PH20029 / PH20067 – THERMAL PHYSICS (Statistical Mechanics)

Problems sheet 1

Accessible microstates and the closed system

Question 1 *toy model 1 - macrostates and microstates (lecture 1)*
An imaginary system has N = 3 distinguishable particles (say red, green and blue), each of which can occupy one of a ladder of equally-spaced single-particle energy levels E = 0, 1, 2, 3... There is no upper limit to E.

A microstate of the system is defined by specifying the energy E of each particle, while a macrostate is defined by the total internal energy U. For example microstate (2,1,1) means red has E = 2, green has E = 1 and blue has E = 1; the macrostate in this case has U = 4. Note that, because the particles are distinguishable, (2,1,1) and (1,2,1) are distinct microstates. However they both yield the same macrostate, U = 4, as indeed does (3,0,1). For the macrostate U = 4, these are three examples of *accessible microstates*.

List *all* the accessible microstates when the macrostate energy is U = 4. Think of a systematic way of identifying them, so that you don't miss any out. You should find W = 15 of them.

Repeat the exercise for U = 0, 1, 2, 3, 5 and 6. Tabulate and then plot W and ln$W$ against U. Comment on how temperature T changes as energy is added to the system, given that $1/k_BT = \partial \ln W / \partial U$.

Question 2 *toy model 2 - system with an uppermost level (lecture 1)*
Suppose the ladder of energy levels from Question 1 is bounded at E = 3, ie E = 0, 1, 2 and 3 are the *only* energy levels. What is the maximum possible internal energy U of the system? Repeat the exercises of Question 1 but for all possible values of U up to the maximum.

Question 3 *ordered lists (lecture 1)*
(a) The football Premier League involves 20 teams. In theory, how many different ways could the teams be ranked in the league table at the end of the season?
(b) The bottom three teams in the table are relegated to the Championship. In theory, how many different sets of three teams could be relegated?

Question 4 *entropy and accessible microstates (lecture 2)*
(a) You roll a six-sided die but don't look at it. Calculate the entropy arising from your ignorance of which face is uppermost.
(b) A pin of mass 1 g is dropped from a height of 1 m. The temperature of the floor and the pin is 300 K. The kinetic energy acquired by the pin as it falls is given up to the floor as heat. By what factor is the number of accessible microstates of the floor increased as a result?
(Hint: use classical thermodynamics to work out the entropy change of the floor, assuming that all temperature changes are small.)

Question 5 *entropy of mixing (lecture 2)*

A sample of alloy has N sites occupied by n atoms of type A and $(N - n)$ of type B. Write an expression for the total number of possible arrangements of the two types of atoms. If the numbers of both types of atoms are very large and the formula of the alloy is A_xB_{1-x} (so that fraction x of the atoms are of type A), show that the entropy of mixing is

$$S = -k_B N \left\{ x \ln x + (1-x)\ln(1-x) \right\}$$

Question 6 *approach to mechanical and diffusive equilibrium (lecture 2)*

(a) In the lecture we found that, as two systems approach thermal equilibrium, the system at the lower temperature gains *energy* at the other's expense until their temperatures are equal. By considering the change in entropy as the volume of one system changes, find which system gains *volume* at the other's expense as mechanical equilibrium is approached.

(b) Likewise find which system gains *particles* at the other's expense as *diffusive* equilibrium is approached. (Assume in both cases that the temperatures have already equalised.)

Question 7 *number of accessible microstates (lecture 2)*

The number of accessible microstates for an ideal gas varies with internal energy as $W(U) = AU^{3N/2}$, where A is a constant. Find the temperature T, and hence U as a function of T.

Question 8 *approximations (lecture 3)*

(a) Derive the first three non-zero terms in the Taylor series for $f(x) = \ln(1 + x)$ about $x = 0$.

(b) For $N = 69$, compare the value of $\ln N!$ you get using your calculator to the approximation yielded by Stirling's formula. (The approximation is even better when N is of the order of Avogadro's number, but your calculator won't be able to cope.)

Summary answers

1. W should be an increasing function of U, and T increases with U.
2. T should rise to infinity and become negative between $U = 4$ and $U = 5$. At this point a given particle is more likely to be in a higher level than a lower one, which demands a negative T in the Boltzmann distribution.
3. $20! = 2.4 \times 10^{18}$, 1140
4. (a) 2.47×10^{-23} JK^{-1}, (b) $\exp(2.37 \times 10^{18})$, too big for a calculator...
6. (a) high pressure system gains volume until pressures have equalised, (b) low chemical potential system gains particles until chemical potentials have equalised
7. $U = (3/2)Nk_BT$
8. (a) $\ln(1 + x) = x - x^2/2 + x^3/3 - \ldots$, (b) $\ln 69! = 226.2$, $69 \ln 69 - 69 = 223.2$