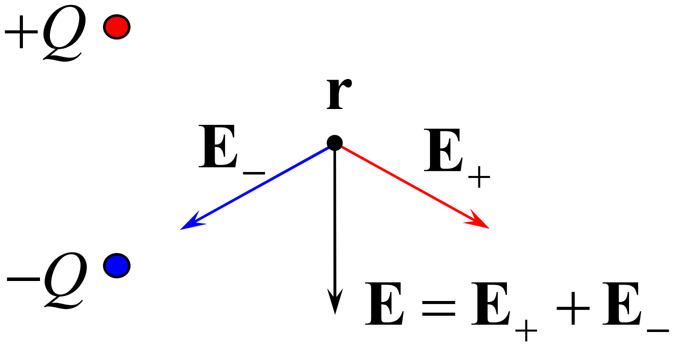
Note that the force on a charge Q_2 at $\mathbf{r}$ is

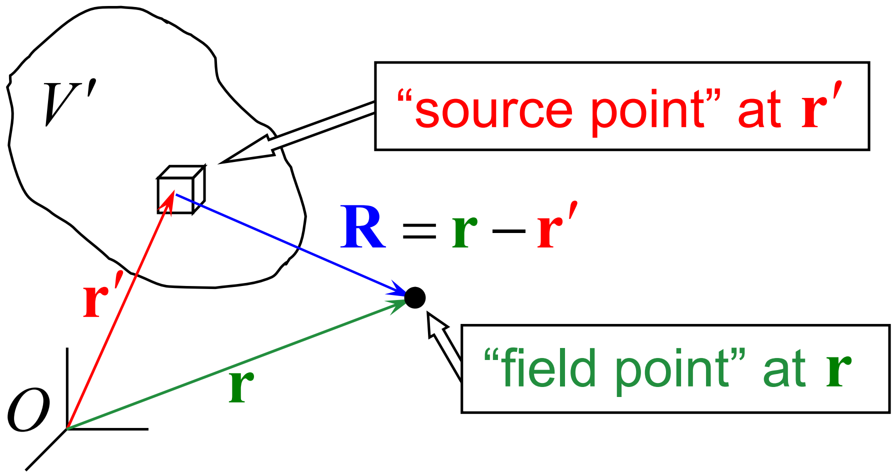
$$\mathbf{F}_e = Q_2\mathbf{E}.$$

The **Principle of Superposition** states that the $\mathbf{E}$-field due to a distribution of charges is the vector sum of the fields due to individual charges:

2 charges:



Use $\rho(\mathbf{r}')$ to describe a general charge distribution inside volume V':

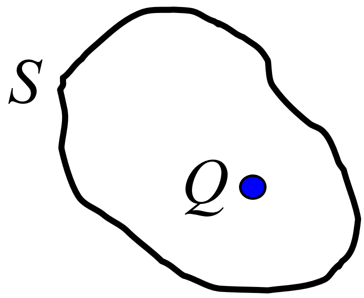


Charge in dV' is $\rho(\mathbf{r}')dV'$ so, by superposition,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\, dV'.$$

Gauss's Law

Gauss's Law, derived from Coulomb's Law, states that if charge Q is **completely** surrounded by a surface S **of any shape**, then



$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}.$$

To show this is true…

With Q at the origin, $\mathbf{E}(\mathbf{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{e}}_r$, so

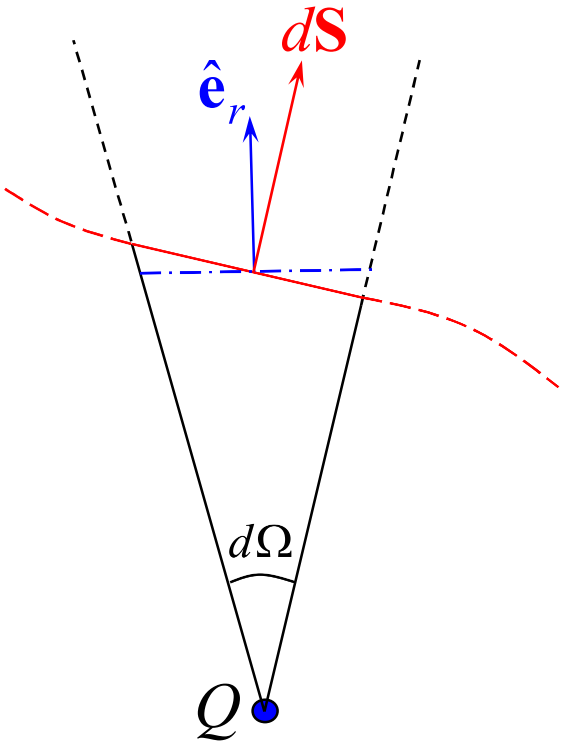
$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0}\oint_S \frac{\hat{\mathbf{e}}_r \cdot d\mathbf{S}}{r^2}.$$

For some shapes (e.g. a sphere, see PS1) this is an easy integral.

For general shapes, use **Solid Angles…**

[Aside on planar and solid angles]

Consider an arbitrary part of a general surface S:



$$\hat{\mathbf{e}}_r \cdot d\mathbf{S} = dS\cos\theta$$
$$= dS',$$
= perpendicular area.

Therefore

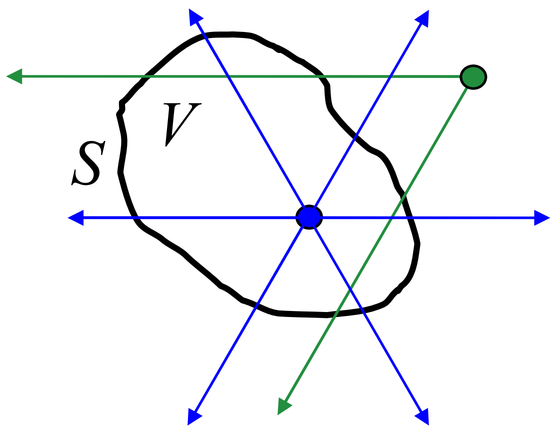
$$\frac{\hat{\mathbf{e}}_r \cdot d\mathbf{S}}{r^2} = \frac{dS'}{r^2} = d\Omega,$$

so

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{4\pi\varepsilon_0}\oint_S d\Omega = \frac{Q}{\varepsilon_0}.$$

- shape of S doesn't matter!

This result is not surprising in terms of flux lines:



Also, any charge **outside** S contributes nothing to the flux integral – all its flux lines enter then leave the enclosed volume V.

Finally, extend to a general charge distribution $\rho(\mathbf{r})$ inside V:

Total charge within $S = \int_V \rho(\mathbf{r})\, dV$, so

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho(\mathbf{r})\, dV.$$

This is **Gauss's Law in integral form.**

Now apply the **Divergence Theorem** to LHS:

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_V [\nabla \cdot \mathbf{E}]\, dV$$

$$\Rightarrow \quad \int_V [\nabla \cdot \mathbf{E}]\, dV = \int_V \left[\frac{\rho(\mathbf{r})}{\varepsilon_0} \right] dV.$$

We have made no assumption about the size and shape of S and V, or the function $\rho(\mathbf{r})$. These volume integrals can only be equal for all cases if the integrands are equal. i.e.:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

This is **Gauss's Law in differential form.**

Notes:

1. Though derived for static charges, the result is always true. This is our first Maxwell Equation.

2. Divergence measures direct sources and sinks of vector fields. So $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$ says

"The direct source of electric field is electric charge"

3. The result is true at all positions $\mathbf{r}$, not just in "V":

Where there is charge, $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$

Away from charge, $\nabla \cdot \mathbf{E} = 0$, though this **doesn't** mean $\mathbf{E} = \mathbf{0}$.