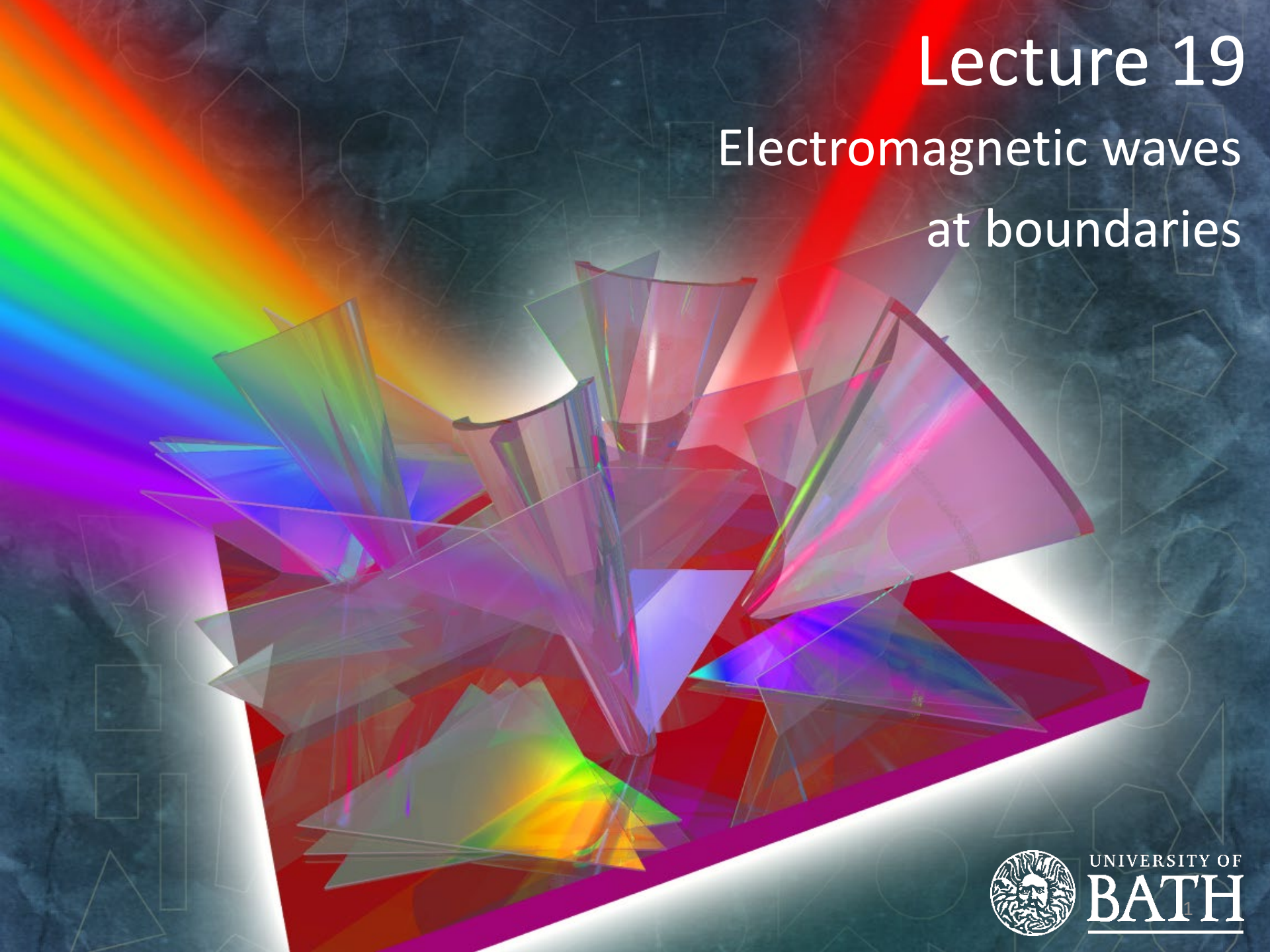


Lecture 19

Electromagnetic waves at boundaries



Summary

At the boundary between two materials, EM wave is partially reflected and partially transmitted.

The plane of incidence is defined by the wave vector of the incident wave and a unit vector normal to the boundary.

On each side of the boundary, the electric and magnetic fields can be resolved into normal and tangential components.

For LIH materials, assuming no surface charges and no surface currents.

	Electric fields	Magnetic fields
Normal components	$\vec{D}_{1n} = \vec{D}_{2n}$	$\vec{B}_{1n} = \vec{B}_{2n}$
Tangential components	$\vec{E}_{1t} = \vec{E}_{2t}$	$\vec{H}_{1t} = \vec{H}_{2t}$

Summary

At normal incidence, the reflection and transmission coefficients are:

$$r_{\parallel/\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \quad t_{\parallel/\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2}{Z_1 + Z_2}$$

Snell's law results from conservation of moment at the interface.

For a general angle of incidence, we distinguish two cases of light polarisation:

- (i) in the plane of incidence (this is P-polarized light)
- (ii) perpendicular to the plane of incidence (this is S-polarized light)

The Fresnel coefficients for P-polarized light are:

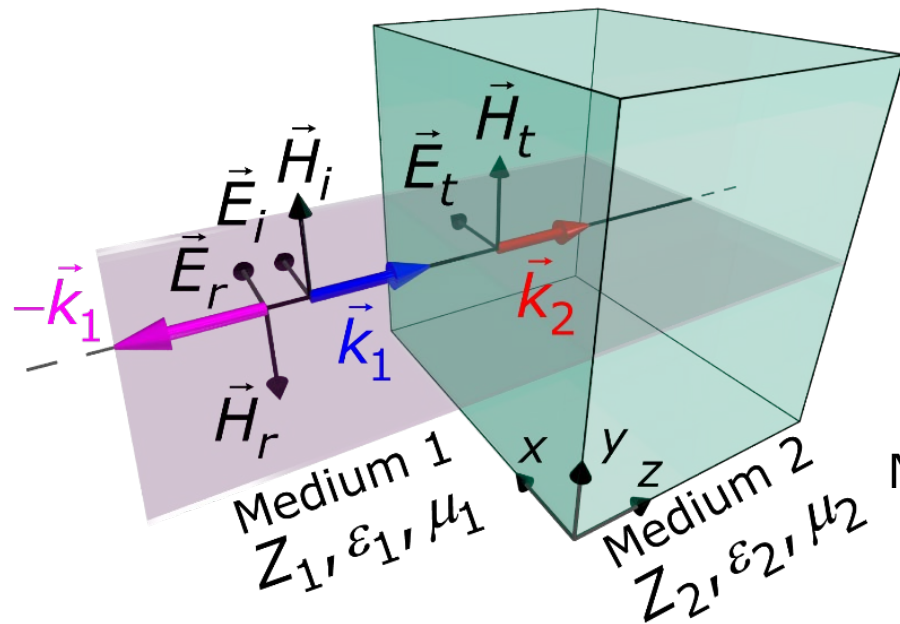
$$r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos \theta_t - Z_1 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i} \quad t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}$$

Today we will see...

In this Lecture we will look at:

- ☐ Electromagnetic waves at normal incidence
- ☐ General incidence at the boundary for S-polarized light
- ☐ Brewster angle
- ☐ The critical angle

Electromagnetic waves at normal incidence



We defined **reflection and transmission coefficients**:

$$r_{\parallel/\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_1 + Z_2}$$

$$t_{\parallel/\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2}{Z_1 + Z_2}$$

Note: $r_{\parallel/\perp} \neq r$ [position] and $t_{\parallel/\perp} \neq t$ [time]

$$Z_1 = E_{i0}/H_{i0}$$

$$Z_1 = E_{r0}/H_{r0}$$

$$Z_2 = E_{t0}/H_{t0}$$

What was polarisation?

Polarisation of an electromagnetic wave

For an EM wave, the plane of oscillations of the electric field defines the direction of **light polarisation**.

Note: Do not confuse the electric polarisation, which we discussed in dielectric materials (a material property!) and the light polarisation, which is a property of electromagnetic wave (a light property!)

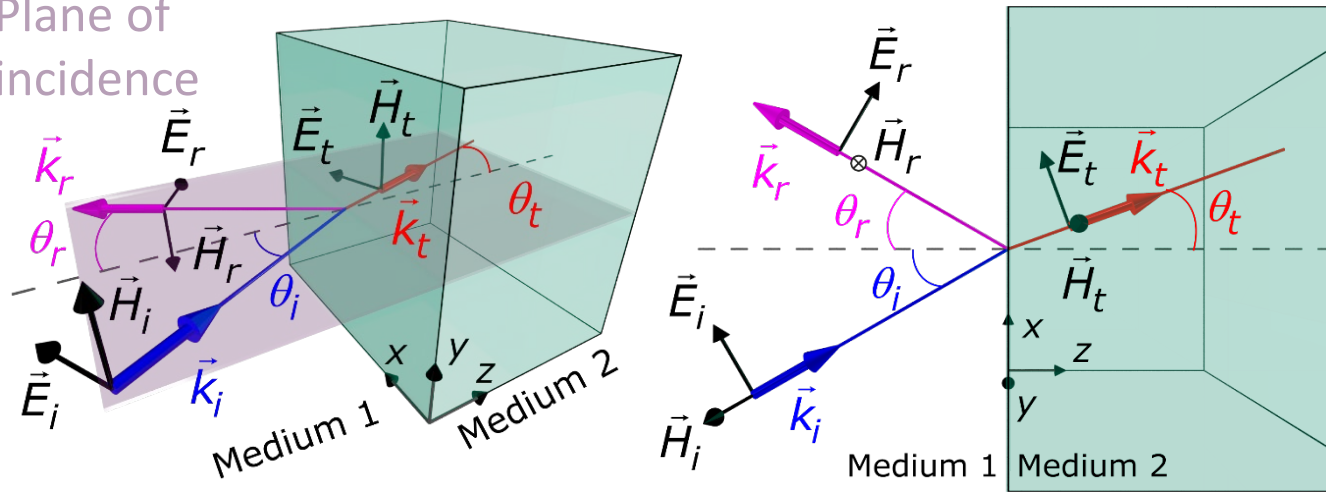
We can consider two cases of light polarisation:

1. The polarisation of light is in the plane of incidence (the magnetic field oscillations are perpendicular to the plane of incidence). This polarisation state is referred to as **P-polarized** (from the German parallel). It can be indicated as (\parallel).
2. The polarisation of light is perpendicular to the plane of incidence (the magnetic field oscillations are in the plane of incidence). This polarisation state is referred to as **S-polarized** (from the German senkrecht). It can be indicated as (\perp).

Let's look at the diagrams more carefully!

General incidence at the boundary for P-polarized light

Plane of incidence



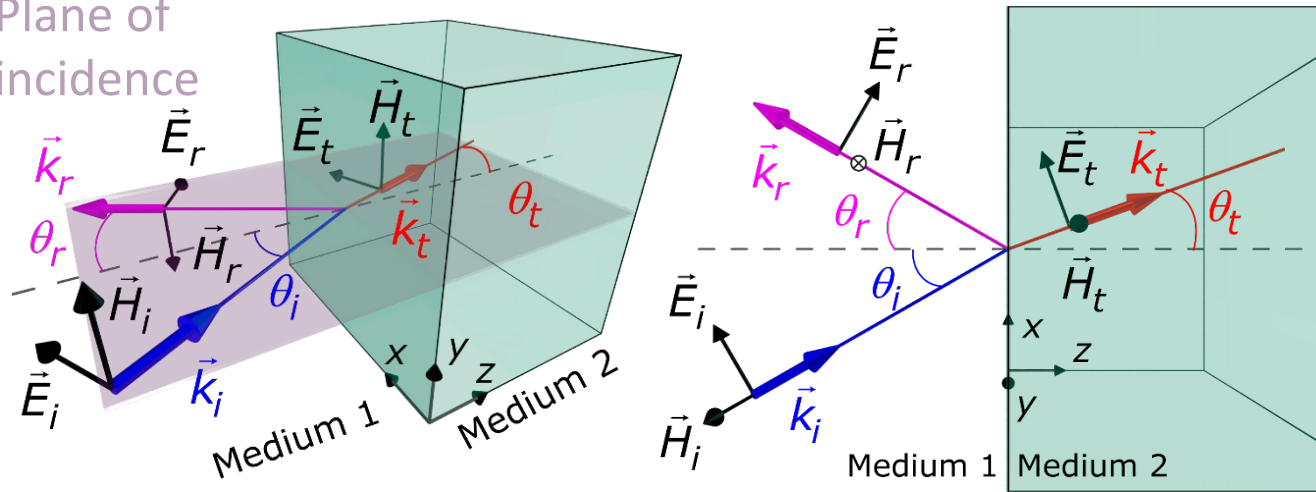
[You need to be able to draw such diagrams]

1. Draw directions of the **wave vectors** according to the propagation directions.
2. Draw the **electric fields**:
 - a. Perpendicular to their respective wave vectors.
 - b. For P-polarized light they are in the plane of the figure [plane of incidence].
 - c. Their components are all either along positive x or negative x. Here, we chose for all to be along positive x.
3. Draw the **magnetic fields** with direction set by:
$$\vec{H} = \frac{1}{\mu\omega} \vec{k} \times \vec{E}$$

Fresnel coefficients for P-polarized light?

General incidence at the boundary for P-polarized light

Plane of incidence



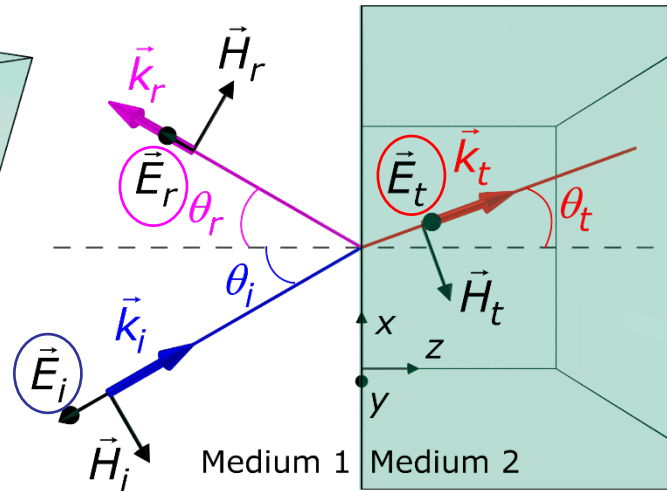
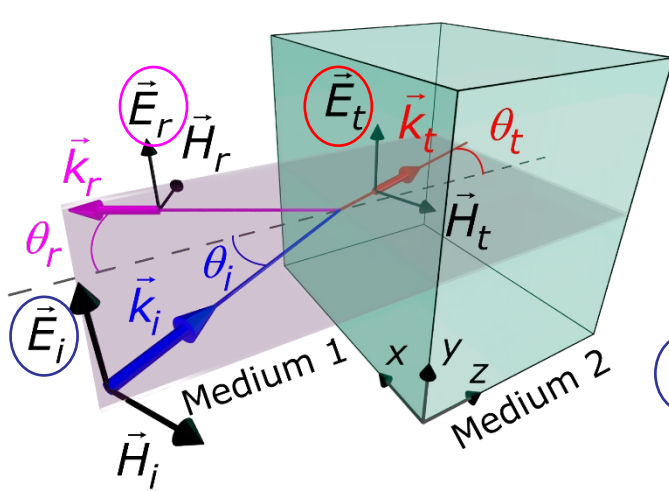
We defined the **Fresnel coefficients for P-polarized light**:

$$r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos \theta_t - Z_1 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}$$

$$t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}$$

How about S-polarized light?

General incidence at the boundary for S-polarized light



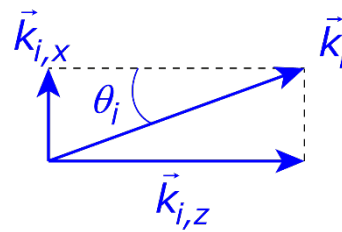
We can resolve all the vectors in the Cartesian coordinate system:

For the E -fields:

$$\vec{E}_i = E_{i0} \hat{y} e^{i(k_1 \sin \theta_i x + k_1 \cos \theta_i z - \omega t)}$$

$$\vec{E}_r = E_{r0} \hat{y} e^{i(k_1 \sin \theta_r x - k_1 \cos \theta_r z - \omega t)}$$

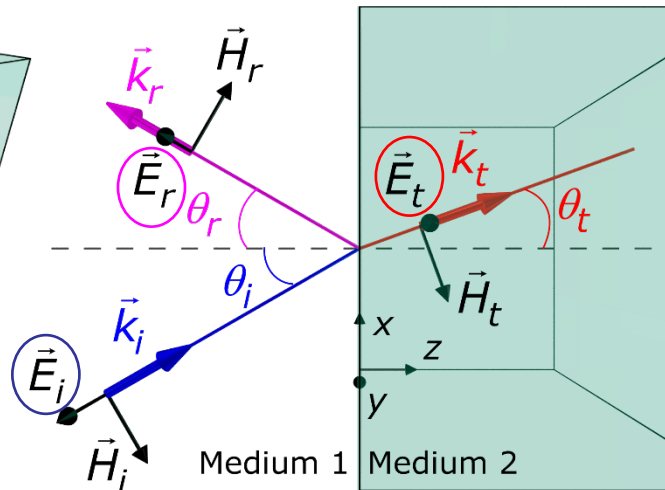
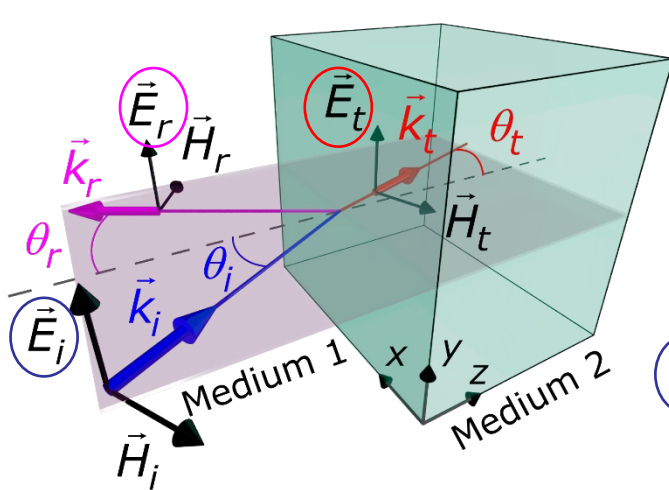
$$\vec{E}_t = E_{t0} \hat{y} e^{i(k_2 \sin \theta_t x + k_2 \cos \theta_t z - \omega t)}$$



$$\begin{cases} \sin \theta_i = \frac{|\vec{k}_{i,x}|}{|\vec{k}_i|} \\ \cos \theta_i = \frac{|\vec{k}_{i,z}|}{|\vec{k}_i|} \end{cases} \rightarrow \begin{cases} |\vec{k}_{i,x}| = |\vec{k}_i| \sin \theta_i \\ |\vec{k}_{i,z}| = |\vec{k}_i| \cos \theta_i \end{cases}$$

Remember E.T.

General incidence at the boundary for S-polarized light



We can resolve all the vectors in the Cartesian coordinate system:



For the E -fields:

$$\vec{E}_i = E_{i0} \hat{y} e^{i(k_1 \sin \theta_i x + k_1 \cos \theta_i z - \omega t)}$$

$$\vec{E}_r = E_{r0} \hat{y} e^{i(k_1 \sin \theta_r x - k_1 \cos \theta_r z - \omega t)}$$

$$\vec{E}_t = E_{t0} \hat{y} e^{i(k_2 \sin \theta_t x + k_2 \cos \theta_t z - \omega t)}$$

We require that: $\vec{E}_1|_{\text{tangential}} = \vec{E}_2|_{\text{tangential}}$

Meaning that: $\vec{E}_1|_y = \vec{E}_2|_y$, so:

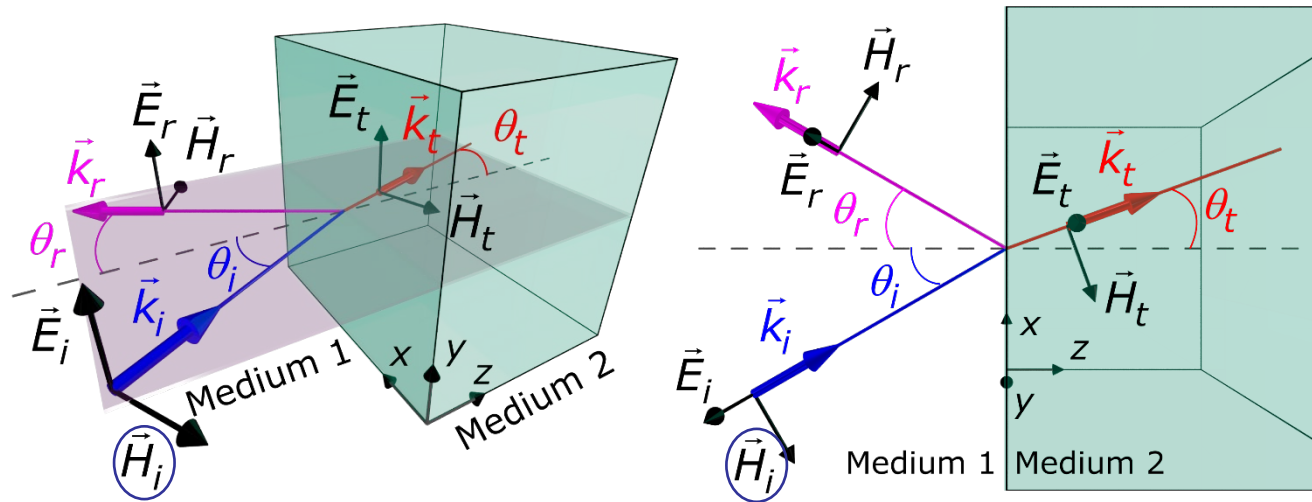
$$(\vec{E}_i + \vec{E}_r)|_y = \vec{E}_t|_y$$

Replacing:

$$E_{i0} + E_{r0} = E_{t0}$$

Now the H -fields:

General incidence at the boundary for S-polarized light

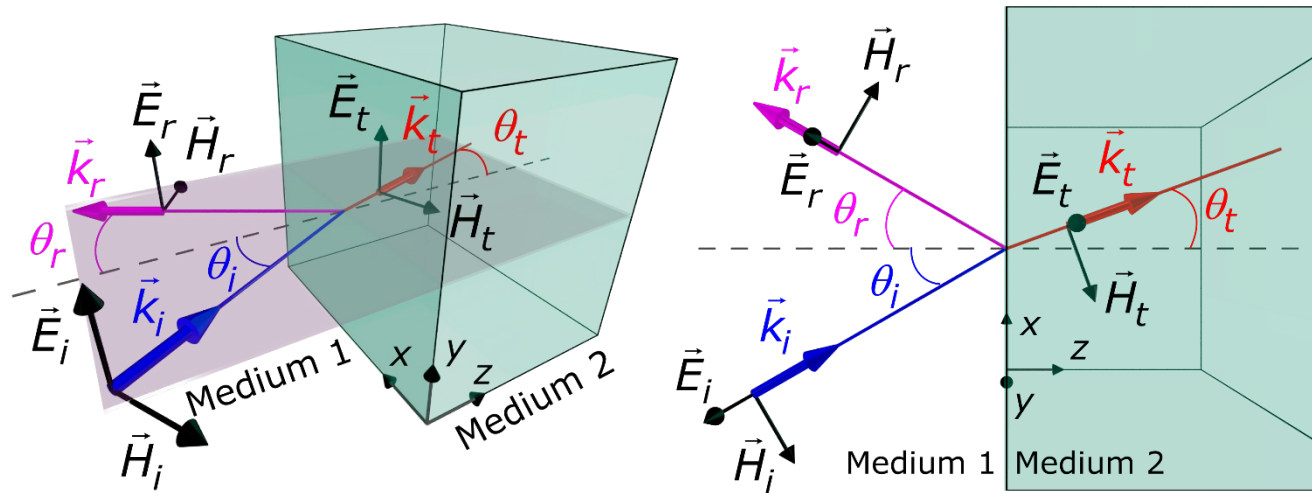


For the H -fields:

$$\vec{H}_i = \frac{E_{i0}}{Z_1} (-\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) e^{i(k_1 \sin \theta_i x + k_1 \cos \theta_i z - \omega t)}$$

$$\begin{cases} \cos \theta_i = \frac{|\vec{H}_{i0,x}|}{|\vec{H}_{i0}|} \\ \sin \theta_i = \frac{|\vec{H}_{i0,z}|}{|\vec{H}_{i0}|} \end{cases} \rightarrow \begin{cases} |\vec{H}_{i0,x}| = |\vec{H}_{i0}| \cos \theta_i = \frac{|\vec{E}_{i0}|}{Z_1} \cos \theta_i \\ |\vec{H}_{i0,z}| = |\vec{H}_{i0}| \sin \theta_i = \frac{|\vec{E}_{i0}|}{Z_1} \sin \theta_i \end{cases}$$

General incidence at the boundary for S-polarized light



For the H -fields:

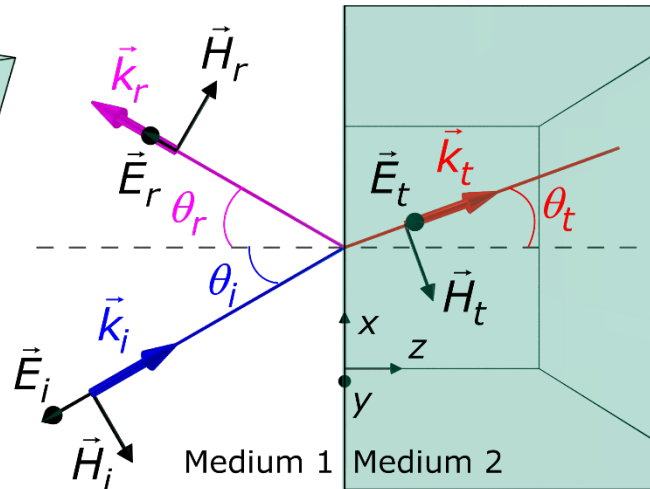
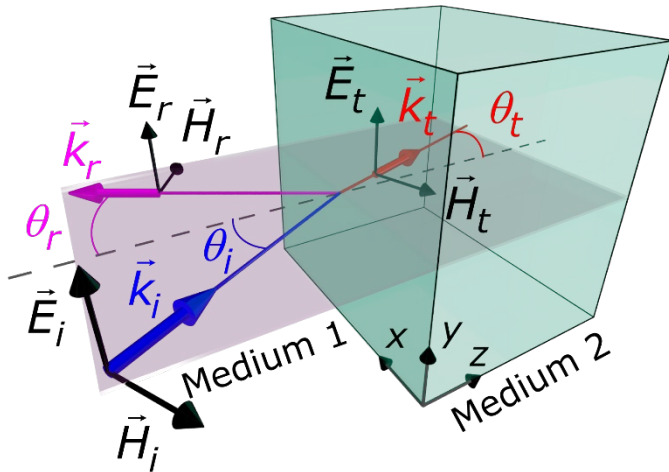
$$\vec{H}_i = \frac{E_{i0}}{Z_1} (-\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) e^{i(k_1 \sin \theta_i x + k_1 \cos \theta_i z - \omega t)}$$

$$\vec{H}_r = \frac{E_{r0}}{Z_1} (\cos \theta_r \hat{x} + \sin \theta_r \hat{z}) e^{i(k_1 \sin \theta_r x - k_1 \cos \theta_r z - \omega t)}$$

$$\vec{H}_t = \frac{E_{t0}}{Z_2} (-\cos \theta_t \hat{x} + \sin \theta_t \hat{z}) e^{i(k_2 \sin \theta_t x + k_2 \cos \theta_t z - \omega t)}$$

We only want the amplitudes

General incidence at the boundary for S-polarized light



WHY BE NORMAL?

We require:

$$\vec{H}_1|_{\text{tangential}} = \vec{H}_2|_{\text{tangential}}$$

Which is:

$$\vec{H}_1|_x = \vec{H}_2|_x$$

Therefore:

$$(\vec{H}_i + \vec{H}_r)|_x = \vec{H}_t|_x$$

For the H -fields:

$$\vec{H}_i = \frac{E_{i0}}{Z_1} (-\cos \theta_i \hat{x} + \sin \theta_i \hat{z}) e^x$$

$$\vec{H}_r = \frac{E_{r0}}{Z_1} (\cos \theta_r \hat{x} + \sin \theta_r \hat{z}) e^y$$

$$\vec{H}_t = \frac{E_{t0}}{Z_2} (-\cos \theta_t \hat{x} + \sin \theta_t \hat{z}) e^z$$

We can now replace:

$$-\frac{E_{i0}}{Z_1} \cos \theta_i \hat{x} + \frac{E_{r0}}{Z_1} \cos \theta_r \hat{x} = -\frac{E_{t0}}{Z_2} \cos \theta_t \hat{x}$$

Dropping the unit vectors:

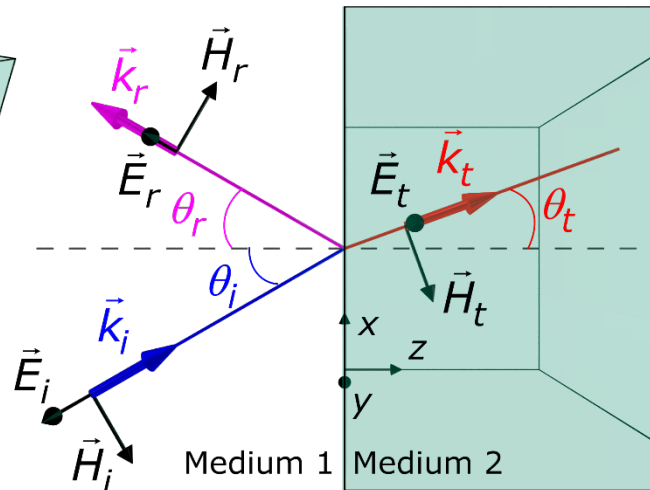
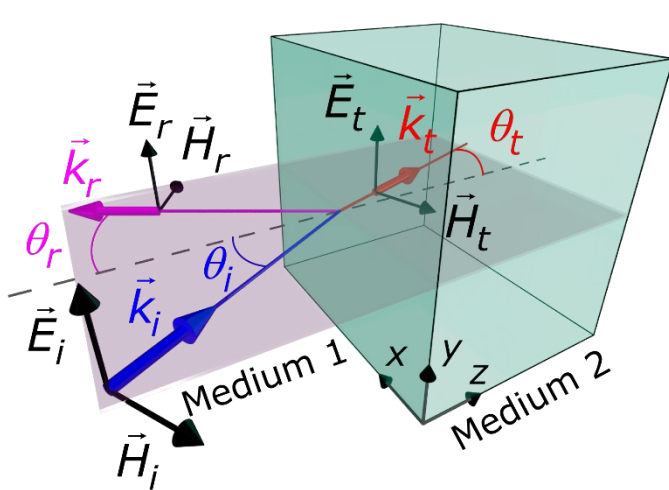
$$-E_{i0} + E_{r0} = -\frac{Z_1}{Z_2} \frac{\cos \theta_t}{\cos \theta_i} E_{t0}$$

Rearranging:

$$E_{i0} - E_{r0} = \frac{Z_1}{Z_2} \frac{\cos \theta_t}{\cos \theta_i} E_{t0}$$

Now we combine the equations for E and H -fields.

General incidence at the boundary for S-polarized light



We obtained:

$$E_{i0} + E_{r0} = E_{t0}$$

And

$$E_{i0} - E_{r0} = \frac{Z_1 \cos \theta_t}{Z_2 \cos \theta_i} E_{t0}$$

Adding them up:

$$2E_{i0} = \left(1 + \frac{Z_1 \cos \theta_t}{Z_2 \cos \theta_i} \right) E_{t0}$$

Which means that: $2E_{i0} = \frac{Z_2 \cos \theta_i + Z_1 \cos \theta_t}{Z_2 \cos \theta_i} E_{t0}$

So: $E_{t0} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} E_{i0}$, and $E_{i0} + E_{r0} = E_{t0} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} E_{i0}$

Hence: $E_{r0} = \left(\frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} - 1 \right) E_{i0} = \left(\frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} \right) E_{i0}$

And we define the **Fresnel coefficients for S-polarized light** as:

$$r_{\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$$

$$t_{\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$$

Example question 1

For non-magnetic materials, show that:

$$r_{\perp} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \quad \text{and} \quad t_{\perp} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_t + \theta_i)}$$

Example question 1

For non-magnetic materials, show that:

$$r_{\perp} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \quad \text{and} \quad t_{\perp} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_t + \theta_i)}$$

Remembering that $r_{\perp} = \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$, we use $Z = \frac{Z_0}{n}$, so:

$$r_{\perp} = \frac{\frac{Z_0}{n_2} \cos \theta_i - \frac{Z_0}{n_1} \cos \theta_t}{\frac{Z_0}{n_2} \cos \theta_i + \frac{Z_0}{n_1} \cos \theta_t} = \frac{\frac{1}{\frac{\sin \theta_i}{\sin \theta_t} n_1} \cos \theta_i - \frac{1}{n_1} \cos \theta_t}{\frac{1}{\frac{\sin \theta_i}{\sin \theta_t} n_1} \cos \theta_i + \frac{1}{n_1} \cos \theta_t} = \frac{\frac{\sin \theta_t \cos \theta_i}{\sin \theta_i} - \frac{\sin \theta_i \cos \theta_t}{\sin \theta_i}}{\frac{\sin \theta_t \cos \theta_i}{\sin \theta_i} + \frac{\sin \theta_i \cos \theta_t}{\sin \theta_i}}$$

Since: $n_1 \sin \theta_i = n_2 \sin \theta_t \rightarrow n_2 = \frac{\sin \theta_i}{\sin \theta_t} n_1$. Next: $r_{\perp} = \frac{\sin \theta_t \cos \theta_i - \sin \theta_i \cos \theta_t}{\sin \theta_t \cos \theta_i + \sin \theta_i \cos \theta_t}$

Since: $\begin{cases} \sin(a + b) = \sin a \cos b + \cos a \sin b \\ \sin(a - b) = \sin a \cos b - \cos a \sin b \end{cases}$. Hence: $r_{\perp} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$

Similarly...

Example question 1

For non-magnetic materials, show that:

$$r_{\perp} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \quad \text{and} \quad t_{\perp} = \frac{2 \sin \theta_t \cos \theta_i}{\sin(\theta_t + \theta_i)}$$

We use again: $n_2 = \frac{\sin \theta_i}{\sin \theta_t} n_1$, and $\begin{cases} \sin(a + b) = \sin a \cos b + \cos a \sin b \\ \sin(a - b) = \sin a \cos b - \cos a \sin b \end{cases}$, to write:

$$t_{\perp} = \frac{2 \frac{Z_0}{n_2} \cos \theta_i}{\frac{Z_0}{n_2} \cos \theta_i + \frac{Z_0}{n_1} \cos \theta_t} = \frac{2 \frac{\cos \theta_i}{\frac{\sin \theta_i}{\sin \theta_t} n_1} n_1}{\frac{\cos \theta_i}{\frac{\sin \theta_i}{\sin \theta_t} n_1} + \frac{\cos \theta_t}{n_1}} = \frac{2 \frac{\cos \theta_i \sin \theta_t}{\sin \theta_i}}{\frac{\cos \theta_i \sin \theta_t}{\sin \theta_i} + \cos \theta_t}$$

We obtain: $t_{\perp} = \frac{2 \cos \theta_i \sin \theta_t}{\cos \theta_i \sin \theta_t + \cos \theta_t \sin \theta_i}$, which leads to: $t_{\perp} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_t + \theta_i)}$

Some angles are special!

The Brewster angle

So far, we have seen:

$$r_{\parallel} = \frac{Z_2 \cos \theta_t - Z_1 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}$$

$$t_{\parallel} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}$$

$$r_{\perp} = \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$$

$$t_{\perp} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$$

If $\cos \theta_i = 0$, $t_{\parallel} = t_{\perp} = 0$

True, if $\theta_i = \pm \frac{\pi}{2} \times (\text{integer})$

Such an angle can be approached but never reached, so there is always transmitted light.

For the reflection coefficients, in non-magnetic media, we have:

$$r_{\parallel} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i}. \quad \text{But if } \theta_t = \frac{\pi}{2} - \theta_i$$

we can use the general identity:

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

$$\text{So, } \sin 2\theta_t = \sin(\pi - 2\theta_i) =$$

$$= \underbrace{\sin \pi}_{=0} \cos 2\theta_i - \underbrace{\cos \pi}_{-1} \sin 2\theta_i = \sin 2\theta_i$$

So, in the numerator: $\sin 2\theta_t - \sin 2\theta_i = 0$

For $\theta_t = \frac{\pi}{2} - \theta_i$, we use $n_1 \sin \theta_i = n_1 \sin \theta_t$

And we obtain:

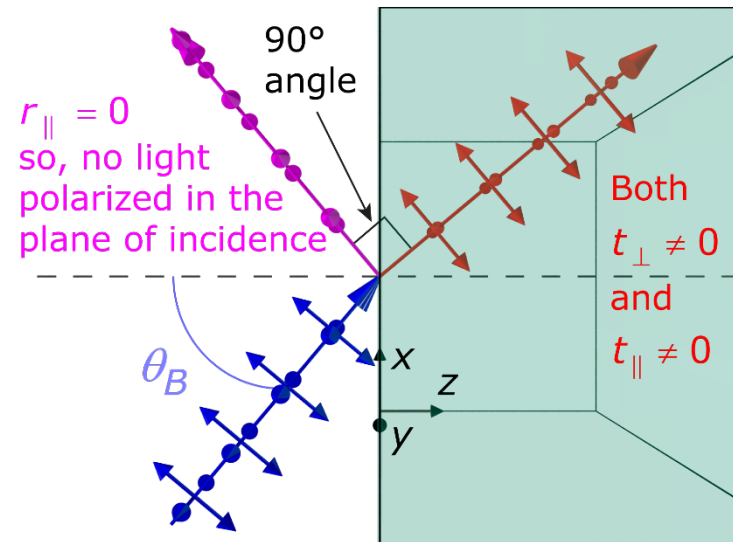
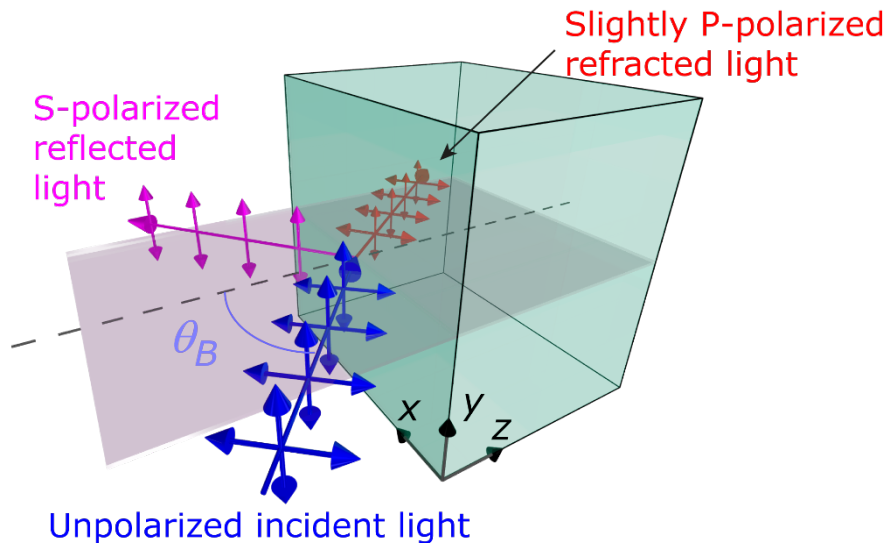
$$n_1 \sin \theta_i = n_2 \sin \left(\frac{\pi}{2} - \theta_i \right) = n_2 \cos \theta_i$$

We combine both sides:

The Brewster angle

We just obtained: $n_1 \sin \theta_i = n_2 \sin \left(\frac{\pi}{2} - \theta_i \right) = n_2 \cos \theta_i$, so $\frac{\sin \theta_i}{\cos \theta_i} = \tan \theta_i = \frac{n_2}{n_1}$

And by definition, the **Brewster angle** is: $\tan \theta_B = \frac{n_2}{n_1}$



This angle only exists for P-polarized light. You can check what happens for S-Polarized light [or see the lecture notes].

There is another important angle.

The critical angle

In the case where light propagates from a medium with higher refractive index into a medium with lower refractive index, consider Snell's law:

$$n_1 \sin \theta_i = n_2 \sin \theta_t$$

If we have $\frac{n_1}{n_2} \sin \theta_i > 1$, then $\sin \theta_t$ is undefined, since the sine function cannot be larger than 1. This is the same as saying that $\sin \theta_i > \frac{n_2}{n_1}$.

Therefore, such angle θ_t cannot exist and there can be no transmitted light.

We define the critical angle θ_c , such as:

$$\sin \theta_c = \frac{n_2}{n_1}$$

where any incidence angle such as $\theta_i > \theta_c$ cannot satisfy Snell's law.

For any such angle, there can be no transmission and this situation is referred to as **total internal reflection**.

Example?

Example question 2

[from Sadiku] A polarised wave is incident from air to polystyrene with $\mu = \mu_0$ and $\varepsilon = 2.6\varepsilon_0$ at Brewster angle. Determine the transmission angle.

Example question 2

[from Sadiku] A polarised wave is incident from air to polystyrene with $\mu = \mu_0$ and $\varepsilon = 2.6\varepsilon_0$ at Brewster angle. Determine the transmission angle.

We have $\tan \theta_B = \frac{n_2}{n_1} = \frac{\sqrt{\mu_2 \varepsilon_2}}{\sqrt{\mu_1 \varepsilon_1}}$, and since both media are non-magnetic,

$$\tan \theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\frac{2.6\varepsilon_0}{\varepsilon_0}} \approx 1.612$$

$$\text{Hence, } \theta_B = \tan^{-1}(1.612) \approx 58.19^\circ$$

Therefore, the incident angle is $\theta_i \approx 58.19^\circ$

Next, we can use Snell's law $n_1 \sin \theta_B = n_2 \sin \theta_t$, and we find

$$\sin \theta_t = \frac{n_1}{n_2} \sin \theta_B = \sqrt{\frac{1}{2.6}} \sin(58.19^\circ) \quad \text{so,} \quad \theta_t = \sin^{-1}(0.555) \approx \underline{\underline{31.8^\circ}}$$

To summarize!

Summary

At the boundary between two materials, EM wave is partially reflected and partially transmitted.

The plane of incidence is defined by the wave vector of the incident wave and a unit vector normal to the boundary.

On each side of the boundary, the electric and magnetic fields can be resolved into normal and tangential components.

For LIH materials, assuming no surface charges and no surface currents.

	Electric fields	Magnetic fields
Normal components	$\vec{D}_{1n} = \vec{D}_{2n}$	$\vec{B}_{1n} = \vec{B}_{2n}$
Tangential components	$\vec{E}_{1t} = \vec{E}_{2t}$	$\vec{H}_{1t} = \vec{H}_{2t}$

Summary

At normal incidence, the reflection and transmission coefficients are:

$$r_{\parallel/\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \quad t_{\parallel/\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2}{Z_1 + Z_2}$$

Snell's law results from conservation of moment at the interface:

For a general angle of incidence, we distinguish two cases of light polarisation:

- (i) in the plane of incidence (this is P-polarized light)
- (ii) perpendicular to the plane of incidence (this is S-polarized light)

The Fresnel coefficients for P- and S-polarized light are:

$$r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos \theta_t - Z_1 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i} \quad t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_t + Z_1 \cos \theta_i}$$

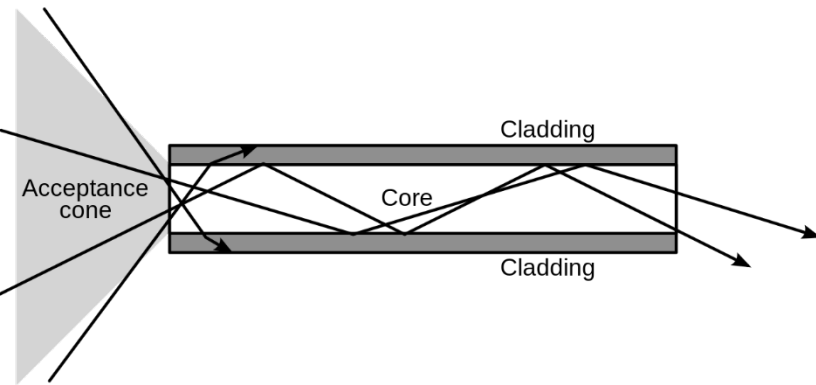
$$r_{\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos \theta_i - Z_1 \cos \theta_t}{Z_2 \cos \theta_i + Z_1 \cos \theta_t} \quad t_{\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos \theta_i}{Z_2 \cos \theta_i + Z_1 \cos \theta_t}$$

Summary

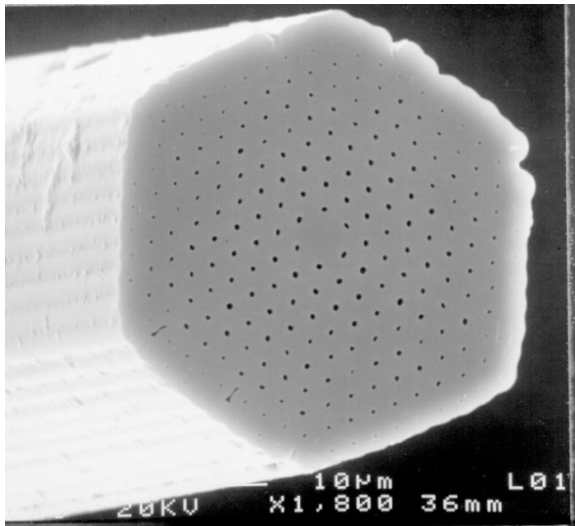
For P-polarized incident lights the Brewster angle ($\tan \theta_B = n_2/n_1$) determines the incidence angle for which the reflected beam is polarized perpendicular to the plane of incidence.

For light with any polarisation, when propagating from a material with higher refractive index into a material with lower refractive index, a critical angle exists ($\sin \theta_c = n_2/n_1$) beyond which the wave experiences total internal reflection.

Structured optical fibers



Conventional optical fibers use critical angles to guide the light (prevent it from escaping via transmission into another medium).



In 1996, scientists at the University of Bath reported a new type of optical waveguide: the **photonic crystal fiber** (PCF). It consisted of a pure silicon dioxide core, surrounded by air holes with a hexagonal symmetry. Its light confinement characteristics exceed those of conventional optical fiber. PCF is now used in **fiber-optic communications**, **fiber lasers**, **nonlinear devices**, **high-power transmission**, **highly sensitive gas sensors**, etc.