PH30030: Quantum Mechanics   Problems Sheet 2

This problems sheet covers section 2 of the course, on angular momentum. Quite a few of the problems are about checking and filling in the details of derivations from the lecture notes – these are designed to give you more practice in handling operators, and to help you understand exactly where the results come from.

1. In the lecture notes we showed that the commutator of $\hat{L}_x$ and $\hat{L}_y$ is $\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\,\hat{L}_z$. Work carefully through the derivation, and make sure that you can follow every step. In a similar way, show that $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\,\hat{L}_x$.

2. In the lectures, we stated that $\left[\hat{L}^2, \hat{L}_z\right] = \left[\hat{L}_x^2, \hat{L}_z\right] + \left[\hat{L}_y^2, \hat{L}_z\right] + \left[\hat{L}_z^2, \hat{L}_z\right] = 0$, because $\left[\hat{L}_x^2, \hat{L}_z\right]$ and $\left[\hat{L}_y^2, \hat{L}_z\right]$ cancel, and $\left[\hat{L}_z^2, \hat{L}_z\right] = 0$. The last one is easy (see question 9 on problems sheet 1) but the cancellation of the first two needs some work. Show that
$$\left[\hat{L}_x^2, \hat{L}_z\right] = -i\hbar\left(\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\right)$$
$$\left[\hat{L}_y^2, \hat{L}_z\right] = +i\hbar\left(\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\right)$$
Note: this is quite tricky, but it can be done just by using the commutation relations $\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\,\hat{L}_z$, $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\,\hat{L}_x$, $\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\,\hat{L}_y$.

3. Work carefully through the derivations in the lecture notes to confirm that
$$\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\,\hat{L}_z$$
$$\left[\hat{L}_z, \hat{L}_+\right] = \hbar\,\hat{L}_+$$
$$\left[\hat{L}_z, \hat{L}_-\right] = -\hbar\,\hat{L}_-$$

4. In the lecture notes we showed that if $\left|\phi_n\right\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue β_n, then $\hat{L}_+\left|\phi_n\right\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $\beta_n + \hbar$. We also stated that $\hat{L}_-\left|\phi_n\right\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $\beta_n - \hbar$. Confirm that this is true.

5. In the lecture notes we started from $\hat{L}_+\left|\phi_{\max}\right\rangle = 0$ to show that $\alpha = \beta_{\max}\left(\beta_{\max} + \hbar\right)$ (see the notes for the meaning of the symbols). Starting from $\hat{L}_-\left|\phi_{\min}\right\rangle = 0$, confirm that $\alpha = \beta_{\min}\left(\beta_{\min} - \hbar\right)$.

6. By following a similar procedure to that we used in the lecture notes for $\hat{L}_z$, it can be shown that $\hat{L}_x$ and $\hat{L}_y$ can be expressed in spherical polars as
$$\hat{L}_x = i\hbar\left(\sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_y = i\hbar\left(-\cos\phi\,\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\,\frac{\partial}{\partial\phi}\right)$$

Show that the ladder operators $\hat{L}_+$ and $\hat{L}_-$ can be expressed in spherical polars as

$$\hat{L}_+ = \hbar \exp(i\phi)\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_- = \hbar \exp(-i\phi)\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

7. Use the results from question 6 to confirm that the ladder operator $\hat{L}_+$ has the expected effect on the eigenfunctions of angular momentum for $\ell = 1$, i.e.,

   a) $\hat{L}_+ |Y_{1-1}\rangle \propto |Y_{10}\rangle$ b) $\hat{L}_+ |Y_{10}\rangle \propto |Y_{11}\rangle$ c) $\hat{L}_+ |Y_{11}\rangle = 0$

   Note: the proportional sign is needed here because the operation of the ladder operators leads to eigenfunctions that are not normalised.

8. In the lecture notes we showed that $\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$. Show, using the Pauli matrices given in the notes, that $\left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x$ and $\left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$.

9. Confirm that the eigenvectors given in the lecture notes for $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ are indeed eigenvectors with the given eigenvalues. Confirm also that, for each component of spin, the two eigenvectors are normalised and orthogonal to each other.

10. In the first section of the course we saw that a wavefunction $|\psi\rangle$ can be expanded in terms of eigenfunctions $|\phi_n\rangle$ as $|\psi\rangle = \sum_n c_n |\phi_n\rangle$, with the coefficients given by $c_m = \langle\phi_m|\psi\rangle$. The probability of measuring the eigenvalue associated with $|\phi_m\rangle$ is $|c_m|^2$. The same ideas work with spin. A general wavefunction can be expressed as $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ where a and b are complex numbers and, to ensure that $|\psi\rangle$ is normalised, we require $|a|^2 + |b|^2 = 1$. We can write $|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$, where $|\phi_1\rangle$ and $|\phi_2\rangle$ are the eigenvectors of one of the spin components, with $c_1 = \langle\phi_1|\psi\rangle$ and $c_2 = \langle\phi_2|\psi\rangle$.

    a) Express $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ in terms of the eigenvectors of $\hat{S}_x$. If a measurement is made of the x component of spin angular momentum, what are the probabilities of measuring $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$?

    b) Repeat a) for both $\hat{S}_y$ and $\hat{S}_z$.