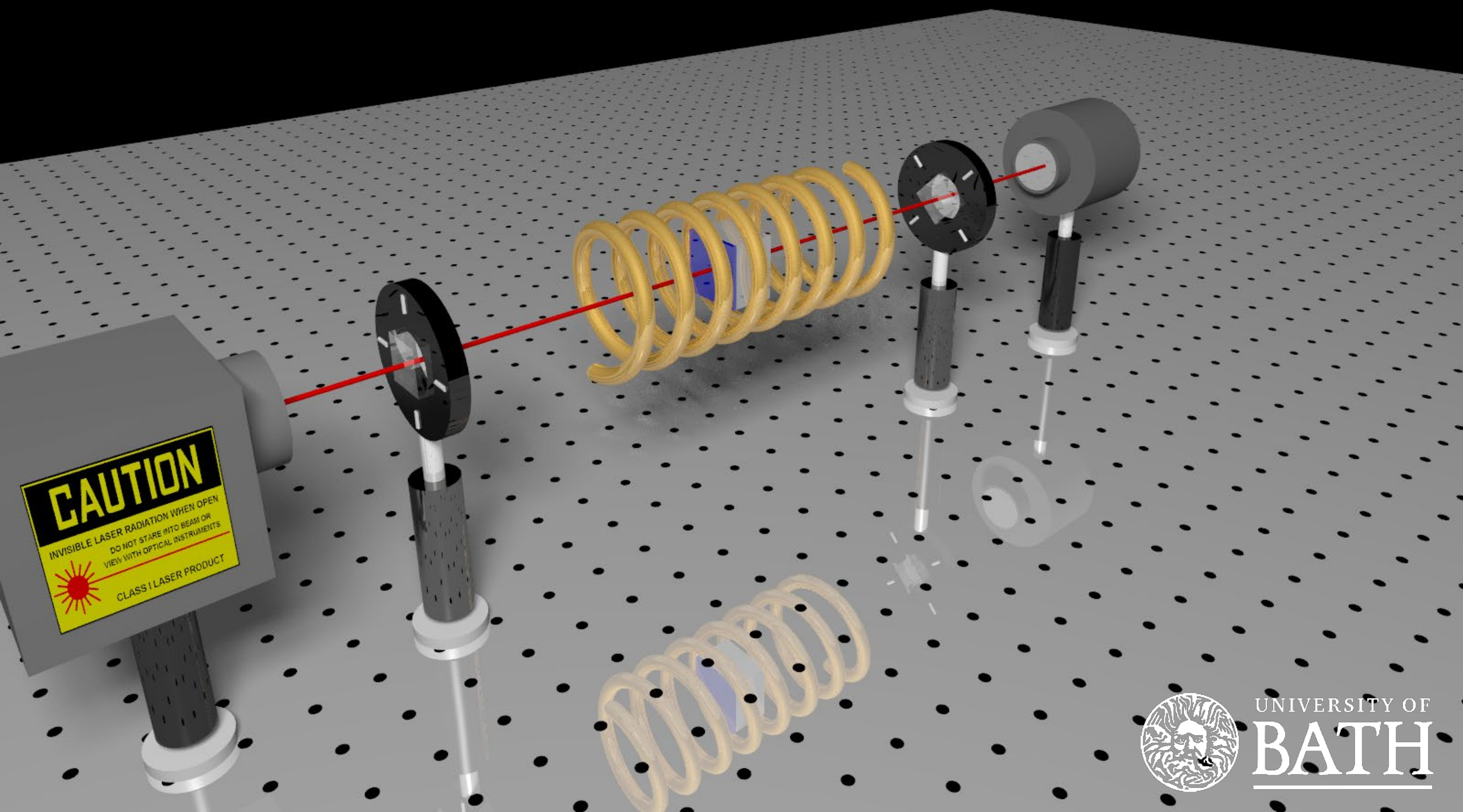
Lecture 13

Magnetic fields in materials



Last time we saw — use this for the A4!

In a dielectric, microscopic dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.

Macroscopically, the combined effect of these dipole moments produces a polarisation, defined as the induced dipole moment per unit volume: $\vec{P} = \frac{\sum \vec{p}}{V}$

In LIH dielectrics, the polarisation originates from the surface charge density $\sigma_p = P$.

In LIH dielectrics: $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ with χ_e being the electric susceptibility of the material.

The electric flux density is $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$

The constitutive relation states that $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

We have a new Maxwell equation that states: $\nabla \cdot \vec{D} = \rho_f$

The relative permeability of the material is given by: $\varepsilon_r = 1 + \chi_e$

Last time we saw

The polarisation has sources, these sources are the bound density of charge:

$$\rho_b = -\nabla \cdot \vec{P}$$

The bound density of charge is related to the free density of charge by:

$$\rho_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\rho_f$$

The surface bound density of charge is related to the surface free density of charge by:

$$\sigma_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\sigma_f$$

The energy per unit volume stored in a dielectric is:

$$w = \frac{1}{2}\vec{D} \cdot \vec{E}$$

How about magnetic materials?

Overview

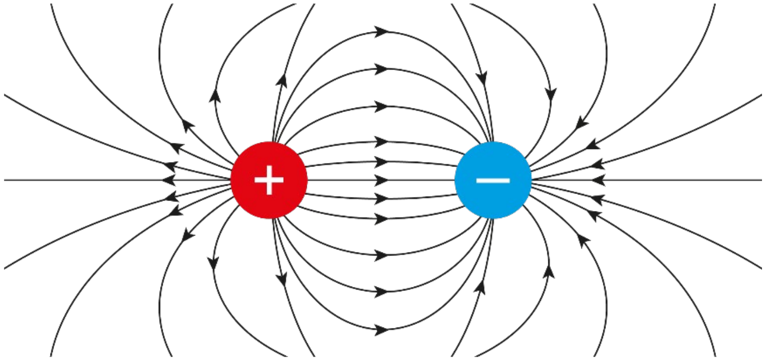
In this Lecture we will look at:

- Magnetic dipole moments
- The magnetic dipole in an externally applied magnetic field
- Macroscopic view: the Magnetisation
- Solenoid in vacuum
- Solenoid with a magnetic core
- The magnetic field strength H
- Magnetic susceptibility and permeability

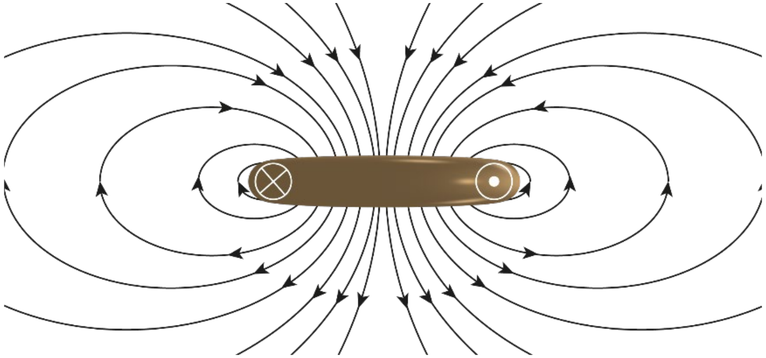
First: magnetic dipole moments!

Magnetic dipole moments

Compare the dipoles:



Electric field lines



Magnetic field lines.

Electric dipoles originate from positive and negative electric charges.

There are no single magnetic ‘charges’; no magnetic monopoles.

Magnetic dipoles originate from current carrying loops.

These loops create magnetic dipoles.

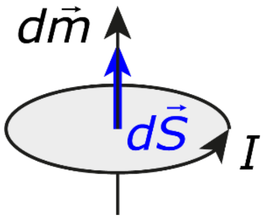
The **magnetic dipole moment** is oriented from south pole to north pole, along the field lines.

Here, the magnetic dipole moment points downwards; if we reverse the current in the loop, it will point upwards.

How do we calculate the magnetic dipole moment?

Magnetic dipole moments

Reversing the current:



Magnetic dipole resulting from a current loop. This loop encloses an oriented surface.

Here we have:

The electric current: I

An oriented element of a surface: $d\vec{S}$

Resulting magnetic dipole moment: $d\vec{m}$

For such a small coil, carrying a current, the magnetic dipole moment is:

$$\boxed{m = I \times (\text{area})}$$

A4!

For small areas, we can write:

$$d\vec{m} = I d\vec{S}$$

How many atoms are magnetic?

Magneto-optical effects: the Faraday effect

79 of the 103 first pure elements carry an atomic moment in the atomic ground state, as shown in table 3.1.

Table 3.1 - Magnetic properties of pure elements in the atomic state

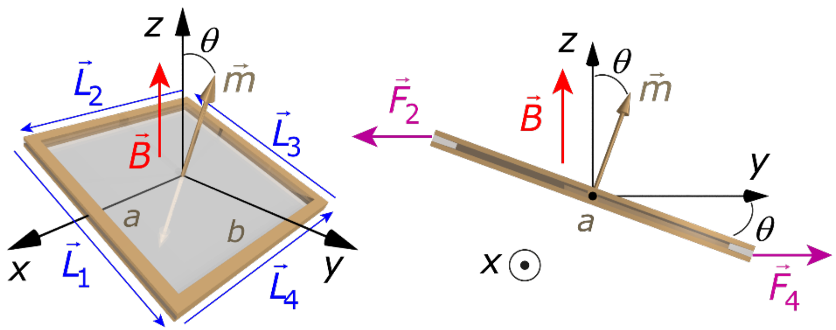
H																	**He**
Li	**Be**											B	**C**	N	O	F	**Ne**
Na	**Mg**											Al	**Si**	P	S	Cl	**Ar**
K	**Ca**	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	**Zn**	Ga	**Ge**	As	Se	Br	**Kr**
Rb	**Sr**	Y	Zr	Nb	Mo	*Tc*	Ru	Rh	**Pd**	Ag	**Cd**	In	**Sn**	Sb	Te	I	**Xe**
Cs	**Ba**	La	Hf	Ta	**W**	Re	Os	Ir	Pt	Au	**Hg**	Tl	**Pb**	Bi	Po	At	**Rn**
Fr	***Ra***	*Ac*															
				Ce	Pr	Nd	*Pm*	**Sm**	Eu	Gd	Tb	Dy	Ho	Er	Tm	**Yb**	Lu
				Th	*Pa*	U	*Np*	*Pu*	*Am*	*Cm*	*Bk*	*Cf*	*Es*	*Fm*	*Md*	*No*	*Lw*

In the atomic ground state, the only ***non*** *magnetic elements (bold framed) are those for which $J = 0$ (see chap. 7): Be…, Zn…, He…, Pd, Yb ($^1S_0$), C… ($^3P_0$), W ($^5D_0$), and Sm ($^7F_0$). The radioactive atoms are shown in italics.*

What if we apply an external magnetic field?

Étienne du Trémolet de Lacheisserie, Damien Gignoux, Michel schlenker, "Magnetism Fundamentals", Springer (2005), p. 84

Applying an external magnetic field:



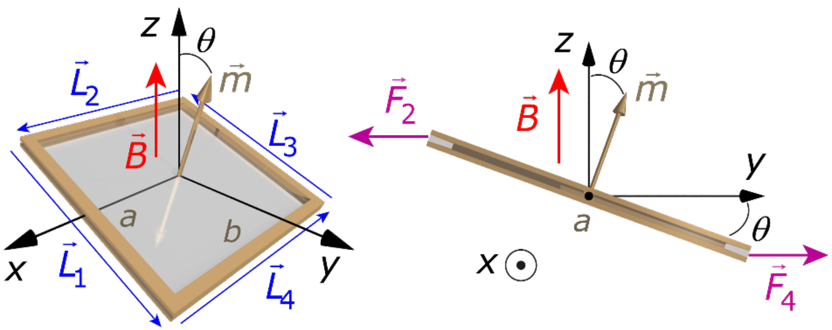
A rectangular current carrying loop in an externally applied magnetic field.

Consider a rectangular loop with current I in a magnetic field $\vec{B}$, that is tilted by an angle θ around the x-axis. The Lorenz force is:

$$\vec{F} = Q\vec{v} \times \vec{B} = Q\frac{\vec{L}}{t} \times \vec{B} = \frac{Q}{t}\vec{L} \times \vec{B} = I\vec{L} \times \vec{B}$$

The magnetic dipole in an externally applied magnetic field

Applying an external magnetic field:



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Now we apply to each side:

$$\vec{F}_1 = I\vec{L}_1 \times \vec{B} = IaB\sin\left(\frac{\pi}{2} - \theta\right)\hat{x}$$

$$\vec{F}_3 = I\vec{L}_3 \times \vec{B} = IaB\sin\left(\frac{\pi}{2} - \theta\right)\left(-\hat{x}\right) = -\vec{F}_1$$

So, the two forces cancel. (2) and (4) are perpendicular to the *B*-field, so

$$\vec{F}_2 = I\vec{L}_2 \times \vec{B} = -IbB\hat{y}$$

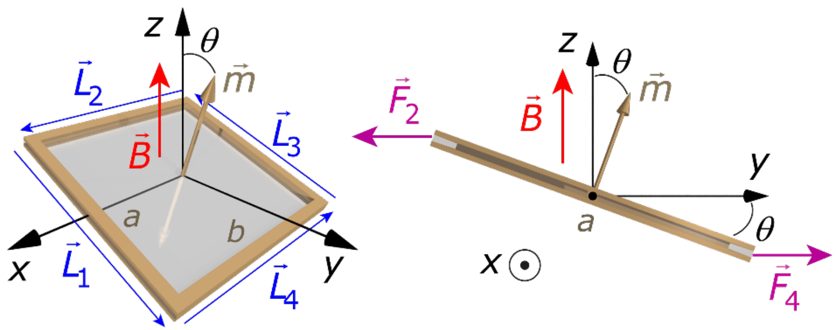
$$\vec{F}_4 = I\vec{L}_4 \times \vec{B} = IbB\hat{y} = -\vec{F}_2$$

Therefore, when the loop is perpendicular to the *B*-field (i.e. theta is zero), all the forces are in balance and the magnetic dipole moment is parallel to the *B*-field.

What if theta is different from zero?

The magnetic dipole in an externally applied magnetic field

Applying an external magnetic field:



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When theta is different from zero, we can calculate the torque:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where:

$$\|\vec{r}\| = \frac{\cancel{b}\ a}{2}$$

Then using:

$$\vec{m} = I\vec{S}$$

$$I = \frac{m}{S}$$

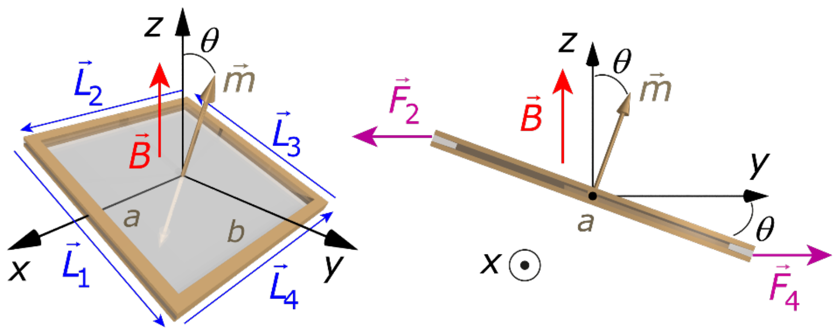
We obtain… try to calculate it:

$$\cancel{\vec{\tau} = \frac{a}{2} \times I\vec{L} \times \vec{B}}$$

$$\cancel{= \frac{a}{2} \times I\vec{L} \times \vec{B}}$$

Example?

Applying an external magnetic field:



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Where:

$$\|\vec{r}\| = \frac{b}{2}$$

Then using:

$$\vec{m} = I\vec{S}$$

We obtain:

$$\vec{\tau} = \|\vec{r}\|\|\vec{F}_2\|\sin\theta + \|\vec{r}\|\|\vec{F}_4\|\sin\theta =$$

$$= 2\frac{a}{2} \cdot IbB\hat{y}(\sin\theta) = IabB(\sin\theta)\hat{y} =$$

$$= mB(\sin\theta)\hat{y} = \vec{m} \times \vec{B}$$

The torque acts to bring $\vec{m}$ parallel to $\vec{B}$.

Example?

Example question

A rectangular coil of area 20 cm^2 carrying current of 20 A lies on the plane $2x + 6y - 3z = 5$, such that the magnetic moment of the coil is directed away from the origin. Calculate its magnetic moment.

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A rectangular coil of area 20 cm^2 carrying current of 20 A lies on the plane $2x + 6y - 3z = 5$, such that the magnetic moment of the coil is directed away from the origin. Calculate its magnetic moment.

The area of the loop is $S = 20\ \text{cm}^2 = 2 \times 10^{-3}\ \text{m}^2$. The current is $I = 20$ A

The magnetic moment is $\vec{m} = I\vec{S} = IS\vec{s}$, where $\vec{s}$ is the normal to the plane surface with function $f(x,y,z) = 2x + 6y - 3z - 5$

Given a 3D surface defined by $f(x,y,z) = 0$, we have $\vec{s} = \pm \frac{\nabla f}{|\nabla f|} = \pm \frac{\nabla f}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$.

Here we chose the + sign because the magnetic moment is directed away from the origin $\vec{s} = \frac{2\vec{s}_x + 6\vec{s}_y - 3\vec{s}_z}{\sqrt{2^2 + 6^2 + 3^2}} = \frac{2\vec{s}_x + 6\vec{s}_y - 3\vec{s}_z}{7}$

Therefore $\vec{m} = IS\frac{2\vec{s}_x + 6\vec{s}_y - 3\vec{s}_z}{7} = \left(\frac{20 \times 2 \times 10^{-3}}{7}\right)\left(2\vec{s}_x + 6\vec{s}_y - 3\vec{s}_z\right)$

$\vec{m} = \underline{\underline{\left(1.14\vec{s}_x + 3.42\vec{s}_y - 1.71\vec{s}_z\right) \times 10^{-2}}}\ \text{Am}^2.$

What about the macro scale?

Macroscopic view: the Magnetisation

The **magnetisation** is defined as the magnetic dipole moment per unit volume:

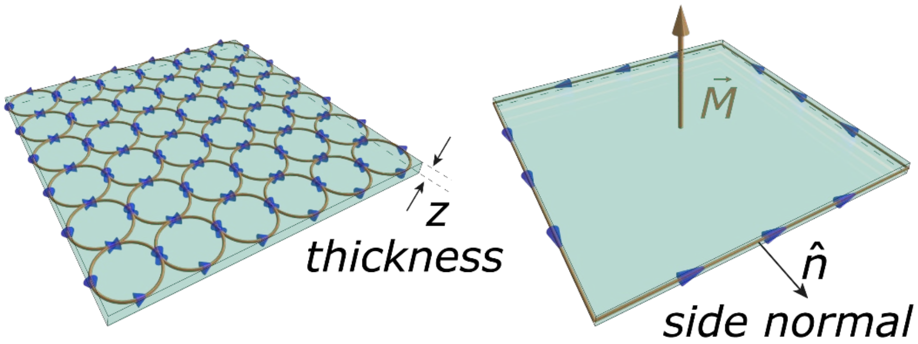
$$\boxed{\vec{M} = \frac{\sum \vec{m}}{V}}$$

A materials is said to be **magnetised** when the magnetization is different from zero.

The units of magnetisation are:

$$\frac{\text{magnetic moment}}{\text{volume}} = \frac{Am^2}{m^3} = Am^{-1}$$

Note: Compare with the electric polarization $\vec{P} = \sum \vec{p}/V$ and with surface charge density σ_b.



A slab of **uniformly magnetised** material of surface *S* and thickness *z*.

We can write: $M = \frac{\sum m}{V} = \frac{IS}{St} = \frac{I}{z}$

We define the **surface current density**:

$$\boxed{k_b = \frac{I}{z}}$$ [in units of current per length]

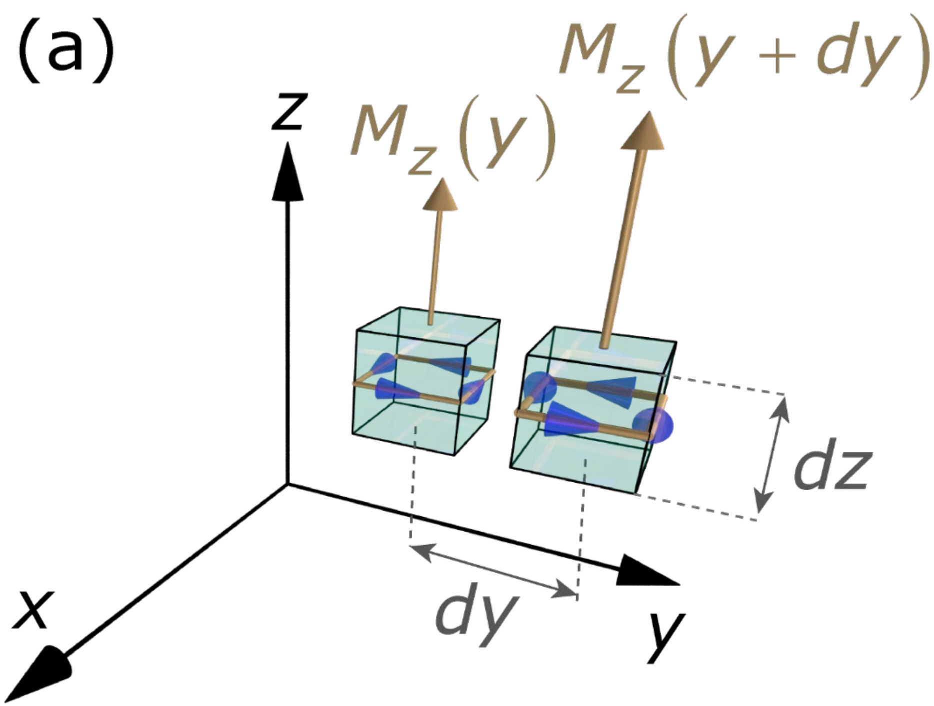
Generally: $\boxed{\vec{k}_b = \vec{M} \times \hat{n}}$

Not quite the same as σ_b but… similar.

What if the magnetisation is not uniform?

Macroscopic view: the Magnetisation

Two neighbouring volume elements:



Two volume elements inside a magnetic material that is **non-uniformly magnetized**.

This is a non-uniformly magnetised material.

As we move along the *y*-axis, the magnetisation (its magnitude) increases.

The magnetisation vector itself is oriented along the *z*-axis.

But we need to examine what happens along the *x*-axis.

Because we found that:

$$I = M \times [\text{thickness}]$$

The current **between the two volumes**, along the *x*-axis almost cancels, but not quite. It is larger along the positive *x*-axis.

Macroscopic view: the Magnetisation

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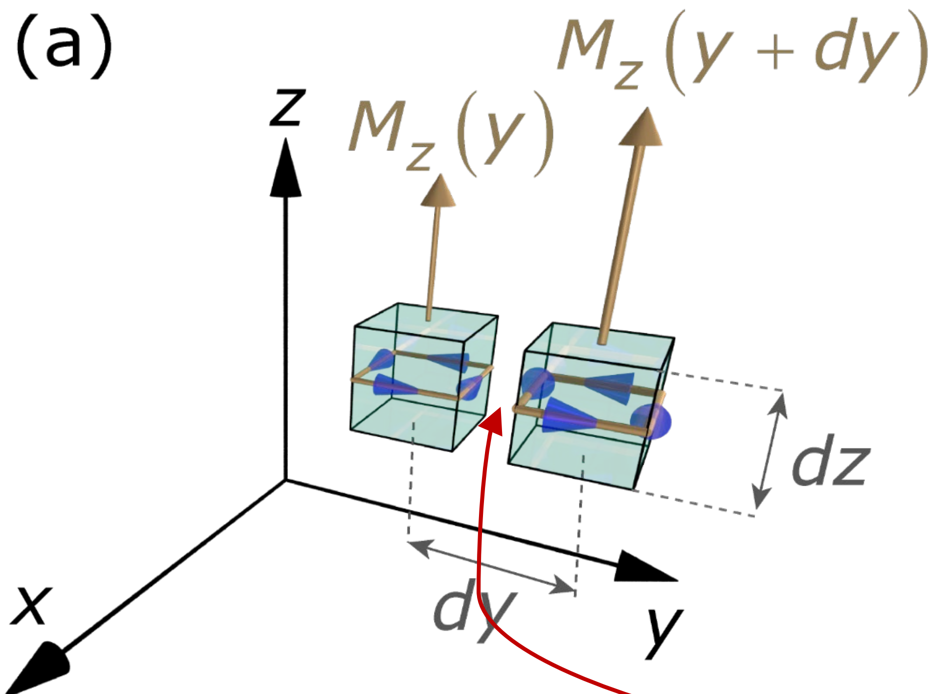

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Because we found that:

$I = M \times [\text{thickness}]$ [same thickness]

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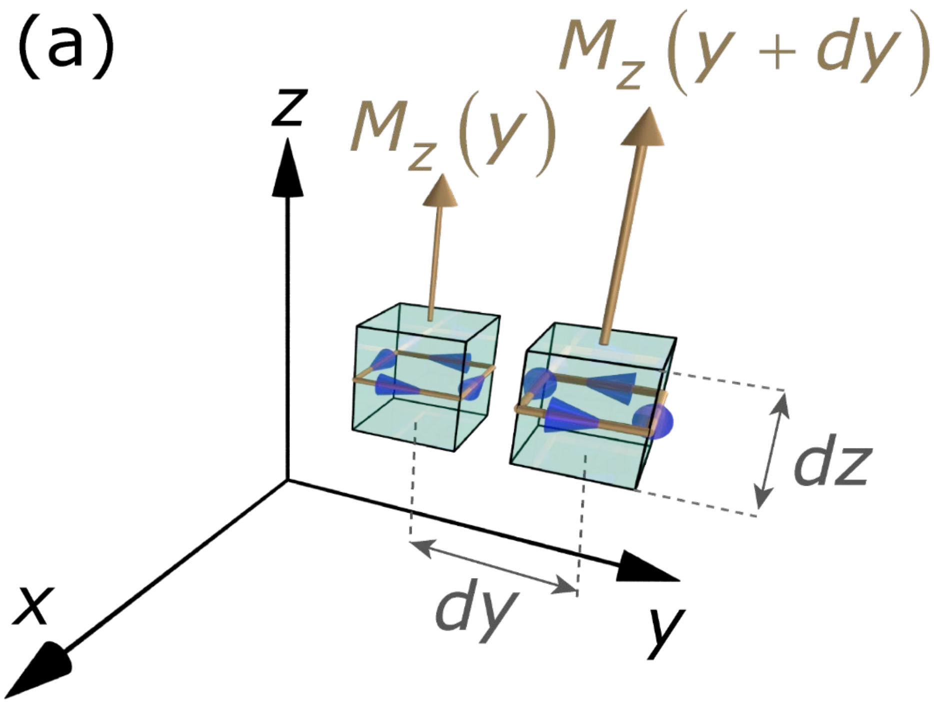
How do we write this in maths?

Macroscopic view: the Magnetisation

Two neighbouring volume elements:



Two volume elements inside a magnetic material that is **non-uniformly magnetized**.

We can write for the current along x:

$$I_x = \left[M_z(y+dy) - M_z(y)\right]dz =$$

$$= \frac{\partial M_z}{\partial y} dy dz$$

since by definition of the derivative:

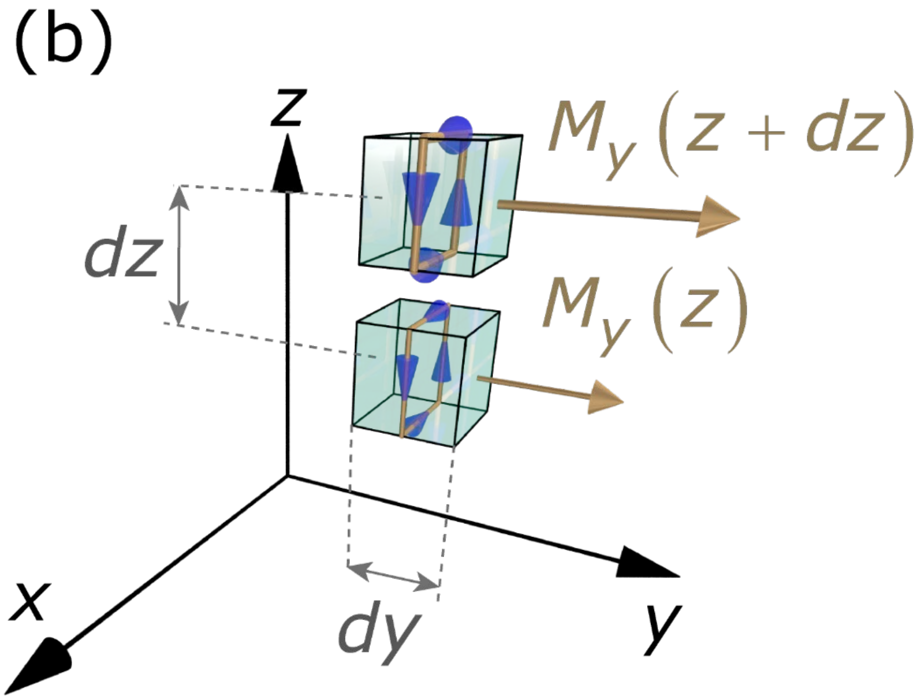
$$\frac{\partial M_z}{\partial y} = \frac{M_z(y+dy) - M_z(y)}{dy}$$

Now we can apply the same reasoning to another set of neighbouring volumes.

How do we write this in maths?

Macroscopic view: the Magnetisation

Two neighbouring volume elements:



Two volume elements inside a magnetic material that is **non-uniformly magnetized**.

We can write for the current along *x*:

$$I_x = \left[M_z(y+dy) - M_z(y)\right]dz =$$

$$= \frac{\partial M_z}{\partial y} dydz$$

Now I_x is larger along the negative *x*.

$$I_x = \left[M_y(z+dz) + M_y(z)\right]dy =$$

$$= -\frac{\partial M_y}{\partial z} dydz$$

Combing the two expressions:

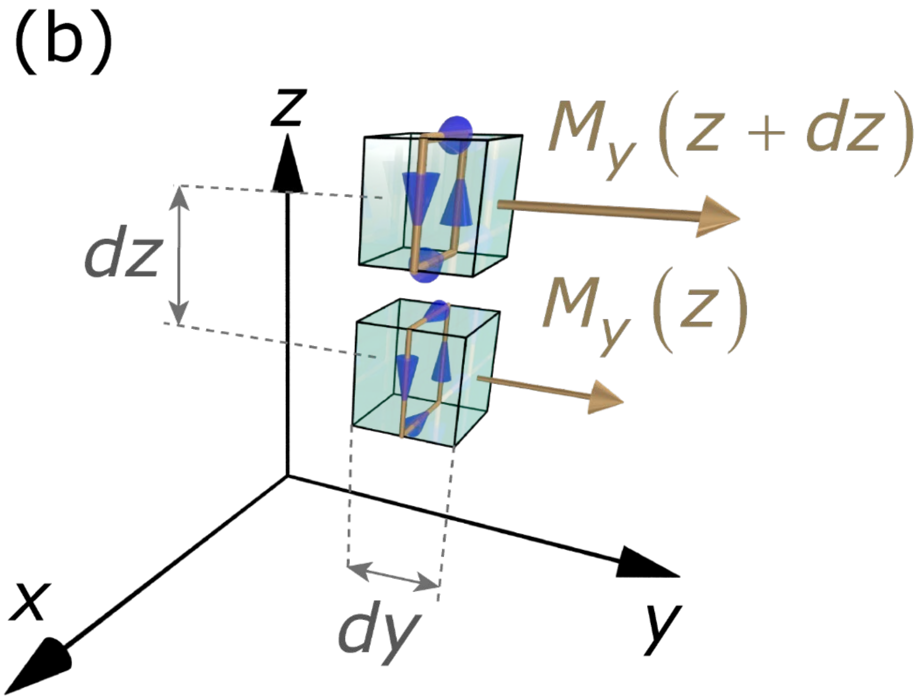
$$I_x = \left[\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right]dydz$$

We obtain the *x* component of the current density vector (because *dydz* is a surface).

How about the other directions, *y* and *z*?

Macroscopic view: the Magnetisation

Two neighbouring volume elements:



Two volume elements inside a magnetic material that is non-uniformly magnetized.

We just found: $I_x = \left[\frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}\right] dydz$

Which is the *x* component of the current density vector:

$$\left(\vec{j}_b\right)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} = \left[\nabla \times \vec{M}\right]_x$$

We can do the same analysis for the *y* and *z* direction and we see that the external magnetic field induces a **bound current density**:

$$\boxed{\vec{j}_b = \nabla \times \vec{M}}$$

to the surface

These currents are a response to the external magnetic field.

Remember a solenoid in vacuum!

Example question

[from Sadiku] The magnetization in a cube of size a is given by:

$\vec{M} = \frac{k_0}{a}\left(-2y\vec{a}_x + x\vec{a}_y\right)$, where k_0 is a constant. Find $\vec{J}_b$.

$$\vec{J}_b = \nabla \times \vec{M} = \frac{k_0}{a} \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2y\vec{a}_x & x\vec{a}_y & 0 \end{vmatrix} = \frac{k_0}{a}\left(\cancel{x - 2\vec{a}_x}\right)\vec{k}$$

[from Sadiku] An infinitely long cylindrical conductor of radius a and permeability $\mu_0\mu_r$ is placed along the z-axis. If the conductor carries a uniformly distributed current I along $\vec{a}_z$, find the magnetization $\vec{M}$ and the bound current density $\vec{J}_b$.

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$\vec{M} = \frac{k_0}{a}\left(-2y\vec{a}_x + x\vec{a}_y\right)$, where k_0 is a constant. Find $\vec{J}_b$.

Answer: $\vec{J}_b = \nabla \times \vec{M} = \frac{k_0}{a}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2y & x & 0 \end{vmatrix} = \frac{k_0}{a}\left(\frac{\partial x}{\partial x} - \frac{\partial(-2y)}{\partial y}\right)\vec{a}_z = \underline{\underline{3\frac{k_0}{a}\vec{a}_z}}$

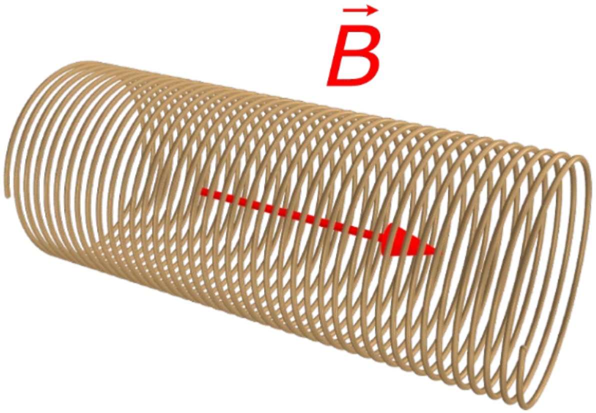
[from Sadiku] An infinitely long cylindrical conductor of radius a and permeability $\mu_0\mu_r$ is placed along the z-axis. If the conductor carries a uniformly distributed current I along $\vec{a}_z$, find the magnetization $\vec{M}$ and the bound current density $\vec{J}_b$.

$$\oint \vec{H}\cdot d\vec{L} = I_{\text{enclosed}} \rightarrow H_\phi \cdot 2\pi r = \frac{\pi r^2}{\pi a^2} I \rightarrow H_\phi = \frac{r}{2\pi a^2} I \qquad \vec{M} = \chi_m \vec{H} = \underline{\underline{(\mu_r - 1)\frac{r}{2\pi a^2} I\vec{a}_z}}$$

$$\vec{J}_b = \nabla \times \vec{M} = \frac{1}{r}\frac{\partial}{\partial r}(rM_\phi)\vec{a}_z = (\mu_r - 1)\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r^2}{2\pi a^2} I\vec{a}_z\right) = \underline{\underline{(\mu_r - 1)\frac{1}{\pi a^2} I\vec{a}_z}}$$

Where we used: $\nabla \times \vec{M} = \left(\frac{1}{r}\frac{\partial M_z}{\partial \phi} - \frac{\partial M_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial M_r}{\partial z} - \frac{\partial M_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rM_\phi) - \frac{\partial M_r}{\partial \phi}\right]\hat{z}$

Solenoid in vacuum



A solenoid in vacuum produces a magnetic field.

Here we have:
N: turns of wire per unit length
I: current through the wire
B: magnetic field inside the solenoid
L: length (assumed to be very long)
A: area

In electromagnetics, the term 'magnetic field' is used for two distinct but closely related vector fields. One is $\vec{B}$, the **magnetic flux density**.

Its units are the tesla. In SI base, its units are $kg \cdot s^2 \cdot A^{-1}$

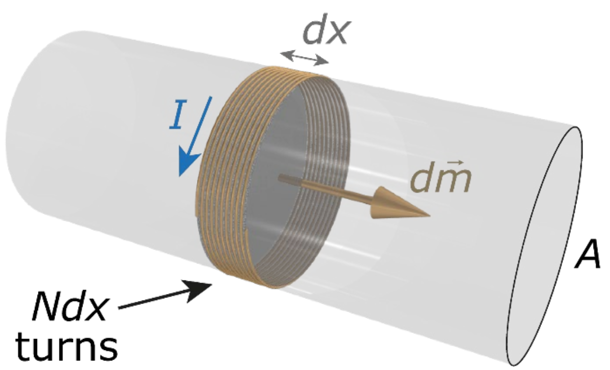
We will find about the other 'magnetic field' later on.

Inside a very long solenoid, in vacuum, the magnetic field is given by:

$$\boxed{B = \mu_0 N I}$$

How about a slice of this solenoid?

Solenoid in vacuum



A slice of thickness *dx* of a solenoid in vacuum.

N: turns of wire per unit length

I: current through the wire

B: magnetic field inside the solenoid

L: length (assumed to be very long)

A: area

Magnetic dipole moment of the slice:

$$d\vec{m} = (\text{current} \times \text{area}) = (NdxI)A =$$

$$= NI \times (\text{volume})$$

But we can also think of the solenoid as equivalent to a magnetised object with a magnetization M_{sol} per unit volume, that is producing the same uniform *B*-field . We can slice that magnetised object as well, and the magnetic dipole moment of that slice would be:

$$d\vec{m} = M_{sol} \times (\text{volume})$$

$M = \frac{\Sigma m}{V}$

From

$$d\vec{m} = M_{sol} \times (\text{volume}) = NI \times (\text{volume})$$

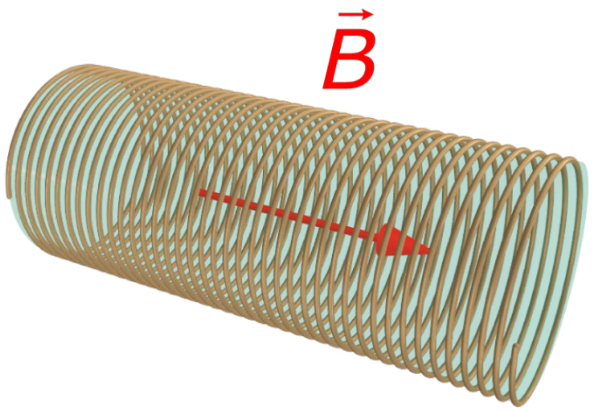
It follows that: $M_{sol} = NI$

And because $B = \mu_0 NI$

We can write: $\boxed{B_{sol}^{vacuum} = \mu_0 M_{sol}}$

What if there was a magnetic core?

Solenoid with a magnetic core



A solenoid with a magnetic core.

We now consider winding a solenoid around a magnetised material with magnetization M_{mat}

This material magnetisation produced a field:

$$B_{mat} = \mu_0 M_{mat}$$

The current in this solenoid is flowing in such a way that the current-induced field $B_{current}$ is parallel to B_{mat}.

The total field is therefore:

$$B = B_{current} + B_{mat}$$

We can rewrite this as:

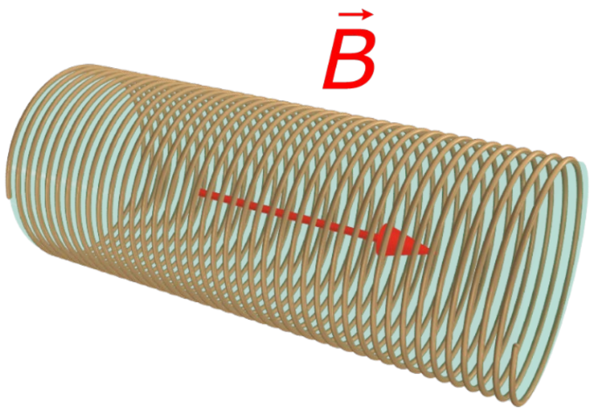
$$B = \mu_0 NI + \mu_0 M_{mat}$$

Then, we can drop the suffix and write

$$B = \mu_0 NI + \mu_0 M$$

Remember that ‘other magnetic field’?

The magnetic field strength H



A solenoid with a magnetic core.

We just found that in a solenoid with a magnetic core, we have:

$$B = \mu_0 NI + \mu_0 M$$

We define the **magnetic field strength** as:

$$\vec{H} = \frac{1}{\mu_0}\vec{B}_{solenoid}$$

The *H*-field is the magnetic flux density that would exist because of the current in the electric circuit alone, i.e. if all the magnetisable material is removed.

$$H = \frac{1}{\mu_0}B_{solenoid} = \frac{\mu_0 NI}{\mu_0} = NI$$

With this definition, we can now replace in:

$$B = \mu_0 NI + \mu_0 M = \mu_0 H + \mu_0 M$$

This leads to another **constitutive equation**:

$$\boxed{\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)}$$

Is there a magnetic equivalent to the electric susceptibility?

Magnetic susceptibility and permeability

Remember that in LIH dielectrics, we have:

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

With χ_e the electric susceptibility.

Similarly for LIH magnetic materials, we have:

$$\boxed{\vec{M} = \chi_m \vec{H}}$$

where χ_m is the **magnetic susceptibility** and $\vec{H}$ is the magnetic field strength.

Remember that in LIH dielectrics, we have:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$$

with ε_r the relative permittivity and $\varepsilon = \varepsilon_0 \varepsilon_r$ the electric permittivity.

Similarly for LIH magnetic materials, we have:

$$\vec{B} = \mu_0 \left(\vec{H} + \chi_m \vec{H} \right) = \mu_0 \left(1 + \chi_m \right) \vec{H} =$$

$$= \mu_0 \mu_r \vec{H} = \mu \vec{H}$$

where μ_r is the **relative permeability** and $\boxed{\mu = \mu_0 \mu_r}$ is the **permeability**.

And we can define: $\boxed{\vec{H} = \frac{1}{\mu} \vec{B}}$

Let's take an example!

Example question

The field surrounding a magnetic dipole is given by

$$\vec{B} = \alpha\left[3\left(\frac{zx}{r^5}\right)\hat{x} + 3\left(\frac{yz}{r^5}\right)\hat{y} + \left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\hat{z}\right] \text{ where } \quad r \equiv \sqrt{x^2 + y^2 + z^2}$$

(a) Write an expression for the magnetic dipole. [easy]

(b) Show that this field satisfies Gauss' law for magnetic fields. [harder]

Example question

The field surrounding a magnetic dipole is given by

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(a) Write an expression for the magnetic dipole. $m = I \times (\text{area})$

(b) Show that this field satisfies Gauss' law for magnetic fields.

$$\nabla \cdot \vec{B} = \alpha\left[3\frac{\partial}{\partial x}\left(\frac{zx}{r^5}\right) + 3\frac{\partial}{\partial y}\left(\frac{yz}{r^5}\right) + \frac{\partial}{\partial z}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\right]$$

$$\nabla \cdot \vec{B} = \alpha\left[3\left(\frac{z}{r^5} - \frac{5xz}{r^6}\frac{\partial r}{\partial x}\right) + 3\left(\frac{z}{r^5} - \frac{5yz}{r^6}\frac{\partial r}{\partial y}\right) + \left(\frac{6z}{r^5} - \frac{15z^2}{r^6}\frac{\partial r}{\partial z} + \frac{3}{r^4}\frac{\partial r}{\partial z}\right)\right]$$

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15z}{r^6}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right) + \frac{3}{r^4}\frac{\partial r}{\partial z}\right] \quad \text{with} \quad \frac{\partial r}{\partial x} = \frac{x}{r}; \frac{\partial r}{\partial y} = \frac{y}{r}; \frac{\partial r}{\partial z} = \frac{z}{r};$$

$$\nabla \cdot \vec{B} = \alpha\left[\frac{12z}{r^5} - \frac{15z}{r^5} + \frac{3z}{r^5}\right] = 0$$

[Full solution in the lecture notes.]

[This is example 1.2 from J. Peatross and M. Ware, Physics of Light and Optics, 2015 Edition]

Summary

Electric current is a source of magnetic fields.

The magnetic dipole moment is current times area: $m = I \times (\text{area})$

The magnetisation is the magnetic dipole moment per unit volume: $\vec{M} = \dfrac{\sum \vec{m}}{V}$

When an external magnetic field is applied to a magnetic dipole moment, the torque acts to bring $\vec{m}$ parallel to $\vec{B}$.

The magnetisation induces a surface current density $\vec{k}_b = \vec{M} \times \hat{n}$ and bound current density $\vec{j}_b = \nabla \times \vec{M}$.

The magnetic flux density results from adding up the magnetic field strength and the magnetisation: $\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$

The magnetic field strength is defined as $\vec{H} = \dfrac{1}{\mu} \vec{B}$.