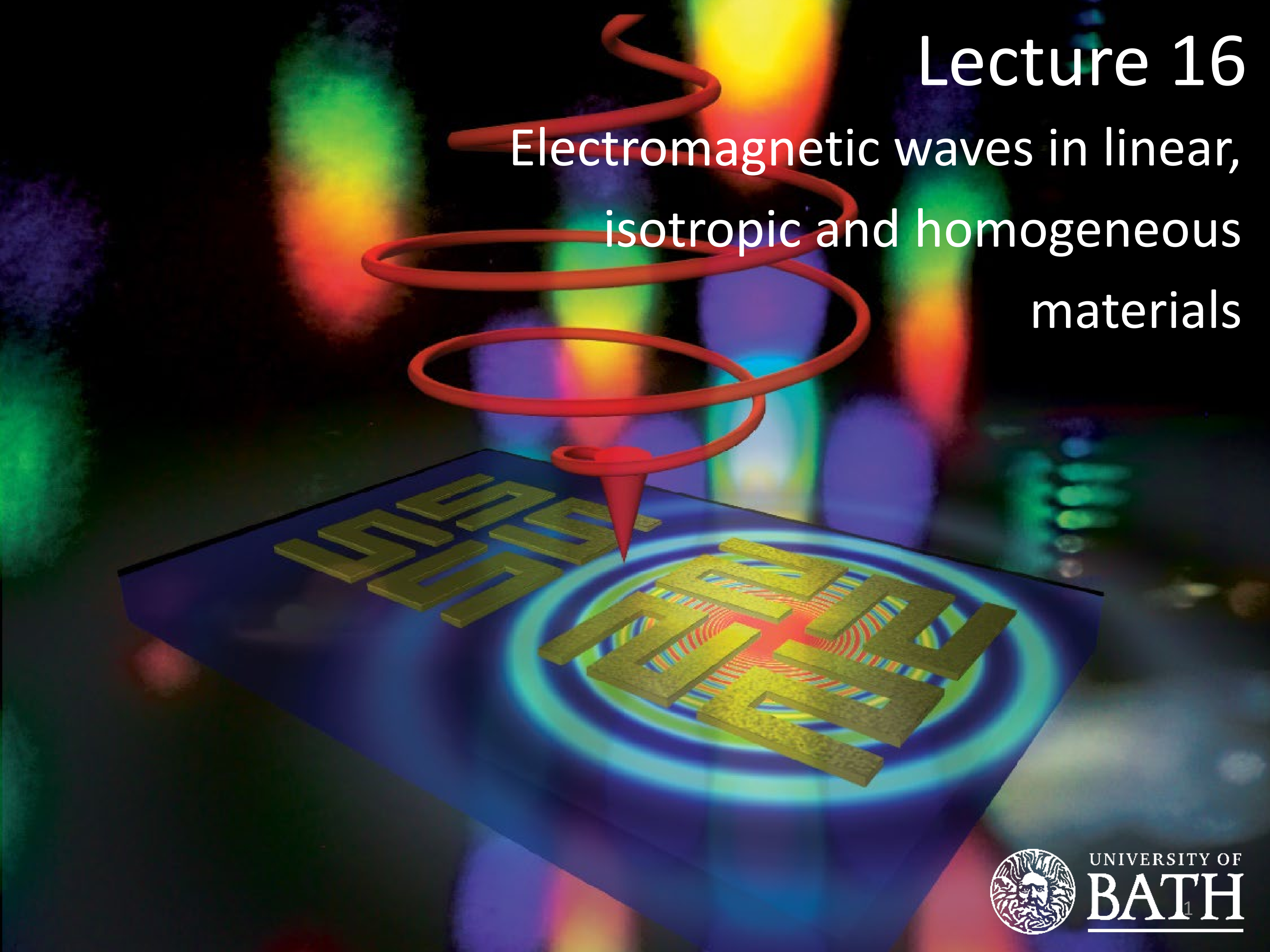
Lecture 16

Electromagnetic waves in linear, isotropic and homogeneous materials



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Last time we saw

In LIH materials, Maxwell's equations become: $\nabla \cdot \vec{D} = \rho_f \qquad \nabla \cdot \vec{B} = 0$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

In an ideal LIH dielectric, the electromagnetic wave propagates similarly to the way it does in free space – it remains a plane wave but its phase velocity is lowered.

$$v_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

The wave equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ becomes $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$, which is the modified wave equation.

A solution to the modified wave equation is $\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$ with $\gamma = -\alpha + i\beta$

So: $\vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)}$ Upon solving, an important ratio appears: $\sigma/\varepsilon\omega$

This ratio is part of the dielectric function of the material: $\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}$

and of its complex refractive index: $\tilde{n} = n + i\eta = \sqrt{\varepsilon(\omega)} = \sqrt{\varepsilon' + i\varepsilon''}$

Confusion?

The modified wave equation

We start with Maxwell's equation:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

and we take the curl.

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right)$$

We can apply a maths formula:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

Without free charges: $\rho_f = 0$

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 0 \qquad \boxed{\nabla \cdot \vec{D} = \rho_f}$$

It follows that:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{E}$$

This is the same as on slide 7.

On the right-hand side:

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \nabla \times \left(-\frac{\partial\left(\mu \vec{H}\right)}{\partial t}\right) =$$

$$= -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) = -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right)$$

We now have (<u>not ideal dielectric</u>):

$\underline{\vec{J}_f = \sigma \vec{E}}$ and $\vec{D} = \varepsilon \vec{E}$

It follows that:

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} =$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

We shorten this.

The modified wave equation

We start with Maxwell's equations

$\nabla \times$

and

$\nabla \times$

We

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

Without free charges: $\rho_f = 0$

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 0 \qquad \boxed{\nabla \cdot \vec{D} = \rho_f}$$

It follows that:

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{E}$$

This is the same as on slide 7.

On the right hand side

Wait what? This is a conductor, with a current density but there is no charge? How is a current flowing without there being any electric charge?

The free charges repel. So, they try to keep away from each other, and they all end up at the surface. Hence, the **volume charge density** inside the conductor is 0. But there is surface charge and therefore **surface current density**.

We now have (<u>not ideal dielectric</u>):

$\underline{\vec{J}_f = \sigma \vec{E}}$ and $\vec{D} = \varepsilon \vec{E}$

It follows that:

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} =$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

We shorten this.

Overview

In this Lecture we will look at:

- The electric and magnetic fields in a conductive medium
- The loss tangent in a conductive medium
- The wave parameters in a conductive medium
- Special case I: Poor conductors (lossy dielectrics)
- Special case II: Good conductors

The electric and magnetic fields in a conductive medium

The complex intrinsic impedance is:

$$Z = \frac{\mu\omega}{k} = \frac{\mu\omega}{\text{wave vector}} = \frac{\mu\omega}{\beta + i\alpha}\frac{\beta - i\alpha}{\beta - i\alpha} = \frac{\mu\omega}{\alpha^2 + \beta^2}(\beta - i\alpha)$$

See slide 13 in lecture 15

We can identify the parts:

$$\mathrm{Re}[Z] = \frac{\mu\omega\beta}{\alpha^2 + \beta^2} \quad \mathrm{Im}[Z] = -\frac{\mu\omega\alpha}{\alpha^2 + \beta^2}$$

Polar form of complex numbers:

[where we used Euler's formula]

$$a + ib = re^{i\phi} = r(\cos\phi + i\sin\phi)$$

$$e^{ix} = \cos x + i\sin x$$

So that

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{b}{a}$$

In our case, we have:

$$\tan\phi = -\frac{\alpha}{\beta} \quad \text{and} \quad Z = |Z|e^{i\phi}$$

Need to tidy up a bit

The electric and magnetic fields in a conductive medium

The complex intrinsic impedance is:

$$Z = \frac{\mu\omega}{\alpha^2 + \beta^2}(\beta - i\alpha)$$

We can identify the parts:

$$\mathrm{Re}[Z] = \frac{\mu\omega\beta}{\alpha^2 + \beta^2} \quad \mathrm{Im}[Z] = -\frac{\mu\omega\alpha}{\alpha^2 + \beta^2}$$

Polar form of complex numbers:

$$a + ib = re^{i\phi} = r(\cos\phi + i\sin\phi)$$

So that

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{b}{a}$$

In our case, we have:

$$\tan\phi = -\frac{\alpha}{\beta} \text{ and } Z = |Z|e^{i\phi}$$

We can also define:

$$\tan\varphi = \tan(-\phi) = -\tan\phi = \frac{\alpha}{\beta}$$

[Keep an eye for this φ]

And in that case:

$$Z = |Z|e^{i\phi} = |Z|e^{-i\varphi}$$ [we use this next slide]

We can write the E and H-fields as:

$$\vec{E} = E(\vec{r})e^{-i\omega t}\hat{n}_1 \text{ and } \vec{H} = H(\vec{r})e^{-i\omega t}\hat{n}_2$$

where $\hat{n}_1$ and $\hat{n}_2$ are unit vectors in the directions of the respective fields.

The complex impedance is then:

$$Z = \frac{E(\vec{r})}{H(\vec{r})}$$

So now we can substitute!

The electric and magnetic fields in a conductive medium

The complex intrinsic impedance is:

$$Z = \frac{E(\vec{r})}{H(\vec{r})}$$

Our *E*-field only depends on *x*:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)} \qquad E(\vec{r}) = \vec{E}_0 e^{\gamma x}$$

So, we can write: (where we omit the vectors $\hat{n}_1$ and $\hat{n}_2$)

$$E(x) = H(x) Z = H(x)|Z| e^{-i\varphi}$$ and just moving around:

$$E(x) = |Z| H(x) e^{-i\varphi}$$

But we recall that:

$$E(x) = E_0 e^{-\alpha x} e^{i(\beta x)}$$

So, we can write:

$$H(x) = \frac{E_0 e^{-\alpha x} e^{i(\beta x)}}{\frac{E_0}{H_0} e^{-i\varphi}} = H_0 e^{-\alpha x} e^{i(\beta x)} e^{i\varphi} = H_0 e^{-\alpha x} e^{i(\beta x + \varphi)}$$

Need to tidy up a bit

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The electric and magnetic fields in a conductive medium

The complex intrinsic impedance is:

$$Z = \frac{E(\vec{r})}{H(\vec{r})}$$

Our *E*-field only depends on *x*:

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$$

So, we can write:

$$E(x) = H(x) Z = H(x)|Z|e^{-i\varphi}$$

$$E(x) = |Z| H(x) e^{-i\varphi}$$

But we recall that:

$$E(x) = E_0 e^{-\alpha x} e^{i(\beta x)}$$

So, we can write:

$$H(x) = H_0 e^{-\alpha x} e^{i(\beta x + \varphi)}$$

Then, from:

$$\begin{cases} \vec{E} = E(x) e^{-i\omega t} \hat{n}_1 \\ \vec{H} = H(x) e^{-i\omega t} \hat{n}_2 \end{cases}$$

$$\begin{cases} \vec{E} = \frac{E_0}{H_0} H_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ \vec{H} = H_0 e^{-\alpha x} e^{i(\beta x - \omega t + \varphi)} \end{cases}$$

We can write with vectors:

$$\boxed{\begin{cases} \vec{E}(x) = |Z| \vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ \vec{H}(x) = \vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t + \varphi)} \end{cases}}$$

where we see that the **electric and magnetic fields of the EM wave are shifted in phase.**

What have we learned about conductive media?

The loss tangent in a conductive medium

So far, we learned that in conductive media:

1. The EM wave decays (described by the decay constant α).
2. The electric and magnetic fields oscillate, but no longer in phase. The phase shift between them is described by the angle φ, which is given by $\tan\varphi = \frac{\alpha}{\beta}$.

Then using trigonometry and substituting for $\tan\varphi$

$$\tan 2\varphi = \frac{2\tan\varphi}{1-(\tan\varphi)^2} = \frac{2\frac{\alpha}{\beta}}{1-\left(\frac{\alpha}{\beta}\right)^2} = \frac{2\frac{\alpha}{\beta}}{\frac{\beta^2}{\beta^2}-\frac{\alpha^2}{\beta^2}} = \frac{2\frac{\alpha}{\beta}\beta^2}{\beta^2-\alpha^2} = \frac{2\alpha\beta}{\beta^2-\alpha^2} = \frac{\sigma}{\omega\varepsilon}$$

We can introduce yet another angle, which defines the **loss tangent** – a quantifier for how good a conductor is:

$$\boxed{\tan\theta = \tan 2\varphi = \frac{\sigma}{\omega\varepsilon} = \left|\frac{\text{ohmic current}}{\text{displacement current}}\right|}$$

How to characterise EM propagation in conductive media?

The wave parameters in a conductive medium

From Maxwell's equation, there are two currents that add up:

$$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} = \sigma \vec{E} - i \omega \varepsilon \vec{E}$$

Here we used:

$$\frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E}$$

Which comes from:

$$\vec{E} \propto e^{-i\omega t}$$

The propagating vector depends on two parameters:

$$\gamma = -\alpha + i\beta$$

But what exactly are these alpha and beta?

We have seen that:

$\gamma^2 = -\mu\varepsilon\omega^2 - i\mu\sigma\omega$ and $\alpha^2 - \beta^2 = -\mu\varepsilon\omega^2$

We can therefore calculate:

$$|\gamma|^2 = \gamma\gamma^* = \sqrt{\gamma^2}\left(\sqrt{\gamma^2}\right)^* =$$

$$= \sqrt{-\mu\varepsilon\omega^2 - i\mu\sigma\omega}\sqrt{-\mu\varepsilon\omega^2 + i\mu\sigma\omega}$$

Combining:

$$|\gamma|^2 = \sqrt{-\left(\mu\varepsilon\omega^2 + i\mu\sigma\omega\right)}\sqrt{-\left(\mu\varepsilon\omega^2 - i\mu\sigma\omega\right)} =$$

$$|\gamma|^2 = \sqrt{\left(\mu\varepsilon\omega^2 + i\mu\sigma\omega\right)\left(\mu\varepsilon\omega^2 - i\mu\sigma\omega\right)}$$

Where we can factorize:

$$|\gamma|^2 = \sqrt{\mu^2\omega^2\left(\varepsilon\omega + i\sigma\right)\left(\varepsilon\omega - i\sigma\right)}$$

Almost there!

The wave parameters in a conductive medium

$$\alpha^2 - \beta^2 = -\mu\varepsilon\omega^2$$

We just obtained:

$$|\gamma|^2 = \sqrt{\mu^2\omega^2\left(\varepsilon\omega + i\sigma\right)\left(\varepsilon\omega - i\sigma\right)}$$

Next, we use:

$$\left(a+b\right)\left(a-b\right) = a^2 - b^2$$

To obtain

$$|\gamma|^2 = \mu\omega\sqrt{\left(\varepsilon\omega\right)^2 - \left(i\sigma\right)^2} =$$

$$= \mu\omega\sqrt{\varepsilon^2\omega^2 + \sigma^2} = \mu\omega\sqrt{\varepsilon^2\omega^2\left(1 + \frac{\sigma^2}{\varepsilon^2\omega^2}\right)}$$

Where we recognise the loss tangent:

$$|\gamma|^2 = \mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}$$

But we can also write:

$$\mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} = |\gamma|^2 = \gamma\gamma^* =$$

$$= \left(-\alpha + i\beta\right)\left(-\alpha - i\beta\right) = \alpha^2 + \beta^2$$

We can add the equations below:

$$\left.\begin{aligned} \beta^2 + \alpha^2 &= \mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} \\ \beta^2 - \alpha^2 &= \mu\varepsilon\omega^2 \end{aligned}\right\} \rightarrow$$

$$\rightarrow 2\beta^2 = \mu\varepsilon\omega^2\left[1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]$$

So, we have an expression for beta:

$$\beta = \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}$$

We identified one parameter!

The wave parameters in a conductive medium

$$\alpha^2 - \beta^2 = -\mu\varepsilon\omega^2$$

We just obtained:

$$|\gamma|^2 = \sqrt{\mu^2\omega^2(\varepsilon\omega + i\sigma)(\varepsilon\omega - i\sigma)}$$

Next, we use:

$$(a+b)(a-b) = a^2 - b^2$$

To obtain

$$|\gamma|^2 = \mu\omega\sqrt{(\varepsilon\omega)^2 - (i\sigma)^2} =$$

$$= \mu\omega\sqrt{\varepsilon^2\omega^2 + \sigma^2} = \mu\omega\sqrt{\varepsilon^2\omega^2\left(1 + \frac{\sigma^2}{\varepsilon^2\omega^2}\right)}$$

Where we recognise the loss tangent:

$$|\gamma|^2 = \mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}$$

But we can also write:

$$\mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} = |\gamma|^2 = \gamma\gamma^* =$$

$$= (-\alpha + i\beta)(-\alpha - i\beta) = \alpha^2 + \beta^2$$

We can add the equations below:

$$\left.\begin{aligned} \beta^2 + \alpha^2 &= \mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} \\ \beta^2 - \alpha^2 &= \mu\varepsilon\omega^2 \end{aligned}\right\} \rightarrow$$

$$\rightarrow 2\alpha^2 = \mu\varepsilon\omega^2\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]$$

So, we have an expression for alpha:

$$\alpha = \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}$$

We identified the other parameter!

The wave parameters in a conductive medium

We obtained for alpha:

$$\alpha = \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[-1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}$$

We obtained for beta:

$$\beta = \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}$$

These expressions are complicated and we do not need to remember them. But they serve to understand the two limiting cases of 'poor' and 'good' conductors:

1. A 'poor' conductor is one where $\frac{\sigma}{\varepsilon\omega} \ll 1$ and the Ohmic current is much smaller than the displacement current.

2. A 'good' conductor is one where $\frac{\sigma}{\varepsilon\omega} \gg 1$ and the Ohmic current is much larger than the displacement current.

Two examples. Then how about EM waves in poor conductors?

Example question

[from Sadiku] At 50 MHz, a lossy dielectric material is characterized by $\mu = 2.1\mu_0$, $\varepsilon = 3.6\varepsilon_0$ and $\sigma = 0.08 \text{ Sm}^{-1}$. If $\vec{E} = 6e^{-\gamma x}\vec{a}_z \text{ Vm}^{-1}$, compute:

(a) γ

(b) The wavelength.

(c) The wave velocity.

Example question

[from Sadiku] At 50 MHz, a lossy dielectric material is characterized by $\mu = 2.1\mu_0$, $\varepsilon = 3.6\varepsilon_0$ and $\sigma = 0.08\ \mathrm{Sm}^{-1}$. If $\vec{E} = 6e^{-\gamma x}\vec{a}_z\ \mathrm{Vm}^{-1}$, compute:
(a) γ
(b) The wavelength.
(c) The wave velocity.

(a) Loss tangent: $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{0.08}{(\pi 10^8)(3.6)\left(\dfrac{10^{-9}}{36\pi}\right)} = \dfrac{0.08}{(3.6)\left(\dfrac{10^{-1}}{36}\right)} = \dfrac{0.08}{(36\times 10^{-1})\left(\dfrac{10^{-1}}{36}\right)} = 8$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} \approx 5.41$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} \approx 6.13$$

Therefore: $\gamma = -\alpha + i\beta \approx \underline{\underline{-5.41 + i6.13}}\ \mathrm{m}^{-1}$

(b) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6.13} \approx \underline{\underline{1.025}}\ \mathrm{m}$

(c) $u = \dfrac{\omega}{\beta} = \dfrac{2\pi \times 50 \times 10^6}{6.13} \approx \underline{\underline{5.10\times 10^7}}\ \mathrm{ms}^{-1}$

[details in the lecture notes]

Example question

[from Sadiku] A lossy material has $\mu = 2.1\mu_0$ and $\varepsilon = 3.6\varepsilon_0$. If at 5 MHz, the phase constant is 10 rad/m, calculate:

(a) The loss tangent.

(b) The conductivity of the material.

(c) The complex permittivity.

(d) The attenuation constant.

Example question

[from Sadiku] A lossy material has $\mu = 2.1\mu_0$ and $\varepsilon = 3.6\varepsilon_0$. If at 5 MHz, the phase constant is 10 rad/m, calculate:
(a) The loss tangent.
(b) The conductivity of the material.
(c) The complex permittivity.
(d) The attenuation constant.

(a) $$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} \rightarrow 10 = \frac{\pi\times10^7\times\sqrt{5}}{3\times10^8}\sqrt{\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} \rightarrow \frac{\sigma}{\varepsilon\omega} \approx \underline{\underline{1823}}$$

(b) $$\sigma = \varepsilon\omega\tan(\theta) = 2\varepsilon_0\omega\tan(\theta) = 2\left(\frac{10^{-9}}{36\pi}\right)\left(\pi\times10^7\right)(1823) \approx \underline{\underline{1.013}}\ \mathrm{Sm^{-1}}$$

(c) $$\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0\omega} = 2\times\left(\frac{10^{-9}}{36\pi}\right) + i\frac{1.013}{\pi\times10^7} \approx \underline{\underline{1.77\times10^{-11} + i3.22\times10^{-8}}}\ \mathrm{Fm^{-1}}$$

(d) $$\frac{\alpha}{\beta} = \sqrt{\frac{-1+\sqrt{1+1823^2}}{+1+\sqrt{1+1823^2}}} \approx \sqrt{\frac{1822}{1824}} \rightarrow \alpha \approx 10\sqrt{\frac{1822}{1824}} \approx \underline{\underline{9.99}}\ \mathrm{m^{-1}}$$

[details in the lecture notes]

Special case I: Poor conductors (lossy dielectrics)

1. Loss tangent: $\tan\theta = \dfrac{\sigma}{\varepsilon\omega} \ll 1 \rightarrow \theta \approx 0$

2. Phase angle: $\varphi = \dfrac{1}{2}\theta \approx 0$

3. Propagation parameters: $\sqrt{1+\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2} = \left[1+\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{2}} = (1+x)^{\frac{1}{2}}$ $\qquad x = \left(\dfrac{\sigma}{\varepsilon\omega}\right)^2$

We use the binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + nx^{n-1} + x^n$

Here we have $\dfrac{\sigma}{\varepsilon\omega} \ll 1$ so: $\sqrt{1+\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2} \approx 1 + \dfrac{1}{2}\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2$ since the other terms are tiny.

Then we can write: $\alpha \approx \sqrt{\dfrac{1}{2}\mu\varepsilon\omega^2\left[-1+1+\dfrac{1}{2}\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2\right]} = \omega\sqrt{\dfrac{\mu\varepsilon}{4}\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2}$

$\alpha \approx \cancel{\omega}\sqrt{\dfrac{\mu\cancel{\varepsilon}}{4}\dfrac{\sigma^2}{\varepsilon^{\cancel{2}}\cancel{\omega}^{\cancel{2}}}} = \dfrac{1}{2}\sigma\sqrt{\dfrac{\mu}{\varepsilon}}$

For a perfect dielectric $\sigma = 0$ and $\alpha = 0$, i.e. no exponential decay for the EM wave in the material.

We can do the same analysis for beta.

Special case I: Poor conductors (lossy dielectrics)

1. Loss tangent: $\tan\theta = \dfrac{\sigma}{\varepsilon\omega} \ll 1 \rightarrow \theta \approx 0$

2. Phase angle: $\varphi = \dfrac{1}{2}\theta \approx 0$

3. Propagation parameters: $\alpha \approx \omega\sqrt{\dfrac{\mu\varepsilon}{4}\dfrac{\sigma^2}{\varepsilon^2\omega^2}} = \dfrac{1}{2}\sigma\sqrt{\dfrac{\mu}{\varepsilon}}$

$$\beta \approx \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[+1+1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]} = \omega\sqrt{\frac{\mu\varepsilon}{\cancel{2}}\cancel{2}\left[1+\underbrace{\frac{1}{4}\left(\frac{\sigma}{\varepsilon\omega}\right)^2}_{\approx 0}\right]} \approx$$

$$\beta \approx \omega\sqrt{\mu\varepsilon}$$

4. The phase velocity: $v_p = \dfrac{\omega}{\text{wave vector}}$, where we have $\gamma = -\alpha + i\beta$

Wo, we have $\vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)}$, where it is clear that alpha does not affect the phase.

To find the phase velocity, we only take beta: $v_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{\omega\sqrt{\mu\varepsilon}} = \dfrac{1}{\sqrt{\mu\varepsilon}}$

How about the impedance?

Special case I: Poor conductors (lossy dielectrics)

1. Loss tangent: $\tan\theta = \dfrac{\sigma}{\varepsilon\omega} \ll 1 \rightarrow \theta \approx 0$

2. Phase angle: $\varphi = \dfrac{1}{2}\theta \approx 0$

3. Propagation parameters: $\alpha \approx \omega\sqrt{\dfrac{\mu\varepsilon}{4}\dfrac{\sigma^2}{\varepsilon^2\omega^2}} = \dfrac{1}{2}\sigma\sqrt{\dfrac{\mu}{\varepsilon}}$ $\qquad \beta \approx \omega\sqrt{\mu\varepsilon}$

4. The phase velocity: $v_p = \dfrac{\omega}{\beta} = \dfrac{\omega}{\omega\sqrt{\mu\varepsilon}} = \dfrac{1}{\sqrt{\mu\varepsilon}}$

5. **The intrinsic impedance** of the medium is: $Z = \sqrt{\dfrac{\mu}{\varepsilon\left(1 + i\dfrac{\sigma}{\varepsilon\omega}\right)}} \approx \sqrt{\dfrac{\mu}{\varepsilon}}$

See slide 24 in lecture 15

6. The electric and magnetic fields:

Given that: $\varphi = \dfrac{1}{2}\theta \approx 0$, we can write:
$$\begin{cases} \vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ \vec{H} = \vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \end{cases}$$

How about good conductors?

Special case II: Good conductors

1. Loss tangent: $\tan\theta = \dfrac{\sigma}{\varepsilon\omega} \gg 1 \rightarrow \theta \approx \dfrac{\pi}{2}$

2. Phase angle: $\varphi = \dfrac{1}{2}\theta \approx \dfrac{\pi}{4}$

3. Propagation parameters: for $\dfrac{\sigma}{\varepsilon\omega} \gg 1$, we have $\sqrt{1+\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2} \approx \dfrac{\sigma}{\varepsilon\omega}$, so

$$\alpha \approx \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[-1+\frac{\sigma}{\varepsilon\omega}\right]} \approx \sqrt{\frac{\omega^{\cancel{2}}\mu\cancel{\varepsilon}}{2}\frac{\sigma}{\cancel{\varepsilon\omega}}} = \sqrt{\frac{\mu\sigma\omega}{2}}$$

and

$$\beta \approx \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[+1+\left(\frac{\sigma}{\varepsilon\omega}\right)\right]} \approx \sqrt{\frac{\omega^{\cancel{2}}\mu\cancel{\varepsilon}}{2}\frac{\sigma}{\cancel{\varepsilon\omega}}} = \sqrt{\frac{\mu\sigma\omega}{2}}$$

Therefore, for good conductors: $\alpha \approx \beta \approx \sqrt{\dfrac{\mu\sigma\omega}{2}}$

How about the phase velocity?

Special case II: Good conductors

1. Loss tangent: $\tan\theta = \dfrac{\sigma}{\varepsilon\omega} \gg 1 \to \theta \approx \dfrac{\pi}{2}$

2. Phase angle: $\varphi = \dfrac{1}{2}\theta \approx \dfrac{\pi}{4}$

3. Propagation parameters: $\alpha \approx \beta \approx \sqrt{\dfrac{\mu\sigma\omega}{2}}$

4. The phase velocity: $v_p = \dfrac{\omega}{\text{wave vector}}$, so $v_p = \dfrac{\omega}{\beta} = \dfrac{\sqrt{\omega^2}}{\sqrt{\dfrac{\mu\sigma\omega}{2}}} = \omega\sqrt{\dfrac{2}{\mu\sigma\omega}} = \omega\delta$

where $\boxed{\delta = \sqrt{\dfrac{2}{\mu\sigma\omega}} = \alpha^{-1}}$ is the **skin depth**.

In a good conductor, there is **dispersion**, as the phase velocity depends on frequency.

Units?

Note: by definition, **dispersion** is the phenomenon in which the phase velocity of a wave depends on its frequency. In practice, this often means that light is separated by individual colours as it travels through a material.

Special case II: Good conductors

1. Loss tangent: $\tan\theta = \frac{\sigma}{\varepsilon\omega} \gg 1 \rightarrow \theta \approx \frac{\pi}{2}$
2. Phase angle: $\varphi = \frac{1}{2}\theta \approx \frac{\pi}{4}$
3. Propagation parameters: $\alpha \approx \beta \approx \sqrt{\frac{\mu\sigma\omega}{2}}$
4. The phase velocity: $v_p = \frac{\omega}{\text{wave vector}}$, so $v_p = \frac{\omega}{\beta} = \frac{\sqrt{\omega^2}}{\sqrt{\frac{\mu\sigma\omega}{2}}} = \omega\sqrt{\frac{2}{\mu\sigma\omega}} = \omega\delta$

where $\boxed{\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \alpha^{-1}}$ is the **skin depth**, in units of meters.

$$\delta = \sqrt{\frac{2}{\mu\sigma 2\pi f}} \propto \sqrt{\frac{2\left[kg \cdot m^3 \cdot s^{-3} \cdot A^{-2}\right]}{\left[\left(kg \cdot m^2 \cdot s^{-2} \cdot A^{-2}\right) \cdot \left(m^{-1}\right)\right] 2\pi \left[s^{-1}\right]}} \propto \sqrt{\frac{[m]^2}{\pi}} \propto [m]$$

How about the impedance?

Special case II: Good conductors

1. Loss tangent: $\tan\theta = \dfrac{\sigma}{\varepsilon\omega} \gg 1 \rightarrow \theta \approx \dfrac{\pi}{2}$

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3. Propagation parameters: $\alpha \approx \beta \approx \sqrt{\dfrac{\mu\sigma\omega}{2}}$

4. The phase velocity: $v_p = \sqrt{\dfrac{2\omega}{\mu\sigma}} = \omega\delta$, with $\delta = \sqrt{\dfrac{2}{\mu\sigma\omega}} = \alpha^{-1}$

5. The **intrinsic impedance** of the medium is: $Z = |Z|e^{i\phi} = |Z|e^{-i\varphi} = |Z|e^{-i\frac{\pi}{4}}$

where $|Z|^2 = ZZ^*$, so $|Z| = \left[ZZ^*\right]^{\frac{1}{2}}$ and therefore:

$$|Z| = \left[\sqrt{\frac{\mu}{\varepsilon\left(1 + i\dfrac{\sigma}{\varepsilon\omega}\right)}}\sqrt{\frac{\mu}{\varepsilon\left(1 - i\dfrac{\sigma}{\varepsilon\omega}\right)}}\right]^{\frac{1}{2}} \approx \left[\sqrt{\frac{\mu}{\cancel{\varepsilon}\left(i\dfrac{\sigma}{\cancel{\varepsilon}\omega}\right)}}\sqrt{\frac{\mu}{\cancel{\varepsilon}\left(-i\dfrac{\sigma}{\cancel{\varepsilon}\omega}\right)}}\right]^{\frac{1}{2}} \approx \sqrt{\frac{\mu\omega}{\sigma}}$$

We need to tidy up here

Special case II: Good conductors

1. Loss tangent: $\tan\theta = \dfrac{\sigma}{\varepsilon\omega} \gg 1 \to \theta \approx \dfrac{\pi}{2}$

2. Phase angle: $\varphi = \dfrac{1}{2}\theta \approx \dfrac{\pi}{4}$

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where $|Z|^2 = ZZ^*$, so $|Z| = \left[ZZ^*\right]^{\frac{1}{2}}$ and therefore: $|Z| \approx \sqrt{\dfrac{\mu\omega}{\sigma}}$

Combining: $Z \approx \sqrt{\dfrac{\mu\omega}{\sigma}}e^{-i\frac{\pi}{4}} = \dfrac{\sqrt{2}}{\sigma\delta}e^{-i\frac{\pi}{4}}$, because $\sqrt{\mu\omega} = \dfrac{1}{\delta}\sqrt{\dfrac{2}{\sigma}}$

Finally, for the electric and magnetic fields?

Special case II: Good conductors

1. Loss tangent: $\tan\theta = \frac{\sigma}{\varepsilon\omega} \gg 1 \rightarrow \theta \approx \frac{\pi}{2}$

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3. Propagation parameters: $\alpha \approx \beta \approx \sqrt{\frac{\mu\sigma\omega}{2}}$

4. The phase velocity: $v_p = \sqrt{\frac{2\omega}{\mu\sigma}} = \omega\delta$, with $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \alpha^{-1}$

5. The intrinsic impedance of the medium is: $Z \approx \sqrt{\frac{\mu\omega}{\sigma}} e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{\sigma\delta} e^{-i\frac{\pi}{4}}$

6. The electric and magnetic fields: because $\varphi = \frac{1}{2}\theta \approx \frac{\pi}{4}$ the *E* and *H*-fields are 45° out of phase.

The wave attenuates very quickly, as $e^{-\alpha x} = e^{-x/\delta}$

After a distance of λ, the amplitude drops by:

$$e^{-\lambda/\delta} = e^{-2\pi/(\beta\delta)} = e^{-2\pi} \approx 1/535$$

So:
$$\begin{cases} \vec{E} = \vec{E}_0 e^{-\frac{x}{\delta}} e^{i\left(\frac{x}{\delta} - \omega t\right)} \\ \vec{H} = \vec{H}_0 e^{-\frac{x}{\delta}} e^{i\left(\frac{x}{\delta} - \omega t + \frac{\pi}{4}\right)} \end{cases}$$

Next, example and summary.

Example question

[from Sadiku] Calculate the skin depth and the velocity of propagation for a uniform plane wave at frequency 6 MHz travelling in polyvinylchloride ($\mu_r = 1$, $\varepsilon_r = 4$ and $\tan\theta = 7 \times 10^{-2}$).

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Example question

[from Sadiku] Calculate the skin depth and the velocity of propagation for a uniform plane wave at frequency 6 MHz travelling in polyvinylchloride ($\mu_r = 1$, $\varepsilon_r = 4$ and $\tan\theta = 7\times10^{-2}$).

From the lectures
$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{2\pi f}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[-1+\sqrt{1+49\times10^{-4}}\right]}$$

$$\alpha = \frac{2\pi\times6\times10^6}{c}\sqrt{\frac{1\times4}{2}\left[-1+\sqrt{1+49\times10^{-4}}\right]} \approx 8.8\times10^{-3}$$

Therefore the skin depth is $\delta = \alpha^{-1} \approx \underline{\underline{113.75}}$ m.

Also from the lectures
$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{2\pi f}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[1+\sqrt{1+49\times10^{-4}}\right]} \approx 0.25$$

The wave velocity is then: $u = \dfrac{\omega}{\beta} = \dfrac{2\pi\times6\times10^6}{0.25} = \underline{\underline{1.5\times10^8}}\ \text{ms}^{-1}$

Summary

In LIH materials, Maxwell's equations become: $\nabla \cdot \vec{D} = \rho_f \qquad \nabla \cdot \vec{B} = 0$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

In an ideal LIH dielectric, the electromagnetic wave propagates similarly to the way it does in free space – it remains a plane wave but its phase velocity is lowered.

$$v_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

The wave equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ becomes $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$, which is the modified wave equation.

A solution to the modified wave equation is $\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$ with $\gamma = -\alpha + i\beta$

So: $\vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)}$ Upon solving, an important ratio appears: $\sigma / \varepsilon\omega$

This ratio is part of the dielectric function of the material: $\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}$

Summary

and of its complex refractive index: $\tilde{n} = n + i\eta = \sqrt{\varepsilon(\omega)} = \sqrt{\varepsilon' + i\varepsilon''}$

This ratio is the loss tangent: $\tan\theta = \tan 2\varphi = \dfrac{\sigma}{\omega\varepsilon} = \left|\dfrac{\text{ohmic current}}{\text{displacement current}}\right|$

When an EM wave propagates in a lossy dielectric, its amplitude decays at a rate α. The electric and magnetic fields oscillate with a phase shift described by the angle φ in $\tan\varphi = \alpha/\beta$.

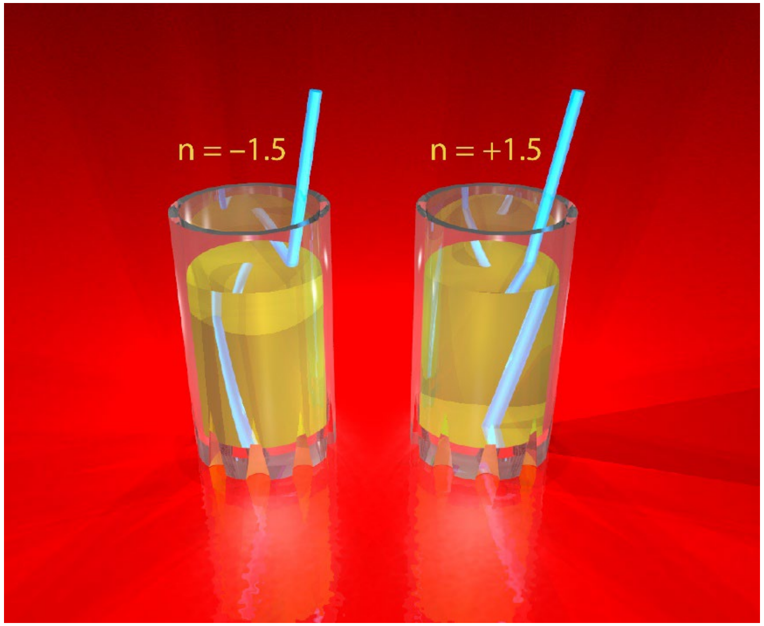
Altogether we have: $\begin{cases} \vec{E}(x) = |Z|\vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ \vec{H}(x) = \vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t + \varphi)} \end{cases}$

In poor conductors, we have $\dfrac{\sigma}{\varepsilon\omega} \ll 1$. The EM plane wave decays slowly. The *E*- and *H*-fields are in phase.

In good conductors, we have $\dfrac{\sigma}{\varepsilon\omega} \gg 1$. The EM wave decays rapidly. The *E*- and *H*-fields are no longer in phase.

We defined the skin depth as: $\delta = \sqrt{\dfrac{2}{\mu\sigma\omega}} = \alpha^{-1}$

Negative refractive index in chiral metamaterials



Constitutive relations in chiral metamaterials :

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E} + i\xi\sqrt{\mu_0 \varepsilon_0}\,\vec{H}$$

$$\vec{B} = \mu_r \mu_0 \vec{H} - i\xi\sqrt{\mu_0 \varepsilon_0}\,\vec{E}$$

$\vec{D}$: electric flux density
$\vec{B}$: magnetic flux density
$\vec{E}$: electric field
$\vec{H}$: magnetic field

ε_r: the relative permittivity
ε_0: the permittivity of vacuum
μ_r: the relative permeability
μ_0: the permeability of vacuum
ζ: the chirality parameter

The refractive index of circularly polarized light is: $n^{\pm} = n \pm \xi$, where $n = \sqrt{\varepsilon_r \mu_r}$.

Consequently, a large $|\xi|$ leads directly to negative refractive index for one of the circularly-polarized electromagnetic waves in chiral (meta)materials.

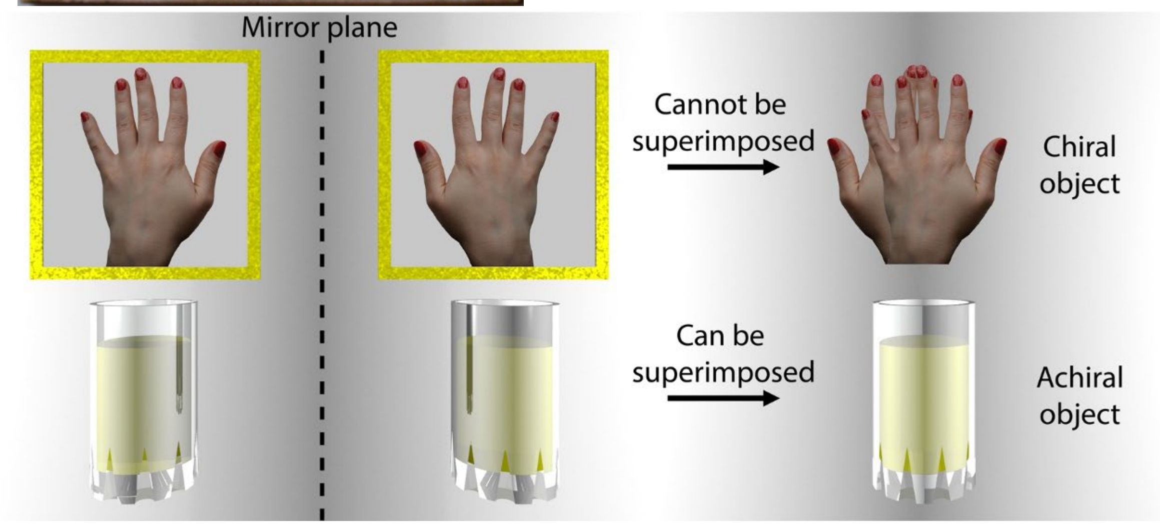
But what is chirality?

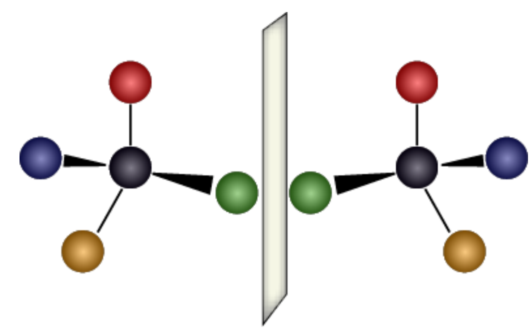
J.B. Pendry, “A chiral route to negative refraction,” Science **306**, 1353 (2004).

Chirality and the mirror

In his Baltimore Lectures on Molecular Dynamics and the wave theory of light, Lord Kelvin defined chirality as follows: "I call any geometric figure, or group of points, chiral, and say it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself“.

Lord Kelvin, in Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light, Clay and Sons: London, 1904, p.449.



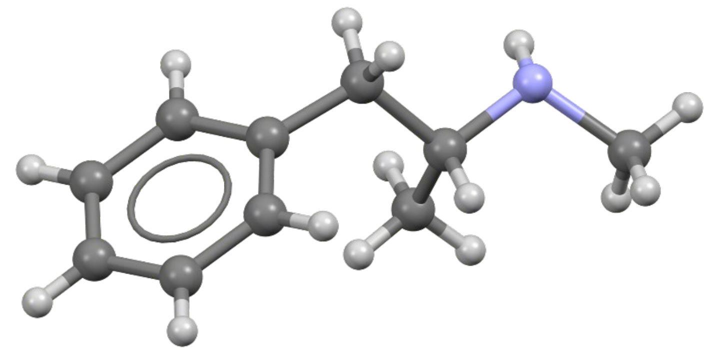


Meds & evil twins

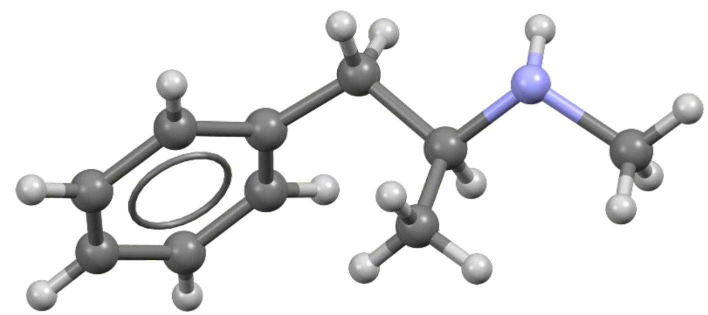
The scientist’s evil image in the mirror

In the episode “Cat’s in the Bag…” (S1E02) of the television show Breaking Bad, there is a good introduction to chirality.

R-methamphetamine



S-methamphetamine



Chirality is used as a metaphor for the transformation that the main character Walter White undergoes.