$-$ sign because the *outward* pointing normal is in the $-z$ direction for this face. The 2nd equation arises from a Taylor expansion of the 1st, and becomes exact in the limit $\delta z \to 0$.

Flux from the blue face at $z + \delta z / 2$ is

$$\begin{aligned} \text{Flux} &= \left[a_z(x, y, z + \frac{\delta z}{2}) \right] \times \delta x \delta y \\ &\approx \left[a_z(x, y, z) + \frac{\delta z}{2} \frac{\partial a_z}{\partial z}(x, y, z) \right] \times \delta x \delta y. \end{aligned} \tag{2}$$

The total flux from the two faces is (1) + (2):

$$\text{Net flux} = \frac{\partial a_z}{\partial z} \delta z \times \delta x \delta y = \frac{\partial a_z}{\partial z} \times \delta V. \tag{3}$$

Now do the same thing to the other two pairs of faces to give (for whole box):

$$\text{Net flux} = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right) \times \delta V = \nabla \cdot \mathbf{a} \times \delta V. \tag{4}$$

This shows that the divergence of a vector field $\mathbf{a}$ is very closely related to the local flux of the vector field.

For many important vector fields in Nature the net flux from a region of space must be *zero* unless there are *sources* (or *sinks*) of the field within the region. eg, for some fluids $\nabla\cdot\mathbf{v}=0$ away from sources or sinks of fluid. For electrostatic fields $\nabla\cdot\mathbf{E}=0$ unless there are charges at that point of space, which act as sources of electric field. For magnetic fields $\nabla\cdot\mathbf{B}=0$ *everywhere in space*, because we know of no isolated magnetic charge.

Such fields, with the property $\nabla\cdot\mathbf{a}=0$ are called *solenoidal* fields.

Example

Which of these fields could represent a physical magnetic field?

$$\mathbf{B}_1 = x^2y\mathbf{i} - xy^2\mathbf{j}$$
$$\mathbf{B}_2 = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$$

$\nabla\cdot\mathbf{B}_1 = 2xy - 2xy = 0$ everywhere in space. Therefore $\mathbf{B}_1$ could, in principle, be a magnetic field.

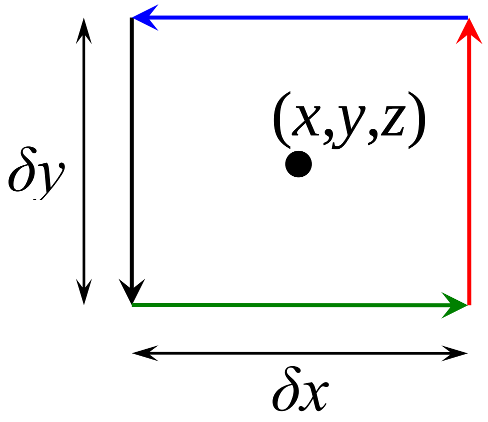
$\nabla\cdot\mathbf{B}_2 = y + z + x$. This is NOT zero everywhere, so $\mathbf{B}_2$ cannot be a magnetic field.

2.2.2 Interpretation of curl

Concentrate on the z-component of $\nabla \times \mathbf{a}$ given by

$$(\nabla \times \mathbf{a})_z = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}.$$

Other components will follow by analogy. We look at a small rectangular loop of area $\delta A = \delta x \times \delta y$ centred at position (x, y, z), as shown below.



A vector field $\mathbf{a}(\mathbf{r})$ is assumed to thread this loop. We define the *circulation* of $\mathbf{a}(\mathbf{r})$ around the loop as:

Circulation = Component of $\mathbf{a}(\mathbf{r})$ along loop x Distance along loop going *anti-clockwise* around loop

The total circulation is built up from the contributions from the four branches of the loop as follows:

$$\text{Circulation} = a_x(x, y - \frac{\delta y}{2}, z)\delta x + a_y(x + \frac{\delta x}{2}, y, z)\delta y$$
$$- a_x(x, y + \frac{\delta y}{2}, z)\delta x - a_y(x - \frac{\delta x}{2}, y, z)\delta y.$$

As we saw earlier in the interpretation of $\nabla \cdot \mathbf{a}$, these expressions can be Taylor-expanded to give

$$\text{Circulation} = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \delta x \delta y = (\nabla \times \mathbf{a})_z \delta A.$$

The circulation around a loop in the plane normal to z can therefore be seen to be proportional to the z component of $\nabla \times \mathbf{a}$, and similar results can be obtained for the x and y components.

For a vector field to have a non-zero circulation (and hence a non-zero curl) it must possess a local *rotational* element. One way to think about this for a fluid flow $\mathbf{v}(\mathbf{r})$, for example, is to ask whether a small paddle placed in the flow would rotate. If it does, $\mathbf{v}(\mathbf{r})$ has a non-zero curl; if not $\nabla \times \mathbf{v} = \mathbf{0}$. In this respect, the paddle wheel can be thought of as a "curl-meter".

Several important vector fields in Nature are said to be *irrotational*, which means that their curl is zero everywhere in space. This is true of some fluid flows, all electric fields in electrostatics, and some magnetic fields.

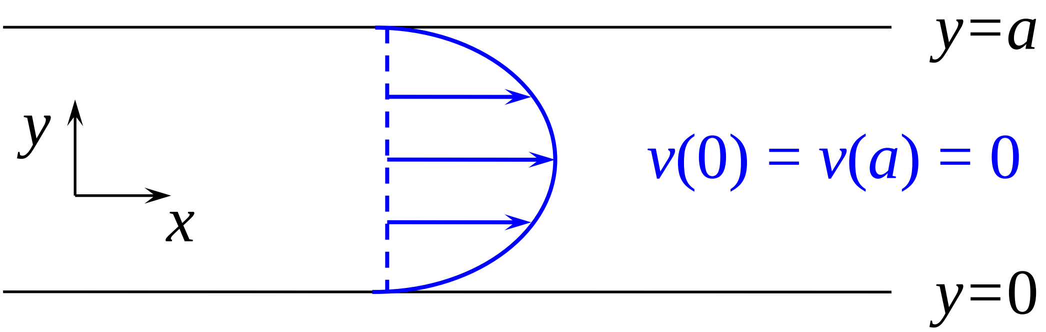
To summarise:

$\nabla \cdot \mathbf{a}$ measures local **flux** in a vector field $\mathbf{a}(\mathbf{r})$

$\nabla \times \mathbf{a}$ measures local **circulation** in $\mathbf{a}(\mathbf{r})$

Example

Fluid flowing along a narrow channel tends to adopt the following velocity profile:



The velocity of the fluid $\mathbf{v}(\mathbf{r})$ is a vector field, given by

$$\mathbf{v} = c[y(a-y)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}]$$

where c is a constant (>0). Calculate $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$ and interpret your results.

2.3 Triple Products involving ∇

grad (ie ∇) acts on a **scalar** field ϕ and produces a **vector** field $\nabla\phi$.

div (ie $\nabla\cdot$) acts on a **vector** field $\mathbf{a}$ and produces a **scalar** field $\nabla\cdot\mathbf{a}$.

curl (ie $\nabla\times$) acts on a **vector** field $\mathbf{a}$ and produces a **vector** field $\nabla\times\mathbf{a}$.

There are lots of ways in which grad, div and curl can be combined to form triple products.

Some of these are useful!

1. Consider $\mathrm{div}(\mathrm{grad}\phi) = \nabla\cdot\nabla\phi$

$$\mathrm{div}(\mathrm{grad}\phi) = \mathrm{div}\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad \equiv \quad \nabla^2\phi$$

∇^2 is a new operator called the **Laplacian operator** (or “del squared”)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

∇^2 is important; it turns up in many of the major (3d) equations of science, as we shall see later in the unit.

2. $\mathrm{curl}(\mathrm{grad}\,\phi) = \nabla \times \nabla \phi$

$$\nabla \times \nabla \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \mathbf{0} \quad \textbf{always}$$

This is a very handy "vector identity" – we will use it very soon. It tells us that the curl of the gradient of **any** scalar field is zero.

3. $\mathrm{div}(\mathrm{curl}\,\mathbf{a}) = \nabla \cdot (\nabla \times \mathbf{a})$

This is another handy vector identity – see if you can prove for yourself that

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \quad \textbf{always}$$

4. For lots more examples, see p18 of

“Basic Formulae and Statistical Tables”