Chapter 2

Microscopic view: the magnetic dipole

Figure 2.1 shows the magnetic field lines (compare with electric dipole).

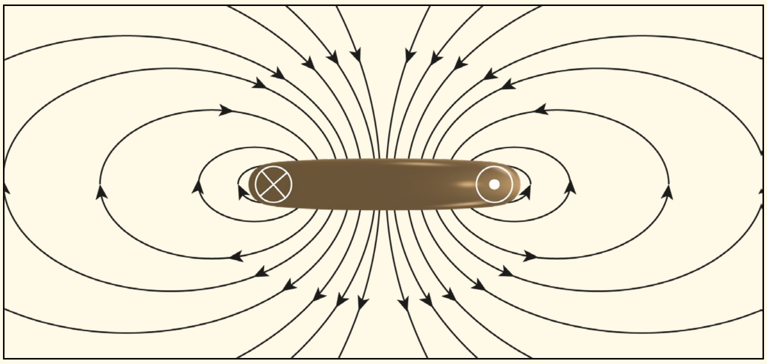


Figure 2.1. Magnetic field lines.

The magnetic moment is oriented from the south pole to the north pole.

The magnetic dipole moment is a vector with direction perpendicular to the current carrying loop; the direction is given by the right-hand-rule.

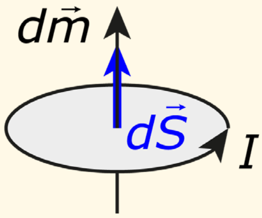


Figure 2.2. Magnetic dipole resulting from a current loop. This loop encloses an oriented surface.

A small permanent bar magnet represents a magnetic dipole. But a small coil of current-carrying wire also represents a magnetic dipole. For such a small coil, carrying a current I, the **magnetic dipole moment** is:

$$\boxed{m = I \times (\text{area})} \qquad \text{Eq.2. 1}$$

For small areas, we can write:

$$d\vec{m} = I d\vec{S} \,. \qquad \text{Eq.2. 2}$$

The magnetic dipole in an externally applied magnetic field

Consider a rectangular loop with current I in a magnetic field $\vec{B}$, that is tilted by an angle θ around the x-axis.

The Lorenz force is:

$$\vec{F} = Q\vec{v} \times \vec{B} = Q\frac{\vec{L}}{t} \times \vec{B} = \frac{Q}{t}\vec{L} \times \vec{B} = I\vec{L} \times \vec{B}. \qquad \text{Eq.2. 3}$$

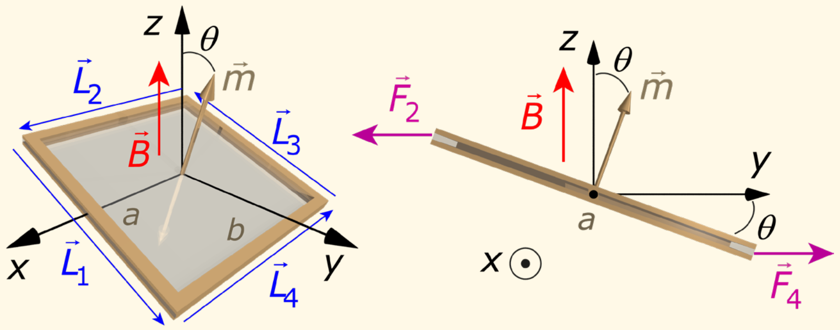


Figure 2.3. A rectangular current carrying loop in an externally applied magnetic field.

Using the numbering of the sides of the loop as above, (1) and (3) are not perpendicular to $\vec{B}$, instead they are tilted by the angle θ. Therefore, we have:

$$\begin{aligned} \vec{F}_1 &= I\vec{L}_1 \times \vec{B} = IaB\sin\left(\frac{\pi}{2} - \theta\right)\hat{x} \\ \vec{F}_3 &= I\vec{L}_3 \times \vec{B} = IaB\sin\left(\frac{\pi}{2} - \theta\right)\left(-\hat{x}\right) = -\vec{F}_1 \end{aligned} \qquad \text{Eq.2. 4}$$

So, the two forces cancel. (2) and (4) are perpendicular to $\vec{B}$, so

$$\begin{aligned} \vec{F}_2 &= I\vec{L}_2 \times \vec{B} = -IbB\hat{y} \\ \vec{F}_4 &= I\vec{L}_4 \times \vec{B} = IbB\hat{y} = -\vec{F}_2 \end{aligned}. \qquad \text{Eq.2. 5}$$

In other words, when the loop is perpendicular to $\vec{B}$ (i.e. $\theta = 0$), all the forces are in balance and $\vec{m}$ is parallel to $\vec{B}$.

However, when $\theta \neq 0$, components perpendicular to the loop act to align it perpendicular to $\vec{B}$. We can calculate the torque:

$$\vec{\tau} = \vec{r} \times \vec{F}, \qquad \text{Eq.2. 6}$$

where

$$\left\|\vec{r}\right\| = \frac{a}{2}. \qquad \text{Eq.2. 7}$$

Using $\vec{m} = I\vec{S}$, we then obtain:

$$\begin{aligned} \vec{\tau} &= \|\vec{r}\|\|\vec{F}_2\|\sin\theta + \|\vec{r}\|\|\vec{F}_4\|\sin\theta = 2\frac{a}{2}\cdot IbB\hat{y}\left(\sin\theta\right) = \\ &= IabB\left(\sin\theta\right)\hat{y} = mB\left(\sin\theta\right)\hat{y} = \vec{m}\times\vec{B} \end{aligned} . \qquad \text{Eq.2. 8}$$

And we see that the torque acts to bring $\vec{m}$ parallel to $\vec{B}$.

Macroscopic view: the magnetisation

The individual microscopic magnetic dipole moments can combine to produce a macroscopic effect – the overall magnetisation vector ($\vec{M}$).

The **magnetisation** is defined as the magnetic dipole moment per unit volume:

$$\boxed{\vec{M} = \frac{\sum\vec{m}}{V}}. \qquad \text{Eq.2. 9}$$

When $\vec{M} \neq 0$, the material is said to be magnetised. The units of $\vec{M}$ are $\frac{\text{magnetic moment}}{\text{volume}} = \frac{Am^2}{m^3} = Am^{-1}$.

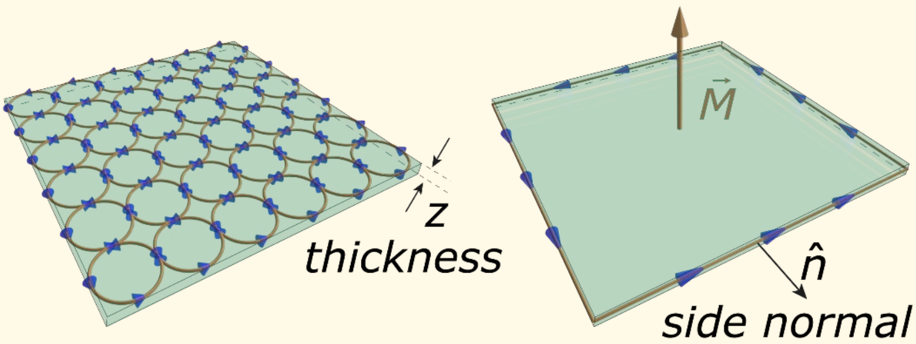


Figure 2.4. A slab of uniformly magnetised material of surface *S* and of thickness *z*.

For a slab of uniformly magnetised materials with thickness *z*, we can write:

$$M = \frac{\sum m}{V} = \frac{IS}{Sz} = \frac{I}{z}, \qquad \text{Eq.2. 10}$$

with S being the surface and where I is the side current. We can then define a **surface current density**:

$$\boxed{k_m = \frac{I}{z}}, \qquad \text{Eq.2. 11}$$

which is equivalent to the surface (bound) charge density σ_b induced by electric fields. Note that k_m has unusual dimensions (current per length, rather than current per area).
Generally:

$$\boxed{\vec{k}_m = \vec{M} \times \hat{n}}. \qquad \text{Eq.2. 12}$$

So far, we assumed uniform magnetisaiton inside the material. Next, we consider two neighbouring volume elements inside the material that is non-uniformly magnetised.

Currents flow along the edges of these volume elements. The magnetisation increases away from the origin.

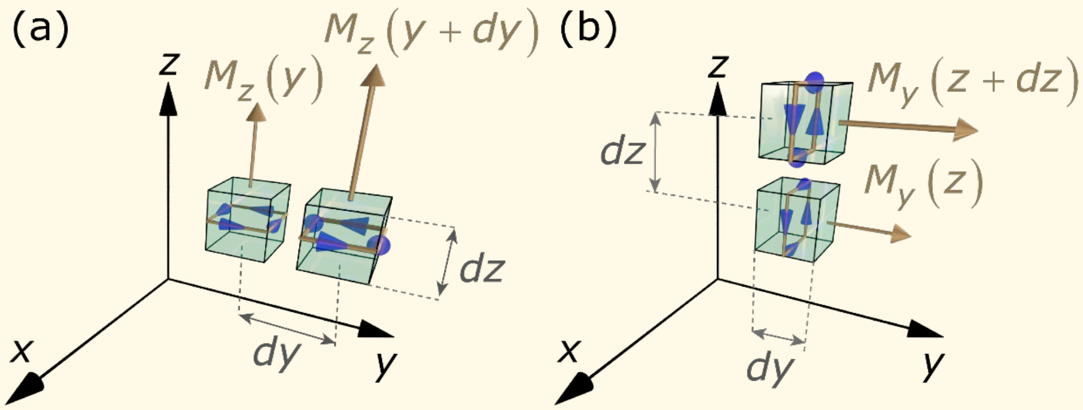


Figure 2.5 Two neighbouring volume elements inside a magnetic material that is non-uniformly magnetised.

Because we have found that $I = M \times [\text{thickness}]$, in (a), the current between the two volumes, along the *x* direction almost cancels, but not quite. It is larger in the positive *x* direction. We can write this as:

$$I_x = \left[M_z(y+dy) - M_z(y)\right]dz = \frac{\partial M_z}{\partial y}dydz, \qquad \text{Eq.2. 13}$$

since by definition of the derivative:

$$\frac{\partial M_z}{\partial y} = \frac{M_z(y+dy) - M_z(y)}{dy}. \qquad \text{Eq.2. 14}$$

Similarly, in (b), the current between the two volumes along the *x* direction is larger in the negative *x* direction. This can be written as:

$$I_x = \left[M_y(z+dz) + M_y(z)\right]dy = -\frac{\partial M_y}{\partial z}dydz. \qquad \text{Eq.2. 15}$$

By combining the two expressions for the current along the x direction:

$$I_x = \left[\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right] dydz\,, \qquad \text{Eq.2. 16}$$

we obtain the x component of the current density vector (because *dydz* is a surface):

$$\left(\vec{J}_m\right)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} = \left[\nabla \times \vec{M}\right]_x. \qquad \text{Eq.2. 17}$$

We can do the same analysis for the y and z direction and we see that the external magnetic field induces a **magnetic bound current density**:

$$\boxed{\vec{J}_m = \nabla \times \vec{M}}. \qquad \text{Eq.2. 18}$$

These currents are a response to the external magnetic field.

<u>Note:</u> In Sadiku's book, $\vec{k}_m$ and $\vec{J}_m$ are noted as $\vec{k}_b$ and $\vec{J}_b$, where the b stands for 'bound' or 'magnetisation'. But this can be confusing with the bound current density in dielectric materials. In the book by Lorrain, Corson and Lorrain, they are noted as $\vec{\alpha}_e$ and $\vec{J}_e$, where the e stands for 'equivalent'. Feynman uses $\vec{J}_{mag}$, which we shorten to $\vec{J}_m$ here.

Solenoid in vacuum

We consider a solenoid in vacuum.

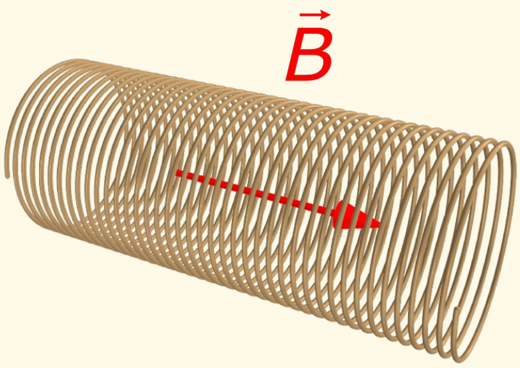


Figure 2.6. A solenoid in vacuum produces a magnetic field.

n: turns of wire per unit length
I: current through the wire
B: magnetic field inside the solenoid
L: length (assumed to be very long)

A: area

In electromagnetics, the term "**magnetic field**" is used for two distinct but closely related vector fields. One is $\vec{B}$, the **magnetic flux density**, which is measured in tesla (in SI base units: $kg \cdot s^2 \cdot A^{-1}$). We will find about the other later on.

Inside a very long solenoid, the uniform magnetic field is:

$$\boxed{B = \mu_0 nI}. \qquad \text{Eq.2. 19}$$

Consider a slice of thickness dx.

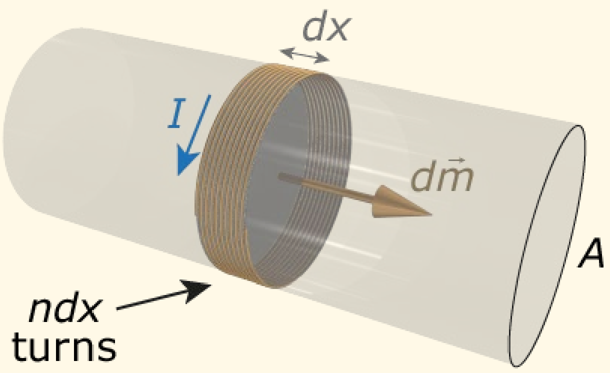


Figure 2.7. A slice of thickness *dx* of a solenoid in vacuum.

The magnetic dipole moment of the slice is:

$$d\vec{m} = (\text{current} \times \text{area}) = (ndxI)A = nI \times (\text{volume}) \qquad \text{Eq.2. 20}$$

But we can also think of the solenoid as equivalent to a magnetised object with a magnetisation M_{sol} per unit volume, that is producing the same uniform magnetic field $\vec{B}$. We can slice that magnetised object as well, and the magnetic dipole moment of that slice would be:

$$d\vec{m} = M_{sol} \times (\text{volume}). \qquad \text{Eq.2. 21}$$

From

$$d\vec{m} = M_{sol} \times (\text{volume}) = nI \times (\text{volume}), \qquad \text{Eq.2. 22}$$

it follows that

$$M_{sol} = nI. \qquad \text{Eq.2. 23}$$

And, because

$$B = \mu_0 nI, \qquad \text{Eq.2. 24}$$

we can write:

$$\boxed{B_{sol}^{vacuum} = \mu_0 M_{sol}}. \qquad \text{Eq.2. 25}$$

Solenoid with a magnetic core:

We will now consider winding a solenoid around a magnetised material with magnetisation M_{mat}. This materials magnetisation produced a field:

$$B_{mat} = \mu_0 M_{mat}. \quad \text{Eq.2. 26}$$

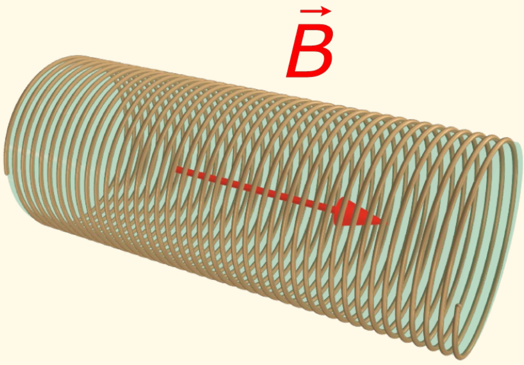


Figure 2.8. A solenoid with a magnetic core.

The current in this solenoid is flowing in such a way that the current-induced field $B_{current}$ is parallel to B_{mat}.

The two field combine to produce a single field:

$$B = B_{current} + B_{mat}. \quad \text{Eq.2. 27}$$

We can rewrite this as:

$$B = \mu_0 nI + \mu_0 M_{mat}. \quad \text{Eq.2. 28}$$

We can drop the suffix and write:

$$B = \mu_0 nI + \mu_0 M. \quad \text{Eq.2. 29}$$

The magnetic field strength H:

We define the **magnetic field strength** as $1/\mu_0$ the magnetic flux density that would exist because of the current in the electric circuit **alone**, i.e. if all the magnetisable material is **removed**.

We can write this as:

$$H = \frac{1}{\mu_0} B_{solenoid} = \frac{\mu_0 nI}{\mu_0} = nI. \quad \text{Eq.2. 30}$$

With this definition, we can now replace in:

$$B = \mu_0 nI + \mu_0 M = \mu_0 H + \mu_0 M. \quad \text{Eq.2. 31}$$

This leads to another **constitutive equation**:

$$\boxed{\vec{B} = \mu_0 \left(\vec{H} + \vec{M}\right)}. \quad \text{Eq.2. 32}$$

Magnetic susceptibility and permeability

Remember that in LIH dielectrics, we have:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \,, \qquad \text{Eq.2. 33}$$

with χ_e the electric susceptibility. Similarly, for LIH magnetic materials, we have:

$$\boxed{\vec{M} = \chi_m \vec{H}} \,, \qquad \text{Eq.2. 34}$$

where χ_m is the **magnetic susceptibility** and $\vec{H}$ is the magnetic field strength.

Remember that in LIH dielectrics, we have:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \,, \qquad \text{Eq.2. 35}$$

with ε_r the relative permittivity and $\varepsilon = \varepsilon_0 \varepsilon_r$ the electric permittivity. Similarly, for LIH magnetic materials, we have:

$$\vec{B} = \mu_0 \left(\vec{H} + \chi_m \vec{H} \right) = \mu_0 \left(1 + \chi_m \right) \vec{H} = \mu_0 \mu_r \vec{H} = \mu \vec{H} \,, \qquad \text{Eq.2. 36}$$

where μ_r is the **relative permeability** and $\boxed{\mu = \mu_0 \mu_r}$ is the **permeability**.

Linear, isotropic and homogeneous magnetic materials

A magnetic material is **linear** if the magnetic permeability and susceptibility do not change with the magnitude of the applied magnetic field.

A magnetic material is **isotropic** if the magnetic permeability and susceptibility do not change with the direction of the magnetic field in the material. In other words, all directions in the material are equivalent.

A dielectric material is **homogeneous** if the magnetic permeability and susceptibility do not change from point to point in the material. In other words, they are independent of the coordinates. We can say that all locations in the material are equivalent.

In vacuum, $\chi_m = 0$ and $\mu_r = 1$. In air, we will take $\mu_r = 1$ as well.

Classification of magnetic materials

What are the main types of magnetic material?

Magnetic materials can be classified according to their magnetic susceptibility χ_m.

In **diamagnetic** materials, the magnetic response opposes the externally applied magnetic field. For diamagnets, typically, $\chi_m \approx -10^{-5}$. Examples include gold, silver, copper, bismuth and beryllium. Superconductors form a special group of materials that are also diamagnetic, with $\chi_m \approx -1$.

In **paramagnetic** materials, the magnetic response is weak but aligned parallel with the direction of the externally applied magnetic field. For paramagnets, typically, $\chi_m \approx 10^{-3}$ to 10^{-5}. Examples of paramagnets are platinum, aluminium and manganese.

In **ferromagnetic** materials, the magnetics response is also aligned with the externally applied field but this response is very strong. For ferromagnets, typically, $\chi_m \approx 50$ to $10{,}000$. Examples of ferromagnets include iron, cobalt, nickel and many rare earth elements.

Enrichment: There are also other types of magnetic materials: ferrimagnets, antiferromagnets, helimagnets and superparamagnets.

Magnetic type	Element	χ_m
Diamagnets	Au	-2.74×10^{-6}
	Ag	-2.02×10^{-6}
	Cu	-0.77×10^{-6}
	Be	-1.85×10^{-6}
	Bi	-1.31×10^{-6}
	Ge	-0.56×10^{-6}
Paramagnets	Pt	21.04×10^{-6}
	Al	1.65×10^{-6}
	W	6.18×10^{-6}
	Mn	66.10×10^{-6}

Diamagnetism:

We start by remembering the direction of the magnetic moment that is produced by a current loop.

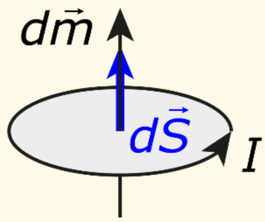


Figure 2.9. The direction of the magnetic dipole *m* with respect to the direction of the current *I* and the surface element *dS*.

Next we consider an electron circulating around a nucleus. We are going to calculate the magnetic moment and then balance the forces on the electron to see what relationships will emerge.

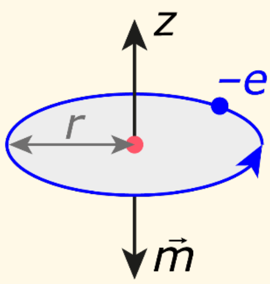


Figure 2.10. The direction of the magnetic dipole moment *m* with respect to the direction of the electron motion.

The current flows in the direction opposite to the electron's motion. We consider the current:

$$I = \frac{Q}{T} = \frac{e}{T} = \frac{ev}{2\pi r}, \qquad \text{Eq.2. 37}$$

where we have:

$$v = \frac{2\pi r}{T}, \qquad \text{Eq.2. 38}$$

which is the linear velocity of the electron (from A-Level Physics).

The magnetic dipole moment is:

$$m = I \times (\text{area}) = \frac{ev}{2\pi r}\pi r^2 = \frac{1}{2}evr\,. \qquad \text{Eq.2. 39}$$

We can include the directions:

$$\vec{m} = -\frac{1}{2}evr\hat{z}, \qquad \text{Eq.2. 40}$$

bearing in mind that on the diagram, the direction of $\vec{m}$ (obtained from the right-hand-rule) is opposite to that of $\hat{z}$.

In the absence of magnetic field, we can balance the forces on the electron: there is the Coulomb force and the centripetal force (A Level Physics), so

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m_e\frac{v^2}{r}, \qquad \text{Eq.2. 41}$$

where the right hand term is the mass of the electron (m_e) times the acceleration.

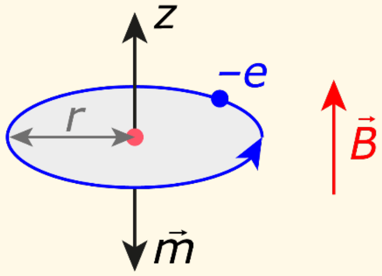


Figure 2.11. Directions of the magnetic moment *m* and the magnetic flux density *B*.

If a magnetic field is present, we need to include the Lorentz force and the velocity of the electron changes from *v* to *v'*:

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} + ev'B = m_e\frac{(v')^2}{r}. \qquad \text{Eq.2. 42}$$

We now combine both equations and obtain:

$$\frac{m_e}{r}\left[(v')^2 - v^2\right] = ev'B, \qquad \text{Eq.2. 43}$$

which requires $v' > v$. We can write that as $v' = v + \Delta v$, rewriting: $v = v' - \Delta v$. We take Δv to be really small, so that $(\Delta v)^2 \approx 0$. Then, replacing:

$$\frac{m_e}{r}\left[(v')^2 - (v'-\Delta v)^2\right] \approx \frac{m_e}{r}\left[(v')^2 - (v')^2 + 2v'\Delta v\right] = \frac{m_e}{r}2v'\Delta v$$

Eq.2. 44

and

$$\frac{m_e}{r}2v'\Delta v = ev'B. \qquad \text{Eq.2. 45}$$

It follows that:

$$\Delta v = \frac{eBr}{2m_e}. \qquad \text{Eq.2. 46}$$

We can use this result in:

$$\vec{m} = -\frac{1}{2}evr\hat{z}, \qquad \text{Eq.2. 47}$$

so that

$$\Delta\vec{m} = -\frac{1}{2}e\Delta vr\hat{z} = -\frac{1}{2}e\left(\frac{eBr}{2m_e}\right)r\hat{z} = -\frac{1}{4}\frac{e^2r^2}{m_e}B\hat{z} = -\frac{1}{4}\frac{e^2r^2}{m_e}\vec{B}.$$

Eq.2. 48

The change in magnetic dipole moment is opposite to the direction of $\vec{B}$.

Therefore, when $\vec{B}$ is added, it results in a Lorenz force that adds to the Coulomb force (that attracts the electron to the nucleus). As a result, the velocity of the electron changes and, in turn, the magnetic dipole moment is reduced by a small amount. This response is called **diamagnetic**.

Diamagnetism describes the repulsion of a material by an external magnetic field due to inducted magnetic moments opposite to the direction of the magnetic field.

[Remember Le Chatelier's principle from GCSE chemistry? When a simple system in thermodynamic equilibrium is subjected to a change in concentration, temperature, volume, or pressure, the system changes to a new equilibrium, and this change partly counteracts the applied change.]

Energy stored in a magnetic material

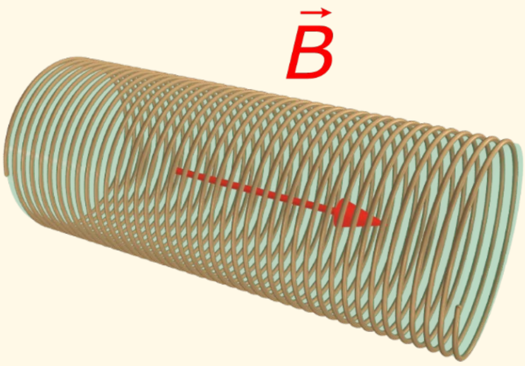


Figure 2.12. A solenoid with a magnetic core.

Self inductance is defined as the induction of a voltage in a current-carrying wire when the current in the wire itself is changing.

$$\boxed{L\left(\text{self inductance}\right)=\frac{\Phi_{m,total}\left(\text{total flux linked through circuit}\right)}{I\left(\text{current flowing in circuit}\right)}}$$

Eq.2. 49

It is measured in henry (SI units). The relative permeability of a material is then:

$$\mu_r=\frac{L_{\text{material}}}{L_{\text{vacuum}}} \qquad \text{Eq.2. 50}$$

From A Level Physics, we know that:

$$\Phi_m=\vec{B}\cdot\vec{A}. \qquad \text{Eq.2. 51}$$

For a solenoid, the vectors $\vec{B}$ and $\vec{A}$ are parallel. Moreover, if we define $n=N/l$ as the number of turns N per unit length l (do not confuse length l with current I) we can write:

$$N=nl \qquad \text{Eq.2. 52}$$

and

$$\Phi_{m,total}=N\Phi_m=NBA=nBAl. \qquad \text{Eq.2. 53}$$

Moreover, inside a very long solenoid, we saw that

$$B=\mu_0 nI. \qquad \text{Eq.2. 54}$$

Therefore, we can write:

$$\Phi_{m,total}=n\left(\mu_0 nI\right)Al=\mu_0 An^2 Il \qquad \text{Eq.2. 55}$$

It follows that the self inductance in vacuum is:

$$L_{\text{vacuum}}=\frac{\Phi_{m,total}}{I}=\mu_0 An^2 l. \qquad \text{Eq.2. 56}$$

Then from

$$\mu_r=\frac{L_{\text{material}}}{L_{\text{vacuum}}}, \qquad \text{Eq.2. 57}$$

it follows that

$$\boxed{L_{\text{material}}=\mu_0\mu_r An^2 l}. \qquad \text{Eq.2. 58}$$

The energy required to establish a current I in the solenoid is:

$$E=\frac{1}{2}LI^2. \qquad \text{Eq.2. 59}$$

This is therefore the energy stored in the magnetic field. We can replace:

$$E=\frac{1}{2}\left(\mu An^2 l\right)I^2=\frac{1}{2}\left(nI\right)\left(\mu nI\right)Al=\frac{1}{2}\left(H\right)\left(B\right)Al. \qquad \text{Eq.2. 60}$$

So, for a volume $V=A\left(\text{area}\right)\times l\left(\text{length}\right)$, the energy stored is:

$$w=\frac{1}{2}HB. \qquad \text{Eq.2. 61}$$

This equation generalises to:

$$\boxed{w = \frac{1}{2}\vec{H}\cdot\vec{B}}. \quad \text{Eq.2. 62}$$

Summary:

Electric current is a source of magnetic fields.

The magnetic dipole moment is current times area: $m = I \times (\text{area})$.

The magnetisation is the magnetic dipole moment per unit volume: $\vec{M} = \frac{\sum \vec{m}}{V}$.

When an external magnetic field is applied to a magnetic dipole moment, the torque acts to bring $\vec{m}$ parallel to $\vec{B}$.

The magnetisation induces a surface current density $\vec{k}_m = \vec{M} \times \hat{n}$ and bound current density $\vec{J}_m = \nabla \times \vec{M}$.

The magnetic flux density results from adding up the magnetic field strength and the magnetisation: $\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$.

The magnetic field strength is $1/\mu_0$ of the magnetic flux density: $\vec{H} = \frac{1}{\mu}\vec{B}$.

In LIH materials: $\vec{M} = \chi_m \vec{H}$, where $\chi_m = \mu_r - 1$.

Examples of magnetic order in magnetic materials include: diamagnetic, paramagnetic and ferromagnetic.

In diamagnetic materials, the magnetic dipole moment is opposite to the direction of $\vec{B}$.

The energy stored in a magnetic material is $w = \frac{1}{2}\vec{H}\cdot\vec{B}$.

Example question 1

In a certain homogeneous isotropic medium for which $\mu_r = 4$, the magnetic is $\vec{B} = 4\hat{x} - 3\hat{y} + 15\hat{z}$ mT. Calculate:

(a) the magnetic susceptibility χ_m,
(b) the magnetic field intensity $\vec{H}$,
(c) the magnetisation $\vec{M}$.

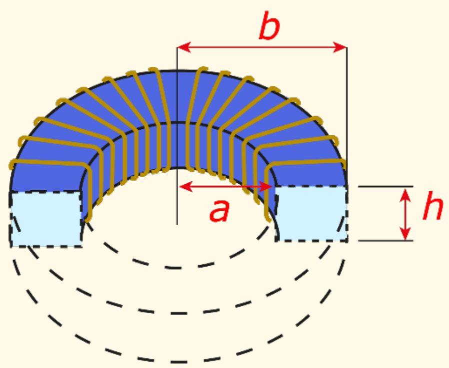
Answer:
(a) From $\mu_r = 1 + \chi_m$, we deduce that $\chi_m = \mu_r - 1 = (4) - 1 = 3$.
(b) From $\vec{B} = \mu_0 \mu_r \vec{H}$, we write

$$\vec{H} = \frac{1}{\mu_0 \mu_r}\vec{B} = \frac{1}{\mu_0 (4)}(4\hat{x} - 3\hat{y} + 15\hat{z}) = \frac{1}{\mu_0}\left(\hat{x} - \frac{3}{4}\hat{y} + \frac{15}{4}\hat{z}\right) \times 10^{-3}\,\text{Am}^{-1}.$$

(c) From $\vec{M} = \chi_m \vec{H}$, we get

$$\vec{M} = (3)\frac{1}{\mu_0}\left(\hat{x} - \frac{3}{4}\hat{y} + \frac{15}{4}\hat{z}\right) \times 10^{-3} = \frac{1}{\mu_0}\left(3\hat{x} - \frac{9}{4}\hat{y} + \frac{45}{4}\hat{z}\right) \times 10^{-3}\,\text{Am}^{-1}.$$

Example question 2



Consider a toroidal solenoid with a rectangular cross-section as shown in the figure and the total number of turns *N*. It is filled with a material with relative permeability μ_r. The current passing through the circuit is *I*. Find:
(a) the fields *B*, *H* and *M*,
(b) the energy stored in the magnetic field.

Answer: We can use Ampère's theorem and for that we need to choose a circular loop. It should be in the plane of the solenoid and inside it. (Otherwise: if smaller than *a*, no current enclosed; if larger than *b*, currents cancel and again no currents enclosed). The number of currents enclosed are set by all the wires perpendicular to *a*, so they do not depend on *h* or *r*. We can write:

$$\oint \vec{H} \cdot d\vec{L} = H 2\pi r = NI, \text{ so } H = \frac{NI}{2\pi r}, \text{ for } a < r < b.$$

We can now use $\vec{B} = \mu_0 \mu_r \vec{H}$ to find $B = \mu_0 \mu_r \dfrac{NI}{2\pi r}$.

Similarly, from $\vec{M} = \chi_m \vec{H}$, we have $M = (\mu_r - 1)\dfrac{NI}{2\pi r}$.

The energy stored can be calculated from $w = \dfrac{1}{2}\vec{H}\cdot\vec{B}$, which we integrate between a and b:

$$W = \frac{1}{2}\int_a^b r dr d\varphi h \frac{NI}{2\pi r}\mu_0\mu_r \frac{NI}{2\pi r} = $$

$$= \frac{1}{2}\mu_0\mu_r\left(\frac{NI}{2\pi}\right)^2 2\pi h\int_a^b \frac{1}{r}dr = \frac{\mu_0\mu_r h(NI)^2}{4\pi}\ln\frac{b}{a}.$$

Example question 3

The field surrounding a magnetic dipole is given by

$$\vec{B} = \alpha\left[3\left(\frac{zx}{r^5}\right)\hat{x} + 3\left(\frac{yz}{r^5}\right)\hat{y} + \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\hat{z}\right],$$ where

$r \equiv \sqrt{x^2 + y^2 + z^2}$.

(a) Write an expression for the magnetic dipole.

(b) Show that this field satisfies Gauss' law for magnetic fields.

Answer:

(a) The magnetic dipole moment is current (I) times area (A). We can write $m = I \times A$ (or $\vec{m} = I\vec{S}$, with surface S).

(b) We need to find that $\nabla\cdot\vec{B} = 0$, so we have to calculate $\nabla\cdot\vec{B}$.

$$\nabla\cdot\vec{B} = \alpha\left[3\frac{\partial}{\partial x}\left(\frac{zx}{r^5}\right) + 3\frac{\partial}{\partial y}\left(\frac{yz}{r^5}\right) + \frac{\partial}{\partial z}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\right]$$

Here, we can use $\dfrac{\partial}{\partial x}(ab) = a\dfrac{\partial b}{\partial x} + b\dfrac{\partial a}{\partial x}$ with $a = zx$ and $b = r^{-5}$.

$$\frac{\partial}{\partial x}\left(\frac{zx}{r^5}\right) = r^{-5}\frac{\partial}{\partial x}(zx) + zx\frac{\partial}{\partial x}\left(r^{-5}\right)$$

where $\dfrac{\partial}{\partial x}\left(r^{-5}\right) = \dfrac{\partial}{\partial r}\left(r^{-5}\right)\dfrac{\partial r}{\partial x} = -5r^{-6}\dfrac{\partial r}{\partial x}$.

So $\frac{\partial}{\partial x}\left(\frac{zx}{r^5}\right) = r^{-5}\frac{\partial}{\partial x}(zx) + zx\left(-5r^{-6}\frac{\partial r}{\partial x}\right) = \frac{z}{r^5} - \frac{5zx}{r^6}\frac{\partial r}{\partial x}$.

Similarly, we can calculate $\frac{\partial}{\partial y}\left(\frac{yz}{r^5}\right) = \frac{z}{r^5} - \frac{5yz}{r^6}\frac{\partial r}{\partial y}$ and

$\frac{\partial}{\partial z}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right) = \frac{6z}{r^5} - \frac{15z^2}{r^6}\frac{\partial r}{\partial z}$.

$$\nabla \cdot \vec{B} = \alpha\left[3\left(\frac{z}{r^5} - \frac{5xz}{r^6}\frac{\partial r}{\partial x}\right) + 3\left(\frac{z}{r^5} - \frac{5yz}{r^6}\frac{\partial r}{\partial y}\right) + \left(\frac{6z}{r^5} - \frac{15z^2}{r^6}\frac{\partial r}{\partial z} + \frac{3}{r^4}\frac{\partial r}{\partial z}\right)\right]$$

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15z}{r^6}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right) + \frac{3}{r^4}\frac{\partial r}{\partial z}\right]$$

Now, we need to find $\frac{\partial r}{\partial x}$.

Let us define R such that $R = x^2 + y^2 + z^2$, then $r \equiv \sqrt{R} = R^{1/2}$.

So, we can write $\frac{\partial r}{\partial x} = \frac{\partial r}{\partial R}\frac{\partial R}{\partial x}$. First $\frac{\partial r}{\partial R} = \frac{\partial}{\partial R}\left(R^{1/2}\right) = \frac{1}{2}R^{-1/2}$. Next,

$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right) = 2x$. Hence,

$$\frac{\partial r}{\partial x} = \frac{\partial r}{\partial R}\frac{\partial R}{\partial x} = \frac{1}{2}R^{-1/2}2x = xR^{-1/2} = \frac{x}{R^{1/2}} = \frac{x}{\sqrt{R}} = \frac{x}{r}.$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now can continue:

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15z}{r^6}\left(x\frac{x}{r} + y\frac{y}{r} + z\frac{z}{r}\right) + \frac{3}{r^4}\frac{z}{r}\right]$$

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15z}{r^6}\left(\frac{x^2 + y^2 + z^2}{r}\right) + \frac{3z}{r^5}\right]$$

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15z}{r^6}\left(\frac{r^2}{r}\right) + \frac{3z}{r^5}\right]$$

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15zr}{r^6} + \frac{3z}{r^5}\right]$$

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15z}{r^5} + \frac{3z}{r^5}\right] = 0$$

[This is example 1.2 from J. Peatross and M. Ware, Physics of Light and Optics, 2015 Edition]

Example question 4

A rectangular coil of area 20 cm2 carrying current of 20 A lies on the plane $2x+6y-3z=5$, such that the magnetic moment of the coil is directed away from the origin. Calculate its magnetic moment.

Answer:

The area of the loop is $S = 20\ \text{cm}^2 = 2\times10^{-3}\ \text{m}^2$.
The current is $I = 20\ \text{A}$.
The magnetic moment is therefore $\vec{m} = I\vec{S} = IS\vec{s}$, where $\vec{s}$ is the normal to the plane surface with function $f(x,y,z) = 2x+6y-3z-5$.
Given a 3D surface defined by $f(x,y,z)=0$, we have

$$\vec{s} = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{\nabla f}{\sqrt{f_x^2+f_y^2+f_z^2}}.$$

Here $\vec{s} = \dfrac{2\vec{s}_x+6\vec{s}_y-3\vec{s}_z}{\sqrt{2^2+6^2+3^2}} = \dfrac{2\vec{s}_x+6\vec{s}_y-3\vec{s}_z}{7}$, where we chose the + sign because the magnetic moment is directed away from the origin.

Therefore $\vec{m} = IS\dfrac{2\vec{s}_x+6\vec{s}_y-3\vec{s}_z}{7} = \left(\dfrac{20\times2\times10^{-3}}{7}\right)\left(2\vec{s}_x+6\vec{s}_y-3\vec{s}_z\right)$

$$\vec{m} = \left(1.14\vec{s}_x+3.42\vec{s}_y-1.71\vec{s}_z\right)\times10^{-2}\ \text{Am}^2.$$

Example question 5

A very long solenoid with a cross section of 3×3 cm has an iron core ($\mu_r = 1000$) and 3000 turns per meter. If carries a current of 200 mA. Find the following:

(a) Its self-inductance per meter.
(b) The energy per meter stored in its field.

Answer:

(a) From the formula $L_{\text{material}} = \mu_0\mu_r AN^2 l$, we can substitute:

$$L_{\text{material}} = \left(4\pi10^{-7}\right)(1000)\left(9\times10^{-4}\right)(3000)^2(1) \approx 10.18\ \text{Mm}^{-1}.$$

(b) From the formula $E = \frac{1}{2} L \cdot I^2$, we can substitute:

$E = \frac{1}{2}(10.18)(0.2)^2 \approx 0.20$ Jm^{-1}.

Example question 6

[from Sadiku] The magnetization in a cube of size a is given by $\vec{M} = \frac{k_0}{a}\left(-2y\vec{a}_x + x\vec{a}_y\right)$, where k_0 is a constant. Find $\vec{J}_m$.

Answer:

$$\vec{J}_m = \nabla \times \vec{M} = \frac{k_0}{a}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & x & 0 \end{vmatrix} = \frac{k_0}{a}\left(\frac{\partial x}{\partial x} - \frac{\partial(-2y)}{\partial y}\right)\vec{a}_z = 3\frac{k_0}{a}\vec{a}_z .$$

Example question 7

[from Sadiku] An infinitely long cylindrical conductor of radius a and permeability $\mu_0\mu_r$ is placed along the z-axis. If the conductor carries a uniformly distributed current I along $\vec{a}_z$, find the magnetisation $\vec{M}$ and the bound current density $\vec{J}_m$.

Answer:

From $\oint \vec{H} \cdot d\vec{L} = I_{\text{enclosed}}$ we obtain $H_\phi \cdot 2\pi r = \frac{\pi r^2}{\pi a^2} I$. So, $H_\phi = \frac{r}{2\pi a^2} I$.

Hence, $\vec{M} = \chi_m \vec{H} = (\mu_r - 1)\frac{r}{2\pi a^2} I\vec{a}_z$.

Next,

$$\vec{J}_m = \nabla \times \vec{M} = \frac{1}{r}\frac{\partial}{\partial r}\left(rM_\phi\right)\vec{a}_z = (\mu_r - 1)\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r^2}{2\pi a^2} I\vec{a}_z\right) = (\mu_r - 1)\frac{1}{\pi a^2} I\vec{a}_z ,$$

Where we used the general formula:

$$\nabla \times \vec{M} = \left(\frac{1}{r}\frac{\partial M_z}{\partial \phi} - \frac{\partial M_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial M_r}{\partial z} - \frac{\partial M_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rM_\phi\right) - \frac{\partial M_r}{\partial \phi}\right]\hat{z} .$$