A \vec{J} \cdot d\vec{A} = \int_V \left(\nabla \cdot \vec{J}\right) dv$$

Substituting:

$$-\frac{d}{dt}\int_V \rho_b dv = -\int_V \left(\frac{\partial \rho_b}{\partial t}\right) dv = -\int_V \left(\frac{\partial\left(-\nabla \cdot \vec{P}\right)}{\partial t}\right) dv = \int_V \left(\nabla \cdot \frac{\partial \vec{P}}{\partial t}\right) dv = \int_V \left(\nabla \cdot \vec{J}_b\right) dv$$

Bound current density and the continuity equation

We just saw that:

$$Q_b = \int_V \rho_b \cdot dv = -\int_V \left(\nabla \cdot \vec{P}\right) dv$$

Under the influence of a time dependent electric field (such as the one in a light wave), by definition, the current flowing through a surface A is related to the **current density** $\vec{J}$ by :

$$\frac{dQ_b}{dt} = I = \int_A \vec{J} \cdot d\vec{A} = \int_V \left(\nabla \cdot \vec{J}\right) dv$$

Substituting:

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Therefore:

$$\vec{J}_b = \frac{\partial \vec{P}}{\partial t}$$

This means that the motion of bound charges results in a polarisation or **'bound' current density** $\vec{J}_b$.

We also deduce that:

$$\boxed{\nabla \cdot \vec{J}_b = -\frac{\partial \rho_b}{\partial t}}$$

which is called the **Continuity Equation**.

We discussed bound charges; how about free charges?

Free charges and the electric flux density

The total charge in a dielectric is:

$$\rho = \rho_b + \rho_f$$

The bound charges result from polarisation. The free charges do not.

From Maxwell's law:

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f$$

Therefore: $\nabla \cdot \left(\varepsilon_0 \vec{E} + \vec{P}\right) = \rho_f$

It looks like the free charges are the source of some field!

Let's define: $\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$

This is the **electric flux density** (aka **the electric displacement**).

Note: Water is a dielectric. It would have bound charges. But mixed in the water are some ion, which are free charges. Also, undoped Si can be a dielectric material, but it could have defects that constitute free charges.

Free charges and the electric flux density

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This is the **electric flux density** (aka **the electric displacement**)

Note: The equation $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ is known as a **Constitutive Equation**.

Now we can write Maxwell's law as:

$\boxed{\nabla \cdot \vec{D} = \rho_f}$ Compare to $\nabla \cdot \vec{E} = \rho$

The total free charge is now:

$$Q_f = \int_V \rho_f \cdot dv = \int_V \left(\nabla \cdot \vec{D} \right) dv = \int_A \vec{D} \cdot d\vec{A}$$

We can also write:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 \left(1 + \chi_e \right) \vec{E}$$

Replacing with: $\varepsilon_r = 1 + \chi_e$

We obtain: $\boxed{\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}}$

In free space $\vec{P} = 0$; $\varepsilon_r = 1$ so $\vec{D} = \varepsilon_0 \vec{E}$.

Examples?

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Example question

[Old exam question] A material has an electrical susceptibility χ_e of 3.5. Calculate the magnitude of the electric dipole moment per unit volume, i.e. the polarization *P*, and the electric displacement, *D*, if the electric field *E* is 15 $\mathrm{Vm^{-1}}$.

Example question

[Old exam question] A material has an electrical susceptibility χ_e of 3.5. Calculate the magnitude of the electric dipole moment per unit volume, i.e. the polarization *P*, and the electric displacement, *D*, if the electric field *E* is 15 Vm^{-1}.

Here, we have: $P = \varepsilon_0 \chi_e E$, $D = \varepsilon_0 \varepsilon_r E = \varepsilon_0 \left(1 + \chi_e\right) E$ and $D = \varepsilon_0 E + P$

For the polarisation:

$$P = \varepsilon_0 \chi_e E = \left(8.85 \times 10^{-12}\right)(3.5)(15) \approx 4.7 \times 10^{-10}\ \mathrm{Cm}^{-2}$$

Then, for the electric flux density, we can use either

$$D = \varepsilon_0 \varepsilon_r E = \varepsilon_0 \left(1 + \chi_e\right) E = \left(8.85 \times 10^{-12}\right)(4.5)(15) \approx 6.0 \times 10^{-10}\ \mathrm{Cm}^{-2}$$

or,

$$D = \varepsilon_0 E + P = \left(8.85 \times 10^{-12}\right)(15) + 4.7 \times 10^{-10} \approx 6.0 \times 10^{-10}\ \mathrm{Cm}^{-2}$$

[One mark is awarded for correct equations and one mark each for the correct numerical answers.]

How are free and bound charges related?

Free charges versus bound charges

In LIH dielectrics, we have a **free volume charge density** (ρ_f) and a **bound volume charge density** (ρ_b).

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \left(1 - \frac{1}{\varepsilon_r}\right)\vec{D}$$

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

Apply div to both sides:

$$\nabla \cdot \vec{P} = \left(1 - \frac{1}{\varepsilon_r}\right)\nabla \cdot \vec{D}$$

$$\boxed{\rho_b = -\nabla \cdot \vec{P}}$$

$$\boxed{\nabla \cdot \vec{D} = \rho_f}$$

Which leads to:

$$\boxed{\rho_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\rho_f}$$

And we can calculate the total ρ :

$$\rho = \rho_b + \rho_f = -\left(1 - \frac{1}{\varepsilon_r}\right)\rho_f + \rho_f = \frac{\rho_f}{\varepsilon_r}$$

Free charges versus bound charges

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And we can calculate the total ρ :

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We also have a **free surface charge density (**σ_f**)** and a **bound surface charge density** (σ_b).

$$\sigma_f = D = \varepsilon_0\varepsilon_r E = \varepsilon_0 E + P = \varepsilon_0 E - \sigma_b$$

Leading to: $E = \dfrac{\sigma_f + \sigma_b}{\varepsilon_0}$

Since $D = \varepsilon_0\varepsilon_r E$, $\dfrac{D}{\varepsilon_0\varepsilon_r} = \dfrac{\sigma_f}{\varepsilon_0\varepsilon_r} = \dfrac{\sigma_f + \sigma_b}{\varepsilon_0}$

Therefore: $\sigma_f = \varepsilon_r\left(\sigma_f + \sigma_b\right)$

Rearranging $\boxed{\sigma_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\sigma_f}$

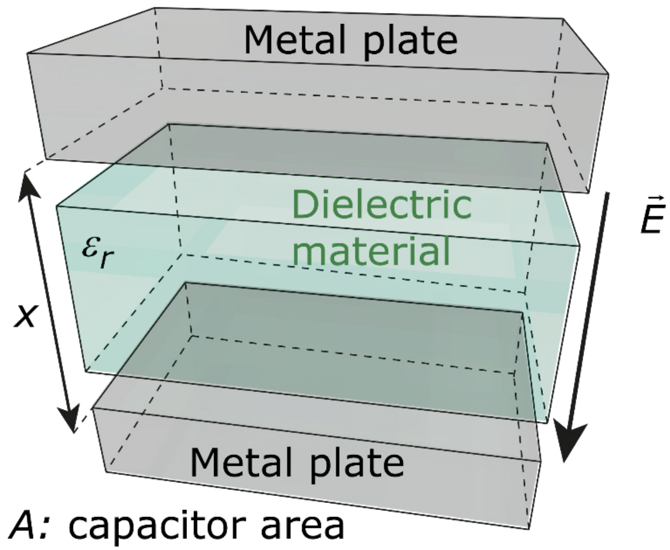
And we can calculate the total σ:

$$\sigma = \sigma_b + \sigma_f = -\left(1 - \frac{1}{\varepsilon_r}\right)\sigma_f + \sigma_f = \frac{\sigma_f}{\varepsilon_r}$$

What is the energy stored in a dielectric?

The energy stored in a dielectric

We consider a parallel plate capacitor:



Work required to charge the capacitor:

$$w = \frac{1}{2}CV^2$$

We also have: $C = \frac{A\varepsilon_0\varepsilon_r}{x}$ and $E = \frac{V}{x}$

Then, using: $\vec{D} = \varepsilon_0\varepsilon_r\vec{E}$ and $V = Ex$

$$\frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\varepsilon_r\varepsilon_0 A}{x}\right)(Ex)^2 =$$

$$= \frac{1}{2}(\varepsilon_r\varepsilon_0 E)E(Ax) = \frac{1}{2}DE \times volume$$

Therefore, the energy (w) stored per unit volume v is: $\boxed{w = \frac{1}{2}\vec{D}\cdot\vec{E}}$

And we have the **energy stored**: $W = \int_V w dv$

Let's have a look at some examples.

Example question

[2005 Exam Question] A charge of uniform charge density ρ is distributed throughout a medium with $\varepsilon_r = 1$ throughout a sphere of radius R. The electric displacement ($\vec{D}$) inside the sphere is given by $\vec{D} = \frac{\rho r}{3}\hat{r}$ for r < R.

Find the energy stored inside the charge distribution (i.e. for $r \leq R$). (3)

Example question

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Find the energy stored inside the charge distribution (i.e. for $r \le R$). (3)

We have $\vec{D} = \frac{\rho r}{3}\hat{r}$ and we know that $\vec{D} = \varepsilon\vec{E} = \varepsilon_0\varepsilon_r\vec{E}$. [1 point]

So, we have $\vec{E} = \frac{\rho r}{3\varepsilon_0\varepsilon_r}\hat{r}$ and since $\varepsilon_r = 1$, we can write $\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r}$.

The energy density is $w = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\left(\frac{\rho r}{3}\hat{r}\right)\cdot\left(\frac{\rho r}{3\varepsilon_0}\hat{r}\right) = \frac{\rho^2 r^2}{18\varepsilon_0}$. [1 point]

Integrate over the volume of the sphere to find the stored energy. [1 point]

$$W = \int_0^R w\,dV = \int_0^R \frac{\rho^2 r^2}{18\varepsilon_0}\left(4\pi r^2 dr\right) = \frac{4\pi\rho^2}{18\varepsilon_0}\int_0^R r^4 dr = \frac{4\pi\rho^2}{18\varepsilon_0}\left[\frac{r^5}{5}\right]_0^R = \frac{2\pi\rho^2 R^5}{45\varepsilon_0}\ \text{J}$$

Example question

[2016 Exam Question] Describe what it means that, with respect to electric and magnetic fields, a medium is:

(a) linear (1)

(b) isotropic (1)

(c) homogeneous (1)

Example question

[2016 Exam Question] Describe what it means that, with respect to electric and magnetic fields, a medium is:
(a) linear (1)
(b) isotropic (1)
(c) homogeneous (1)

(a) For the electric fields, the polarization density $\vec{P}$ (and the displacement field $\vec{D}$) is proportional to the external electric field $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ (and $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$). For the magnetic fields, the magnetization $\vec{M}$ (magnetic field $\vec{B}$) is proportional to the magnetic field intensity $\vec{M} = \mu_0 \chi_m \vec{H}$ (and $\vec{B} = \mu_0 \mu_r \vec{H}$).

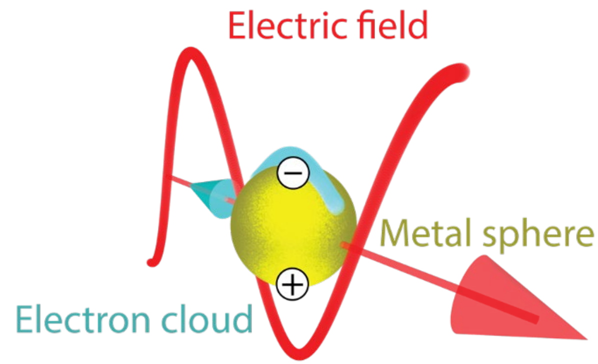
(b) At a set point in the medium ε_r (or χ_e) and μ_r (or χ_m) do not depend on the direction of the applied external field $\vec{E}$ and $\vec{H}$, respectively.

(c) The relative permittivity ε_r (electric susceptibility χ_e) and relative permeability μ_r (magnetic susceptibility χ_m) do not depend on the position within the medium.

What about nonlinear materials?

Optical Harmonic Generation

In linear optics: $\mathbf{P} = \chi^{(1)} \cdot \mathbf{E}$



For more intense electromagnetic fields:

$$\mathbf{P} = \chi^{(1)} \cdot \mathbf{E} + \chi^{(2)} : \mathbf{EE} + \chi^{(3)} \vdots \mathbf{EEE} + \ldots$$

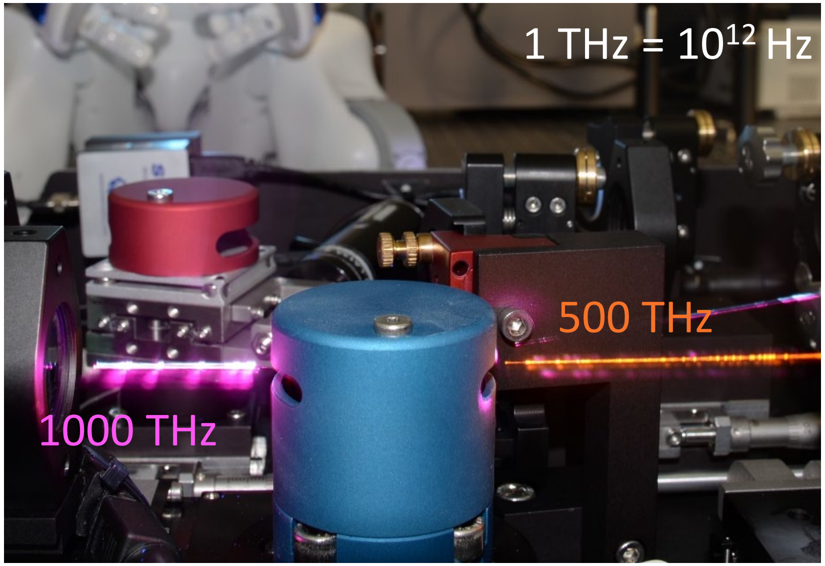
The induced polarization contains higher harmonics:

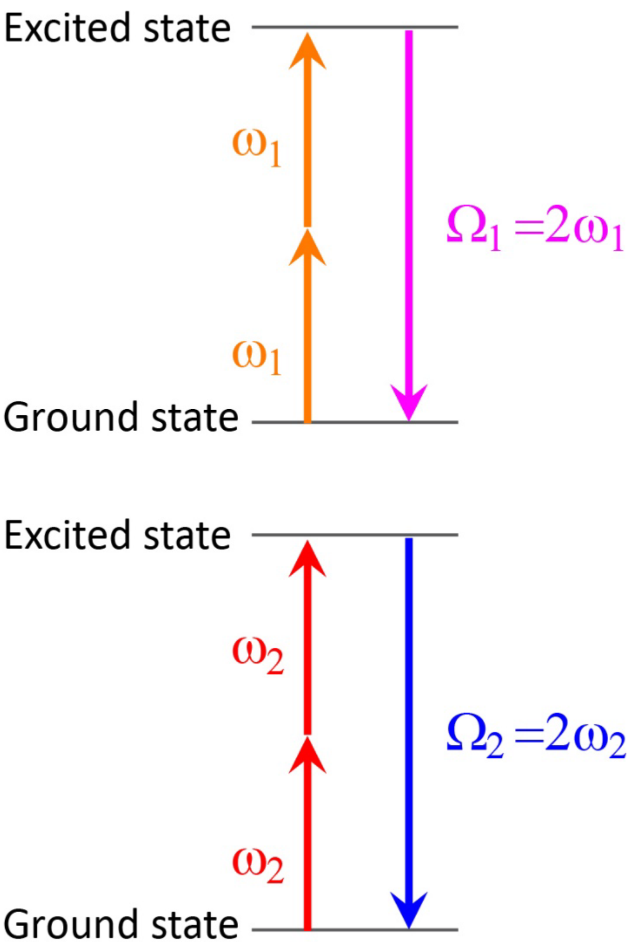
$$\mathbf{P} = \mathbf{P}(0) + \mathbf{P}(\omega) + \mathbf{P}(2\omega) + \mathbf{P}(3\omega) + \ldots$$

where $\mathbf{P}_i(2\omega) = \chi^{(2)}_{ijk} : \mathbf{E}_j(\omega)\mathbf{E}_k(\omega)$ and $\mathbf{P}_i(2\omega) = \chi^{(3)}_{ijkl} \vdots \mathbf{E}_j(\omega)\mathbf{E}_k(\omega)\mathbf{E}_l(\omega)$ etc.

Second harmonic generation in practice?

Second Harmonic Generation (SHG)





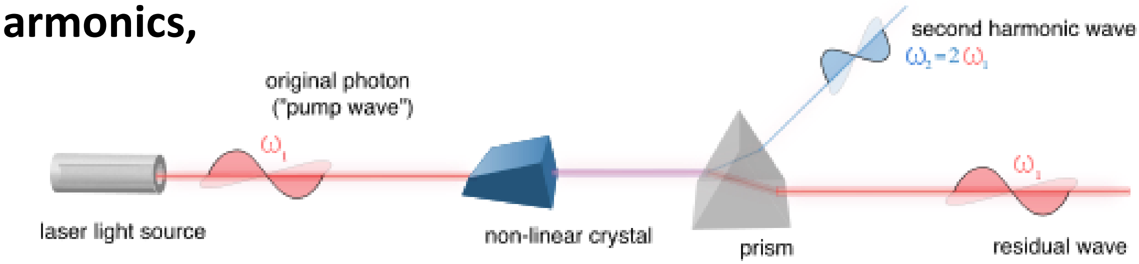
SHG is used for:

- Microscopy
- Frequency conversion in lasers
- Materials analysis: externally applied magnetic and electric fields, chirality, strain, localized surface electric fields, surface roughness, etc.

How did it all begin?

The spotless entrance of Second Harmonic Generation

P. A. Franken, A. E. Hill, C. W. Peters, and G. Weinreich, **Generation of Optical Harmonics,** Phys. Rev. Lett. 7, 118–119 (1961)



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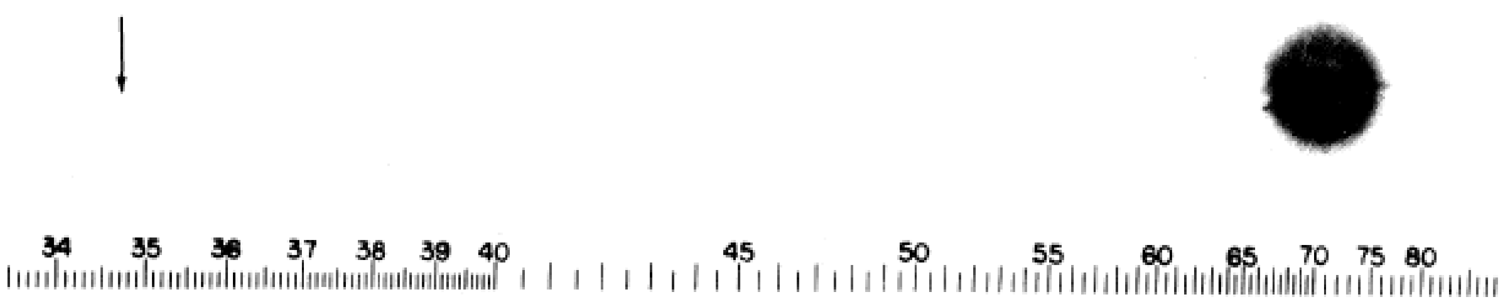

FIG. 1. A direct reproduction of the first plate in which there was an indication of second harmonic. The wavelength scale is in units of 100 A. The arrow at 3472 A indicates the small but dense image produced by the second harmonic. The image of the primary beam at 6943 A is very large due to halation.

In summary

Summary

In a dielectric, microscopic dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.

Macroscopically, the combined effect of these dipole moments produces a polarisation, defined as the induced dipole moment per unit volume: $\vec{P} = \frac{\sum \vec{p}}{V}$

In LIH dielectrics, the polarisation originates from the surface charge density $\sigma_p = P$.

In LIH dielectrics: $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ with χ_e being the electric susceptibility of the material.

The electric flux density is $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$

The constitutive relation states that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

We have a new Maxwell equation that states: $\nabla \cdot \vec{D} = \rho_f$

The relative permeability of the material is given by: $\varepsilon_r = 1 + \chi_e$

Summary

The polarisation has sources, these sources are the bound density of charge:

$$\rho_b = -\nabla \cdot \vec{P}$$

The bound density of charge is related to the free density of charge by:

$$\rho_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\rho_f$$

The surface bound density of charge is related to the surface free density of charge by:

$$\sigma_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\sigma_f$$

The energy per unit volume stored in a dielectric is:

$$w = \frac{1}{2}\vec{D} \cdot \vec{E}$$

How about magnetic materials?