Chapter 4: Electromagnetic waves at boundaries

Lecture 7 – Boundary conditions

We know how EM waves propagate in a bulk material. However, what happens when an EM wave encounters a boundary between two materials? As part of our discussion, we shall see how (some of) the principles of ray (geometrical) optics arise from Maxwell's equations.

We start our discussion by considering electric fields in the vicinity of a (sharp) boundary between two dielectric materials. In our treatment, we will always assume that the boundary does not contain any free charges and that no surface currents flow at the boundary.

We will be using the divergence and Stoke's theorems.

The **divergence theorem**: $\int_A \vec{E} \cdot d\vec{A} = \int_V \left(\nabla \cdot \vec{E}\right) dv$, where A is the area of a closed surface that bounds the volume V.

Stoke's theorem: $\oint_L \vec{E} \cdot d\vec{L} = \int_A \left(\nabla \times \vec{E}\right) \cdot d\vec{A}$, where L is the closed path that encloses a surface area A.

Boundary conditions for the electric flux density

What happens to the electric fields as we cross a boundary between two materials?

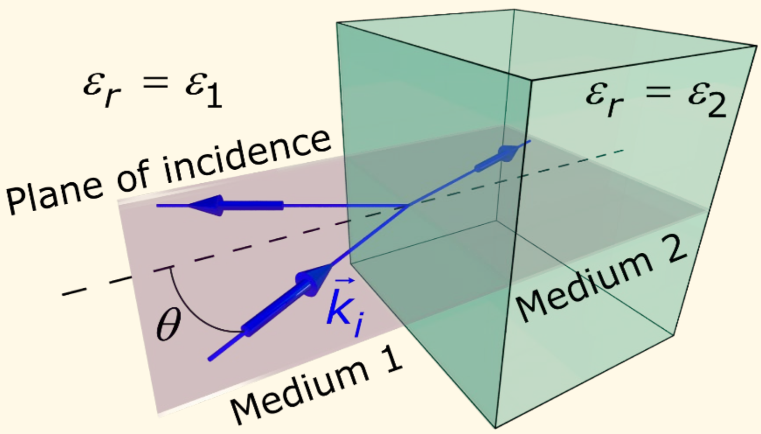


Figure 4.1. The plane of incidence and the boundary between two media.

The interface is free of charge and, in LIH materials, we have:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}\,, \qquad \text{Eq. 4. 1}$$

so we can easily switch between $\vec{D}$ and $\vec{E}$.

Since we have no free charges, $\nabla \cdot \vec{D} = \rho_f$ becomes $\nabla \cdot \vec{D} = 0$. We can now integrate both sides over any volume V:

$$\nabla \cdot \vec{D} = 0 \rightarrow \int_V \left(\nabla \cdot \vec{D}\right) dV = \int_V (0)\, dV = 0\,. \qquad \text{Eq. 4. 2}$$

Next, we can apply the divergence theorem:

$$\int_V \left(\nabla \cdot \vec{D}\right) dV = 0 \rightarrow \int_A \vec{D} \cdot d\vec{A} = 0\,, \qquad \text{Eq. 4. 3}$$

where A is the surface of V.

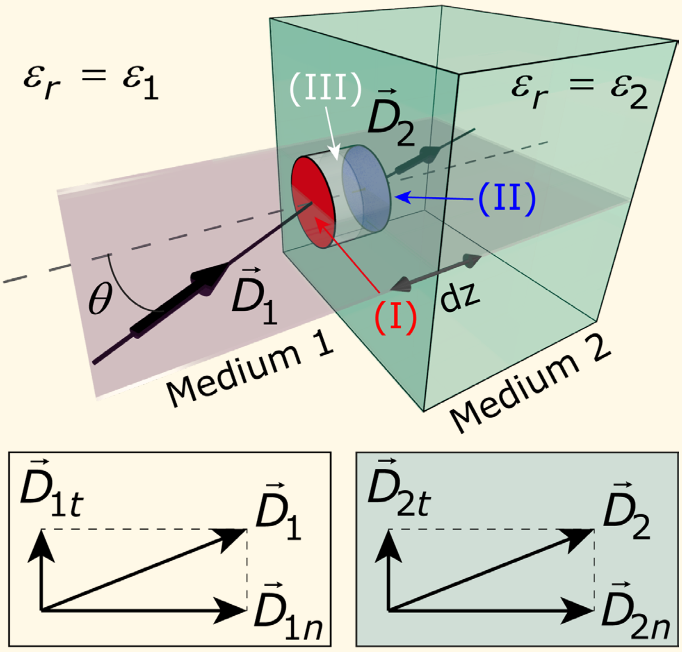


Figure 4.2. The electric flux density vectors ($\vec{D}$) at the boundary between two media. These vectors can be resolved into normal and tangential components. We consider a cylindrical Gaussian surface crossing the boundary.

We now consider two vectors $\vec{D}$, in the two different media. These vectors are $\vec{D}_1$ and. Moreover, each one of these vectors has two components.

The **normal component** $\vec{D}_{1n}$ is perpendicular to the boundary between the two materials.

The **tangential component** $\vec{D}_{1t}$ is parallel to the boundary between the two materials.

We can draw a cylindrical Gaussian surface crossing the boundary, where we can identify three surfaces that are parts of the cylinder.

We then apply Gauss's law:

$$0 = \oint_A \vec{D} \cdot d\vec{A} =$$

$$= \int_{A_I} \left(\vec{D}_{1n} + \vec{D}_{1t}\right) \cdot d\vec{A}_I + \int_{A_{II}} \left(\vec{D}_{2n} + \vec{D}_{2t}\right) \cdot d\vec{A}_{II} + \int_{A_{III}} \left(\vec{D}_{1n} + \vec{D}_{1t} + \vec{D}_{2n} + \vec{D}_{2t}\right) \cdot d\vec{A}_{III}$$

Eq. 4. 4

We see that

$$\vec{D}_{1t} \cdot d\vec{A}_I = 0, \qquad \text{Eq. 4. 5}$$

because they are perpendicular. By integrating over a continuous electric field density field:

$$\int_{A_I} \vec{D}_{1n} \cdot d\vec{A}_I = \vec{D}_{1n} \cdot \int_{A_I} d\vec{A}_I = \vec{D}_{1n} \cdot \vec{A}_I = -D_{1n}A_I \qquad \text{Eq. 4. 6}$$

(because $\vec{D}_{1n}$ and $\vec{A}_I$ are antiparallel).

Similarly,

$$\vec{D}_{2t} \cdot d\vec{A}_2 = 0 \qquad \text{Eq. 4. 7}$$

and by integrating over a continuous electric field density field:

$$\int_{A_{II}} \vec{D}_{2n} \cdot d\vec{A}_{II} = \vec{D}_{2n} \cdot \vec{A}_{II} = D_{2n}A_{II} \qquad \text{Eq. 4. 8}$$

(because $\vec{D}_{2n}$ and $\vec{A}_{II}$ are parallel).

When we decrease the height of the Gaussian cylinder, area A_{III} decreases as well. For an infinitely short cylinder (right at the surface), this area tends to zero. Therefore:

$$\oint_{A_{III}} \left(\vec{D}_{1n} + \vec{D}_{1t} + \vec{D}_{2n} + \vec{D}_{2t}\right) \cdot d\vec{A}_{III} \approx 0. \qquad \text{Eq. 4. 9}$$

Moreover, areas A_I and A_{II} are of the same size, so

$$D_{2n}A_{II} = D_{2n}A_I. \qquad \text{Eq. 4. 10}$$

We also assume that $\vec{D}$ is constant across the boundary.

Going back to Gauss's law, we now have:

$$0 = \oint_A \vec{D} \cdot d\vec{A} = -D_{1n}A_I + D_{2n}A_I, \qquad \text{Eq. 4. 11}$$

which means

$$\left(D_{1n} - D_{2n}\right) A_I = 0, \qquad \text{Eq. 4. 12}$$

so $D_{1n} = D_{2n}$. As those are parallel and similarly oriented, we can write:

$$\boxed{\vec{D}_{1n} = \vec{D}_{2n}}. \qquad \text{Eq. 4. 13}$$

The **normal component** of $\vec{D}$ is **continuous** across the boundary.

Boundary conditions for the electric field strength

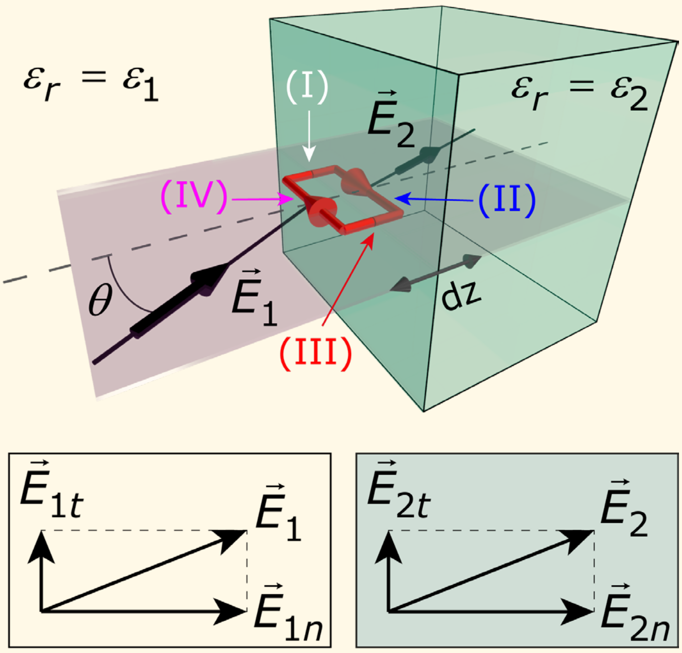


Figure 4.3. The electric field strength vectors ($\vec{E}$) at the boundary between two media. These vectors can be resolved into normal and tangential components. We consider a rectangular loop across the boundary, its four sides are labelled with Roman numerals.

We can also look at (ME3V): $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$.

We can integrate this equation over any surface A that is within a loop crossing the boundary.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \int_A \left(\nabla \times \vec{E}\right) d\vec{A} = \int_A \left(-\frac{\partial \vec{B}}{\partial t}\right) d\vec{A} = -\frac{\partial}{\partial t}\int_A \vec{B} \cdot d\vec{A}. \qquad \text{Eq. 4. 14}$$

We then apply Stoke's theorem:

$$\oint_L \vec{E} \cdot d\vec{L} = -\frac{\partial}{\partial t}\int_A \vec{B} \cdot d\vec{A}. \qquad \text{Eq. 4. 15}$$

We now consider two vectors $\vec{E}$, in the two different media. These vectors are $\vec{E}_1$ and $\vec{E}_2$. Moreover, each one of these vectors has two components.

The **normal component** $\vec{E}_{1n}$ is perpendicular to the boundary between the two materials.

The **tangential component** $\vec{E}_{1t}$ is parallel to the boundary between the two materials.

We have numbered the 4 segments of the loop as L_I, L_{II}, L_{III} and L_{IV}. We then can write:

$$\oint_L \vec{E}\cdot d\vec{L} = \int_{L_{IV}} \left(\vec{E}_{1n}+\vec{E}_{1t}\right)\cdot d\vec{L}_{IV} + \int_{L_{II}} \left(\vec{E}_{2n}+\vec{E}_{2t}\right)\cdot d\vec{L}_{II} + \int_{L_I} \left(\vec{E}_{1n}+\vec{E}_{1t}+\vec{E}_{2n}+\vec{E}_{2t}\right)\cdot d\vec{L}_I + \int_{L_{III}} \left(\vec{E}_{1n}+\vec{E}_{1t}+\vec{E}_{2n}+\vec{E}_{2t}\right)\cdot d\vec{L}_{III} \quad \text{Eq. 4. 16}$$

The length vectors are oriented along the loop and following the arrows. Therefore:

$$\vec{E}_{1n}\cdot d\vec{L}_{IV} = 0 \quad \text{Eq. 4. 17}$$

and

$$\vec{E}_{2n}\cdot d\vec{L}_{II} = 0 . \quad \text{Eq. 4. 18}$$

Moreover, as we shrink the loop on both sides towards the boundary, the lengths I and III tend to zero. Therefore,

$$\int_{L_I} \left(\vec{E}_{1n}+\vec{E}_{1t}+\vec{E}_{2n}+\vec{E}_{2t}\right)\cdot d\vec{L}_I \approx 0 \quad \text{Eq. 4. 19}$$

and

$$\int_{L_{III}} \left(\vec{E}_{1n}+\vec{E}_{1t}+\vec{E}_{2n}+\vec{E}_{2t}\right)\cdot d\vec{L}_{III} \approx 0 . \quad \text{Eq. 4. 20}$$

Altogether, we now have:

$$\oint_L \vec{E}\cdot d\vec{L} = \int_{L_{IV}} \vec{E}_{1t}\cdot d\vec{L}_{IV} + \int_{L_{II}} \vec{E}_{2t}\cdot d\vec{L}_{II} . \quad \text{Eq. 4. 21}$$

We note that the length of segments II and IV are the same but the directions are opposite.

We assume the electric field is continuous, then

$$\int_{L_{II}} \vec{E}_{2t}\cdot d\vec{L}_{II} = \vec{E}_{2t}\cdot \int_{L_{II}} d\vec{L}_{II} = \vec{E}_{2t}\cdot \vec{L}_{II} = -E_{2t}L_{II} \quad \text{Eq. 4. 22}$$

(because $\vec{E}_{2t}$ and $\vec{L}_{II}$ are antiparallel).

Also,

$$\int_{L_{IV}} \vec{E}_{1t}\cdot d\vec{L}_{IV} = \vec{E}_{1t}\cdot \int_{L_{IV}} d\vec{L}_{IV} = \vec{E}_{1t}\cdot \vec{L}_{IV} = E_{1t}L_{IV} \quad \text{Eq. 4. 23}$$

(because $\vec{E}_{1t}$ and $\vec{L}_{IV}$ are parallel).

Moreover, the segments II and IV are equal in length:

$$L_{II} = L_{IV} \quad \text{Eq. 4. 24}$$

Going back to Stoke's theorem:

$$\oint_L \vec{E} \cdot d\vec{L} = -E_{2t}L_{II} + E_{1t}L_{II} = -\frac{\partial}{\partial t}\int_A \vec{B} \cdot d\vec{A}. \qquad \text{Eq. 4. 25}$$

When the loop shrinks towards the boundary, it becomes infinitely thin and the area it encloses tends to zero. As a result:

$$\int_A \vec{B} \cdot d\vec{A} \approx 0. \qquad \text{Eq. 4. 26}$$

We are now left with

$$-E_{2t}L_{II} + E_{1t}L_{II} = 0 \rightarrow \left(E_{2t} - E_{1t}\right)L_{II} = 0. \qquad \text{Eq. 4. 27}$$

Finally: $E_{2t} = E_{1t}$ and because they are parallel, we can write:

$$\boxed{\vec{E}_{1t} = \vec{E}_{2t}}. \qquad \text{Eq. 4. 28}$$

The **tangential component** of $\vec{E}$ is **continuous** across the boundary.

Boundary conditions for the magnetic flux density

As for the electric fields, here we assume that the interface is free of surface charges and surface currents. For LIH materials, we have

$$\vec{B} = \mu_0\mu_r\vec{H} = \mu\vec{H}. \qquad \text{Eq. 4. 29}$$

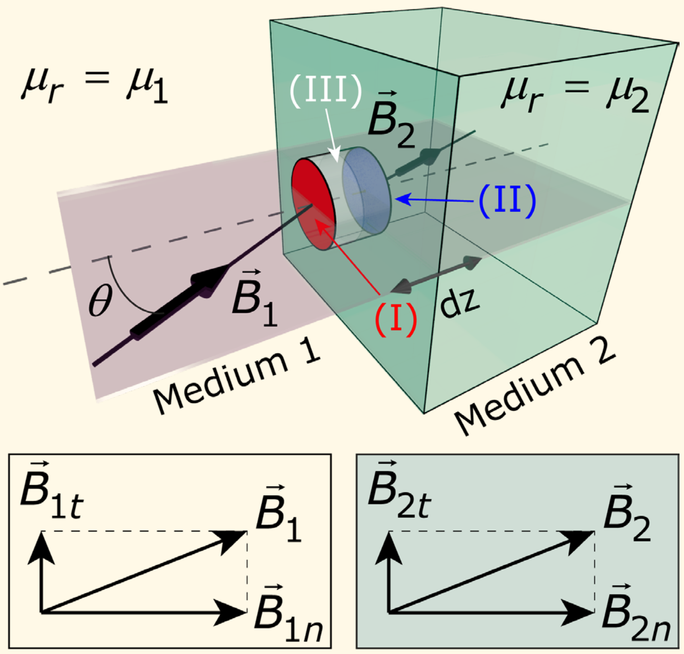


Figure 4.4. The magnetic flux density vectors ($\vec{B}$) at the boundary between two media. These vectors can be resolved into normal and tangential components. We consider a cylindrical Gaussian surface crossing the boundary.

We now consider two vectors $\vec{B}$, in the two different media. These vectors are $\vec{B}_1$ and $\vec{B}_2$. Moreover, each one of these vectors has two components.

The **normal component** $\vec{B}_{1n}$ is perpendicular to the boundary between the two materials.

The **tangential component** $\vec{B}_{1t}$ is parallel to the boundary between the two materials.

From Maxwell's equations, we know that $\nabla \cdot \vec{B} = 0$.

We can now integrate both sides over any volume V:

$$\int_V \left(\nabla \cdot \vec{B}\right) dV = \int_V (0)\, dV = 0. \qquad \text{Eq. 4. 30}$$

Next, we can apply the divergence theorem:

$$\int_V \left(\nabla \cdot \vec{B}\right) dV = 0 \rightarrow \int_A \vec{B} \cdot d\vec{A} = 0, \qquad \text{Eq. 4. 31}$$

where A is the surface of V.

We can draw a cylindrical Gaussian surface A crossing the boundary, where we can identify three surfaces that are parts of the cylinder.

Following the same logic as in the case of the electric flux density, we end up with:

$$0 = \oint_A \vec{B} \cdot d\vec{A} = -B_{1n} A_I + B_{2n} A_I, \qquad \text{Eq. 4. 32}$$

which leads to

$$\left(B_{1n} - B_{2n}\right) A_I = 0 \qquad \text{Eq. 4. 33}$$

and $B_{1n} = B_{2n}$. As they are parallel and directed similarly, we obtain:

$$\boxed{\vec{B}_{1n} = \vec{B}_{2n}} \qquad \text{Eq. 4. 34}$$

The **normal component** of $\vec{B}$ is **continuous** across the boundary.

Boundary conditions for the magnetic field strength

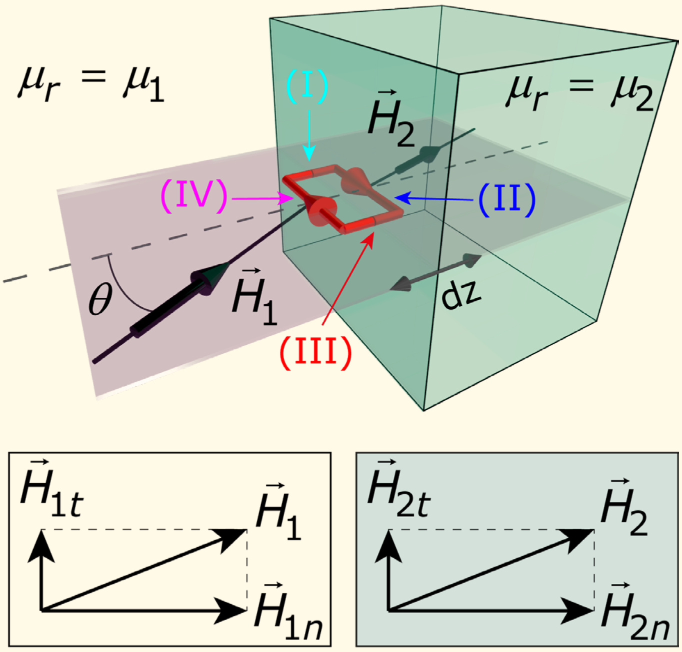


Figure 4.5. The magnetic field strength vectors ($\vec{B}$) at the boundary between two media. These vectors can be resolved into normal and tangential components. We consider a rectangular loop across the boundary, its four sides are labelled with Roman numerals.

Here, we are going to use (ME4$_M$), which is $\nabla \times \vec{H} = \vec{J}_f + \dfrac{\partial \vec{D}}{\partial t}$.

We can integrate both sides over a surface that is enclosed by a rectangular loop across the boundary:

$$\int_A \left(\nabla \times \vec{H}\right) \cdot d\vec{A} = \int_A \left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{A} \qquad \text{Eq. 4. 35}$$

From Stoke's theorem:

$$\int_A \left(\nabla \times \vec{H}\right) \cdot d\vec{A} = \oint_L \vec{H} \cdot d\vec{L} \qquad \text{Eq. 4. 36}$$

Just as we did with the electric field strength, we can decompose the loop and evaluate each segment with respect to the normal and tangential components of the magnetic field strength vector.
We obtain:

$$\oint_L \vec{H} \cdot d\vec{L} = -H_{2t} L_{II} + H_{1t} L_{IV} = \int_A \left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{A}\,. \qquad \text{Eq. 4. 37}$$

Upon shrinking the loop towards the boundary, the area enclosed tends to zero and therefore:

$$\int_A \left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{A} \approx 0 . \qquad \text{Eq. 4. 38}$$

Therefore, $H_{2t} = H_{1t}$ and we can write

$$\boxed{\vec{H}_{1t} = \vec{H}_{2t}} . \qquad \text{Eq. 4. 39}$$

because these vectors are parallel and similarly oriented.
The **tangential component** of $\vec{H}$ is **continuous** across the boundary.

Boundary conditions summary

For LIH materials, assuming no surface charges and no surface currents.

	Electric fields	Magnetic fields
Normal components	$\vec{D}_{1n} = \vec{D}_{2n}$	$\vec{B}_{1n} = \vec{B}_{2n}$
Tangential components	$\vec{E}_{1t} = \vec{E}_{2t}$	$\vec{H}_{1t} = \vec{H}_{2t}$

Lecture 8 – Fresnel coefficients

Why when you look in the window during daytime you can see outside but when you look in the window during night time you can see yourself?

Our experience shows us that at the boundary between two materials, some light is reflected and some light is transmitted. But how much?

We will consider a plane EM wave.

Reminder: we can use the following equations to determine the directions of $\vec{k}$, $\vec{E}$ and $\vec{H}$:

$$\vec{k} \times \vec{E} = \mu\omega\vec{H} , \qquad \text{Eq. 4. 40}$$

$$\vec{k} \cdot \vec{E} = 0 , \qquad \text{Eq. 4. 41}$$

$$\vec{k} \cdot \vec{H} = 0 . \qquad \text{Eq. 4. 42}$$

Electromagnetic waves at normal incidence

We consider a boundary between two materials, placed at $z = 0$. An incident EM wave ($\vec{E}_i$) is travelling along z.

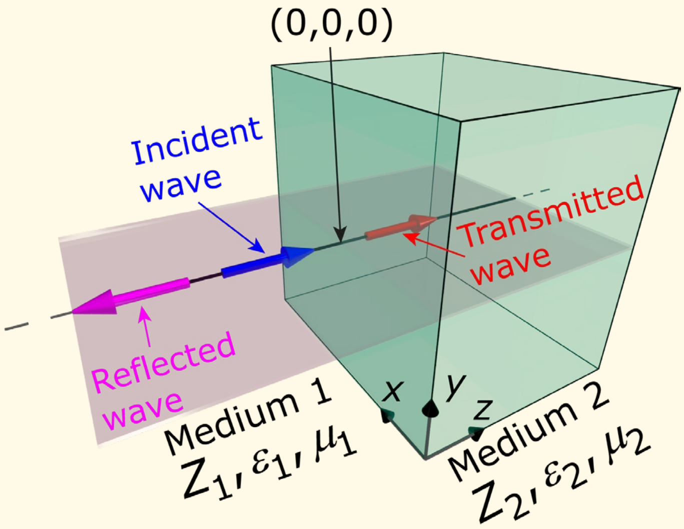


Figure 4.6. An electromagnetic wave arriving at the boundary between two materials, at normal incidence, is partially reflected and partially transmitted.

It hits the boundary at an incidence angle of $\theta_i = 0$ (along the normal to the boundary).

One part of the wave will be reflected ($\vec{E}_r$). It will travel in medium 1 but backwards, i.e. along $-z$.

Another part of the wave will be transmitted ($\vec{E}_t$). It will travel in medium 2 and forwards, i.e. along $+z$.

For each EM wave, the wave vector $\vec{k}$ points along its direction of propagation.

The field vectors $\vec{E}$ and $\vec{H}$ are perpendicular to $\vec{k}$ and they are therefore pointing in the plane of the boundary. This means that they are both tangential.

Some relation must exist between $\vec{E}_i$, $\vec{E}_r$ and $\vec{E}_t$. This relation will be true at the boundary, for all points $r_{boundary}$ and at all times t. Such a relation is only possible if $\vec{E}_i$, $\vec{E}_r$ and $\vec{E}_t$ are identical functions of $r_{boundary}$ and of t. This can only be the case if the three angular frequencies ω_i, ω_r and ω_t are identical: $\omega_i = \omega_r = \omega_t = \omega$. So all three EM wave have the same frequency. Another way to think about this identity is to consider that all three waves are associated with the motion of the same electron vibrations at the boundary.

We chose the electric field of the incident wave to be directed along positive x:

$$\vec{E}_i = E_{i0} e^{i(k_1 z - \omega t)} \hat{x}\,, \qquad \text{Eq. 4. 43}$$

because $\vec{k}$ is in medium 1 and it only has a component along z, travelling along positive z.

Because of:

$$\vec{k} \times \vec{E} = \mu\omega\vec{H}\,, \qquad \text{Eq. 4. 44}$$

$\vec{H}$ must point along y. Therefore:

$$\vec{H}_i = H_{i0} e^{i(k_1 z - \omega t)} \hat{y}\,. \qquad \text{Eq. 4. 45}$$

We can now use the impedance in medium 1, which is

$$Z_1 = E_{i0}/H_{i0}\,, \qquad \text{Eq. 4. 46}$$

to write:

$$\vec{H}_i = \frac{E_{i0}}{Z_1} e^{i(k_1 z - \omega t)} \hat{y}\,. \qquad \text{Eq. 4. 47}$$

So, for the **incident wave**, we have:

$$\vec{E}_i = E_{i0} e^{i(k_1 z - \omega t)} \hat{x} \qquad \text{Eq. 4. 48}$$

and

$$\vec{H}_i = \frac{E_{i0}}{Z_1} e^{i(k_1 z - \omega t)} \hat{y}\,. \qquad \text{Eq. 4. 49}$$

For the reflected wave,

$$\vec{E}_r = E_{r0} e^{i(-k_1 z - \omega t)} \hat{x}\,, \qquad \text{Eq. 4. 50}$$

because $\vec{k}$ is in medium 1 and it only has a component along z, travelling along negative z. Note that the electric field is again oriented along positive x, due to continuity at the boundary.

Because of

$$\vec{k} \times \vec{E} = \mu\omega\vec{H}\,, \qquad \text{Eq. 4. 51}$$

$\vec{H}$ must now point along negative y. Therefore:

$$\vec{H}_r = -H_{r0} e^{i(-k_1 z - \omega t)} \hat{y}\,. \qquad \text{Eq. 4. 52}$$

We can again use the impedance in medium 1, which is

$$Z_1 = E_{r0}/H_{r0}\,, \qquad \text{Eq. 4. 53}$$

to write:

$$\vec{H}_r = -\frac{E_{r0}}{Z_1} e^{i(-k_1 z - \omega t)} \hat{y}\,. \qquad \text{Eq. 4. 54}$$

So, for the **reflected wave**, we have:

$$\vec{E}_r = E_{r0} e^{i(-k_1 z - \omega t)} \hat{x} \qquad \text{Eq. 4. 55}$$

and

$$\vec{H}_r = -\frac{E_{r0}}{Z_1} e^{i(-k_1 z - \omega t)} \hat{y} . \qquad \text{Eq. 4. 56}$$

For the transmitted wave,

$$\vec{E}_t = E_{t0} e^{i(k_2 z - \omega t)} \hat{x} , \qquad \text{Eq. 4. 57}$$

because $\vec{k}$ is in medium 2 and it only has a component along z, travelling along positive z. Note that the electric field is again oriented along positive x, due to continuity at the boundary. Therefore, the $\vec{H}$ field has the same direction as the incident wave and this time we can use the impedance in medium 2, which is

$$Z_2 = E_{t0} / H_{t0} , \qquad \text{Eq. 4. 58}$$

to write:

$$\vec{H}_t = \frac{E_{t0}}{Z_2} e^{i(k_2 z - \omega t)} \hat{y} . \qquad \text{Eq. 4. 59}$$

So, for the **transmitted wave**, we have:

$$\vec{E}_t = E_{t0} e^{i(k_2 z - \omega t)} \hat{x} \qquad \text{Eq. 4. 60}$$

and

$$\vec{H}_t = \frac{E_{t0}}{Z_2} e^{i(k_2 z - \omega t)} \hat{y} . \qquad \text{Eq. 4. 61}$$

Reflection and transmission for normal incidence

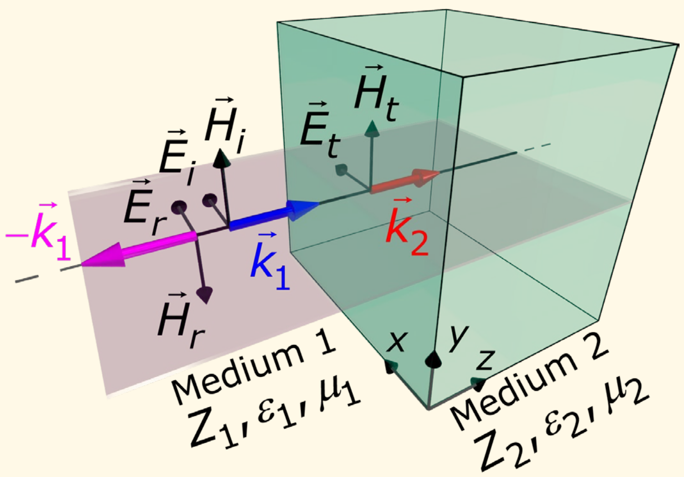


Figure 4.7. Electric ($\vec{E}$) and magnetic ($\vec{H}$) field strength components, together with the wave vectors ($\vec{k}$), for incident, reflected and transmitted electromagnetic waves, in the case of normal incidence.

The total fields are:

In medium 1:

$$\begin{cases} \vec{E}_1 = \vec{E}_i + \vec{E}_r \\ \vec{H}_1 = \vec{H}_i + \vec{H}_r \end{cases}. \qquad \text{Eq. 4. 62}$$

In medium 2:

$$\begin{cases} \vec{E}_2 = \vec{E}_t \\ \vec{H}_2 = \vec{H}_t \end{cases}. \qquad \text{Eq. 4. 63}$$

At the boundary:

$$\begin{cases} \vec{E}_1 = \vec{E}_2 \\ \vec{H}_1 = \vec{H}_2 \end{cases}, \qquad \text{Eq. 4. 64}$$

since all the vector components are tangential and since tangential components are continuous.

Therefore, at the boundary, where $z = 0$:

$$\begin{cases} E_{i0}e^{i(k_1 z-\omega t)}\hat{x} + E_{r0}e^{i(-k_1 z-\omega t)}\hat{x} = E_{t0}e^{i(k_2 z-\omega t)}\hat{x} \\ \dfrac{E_{i0}}{Z_1}e^{i(k_1 z-\omega t)}\hat{y} - \dfrac{E_{r0}}{Z_1}e^{i(-k_1 z-\omega t)}\hat{y} = \dfrac{E_{t0}}{Z_2}e^{i(k_2 z-\omega t)}\hat{y} \end{cases}. \qquad \text{Eq. 4. 65}$$

We can drop the common term $e^{i(-\omega t)}$ from all equations, as well as the x and y unit vectors. We obtain:

$$\begin{cases} E_{i0}e^{i(k_1 z)} + E_{r0}e^{i(-k_1 z)} = E_{t0}e^{i(k_2 z)} \\ \dfrac{E_{i0}}{Z_1}e^{i(k_1 z)} - \dfrac{E_{r0}}{Z_1}e^{i(-k_1 z)} = \dfrac{E_{t0}}{Z_2}e^{i(k_2 z)} \end{cases}. \qquad \text{Eq. 4. 66}$$

Moreover, at the boundary $z = 0$ and therefore

$$e^{i(k_1 z)} = e^{i(k_2 z)} = 1. \qquad \text{Eq. 4. 67}$$

So:

$$\begin{cases} E_{i0} + E_{r0} = E_{t0} \\ \dfrac{E_{i0}}{Z_1} - \dfrac{E_{r0}}{Z_1} = \dfrac{E_{t0}}{Z_2} \end{cases} \rightarrow \begin{cases} E_{i0} + E_{r0} = E_{t0} \\ E_{i0} - E_{r0} = \dfrac{Z_1}{Z_2}E_{t0} \end{cases}. \qquad \text{Eq. 4. 68}$$

We can add up the equations:

$$2E_{i0} = E_{t0} + \frac{Z_1}{Z_2}E_{t0} = \left(1 + \frac{Z_1}{Z_2}\right)E_{t0} = \frac{Z_2 + Z_1}{Z_2}E_{t0}, \qquad \text{Eq. 4. 69}$$

which leads to:

$$E_{t0} = \frac{2Z_2}{Z_2 + Z_1}E_{i0}. \qquad \text{Eq. 4. 70}$$

Then, from

$$E_{i0} + E_{r0} = E_{t0}, \qquad \text{Eq. 4. 71}$$

we have

$$E_{r0} = E_{t0} - E_{i0} = \frac{2Z_2}{Z_1 + Z_2}E_{i0} - E_{i0} = \frac{2Z_2}{Z_1 + Z_2}E_{i0} - \frac{Z_1 + Z_2}{Z_1 + Z_2}E_{i0} =$$

$$E_{r0} = \frac{2Z_2 - Z_1 - Z_2}{Z_1 + Z_2}E_{i0} \quad \text{Eq. 4. 72}$$

So, we have

$$E_{r0} = \frac{Z_2 - Z_1}{Z_1 + Z_2}E_{i0}. \quad \text{Eq. 4. 73}$$

We now define the **reflection coefficient:**

$$\boxed{r_{\parallel/\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_1 + Z_2}}. \quad \text{Eq. 4. 74}$$

We also define the **transmission coefficient:**

$$\boxed{t_{\parallel/\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2}{Z_1 + Z_2}}. \quad \text{Eq. 4. 75}$$

Note that both $r_{\parallel/\perp}$ and $t_{\parallel/\perp}$ can be complex because the impedance of a materials can be complex. [Here, we use the subscript $\parallel / \perp$ to distinguish the coefficients from the position *r* and the time *t*.]

With our new definitions we can write:

$$E_{t0} = t_{\parallel/\perp}E_{i0} \quad \text{Eq. 4. 76}$$

and

$$E_{r0} = r_{\parallel/\perp}E_{i0}. \quad \text{Eq. 4. 77}$$

General case of incidence at the boundary and the law of reflection

In this case the wave vectors are no longer all parallel and instead they form various angles with the normal to the boundary. However, these wave vectors can all be **resolved** (GCSE Physics). Since we know how to treat the case of normal incidence, upon resolving the vectors, we chose one of the components to be along the normal to the boundary. The other component must be perpendicular.

We now consider the case where the angle of incidence $\theta_i \neq 0$.

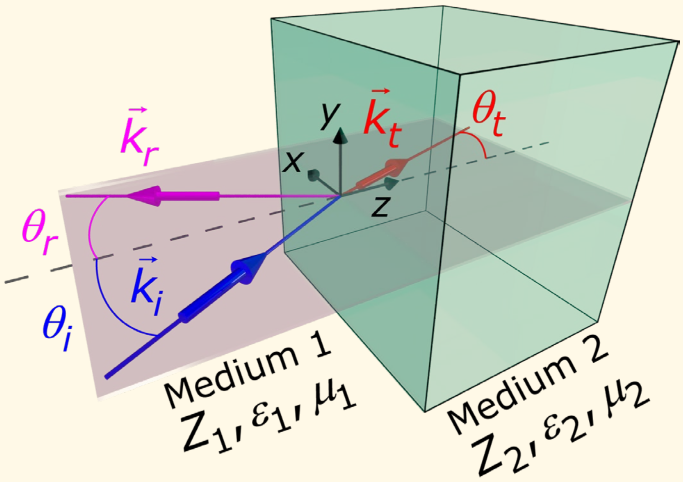


Figure 4.7. The place of incidence is defined by the wave vector of the incident wave ($\vec{k}_i$) and a unit vector normal to the boundary.

The **plane of incidence** is defined by the wave vector of the incident wave and a unit vector normal to the boundary. The plane of incidence is also defined as the plane, where $\vec{k}_i$, $\vec{k}_r$ and $\vec{k}_t$ are **coplanar**.

At the boundary, the phases of all three waves are identical (it is the same oscillations at the interface). The phase velocity is $v_p = \omega/k$, but for all three waves we have. $\omega_i = \omega_r = \omega_t = \omega$. For the three waves of the form $e^{(\vec{k}\cdot\vec{r}-\omega t)}$, it follows that

$$\begin{gathered}\left(\vec{k}_i \cdot \vec{r}\right)_{\text{interface}} = \left(\vec{k}_r \cdot \vec{r}\right)_{\text{interface}} = \left(\vec{k}_t \cdot \vec{r}\right)_{\text{interface}}. \\ \left(k_{ix}x + k_{iz}z\right) = \left(k_{rx}x + k_{rz}z\right) = \left(k_{tx}x + k_{tz}z\right)\end{gathered} \qquad \text{Eq. 4. 78}$$

And therefore:

$$\begin{cases} k_{i,x} = k_{r,x} = k_{t,x} = k_x \\ k_{i,y} = k_{r,y} = k_{t,y} = k_y \end{cases}. \qquad \text{Eq. 4. 79}$$

So, the magnitudes of the wave vector before and after reflection are the same; remembering that in vacuum:

$$\left|\vec{k}\right| = \frac{2\pi}{\lambda} = \frac{\omega T}{\lambda} = \frac{\omega}{c} = \omega\sqrt{\varepsilon_0\mu_0}. \qquad \text{Eq. 4. 80}$$

Opposite directions of travel mean that $k_{r,z} = -k_{i,z}$. But note that the length of the wave vector changes in transmission, because while ω stays the same, ε_r and μ_r do not (they are part of the refractive index $n = \sqrt{\varepsilon_r \mu_r}$, which changes at the boundary). We remember that:

$$v_p = \frac{\omega}{|\vec{k}|} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}} = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{c}{n}, \qquad \text{Eq. 4. 81}$$

so:

$$|\vec{k}| = \frac{\omega}{c} n . \qquad \text{Eq. 4. 82}$$

This is the **dispersion relation** for a homogeneous medium.
Since $k_{i,x} = k_{r,x}$, $k_{i,y} = k_{r,y}$ and $k_{r,z} = -k_{i,z}$, it follows that the angle of incidence equals the angle of reflection: $\theta_i = \theta_r$, which is the **law of reflection**.

General case of incidence at the boundary and Snell's law

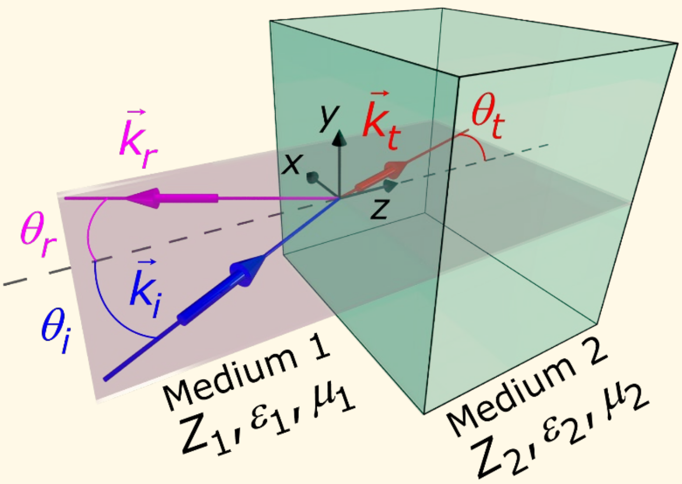
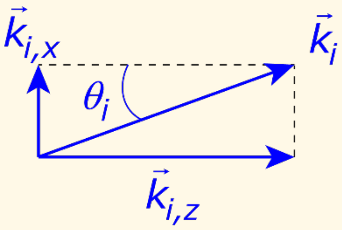
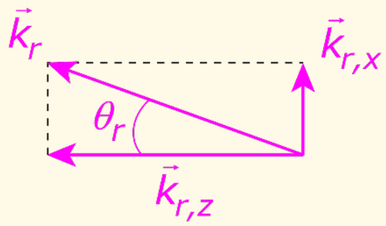


Figure 4.8. The wave vectors at the boundary between two materials.

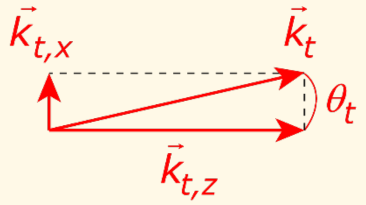
In medium 2, the relation $k = \omega\sqrt{\mu_r \varepsilon_r}/c$ becomes $k_t = \omega\sqrt{\mu_{r2}\varepsilon_{r2}}/c$. We can resolve the wave vectors:



$$\begin{cases} |\vec{k}_{i,x}| = |\vec{k}_i| \sin\theta_i \\ |\vec{k}_{i,z}| = |\vec{k}_i| \cos\theta_i \end{cases}$$



$$\begin{cases} \left|\vec{k}_{r,x}\right| = \left|\vec{k}_r\right| \sin\theta_r = \left|\vec{k}_i\right| \sin\theta_i \\ \left|\vec{k}_{r,z}\right| = \left|\vec{k}_r\right| \cos\theta_r = \left|\vec{k}_i\right| \cos\theta_i \end{cases}$$



$$\begin{cases} \left|\vec{k}_{t,x}\right| = \left|\vec{k}_t\right| \sin\theta_t = \dfrac{\omega}{c}\sqrt{\varepsilon_{r2}\mu_{r2}}\,\sin\theta_t \\ \left|\vec{k}_{t,z}\right| = \left|\vec{k}_t\right| \cos\theta_t = \dfrac{\omega}{c}\sqrt{\varepsilon_{r2}\mu_{r2}}\,\cos\theta_t \end{cases}$$

From the momentum conservation at the boundary, we saw that: $k_{i,x} = k_{r,x} = k_{t,x}$, so $k_{i,x} = k_{t,x}$, which is

$$\left|\vec{k}_i\right| \sin\theta_i = \left|\vec{k}_t\right| \sin\theta_t . \qquad \text{Eq. 4. 83}$$

We can now use $k_i = \omega\sqrt{\mu_{r1}\varepsilon_{r1}}/c$ and $k_t = \omega\sqrt{\mu_{r2}\varepsilon_{r2}}/c$ to get:

$$\frac{\omega}{c}\sqrt{\mu_{r1}\varepsilon_{r1}}\,\sin\theta_i = \frac{\omega}{c}\sqrt{\mu_{r2}\varepsilon_{r2}}\,\sin\theta_t , \qquad \text{Eq. 4. 84}$$

which is

$$\sqrt{\mu_{r1}\varepsilon_{r1}}\,\sin\theta_i = \sqrt{\mu_{r2}\varepsilon_{r2}}\,\sin\theta_t \qquad \text{Eq. 4. 85}$$

and we obtain **Snell's law**:

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t} . \qquad \text{Eq. 4. 86}$$

Polarisation of an electromagnetic wave

For an EM wave, the plane of oscillations of the electric field defines the direction of **light polarisation**. This term should not be confused with the polarisation density $\vec{P}$ that we saw in dielectric materials. We can consider two cases of light polarisation:

1. The polarisation of light is in the plane of incidence (the magnetic field oscillations are perpendicular to the plane of incidence). This polarisation state is referred to as **P-polarized** (from the German parallel). It can be indicated as ($\parallel$).
2. The polarisation of light is perpendicular to the plane of incidence (the magnetic field oscillations are in the plane of incidence). This polarisation state is referred to as **S-polarized** (from the German senkrecht). It can be indicated as ($\perp$).

General case of incidence at the boundary: P-polarized light

It is important to able to draw this diagram:

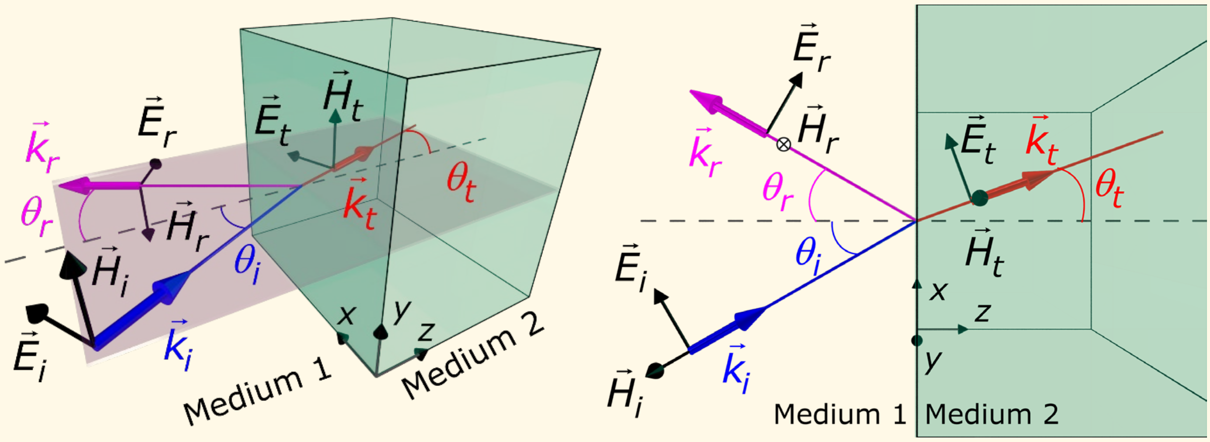


Figure 4.9. For P-polarized light, electric ($\vec{E}$) and magnetic ($\vec{H}$) field strength components, together with the wave vectors ($\vec{k}$), for incident, reflected and transmitted electromagnetic waves, in the case of incidence at angle $\theta_i \neq 0$.

1. Draw directions of the wave vectors according to the propagation directions.
2. Draw the electric fields:
   a. Perpendicular to their respective wave vectors.
   b. For P-polarized light they are in the plane of the figure, which is also the plane of incidence.
   c. Their components are all either along positive x or negative x. Here, we chose for all to be along positive x.
3. Draw the magnetic fields with direction set by:

$$\vec{H} = \frac{1}{\mu\omega}\vec{k}\times\vec{E}. \qquad \text{Eq. 4. 87}$$

Resolving all the vectors in the Cartesian coordinate system, we can write:

$$\vec{E}_i = \left(E_{i0,x} + E_{i0,z}\right)e^{i\left(k_{i,x}x + k_{i,z}z - \omega t\right)}$$
$$\vec{E}_i = E_{i0}\left(\cos\theta_i\hat{x} - \sin\theta_i\hat{z}\right)e^{i\left(k_1\sin\theta_i x + k_1\cos\theta_i z - \omega t\right)}. \qquad \text{Eq. 4. 88}$$

Similarly:

$$\vec{E}_r = \left(E_{r0,x} + E_{r0,z}\right)e^{i\left(k_{r,x}x - k_{r,z}z - \omega t\right)}$$
$$\vec{E}_r = E_{r0}\left(\cos\theta_r\hat{x} + \sin\theta_r\hat{z}\right)e^{i\left(k_1\sin\theta_r x - k_1\cos\theta_r z - \omega t\right)} \qquad \text{Eq. 4. 89}$$

and:

$$\vec{E}_t = \left(E_{t0,x} + E_{t0,z}\right) e^{i\left(k_{t,x}x + k_{t,z}z - \omega t\right)}$$
$$\vec{E}_t = E_{t0}\left(\cos\theta_t \hat{x} - \sin\theta_t \hat{z}\right) e^{i\left(k_2 \sin\theta_t x + k_2 \cos\theta_t z - \omega t\right)} . \qquad \text{Eq. 4. 90}$$

Next we calculate the magnetic fields, using the impedances in the two media. Their amplitudes are only along the y direction.

$$\vec{H}_i = \frac{E_{i0}}{Z_1} \hat{y} e^{i\left(k_1 \sin\theta_i x + k_1 \cos\theta_i z - \omega t\right)}, \qquad \text{Eq. 4. 91}$$

with the same k as $\vec{E}_i$ and same oscillatory part. In reflection:

$$\vec{H}_r = \frac{E_{r0}}{Z_1}\left(-\hat{y}\right) e^{i\left(k_1 \sin\theta_r x - k_1 \cos\theta_r z - \omega t\right)}, \qquad \text{Eq. 4. 92}$$

with again the same k as $\vec{E}_r$ and same oscillatory part. In transmission:

$$\vec{H}_t = \frac{E_{t0}}{Z_1} \hat{y} e^{i\left(k_2 \sin\theta_t x + k_2 \cos\theta_t z - \omega t\right)}, \qquad \text{Eq. 4. 93}$$

with once again the same k as $\vec{E}_t$ and same oscillatory part.

We require that the tangential components of $\vec{E}$ and $\vec{H}$ be continuous at the boundary. The boundary is at $z = 0$. Moreover, we can chose that the EM wave impacts the boundary at $x = 0$.

$$\vec{E}_1\Big|_{\text{tangential}} = \vec{E}_2\Big|_{\text{tangential}} \qquad \text{Eq. 4. 94}$$

means that:

$$\vec{E}_1\Big|_x = \vec{E}_2\Big|_x, \qquad \text{Eq. 4. 95}$$

which is

$$\left(\vec{E}_i + \vec{E}_r\right)\Big|_x = \vec{E}_t\Big|_x . \qquad \text{Eq. 4. 96}$$

We can replace with the terms we obtained above:

$$E_{i0}\cos\theta_i \hat{x} + E_{r0}\cos\theta_r \hat{x} = E_{t0}\cos\theta_t \hat{x}, \qquad \text{Eq. 4. 97}$$

and we can drop the unit vectors as we know that $\theta_i = \theta_r$, so

$$\cos\theta_i = \cos\theta_r, \qquad \text{Eq. 4. 98}$$

which leads to:

$$E_{i0} + E_{r0} = E_{t0}\frac{\cos\theta_t}{\cos\theta_i} . \qquad \text{Eq. 4. 99}$$

Moreover,

$$\vec{H}_1\Big|_{\text{tangential}} = \vec{H}_2\Big|_{\text{tangential}} \qquad \text{Eq. 4. 100}$$

means that:

$$\vec{H}_1\Big|_y = \vec{H}_2\Big|_y, \qquad \text{Eq. 4. 101}$$

which is

$$\left(\vec{H}_i + \vec{H}_r\right)\Big|_y = \vec{H}_t\Big|_y. \qquad \text{Eq. 4. 102}$$

We can replace with the terms we obtained above:

$$\left(\frac{E_{i0}}{Z_1} - \frac{E_{r0}}{Z_1}\right)\hat{y} = \frac{E_{t0}}{Z_2}\hat{y}. \qquad \text{Eq. 4. 103}$$

Dropping the unit vector, this becomes:

$$E_{i0} - E_{r0} = \frac{Z_1}{Z_2}E_{t0}. \qquad \text{Eq. 4. 104}$$

Combining the equations by adding them together:

$$\begin{cases} E_{i0} + E_{r0} = E_{t0}\dfrac{\cos\theta_t}{\cos\theta_i} \\ E_{i0} - E_{r0} = \dfrac{Z_1}{Z_2}E_{t0} \end{cases} \rightarrow 2E_{i0} = \left(\frac{\cos\theta_t}{\cos\theta_i} + \frac{Z_1}{Z_2}\right)E_{t0} = \left(\frac{Z_2\cos\theta_t + Z_1\cos\theta_i}{Z_2\cos\theta_i}\right)E_{t0}$$

Eq. 4. 105

and

$$E_{t0} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0}. \qquad \text{Eq. 4. 106}$$

It follows that:

$$E_{i0} - E_{r0} = \frac{Z_1}{Z_2}E_{t0} = \left(\frac{Z_1}{Z_2}\right)\frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0} \qquad \text{Eq. 4. 107}$$

$$E_{i0} - \frac{2Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0} = E_{r0} \rightarrow E_{r0} = \left(1 - \frac{2Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}\right)E_{i0}$$

Eq. 4. 108

$$E_{r0} = \frac{Z_2\cos\theta_t + Z_1\cos\theta_i - 2Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0}.$$

Eq. 4. 109

So,

$$E_{r0} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0}. \qquad \text{Eq. 4. 110}$$

We can now define the **Fresnel coefficients for P-polarized light**:

$$\boxed{r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}} \qquad \text{Eq. 4. 111}$$

and

$$\boxed{t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}}. \qquad \text{Eq. 4. 112}$$

Example question 1:

For non-magnetic materials, show that:

$$r_{\parallel} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \text{ and } t_{\parallel} = \frac{4\sin\theta_t\cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}.$$

Answer: Remembering that

$$Z = \sqrt{\frac{\mu_r}{\varepsilon_r}}Z_0, \qquad \text{Eq. 4. 113}$$

for non-magnetic materials, where $\mu_r = 1$, we have

$$Z = \frac{Z_0}{\sqrt{\varepsilon_r}} = \frac{Z_0}{n}. \qquad \text{Eq. 4. 114}$$

We can replace in the Fresnel coefficients:

$$r_{\parallel} = \frac{\frac{Z_0}{n_2}\cos\theta_t - \frac{Z_0}{n_1}\cos\theta_i}{\frac{Z_0}{n_2}\cos\theta_t + \frac{Z_0}{n_1}\cos\theta_i} = \frac{\frac{\cos\theta_t}{n_2} - \frac{\cos\theta_i}{n_1}}{\frac{\cos\theta_t}{n_2} + \frac{\cos\theta_i}{n_1}}. \qquad \text{Eq. 4. 115}$$

We then use Snell's law

$$n_1\sin\theta_i = n_2\sin\theta_t \to n_2 = \frac{\sin\theta_i}{\sin\theta_t}n_1, \qquad \text{Eq. 4. 116}$$

so that

$$r_{\parallel} = \frac{\frac{Z_0}{\frac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_t - \frac{Z_0}{n_1}\cos\theta_i}{\frac{Z_0}{\frac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_t + \frac{Z_0}{n_1}\cos\theta_i} = \frac{\frac{\sin\theta_t\cos\theta_t}{\sin\theta_i} - \cos\theta_i}{\frac{\sin\theta_t\cos\theta_t}{\sin\theta_i} + \cos\theta_i} = \qquad \text{Eq. 4. 117}$$

$$= \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i} = \frac{2\sin\theta_t\cos\theta_t - 2\sin\theta_i\cos\theta_i}{2\sin\theta_t\cos\theta_t + 2\sin\theta_i\cos\theta_i}$$

We then use

$$\begin{cases} \sin(a+b) = \sin a\cos b + \cos a\sin b \\ \sin(2a) = 2\sin a\cos a \end{cases}, \qquad \text{Eq. 4. 118}$$

to obtain:

$$\boxed{r_{\parallel} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i}}. \qquad \text{Eq. 4. 119}$$

Similarly,

$$t_{\parallel} = \frac{2\frac{Z_0}{n_2}\cos\theta_i}{\frac{Z_0}{n_2}\cos\theta_t + \frac{Z_0}{n_1}\cos\theta_i} = \frac{\frac{2}{\frac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_i}{\frac{1}{\frac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_t + \frac{1}{n_1}\cos\theta_i} =$$

$$= \frac{2\frac{\sin\theta_t\cos\theta_i}{\sin\theta_i}}{\frac{\sin\theta_t\cos\theta_t}{\sin\theta_i} + \cos\theta_i} = \frac{2\sin\theta_t\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i} = \qquad \text{Eq. 4. 120}$$

$$= \frac{4\sin\theta_t\cos\theta_i}{2\sin\theta_t\cos\theta_t + 2\sin\theta_i\cos\theta_i}$$

And we can write:

$$\boxed{t_{\parallel} = \frac{4\sin\theta_t\cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}}. \qquad \text{Eq. 4. 121}$$

General case of incidence at the boundary: S-polarized light

We follow the same logic as in the case for P-polarized light. Here, we chose to point all the electric field strength vectors along positive y.

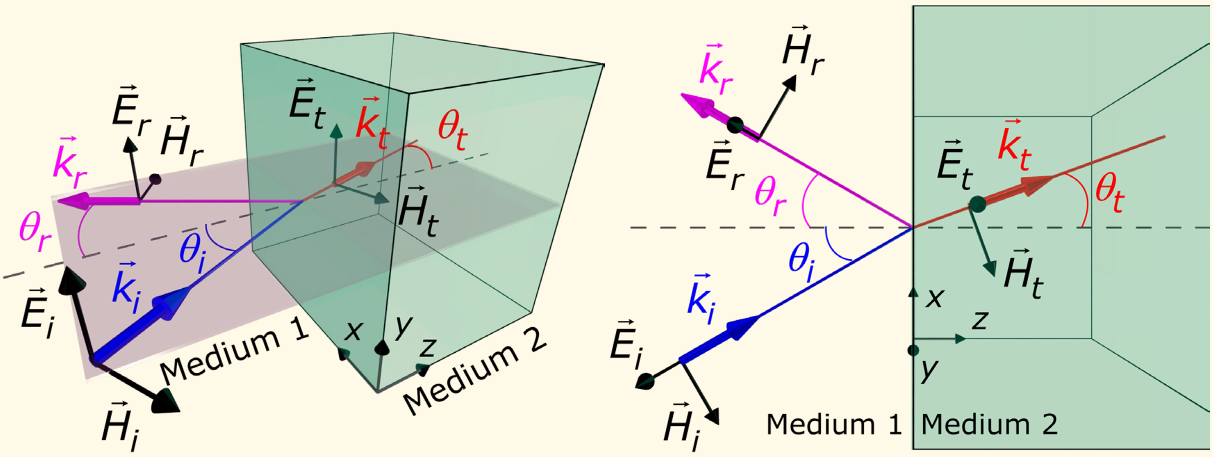


Figure 4.10. For S-polarized light, electric ($\vec{E}$) and magnetic ($\vec{H}$) field strength components, together with the wave vectors ($\vec{k}$), for incident,

reflected and transmitted electromagnetic waves, in the case of incidence at angle $\theta_i \neq 0$.

The wave vectors are the same, so the oscillatory part of the electric fields is the same.

$$\vec{E}_i = E_{i0}\hat{y}e^{i(k_1 \sin\theta_i x + k_1 \cos\theta_i z - \omega t)} \quad \text{Eq. 4. 122}$$

$$\vec{E}_r = E_{r0}\hat{y}e^{i(k_1 \sin\theta_r x - k_1 \cos\theta_r z - \omega t)} \quad \text{Eq. 4. 123}$$

$$\vec{E}_t = E_{t0}\hat{y}e^{i(k_2 \sin\theta_t x + k_2 \cos\theta_t z - \omega t)} \quad \text{Eq. 4. 124}$$

The magnetic fields are:

$$\vec{H}_i = \frac{E_{i0}}{Z_1}\left(-\cos\theta_i\hat{x} + \sin\theta_i\hat{z}\right)e^{i(k_1 \sin\theta_i x + k_1 \cos\theta_i z - \omega t)} \quad \text{Eq. 4. 125}$$

$$\vec{H}_r = \frac{E_{r0}}{Z_1}\left(\cos\theta_r\hat{x} + \sin\theta_r\hat{z}\right)e^{i(k_1 \sin\theta_r x - k_1 \cos\theta_r z - \omega t)} \quad \text{Eq. 4. 126}$$

$$\vec{H}_t = \frac{E_{t0}}{Z_2}\left(-\cos\theta_t\hat{x} + \sin\theta_t\hat{z}\right)e^{i(k_2 \sin\theta_t x + k_2 \cos\theta_t z - \omega t)} \quad \text{Eq. 4. 127}$$

Again, we require that the tangential components of $\vec{E}$ and $\vec{H}$ be continuous at the boundary. The boundary is at $z = 0$. Moreover, we can chose that the EM wave impacts the boundary at $x = 0$. We therefore drop the oscillatory terms.

$$\vec{E}_1\big|_{\text{tangential}} = \vec{E}_2\big|_{\text{tangential}} \quad \text{Eq. 4. 128}$$

means that:

$$\vec{E}_1\big|_y = \vec{E}_2\big|_y, \quad \text{Eq. 4. 129}$$

which is

$$\left(\vec{E}_i + \vec{E}_r\right)\big|_y = \vec{E}_t\big|_y. \quad \text{Eq. 4. 130}$$

We can replace with the terms we obtained above:

$$E_{i0} + E_{r0} = E_{t0} \quad \text{Eq. 4. 131}$$

Moreover,

$$\vec{H}_1\big|_{\text{tangential}} = \vec{H}_2\big|_{\text{tangential}} \quad \text{Eq. 4. 132}$$

means that:

$$\vec{H}_1\big|_x = \vec{H}_2\big|_x, \quad \text{Eq. 4. 133}$$

which is

$$\left(\vec{H}_i + \vec{H}_r\right)\big|_x = \vec{H}_t\big|_x. \quad \text{Eq. 4. 134}$$

We can again replace with the terms we obtained above:

$$-\frac{E_{i0}}{Z_1}\cos\theta_i\hat{x} + \frac{E_{r0}}{Z_1}\cos\theta_r\hat{x} = -\frac{E_{t0}}{Z_2}\cos\theta_t\hat{x} \qquad \text{Eq. 4. 135}$$

dropping the unit vector, using $\theta_i = \theta_r$, so $\cos\theta_i = \cos\theta_r$ this becomes:

$$-E_{i0} + E_{r0} = -\frac{Z_1}{Z_2}\frac{\cos\theta_t}{\cos\theta_i}E_{t0}, \qquad \text{Eq. 4. 136}$$

i.e.

$$E_{i0} - E_{r0} = \frac{Z_1}{Z_2}\frac{\cos\theta_t}{\cos\theta_i}E_{t0}. \qquad \text{Eq. 4. 137}$$

Combining the two equation by adding them:

$$\begin{cases} E_{i0} + E_{r0} = E_{t0} \\ E_{i0} - E_{r0} = \dfrac{Z_1}{Z_2}\dfrac{\cos\theta_t}{\cos\theta_i}E_{t0} \end{cases} \rightarrow 2E_{i0} = \left(1 + \frac{Z_1}{Z_2}\frac{\cos\theta_t}{\cos\theta_i}\right)E_{t0} \qquad \text{Eq. 4. 138}$$

$$2E_{i0} = \frac{Z_2\cos\theta_i + Z_1\cos\theta_t}{Z_2\cos\theta_i}E_{t0}. \qquad \text{Eq. 4. 139}$$

Which means that:

$$E_{t0} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}E_{i0}. \qquad \text{Eq. 4. 140}$$

We can now replace in

$$E_{i0} + E_{r0} = E_{t0} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}E_{i0}. \qquad \text{Eq. 4. 141}$$

So,

$$E_{r0} = \left(\frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t} - 1\right)E_{i0} = \left(\frac{2Z_2\cos\theta_i - Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}\right)E_{i0}.$$

Eq. 4. 142

And we obtain:

$$E_{r0} = \left(\frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}\right)E_{i0}. \qquad \text{Eq. 4. 143}$$

We can now define the **Fresnel coefficients for S-polarized light**:

$$\boxed{r_\perp \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}} \qquad \text{Eq. 4. 144}$$

and

$$\boxed{t_\perp \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}}. \qquad \text{Eq. 4. 145}$$

Example question 2

Show that for non-magnetic materials:

$$r_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \text{ and } t_\perp = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_t + \theta_i)}.$$

Answer: Starting with

$$r_\perp = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}, \quad \text{Eq. 4. 146}$$

we use

$$Z = \frac{Z_0}{n}. \quad \text{Eq. 4. 147}$$

Replacing:

$$r_\perp = \frac{\frac{Z_0}{n_2}\cos\theta_i - \frac{Z_0}{n_1}\cos\theta_t}{\frac{Z_0}{n_2}\cos\theta_i + \frac{Z_0}{n_1}\cos\theta_t} = \frac{\frac{1}{\frac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_i - \frac{1}{n_1}\cos\theta_t}{\frac{1}{\frac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_i + \frac{1}{n_1}\cos\theta_t} \quad \text{Eq. 4. 148}$$

$$r_\perp = \frac{\frac{\sin\theta_t\cos\theta_i}{\sin\theta_i} - \frac{\sin\theta_i}{\sin\theta_i}\cos\theta_t}{\frac{\sin\theta_t\cos\theta_i}{\sin\theta_i} + \frac{\sin\theta_i}{\sin\theta_i}\cos\theta_t} = \frac{\sin\theta_t\cos\theta_i - \sin\theta_i\cos\theta_t}{\sin\theta_t\cos\theta_i + \sin\theta_i\cos\theta_t}$$

$$\boxed{r_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}}, \quad \text{Eq. 4. 149}$$

where we used

$$n_1\sin\theta_i = n_2\sin\theta_t \rightarrow n_2 = \frac{\sin\theta_i}{\sin\theta_t}n_1 \quad \text{Eq. 4. 150}$$

and

$$\begin{cases} \sin(a+b) = \sin a\cos b + \cos a\sin b \\ \sin(a-b) = \sin a\cos b - \cos a\sin b \end{cases}. \quad \text{Eq. 4. 151}$$

Similarly,

$$t_{\perp} = \frac{2\frac{Z_0}{n_2}\cos\theta_i}{\frac{Z_0}{n_2}\cos\theta_i + \frac{Z_0}{n_1}\cos\theta_t} = \frac{2\frac{\cos\theta_i}{\frac{\sin\theta_i}{\sin\theta_t}n_1}}{\frac{\cos\theta_i}{\frac{\sin\theta_i}{\sin\theta_t}n_1} + \frac{\cos\theta_t}{n_1}}, \quad \text{Eq. 4. 152}$$

$$t_{\perp} = \frac{2\frac{\cos\theta_i\sin\theta_t}{\sin\theta_i}}{\frac{\cos\theta_i\sin\theta_t}{\sin\theta_i} + \cos\theta_t} = \frac{2\cos\theta_i\sin\theta_t}{\cos\theta_i\sin\theta_t + \cos\theta_t\sin\theta_i}$$

which leads to:

$$\boxed{t_{\perp} = \frac{2\cos\theta_i\sin\theta_t}{\sin\left(\theta_t + \theta_i\right)}}. \quad \text{Eq. 4. 153}$$

The Brewster angle

We have obtained four equations, two for the case of P-polarized light and two others for the case of S-polarized light:

$$\boxed{\begin{aligned} r_{\parallel} &= \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i} \text{ and } t_{\parallel} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i} \\ r_{\perp} &= \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t} \text{ and } t_{\perp} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t} \end{aligned}}$$

In the case of the transmission coefficients, we see that they can become zero (no transmission) when $\cos\theta_i = 0$, meaning when $\theta_i = \pm\frac{\pi}{2}\times(\text{integer})$. Such an angle can be approached (**grazing incidence**) but it can never be reached. Therefore, some light is always transmitted.

In the case of the reflection coefficients, a zero can be reached in the numerator.

For instance, in non-magnetic media,

$$r_{\parallel} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i}. \quad \text{Eq. 4. 154}$$

This could be zero if $\theta_t = \theta_i$, but this makes no sense as it conflicts with Snell's law, unless $\theta_t = \theta_i = 0$, which is not particularly interesting.

But, if

$$\theta_t = \frac{\pi}{2} - \theta_i, \quad \text{Eq. 4. 155}$$

then we use

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \quad \text{Eq. 4. 156}$$

and we obtain:

$$\sin 2\theta_t = \sin(\pi - 2\theta_i) = \underbrace{\sin\pi}_{=0}\cos 2\theta_i - \underbrace{\cos\pi}_{-1}\sin 2\theta_i = \sin 2\theta_i. \quad \text{Eq. 4. 157}$$

So, for the numerator:

$$\sin 2\theta_t - \sin 2\theta_i = 0. \quad \text{Eq. 4. 158}$$

For $\theta_t = \frac{\pi}{2} - \theta_i$, Snell's law $n_1 \sin\theta_i = n_1 \sin\theta_t$ leads to:

$$n_1 \sin\theta_i = n_2 \sin\left(\frac{\pi}{2} - \theta_i\right) = n_2 \cos\theta_i, \quad \text{Eq. 4. 159}$$

so

$$\frac{\sin\theta_i}{\cos\theta_i} = \tan\theta_i = \frac{n_2}{n_1}. \quad \text{Eq. 4. 160}$$

And by definition, the **Brewster angle** is:

$$\boxed{\tan\theta_B = \frac{n_2}{n_1}}, \quad \text{Eq. 4. 161}$$

but only for P-polarized light.

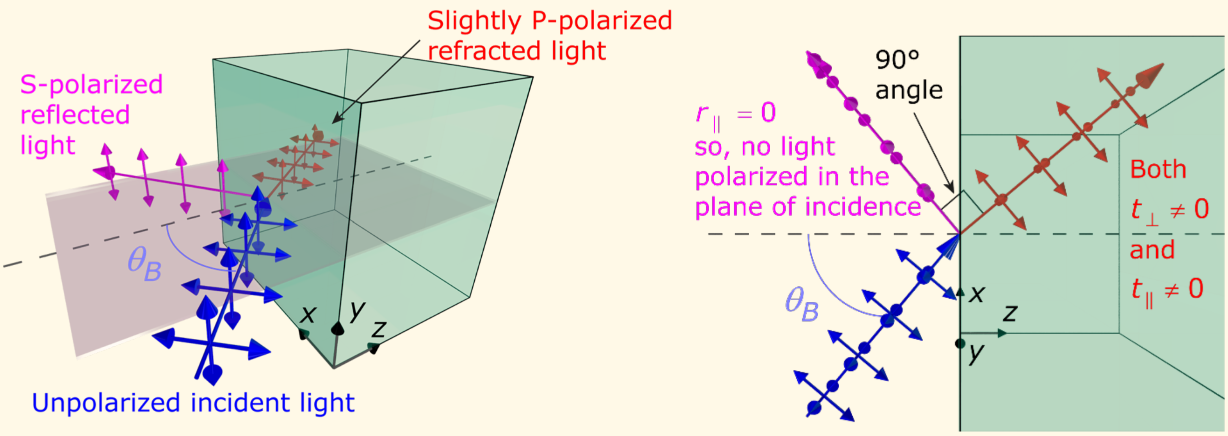


Figure 4.11. For unpolarized light, incident at the Brewster angle (θ_B), the reflected wave is S-polarized and the transmitted wave is mostly P-polarized.

For S-polarized light we have

$$r_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}, \quad \text{Eq. 4. 162}$$

which is zero only when $\theta_t = \theta_i$ and therefore $n_1 = n_2$, to satisfy Snell's law (which is not much of a boundary). It follows that the Brewster angle does not exist for S-polarized light.

The critical angle

In the case where light propagates from a medium with higher refractive index into a medium with lower refractive index, consider Snell's law:

$$n_1 \sin\theta_i = n_2 \sin\theta_t . \qquad \text{Eq. 4. 163}$$

If $\frac{n_1}{n_2}\sin\theta_i > 1$, then $\sin\theta_t$ is undefined, since the sine function cannot be larger than 1. Therefore, such angle θ_t does not exist. We define the **critical angle** θ_c as:

$$\boxed{\sin\theta_c = \frac{n_2}{n_1}}, \qquad \text{Eq. 4. 164}$$

where any incidence angle such as $\theta_i > \theta_c$ cannot satisfy Snell's law. For any such angle, there can be no transmission and this situation is referred to as **total internal reflection**.

Summary

At the boundary between two materials, EM wave is partially reflected and partially transmitted.

The plane of incidence is defined by the wave vector of the incident wave and a unit vector normal to the boundary.

On each side of the boundary, the electric and magnetic fields can be resolved into normal and tangential components.

For LIH materials, assuming no surface charges and no surface currents.

	Electric fields	Magnetic fields
Normal components	$\vec{D}_{1n} = \vec{D}_{2n}$	$\vec{B}_{1n} = \vec{B}_{2n}$
Tangential components	$\vec{E}_{1t} = \vec{E}_{2t}$	$\vec{H}_{1t} = \vec{H}_{2t}$

At normal incidence, the reflection and transmission coefficients are

$r_{\parallel/\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_1 + Z_2}$ and $t_{\parallel/\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2}{Z_1 + Z_2}$.

Snell's law results from conservation of moment at the interface: $n_1 \sin\theta_i = n_2 \sin\theta_t$.

For a general angle of incidence, we distinguish two cases of wave's electric field oscillating:
(i) in the plane of incidence (this is P-polarized light)
(ii) perpendicular to the plane of incidence (this is S-polarized light)

The Fresnel coefficients for P- and S-polarized light are:

$$r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i} \text{ and } t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}$$
$$r_{\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t} \text{ and } t_{\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}$$

For P-polarized incident lights the Brewster angle ($\tan\theta_B = n_2/n_1$) determines the incidence angle for which the reflected beam is polarized perpendicular to the plane of incidence.

For light with any polarisation, when propagating from a material with higher refractive index into a material with lower refractive index, a critical angle exists ($\sin\theta_c = n_2/n_1$) beyond which the wave experiences total internal reflection.

Example question 3

[from Sadiku] [harder] The region y≤0 consists of a perfect conductor, while the region y≥0 is a LIH dielectric medium $\varepsilon_{1r} = 2$. If there is a surface charge (σ_p) of 2 nC/m^2 on the conductor, determine $\vec{E}$ and $\vec{D}$ at:
(a) A(3,-2,2)
(b) B(-4,1,5)

Answer:
(a) The point A(3,-2,2) is in the conductor since y=-2<0 at A. Hence, $\vec{E} = 0 = \vec{D}$.

(b) The point B(-4,1,5) is in the dielectric medium since y=1>0 at B.
If the surface carries charge (Q), we can use the same pillbox (Gaussian cylinder) reasoning as we did for "Boundary conditions for the electric flux density", and we find that
$\Delta Q = \sigma_p A_I = \oint_A \vec{D} \cdot d\vec{A} = -D_{1n}A_I + D_{2n}A_I = \left(-D_{1n} + D_{2n}\right) A_I$.
Hence, since D=0 on the conductor side, inside the dielectric, $D_n = \sigma_p = 2$ nCm^{-2}. Therefore, $\vec{D} = 2\vec{a}_y$ nCm^{-2} and
$\vec{E} = \dfrac{\vec{D}}{\varepsilon_0 \varepsilon_r} = 2 \times 10^{-9} \times \dfrac{36\pi}{2} \times 10^9 \vec{a}_y = 36\pi\vec{a}_y = \underline{\underline{113.1\vec{a}_y}}$ V/m.

Example question 4

[from Sadiku] The region 1, described by 3x+4y≥10, is free space. The region 2, described by 3x+4 ≤10 is a magnetic material for which $\mu = 10\mu_0$. Assuming that the boundary between the material and free space is current free, find $\vec{B}_2$ if $\vec{B}_1 = 0.1\vec{a}_x + 0.4\vec{a}_y + 0.2\vec{a}_z$ Wbm^{-2}.

Answer:
If we let the surface of the plane be described by $f\left(x,y\right) = 3x + 4y - 10$, a unit vector normal to the plane is given by
$\vec{a}_n = \dfrac{\nabla f}{\left|\nabla f\right|} = \dfrac{3\vec{a}_x + 4\vec{a}_y}{\sqrt{3^2 + 4^2}} = \dfrac{3\vec{a}_x + 4\vec{a}_y}{5}$. We remember that the vector resolution of any vector $\vec{a}$ in the direction of $\vec{b}$ is:
$\vec{a}_1 = \left(\vec{a} \cdot \hat{b}\right)\hat{b} = \dfrac{\vec{a} \cdot \vec{b}}{\left\|\vec{b}\right\|}\dfrac{\vec{b}}{\left\|\vec{b}\right\|} = \dfrac{\vec{a} \cdot \vec{b}}{\left\|\vec{b}\right\|^2}\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\vec{b}$.

Therefore, $\vec{B}_{1n} = \left(\vec{B}_1 \cdot \vec{a}_n\right)\vec{a}_n = \left[(0.1,0.4,0.2) \cdot \dfrac{(3,4,0)}{5}\right]\dfrac{(3,4,0)}{5}$ and

$\vec{B}_{1n} = \dfrac{(0.3 + 1.6 + 0)/5}{(9 + 16 + 0)/25}\dfrac{(3,4,0)}{5} = \dfrac{1.9}{25}5\dfrac{(3,4,0)}{5} = 0.228\vec{a}_x + 0.304\vec{a}_y = \vec{B}_{2n}$.

$\vec{B}_{1t} = \vec{B}_1 - \vec{B}_{1n} = (0.1,0.4,0.2)(0.228,0.304,0)$

$\vec{B}_{1t} = -0.128\vec{a}_x + 0.096\vec{a}_y + 0.2\vec{a}_z$

Moreover, we know that $\vec{H}_{1t} = \vec{H}_{2t}$ and that in general $\vec{B} = \mu\vec{H}$, so $\frac{\vec{B}_{1t}}{\mu_1} = \frac{\vec{B}_{2t}}{\mu_2}$. Hence, $\vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t} = 10\vec{B}_{1t} = -1.28\vec{a}_x + 0.96\vec{a}_y + 2\vec{a}_z$.

So, $\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} = \underline{\underline{-1.052\vec{a}_x + 1.264\vec{a}_y + 2\vec{a}_z}}$ Wbm^{-2}.

Example question 5

[from Sadiku] The plane z=0 separates air (region 1, z≥0, $\mu = \mu_0$) from iron (region 2, z≤0, $\mu = 200\mu_0$). Given that $\vec{H} = 10\vec{a}_x + 15\vec{a}_y - 3\vec{a}_z$ Am^{-1} in air, find $\vec{B}$ in iron and the angle it makes with the interface.

Answer:

We have $\vec{H}_{1n} = -3\vec{a}_z$ and $\vec{H}_{1t} = \vec{H}_1 - \vec{H}_{1n} = 10\vec{a}_x + 15\vec{a}_y$.

$\vec{H}_{2t} = \vec{H}_{1t} = 10\vec{a}_x + 15\vec{a}_y$

$\vec{B}_{1n} = \vec{B}_{2n}$ and $\vec{B} = \mu\vec{H}$, so $\vec{H}_{2n} = \frac{\mu_1}{\mu_2}\vec{H}_{1n} = \frac{1}{200}\left(-3\vec{a}_z\right) = -0.015\vec{a}_z$.

$\vec{H}_2 = \vec{H}_{2n} + \vec{H}_{2t} = 10\vec{a}_x + 15\vec{a}_y - 0.015\vec{a}_z$

$\vec{B}_2 = \mu_2\vec{H}_2 = 200 \times 4\pi \times 10^{-7}\left(10, 15, -0.015\right)$

$\vec{B}_2 = \underline{\underline{2.51\vec{a}_x + 3.77\vec{a}_y - 0.0037\vec{a}_z}}$ mWbm^{-2}.

And $\tan\alpha = \frac{\left\|\vec{B}_{2n}\right\|}{\left\|\vec{B}_{2t}\right\|}$, so $\alpha = \tan^{-1}\frac{\left\|\mu_2\vec{H}_{2n}\right\|}{\left\|\mu_2\vec{H}_{2t}\right\|} = \tan^{-1}\frac{(0.015)}{\sqrt{10^2 + 15^2}} \approx \underline{\underline{0.048^\circ}}$.

Example question 6

[from Sadiku] A polarised wave is incident from air to polystyrene with $\mu = \mu_0$ and $\varepsilon = 2.6\varepsilon_0$ at Brewster angle. Determine the transmission angle.

Answer:

We have $\tan\theta_B = \frac{n_2}{n_1} = \frac{\sqrt{\mu_{r2}\varepsilon_{r2}}}{\sqrt{\mu_{r1}\varepsilon_{r1}}}$ and since both media are non-magnetic, $\tan\theta_B = \sqrt{\frac{\varepsilon_{r2}}{\varepsilon_{r1}}} = \sqrt{\frac{2.6\varepsilon_0}{\varepsilon_0}} \approx 1.612$.

Hence, $\theta_B = \tan^{-1}(1.612) \approx 58.19°$.

Therefore the incident angle is $\theta_i \approx 58.19°$.

Next, we can use Snell's law: $n_1 \sin\theta_B = n_2 \sin\theta_t$ and we find

$\sin\theta_t = \dfrac{n_1}{n_2}\sin\theta_B = \sqrt{\dfrac{1}{2.6}}\sin(58.19°)$ and $\theta_t = \sin^{-1}(0.555) \approx \underline{\underline{31.8}}°$.