3.1.4 Conservative Fields and Potential Functions

When does the value of a TLI **not** depend on path?

Answer: For **conservative** force fields, where the work done in moving from A to B depends **only on where** A **and** B **are** and **not** on the path between them.

Examples:

Gravity is a conservative force field.
Friction is **not** conservative.

If $\mathbf{F}(\mathbf{r})$ is conservative, $\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for **all** closed paths C.

How can we determine if a field is conservative?

Theorem:

A force field $\mathbf{F}(\mathbf{r})$ is conservative in a region of space if, at all points in that region, it can be written as the gradient of a scalar field.

ie $\mathbf{F}(\mathbf{r})$ is conservative if $\mathbf{F} = \nabla \phi$

Since for **any** scalar field $\nabla \times \nabla \phi = \mathbf{0}$, this means

$\mathbf{F}(\mathbf{r})$ is conservative if $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere.

So, a conservative field $\mathbf{F}(\mathbf{r})$ is associated with a scalar field $\phi(\mathbf{r})$ through the equation $\mathbf{F} = \nabla\phi$.

$\phi(\mathbf{r})$ is called the "potential function" of $\mathbf{F}(\mathbf{r})$

For a conservative field, the tangential line integral along **any** path between 2 points is given by the **potential difference** between the points:

$$\int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_A^B \nabla\phi \cdot d\mathbf{r} = \phi_B - \phi_A$$

[Aside: to see this is true, compare with the more familiar result

$$\int_A^B \frac{dy}{dx} dx = [y]_A^B = y_B - y_A.$$

Here $\nabla\phi \cdot d\mathbf{r}$ is the component of $\nabla\phi$ along the path. Let the coordinate along the line be s. Then

$$\nabla\phi \cdot d\mathbf{r} = \frac{d\phi}{ds} ds,$$

so

$$\int_A^B \nabla\phi \cdot d\mathbf{r} = \int_A^B \frac{d\phi}{ds} ds = [\phi]_A^B = \phi_B - \phi_A.]$$

Example:

Gravity is a conservative force.

At (or near to) the surface of the Earth, the gravitational acceleration is $g \approx 9.8\,\mathrm{ms}^{-2}$, and the force attracting an object of mass m to the Earth is

$$\mathbf{F}(\mathbf{r}) = -mg\mathbf{k}.$$

[Check that $\nabla \times \mathbf{F} = \mathbf{0}$]

The potential function is

$$\phi(\mathbf{r}) = mgz$$

[Check that $\mathbf{F} = -\nabla\phi$].

So, the work done against the Earth's gravitation attraction moving an object from A to B is

$$\phi_B - \phi_A = mg(z_B - z_A),$$

irrespective of the path taken.

Finding potential functions of conservative fields

First make sure $\mathbf{F}(\mathbf{r})$ **is** conservative.
Then find $\phi(\mathbf{r})$ by "partial integration" of each component of $\mathbf{F}$.

Illustrate by example:

(i) Show that $\mathbf{F} = \mathbf{i} - z\mathbf{j} - y\mathbf{k}$ is conservative.

(ii) Find $\phi(\mathbf{r})$ such that $\mathbf{F} = \nabla\phi$.

(iii) Evaluate $\int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ along any path between $A = (2,1,1)$ and $B = (4,-1,1)$.

(i) Need to show that $\nabla \times \mathbf{F} = \mathbf{0}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & -z & -y \end{vmatrix} = \mathbf{i}(-1+1) - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

This shows $\mathbf{F} = \mathbf{i} - z\mathbf{j} - y\mathbf{k}$ is conservative.

So potential function $\phi(\mathbf{r})$ must exist.

(ii) $\mathbf{F} = \nabla \phi$, so $\begin{cases} F_x = 1 & = \dfrac{\partial \phi}{\partial x} \\ F_y = -z & = \dfrac{\partial \phi}{\partial y} \\ F_z = -y & = \dfrac{\partial \phi}{\partial z} \end{cases}$

Integrate each of these equations…

$$\frac{\partial \phi}{\partial x} = 1, \quad \therefore \quad \phi = x + c_1 + f_1(y, z)$$

where c_1 is unknown constant and
$f_1(y, z)$ is unknown function of y and z.

Similarly,

$$\frac{\partial \phi}{\partial y} = -z, \quad \therefore \quad \phi = -zy + c_2 + f_2(x, z)$$

and

$$\frac{\partial \phi}{\partial z} = -y, \quad \therefore \quad \phi = -zy + c_3 + f_3(x, y)$$

Now inspect these 3 ways of writing $\phi(\mathbf{r})$ to see that

$$\phi(\mathbf{r}) = x - zy + c$$

(iii) As $\mathbf{F}(\mathbf{r})$ is conservative, the line integral may be found from the potential difference:

$$\phi(2,1,1) = 1 + c \quad \text{and} \quad \phi(4,-1,1) = 5 + c.$$

$$\therefore \qquad \int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \phi_B - \phi_A = 4$$

[If you don't yet believe, choose a path between A & B, parameterise it, & integrate to check you get this answer.]

3.1.5 Other types of line integral

So far, only considered "tangential" line integrals

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}.$$

Other line integrals are possible and useful. eg

$$\int_C \phi(\mathbf{r})\, dr \quad \text{and} \quad \int_C \mathbf{F}(\mathbf{r}) \times d\mathbf{r}$$

(where $dr = |d\mathbf{r}|$)

3.2 Surface integrals

Integrals over a surface S include things like

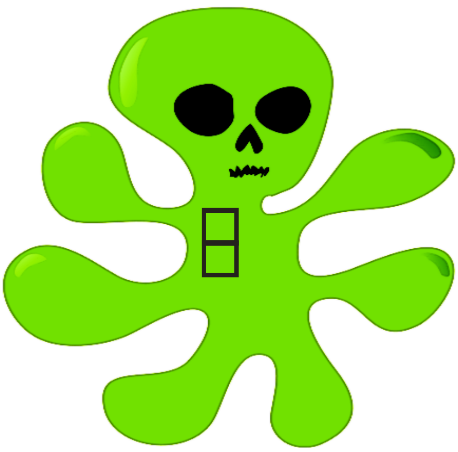
$$\int_S \psi(\mathbf{r})\,dS, \quad \int_S \mathbf{F}(\mathbf{r})\,dS \quad \int_S \psi(\mathbf{r})\,d\mathbf{S}, \quad \int_S \mathbf{F}(\mathbf{r})\cdot d\mathbf{S}.$$

Start with $\int_S \psi(\mathbf{r})\,dS$. When does this arise?

Eg suppose we have a charged 2-d plate, area S, in the xy plane, with charge density $\sigma(x,y)$. What is the total charge Q on the plate?

(a) σ constant: $Q = \sigma S$

(b) $\sigma = \sigma(x,y)$ and / or S an awkward shape:



Divide S into infinitesimal patches of area dS. In one patch, $dQ = \sigma(x,y)dS$.

Add up charge on all patches:

$$Q = \int_S \sigma dS$$

How do we evaluate $Q = \int_S \sigma dS$?

In this unit, all surfaces will have 1 coordinate fixed.

In Cartesians, x, y or z will be fixed. Then

$$dS = dydz \quad \text{or} \quad dS = dxdz \quad \text{or} \quad dS = dxdy$$