PH30030: Quantum Mechanics   Problems Sheet 5

This problems sheet covers section 5 of the course, on the variational method.

1. Use the trial wavefunction $\psi(x) = Ax\exp(-\lambda x)$ ($x \geq 0$) to estimate the ground state energy of a particle of mass *m* in the one-dimensional triangular well

   $$V(x) = \infty \quad \text{for} \quad x < 0$$

   $$V(x) = \frac{V_0 x}{a} \quad \text{for} \quad x > 0$$

   Note: $\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$

2. Explain why the parabolic function $\psi(x) = Ax(a-x)$ is a suitable trial ground state wavefunction for a particle of mass *m* in a 1D box with

   $$V(x) = \infty \quad \text{for} \quad 0 > x > a$$

   $$V(x) = 0 \quad \text{for} \quad 0 \leq x \leq a$$

   Use this wavefunction to estimate the ground state energy of the particle.

3. Estimate the ground state energy of a 1D harmonic oscillator described by the Hamiltonian

   $$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

   using the Gaussian trial wavefunction $\psi(x) = Ae^{-bx^2}$.

   Note: $\int_{-\infty}^{\infty} e^{-ax^2} dx = \left(\frac{\pi}{a}\right)^{\frac{1}{2}}$ for $a > 0$; $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{\frac{1}{2}}$ for $a > 0$.

4. Estimate the ground state energy of a particle in 1D if the potential energy is given by $V(x) = -\alpha\delta(x)$, where α is a constant, using the Gaussian trial wavefunction $\psi(x) = Ae^{-bx^2}$. The delta function is defined by $\delta(x) = 0$ if $x \neq 0$, $\delta(x) = \infty$ if $x = 0$ and $\int_{-\infty}^{\infty} \delta(x) dx = 1$.

5. Estimate the ground state energy of a particle in 1D for the quartic potential $V(x) = \alpha x^4$, where α is a constant, using the Gaussian trial wavefunction $\psi(x) = Ae^{-bx^2}$.

   Note: $\int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{3}{4}\left(\frac{\pi}{a^5}\right)^{\frac{1}{2}}$ for $a > 0$.