## Problem Sheet 4 - Answers

1. $F(x) = A\cos x + B\sin x$ is the general solution. The BCs determine $A$ and $B$: $F(0) = 0 \to A = 0$; $F(1) = 1 \to B = 1/\sin(1)$, so the exact solution at $x = \frac{1}{2}$ is

$$F(0.5) = \frac{\sin(\frac{1}{2})}{\sin(1)} \approx 0.5697.$$

2. At a general point $i$ on a regular grid of spacing $h$, using centred differences for the 2nd derivative, our difference equation approximation to equation (1) is

$$\frac{1}{h^2}\{f_{i+1} + f_{i-1} - 2f_i\} + f_i = 0.$$

For this problem, the boundaries are at $x = 0$ and $x = 1$, where we know the solution.

If $h = 0.5$, we have grid points at $x = 0$, $x = 0.5$, $x = 1$. Denoting $f(0.5) = f_1$ gives us a single difference equation to solve for $f_1$:

$$\frac{1}{0.5^2}\{f(1) + f(0) - 2f_1\} + f_1 = 0.$$

Inserting the BCs gives

$$4 - 8f_1 + f_1 = 0 \to f_1 = \frac{4}{7} \approx 0.5714.$$

This is our $h = 0.5$ approximation to $F(0.5)$; it has (absolute) error $\varepsilon \approx |0.5714 - 0.5697| = 0.0017$.

If $h = 0.25$, we should expect a more accurate approximation to $F(0.5)$; if these values of $h$ are already "small", our $O(h^2)$ discretisation should give us an error $\varepsilon$ about 4 times smaller. For this $h$, we have 3 gridpoints where we don't know the answer; these are at $x_1 = 0.25$ $x_2 = 0.5$ and $x_3 = 0.75$. We are after the value of $f_2$, but will need to find all 3 unknowns simultaneously. To do this, we write down difference equations for $i = 1, 2, 3$, inserting boundary conditions where appropriate:

$$\begin{aligned}
\text{at } x_1 &: \quad 16(0 + f_2 - 2f_1) + f_1 = 0 \\
\text{at } x_2 &: \quad 16(f_1 + f_3 - 2f_2) + f_2 = 0 \\
\text{at } x_3 &: \quad 16(f_2 + 1 - 2f_3) + f_3 = 0
\end{aligned}$$

Solve these three equations simultaneously (not too hard by hand) to get $f_2 = \frac{256}{449} \approx 0.5702$. This is our $h = 0.25$ approximation to $F(0.5)$, with error $\varepsilon \approx |0.5702 - 0.5697| = 0.0005$.

If $h = 0.125$, we still have a gridpont at $x = 0.5$ which we can use to approximate $F(0.5)$. We have 6 simultaneous difference equations, one for each of the 6 "internal" (i.e. non-boundary) grid points. A typical one looks like

$$64\,\{f_{i+1} + f_{i-1} - 2f_i\} + f_i = 0.$$

I couldn't be bothered to try and solve 6 equations simultaneously by hand. By computer I got $f(0.5) \approx 0.5698$, with $\varepsilon \approx 0.0001$.

If you do choose to write some code to explore smaller values of $h$, make sure one of the (regularly-spaced) gridpoints is at $x = 0.5$. Plot $\log \varepsilon$ against $\log h$ - this should look linear, with slope 2.

3. Cylindrical symmetry means there is no variation (of the solution) as we swing around the axis. So $\Psi$ cannot depend on angle $\phi$, meaning $\frac{\partial \Psi}{\partial \phi} = 0$. This means we can ignore the second term in $\nabla^2 \Psi$, which now depends only on the 2 (orthogonal) coordinates $\rho$ and $z$. We can make regular grids in these 2 coordinates. For example, choosing a grid spacing $h$ for both coordinates, and rewriting the first term as

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Psi}{\partial \rho}\right) = \frac{1}{\rho}\left(\rho \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{\partial \Psi}{\partial \rho}\right) = \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho},$$

we get the centred-difference approximation at a general point $(i, j)$:

$$\frac{1}{h^2}(\psi_{i+1,j} + \psi_{i-1,j} - 2\psi_{i,j}) + \frac{1}{2h\rho_i}(\psi_{i+1,j} - \psi_{i-1,j}) + \frac{1}{h^2}(\psi_{i,j+1} + \psi_{i,j-1} - 2\psi_{i,j}) = 0.$$

4. This is very similar to the 1d example in the notes...

Let $f(x) = f_o(x) + \eta(x)$. $f_o(x)$ is the (usually unknown) function that minimises $I$; $\eta(x)$ is a small variation. Substitute this expression for $f(x)$ into the functional given in the question:

$$I[f] = \int_0^1 \left[\left(\frac{df_o}{dx} + \frac{d\eta}{dx}\right)^2 - (f_o + \eta)^2\right] dx.$$

Expand out the squares to get

$$I[f] = \int_0^1 \left[\left(\frac{df_o}{dx}\right)^2 - f_o^2 + 2\frac{df_o}{dx}\frac{d\eta}{dx} + \left(\frac{d\eta}{dx}\right)^2 - 2f_o\eta - \eta^2\right] dx.$$

Now we start simplifying. We ignore the (2) terms that are $O(\eta^2)$, so we can argue that, to first order, the difference $\delta I = I[f] - I[f_o]$ is zero. Then we have

$$2\int_0^1 \left[\frac{df_o}{dx}\frac{d\eta}{dx} - f_o\eta\right] dx = 0.$$

Next use integration by parts on the first term; with $v = f_o'$ and $u = \eta$ we have

$$2\left[\eta \frac{df_o}{dx}\right]_0^1 - 2\int_0^1 \eta \frac{d^2 f_o}{dx^2}\, dx - 2\int_0^1 \eta f_o\, dx = 0.$$

For our problem, the first term will be 0 if (as we must) we set $\eta = 0$ at the boundaries $x = 0$ and $x = 1$. So we now know that the solution $f_o$ satisfies the equation

$$-2\int_0^1 \eta(f_o'' + f_o)\, dx = 0.$$

Since $\eta \neq 0$ in general (away from our boundaries), we must have $f_o'' + f_o = 0$. In other words, $f_0$ is $F$, the solution to equation (1).

The rest of this question is about evaluating the integral $I$ for a couple of functions, the first of which is the exact solution - this never happens in a real implementation of the FEM!

The exact solution is $F(x) = \sin(x)/\sin(1)$. Therefore

$$I[F] = \frac{\int_0^1 (\cos^2 x - \sin^2 x)\, dx}{\sin^2(1)} = \frac{\int_0^1 (\cos 2x)\, dx}{\sin^2(1)} = \frac{\left[\frac{1}{2}\sin 2x\right]_0^1}{\sin^2(1)} \approx 0.642.$$

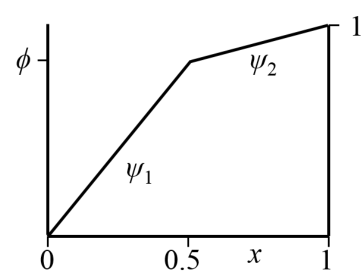
Note that the *value* of $I[F]$ is not in itself particlularly interesting; what matters is that all other (allowed) functions should give a bigger value of the integral...

For the trial function $f(x) = x$, $f'(x) = 1$, so

$$I[x] = \int_0^1 \left[1 - x^2\right] dx = \frac{2}{3}$$

which is, as expected, greater than $I[F]$.

5. Let the 2 element functions be $\psi_1$ and $\psi_2$. A quick sketch should help you see that $\psi_1 = 2\phi x$ and $\psi_2 = 2(1-\phi)x + (2\phi - 1)$:




The functional $I = I[\psi_1] + I[\psi_2]$:

$$I = 4\phi^2 \int_0^{\frac{1}{2}} (1 - x^2)\, dx + \int_{\frac{1}{2}}^1 \left[4(1-\phi)^2 - \{2(1-\phi)x + (2\phi - 1)\}^2\right] dx.$$

A bit of bracket-expanding and integrating eventually leads to $I = \frac{11}{3}\phi^2 - \frac{25}{6}\phi + \frac{11}{6}$. This is a quadratic equation in the single node value $\phi$. To find the FEM solution we minimise $I$ with respect to $\phi$. We set

$$\frac{dI}{d\phi} = 0 = \frac{22}{3}\phi - \frac{25}{6},$$

for which $44\phi = 25$, so $\phi = \frac{25}{44} \approx 0.5682$. If we substitute this value of $\phi$ into the equation for $I(\phi)$ we get $I = 0.6496$. Note that this is $> I[F]$, as it must be; more importantly, the use of two elements gives a better answer (smaller $I$) than the 1-element solution in Q4, $I[x] = \frac{2}{3}$.