University of Bath

Department of Physics

PH30030

Quantum Mechanics

19th January 2023
13:30 – 16:30
180 minutes

Answer all questions

Only calculators provided by the University may be used

University Formula Book with fundamental Constant tables provided by the University may be used.

During this exam you are not permitted to communicate with any person(s) except an invigilator or an assigned support worker.

You must not have any unauthorised devices or materials with you.

You must keep your Library card on your desk at all times.

Please fill in the details on the front of your answer book/cover and sign in the section on the right of your answer book/cover, peel away adhesive strip and seal.

Take care to enter the correct candidate number as detailed on your desk label.

Do not turn over your question paper until instructed to by the chief invigilator.

1. (a) Give the definition of a Hermitian operator. (1)

(b) Prove that the eigenvalues of Hermitian operators are real. (4)

2. A particle moving along the x axis in a region of space where the potential energy is zero everywhere is described by the wavefunction $\psi(x) = A\exp(ikx)$ where A is a normalisation constant.

(a) Find the momentum and total energy eigenvalues. Show that these values are consistent with the de Broglie relation. $\hbar k$; $\hbar^2 k^2/2m$ (3)

(b) By confining the particle to a box of finite size, find the normalisation constant A. (1)

3. Consider the observable associated with the operator $\hat{Q}$. Show that the observables associated with the operators $\hat{Q}$, $\hat{Q}^2$, $\hat{Q}^3$ etc., are all compatible. Hence show that the linear momentum of a particle in one-dimension can always be measured compatibly with the non-relativistic kinetic energy. (4)

4. For a spin-half system, the spin operator $\hat{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$, where $\hat{S}_i = \frac{\hbar}{2}\sigma_i$ and the Pauli spin matrices are given by $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The eigenvectors of $\hat{S}_z$ are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues of $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$, respectively.

(a) For the orbital angular momentum, the ladder operators are defined by $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$.

(i) Using this formalism, obtain matrices representing the raising and lowering operators for a spin-half system. $\hat{S}_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\hat{S}_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (2)

(ii) By applying these operators to the eigenstates of $\hat{S}_z$, verify that they act as ladder operators. (4)

(iii) Verify that both eigenstates of $\hat{S}_z$ are also eigenstates of $\hat{S}^2$ and correspond to the same eigenvalue of this operator. (3)

(b) Calculate the expectation values $\langle S_x{}^2 \rangle$ and $\langle S_y{}^2 \rangle$ for a spin-half particle known to be in an eigenstate of $\hat{S}_z$. Show that their product is consistent with the uncertainty principle, which states that

$$\langle S_x{}^2 \rangle \langle S_y{}^2 \rangle \geq \frac{1}{4}\hbar^2 \langle S_z{}^2 \rangle . \qquad (4)$$

5. Consider two non-interacting spin-half fermions in a one-dimensional infinite square well potential.

(a) Describe the symmetry condition for the two-particle wavefunction and how this wavefunction can be built from spatial- and spin-dependent parts. (2)

(b) The orthogonal and normalised one-particle states are given by $u_n(x_i)$ and the spin states are given by α_i and β_i, with i = 1, 2 numbering the particles. Find the ground state wavefunction if the spins of the fermions are

(i) parallel $\phi(1,2) = u_-(1,2)\,\chi_{1M_s}(1,2)$ (3)

(ii) anti-parallel $\phi(1,2) = u_+(1,2)\,\chi_{00}(1,2)$ (3)

Note that: -

$$\chi_{00}(1,2) = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 - \alpha_2\beta_1),$$

$$\chi_{10}(1,2) = \frac{1}{\sqrt{2}}(\alpha_1\beta_2 + \alpha_2\beta_1), \quad \chi_{11}(1,2) = \alpha_1\alpha_2, \quad \chi_{1-1}(1,2) = \beta_1\beta_2$$

(c) For both the parallel and anti-parallel spin states, describe what happens to the ground state wavefunction if both particles try to occupy the same region of space.

(4)

6. The potential energy of a one-dimensional harmonic oscillator at position x about the mean $\langle x \rangle$ is given by

$$V(x) = \frac{1}{2}k\left(x - \langle x \rangle\right)^2$$

where $k = m\omega^2$ is the spring constant.

(a) Explain why energy quantisation is not observed for a macroscopic system. (2)

(b) For a quantum system: -

(i) Briefly explain why $\langle \hat{p}_x \rangle = 0$. (1)

(ii) Use the uncertainty relation to place a lower bound on the mean kinetic energy of the oscillator in terms of Δx. $\langle \hat{T} \rangle \geq \hbar^2 / 8m(\Delta x)^2$ (5)

(iii) Hence, derive the lower bound on the total energy of the oscillator in terms of Δx. $\langle \hat{H} \rangle_{min} = \hbar^2 / 8m(\Delta x)^2 + m\omega^2(\Delta x)^2/2$ (2)

(iv) Use this expression to estimate the zero-point energy of the harmonic oscillator. $\hbar\omega/2$ (4)

(c) Discuss briefly how time-independent perturbation theory is used in the approximate evaluation of the ground-state energies of non-degenerate systems, for which the exact quantum mechanical solutions cannot be found. (4)

(d) Consider an *anharmonic* oscillator with mean position $\langle x \rangle = 0$ described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^{\,2}}{2m} + \frac{1}{2}kx^2(1+\alpha x^2)$$

where α is a constant. Derive an approximate (first-order) expression for the ground state energy of this oscillator, given that the wave function for the harmonic oscillator is $\phi_0(x) = (m\omega/\pi\hbar)^{1/4}\exp\left(-m\omega x^2/2\hbar\right)$. (4)

Note: $\displaystyle\int_{-\infty}^{\infty} x^4 e^{-bx^2} dx = \frac{3}{4}\left[\frac{\pi}{b^5}\right]^{1/2}$ $\qquad E_0 = \dfrac{\hbar\omega}{2} + \dfrac{3}{8}\alpha\dfrac{\hbar^2}{m}$

THE END

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