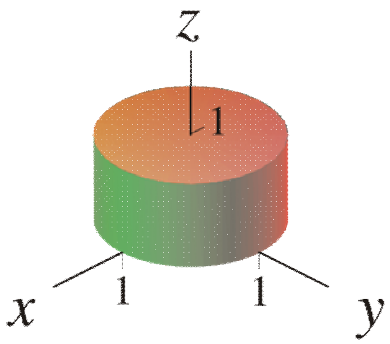
Example of the use of Gauss' Divergence Theorem

Recall worked example:



$$\mathbf{F} = \rho^2 z \hat{\mathbf{e}}_z .$$

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}.$$

According to GDT, should get $\int_V \nabla \cdot \mathbf{F} \, dV = \frac{\pi}{2}$.

Check: $\nabla \cdot \mathbf{F} = \nabla \cdot \left(\rho^2 z \hat{\mathbf{e}}_z\right) = \frac{\partial}{\partial z}(\rho^2 z) = \rho^2$

(using formula for divergence in cylindrical polars)

We also have $dV = \rho \, d\rho \, d\phi \, dz$ in cylindrical polars, and ranges

$$0 \le \rho \le 1 \qquad 0 \le \phi \le 2\pi \qquad 0 \le z \le 1$$

in this case. Therefore,

$$\int_V \nabla \cdot \mathbf{F} \, dV = \int_{z=0}^{1} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{1} \rho^3 \, d\rho \, d\phi \, dz$$

$$= 1 \times 2\pi \times \left[\frac{\rho^4}{4}\right]_0^1 = \frac{\pi}{2}$$

Second example:

Evaluate $I = \oint_S \mathbf{r} \cdot d\mathbf{S}$ where S encloses some volume V.

Using Divergence Theorem, $I = \int_V \nabla \cdot \mathbf{r}\, dV$.

$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$, so $I = \int_V 3\, dV = 3V$.

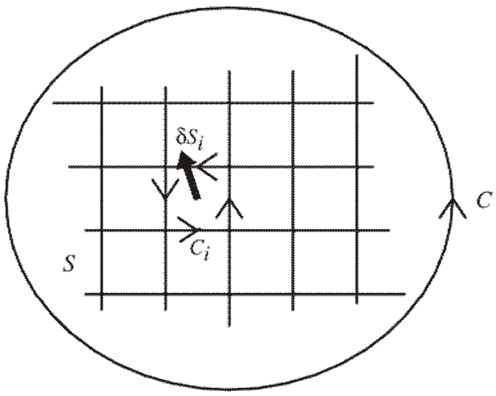
7.2 Stokes' Theorem

If $\mathbf{F}$ is a (well behaved) vector field, in a region containing a surface S bounded by the closed curve C, then

$$\int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Proof is similar to that of the divergence theorem…

Divide S into small elemental surfaces $d\mathbf{S}_i = \hat{\mathbf{n}}_i dS_i$ bounded by curves C_i.



From our previous derivation of $\operatorname{curl} \mathbf{F}$,

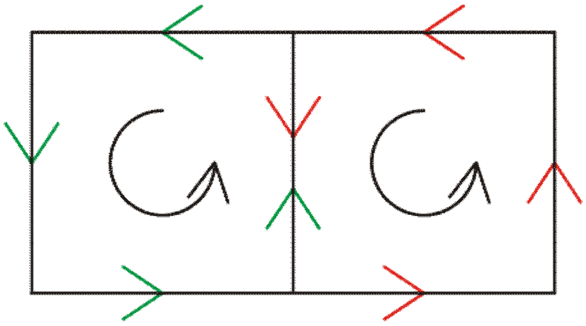
$$\operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}_i \simeq \frac{1}{\delta S_i} \oint_{C_i} \mathbf{F} \cdot d\mathbf{r} \quad \text{or} \quad \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}_i \delta S_i \simeq \oint_{C_i} \mathbf{F} \cdot d\mathbf{r}$$

(correct to leading order, and exact in limit of vanishing δS_i).

Now add the contributions from all the elements:

$$\sum_i \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}_i \delta S_i = \sum_i \operatorname{curl} \mathbf{F} \cdot \delta \mathbf{S}_i \xrightarrow[\delta S_i \to 0]{} \int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

= sum of circulation around all the curves



But flow around each edge of C_i is cancelled by equal & opposite contribution from neighbouring element, *unless the edge forms part of the outer boundary* C.

Therefore

$$\text{sum of circulations} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$\text{and} \quad \int_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Example

Show that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 2A$ where A is the area in the xy plane enclosed by curve C and $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$.

Can’t show directly. Use Stokes’ theorem:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = +2\mathbf{k}, \text{ so}$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_S 2\mathbf{k} \cdot dxdy\mathbf{k} = 2\int_S dxdy = 2A$$

7.3 Alternate forms

Let vector field $\mathbf{F} = \mathbf{a}\psi$ where $\mathbf{a}$ is a non-zero constant vector and $\psi(\mathbf{r})$ a scalar field. Then

$$\nabla \cdot \mathbf{F} = \nabla \cdot (\mathbf{a}\psi) = (\underbrace{\nabla \cdot \mathbf{a}}_{=0})\psi + \mathbf{a} \cdot (\nabla \psi),$$

so the Divergence Theorem becomes

$$\begin{aligned}\int_V \nabla \cdot \mathbf{F}\, dV &= \mathbf{a} \cdot \int_V \nabla \psi\, dV \\ &= \oint_S \mathbf{F} \cdot d\mathbf{S} = \mathbf{a} \cdot \oint_S \psi\, d\mathbf{S}.\end{aligned}$$

For this to be true for arbitrary $\mathbf{a}$ we must have

$$\int_V \nabla \psi\, dV = \oint_S \psi\, d\mathbf{S}$$ - alternate form of Divergence Theorem.

Exactly the same trick (writing $\mathbf{F} = \mathbf{a}\psi$) yields the

Alternate form of Stokes' Theorem:

$$-\int_S \nabla \psi \times d\mathbf{S} = \oint_C \psi\, d\mathbf{r}$$

7.4 Green's Theorems

1st: $\int_V \left[\varphi \nabla^2 \psi + (\nabla \varphi) \cdot (\nabla \psi) \right] dV = \oint_S \varphi \nabla \psi \cdot d\mathbf{S}$

2nd: $\int_V \left[\varphi \nabla^2 \psi - \psi \nabla^2 \varphi \right] dV = \oint_S \left(\varphi \nabla \psi - \psi \nabla \varphi \right) \cdot d\mathbf{S}$

S encloses V.

Proof of GFT:

$\nabla \cdot (\varphi \nabla \psi) = (\nabla \varphi) \cdot (\nabla \psi) + \varphi \nabla^2 \psi$ - vector identity.

Integrate both sides over volume

$$LHS \to \int_V \nabla \cdot (\varphi \nabla \psi) \, dV = \oint_S \varphi \nabla \psi \cdot d\mathbf{S} \quad \text{(Div theorem)}$$

$$RHS \to \int_V \left[\varphi \nabla^2 \psi + (\nabla \varphi) \cdot (\nabla \psi) \right] dV$$

Equating these gives Green's First Theorem.

Proof of GST:

Swap ψ and φ in GFT and subtract from original version.

[Note analogy with

$$\int_a^b u \frac{d^2 v}{dx^2} dx = \left[u \frac{dv}{dx} \right]_a^b - \int_a^b \frac{du}{dx} \frac{dv}{dx} dx.$$

Swap u and v and subtract:

$$\int_a^b \left[u \frac{d^2 v}{dx^2} - v \frac{d^2 u}{dx^2} \right] dx = \left[u \frac{dv}{dx} - v \frac{du}{dx} \right]_a^b$$]

7.5 Divergence Theorem & Equations of Science

An important application of Vector Integral Theorems is in deriving equations describing how quantities vary, when conservation laws require a balance between flow through a surface and changes within a region.

Example

It is an empirical fact (first noted by J.B. Fourier) that the flow of heat across a surface is proportional to the temperature gradient

$$\rightarrow \text{heatflow} = -\kappa \nabla T(\mathbf{r}, t).$$

Constant κ is called the thermal conductivity.

It is also an empirical fact that the rate of change in heat content of a volume is proportional to the rate of change in temperature.

$$\rightarrow \text{rate of increase in heat content} = \rho C \frac{\partial T}{\partial t}$$

ρ is the density and C the specific heat capacity.

Conservation of energy for a volume V requires the rate of heat generation ($P(\mathbf{r})$ per unit volume), minus the rate of heat loss through the surrounding surface, equals the rate of increase in heat content.

$$\int_V P(\mathbf{r})\,dV + \oint_S \kappa\nabla T\cdot d\mathbf{S} = \int_V \rho\, C\frac{\partial T}{\partial t}\,dV.$$

A governing equation, but not nice. Use divergence theorem to convert the second integral:

$$\oint_S \kappa\nabla T\cdot d\mathbf{S} = \int_V \nabla\cdot\left(\kappa\nabla T\right)dV$$

Substituting and combining,

$$\int_V \left[P + \nabla\cdot\left(\kappa\nabla T\right) - \rho C\frac{\partial T}{\partial t}\right]dV = 0$$

We have not specified volume V. This expression must $\therefore$ be true for any V. This is only possible if

$$\nabla\cdot\left(\kappa\nabla T\right) = \rho C\frac{\partial T}{\partial t} - P$$

which is the diffusion equation in the presence of a source. Solving this equation with suitable boundary conditions (easier than integral equation) allows us to predict the temperature in space and time.

This example shows the role of the Divergence Theorem in transforming experimental observations into differential equations. Similar analysis applies to other situations such as diffusion of a chemical.