7. Vector Integral Theorems

In this last section, we will look at some theorems which relate one type of integral to another.
They have at least 2 uses:

1. Simplify evaluation of integrals
2. Aid derivation of differential field equations.

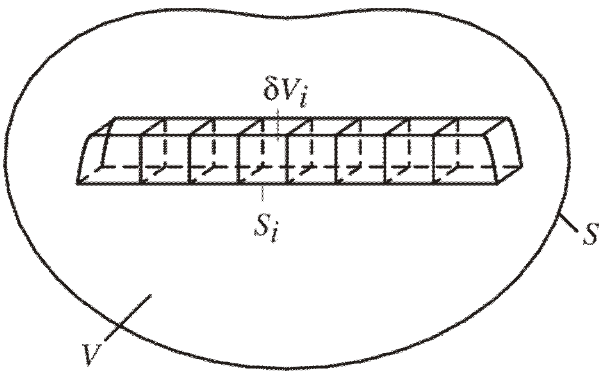
7.1 Gauss' Divergence Theorem

If $\mathbf{F}$ is a continuous vector field, in a volume V enclosed by surface S, then

$$\int_V \operatorname{div}\mathbf{F}\, dV = \oint_S \mathbf{F}\cdot d\mathbf{S}$$

Proof.

Divide volume V into a system of elementary volumes $\{\delta V_i\}$, as shown.



From our previous derivation of an expression for $\operatorname{div}\mathbf{F}$, for the small volume δV_i with bounding surface S_i

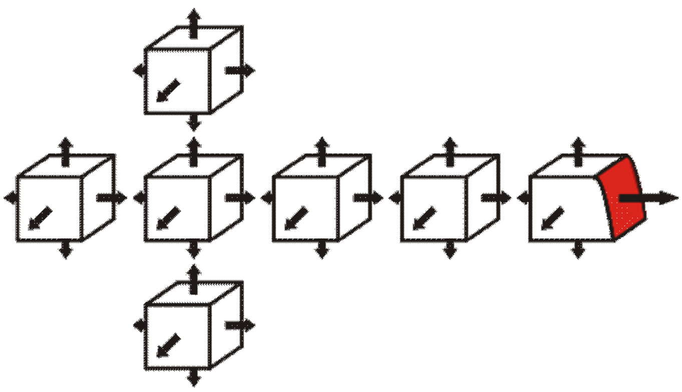
$$\operatorname{div}\mathbf{F} \simeq \frac{1}{\delta V_i}\oint_{S_i}\mathbf{F}\cdot d\mathbf{S} \qquad \text{or} \qquad \operatorname{div}\mathbf{F}\,\delta V_i \simeq \oint_{S_i}\mathbf{F}\cdot d\mathbf{S}$$

- correct to first order in δV_i; exact as $\delta V_i \to 0$.
- Says $\operatorname{div}\mathbf{F}\,\delta V_i \approx$ the total outward flux of $\mathbf{F}$ across the surface S_i.

Add contributions from all the elemental regions δV_i:

$$\sum_i \operatorname{div}\mathbf{F}\,\delta V_i \xrightarrow[\delta V_i \to 0]{} \int_V \operatorname{div}\mathbf{F}\,dV$$

= total outward flux of $\mathbf{F}$ across all the surfaces of all the elements δV_i $= \sum_i \oint_{S_i}\mathbf{F}\cdot d\mathbf{S}$



The flux through any internal face is equal & opposite to the flux through the shared face of the adjacent block .

The only exceptions are faces which form part of the external surface S, as these are not shared with any neighbouring block.

Adding all the fluxes, we get $\sum_i \oint_{S_i} \mathbf{F} \cdot d\mathbf{S} = \oint_S \mathbf{F} \cdot d\mathbf{S}$

and so

$$\int_V \operatorname{div} \mathbf{F} \, dV = \oint_S \mathbf{F} \cdot d\mathbf{S}$$

The total divergence of $\mathbf{F}$ from a volume V equals the total outward flux of $\mathbf{F}$ through the surface enclosing V.

[Note the analogy with

$$\int_a^b \frac{dF}{dx} dx = F(x)\Big|_a^b = F(b) - F(a)]$$