NUCLEAR MODELS

1. *The two light nuclei ${}^{11}_{5}B$ and ${}^{11}_{6}C$ are a pair of mirror nuclei: the number of protons in one equals the number of neutrons in the other. The binding energy of the nuclei ${}^{11}_{5}B$ and ${}^{11}_{6}C$ are 76.205 MeV and 73.443 MeV respectively. Assuming that the difference is due entirely to Coulomb effects and that the proton charge is uniformly distributed through a sphere of radius R_c (identical for both nuclei), find R_c. This was an early way of estimating the size of the nucleus.*

[3.44 fm]

The potential energy of a uniformly charged sphere is : $\frac{3}{5}\frac{Z^2\, e^2}{4\pi\,\varepsilon_0\, R_C}$

where $R_{nuc} = R_C$ is the radius of the sphere.

The difference in potential energy is: $\frac{3}{5}\left(Z_1^2 - Z_2^2\right)\frac{e^2}{4\pi\,\varepsilon_0\, R_C}$

where $Z_1 = 6$ and $Z_2 = 5$ (and assuming the radius is the same).

The difference is therefore:

$$\Delta B.E. = \frac{3}{5}\frac{(Z_1^2 - Z_2^2)e^2}{4\pi\epsilon_0} \times \frac{1}{R_C}$$

Or:

$$R_C = \frac{3}{5} \times (Z_1^2 - Z_2^2) \times \frac{e^2}{4\pi\epsilon_0} \times \frac{1}{\Delta B.E.}$$

$$R_C = \frac{3}{5} \times (6^2 - 5^2) \times 1.44 MeV.fm \times \frac{1}{(76.205 - 73.443)\ MeV}$$

Hence: $R_C = 3.44$ fm

2. * *Use the semi-empirical mass formula to predict which of the following nuclei you would expect to be β-stable:*

$${}^{183}_{73}Ta \qquad {}^{183}_{74}W$$

given that $(m_n\text{-}m_p\text{-}m_e)c^2 = 0.8$ MeV.

From the semi-empirical mass formula, we can find the atomic number Z_{min} of the lightest isobar, using the formula also derived in the lecture:

$$Z_{\min} = \left[\frac{4s + \left(m_n - m_p - m_e\right)c^2}{4s + d\ A^{\frac{2}{3}}}\right]\frac{A}{2}, \text{ where: } \begin{cases} 4s = 92.8 \text{ MeV} \\ d = 0.714 \text{ MeV} \\ \left(m_n - m_p - m_e\right)c^2 = 0.8 \text{ MeV} \end{cases}$$

For $A = 183$, $Z_{\min} = \left[\frac{93.6}{92.8 + 0.714(183)^{\frac{2}{3}}}\right] 91.5 = 73.9$

Therefore, we would expect ${}^{183}_{74}W$ to be β-stable.

3. * *Which of the following nuclei would you expect to be β-stable?*

$$^{190}_{78}Pt \qquad ^{190}_{76}Os \qquad ^{190}_{74}W.$$

This is answered exactly as Question 2: $Z_{\min} = \left[\dfrac{93.6}{92.8 + 0.714(190)^{\frac{2}{3}}}\right] \times 95 = 76.4$

We would therefore expect $^{190}_{76}Os$ to be β-stable.

4. * *Verify that if the most stable isobar has a neutron to proton ratio given by*

$$\frac{N}{Z} = 1 + \frac{dA^{2/3}}{2s}$$

then the binding energy per nucleon (neglecting the pairing term) is given by

$$\frac{B}{A} = a - \frac{b}{A^{1/3}} - \frac{sdA^{2/3}}{4s + dA^{2/3}}.$$

We need to express *B/A* as a function of *A* only, i.e. replace *N* and *Z* with functions of *A* only. The equation of the semi-empirical mass formula contains a term in $(N-Z)$.

We can use the help from the first equation: $\dfrac{N}{Z} = 1 + \dfrac{d\,A^{\frac{2}{3}}}{2s}$.

This gives $N = Z + Z\,\dfrac{d\,A^{\frac{2}{3}}}{2s} \Leftrightarrow N - Z = Z\,\dfrac{d\,A^{\frac{2}{3}}}{2s}$

We can also use the fact that: $A = N + Z = 2Z + Z\dfrac{dA^{2/3}}{2s}$, meaning that: $\dfrac{Z}{A} = \dfrac{2s}{\left(4s + dA^{2/3}\right)}$

The binding energy per nucleon (neglecting the pairing term δ) can be written as:

$$\frac{B}{A} = a - \frac{b}{A^{1/3}} - s\frac{(N-Z)^2}{A^2} - \frac{dZ^2}{A^{4/3}}$$

$$= a - \frac{b}{A^{1/3}} - Z^2\left[\frac{s}{A^2}\left(\frac{N}{Z} - 1\right)^2 + \frac{d}{A^{4/3}}\right]$$

Substituting for *Z* and *(N – Z)* gives:

$$\frac{B}{A} = a - \frac{b}{A^{1/3}} - \frac{4s^2A^2}{\left(4s + dA^{2/3}\right)^2}\left(\frac{\cancel{s}}{\cancel{A^2}}\left(\frac{d^2\cancel{A^{4/3}}}{4s^2}\right) + \frac{d}{A^{4/3}}\right)$$

$$= a - \frac{b}{A^{1/3}} - \frac{4s^2A^2}{\left(4s + dA^{2/3}\right)^2}\,\frac{d\left(4s + dA^{2/3}\right)}{4sA^{4/3}}$$

The binding energy per nucleon of the most stable nuclei is therefore:

$$\frac{B}{A} = a - \frac{b}{A^{1/3}} - \frac{sdA^{2/3}}{\left(4s + dA^{2/3}\right)}$$

5. * *The lowest few energy levels in the shell model are*

$$1s_{1/2} \quad 1p_{3/2} \quad 1p_{1/2} \quad 1d_{5/2} \quad 2s_{1/2} \quad 1d_{3/2}.$$

How many nucleons can be accommodated in each level? Predict the spins of the following nuclei:

$${}^{4}_{2}He \quad {}^{17}_{8}O \quad {}^{35}_{17}Cl \quad {}^{15}_{7}N \quad {}^{11}_{5}B \text{ and } {}^{11}_{5}B^{*} \text{(in the first excited state).}$$

The number of nucleons in each level is $2j + 1$ (cf. lecture notes).
Therefore:

Level	$1s_{1/2}$	$1p_{3/2}$	$1p_{1/2}$	$1d_{5/2}$	$2s_{1/2}$	$1d_{3/2}$
j	1/2	3/2	1/2	5/2	1/2	3/2
No. of nucleons	2	4	2	6	2	4

We saw in the notes that pairs of nucleons have a total spin contribution of 0. And we saw that unpaired nucleons would give the spin of their level to the full nucleus.

${}^{4}_{2}He$: 2 protons, 2 neutrons $\Rightarrow$ both $1s_{1/2}$ shells are filled: $J = 0$

${}^{17}_{8}O$: 8 protons $\Rightarrow 1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$ all filled

9 neutrons $\Rightarrow$ $1s_{1/2}$, $1p_{3/2}$, $1p_{1/2}$ are all filled, and there is 1 extra neutron in $1d_{5/2}$
Therefore $J = 5/2$

${}^{35}_{17}Cl$: 17 protons: one unpaired proton in $1d_{3/2}$ level

18 neutrons
Therefore $J = 3/2$

${}^{15}_{7}N$: one unpaired proton in $1p_{1/2}$ level $\Rightarrow J = 1/2$

${}^{11}_{5}B$: one unpaired proton in $1p_{3/2}$ level $\Rightarrow J = 3/2$

${}^{11}_{5}B^{*}$: first excited state corresponds to an unpaired proton being raised to the next energy level (i.e. $1p_{1/2}$) $\Rightarrow J = 1/2$