2. Differentiation of Scalar and Vector Fields

So far, we have looked at vectors which depend on one scalar, such as $\mathbf{r}(t)$, the position vector of an object as a function of time.

Now we consider physical quantities which vary with **position**. These quantities are called **fields** and we can have both **scalar** and **vector** fields.

Scalar fields

Consider a scalar quantity ϕ which depends on position in space. We write this as

$$\phi(\mathbf{r}) \quad \text{or} \quad \phi(x, y, z)$$

ϕ is a **scalar field**.

Examples:

Temperature in a room	$T(\mathbf{r})$ or $T(x, y, z)$	3d
Air pressure at Earth's surface	$P(\mathbf{r})$ or $P(x, y)$	2d
Height of land above sea level	$h(\mathbf{r})$ or $h(x, y)$	2d
Electric potential	$V(\mathbf{r})$ or $V(x, y, z)$	3d

Scalar fields are often represented by **contour maps** which show curves (for 2d fields) or surfaces (3d) which joint points in space that have the same value of the scalar function.

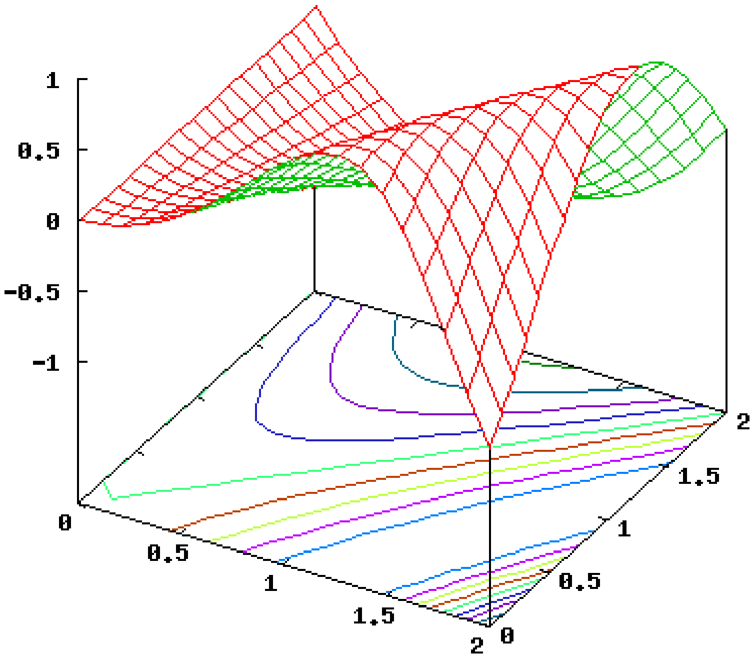
Visualisation is relatively easy for 2d scalar fields (eg weather maps, contour maps) but much trickier for 3d scalar fields.

Example

$$\phi = \sin(x^2 - xy)$$

Surface plot - perspective view of $\phi(x, y)$

Contour plot - contours join points (x,y) of equal ϕ



Vector fields

Consider a vector quantity $\mathbf{a}$ which depends on position in space. Both the **magnitude** and **direction** of $\mathbf{a}$ may vary. We write

$$\mathbf{a}(\mathbf{r}) \quad \text{or} \quad \mathbf{a}(x, y, z) \text{ or}$$

$$a_x(\mathbf{r})\mathbf{i} + a_y(\mathbf{r})\mathbf{j} + a_z(\mathbf{r})\mathbf{k} \quad \text{or}$$

$$a_x(x, y, z)\mathbf{i} + a_y(x, y, z)\mathbf{j} + a_z(x, y, z)\mathbf{k}$$

$\mathbf{a}$ is a **vector field**.

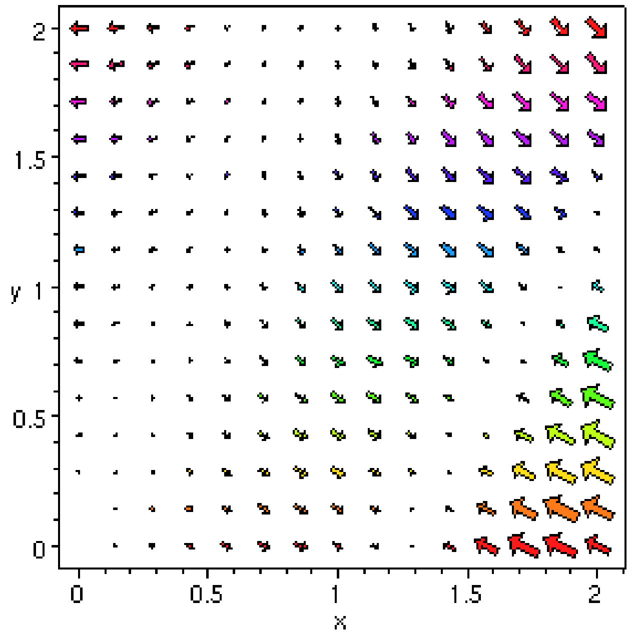
Examples:

Wind velocity	$\mathbf{v}(\mathbf{r})$ or $\mathbf{v}(x, y, z)$	3d
Electric field	$\mathbf{E}(\mathbf{r})$ or $\mathbf{E}(x, y, z)$	3d
Magnetic field	$\mathbf{B}(\mathbf{r})$ or $\mathbf{B}(x, y, z)$	3d

Visualisation of vector fields is quite tricky – arrows and flux lines can sometimes help:

Example

$$\mathbf{F} = (2x - y)\cos(x^2 - xy)\,\mathbf{i} - x\cos(x^2 - xy)\,\mathbf{j}$$



Size of arrow indicates magnitude of $\mathbf{F}$

Flux lines follow direction of arrows

Differentiation of Scalar and Vector Fields

Now that our physical quantities depend on more than one variable, we will have to use **partial differentiation**.

We get

$$\frac{\partial \phi}{\partial x}, \quad \frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial z}$$ for a scalar field, and

$$\frac{\partial \mathbf{a}}{\partial x}, \quad \frac{\partial \mathbf{a}}{\partial y}, \quad \frac{\partial \mathbf{a}}{\partial z}$$ for a vector field, where

$$\frac{\partial \mathbf{a}}{\partial x} = \frac{\partial a_x}{\partial x}\mathbf{i} + \frac{\partial a_y}{\partial x}\mathbf{j} + \frac{\partial a_z}{\partial x}\mathbf{k}$$

$$\frac{\partial \mathbf{a}}{\partial y} = \frac{\partial a_x}{\partial y}\mathbf{i} + \frac{\partial a_y}{\partial y}\mathbf{j} + \frac{\partial a_z}{\partial y}\mathbf{k}$$

$$\frac{\partial \mathbf{a}}{\partial z} = \frac{\partial a_x}{\partial z}\mathbf{i} + \frac{\partial a_y}{\partial z}\mathbf{j} + \frac{\partial a_z}{\partial z}\mathbf{k}$$

Note that each component of $\mathbf{a}$ depends on x, y and z in general.

2.1 Gradient of a Scalar Field

Consider a scalar field $\phi(\mathbf{r}) = \phi(x, y, z)$.

What is the change in ϕ when we move a small distance $\delta\mathbf{r}$?

Change will depend both on **magnitude** and **direction** of $\delta\mathbf{r}$.

(Think about a contour map.)

We write

$$\delta\mathbf{r} = \delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k} \quad ,$$

so the change in ϕ is

$$\delta\phi = \phi(x + \delta x, y + \delta y, z + \delta z) - \phi(x, y, z).$$

Definition of partial derivative:

$$\frac{\partial\phi}{\partial x} = \lim_{\delta x \to 0}\left[\frac{\phi(x + \delta x, y, z) - \phi(x, y, z)}{\delta x}\right],$$

so $\phi(x + \delta x, y, z) - \phi(x, y, z) \approx \dfrac{\partial\phi}{\partial x}\delta x.$

This is the change in ϕ due to a small change in x. We get a similar result due to small changes in y and z. The total change in ϕ is the sum of these:

$$\delta\phi \approx \frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z\,.$$

We now **define** a **vector** called $\mathrm{grad}\phi$ (or later, $\nabla\phi$) so that

$$\mathrm{grad}\phi \equiv \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\,.$$

[Note - $\mathrm{grad}\phi$ is a vector field.]

We now see that $\delta\phi$ can be written as:

$$\delta\phi \approx \mathrm{grad}\phi \cdot \delta\mathbf{r} \qquad (1)$$

[dot or scalar product]

If we write the small displacement $\delta\mathbf{r}$ as

$$\delta\mathbf{r} = \hat{\mathbf{s}}\,\delta s \text{ where } \begin{cases} \hat{\mathbf{s}} = \text{unit vector in direction of } \delta\mathbf{r} \\ \delta s = \text{magnitude of } \delta\mathbf{r} \end{cases}$$

then equation (1) becomes

$$\delta\phi \approx \mathrm{grad}\,\phi \cdot \hat{\mathbf{s}}\,\delta s$$

or

$$\delta\phi \approx \left|\mathrm{grad}\,\phi\right| \cos\theta\, \delta s$$

Here θ is the angle between $\mathrm{grad}\,\phi$ and $\delta\mathbf{r}$.

The DIRECTIONAL DERIVATIVE of ϕ in the direction of $\hat{\mathbf{s}}$ is

$$\frac{d\phi}{ds} = \lim_{\delta s \to 0} \frac{\delta\phi}{\delta s} = \mathrm{grad}\,\phi \cdot \hat{\mathbf{s}} = \left|\mathrm{grad}\,\phi\right| \cos\theta. \quad (2)$$

ie using $\mathrm{grad}\,\phi$ we can find the derivative of ϕ in any chosen direction.

Equation (2) shows that this gradient is maximised if $\hat{\mathbf{s}}$ is parallel to $\mathrm{grad}\,\phi$, **so**

$$\mathrm{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

always tells us the magnitude and direction of the steepest gradient of ϕ

Also, when $\hat{\mathbf{s}}$ is perpendicular to $\mathrm{grad}\phi$, $\frac{d\phi}{ds} = 0$, so ϕ is **constant** in this direction. This direction must therefore be a tangent to the local contour line (2d) or surface (3d). It follows from this that

$\mathrm{grad}\phi$ **is always normal to lines (2d) or surfaces (3d) of constant** ϕ