University of Bath

Department of Physics

Year 3

PH30030 – Quantum Mechanics

Wednesday, 23rd January 2019, 16:30 to 18:30

Answer ALL questions

The only calculators that may be used are those supplied by the University.

Please fill in your name and sign the section on the right of your answer book, peel away adhesive strip and seal.

Take care to enter the correct candidate number as detailed on your desk label.

CANDIDATES MUST NOT TURN OVER THE PAGE AND READ THE EXAMINATION PAPER UNTIL THE CHIEF INVIGILATOR GIVES PERMISSION TO DO SO.

1. A particle is moving in 1D.

(a) Show that the momentum and total energy of the particle can be measured simultaneously only if the potential energy is constant everywhere. [6]

(b) If the potential energy is zero, show that $\hat{p}_x$ and $\hat{H}$ have the common set of eigenfunctions $\phi(x) \propto \exp(ikx)$, and find the corresponding eigenvalues.

$\hbar k$; $\hbar^2 k^2/2m$

[2]

(c) By confining the particle to a large but finite box, normalise these eigenfunctions.

[1]

2. (a) If $\Delta A^2 = \left(\hat{A} - \langle \hat{A} \rangle\right)^2$, where ΔA is the "uncertainty" associated with operator $\hat{A}$, show that

$$\Delta A^2 = \int dx\, \psi^*(x)\, \Delta A^2\, \psi(x) = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$$

where ψ is a normalised wavefunction. [4]

(b) A particle moving in 1D free space is described at time $t = 0$ by the normalised Gaussian wavepacket

$$\psi(x,0) = \left(\frac{a}{\pi}\right)^{1/4} \exp\left(-\frac{ax^2}{2}\right).$$

Find the uncertainties in the particle position Δx and momentum Δp_x, and relate them to the uncertainty principle. $\Delta x = \sqrt{\frac{1}{2a}}$; $\Delta p_x = \hbar\sqrt{\frac{a}{2}}$

[8]

Note: $\int_{-\infty}^{\infty} dx\, x^2 \exp(-ax^2) = \frac{1}{2a}\left(\frac{\pi}{a}\right)^{1/2}$ and $\int_{-\infty}^{\infty} dx \exp(-ax^2) = \left(\frac{\pi}{a}\right)^{1/2}$

3. (a) If $|\phi_n\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue β_n, describe the effect of the ladder operators $\hat{L}_+ = \hat{L}_x + i\hat{L}_y$ and $\hat{L}_- = \hat{L}_x - i\hat{L}_y$ when acting on $|\phi_n\rangle$. What is the effect of $\hat{L}_+$ and $\hat{L}_-$ on the eigenvalues of $\hat{L}^2$?

[3]

(b) In spherical polar coordinates, the $\hat{L}_x$ and $\hat{L}_y$ operators are given by

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

Show that the ladder operators $\hat{L}_+$ and $\hat{L}_-$ can be expressed in spherical polar coordinates as

$$\hat{L}_+ = \hbar\exp(i\phi)\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$
$$\hat{L}_- = \hbar\exp(-i\phi)\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

[4]

(c) The eigenfunctions of angular momentum $|Y_{\ell m}\rangle$ for $\ell = 1$ are

$$Y_{10}(\theta,\phi) \propto \cos\theta$$

$$Y_{1\pm1}(\theta,\phi) \propto \sin\theta\exp(\pm i\phi).$$

Find the effect of the ladder operator $\hat{L}_+$ as defined in (b) on each of these eigenfunctions.

[6]

4. An experiment is performed on an incident beam of spin half particles. The apparatus is designed to measure the spin component in the x-y plane at an angle ϕ to the x axis. The operator $\hat{S}_\phi$ for this component is given, by analogy with the classical expression for a vector component, as

$$\hat{S}_\phi = \hat{S}_x \cos\phi + \hat{S}_y \sin\phi .$$

(a) Show that $\hat{S}_\phi = \frac{\hbar}{2}\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$. [2]

(b) Find the eigenvalues α of the operator $\hat{S}_\phi$. $\alpha = \pm\frac{\hbar}{2}$ [2]

(c) Show that normalised and orthogonal eigenvectors of $\hat{S}_\phi$ are given by

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\phi} \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\phi} \end{pmatrix}$, and identify the corresponding eigenvalues. [4]

(d) The beam is in a state described by the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The emergent beam is observed to have two components that correspond to the eigenstates of $\hat{S}_\phi$.

(i) Write an equation that relates the eigenvector of the incident beam to the eigenvectors of $\hat{S}_\phi$. $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_\uparrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix} + c_\downarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -e^{i\alpha} \end{pmatrix}$ [4] $c_\uparrow = c_\downarrow = \frac{1}{\sqrt{2}}$

(ii) Find numerical values for the probability of observing the eigenvalues corresponding to $\hat{S}_\phi$. $|c_\uparrow|^2 = \frac{1}{2}$; $|c_\downarrow|^2 = \frac{1}{2}$ [2]

Notes: $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

If $\hat{A}|a\rangle = \alpha|a\rangle$ where $\hat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, then $\det\left(\hat{A} - \alpha I\right) = 0$ where I is the identity matrix.

5. (a) Let $\hat{H}_0$ be the Hamiltonian for an unperturbed system that is described by the time independent Schrödinger equation $\hat{H}_0 |\phi_{0n}\rangle = E_{0n} |\phi_{0n}\rangle$ for which the non-degenerate normalised and orthogonal eigenfunctions $|\phi_{0n}\rangle$ and eigenvalues E_{0n} are known. When a perturbation $\hat{H}'$ is applied, the Hamiltonian for the perturbed system can be written as $\hat{H} = \hat{H}_0 + \lambda \hat{H}'$. Look for solutions of $\hat{H}|\phi_n\rangle = E_n |\phi_n\rangle$ in the form $|\phi_n\rangle = |\phi_{0n}\rangle + \lambda |\phi_{1n}\rangle + \cdots$ and $E_n = E_{0n} + \lambda E_{1n} + \cdots$. Hence show that the first order correction for the energy eigenvalue is given by

$E_{1n} = \langle \phi_{0n} | \hat{H}' | \phi_{0n} \rangle$. [6]

(b) The Hamiltonian for a 1D anharmonic oscillator of mass m and angular frequency ω is given by

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 \left(1 + \alpha x^2\right),$$

where x is the displacement and the term in α represents a perturbation. If the ground state energy of the harmonic oscillator is $\hbar\omega/2$ and the corresponding eigenfunction is given by $\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(\frac{-m\omega x^2}{2\hbar}\right)$, derive an approximate (first-order) expression for the ground state energy of the anharmonic oscillator.

$E_0 = \frac{1}{2}\hbar\omega + \frac{3}{8}\alpha\frac{\hbar^2}{m}$

[4]

Note: $\int_{-\infty}^{\infty} dx\, x^4 \exp\left(-bx^2\right) = \frac{3}{4}\left(\frac{\pi}{b^5}\right)^{1/2}$

(c) An experimentalist wishes to observe the harmonic vibrational (phonon) modes in solid argon. Explain whether they should work at low or high temperature.

[2]

(PSS)