10. For the momentum and total energy to be measured simultaneously, the operators $\hat{p}$ and $\hat{H}$ must commute. In 1D we have $\hat{p} = -i\hbar \frac{d}{dx}$ and $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$. The commutator is $\left[\hat{p}, \hat{H}\right] = \frac{i\hbar^3}{2m}\left(\frac{d}{dx}\frac{d^2}{dx^2} - \frac{d^2}{dx^2}\frac{d}{dx}\right) - i\hbar\left(\frac{d}{dx}V(x) - V(x)\frac{d}{dx}\right)$. The first bracket is zero. If we apply the second bracket to an arbitrary function $f(x)$ we get $\frac{d}{dx}\left(V(x)f(x)\right) - V(x)\frac{d}{dx}f(x) = \frac{dV}{dx}f(x)$. Therefore $\left[\hat{p}, \hat{H}\right] = -i\hbar\frac{dV}{dx}$. $\hat{p}$ and $\hat{H}$ commute only if $\frac{dV}{dx} = 0$, ie if $V(x)$ is constant.

11. For this question, we make extensive use of the useful integrals.

$$\left\langle \hat{x} \right\rangle = \frac{2}{a}\int_0^a dx\, x \sin^2\left(\frac{\pi x}{a}\right) = \frac{a}{2}$$ (ie the average position of the particle is the centre of the well)

$$\left\langle \hat{x}^2 \right\rangle = \frac{2}{a}\int_0^a dx\, x^2 \sin^2\left(\frac{\pi x}{a}\right) = \frac{a^2\left(2\pi^2 - 3\right)}{6\pi^2}$$

$$\left\langle \hat{p}_x \right\rangle = -\frac{2i\hbar}{a}\int_0^a dx \sin\left(\frac{\pi x}{a}\right)\frac{d}{dx}\sin\left(\frac{\pi x}{a}\right) = -\frac{2i\pi\hbar}{a^2}\int_0^a dx \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right) = 0$$

$$\left\langle \hat{p}_x^2 \right\rangle = -\frac{2\hbar^2}{a}\int_0^a dx \sin\left(\frac{\pi x}{a}\right)\frac{d^2}{dx^2}\sin\left(\frac{\pi x}{a}\right) = \frac{2\pi^2\hbar^2}{a^3}\int_0^a dx \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right) = \frac{\pi^2\hbar^2}{a^2}$$

From the lecture notes, $\Delta A^2 = \int \psi^*\left(\hat{A} - \left\langle \hat{A} \right\rangle\right)^2 \psi = \int \psi^*\left(\hat{A}^2 - 2\left\langle \hat{A} \right\rangle \hat{A} + \left\langle \hat{A} \right\rangle^2\right)\psi = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2$. Using this, we find $\Delta x^2 = a^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right)$, so $\Delta x = 0.18a$. We also find $\Delta p_x^2 = \frac{\pi^2\hbar^2}{a^2}$, so $\Delta p_x = 3.14\frac{\hbar}{a}$. This gives $\Delta x \Delta p_x = 0.57\hbar$, which is consistent with the uncertainty principle.

12. a) The probability of measuring both $E_1 = \frac{\hbar^2\pi^2}{2ma^2}$ and $E_2 = \frac{4\hbar^2\pi^2}{2ma^2}$ is $\frac{1}{2}$.

b) $\psi(x,t) = \frac{1}{\sqrt{2}}\,\phi_1(x)\,\exp\left(-iE_1\,t/\hbar\right) + \frac{1}{\sqrt{2}}\,\phi_2(x)\,\exp\left(-iE_2\,t/\hbar\right)$ with $\phi_1(x)$ and $\phi_2(x)$ given in the question, and E_1 and E_2 given above.

c) We want to calculate $\left\langle \hat{H} \right\rangle = \int_0^a dx\,\psi^*(x,t)\,\hat{H}\,\psi(x,t)$. Because $\phi_1(x)$ and $\phi_2(x)$ are energy eigenfunctions $\hat{H}\,\psi(x,t) = \frac{E_1}{\sqrt{2}}\,\phi_1(x)\,\exp\left(-iE_1\,t/\hbar\right) + \frac{E_2}{\sqrt{2}}\,\phi_2(x)\,\exp\left(-iE_2\,t/\hbar\right)$. Because $\phi_1(x)$ and $\phi_2(x)$ are normalised and orthogonal to each other, the integrals are easy and we find $\left\langle \hat{H} \right\rangle = \frac{E_1}{2} + \frac{E_2}{2}$. Note that this is independent of time, and is consistent with the result found in part a).

d) We want to calculate $\left\langle \hat{x} \right\rangle = \int_0^a dx\,\psi^*(x,t)\,x\,\psi(x,t)$. The algebra is messy, but we eventually find

$$\left\langle \hat{x} \right\rangle = \frac{1}{a}\int_0^a dx\,x\sin^2\left(\frac{\pi x}{a}\right) + \frac{1}{a}\int_0^a dx\,x\sin^2\left(\frac{2\pi x}{a}\right) + \frac{2}{a}\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)\int_0^a dx\,x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right).$$

Using the useful integrals, we find $\left\langle \hat{x} \right\rangle = \frac{a}{2} - \frac{16a}{9\pi^2}\cos\left(\frac{(E_1 - E_2)t}{\hbar}\right)$. Note how the average position of the particle oscillates about the centre of the well (ie $a/2$), with a frequency determined by the difference in the energy levels of $\phi_1(x)$ and $\phi_2(x)$.

13. a) We want to calculate $c(k) = \frac{1}{2\pi}\sqrt{\frac{2}{a}}\int_0^a dx\,\sin\left(\frac{\pi x}{a}\right)\exp\left(-ikx\right)$. The useful integrals give

$$c(k) = \frac{1}{2\pi}\sqrt{\frac{2}{a}}\;\frac{\pi a\left(1+\exp\left(-ika\right)\right)}{\pi^2 - k^2a^2}.$$

b) We find that $\left|c(k)\right|^2 \propto \frac{\cos^2\left(\frac{ka}{2}\right)}{\left(\pi^2 - k^2a^2\right)^2}$. This is symmetrical about $k = 0$, so we are equally likely to measure positive and negative momenta. This is consistent with $\left\langle \hat{p}_x \right\rangle = 0$ in question 11. $\left|c(k)\right|^2$ is strongly peaked for $k = \pm\frac{\pi}{a}$. Given that $p = \hbar k$, this is consistent with $\left\langle \hat{p}_x^2 \right\rangle = \frac{\pi^2\hbar^2}{a^2}$ in question 11.

14. From the lecture notes, we know that the width of a Gaussian wavepacket varies with time as $\Delta x(t)=\sqrt{\frac{1}{2a}+\frac{\hbar^2 at^2}{2m^2}}$. At $t=0$, $\Delta x=\sqrt{\frac{1}{2a}}=1\,\text{Å}$.

a) We need $\frac{\hbar^2 at^2}{2m^2}=\frac{3}{2a}$, ie $t=\frac{\sqrt{3}m}{\hbar a}$. Putting in the numbers gives $t\approx 3\times 10^{-16}$ s.

b) The second term under the square root in $\Delta x(t)$ dominates in this case, so $\Delta x=\frac{\hbar t\sqrt{a}}{\sqrt{2}m}$.

Putting in the numbers gives $\Delta x\approx 6\times 10^{5}$ m.