Homogeneous: all terms involve dependent var &/or its derivs → **Linear** $L(a\psi + b\Phi) = aL(\psi) + bL(\Phi)$

$\frac{\partial \psi^2}{\partial x^2} - \frac{1}{k}\frac{\partial T}{\partial t} = 0$ ✓ $\quad R\frac{\partial Q}{\partial t} + \frac{Q}{C} = V$ ✗ $\quad \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$ | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ✓

$\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial t} - \sin x = 0$ ✗ $\quad i\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\psi}{\partial x^2} + U|\psi|^2\psi$ | $u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 - u^2 = 0$ ✗

can be

- commutative Σ $L+M = M+L$
- associative $(L+M)+N = L+(M+N)$ $\quad L(LM)N = L(MN)$
- distributive $L(c_1M + c_2N) = c_1LM + c_2LN$
- commutative (c const coeff) $LM = ML$

PDE linear if: dep var & derivs occur only first pwr & no products dependent on var & it's derivs

$L(u) = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$

$L(u) = G(x,y) = 0$ ∴ homog

$\neq 0$ ∴ inhom

PDEs

- elliptic $B^2 - 4AC < 0$ steady-state
- hyperbolic $B^2 - 4AC > 0$ vibrat°
- parabolic $B^2 - 4AC = 0$ transit°

- Poisson $\nabla^2\Phi = -S(\vec{r})$
- Laplace $\nabla^2\Phi = 0$
- Diffus° $\nabla^2\Phi - \frac{1}{k}\frac{\partial\Phi}{\partial t} = -S(\vec{r},t)$
- wave $\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -S(\vec{r},t)$
- Helmholtz $\nabla^2\Phi + \lambda\Phi = 0$
- Schröd $-\frac{\hbar^2}{2m}\nabla^2\Phi + V(\vec{r})\Phi = i\hbar\frac{\partial\Phi}{\partial t} = E\Phi$

Div th $\int_S \vec{A}\,ds = \int_V \nabla\cdot\vec{A}\,dV$

Diffus° chem species

Change tot amount species: $\Delta M = \delta t\int_V Q(\vec{r},t)\,dV$

flow vector: $\vec{J}$

Change due diffus°: $\Delta M_2 = \delta t\int_S -\vec{J}\cdot d\vec{s} = \delta t\int_V \nabla\cdot J\,dV$

tot change: $\Delta M = M_1 + M_2$

we know $\Delta M = \delta t\int_V \frac{\partial\rho}{\partial t}dV$

$\therefore \int_V \frac{\partial\rho}{\partial t}dV = \int_V Q(\vec{r},t)\,dV + \delta t\int_V \nabla^2 p\,dV$

$\frac{\partial\rho}{\partial t} = Q(\vec{r},t) + D\nabla^2 p$

$\nabla^2 p - \frac{1}{D}\frac{\partial p}{\partial t} = -\frac{Q(\vec{r},t)}{D}$

Temp distribut°

$\nabla^2 T - \frac{\rho c}{k}\frac{\partial T}{\partial t} = -\frac{P(\vec{r})}{k}$

steady state: $\frac{\partial\rho}{\partial t} = 0$, $\frac{\partial T}{\partial t} = 0$

$\therefore \nabla^2 p = -\frac{Q(\vec{r})}{D}$

$\nabla^2 T = -P(\vec{r})/k$

Poisson!

Sling ODEs

Homogeneous

$0 = \frac{d^2y}{dx^2} + P(x)\frac{dy}{d} + Q(x)y$

2 solut°s → $y(x) = c_1y_1(x) + c_2y_2(x)$

ex: $P(x) = 0$, $Q(x) = \omega^2$ $\quad \frac{d^2y}{dx^2} + \omega^2 y = 0$

$P(x) = \frac{2}{x}$, $Q(x) = 1 - \frac{n}{x^2}$ $\quad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$

Inhomog $F(x) = \cdots$

y_p = sol ∴ General sol → $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

Pwr series (needs more work)

$\frac{d^2y}{dx^2} + \omega^2 y = 0$

$\Rightarrow y(x) = \sum_{s=0}^{\infty} a_s x^{3+q}$

recurrence relat°: $a_{s+2} = \frac{-\omega^2}{(q+s+2)(q+s+1)}a_s$

Boundary condit°s

ex: wave eq^n: $\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c}\frac{\partial^2\phi}{\partial t^2} = 0$

$L(\phi) = 0$

$L = \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

General sol^n: $\phi(x,t) = \sum_k [A_k e^{ikx} + B_k e^{-ikx}]$ (ikx = kωt)

BC: $\phi(x,0) = \sin 3x$ & $\frac{\partial\phi(x,0)}{\partial t} = 0$

$\Rightarrow \phi(x,0) = \sum_k [-ikc\,A_k e^{ikx} + ikc\,B_k e^{-ikx}]$

$= \sum_k -ikc(A_k - B_k)e^{ikx} = 0$

$\phi(x,t) = \sum_k A_k e^{ikx}(e^{-ikct} + e^{ikct})$

BC: $\phi(x,0) = \sum_k 2A_k e^{ikx} = \sin 3x = \frac{e^{i3x}}{2i} - \frac{e^{-i3x}}{2i}$

$2A_3 = 1/2i$

$2A_{-3} = -1/2i$

other $A_k = 0$

$\phi(x,t) = \sum_k 2A_k e^{ikx}\cos(kct) = \sin(3x)\cos(3ct)$

Separat° of var

1D wave eq^n: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

$u(x,t) = X(x)T(t)$

$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 X}{dx^2}T$, $\frac{\partial^2 u}{\partial t^2} = X\frac{d^2T}{dt^2}$ $\Big\}$ $\frac{d^2X}{dx^2} - \frac{1}{c^2}X\frac{d^2T}{dt^2} = 0$

$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -k^2$

Possible solut°s:

$\frac{d^2X}{dx^2} = -k^2X \qquad \frac{d^2T}{dt^2} = -k^2c^2T$

General solut°s:

$\begin{cases} X(x) = a_k e^{ikx} + b_k e^{-ikx} \\ T(t) = c_k e^{ikct} + d_k e^{-ikct} \end{cases}$

- Divergent: $S_i \to \infty$ as $i \to \infty$
- Convergent: $S = \lim_{i\to\infty} S_i$
- Oscillatory: partial sums oscillate but never converge

Ts ts

- $\lim_{n\to\infty} u_n \neq 0$ ∴ div
- ratio test: $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2}\right) = 1$ $\begin{cases} <1 \to \text{conv} \\ >1 \to \text{div} \\ =1 \to \text{undet} \end{cases}$
- bigger series converges → so does smaller
  smaller series div → does bigger

1D diffus°

$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c}\frac{\partial p}{\partial t}$

$x=0 \qquad x=L$

$p(x,t) = X(x)T(t)$

$T\,d^2X$

G.S: $p(x,t) = X(x)T(t) = \sum_a (A_{an}\sin ax + B_{an}\cos ax)e^{-a^2ct}$

1D-diffus° eq^n: $\frac{\partial^2 p}{\partial x^2} = \frac{1}{c}\frac{\partial p}{\partial t}$

$\mathcal{F}[p(x,t)] = P(\omega,t) = \int_{-\infty}^{\infty} p(x,t)e^{-i\omega x}dx$

$\mathcal{F}\left[\frac{\partial^2 p(x,t)}{\partial x^2}\right] = -\omega^2 P(\omega,t)$

$\mathcal{F}\left[\frac{1}{c}\frac{\partial p}{\partial t}\right] = \frac{1}{c}\frac{\partial P(\omega,t)}{\partial t}$

$-\omega^2 P(\omega,t) = \frac{1}{c}\frac{\partial P(\omega,t)}{\partial t}$ ...

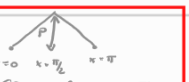
Fourier Transform

$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$

- Absolutely $\sum_n u_n$ & $\sum_n |u_n|$ both converge
- Conditionally $\sum_n u_n$ conv, $\sum_n |u_n|$ div

Displacement profile $t=0$



$u(x,0) = \sum_n A_n\sin(nx) = \begin{cases} P_x & \text{for } 0 \le x \le \pi/2 \\ P(\pi - x) & \text{for } \pi/2 \le x \le \pi \end{cases}$

$A_n = \frac{2}{\pi}\times\int_0^{\pi}\sin(nx)f(x)\,dx$

profile of spring $t=0$

symmetry $f(x)$: integral will be non-zero for odd values

$A_n = \frac{2}{\pi}\cdot 2\cdot\int_0^{\pi/2}\sin(nx)\left[\frac{2P_x}{\pi}\right]dx$

integral restricted to $0 \le x \le \pi/2$ $\quad$ n odd only

$I = \int_0^{\pi/2} x\sin(nx)\,dx \xrightarrow{\text{int by parts}} I = \frac{1}{n^2}[\sin(nx)]_0^{\pi/2}$

Now $[\sin(nx)]_0^{\pi/2} = 1$ for $n = 1,5,9,\ldots$

$[\sin(nx)]_0^{\pi/2} = -1$ for $n = 3,7,11,15,\ldots$

$[\sin(nx)]_0^{\pi/2} = 0$ for even values of n

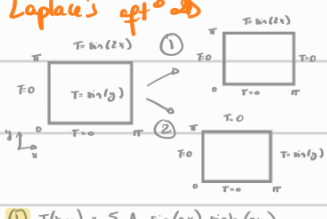
⇓

general express°: $[\sin(nx)]_0^{\pi/2} = (-1)^m$, $m = \frac{n-1}{2}$ (n always odd)

coeffs of n:

$\therefore I = \frac{(-1)^m}{n^2} \to A_n = \frac{2}{\pi}\cdot 2\,\frac{2P}{\pi}\cdot\frac{(-1)^m}{m^2} = \frac{8P}{\pi^2}\cdot\frac{(-1)^m}{(2m+1)^2}$

Laplace's eq^n ex



① $T(x,y) = \sum_n A_n\sin(nx)\sinh(ny)$

We know $T(x,\pi) = \sin(2x) = \sum A_n\sin(nx)\sinh(ny)$

$\therefore A_n = 0$ for $n \neq 2$, when $n = 2$: $1 = A_2\sinh(2\pi)$

$T(x,y) = \sum_n A_n\sin(nx)\sinh(ny)$ $\quad A_2 = 1/\sinh(2\pi)$

$= \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}$

② by analogy & using & noting $\sin(2x)$ as top boundary condit°

$T(x,y) = \frac{\sin(y)\sinh(x)}{\sinh(\pi)}$

full solut°: $T(x,y) = \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)} + \frac{\sin(y)\sinh(x)}{\sinh(\pi)}$

AND 3D vrs° after taking cauchy Riem

+ ex 3 pb 2

Point → ordinary

$P(x)$ & $Q(x)$ finite at $x = x_0$

singular P & Q div at $x = x_0$

- **regular** $(x-x_0)P$ & $(x-x_0)^2Q$ remain finite as $x \to x_0$
- **irregular** ... approach infinity ... expansion may fail

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

- sol^n: a indicial q_1, q_2
- $q_1 - q_2 \to$ int $\begin{cases} q_1 \to \text{sol}\ q_1 \\ q_2 \to \text{sol}\ q_2 \end{cases}$
- $q_1 - q_2 =$ int → larger q will give a sol, smaller may not
- $q_1 - q_2 \neq$ int → q_1, q_2 → 2 sols

— Chap 7: spherical harmonics