Problem Sheet 2 – Differentiation of Scalar and Vector Fields

The idea of this sheet is to get you used to finding various derivatives of scalar and vector fields, using Cartesian coordinates.

1. Find gradϕ for the following scalar fields

   (a) $\phi = x + y + z$

   (b) $\phi = xyz$

2. Throughout this question, the scalar field $\phi = x^2y + y^2z + z^2x$.

   (a) Write down an expression for ϕ if $z = 0$. Use this to sketch equipotential contours (ϕ =constant) in the plane $z = 0$ for some constants of your choice.

   (b) Find gradϕ.

   (c) Find the magnitude and direction of the steepest gradient of ϕ at (1,1,2).

   (d) Find the gradient of ϕ at (1,1,2) in the (0,1,1) direction.

   (e) Find the equation of the plane which is tangential to the equipotential surface of ϕ at (1,1,2). [Recall that a plane can be defined as the set of position vectors $\mathbf{r}$ for which $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, where $\mathbf{n}$ is a vector normal to the plane, and $\mathbf{a}$ the position vector of a point in the plane. Expand the scalar products to write the plane in the more familar form $ax + by + cz = d$ where a, b, c and d are constants.]

3. Find $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ for the following vector fields

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

   (b) $\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$

4. If $\psi(\mathbf{r}) = \exp[i(2x + 3y - z)]$, find the value of $\nabla^2\psi$ at the general point (x, y, z).

5. Show that the triple product $\nabla \cdot (\nabla \times \mathbf{a})$ is zero for every "well-behaved" vector field $\mathbf{a}$. [HINT: Write $\mathbf{a}$ in component form (a_x, a_y, a_z). By "well-behaved" I mean that all components of $\mathbf{a}$ are everywhere differentiable as many times as needed.]