PH30030: Quantum Mechanics Problems Sheet 2 Solutions

Note: $\underline{L} = \underline{r} \times \underline{p} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \underline{i}\left(yp_z - zp_y\right) - \underline{j}\left(xp_z - zp_x\right) + \underline{k}\left(xp_y - yp_x\right)$

1. $\left[\hat{L}_y, \hat{L}_z\right] = (\hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z)\left(\hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x\right) - \left(\hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x\right)(\hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z)$. Multiplying out gives
$\left[\hat{L}_y, \hat{L}_z\right] = \hat{z}\,\hat{p}_x\,\hat{x}\,\hat{p}_y - \hat{z}\,\hat{p}_x\,\hat{y}\,\hat{p}_x - \hat{x}\,\hat{p}_z\,\hat{x}\,\hat{p}_y + \hat{x}\,\hat{p}_z\,\hat{y}\,\hat{p}_x - \hat{x}\,\hat{p}_y\,\hat{z}\,\hat{p}_x + \hat{x}\,\hat{p}_y\,\hat{x}\,\hat{p}_z + \hat{y}\,\hat{p}_x\,\hat{z}\,\hat{p}_x - \hat{y}\,\hat{p}_x\,\hat{x}\,\hat{p}_z$
The only non-commuting operators here are $\hat{x}$ and $\hat{p}_x$ ($[\hat{x}, \hat{p}_x] = i\hbar$), so the 2nd and 7th terms cancel, and the 3rd and 6th terms cancel. The remaining terms give

$$\begin{aligned}\left[\hat{L}_y, \hat{L}_z\right] &= \hat{z}\,\hat{p}_x\,\hat{x}\,\hat{p}_y + \hat{x}\,\hat{p}_z\,\hat{y}\,\hat{p}_x - \hat{x}\,\hat{p}_y\,\hat{z}\,\hat{p}_x - \hat{y}\,\hat{p}_x\,\hat{x}\,\hat{p}_z \\ &= \hat{z}\,\hat{p}_y\left(\hat{p}_x\,\hat{x} - \hat{x}\,\hat{p}_x\right) + \hat{y}\,\hat{p}_z\left(\hat{x}\,\hat{p}_x - \hat{p}_x\,\hat{x}\right) \\ &= \left(\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y\right)\left(\hat{x}\,\hat{p}_x - \hat{p}_x\,\hat{x}\right)\end{aligned}$$

By definition, $\left(\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y\right) = \hat{L}_x$ (see notes). $\left(\hat{x}\,\hat{p}_x - \hat{p}_x\,\hat{x}\right)$ is the commutator of $\hat{x}$ and $\hat{p}_x$, which was evaluated in section 1.6 of the lecture notes. We find $\left(\hat{x}\,\hat{p}_x - \hat{p}_x\,\hat{x}\right) = i\hbar$ and so $\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\,\hat{L}_x$. We cannot measure simultaneously the *y* and *z* components of the orbital angular momentum.

2. $\left[\hat{L}_x^2, \hat{L}_z\right] = \hat{L}_x\,\hat{L}_x\,\hat{L}_z - \hat{L}_z\,\hat{L}_x\,\hat{L}_x$. The commutation relation $\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\,\hat{L}_y$ can be written as $\hat{L}_z\,\hat{L}_x - \hat{L}_x\,\hat{L}_z = i\hbar\,\hat{L}_y$, from which we find $\hat{L}_z\,\hat{L}_x = i\hbar\,L_y + \hat{L}_x\,\hat{L}_z$ and $\hat{L}_x\,\hat{L}_z = -i\hbar\,L_y + \hat{L}_z\,\hat{L}_x$.
These two expressions are substituted into

$$\begin{aligned}\left[\hat{L}_x^2, \hat{L}_z\right] &= \hat{L}_x\,\hat{L}_x\,\hat{L}_z - \hat{L}_z\,\hat{L}_x\,\hat{L}_x \\ &= \hat{L}_x\left(-i\hbar\,L_y + \hat{L}_z\,\hat{L}_x\right) - \left(i\hbar\,L_y + \hat{L}_x\,\hat{L}_z\right)\hat{L}_x \\ &= -i\hbar\left(\hat{L}_x\,\hat{L}_y + \hat{L}_y\,\hat{L}_x\right),\end{aligned}$$

as required. $\left[\hat{L}_y^2, \hat{L}_z\right]$ can be worked through in a similar way – I'll leave this one to you.

3. $\hat{L}_+\,\hat{L}_- = \left(\hat{L}_x + i\hat{L}_y\right)\left(\hat{L}_x - i\hat{L}_y\right) = \hat{L}_x^2 + \hat{L}_y^2 - i\,\hat{L}_x\,\hat{L}_y + i\,\hat{L}_y\,\hat{L}_x$. But $\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \hat{L}^2$ and $\hat{L}_x\,\hat{L}_y - \hat{L}_y\,\hat{L}_x = \left[\hat{L}_x, \hat{L}_y\right] = i\hbar\,\hat{L}_z$. So,

$$\hat{L}_+\,\hat{L}_- = \hat{L}^2 - \hat{L}_z^2 - i\left(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x\right) = \hat{L}^2 - \hat{L}_z^2 + \hbar\,\hat{L}_z.$$

Similarly, $\hat{L}_-\,\hat{L}_+ = \left(\hat{L}_x - i\hat{L}_y\right)\left(\hat{L}_x + i\hat{L}_y\right) = \hat{L}_x^2 + \hat{L}_y^2 + i\,\hat{L}_x\,\hat{L}_y - i\,\hat{L}_y\,\hat{L}_x = \hat{L}^2 - \hat{L}_z^2 - \hbar\,\hat{L}_z$.

So, $\left[\hat{L}_+, \hat{L}_-\right] = \hat{L}_+\,\hat{L}_- - \hat{L}_-\,\hat{L}_+ = 2\hbar\,\hat{L}_z$.

$\left[\hat{L}_z,\hat{L}_+\right]=\left[\hat{L}_z,\hat{L}_x+i\hat{L}_y\right]=\left[\hat{L}_z,\hat{L}_x\right]+i\left[\hat{L}_z,\hat{L}_y\right]=\left[\hat{L}_z,\hat{L}_x\right]-i\left[\hat{L}_y,\hat{L}_z\right]$. The angular momentum commutators $\left[\hat{L}_z,\hat{L}_x\right]=i\hbar\hat{L}_y$ and $\left[\hat{L}_y,\hat{L}_z\right]=i\hbar\hat{L}_x$.

So, $\left[\hat{L}_z,\hat{L}_+\right]=i\hbar\,\hat{L}_y-i\,i\hbar\,\hat{L}_x=\hbar\left(\hat{L}_x+i\hat{L}_y\right)=\hbar\,\hat{L}_+$.

$\left[\hat{L}_z,\hat{L}_-\right]=\left[\hat{L}_z,\hat{L}_x-i\hat{L}_y\right]=\left[\hat{L}_z,\hat{L}_x\right]-i\left[\hat{L}_z,\hat{L}_y\right]=\left[\hat{L}_z,\hat{L}_x\right]+i\left[\hat{L}_y,\hat{L}_z\right]$. From the angular momentum commutators we get $\left[\hat{L}_z,\hat{L}_-\right]=i\hbar\,\hat{L}_y+i\,i\hbar\,\hat{L}_x=-\hbar\left(\hat{L}_x-i\hat{L}_y\right)=-\hbar\,\hat{L}_-$.

4. Start from $\hat{L}_z\left|\phi_n\right\rangle=\beta_n\left|\phi_n\right\rangle$. Operate with $\hat{L}_-$ on both sides to give $\hat{L}_-\,\hat{L}_z\left|\phi_n\right\rangle=\beta_n\,\hat{L}_-\left|\phi_n\right\rangle$.
From the commutation relation $\left[\hat{L}_z,\hat{L}_-\right]=\hat{L}_z\,\hat{L}_--\hat{L}_-\,\hat{L}_z=-\hbar\,\hat{L}_-$ (question 3) we find $\hat{L}_-\,\hat{L}_z=\hbar\,\hat{L}_-+\hat{L}_z\,\hat{L}_-$.
Substituting this in, we find $\hbar\,\hat{L}_-\left|\phi_n\right\rangle+\hat{L}_z\,\hat{L}_-\left|\phi_n\right\rangle=\beta_n\,\hat{L}_-\left|\phi_n\right\rangle$, which can be re-arranged to give $\hat{L}_z\left(\hat{L}_-\left|\phi_n\right\rangle\right)=\left(\beta_n-\hbar\right)\left(\hat{L}_-\left|\phi_n\right\rangle\right)$.
This shows that $\hat{L}_-\left|\phi_n\right\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $\left(\beta_n-\hbar\right)$.

5. Start from $\hat{L}_-\left|\phi_{\min}\right\rangle=0$ (at bottom of ladder of eigenvalues of $\hat{L}_z$), so $\hat{L}_+\,\hat{L}_-\left|\phi_{\min}\right\rangle=0$.
In the notes (and in question 3) we showed that $\hat{L}_+\,\hat{L}_-=\hat{L}^2-\hat{L}_z^2+\hbar\,\hat{L}_z$.
So $\left(\hat{L}^2-\hat{L}_z^2+\hbar\,\hat{L}_z\right)\left|\phi_{\min}\right\rangle=0$.
But $\hat{L}^2\left|\phi_{\min}\right\rangle=\alpha\left|\phi_{\min}\right\rangle$ and $\hat{L}_z\left|\phi_{\min}\right\rangle=\beta_{\min}\left|\phi_{\min}\right\rangle$, so $\left(\alpha-\beta_{\min}^2+\hbar\,\beta_{\min}\right)\left|\phi_{\min}\right\rangle=0$.
Therefore, $\alpha=\beta_{\min}\left(\beta_{\min}-\hbar\right)$.

6. $\hat{L}_+=\hat{L}_x+i\hat{L}_y=i\hbar\sin\phi\dfrac{\partial}{\partial\theta}+i\hbar\cot\theta\cos\phi\dfrac{\partial}{\partial\phi}+\hbar\cos\phi\dfrac{\partial}{\partial\theta}-\hbar\cot\theta\sin\phi\dfrac{\partial}{\partial\phi}$.
Collecting terms, we get $\hat{L}_+=\hbar\left(\cos\phi+i\sin\phi\right)\dfrac{\partial}{\partial\theta}+i\hbar\left(\cos\phi+i\sin\phi\right)\cot\theta\dfrac{\partial}{\partial\phi}$.
But $\exp\left(i\phi\right)=\cos\phi+i\sin\phi$, so $\hat{L}_+=\hbar\exp\left(i\phi\right)\left(\dfrac{\partial}{\partial\theta}+i\cot\theta\dfrac{\partial}{\partial\phi}\right)$.

$$\hat{L}_{-} = \hat{L}_x - i\hat{L}_y = i\hbar\sin\phi\frac{\partial}{\partial\theta} + i\hbar\cot\theta\cos\phi\frac{\partial}{\partial\phi} - \hbar\cos\phi\frac{\partial}{\partial\theta} + \hbar\cot\theta\sin\phi\frac{\partial}{\partial\phi}.$$

Collecting terms, we get $\hat{L}_{-} = -\hbar\left(\cos\phi - i\sin\phi\right)\frac{\partial}{\partial\theta} + i\hbar\left(\cos\phi - i\sin\phi\right)\cot\theta\frac{\partial}{\partial\phi}$.

But $\exp(-i\phi) = \cos\phi - i\sin\phi$, so $\hat{L}_{-} = \hbar\exp(-i\phi)\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$.

$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1\pm1}(\theta,\phi$

7. a) From the notes, $|Y_{1-1}\rangle \propto \sin\theta\exp(-i\phi)$, so $\hat{L}_{+}|Y_{1-1}\rangle \propto \exp(i\phi)\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(\sin\theta\exp(-i\phi)\right)$. Doing the differentiations gives $\hat{L}_{+}|Y_{1-1}\rangle \propto \exp(i\phi)\left(\cos\theta\exp(-i\phi) + \cot\theta\sin\theta\exp(-i\phi)\right)$, which is proportional to $\cos\theta$, as required.

$\cot\theta = \cos\theta/\sin\theta$

b) From the notes, $|Y_{10}\rangle \propto \cos\theta$, so $\hat{L}_{+}|Y_{10}\rangle \propto \exp(i\phi)\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\cos\theta$. Doing the differentiation gives $\hat{L}_{+}|Y_{10}\rangle \propto \exp(i\phi)\sin\theta$, which is proportional to $|Y_{11}\rangle$, as required.

c) From the notes, $|Y_{11}\rangle \propto \sin\theta\exp(i\phi)$, so $\hat{L}_{+}|Y_{11}\rangle \propto \exp(i\phi)\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\sin\theta\exp(i\phi)$. Doing the differentiation gives $\hat{L}_{+}|Y_{11}\rangle \propto \exp(i\phi)\left(\cos\theta\exp(i\phi) - \cot\theta\sin\theta\exp(i\phi)\right) = 0$, as required (at top of ladder of eigenvalues of $\hat{L}_z$).

8. $\left[\hat{S}_y, \hat{S}_z\right] = \hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y = \frac{\hbar^2}{4}\left(\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\right)$. Multiplying out the matrices gives $\left[\hat{S}_y, \hat{S}_z\right] = \frac{\hbar^2}{4}\left(\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}\right) = \frac{\hbar^2}{4}\begin{pmatrix}0 & 2i\\ 2i & 0\end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = i\hbar\hat{S}_x$, as required. We cannot measure simultaneously the *y* and *z* components of spin.

$\left[\hat{S}_z, \hat{S}_x\right] = \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z = \frac{\hbar^2}{4}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$. Multiplying out the matrices gives $\left[\hat{S}_z, \hat{S}_x\right] = \frac{\hbar^2}{4}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right) = \frac{\hbar^2}{4}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix} = i\hbar\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = i\hbar\hat{S}_y$, as required. We cannot measure simultaneously the *y* and *z* components of spin.

9. For $\hat{S}_x$ we have $\hat{S}_x\left|\hat{S}_x;\uparrow\right\rangle = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{\hbar}{2}\left|\hat{S}_x;\uparrow\right\rangle$ and

$\hat{S}_x\left|\hat{S}_x;\downarrow\right\rangle = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = -\frac{\hbar}{2}\left|\hat{S}_x;\downarrow\right\rangle$, as required.

For $\hat{S}_y$ we have $\frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ and

$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\i\end{pmatrix} = -\frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$, as required.

For $\hat{S}_z$ we have $\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\0\end{pmatrix}$ and $\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix}$, as required.

As discussed in section 1.11 of the lecture notes, when we are using a matrix representation the kets are column vectors and the bras are row vectors.

For $\hat{S}_x$ the eigenvectors kets are $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ and the respective eigenvector bras are $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}$.

For $\hat{S}_y$ the eigenvector kets are $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ and the respective eigenvector bras are $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}$, where the complex conjugation should be noted.

For $\hat{S}_z$ the eigenvectors kets are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ and the respective eigenvector bras are $\begin{pmatrix}1 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 1\end{pmatrix}$.

The normalisation and orthogonality properties then follow by matrix multiplication.

For example,

$$\left\langle\hat{S}_y;\uparrow\middle|\hat{S}_y;\uparrow\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = 1$$

$$\left\langle\hat{S}_x;\downarrow\middle|\hat{S}_x;\uparrow\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = 0$$

10. a) For $\hat{S}_x$, $|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ so $c_1 = \langle\phi_1|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{a+b}{\sqrt{2}}$ and

$|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ so $c_2 = \langle\phi_2|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{a-b}{\sqrt{2}}$.

Also, $\langle\psi|\psi\rangle = \begin{pmatrix}a^* & b^*\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = |a|^2 + |b|^2 = 1$.

The probability of measuring $+\frac{\hbar}{2}$ is therefore $\left|\frac{a+b}{\sqrt{2}}\right|^2 = \frac{|a|^2+|b|^2+ab^*+a^*b}{2} = \frac{1}{2} + \mathrm{Re}\left(ab^*\right)$.

Note: $\left(ab^* + a^*b\right) = \left(a_1 + ia_2\right)\left(b_1 - ib_2\right) + \left(a_1 - ia_2\right)\left(b_1 + ib_2\right) = 2\left(a_1b_1 + a_2b_2\right)$ and

$\mathrm{Re}\left(ab^*\right) = \mathrm{Re}\left[\left(a_1 + ia_2\right)\left(b_1 - ib_2\right)\right] = a_1b_1 + a_2b_2$. So, $\left(ab^* + a^*b\right)/2 = a_1b_1 + a_2b_2 = \mathrm{Re}\left(ab^*\right)$

The probability of measuring $-\frac{\hbar}{2}$ is $\left|\frac{a-b}{\sqrt{2}}\right|^2 = \frac{|a|^2+|b|^2-ab^*-a^*b}{2} = \frac{1}{2} - \mathrm{Re}\left(ab^*\right)$.

b) For $\hat{S}_y$, $|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ so $c_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{a-ib}{\sqrt{2}}$ and

$|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ so $c_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \frac{a+ib}{\sqrt{2}}$.

The probability of measuring $+\frac{\hbar}{2}$ is therefore

$$\left|\frac{a-ib}{\sqrt{2}}\right|^2 = \frac{|a|^2+|b|^2+i\,ab^*-i\,a^*b}{2} = \frac{1}{2} - \mathrm{Im}\left(ab^*\right)$$

The probability of measuring $-\frac{\hbar}{2}$ is $\left|\frac{a+ib}{\sqrt{2}}\right|^2 = \frac{|a|^2+|b|^2-i\,ab^*+i\,a^*b}{2} = \frac{1}{2} + \mathrm{Im}\left(ab^*\right)$.

c) For $\hat{S}_z$, $|\phi_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ so $c_1 = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = a$ and

$|\phi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ so $c_2 = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = b$.

The probability of measuring $+\frac{\hbar}{2}$ is therefore $|a|^2$ and the probability of measuring $-\frac{\hbar}{2}$ is $|b|^2$.