PH20014: Electromagnetism 1

Problem Sheet 2

Standard problems

Question 1.

The electric field $\vec{E}$ associated with an electromagnetic plane wave propagating in a lossy medium is of the form

$$\vec{E} = 50\hat{y}e^{-\alpha x}e^{i(\beta x - \omega t)}\ \mathrm{Vm^{-1}},$$

in our usual notation. If $\varepsilon_r = 3.0$, $\mu_r = 7.0$, $\sigma = 6.0$ ohms^{-1}m^{-1} at the frequency 6.0 MHz, show that the medium can be considered as a good conductor and hence find α and β. Then, calculate

(a) the phase velocity,
(b) the wavelength,
(c) the amplitude of the electric field $\vec{E}$ at x = 0.1 m,
(d) the amplitude of the $\vec{H}$ field at x = 0.1 m.

Compare the results of (a) and (b) with those that would be obtained if the conductivity were zero.

Answer: The loss tangent $\tan\theta = \frac{\sigma}{\omega\varepsilon}$ provides a qualitative statement on how good a conductor the material is. In this case, $\frac{\sigma}{\omega\varepsilon} = \frac{\sigma}{2\pi f\,\varepsilon_0\varepsilon_r} \approx 6000 \gg 1$, where f is the actual frequency given in the problem (all the units cancel out and the ratio is dimensionless; notice that $1\mathrm{S}=1\Omega^{-1}=\frac{\mathrm{A}}{\mathrm{V}}=\frac{\mathrm{C}}{\mathrm{Vs}}$ and $1\mathrm{F}=\frac{\mathrm{C}}{\mathrm{V}}$), and hence the material can be treated as a good conductor. Using the formulae applicable in the 'good conductor' regime (see lectures: $\alpha \approx \beta \approx \sqrt{\frac{\mu\sigma\omega}{2}}$), we get $\alpha = \beta = \sqrt{\frac{\mu_0\mu_r\sigma 2\pi f}{2}} \approx \underline{\underline{31.5}}$ m^{-1}.

(a) From the lectures: $v_p = \frac{\omega}{\text{wave vector}}$ and $v_p = \frac{\omega}{\beta} = \frac{\sqrt{\omega^2}}{\sqrt{\frac{\mu\sigma\omega}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}} = \omega\delta$. So, here we can write: $v_p = \sqrt{\frac{2(2\pi f)}{\mu_0\mu_r\sigma}} \approx \underline{\underline{1.2\times10^6}}$ m/s.

(b) For the wavelength, we have $\lambda = \frac{2\pi}{\text{wave vector}}$, so here $\lambda = \frac{2\pi}{\beta} \approx \underline{\underline{0.2}}$ m.

(c) For the amplitude of $\vec{E}$, we just need to replace: $\vec{E} = 50\hat{y}e^{-\alpha x}e^{i(\beta x-\omega t)} = \vec{E}_0 e^{i(\beta x-\omega t)}$, so $\vec{E}_0 = 50\hat{y}e^{-\alpha x}$. We have found that $\alpha \approx 31.5$, so $\left|\vec{E}_0\right| = 50e^{-(31.5)\cdot(0.1)} \approx \underline{\underline{2.14}}$ V/m.

(d) From the lectures, we have $Z \approx \sqrt{\frac{\mu\omega}{\sigma}}e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{\sigma\delta}e^{-i\frac{\pi}{4}}$ and, of course, $Z = \frac{E_0}{H_0}$, so

$\frac{E_0}{H_0} = \sqrt{\frac{\mu\omega}{\sigma}}e^{-i\frac{\pi}{4}}$ and $H_0 = \sqrt{\frac{\sigma}{\mu\omega}}e^{i\frac{\pi}{4}}E_0 = \sqrt{\frac{\sigma}{\mu_0\mu_r 2\pi f}}e^{i\frac{\pi}{4}}E_0$. We now replace:

$$H_0(x=0.1) = \sqrt{\frac{(6.0)}{\left(4\pi\times10^{-7}\right)(7.0)2\pi\left(6.0\times10^6\right)}}(2.14) \approx \underline{\underline{0.29}}\ \text{A/m}.$$

If conductivity $\sigma = 0$, the material becomes a perfect insulator (dielectric). Then, the phase velocity $v_p = \frac{c}{n} = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{299792458}{\sqrt{3 \cdot 7}} \approx 65.4 \times 10^6$ m/s, over 50 times higher than for the case $\sigma \neq 0$. As a consequence, the wavelength would also be considerably different, $\lambda = \frac{v_p}{f} \approx \underline{\underline{11}}$ m.

Question 2.

Deduce the conductivity of a medium of relative permittivity equal to five, if the magnitude of the conduction and displacement current densities in it are equal when a monochromatic plane wave of frequency 10^8 Hz is propagated. What is the attenuation per metre of the amplitude of a plane wave of frequency 10^6 Hz when propagated in such a medium? The relative permeability is unity.

Answer: We have $\varepsilon_r = 5$, $\mu_r = 1$ and $f = 10^8$ Hz, so $\omega = \frac{2\pi}{T} = 2\pi f = 2\pi \cdot 10^8$. From the lectures we remember:

1. A 'poor' conductor is one where $\frac{\sigma}{\varepsilon\omega} << 1$ and the Ohmic [or conduction] current is much smaller than the displacement current.
2. A 'good' conductor is one where $\frac{\sigma}{\varepsilon\omega} >> 1$ and the Ohmic [or conduction] current is much larger than the displacement current.

So, when the conduction and displacement current densities are equal: $\frac{\sigma}{\varepsilon\omega} = 1$.

Hence, $\sigma = \varepsilon\omega = \varepsilon_0 \varepsilon_r \omega = \left(8.85 \times 10^{-12}\right) 5 \cdot 2\pi \cdot 10^8 \approx \underline{\underline{27.8 \times 10^{-3}}}$ S/m. For a lower frequency $f_1 = 10^6$ Hz, assuming that the conductivity and ε_r have not changed, the loss tangent will increase (as we divide by a smaller number), so that we have $\frac{\sigma}{\varepsilon\omega_1} = \frac{\sigma}{\varepsilon 2\pi f_1} \gg 1$. Hence, we use the expression for α for the 'good conductor' limit [from the lectures $\alpha \approx \beta \approx \sqrt{\frac{\mu\sigma\omega}{2}}$] and we obtain

$$\alpha = \sqrt{\frac{\mu\sigma\omega_1}{2}} = \sqrt{\frac{\mu_0 \mu_r \sigma 2\pi f_1}{2}} = \sqrt{\frac{\left(4\pi \times 10^{-7}\right)(1)\left(27.8 \times 10^{-3}\right) 2\pi \left(10^6\right)}{2}} \approx \underline{\underline{0.33}} \text{ m}^{-1}.$$

Question 3.

(a) Show that the time-average of the energy density in a monochromatic (linearly-polarized) plane wave moving in an isotropic **non-conducting medium** is distributed equally between the electric and magnetic fields.
(b) In comparison, show that in a **conducting medium**, the time-average of energy density in the magnetic field is greater than in the electric field.
[Hint: When comparing the energy densities in the magnetic and electric fields, use the internal impedance of the medium Z which conveniently expresses the ratio of the magnitudes of the two fields.]

Answer:

(a) For a non-conducting medium, we have the following ratio of energies carried by the electric and magnetic fields:

$$\frac{\langle U_{electric} \rangle}{\langle U_{magnetic} \rangle},$$

where the angular brackets denote time-averaging. Notice that, strictly speaking, we have used energy densities and not energies but this does not make a difference as both energy densities are integrated over the same volume. Using the expressions for energy densities we obtained when studying static fields, we have:

$$\frac{\langle U_{electric}\rangle}{\langle U_{magnetic}\rangle} = \frac{\left\langle \frac{1}{2}\vec{D}\cdot\vec{E}\right\rangle}{\left\langle \frac{1}{2}\vec{H}\cdot\vec{B}\right\rangle} = \frac{\varepsilon\left\langle \vec{E}^2\right\rangle}{\mu\left\langle \vec{H}^2\right\rangle}.$$

Formally, the time-average for a field $\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$ is given by $\left\langle \vec{E}^2\right\rangle = \frac{1}{2}\left[\vec{E}\cdot\vec{E}^*\right]$, where $\Re$ means we should only take the real part of the expression in square brackets. In our case, this gives $\left\langle \vec{E}^2\right\rangle = \frac{1}{2}\left|\vec{E}\right|^2$ and using the impedance Z of the medium:

$$\frac{\langle U_{electric}\rangle}{\langle U_{magnetic}\rangle} = \frac{\left\langle \frac{1}{2}\vec{D}\cdot\vec{E}\right\rangle}{\left\langle \frac{1}{2}\vec{H}\cdot\vec{B}\right\rangle} = \frac{\varepsilon\frac{1}{2}\left|\vec{E}\right|^2}{\mu\frac{1}{2}\left|\vec{H}\right|^2} = \frac{\varepsilon}{\mu}\left|Z\right|^2 = \frac{\varepsilon}{\mu}\frac{\mu}{\varepsilon} = \underline{\underline{1}}$$

where we used the fact that for a non-conducting medium $Z = \sqrt{\frac{\mu}{\varepsilon}}$.

(b) From above, we already know that $\frac{\langle U_{electric}\rangle}{\langle U_{magnetic}\rangle} = \frac{\varepsilon}{\mu}\left|Z\right|^2$. In a conducting medium, we have in general complex impedance. From the lectures:

$$Z = \sqrt{\frac{\mu^2\omega^2}{\mu\varepsilon\omega^2\left(1 + i\frac{\mu\sigma\omega}{\mu\varepsilon\omega^2}\right)}} = \sqrt{\frac{\mu}{\varepsilon\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)}}.$$

Hence, $\frac{\langle U_{electric}\rangle}{\langle U_{magnetic}\rangle} = \frac{\varepsilon}{\mu}\left|Z\right|^2 = \frac{\varepsilon}{\mu}\left|Z\cdot Z^*\right|^2 = \frac{\varepsilon}{\mu}\sqrt{\frac{\mu}{\varepsilon\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)}}\sqrt{\frac{\mu}{\varepsilon\left(1 - i\frac{\sigma}{\varepsilon\omega}\right)}} = \underline{\underline{\frac{1}{\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}}}.$

Because the denominator $\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \gg 1$, the ratio above indicates that more energy is stored in the magnetic field than in the electric field.

Question 4.

[2016 Exam question] Consider a conducting medium with conductivity σ, with no free charges and obeying Ohm's law.

(a) Using the appropriate Maxwell's equations, relations between $\vec{D}$ and $\vec{E}$, $\vec{B}$ and $\vec{H}$ in a medium and the identity $\nabla\times\left[\nabla\times\vec{F}(\vec{r})\right]=\nabla\left(\nabla\cdot\vec{F}(\vec{r})\right)-\nabla^2\vec{F}(\vec{r})$, derive the modified wave equation, $\nabla^2\vec{E}=\mu_r\mu_0\varepsilon_r\varepsilon_0\dfrac{\partial^2\vec{E}}{\partial t^2}+\mu_r\mu_0\sigma\dfrac{\partial\vec{E}}{\partial t}$.

(b) Show that $\vec{E}=\vec{E}_0e^{-\alpha x}e^{i(\omega t-\beta x)}$ is a solution of the above equation with $\dfrac{2\alpha\beta}{\beta^2-\alpha^2}=\dfrac{\sigma}{\omega\varepsilon_0\varepsilon_r}$.

(c) For an electromagnetic wave with frequency 25 kHz, the loss tangent of sea water ($\varepsilon_r=80$, $\mu_r=1$) is approximately 36,000. What is the skin depth for this wave?

Answer:

(a) Can be found in lecture notes, see section 'Modified wave equation in a conducting medium', Eq. 3.29 to Eq. 3.36.

(b) Can be found in lecture notes, see section 'Solving the modified wave equation', Eq. 3.37 to Eq. 3.44.

(c) The loss tangent $\tan\theta=\dfrac{\sigma}{\varepsilon\omega}\approx 36\times10^3\gg1$, so we are in the 'good conductor' regime. From the lectures, the skin depth is $\delta=\sqrt{\dfrac{2}{\mu\sigma\omega}}=\alpha^{-1}$, so here $\delta=\sqrt{\dfrac{2}{\mu_0\mu_r\sigma 2\pi f}}$ with f the frequency in Hz as given in the text of the problem. Using $\sigma=\varepsilon_0\varepsilon_r 2\pi f\tan\theta$, we get:

$$\delta=\sqrt{\frac{2}{\mu_0\mu_r\left(\varepsilon_0\varepsilon_r 2\pi f\tan\theta\right)2\pi f}}=\frac{c}{2\pi f}\sqrt{\frac{2}{\varepsilon_r\mu_r\tan\theta}}=\frac{299792458}{2\pi\left(25\times10^3\right)}\sqrt{\frac{2}{80\left(36\times10^3\right)}}\approx\underline{\underline{1.59}}\text{ m.}$$

Advanced problem

Question 5.

From the first half of the unit, you know that the energy flow in electromagnetism is described by Poynting vector $\vec{S}=\vec{E}\times\vec{H}$. Show that for a conducting LIH material, the following (Poynting's) theorem is true:

$$\oiint_S\left(\vec{E}\times\vec{H}\right)\cdot d\vec{a}=-\frac{\partial}{\partial t}\iiint_V\left(\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right)dV-\iiint_V\sigma E^2 dv\,.$$

Explain the physical meaning of all the terms in the theorem.

Answer: For a conductor, we can express Ohm's law as $\vec{J}_f=\sigma\vec{E}$. Using this, we rewrite two of Maxwell's equations into the following form ($\mu=\mu_0\mu_r$ and $\varepsilon=\varepsilon_0\varepsilon_r$)

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=-\mu\frac{\partial\vec{H}}{\partial t}\qquad(1)$$

$$\nabla\times\vec{H}=\frac{\partial\vec{D}}{\partial t}+\vec{J}_f\rightarrow\nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}+\sigma\vec{E}\qquad(2)$$

Let us take a scalar product of $\vec{E}$ with the second one, i.e. with (2):

$$\vec{E}\cdot\left(\nabla\times\vec{H}\right)=\varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\sigma\vec{E}^2\,. \tag{3}$$

We can use the vector identity $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$, with $\vec{A}=\vec{H}$ and $\vec{B}=\vec{E}$, which is $\nabla\cdot\left(\vec{H}\times\vec{E}\right)=\vec{E}\cdot\left(\nabla\times\vec{H}\right)-\vec{H}\cdot\left(\nabla\times\vec{E}\right)$. This expression leads to:

$$\vec{E}\cdot\left(\nabla\times\vec{H}\right)=\underline{\vec{H}\cdot\left(\nabla\times\vec{E}\right)}+\nabla\cdot\left(\vec{H}\times\vec{E}\right)=\varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}+\sigma\vec{E}^2\,. \tag{4}$$

But, $\nabla\times\vec{E}=-\mu\dfrac{\partial\vec{H}}{\partial t}$, so $\vec{H}\cdot\left(\nabla\times\vec{E}\right)=-\mu\vec{H}\cdot\dfrac{\partial\vec{H}}{\partial t}$. We also know that $\dfrac{\partial\left(\vec{H}\cdot\vec{H}\right)}{\partial t}=\vec{H}\cdot\dfrac{\partial\vec{H}}{\partial t}+\vec{H}\cdot\dfrac{\partial\vec{H}}{\partial t}=2\vec{H}\cdot\dfrac{\partial\vec{H}}{\partial t}$, so we can write:

$$\vec{H}\cdot\left(\nabla\times\vec{E}\right)=\underline{-\frac{1}{2}\mu\frac{\partial\left(\vec{H}\cdot\vec{H}\right)}{\partial t}}\,. \tag{5}$$

Hence, Eq. (4) becomes $\underline{-\dfrac{1}{2}\mu\dfrac{\partial\vec{H}^2}{\partial t}}+\underline{\underline{\nabla\cdot\left(\vec{H}\times\vec{E}\right)}}=\varepsilon\vec{E}\cdot\dfrac{\partial\vec{E}}{\partial t}+\sigma\vec{E}^2$.

Next, we use $\underline{\underline{\nabla\cdot\left(\vec{H}\times\vec{E}\right)}}=\underline{\underline{-\nabla\cdot\left(\vec{E}\times\vec{H}\right)}}$ and $\dfrac{\partial\left(\vec{E}\cdot\vec{E}\right)}{\partial t}=2\vec{E}\cdot\dfrac{\partial\vec{E}}{\partial t}$ to obtain:

$$\underline{-\frac{1}{2}\mu\frac{\partial\vec{H}^2}{\partial t}}\underline{\underline{-\nabla\cdot\left(\vec{E}\times\vec{H}\right)}}=\varepsilon\frac{1}{2}\frac{\partial\vec{E}^2}{\partial t}+\sigma\vec{E}^2\,. \tag{6}$$

We now rearrange the terms

$$\nabla\cdot\left(\vec{E}\times\vec{H}\right)=-\varepsilon\frac{1}{2}\frac{\partial\vec{E}^2}{\partial t}-\frac{1}{2}\mu\frac{\partial\vec{H}^2}{\partial t}-\sigma\vec{E}^2 \tag{7}$$

and integrate over whole volume,

$$\iiint_V\nabla\cdot\left(\vec{E}\times\vec{H}\right)dV=-\frac{\partial}{\partial t}\iiint_V\left(\frac{1}{2}\varepsilon\vec{E}^2+\frac{1}{2}\mu\vec{H}^2\right)dV-\iiint_V\sigma\vec{E}^2dV\,. \tag{8}$$

We apply the divergence theorem to simplify the left-hand side to obtain the final form,

$$\oiint_S\left(\vec{E}\times\vec{H}\right)\cdot d\vec{a}=-\frac{\partial}{\partial t}\iiint_V\left(\frac{1}{2}\varepsilon\vec{E}^2+\frac{1}{2}\mu\vec{H}^2\right)dV-\iiint_V\sigma\vec{E}^2dV \tag{9}$$

The term on the left-hand side expresses the total power leaving the volume through the surface. On the right, the first term describes the rate of decrease of the energy stored in electric and magnetic fields and the second the Ohmic power dissipated due to conductance.