University of Bath

Department of Physics

Year 3

PH30030 – Quantum Mechanics

Friday, 17th January 2020, 09:30 – 11:30

Answer ALL questions

The only calculators that may be used are those supplied by the University.

Please fill in your name and sign the section on the right of your answer book, peel away adhesive strip and seal.

Take care to enter the correct candidate number as detailed on your desk label.

CANDIDATES MUST NOT TURN OVER THE PAGE AND READ THE EXAMINATION PAPER UNTIL THE CHIEF INVIGILATOR GIVES PERMISSION TO DO SO

1. Consider the observable associated with the operator $\hat{Q}$. Show that the observable associated with the operators $\hat{Q}$, $\hat{Q}^2$, $\hat{Q}^3$ etc., are all compatible. Hence show that the linear momentum of a particle in one-dimension can always be measured compatibly with the non-relativistic kinetic energy. (4)

2. At time $t = 0$, the normalised wavefunction of a particle in a one-dimensional infinite square well potential defined between $x = 0$ and $x = a$ is given by

$$\psi(x,0) = c_1\,\phi_1(x) + c_2\,\phi_2(x)$$

where $c_1 = \sqrt{4/5}$ and $c_2 = \sqrt{1/5}$, and $\phi_1(x)$ and $\phi_2(x)$ are the normalised and orthogonal eigenfunctions corresponding to the lowest energy eigenvalues E_1 and E_2, respectively.

(a) Verify that the coefficient c_1 at time $t = 0$ is given by the expression

$$c_1 = \int_0^a dx\,\phi_1^*(x)\psi(x,0). \qquad (2)$$

(b) At $t = 0$, what is the probability of measuring the energy to be E_1 or E_2? (1) $\frac{4}{5}, \frac{1}{5}$

(c) What is the wavefunction at later time t? (2)

(d) What is the expectation value of the energy $\langle \hat{H} \rangle$ at time t? Express your answer in terms of E_1 and E_2. (4)

$\frac{4}{5}E_1 + \frac{1}{5}E_2$

3. The potential energy of a one-dimensional harmonic oscillator of mass m and angular frequency ω, with position x about the mean $\langle \hat{x} \rangle$, is given by

$$V(x) = \frac{1}{2} m\omega^2 \left(x - \langle \hat{x} \rangle\right)^2 .$$

The uncertainty relation between the root mean square displacement of the particle about its mean, Δx, and the root mean square deviation of the particle momentum about its mean, Δp_x, is given by $\Delta p_x \Delta x \geq \hbar/2$.

(a) For this system, briefly explain why $\langle \hat{p}_x \rangle = 0$. (1)

(b) Use the uncertainty relation to express the mean kinetic energy of the oscillator in terms of Δx. $\langle \hat{T} \rangle \geq \hbar^2 / 8m \Delta x^2$ (4)

(c) Hence express the mean total energy of the oscillator in terms of Δx. (2)

(d) Use this expression to estimate the zero-point energy of the harmonic oscillator. (4)

$\langle \hat{H} \rangle_{min} = \frac{\hbar\omega}{2}$

Note: $\Delta A^2 = \left\langle \left(\hat{A} - \langle \hat{A} \rangle\right)^2 \right\rangle$

4. The z component of the electron spin operator $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Show that $|\phi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are normalised eigenvectors of $\hat{S}_z$. Find the corresponding eigenvalues. (3)

(b) A general spin wavefunction can be written as $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$.

Express $|\psi\rangle$ in terms of the eigenvectors of $\hat{S}_z$. (2)

(c) For the wavefunction $|\psi\rangle$, find the probability of measuring the eigenvalues corresponding to $|\phi_1\rangle$ and $|\phi_2\rangle$. (1)

(d) Find the expectation value of $\hat{S}_z$ for $|\psi\rangle$. $\frac{\hbar}{2}\left(|a|^2 - |b|^2\right)$ (2)

5. Consider two non-interacting particles in a one-dimensional infinite potential well described by $V(x)=0$ for $0 \le x \le a$, $V(x)=\infty$ for $x<0$ and $x>a$. The normalised one-particle states are given by $u_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ with $E_n = n^2\frac{\hbar^2\pi^2}{2ma^2}$.

(a) If the particles are indistinguishable bosons find the energy eigenfunctions and eigenvalues for the two-particle ground state and for the first excited state. Find the degeneracy of these states. (5)

(b) If the particles are indistinguishable fermions with parallel spins: -

(i) Find the energy eigenfunction if both particles try to occupy the same state with $n=1$. Briefly describe the physical interpretation of this result. (4)

(ii) Find the energy eigenfunctions and eigenvalue for the two-particle ground state. Comment on the degeneracy of this state. (3)

Note: There is no need to normalise the eigenfunctions.

6. (a) Outline the variational method for calculating the ground state energy of a system. (4)

(b) Estimate the ground state energy of a particle in 1D if the potential energy is given by $V(x)=-\alpha\delta(x)$, where α is a constant and $\delta(x)$ is the Dirac delta function, using the Gaussian trial wavefunction $\psi(x)=Ae^{-bx^2}$. Comment on the sign of the ground state energy. (12)

Note: -

$\langle\hat{T}\rangle = \frac{\hbar^2 b}{2m}$; $\langle\hat{V}\rangle = -\alpha\left(\frac{2b}{\pi}\right)^{1/2}$

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \text{ for } a>0$$

$\langle\hat{H}\rangle_{min} = -\frac{m\alpha^2}{\pi\hbar^2} \geq E_{ground}$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{\frac{1}{2}} \text{ for } a>0$$

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