ANSWERS TO PROBLEM SHEET 3

1. In Cartesian coordinates $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, so the dot product $\mathbf{F} \cdot d\mathbf{r} = ydx + 2xdy$.

(a) Requires 2 integrals. For the first, from $(0,0,0)$ to $(0,1,0)$, parameterise using eg $x = 0$, $y = t$, $z = 0$. Then $dx = 0$, $dy = dt$ and the range of t is 0 to 1. So this integral is $\int_0^1 (t \times 0dt + 2 \times 0dt) = 0$. For the second integral, from $(0,1,0)$ to $(1,1,0)$, parameterise using eg $x = s$, $y = 1$, $z = 0$. Then $dx = ds$, $dy = 0$ and the range of s is 0 to 1. So this integral is $\int_0^1 ds = 1$. Adding, we get 1 as the line integral along this path from A to B.

(b) Along the straight line from $(0,0,0)$ to $(1,1,0)$, parameterise using eg $x = u$, $y = u$, $z = 0$. Then $dx = du$, $dy = du$ and the range of u is 0 to 1. So this integral is $\int_0^1 (u + 2u)du = 3/2$.

(c) Parameterise the arc using an angle; eg $x = \cos\theta$, $y = 1 + \sin\theta$, $z = 0$. Then $dx = -\sin\theta d\theta$, $dy = \cos\theta d\theta$ and the range of θ is $-\pi/2$ to 0 (sketch the arc if you are unsure about the ranges). Then this integral is

$$\int_{-\pi/2}^{0} [(1 + \sin\theta)(-\sin\theta) + 2\cos\theta\cos\theta]\, d\theta = \int_{-\pi/2}^{0} \left[2\cos^2\theta - \sin^2\theta - \sin\theta\right] d\theta = 1 + \frac{\pi}{4}.$$

[To evaluate these integrals, use $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$ and $2\cos^2 u = 1 + \cos 2u$.]

2. First take the dot product. $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, so $\mathbf{A} \cdot d\mathbf{r} = (2xy + 1)dx + (x^2 + 4y)dy$.

(a) eg $x = y = t$, so $dx = dy = dt$, t from 0 to 1. Then

$$I_{(a)} = \int_0^1 (2t^2 + 1 + t^2 + 4t)dt = \int_0^1 (3t^2 + 4t + 1)dt = \left[t^3 + 2t^2 + t\right]_0^1 = 4.$$

(b) eg $x = u$, $y = u^2$, so $dx = du$, $dy = 2udu$, u from 0 to 1. Then

$$I_{(b)} = \int_0^1 (2u^3 + 1)du + (u^2 + 4u^2)2udu = \int_0^1 (12u^3 + 1)du = \left[3u^4 + u\right]_0^1 = 4.$$

These answers are the same because $\mathbf{A}$ is a conservative field. To see this, check

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + 1 & x^2 + 4y & 0 \end{vmatrix} = \mathbf{0}.$$

We therefore know that there is a potential function $\phi(\mathbf{r})$ such that $\mathbf{A} = \nabla\phi(\mathbf{r})$. We have

$$\frac{\partial\phi}{\partial x} = A_x = 2xy + 1 \Rightarrow \phi = x^2y + x + C_1 + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = A_y = x^2 + 4y \Rightarrow \phi = x^2y + 2y^2 + C_2 + f_2(x,z)$$

$$\frac{\partial \phi}{\partial z} = A_z = 0 \Rightarrow \phi = C_3 + f_3(x, y).$$

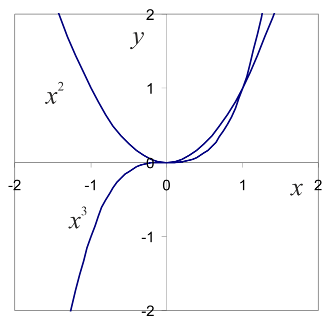
Combining these gives $\phi = x^2y+x+2y^2+C$. Therefore each of our integrals should be the potential difference $\Delta\phi$ between $(1,1,0)$ and $(0,0,0)$. $\phi(1,1,0) = 4+C$, $\phi(0,0,0) = C$, so $\Delta\phi = 4$ as we found.

3. (a)$\nabla \times \mathbf{F} = 0$ so $\mathbf{F}$ is conservative; $\phi(\mathbf{r}) = xy + C$.

(b) $\nabla \times \mathbf{F} = 2(x-y)\mathbf{k}$ so $\mathbf{F}$ is not conservative.

(c) $\nabla \times \mathbf{F} = 0$ so $\mathbf{F}$ is conservative; $\phi(\mathbf{r}) = xy\sin(z) + C$.

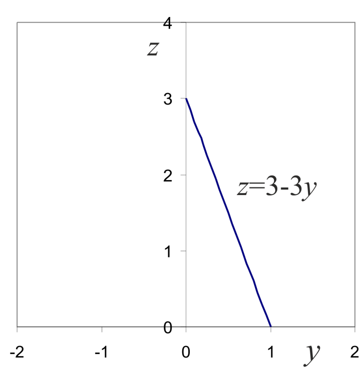
4. A quick sketch shows that the overlap between the functions happens between $x = 0$ and $x = 1$, with $x^3 \leq x^2$ over that range:



It is natural to take vertical strips, so our integration ranges are $x^3 \leq y \leq x^2$ and $0 \leq x \leq 1$. We must do the y-integral first, as it depends on x. Then

$$A = \int_{x=0}^{x=1} \left[\int_{y=x^3}^{y=x^2} dy \right] dx = \int_{x=0}^{x=1} (x^2 - x^3)dx = \left[\frac{1}{3}x^3 - \frac{1}{4}x^4 \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

5. Sketch the triangle in the yz plane.

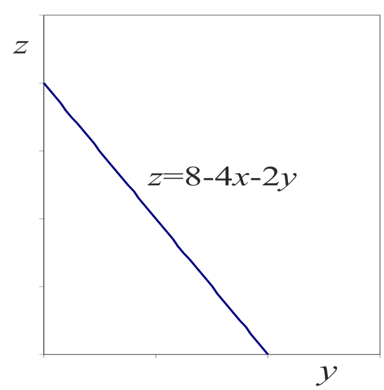


The normals to this plane are in the directions $\pm\mathbf{i}$, so these are the possible directions of the vector surface element $d\mathbf{S}$. As the triangle is an open surface, we have a free choice of plus or minus. Choosing + gives $d\mathbf{S} = \mathbf{i}dydz$. Then $\mathbf{A} \cdot d\mathbf{S} = 3ydydz$.

Taking vertical strips, our integration ranges are $0 \leq z \leq 3 - 3y$ and $0 \leq y \leq 1$. We must do the z-integral first, as it depends on y. Then

$$\int_S \mathbf{A} \cdot d\mathbf{S} = \int_{y=0}^{y=1} \left[\int_{z=0}^{z=3-3y} 3ydz\right] dy = 9\int_{y=0}^{y=1} (y - y^2)dy = 9\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 = 9\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{3}{2}.$$

6. A sketch of the volume should look a lot like the sketch in the example in the lecture notes where we found the volume of a pyramid. In this case, the only difference is that the plane forming the base of the pyramid is $4x + 2y + z = 8$, which intercepts the x-axis at $x = 2$, the y-axis at $y = 4$ and the z-axis at $z = 8$. As ϕ has no dependence on z it will probably pay to do the z integral first. The x^2 in ϕ is likely to lead to nastier terms than the y, so we will leave the x integral to last. [Remember that all this is a choice.] Taking my choices, I am taking vertical strips in a triangle at fixed x (somewhere between $x = 0$ and $x = 2$) in a triangle:



To sort out the integral limits, we need to know where the diagonal line meets the y-axis. The diagonal line is $z = 8 - 4x - 2y$, so when $z = 0$, $y = 4 - 2x$. Now we have our integration ranges: $0 \leq x \leq 2$, $0 \leq y \leq 4 - 2x$ and $0 \leq z \leq 8 - 4x - 2y$. We must do the z-integral first, as it depends on x and y. Then the y integral as it depends on x, then the x integral:

$$I = \int_V \phi dV = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2ydzdydx = 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y(8 - 4x - 2y)dydx.$$

Continuing with the y integral, we get

$$I = 45\int_{x=0}^{2} \frac{1}{3}x^2(4 - 2x)^3dx = 128.$$

To get the last bit, I just expanded out the cube: $(4-2x)^3 = 64-96x+48x^2-8x^3$, then multiplied by the x^2 and integrated.