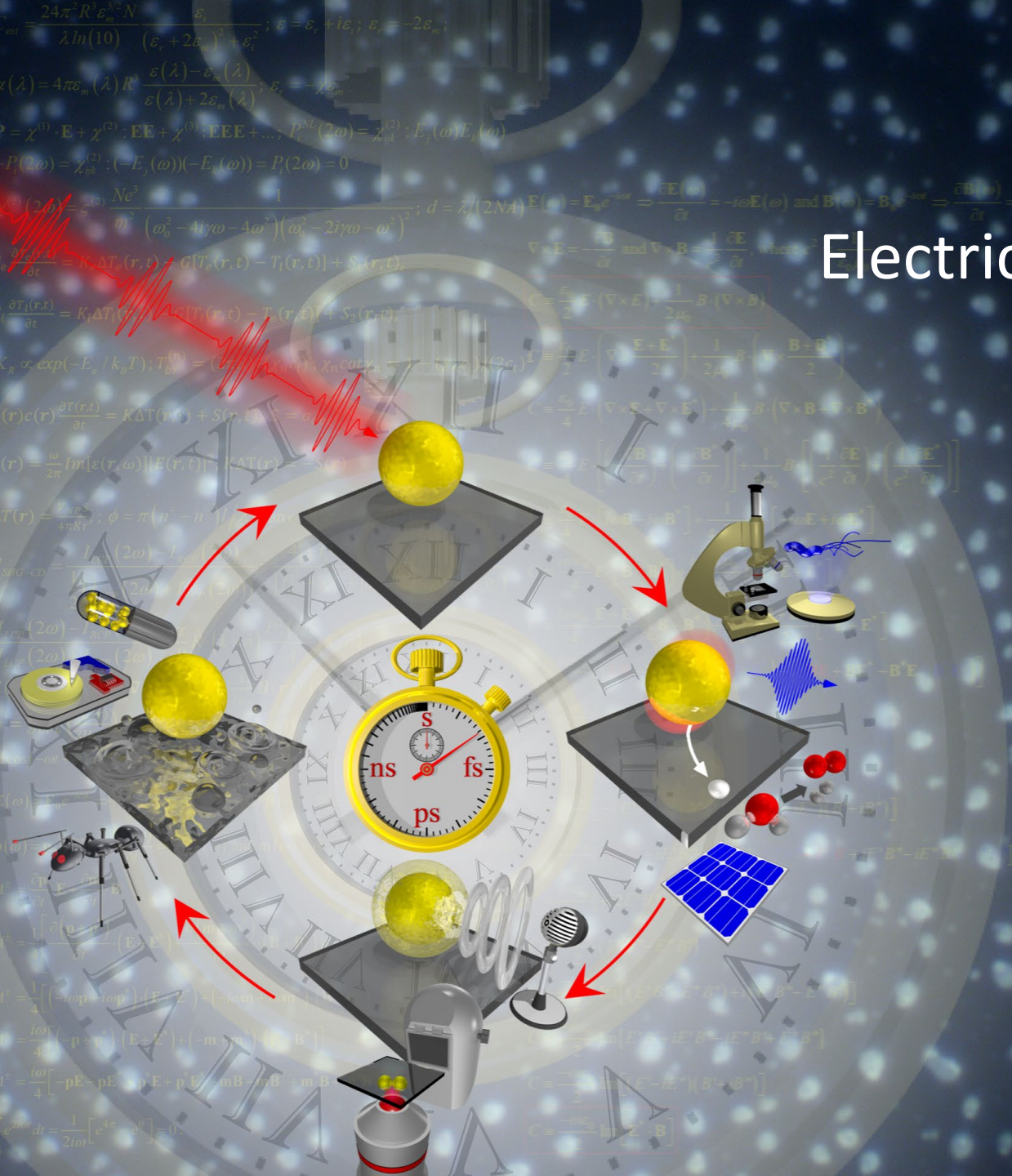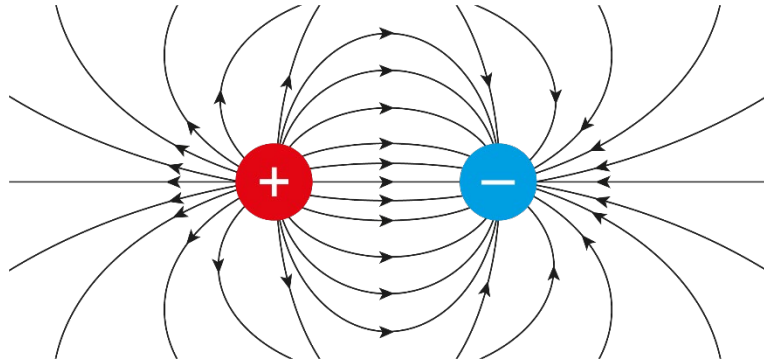# Lecture 11
Electric fields in materials

In this Lecture we will look at:

- ❑ Electric fields in materials

- ❑ Electric dipoles

- ❑ Relative permittivity and electric susceptibility

- ❑ Concepts of linearity, isotropy and homogeneity

- ❑ Electric polarisation and dipole moment
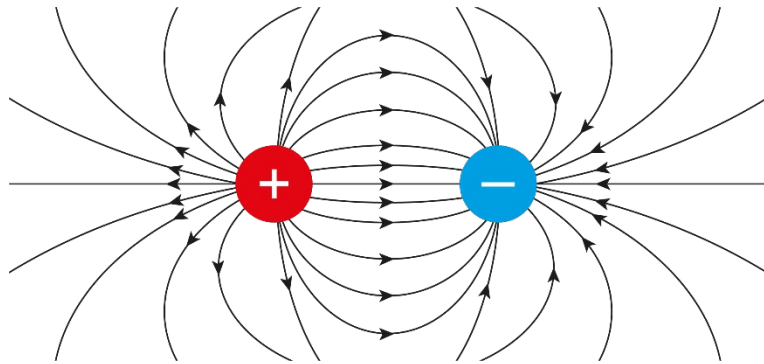
Electric dipole:

Electric field lines

- The previous part of this course dealt with electromagnetic fields in vacuum.

- Now, we consider what happens when materials are present.

- We begin with studying electric fields applied to **dielectric materials**.

- These materials are electric isolators – they typically lack free charges and do not conduct electricity.

- Instead, dielectrics have bound charges that can only slightly shift from their average equilibrium position, when subjected to an electric field.

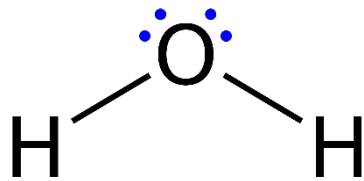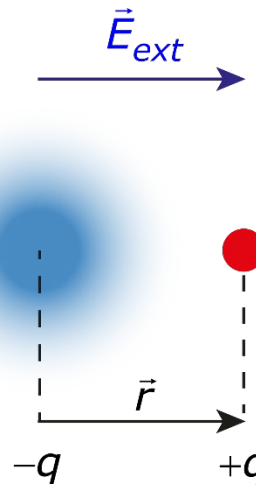Examples of such systems?

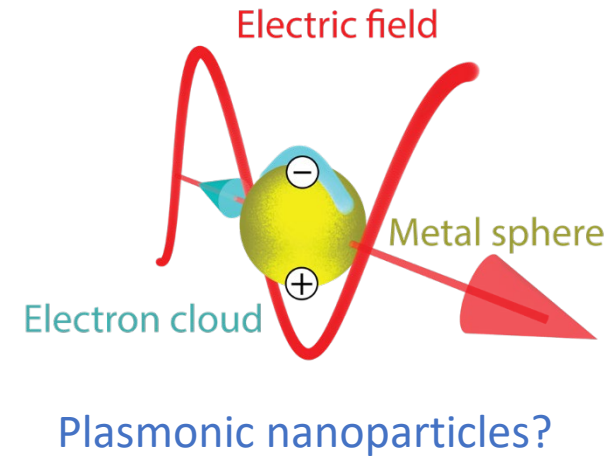UNIVERSITY OF BATH

Electric dipole:

Electric field lines

Water molecules

$\vec{E}_{ext}$

$\vec{r}$

$-q$ $+q$

Atoms

Electric field

Electron cloud

Metal sphere

Plasmonic nanoparticles?

Consider a larger picture!

The electric dipole moment is defined as: [units of Cm]

$$\vec{p} = q\vec{r}$$

Electric fields can induce dipole moments:

$$\vec{p} = \alpha \vec{E}_{ext}$$

Where $\vec{E}_{ext}$ is the externally applied electric field and α is a proportionality term.

+q feels two fields: $\vec{E}_{ext}$ and the local electric field ($\vec{E}_{local}$) created by $-q$.

$\vec{E}_{ext}$

$\vec{r}$

$-q$ $+q$

Charge separation in an atom due to an externally applied electric field.

What happens at equilibrium?

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The electric dipole moment is defined as:
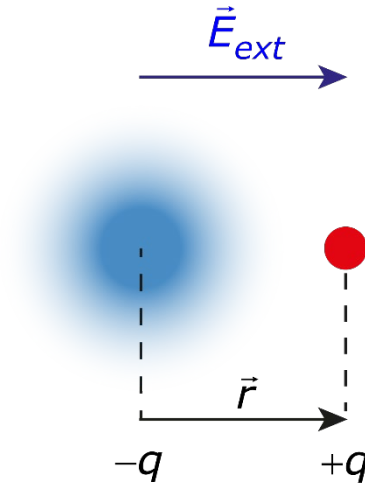
$$\boxed{\vec{p} = q\vec{r}}$$

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Where $\vec{E}_{ext}$ is the externally applied electric field and α is a proportionality term.

$+q$ feels two fields: $\vec{E}_{ext}$ and the local electric field ($\vec{E}_{local}$) created by $-q$.

The charges themselves could be protons and electrons of an atom stretched in an external electric field, the poles of a polar molecule, or two oppositely charged metal spheres.
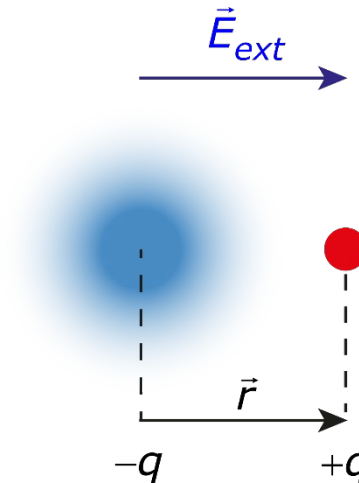
$\vec{E}_{ext}$

$\vec{r}$

$-q$ $+q$

Charge separation in an atom due to an externally applied electric field.

What happens at equilibrium?

At equilibrium, the effects of the two fields on +q are balanced:

$$\vec{E}_{local} + \vec{E}_{ext} = 0$$

$$\boxed{\vec{p} = q\vec{r}}$$

From Coulomb's law

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}r$$

But we need to adapt it here:

$$\vec{E}_{local} = \frac{1}{4\pi\varepsilon_0}\frac{-q}{r^3}\vec{r} = \frac{1}{4\pi\varepsilon_0}\frac{-qr}{r^3}\hat{z} = \frac{-p}{4\pi\varepsilon_0 r^3}\hat{z} = \frac{-\vec{p}}{4\pi\varepsilon_0 r^3}$$
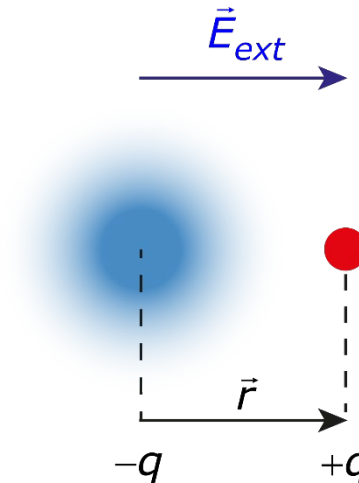
And

$$\vec{E}_{ext} = E_{ext}\hat{z} = -\vec{E}_{local} = \frac{p}{4\pi\varepsilon_0 r^3}\hat{z} = \frac{\vec{p}}{4\pi\varepsilon_0 r^3}$$

And

$$\vec{p} = 4\pi\varepsilon_0 r^3 \vec{E}_{ext}$$

$$\vec{E}_{ext}$$

$\vec{r}$

$-q$ $\qquad$ $+q$

We need to tidy up

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Electric dipoles

At equilibrium, the effects of the two fields on +q are balanced:

$$\vec{E}_{local} + \vec{E}_{ext} = 0$$

From Coulomb's law

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^3}r$$

But we need to adapt it here.

$$\vec{E}_{local} = \frac{-\vec{p}}{4\pi\varepsilon_0 r^3}$$

And

$$\vec{E}_{ext} = E_{ext}\hat{z} = -\vec{E}_{local} = \frac{p}{4\pi\varepsilon_0 r^3}\hat{z}$$

And

$$\vec{p} = 4\pi\varepsilon_0 r^3 \vec{E}_{ext}$$

Since we started with:

$$\vec{p} = \alpha\vec{E}_{ext}$$

We can write:

$$\alpha = 4\pi\varepsilon_0 r^3 = 3\varepsilon_0\left(\frac{4}{3}\pi r^3\right) = 3\varepsilon_0 V$$

where V is the volume of a sphere with radius r.

Therefore:

$$\vec{p} = 3\varepsilon_0 V\vec{E}_{ext}$$

We now have an expression for the induced dipole moments: it depends on the susceptibility of vacuum.
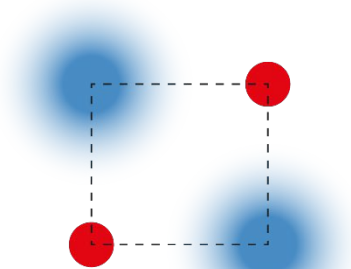
How about… multipoles?

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Take a monopole, displace it through a small distance *r* and replace the original monopole by another with same magnitude but opposite sign. We obtain a **dipole**.

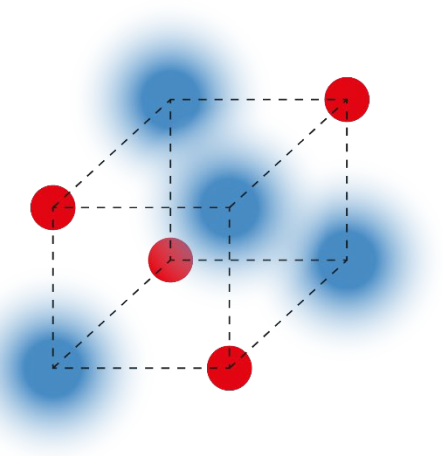Take a dipole, displace it through a small distance *r* and replace the original dipole by another with same magnitude but opposite sign. We obtain a **quadrupole**.
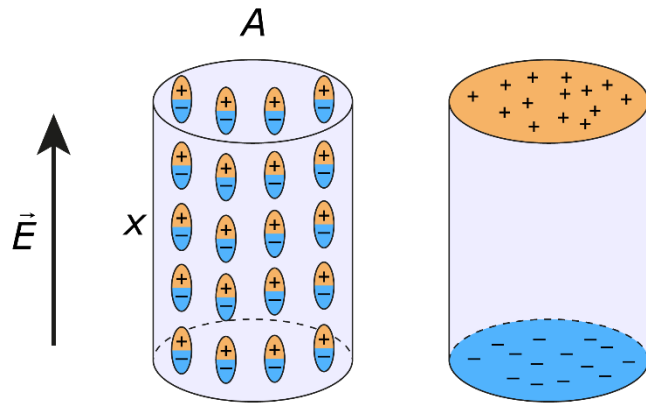
Take a quadrupole, displace it through a small distance *r* and replace the original quadrupole by another with same magnitude but opposite sign. We obtain an **octupole**.

Let's look at many dipoles!

We consider a dielectric cylinder :

A dielectric cylinder of surface A and height x, upon applying an external electric field. Positive and negative charges Q are formed at the surfaces.

Within the material, the + and − charges cancel. However, at the surfaces, there is a net charge. Overall, the material is now polarized, with **polarization vector** $\vec{P}$.
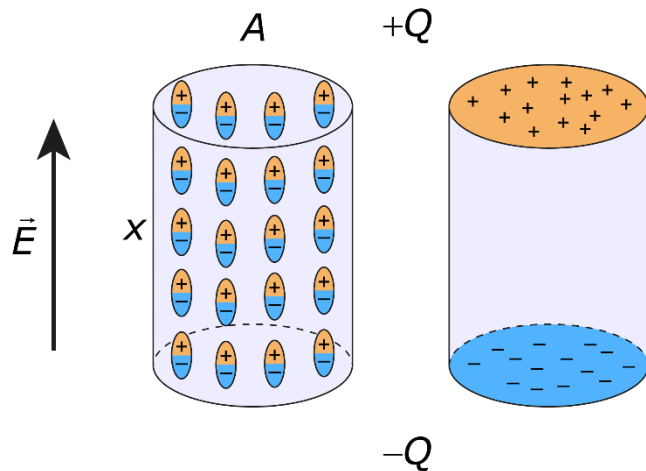
This polarization must depend on the externally applied electric field and on the properties of the dielectric itself; we write this dependence as

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

where χ_e is the **electric susceptibility** of the material; it tells us how polarizable a material is.

What is the resulting charge at the surfaces?

The resulting ==positive== charge is:

$$+Q = +\sigma_p \times (\text{unit area})$$

Where σ_p is the **surface charge density**. Similarly:

$$-Q = -\sigma_p \times (\text{unit area})$$

We can calculate the magnitude of the dipole moment of the whole block in two ways:

$$(\text{dipole moment}) = (\text{polarized charge}) \times (\text{separation}) = \left(\sigma_p A\right) x$$

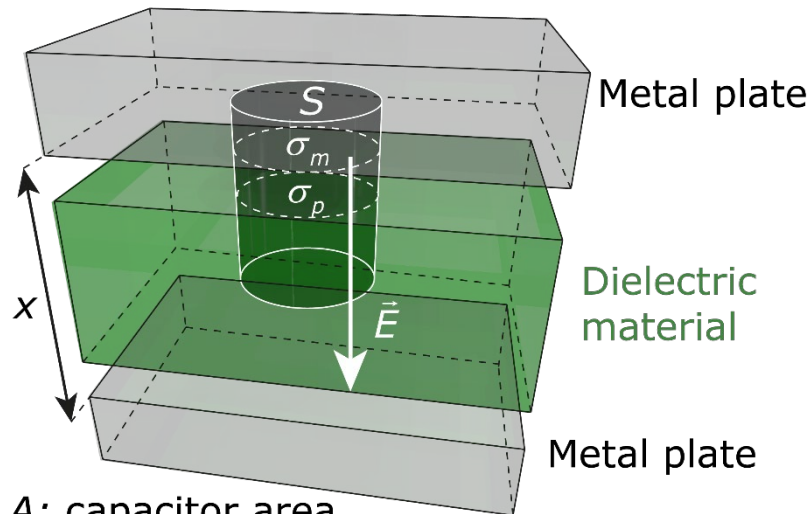$$\boxed{\vec{p} = q\vec{r}}$$

and

$$(\text{dipole moment}) = (\text{dipole moment/unit volume}) \times (\text{volume}) = P(Ax)$$

They must be the same, so: $\boxed{\sigma_p = P}$ and they have the same units! $[\text{Cm}^{-2}]$.

The figure looks like a capacitor! So what if the dielectric is part of a capacitor?

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We consider a plate capacitor:

Metal plate

Dielectric material

x

Metal plate

A: capacitor area

A dielectric materials between two metal plates.

Here we have [dielectric + conductor]:

A: capacitor area

x: thickness

S: area of Gaussian cylinder

ρ : the charge density enclosed in an arbitrary volume V.

$\sigma_m = Q/A$: stored **charge density on the metal surface**.

σ_p : **polarization charge density** (it is opposite to σ_m)
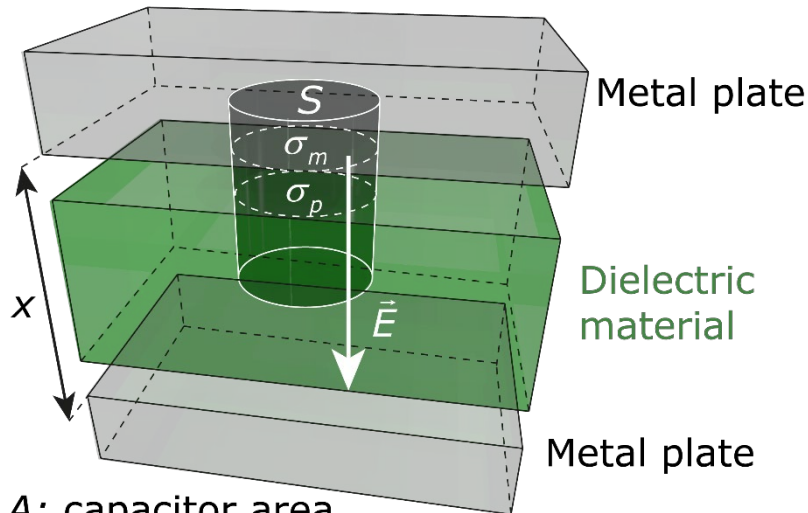
We can apply Gauss' theorem:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dv$$

Gauss' law states that the flux of the electric field out of an arbitrary closed surface is proportional to the electric charge enclosed by the surface, irrespective of how that charge is distributed.   So, what is the electric flux?

We consider a plate capacitor:

Metal plate

Dielectric material

x

$\vec{E}$

Metal plate

A: capacitor area

A dielectric materials between two metal plates.

The *E*-field leaves the cylinder on three sides: side, top and bottom.

Therefore (side+top+bottom):

$$\Phi_e = 0 + 0 + \vec{E} \cdot \vec{S} = \vec{E} \cdot \vec{S}$$

The charge enclosed is (metal and dielectric):

$$\sigma_m S + \left(-\sigma_p\right) S = \left(\sigma_m - \sigma_p\right) S$$

Therefore, by Gauss' law:

$$ES = \frac{\left(\sigma_m - \sigma_p\right) S}{\varepsilon_0} \quad \text{and} \quad E = \frac{\left(\sigma_m - \sigma_p\right)}{\varepsilon_0}$$
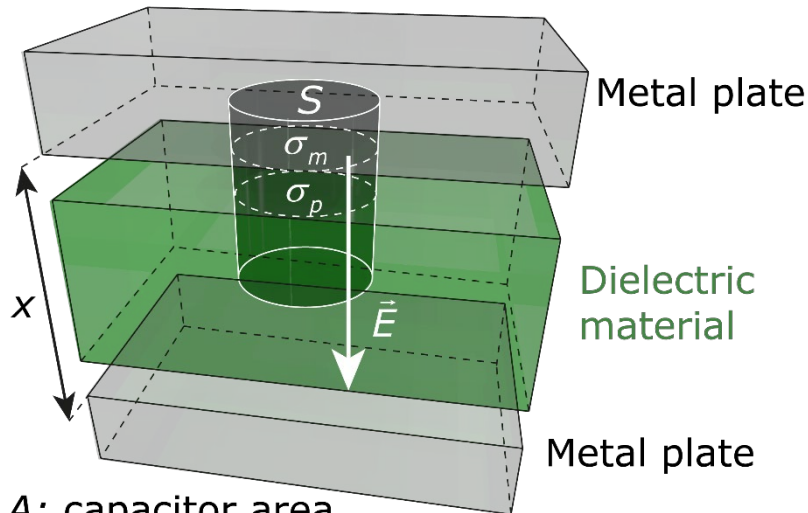
We can re-arrange using the magnitudes of the polarization vector $\vec{P} = \varepsilon_0 \chi_e \vec{E}$

$$\varepsilon_0 E = \sigma_m - \sigma_p = \sigma_m - P = \sigma_m - \varepsilon_0 \chi_e E \quad \text{so that} \quad E = \frac{\sigma_m}{\varepsilon_0 \left(1 + \chi_e\right)}$$

Can we find another expression for the *E*-field?

We consider a plate capacitor:

Metal plate

Dielectric material

Metal plate

A: capacitor area

A dielectric materials between two metal plates.

We just found: $E = \dfrac{\sigma_m}{\varepsilon_0 \left(1 + \chi_e\right)}$

From A-level physics: $E = \dfrac{V}{x}$, so:

$$\frac{V}{x} = \frac{\sigma_m}{\varepsilon_0 \left(1 + \chi_e\right)} = \frac{Q/A}{\varepsilon_0 \left(1 + \chi_e\right)}$$

Rearranging: $\dfrac{Q}{V} = \dfrac{\varepsilon_0 \left(1 + \chi_e\right) A}{x}$

From A-Level physics: $C = \dfrac{Q}{V}$; $C = \dfrac{A \varepsilon_0 \varepsilon_r}{x}$

It follows that: $\dfrac{A \varepsilon_0 \varepsilon_r}{x} = \dfrac{\varepsilon_0 \left(1 + \chi_e\right) A}{x}$

Where we see that $\boxed{\varepsilon_r = 1 + \chi_e}$. This is the **relative permittivity of the material**.

Consider an example question!

A parallel plate capacitor has a capacitance of 112 pF, a plate area of 96.5 cm², and a mica dielectric ($\varepsilon_r = 5.4$). At a 55 V potential difference, calculate:
(a) the electric field strength in the mica;
(b) the magnitude of the free charge on the plates;
(c) the magnitude of the induced surface charge;
(d) the magnitude of the polarisation.

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(a) the electric field strength in the mica;
(b) the magnitude of the free charge on the plates;
(c) the magnitude of the induced surface charge;
(d) the magnitude of the polarisation.

(a) From $C = \dfrac{A\varepsilon_0\varepsilon_r}{x}$ and $E = \dfrac{V}{x}$ we get $E = \dfrac{VC}{A\varepsilon_0\varepsilon_r} \approx 13.4$ kV

(b) From $\sigma_m = E\varepsilon_0\varepsilon_r$ we get: $q_m = E\varepsilon_0\varepsilon_r A \approx 6.18$ nC

(c) From $\sigma_p = \sigma_m - E\varepsilon_0$ we get: $q_p = A\sigma_p = A\left(E\varepsilon_0\varepsilon_r - E\varepsilon_0\right) = AE\varepsilon_0\left(\varepsilon_r - 1\right) \approx 5.03$ nC

(d) From $\sigma_p = P \approx 522$ nC/m^2

[full solution in the lecture notes]

What kind of materials are we interested in?

We will only discuss linear, isotropic and homogeneous (LIH) dielectric materials.

A dielectric material is **linear** if the electric permittivity and the electric susceptibility **do not change with the magnitude of the applied electric field**.

Isotropic: if the electric permittivity and the electric susceptibility **do not change with the direction of the electric field** in the material. In other words, all directions in the material are equivalent.

Homogeneous: if the electric permittivity and the electric susceptibility **do not change from point to point** in the material. In other words, the permittivity is independent of the coordinates. We can say that all locations in the material are equivalent.

Note: Not isotropic materials are called **anisotropic**. In anisotropic materials $\vec{P}$ and $\vec{E}$ are not parallel and are not simply related. For instance, this relationship may be nonlinear and could also depend on the past history of the sample, i.e. there can be a hysteresis.

Can we link polarisation and electric dipole moments?

An dielectric is a collection of positive and negative charges.

Under the influence of the electric field, the negative and positive charges will split. In turn, this is going to influence the electric field and that influence will show up in Maxwell's equations.

We have seen that: $\vec{p} = q_b \vec{r}$

The two charges making up the individual dipole are **bound charges**. This is often indicated with a small b written in subscript. The units are Coulomb meter (Cm).

The **electric polarisation** of a region is defined as the dipole moment per unit volume (V).

$$\vec{P} = \frac{\sum \vec{p}}{V}$$

Because the units of the dipole moment are Cm, the units of polarisation are Cm^{-2}, which is the same unit as a surface charge density, as we saw before.

If all the atoms have the same dipole moment, or if p designates the average dipole moment:

$$\vec{P} = \frac{N\vec{p}}{V} = n\vec{p}$$

where N is the number of molecules and n is the molecular density.

Consider an example question!

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A sample of diamond has a density of atoms of 3.5×10^{29} atoms per cubic meter. Diamond is a non-magnetic material and has a refractive index of 2.4. Carbon has a nucleus with a charge of +6e, surrounded by an electron cloud with charge 6e.

If the diamond is given a polarization of 10^{-7} Cm^{-2}, what is the average separation between the centres of the positive and negative charges?

A sample of diamond has a density of atoms of 3.5×10^{29} atoms per cubic meter. Diamond is a non-magnetic material and has a refractive index of 2.4. Carbon has a nucleus with a charge of +6e, surrounded by an electron cloud with charge 6e.

If the diamond is given a polarization of 10^{-7} Cm^{-2}, what is the average separation between the centres of the positive and negative charges?

The magnitude of the polarization *P* is: $P = nq_b r$

where *n* is the number of atoms per unit volume, $\pm q_b$ is the bound charge of the dipole and *r* is the charge separation of the dipole.

For a carbon atom, $q_b = 6e$. Therefore we find:

$$r = \frac{P}{nq_b} = \frac{P}{6ne} = \frac{10^{-7}}{6\left(3.5 \times 10^{29}\right)\left(1.6 \times 10^{-19}\right)} \approx 3.0 \times 10^{-19} \text{m},$$

which is a very small distance indeed.

Another way to link surface charge and polarisation?

In a dielectric, microscopic dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.
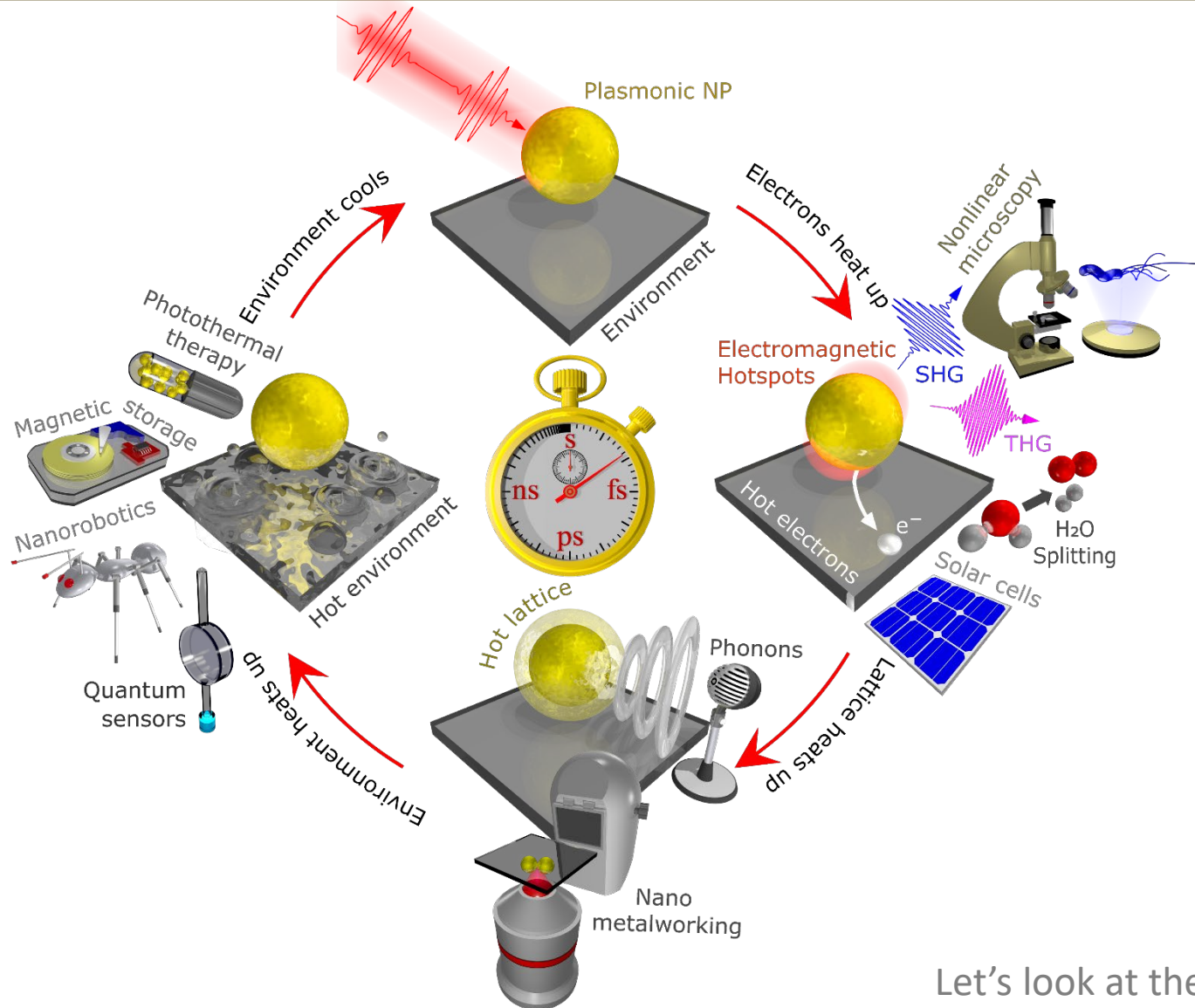
Macroscopically, the combined effect of these dipole moments produces a **polarisation**, defined as the induced dipole moment per unit volume: $\vec{P} = \dfrac{\sum \vec{p}}{V}$

In **LIH dielectrics**, the polarisation originates from the surface charge density $\sigma_p = P$.

In LIH dielectrics: $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ with χ_e being the **electric susceptibility** of the material.

The **relative permeability** of the material is given by: $\varepsilon_r = 1 + \chi_e$

Let's look at the dipoles!