Please write on the exam paper any brief answers to questions (e.g. numerical, simple algebraic and/or brief wording), so that these are made available to the students when the papers go to the Library

DEPARTMENT OF PHYSICS

PH40084

Advanced Quantum Theory

Monday 14th May 2018
16:30 to 18:30
2 hours

Answer ALL questions

The only calculators that may be used are those supplied by the University.

Please fill in you name and sign the section on the right of your answer cover, peel away adhesive strip and seal.

Take care to enter the correct candidate number as detailed on your desk label.

CANDIDATES MUST NOT TURN OVER THE PAGE AND THE EXAMINATION PAPER UNTIL THE CHIEF INVIGILATOR GIVES PERMISSION TO DO SO.

1. **Principle of quantum mechanics**
An operator $\hat{A}$ acts on vectors $|\psi\rangle$ in a Hilbert space $\mathcal{H}$ by mapping them to another vector $|\phi\rangle$ as $\hat{A}: |\psi\rangle \in \mathcal{H} \mapsto |\phi\rangle \in \mathcal{H}$.

(a) Define the adjoint of an operator $\hat{A}$. Use the adjoint to define what *hermitian* and *unitary* operators are. If $\hat{H}$ is hermitian show that $\hat{U} = \exp(-\mathrm{i}\hat{H})$ is unitary. **[4 marks]**

(b) A composite quantum system is made of two subsystem. The first is a with Hilbert space $\mathcal{H}_a$, and the second is b with Hilbert space $\mathcal{H}_b$. If $\hat{A}$ is an operator that acts on $\mathcal{H}_a$, and $\hat{B}$ is an operator that acts on $\mathcal{H}_b$, define elevated versions of these operators that act on $\mathcal{H}_a \otimes \mathcal{H}_b$. Use them to show that elevated operators originating from two different subsystems always commute. **[2 marks]**

(c) A spin-1/2 particle is described by the vector spin operator

$$\hat{\mathbf{s}} = \begin{pmatrix} \hat{s}_x \\ \hat{s}_y \\ \hat{s}_z \end{pmatrix} = \frac{1}{2}\hbar \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix}, \tag{1}$$

in terms of the Pauli operators. Working in the eigenbasis $\{|\uparrow\rangle, |\downarrow\rangle\}$ of $\hat{\sigma}_z$, use the vector representation $|\uparrow\rangle = \binom{1}{0}$, $|\downarrow\rangle = \binom{0}{1}$ and the corresponding standard matrix representations of the Pauli operators to explicitly confirm that

$$[\hat{s}_x, \hat{s}_y] = i\hbar\hat{s}_z. \tag{2}$$

[2 marks]

(d) Suppose we now have two spin-1/2 particles. The total spin vector operator for the pair is $\hat{\mathbf{S}} = \hat{\mathbf{s}}_1 + \hat{\mathbf{s}}_2$. Write down a representation of the product eigenbasis of $\hat{\sigma}_z \otimes \hat{\sigma}_z$ for their Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$. Show that the $\hat{\sigma}_z \otimes \hat{\sigma}_z$ product basis states are eigenstates of $\hat{S}_z$ and state their eigenvalues. **[3 marks]**

(e) The two spins interact within one another via the Hamiltonian

$$\hat{H} = J\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{s}}_2, \tag{3}$$

where J is a coupling strength. Use the commutation relation in Equation (2), which you can assume holds for the cyclic permutations of x, y, z labels as well, and the result from part (d), to show that the z component of the total spin obeys $[\hat{H}, \hat{S}_z] = 0$.
State all the information you can glean about the eigenstates of $\hat{H}$ from this. **[4 marks]**

(f) Take the tensor product of the Pauli matrices to show that the corresponding matrix representation of $\hat{H}$ in the $\hat{\sigma}_z \otimes \hat{\sigma}_z$ product basis is

$$\hat{H} = \frac{\hbar^2 J}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

Using the information you found in part (e), or otherwise, write down a complete set of energy eigenvalues and energy eigenstates of this Hamiltonian.

[**Hint**: Explicit diagonalisation of $\hat{H}$ is not required if you notice that (i) the matrix in Equation (4) is block diagonal, and (ii) the central 2×2 sub-matrix is related to a Pauli matrix whose eigenvalues and eigenvectors you can state.] **[5 marks]**

2. **The Klein-Gordon and Dirac equations**

For an observer in an inertial frame K, with coordinates t, x, a free particle with rest mass m_0 moving in one spatial dimension is described by the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^2} + \frac{m_0^2c^2}{\hbar^2}\right)\psi(x,t) = 0. \tag{5}$$

Another observer, moving at a speed v along the x-axis, defines an inertial frame K' with coordinates t', x'. The Lorentz transformation connecting coordinates in K to those in K' is

$$ct' = \gamma(ct - \mu x), \quad \text{and} \quad x' = \gamma(x - \mu ct), \tag{6}$$

while the inverse transform from K' to K is

$$ct = \gamma(ct' + \mu x'), \quad \text{and} \quad x = \gamma(x' + \mu ct'), \tag{7}$$

where $\gamma = 1/\sqrt{1-\mu^2}$ and $\mu = v/c$.

(a) Starting from Equation (5) and using the Lorentz transformation in Equation (6), show explicitly that the Klein-Gordon equation is Lorentz invariant.

[**Hint**: Begin by computing the Jacobian of the Lorentz transformation $(ct, x) \mapsto (ct', x')$, and then use the chain-rule to express the partial derivatives of $\psi(x,t)$ in terms of those of $\psi'(x',t')$.] **[6 marks]**

(b) A plane-wave in frame K is given as

$$\psi_p(x,t) = Ae^{\mathrm{i}(px-Et)/\hbar}, \tag{8}$$

where A is a normalisation constant. How must p and E be related for $\psi_p(x,t)$ to be a solution of the Klein-Gordon equation? How is a solution $\psi'(x',t')$ to the Klein-Gordon equation in frame K' related to the solution $\psi(x,t)$ in frame K? Use that E and cp Lorentz transform identically to ct and x, respectively, to show that the $\psi_p(x,t)$ in K transforms to

$$\psi'_{p'}(x',t') = Ae^{\mathrm{i}(p'x'-E't')/\hbar}, \tag{9}$$

in frame K'. **[5 marks]**

The Dirac equation in three spatial dimensions for a spin-1/2 free particle with rest mass m_0 is

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \hat{H}\psi(\mathbf{r},t), \quad \text{with} \quad \hat{H} = \left(c\hat{\boldsymbol{\alpha}}\cdot\hat{\mathbf{p}} + \hat{\beta}m_0c^2\right), \tag{10}$$

where $\psi(\mathbf{r},t)$ is a 4-component spinor, and $\hat{\mathbf{p}}$ is the canonical momentum vector operator. Here $\hat{\beta}$ is a 4×4 matrix and $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_x, \hat{\alpha}_y, \hat{\alpha}_z)$ is a vector of 4×4 matrices. The Dirac spin operator is defined as $\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\boldsymbol{\Sigma}}$ where the vector of operators (4×4 matrices) is

$$\hat{\boldsymbol{\Sigma}} = \begin{pmatrix} \hat{\Sigma}_x \\ \hat{\Sigma}_y \\ \hat{\Sigma}_z \end{pmatrix} = -\mathrm{i}\begin{pmatrix} \hat{\alpha}_y\hat{\alpha}_z \\ \hat{\alpha}_z\hat{\alpha}_x \\ \hat{\alpha}_x\hat{\alpha}_y \end{pmatrix}. \tag{11}$$

(c) Use Equation (11) and the anticommutative properties of the $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ matrices to show that

$$[\hat{S}_x, \hat{H}] = \mathrm{i}\hbar c(\hat{\alpha}_z\hat{p}_y - \hat{\alpha}_y\hat{p}_z). \tag{12}$$

We are told that the components of $\hat{\mathbf{S}}$ obey $[\hat{\mathbf{S}}, \hat{H}] = -\mathrm{i}\hbar c\,\hat{\boldsymbol{\alpha}}\times\hat{\mathbf{p}}$. Show the result in Equation (12) is consistent with the x component of this cross product. **[6 marks]**

(d) Orbital angular momentum is defined by the vector operator $\hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$. For the x component $\hat{L}_x$ we have the commutators $[\hat{L}_x, \hat{p}_x] = 0$, $[\hat{L}_x, \hat{p}_y] = \mathrm{i}\hbar\hat{p}_z$ and $[\hat{L}_x, \hat{p}_z] = -\mathrm{i}\hbar\hat{p}_y$. Use these commutators to show that

$$[\hat{L}_x, \hat{H}] = \mathrm{i}\hbar c(\hat{\alpha}_y\hat{p}_z - \hat{\alpha}_z\hat{p}_y). \tag{13}$$

Given that this result arises from an overall commutator $[\hat{\mathbf{L}}, \hat{H}] = \mathrm{i}\hbar c\,\hat{\boldsymbol{\alpha}}\times\hat{\mathbf{p}}$, and using the analogous result for $\hat{\mathbf{S}}$ from part (c), what can you conclude about the quantities $\hat{\mathbf{S}}$, $\hat{\mathbf{L}}$ and $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ for a relativistic spin-1/2 free particle? **[3 marks]**

3. **Quantum information**

The following information will be useful: A qubit has a computational basis denoted as $|x\rangle$ with $x \in \{0,1\}$. Quantum gates are defined by their action on the computational basis. Commonly used single-qubit gates include the Pauli gates $\hat{\mathrm{Z}}|x\rangle = (-1)^x|x\rangle$, and $\hat{\mathrm{X}}|x\rangle = |x \oplus 1\rangle$, where $\oplus$ is modulo 2 addition, e.g. $1 \oplus 1 = 0$. We also have the Hadamard gate $\hat{\mathrm{H}}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\hat{\mathrm{H}}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. When we have more than one qubit we denote the tensor product computational basis as $|x\rangle|y\rangle$, where $x, y \in \{0,1\}$ for the case of two qubits numbered 1 to 2 from the left to right. We will use a subscript i to denote which qubit a single-qubit gate is applied to, e.g. $\hat{\mathrm{Z}}_i$ to means $\hat{\mathrm{Z}}$ is applied to the ith qubit.

(a) Two parties, Alice and Bob, share a quantum communication channel allowing them to exchange qubits. Describe in detail the steps involved in the BB84 quantum key distribution protocol. Explain in words why the scheme is robust against an eavesdropper Eve using an *intercept-resend* attack. **[7 marks]**

(b) (i) Given a system of two qubits, explain what an *entangled* state is.

(ii) Entanglement is unchanged by applying local unitaries to the qubits. Take the state $|\psi\rangle|\phi\rangle$, where $|\psi\rangle$ and $|\phi\rangle$ are arbitrary states of a single qubit. Show that its entanglement is unchanged if we apply a single qubit unitary $\hat{U}_1$ on the first qubit and $\hat{U}_2$ on the second.

(iii) Consider the two qubit state

(14) $$|\Psi\rangle = \frac{1}{\sqrt{6}}(|0\rangle|0\rangle + |0\rangle|1\rangle) + \sqrt{\frac{2}{6}}(|1\rangle|0\rangle - |1\rangle|1\rangle).$$

Is this $|\Psi\rangle$ entangled? Justify your answer.

(iv) For the state $|\Psi'\rangle = \hat{\mathrm{H}}_2|\Psi\rangle$ show the reduced density operator $\hat{\rho}_1 = \mathrm{tr}_2(|\Psi'\rangle\langle\Psi'|)$ for the first qubit is

(15) $$\hat{\rho}_1 = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|.$$

(v) Quantify the entanglement in $|\Psi'\rangle$ by calculating the von Neumann entropy $S(\hat{\rho}_1) = -\mathrm{tr}[\hat{\rho}_1 \log_2(\hat{\rho}_1)]$.

[**Hint**: Calculating $S(\hat{\rho}_1)$ is simplified by noting that $\hat{\rho}_1$ is a diagonal matrix in the computational basis.]

(vi) Given a Bell state is

(16) $$|\Psi_{00}\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle),$$

how does the amount of entanglement in $|\Psi'\rangle$ compare? **[8 marks]**

(c) Suppose Alice and Bob share many copies of the Bell state $|\Psi_{00}\rangle$. They want to test if they can violate local realism by measuring an observable $\hat{\sigma}_{\mathbf{n}} = n_x\hat{\mathrm{X}} + n_y\hat{\mathrm{Y}} + n_z\hat{\mathrm{Z}}$ for their qubits, defined by an axis $\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y + n_z\mathbf{e}_z$, where $\mathbf{e}_i$ are $i = \{x, y, z\}$ axis Cartesian unit vectors. Specifically, Alice randomly chooses to measure each of her qubits along two axes $\mathbf{a} = -\mathbf{e}_z$ or $\mathbf{a}' = -\mathbf{e}_x$, while Bob randomly chooses to measure his qubits along two axes $\mathbf{b} = -(\mathbf{e}_x + \mathbf{e}_z)/\sqrt{2}$ or $\mathbf{b}' = (\mathbf{e}_z - \mathbf{e}_x)/\sqrt{2}$. Compute the value obtained in their experiment for Bell's inequality bounding the correlations of hidden variable theories

$$\mathbb{B}_{\mathrm{hv}} = \mathbb{E}(\mathbf{a}, \mathbf{b}) + \mathbb{E}(\mathbf{a}', \mathbf{b}) + \mathbb{E}(\mathbf{a}', \mathbf{b}') - \mathbb{E}(\mathbf{a}, \mathbf{b}') \leq 2,$$

where $\mathbb{E}(\mathbf{a}, \mathbf{b})$ denotes the expectation value of Alice measuring along $\mathbf{a}$ and Bob measuring along $\mathbf{b}$. Comment on your answer. **[5 marks]**

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UNIVERSITY OF BATH – DEPARTMENT OF PHYSICS

FUNDAMENTAL CONSTANTS

Note: Numerical values have been rounded to four significant figures.

Quantity	Symbol	Value	Unit	Dimensions
Atomic mass unit	u	1.661×10^{-27}	kg	M
Avogadro constant	N_A	6.022×10^{23}	mol^{-1}	
Bohr magneton ($e\hbar/2m_e$)	μ_B	9.274×10^{-24}	JT^{-1}	$I L^2$
Bohr radius ($4\pi\hbar^2/\mu_o c^2 e^2 m_e$)	a_o	5.292×10^{-11}	m	L
Boltzmann constant	k	1.381×10^{-23}	$J K^{-1}$	$ML^2T^{-2}\theta^{-1}$
Charge of electron (magnitude)	e	1.602×10^{-19}	C	I T
Charge (magnitude)/rest mass ratio (electron)	e/m_e	1.759×10^{11}	$C\ kg^{-1}$	$I M^{-1}T$
Fine–structure constant ($\mu_o ce^2/2h$)	α	7.292×10^{-3}		
	$1/\alpha$	137.0		
Gravitational constant	G	6.672×10^{-11}	$Nm^2\ kg^{-2}$	$M^{-1} L^3 T^{-2}$
Mass ratio, m_p/m_e	m_p/m_e	1836		
Molar gas constant	R	8.314	$J\ mol^{-1}K^{-1}$	$ML^2T^{-2}\theta^{-1}$
Molar volume (ideal gas, STP)	V_m	2.241×10^{-2}	m^3	L^3
Permeability of vacuum	μ_o	$4\pi \times 10^{-7}$	Hm^{-1}	$I^{-2}MLT^{-2}$
Permittivity of vacuum ($1/\mu_o c^2$)	ε_o	8.854×10^{-12}	Fm^{-1}	$I^2M^{-1}L^{-3}T^4$
	$4\pi\varepsilon_o$	1.113×10^{-10}	Fm^{-1}	$I^2M^{-1}L^{-3}T^4$
Planck constant	h	6.626×10^{-34}	Js	ML^2T^{-1}
	$\hbar$	1.055×10^{-34}	Js	ML^2T^{-1}
Rest mass of electron	m_e	9.110×10^{-31}	kg	M
Rest mass of proton	m_p	1.673×10^{-27}	kg	M
Speed of light in vacuum	c	2.998×10^8	ms^{-1}	LT^{-1}
Stefan-Boltzmann constant ($2\pi^5k^4/15h^3c^2$)	σ	5.670×10^{-8}	$Wm^{-2}K^{-4}$	$MT^{-3}\theta^{-4}$