

University of Bath
Department of Physics

Year 3
PH30030 – Quantum Mechanics

Friday, 26th January 2018, 09:30 to 11:30

Answer ALL questions

The only calculators that may be used are those supplied by the University.

*Please fill in your name and sign the section on the right of your answer book,
peel away adhesive strip and seal.*

Take care to enter the correct candidate number as detailed on your desk label.

**CANDIDATES MUST NOT TURN OVER THE PAGE
AND READ THE EXAMINATION PAPER UNTIL THE
CHIEF INVIGILATOR GIVES PERMISSION TO DO SO.**

1. The wave function of a particle in a one-dimensional infinite potential well of width a is given by

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}} \phi_1(x) + c_2 \phi_2(x) + c_3 \phi_3(x),$$

where ϕ_1 , ϕ_2 and ϕ_3 are the orthonormal eigenfunctions of the total energy operator, \hat{H} , corresponding to eigenvalues $E_1 = \frac{\hbar^2 \pi^2}{2ma^2}$, $E_2 = 4 \frac{\hbar^2 \pi^2}{2ma^2}$, $E_3 = 9 \frac{\hbar^2 \pi^2}{2ma^2}$, respectively.

- (a) It has been determined experimentally that the probability of obtaining the energy

E_2 is $P(E_2) = \frac{1}{4}$. Determine the coefficients c_2 and c_3 (assume that the phases of these coefficients are zero). (3)

- (b) Calculate the expectation value $\langle \hat{H} \rangle$. (3)

- (c) What is the form of the wave function $\psi(x, t)$ at a later time t ? (2)

2. (a) Using the expressions from classical mechanics, derive the operators describing the Cartesian components \hat{L}_x , \hat{L}_y and \hat{L}_z , of the orbital angular momentum. (3)

- (b) With the help of the relation $[\hat{z}, \hat{p}_z] = i\hbar$, find the commutation relation for \hat{L}_x and \hat{L}_y . (4)

- (c) Briefly discuss the consequences of this commutation relation for measuring the x - and y - components of the orbital angular momentum of a particle. (1)

3. An operator \hat{A} has normalised eigenfunctions ϕ_1 and ϕ_2 with corresponding distinct eigenvalues α_1 and α_2 . A second operator \hat{B} has two normalised eigenfunctions χ_1 and χ_2 with corresponding distinct eigenvalues β_1 and β_2 . The eigenfunctions of the two operators are related by

$$\phi_1 = \sqrt{\frac{1}{10}} (3\chi_1 + \chi_2),$$

$$\phi_2 = \sqrt{\frac{1}{10}} (\chi_1 - 3\chi_2).$$

- (a) Given that χ_1 and χ_2 are normalised and orthogonal to each other, show that both ϕ_1 and ϕ_2 are also normalised and orthogonal to each other. (4)
- (b) A particle is prepared in a state

$$\psi = \frac{\sqrt{3}}{2} \phi_1 + \frac{1}{2} \phi_2.$$

Two measurements are then performed: first, of the observable associated with the operator \hat{A} and after that of the observable associated with the operator \hat{B} . What is the probability of obtaining β_1 in the second measurement? (4)

4. Consider a particle with spin $s = 1$.

(a) What are the possible outcomes of a measurement of

i. the z -component of spin; (1)

ii. the y -component of spin? (1)

(b) The matrix form of the x -component of spin $s = 1$ operator is

$$\hat{I}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Find the normalised eigenvector corresponding to the eigenvalue $-\hbar$ of the x -

component of spin. (2)

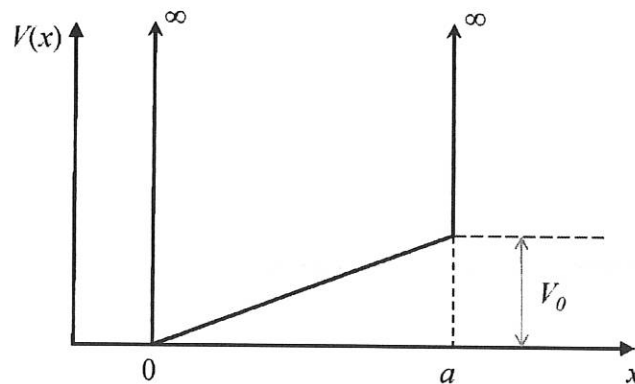
(c) The spin-part of the wave function of a particle with spin $s=1$ is described by the state

$$|\sigma\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}.$$

What is the probability of measuring $-\hbar$ in a measurement of the x -component of

spin? (2)

5. A particle of mass m is confined in a one-dimensional infinite square well of width a , where its normalised eigenfunctions are given by $\phi_{0n}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$ and the corresponding eigenvalues are given by $E_{0n} = n^2 \frac{\pi^2 \hbar^2}{2ma^2}$. Employ first order perturbation theory to estimate the ground state energy if the infinite square well is perturbed by the potential $V(x) = V_0 \frac{x}{a}$ for $0 < x < a$, as shown in the figure. (5)



Note: $\int_0^a dx x \sin^2\left(\frac{m\pi x}{a}\right) = \frac{a^2}{4}$

6. (a) An electron in an unperturbed system described by the Hamiltonian \hat{H}_0 has two orthonormal degenerate states $|\phi_{01}\rangle$ and $|\phi_{02}\rangle$ of energy E_0 . A perturbation \hat{H}' is applied to the system, and solutions to the Schrödinger equation

$(\hat{H}_0 + \hat{H}')|\phi\rangle = E|\phi\rangle$ in the form $|\phi\rangle = a_1|\phi_{01}\rangle + a_2|\phi_{02}\rangle$ are searched for, where a_1 and a_2 are constants. Show that the governing equation can be written as

$$\begin{pmatrix} (E_0 + \hat{H}'_{11}) - E & H'_{12} \\ H'_{21} & (E_0 + \hat{H}'_{22}) - E \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

where $\hat{H}'_{\alpha\beta} = \langle \phi_{0\alpha} | \hat{H}' | \phi_{0\beta} \rangle$. (8)

- (b) Assume that the degenerate states are given by the eigenstates of the spin operator

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ i.e., } |\phi_{01}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |\phi_{02}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \text{ If a magnetic field } \underline{B} = (0, 0, B)$$

is applied to the system, a perturbation of the form

$$\hat{H}' = \frac{e}{mc} \underline{B} \cdot \underline{\hat{S}}$$

is applied where e is the elementary charge, m is electron mass, c is the speed of light, and $\underline{\hat{S}}$ is the spin operator with Cartesian components $\underline{\hat{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$.

- (i) Write an expression for the perturbation in terms of a suitable 2×2 matrix. (1)

- (ii) Hence, evaluate the matrix elements \hat{H}'_{11} , \hat{H}'_{12} , \hat{H}'_{21} and \hat{H}'_{22} . (3)

- (iii) Use the governing equation to find the energy levels of each of the perturbed states. Comment on the effect of the magnetic field. (4)

7. During the performance of a spectroscopic experiment, a hydrogen atom is perturbed by the application of a time dependent electric field. Transitions between initial states i and final states f are observed with a probability $W \propto |H'_{fi}|^2$ where the quantum numbers

$m_i = m_f = 0$ such that $H'_{fi} \propto \int_0^\pi d\theta \sin \theta \cos \theta Y_{\ell_f 0}^*(\theta) Y_{\ell_i 0}(\theta)$. Three of the spherical

harmonics for $m = 0$ are given by $Y_{00} = \sqrt{\frac{1}{4\pi}}$, $Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$ and

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1).$$

- (a) Find the transition probability for the cases when:

(i) $\ell_i = 0, \quad \ell_f = 0$

(ii) $\ell_i = 1, \quad \ell_f = 0$

(iii) $\ell_i = 1, \quad \ell_f = 1$

(iv) $\ell_i = 1, \quad \ell_f = 2$

(5)

- (b) On the basis of these results, deduce the selection rule.

(1)

- (c) How will these transitions manifest themselves in a spectroscopic experiment?

(3)

Note: $\int dx \sin(ax) \cos^m(ax) = -\frac{1}{(m+1)a} \cos^{m+1}(ax)$

(MMK/PSS)

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FUNDAMENTAL CONSTANTS

Note: Numerical values have been rounded to four significant figures.

<u>Quantity</u>	<u>Symbol</u>	<u>Value</u>	<u>Unit</u>	<u>Dimensions</u>
Atomic mass unit	u	1.661×10^{-27}	kg	M
Avogadro constant	N_A	6.022×10^{23}	mol^{-1}	
Bohr magneton ($e\hbar/2m_e$)	μ_B	9.274×10^{-24}	JT^{-1}	I L^2
Bohr radius ($4\pi\hbar^2/\mu_0 c^2 e^2 m_e$)	a_0	5.292×10^{-11}	m	L
Boltzmann constant	k	1.381×10^{-23}	J K^{-1}	$\text{ML}^2\text{T}^{-2}\theta^{-1}$
Charge of electron (magnitude)	e	1.602×10^{-19}	C	I T
Charge (magnitude)/rest mass ratio (electron)	e/m_e	1.759×10^{11}	C kg^{-1}	$\text{I M}^{-1}\text{T}$
Fine-structure constant ($\mu_0 c e^2/2\hbar$)	α	7.292×10^{-3}		
	$1/\alpha$	137.0		
Gravitational constant	G	6.672×10^{-11}	$\text{Nm}^2 \text{kg}^{-2}$	$\text{M}^{-1} \text{L}^3 \text{T}^{-2}$
Mass ratio, m_p/m_e	m_p/m_e	1836		
Molar gas constant	R	8.314	$\text{J mol}^{-1}\text{K}^{-1}$	$\text{ML}^2\text{T}^{-2}\theta^{-1}$
Molar volume (ideal gas, STP)	V_m	2.241×10^{-2}	m^3	L^3
Permeability of vacuum	μ_0	$4\pi \times 10^{-7}$	Hm^{-1}	$\text{I}^2\text{MLT}^{-2}$
Permittivity of vacuum ($1/\mu_0 c^2$)	ϵ_0	8.854×10^{-12}	Fm^{-1}	$\text{I}^2\text{M}^{-1}\text{L}^{-3}\text{T}^4$
	$4\pi\epsilon_0$	1.113×10^{-10}	Fm^{-1}	$\text{I}^2\text{M}^{-1}\text{L}^{-3}\text{T}^4$
Planck constant	h	6.626×10^{-34}	Js	ML^2T^{-1}
	\hbar	1.055×10^{-34}	Js	ML^2T^{-1}
Rest mass of electron	m_e	9.110×10^{-31}	kg	M
Rest mass of proton	m_p	1.673×10^{-27}	kg	M
Speed of light in vacuum	c	2.998×10^8	ms^{-1}	LT^{-1}
Stefan-Boltzmann constant ($2\pi^5 k^4/15h^3 c^2$)	σ	5.670×10^{-8}	$\text{Wm}^{-2}\text{K}^{-4}$	$\text{MT}^{-3}\theta^{-4}$