PH20107 Mathematical Methods for Physics 2

Fourier analysis

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Table of Contents:

Revision: You will need techniques from previous Mathematical Methods for Physics unit, including:

- defining and sketching functions

- symmetry (even and odd functions)

- integration by parts

- sin and cos and their derivatives and integrals

- complex numbers (Re & Im, |z| and arg z, $e^{i\theta} \leftrightarrow$ cos & sin)

Formula book: Many key equations from this notes, and certain Fourier transform pairs you can find in the University Formula Book (section “Fourier Series”)

If you decide to print these notes I recommend 2 pages per sheet, double-sided, black and white.

1. Fourier Series

A Fourier series is the expression of a periodic function as a sum of sine and cosine functions.

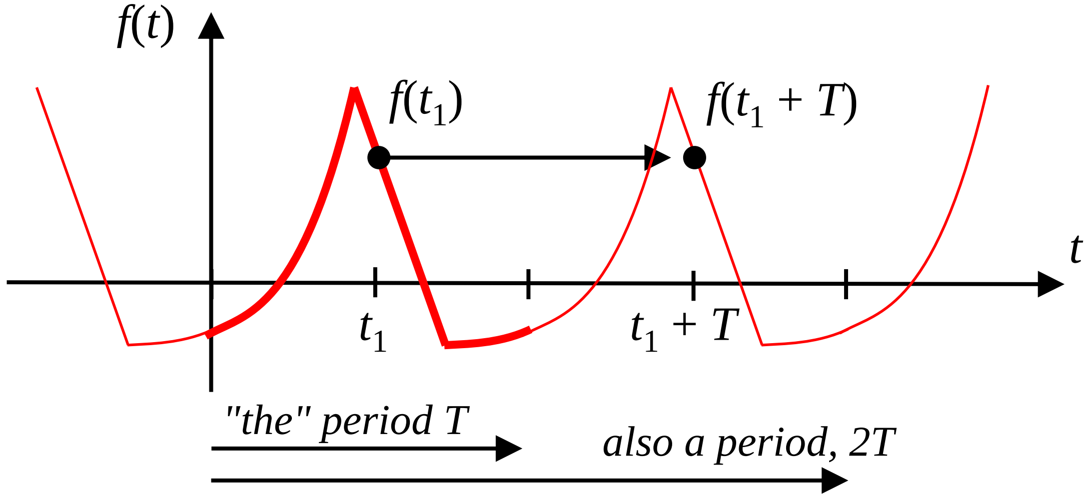
1.1. Periodic Functions (things we need to know about them)

A function $f(t)$ is *periodic* with period T iff

$$\boxed{f(t+T) = f(t) \quad \forall t}$$

[$\forall t$ means "for all values of t"]

eg:



If T is a period, so is any integer × T. The *fundamental* period (or, <u>the</u> period) means the smallest positive value of T.

The fundamental angular frequency ω_0 is

we'll omit the word "angular" from now on

$$\boxed{\omega_0 = \frac{2\pi}{T}}$$

<u>Example</u>

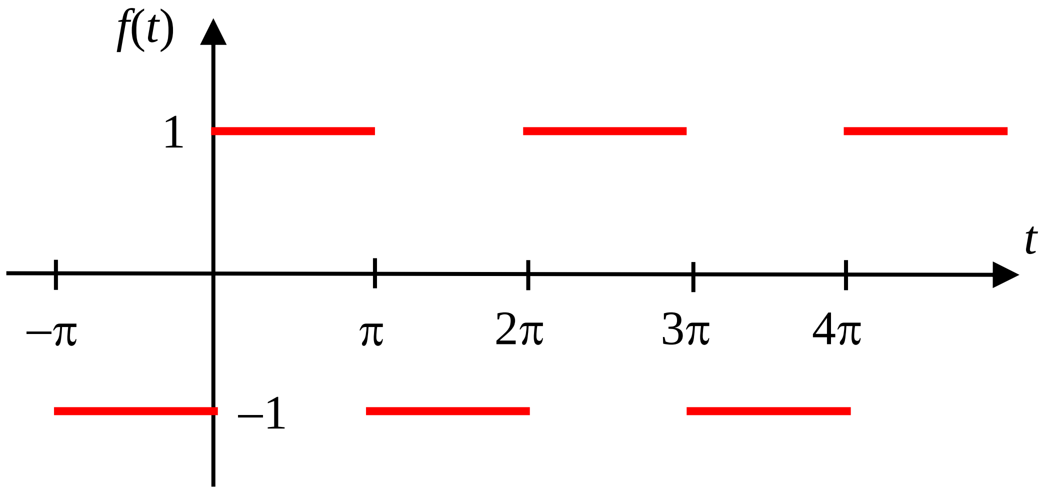
$f(t) = \sin(t)$ has periods 2π, 4π, 6π etc because $\sin(t + 2\pi) = \sin(t)$ etc for all t. "The" period of $f(t)$ is $T = 2\pi$, and $\omega_0 = 2\pi/T = 1$.

Defining a periodic function

Define $f(t)$ within a single period (maybe in pieces), with a statement of its periodicity.

Example

This square wave:



is
$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$$ *definition in a single period (in 2 pieces)*

$$f(t + 2\pi) = f(t) \quad \forall t$$ *statement of periodicity*

Like other functions, periodic functions can be even, odd, or neither even nor odd (revision needed?)

even: $$f(-t) = f(t) \quad \forall t$$

odd: $$f(-t) = -f(t) \quad \forall t$$

Don't confuse even / odd *functions* with even / odd *numbers*!

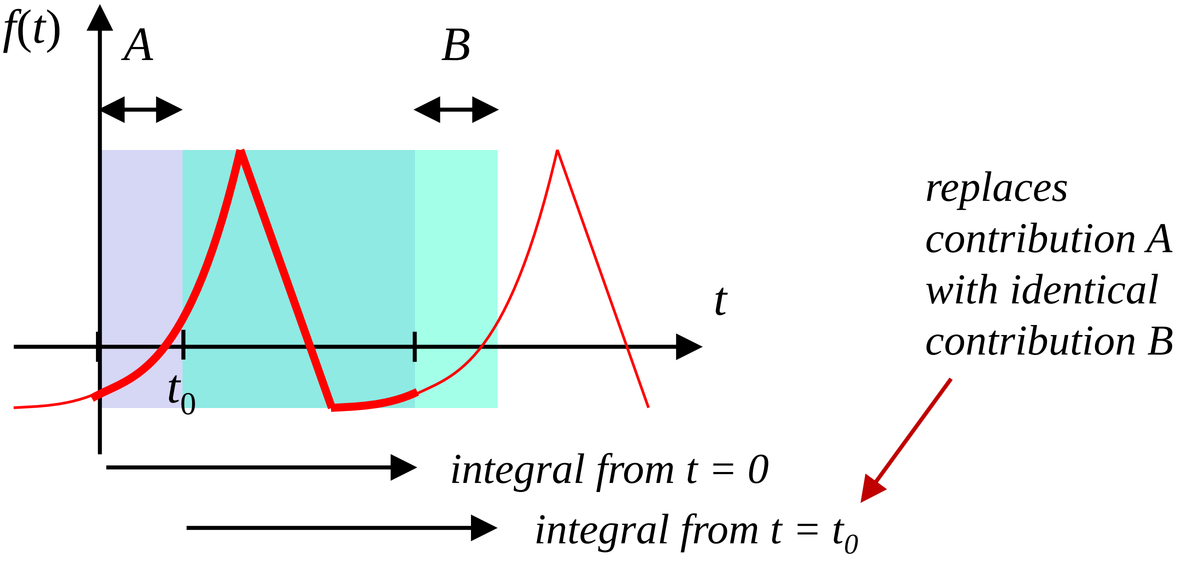
eg, the above square wave is an odd function.

Integrating over a period

The integral of a periodic function over exactly one period does not depend on where the integral starts.

If $\quad f(t+T) = f(t) \quad \forall t$

then $\quad \int_{t_0}^{t_0+T} f(t)dt \quad$ is independent of t_0



When evaluating such an integral, you are *free to choose* the most convenient value for t_0. Usually:

$t_0 = 0$ or $t_0 = -T/2$

$$\int_0^T f(t)dt \qquad\qquad \int_{-T/2}^{T/2} f(t)dt$$

a limit of 0 is often easy to substitute in the indefinite integral

a symmetric range is usually best for even and odd functions

Sinusoidal functions

The simplest periodic functions are sin and cos

$$f(t) = \sin(t) \quad \rightarrow \quad T = 2\pi\ ;\ \omega_0 = 1$$

$$f(t) = \sin(at) \quad \rightarrow \quad T = 2\pi/a\ ;\ \omega_0 = a$$

You need to know special values for sin and cos, eg where they are 0, +1 and –1. Particularly useful results <u>for integer n</u> are:

$$\sin(n\pi) \equiv 0$$

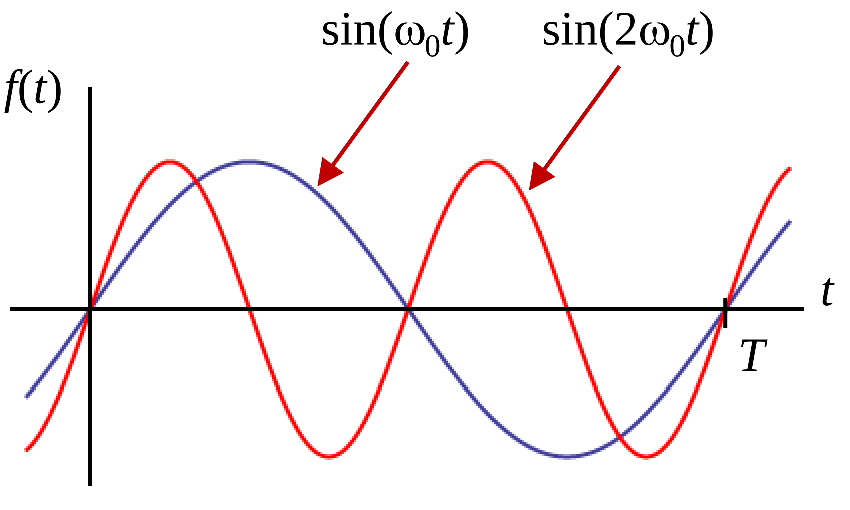
$$\cos(n\pi) \equiv (-1)^n$$

The trigonometric system

The set of all independent sinusoidal functions that share <u>a</u> period of T. Taking $\omega_0 = 2\pi/T$, here are the members of this set:

- $f(t) = \sin(n\omega_0 t) \quad n = 1, 2, 3, \ldots$

an infinite subset which all have <u>a</u> period of T (repeating n times within that period)



Similarly

- $$f(t) = \cos(n\omega_0 t) \quad n = 1, 2, 3, \ldots$$

another infinite subset which all have a period of T.

Finally, when $n = 0$, $\cos(n\omega_0 t) = 1$:

- $$f(t) = 1$$

a single function which also has a period of T. Yes, this constant function is *trivially* periodic, since $f(t + T) = f(t)$ for all t.

Hence the trigonometric system with period T is the infinite set of functions:

- $$\begin{cases} \sin(n\omega_0 t) & n = 1, 2, 3, \ldots \\ \cos(n\omega_0 t) & n = 1, 2, 3, \ldots \\ 1 & \end{cases}$$

The trigonometric system is like a set of "basis vectors", used to resolve an arbitrary periodic function into sinusoidal components as a *Fourier series*.

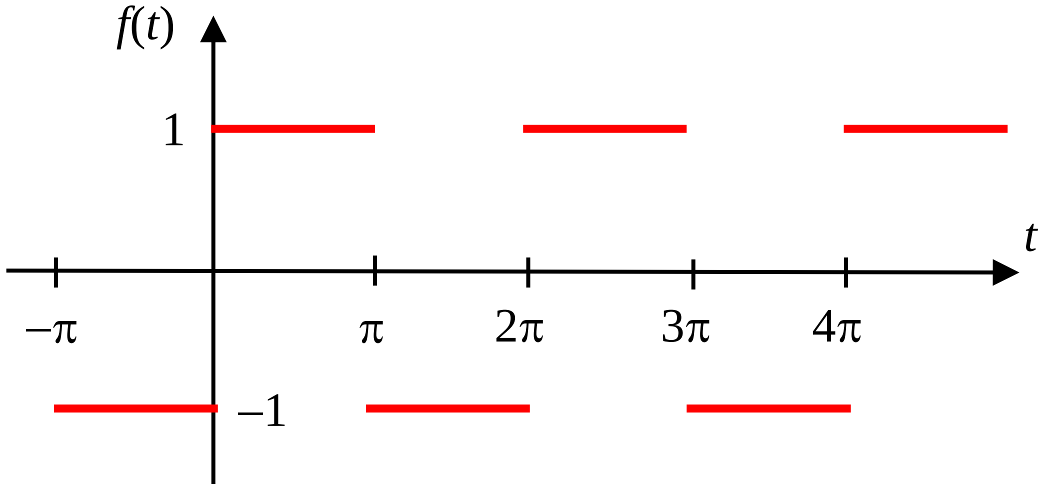
1.2. ***Real Fourier Series***

Fourier's Theorem

Any periodic function can be expressed as a sum of sinusoidal functions with the same period* - a *Fourier series* (FS).

Example

The square wave we looked at before



(with $T = 2\pi$, $\omega_0 = 2\pi/T = 1$) can be expressed as

$$f(t) = \frac{4}{\pi}\left\{\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \ldots\right\}$$

See Appendix A on page 30, which shows how adding each term gives a closer approximation to $f(t)$, and also covers the important topic of the *Gibbs phenomenon*.

** ie, the trigonometric system for the function's period T*

In general, if $f(t)$ has the period T and $\omega_0 = 2\pi/T$ then its FS is

find it in the formula book

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)}$$

sum starts at n = 1

where the *Fourier coefficients* a_n and b_n are

find it in the formula book

$$\boxed{a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)dt \quad n = 0, 1, 2, 3, \ldots}$$

n = 0 gives us a_0

find it in the formula book

$$\boxed{b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots}$$

no b_0 because sin(0) = 0

You prove these expressions in problem sheet 1 but don't memorise them - they're in the formula book! They can be evaluated if you know $f(t)$ and can do the integrals.

The constant (t-independent) term in the FS

$$\frac{a_0}{2} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt$$

has the factor 1/2 so that the expression for a_n works for $n = 0$ as well as $n \neq 0$. Note that it is just the *average* of $f(t)$. It is sometimes called the *d.c. term*, because in electronics it gives the direct current.

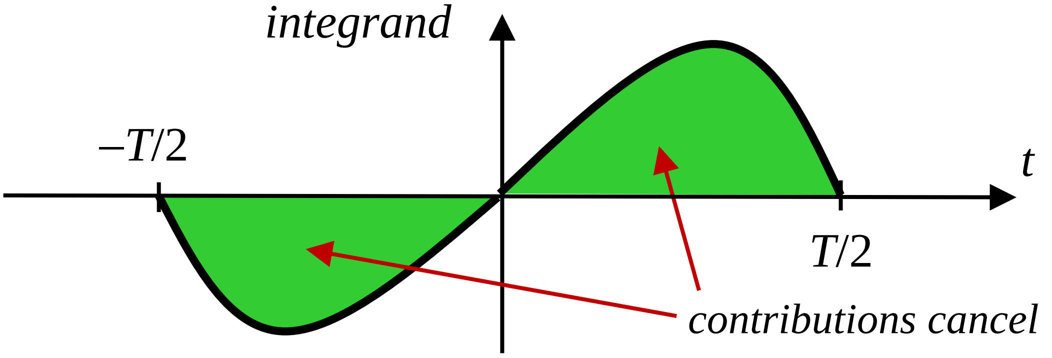
<u>Symmetry</u> (a valuable shortcut)

The effort of finding a_n and b_n is *halved* if $f(t)$ is either even or odd. First choose $t_0 = -T/2$ to make the range of t symmetric about the origin, then consider the symmetry of the *integrand*:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \underbrace{f(t)\cos(n\omega_0 t)dt}_{\textit{the integrand}}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \overbrace{f(t)\sin(n\omega_0 t)dt}$$

If the integrand is *odd*, then $\int_{-T/2}^{T/2} \text{integrand } dt \equiv 0$



So, <u>if $f(t)$ is even</u> then $f(t)\sin(n\omega_0 t)$ is odd
(even function × odd function = odd function) and $\boxed{b_n = 0}$

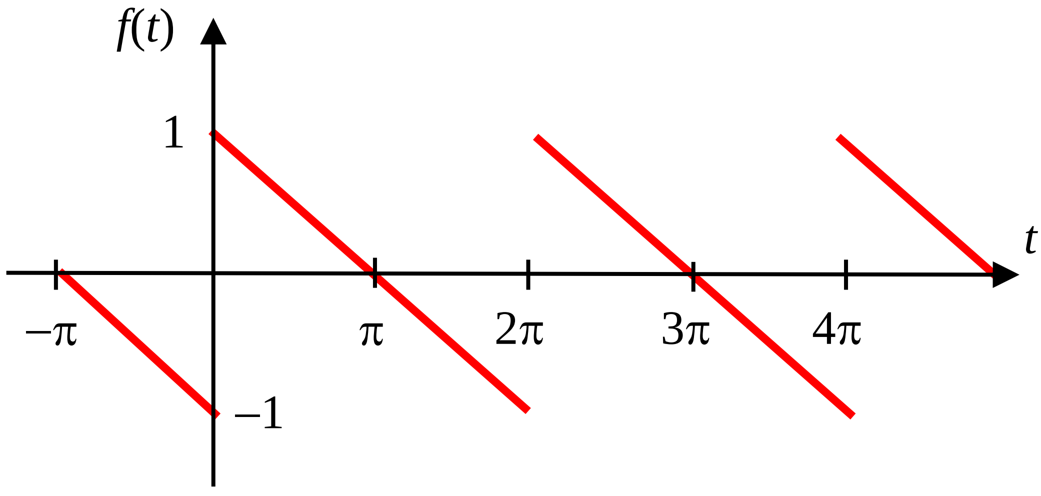
and, <u>if $f(t)$ is odd</u> then $f(t)\cos(n\omega_0 t)$ is odd
(odd function × even function = odd function) and $\boxed{a_n = 0}$

Also (much less useful, but...) if the integrand is *even*, then

$$\int_{-T/2}^{T/2} \text{integrand } dt \equiv 2\int_{0}^{T/2} \text{integrand } dt$$

<u>Example</u> (a simple one, if you know YR1 maths as you should)

Q. Express the following "saw tooth" function *f*(*t*) as a FS.



A. (note: it's an odd function ...)

[The Q gives *f*(*t*) as a graph, so write it algebraically. How? Well, it's a simple straight line from 0 to 2π, so use $y = mx + c$. The slope *m* is $\Delta y/\Delta x = -1/\pi$ (the line goes down from 1 to 0 as *t* goes from 0 to π) and the *y* intercept is 1. So]

$$f(t) = 1 - \frac{t}{\pi} \quad 0 < t < 2\pi$$

$$f(t + 2\pi) = f(t) \quad \forall t$$

[Write down the parameters *T* and ω_0]

$$\text{period } T = 2\pi \quad \Rightarrow \quad \omega_0 = 2\pi/T = 2\pi/2\pi = 1$$

[Now we have everything ready to plug into the integrals for a_n and b_n that we can get from the formula book]

- $$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)dt$$

[But *f*(*t*) is odd, so a_n must be zero. It is sufficient to write:]

f(*t*) is odd, so $a_n = 0$ by symmetry

- $$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)dt$$

[Substitute for $f(t)$, T and ω_0. By choosing $t_0 = 0$ we can use our straight-line expression for $f(t)$, which is valid for $0 < t < 2\pi$. We end up with an integral we can evaluate using YR1 methods]

$$= \frac{1}{\pi}\int_0^{2\pi}\left(1-\frac{t}{\pi}\right)\sin(nt)dt$$

$$= \ldots$$

revise?!

$$= \frac{1}{\pi}\left[\frac{-\cos(nt)}{n} - \frac{\sin(nt)}{\pi n^2} + \frac{t\cos(nt)}{\pi n}\right]_0^{2\pi}$$ [integration by parts]

$$= \frac{2}{\pi n}$$ [using $\cos(2n\pi) \equiv 1$ and $\sin(2n\pi) \equiv 0$ for integer n]

- So $$\begin{cases} a_n = 0 & n = 0,1,2,3,\ldots \\ b_n = \dfrac{2}{\pi n} & n = 1,2,3,\ldots \end{cases}$$

- [Finally substitute into the general FS from the formula book]

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(nt)$$

substitute for ω_0 as well a_n and b_n

$$= \frac{2}{\pi}\left\{\sin(t) + \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t) + \ldots\right\}$$

See Appendix B on page 32.

NB your final answers should *never* contain the symbols T and ω_0 unless they're in the question!

The next example is long and has some important "catches".

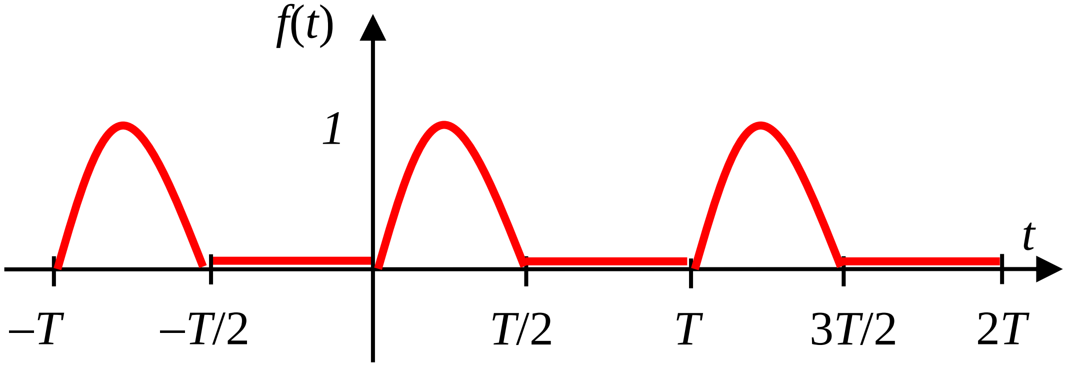
Example

Q.
$$f(t)=\begin{cases}\sin(\omega_0 t) & 0<t<T/2\\ 0 & T/2<t<T\end{cases}$$
$$f(t+T)=f(t)\quad \forall t$$

where $\omega_0 = 2\pi/T$

Sketch $f(t)$ over three periods and express it as a FS.

A.



[It's a good idea to plot either side of $t = 0$ to see if $f(t)$ is even or odd. Unfortunately, in this case it is neither.]

[Note that T and ω_0 are already given in the Q.]

- $$b_n=\frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)dt$$
$$=\frac{2}{T}\int_0^{T/2}\sin(\omega_0 t)\sin(n\omega_0 t)dt+\frac{2}{T}\int_{T/2}^{T}0.\sin(n\omega_0 t)dt$$

[Chose $t_0 = 0$ and substituted for $f(t)$. Because $f(t)$ is defined in two pieces we must break the integral up into two pieces, one for each range in the definition of $f(t)$. Obviously in this case the second piece is zero, but I'm making a point ...]

[To perform the integral we use a trig identity from the formula book to convert the trig product into a sum (see YR1 methods)]

$$b_n = \frac{1}{T}\int_0^{T/2}\left[\cos\{\omega_0(n-1)t\} - \cos\{\omega_0(n+1)t\}\right]dt \qquad *$$

$$= \frac{1}{T}\left[\frac{\sin\{\omega_0(n-1)t\}}{\omega_0(n-1)} - \frac{\sin\{\omega_0(n+1)t\}}{\omega_0(n+1)}\right]_0^{T/2} \qquad \text{if } n \neq 1$$

[NB be very careful here - we are finding a general expression for b_n for n = 1, 2, 3, ... but the last step fails for the special case n = 1 because there's a division by zero! What follows is therefore only valid for the other n values and we must say so. To find the special case b_1 we will return to the last valid step (marked *) later. *Always be alert for special cases like this.*]

$$= \frac{1}{T\omega_0}\left[\frac{\sin\left\{(n-1)\frac{2\pi}{T}\frac{T}{2}\right\}}{(n-1)} - \frac{\sin\left\{(n+1)\frac{2\pi}{T}\frac{T}{2}\right\}}{(n+1)}\right]$$

[Substituted integral limits, sin(0) = 0 and $\omega_0 = 2\pi/T$]

$$= 0 \qquad \text{if } n \neq 1 \qquad [\sin(\text{integer} \times \pi) \equiv 0]$$

[Now do n = 1, from the step* before the division by 0]

$$b_1 = \frac{1}{T}\int_0^{T/2}\left[1 - \cos\{2\omega_0 t\}\right]dt \qquad [* \text{ with } n = 1]$$

$$= \frac{1}{T}\left[t - \frac{\sin\{2\omega_0 t\}}{2\omega_0}\right]_0^{T/2}$$

$$= \frac{1}{T}\left[\frac{T}{2} - \frac{\sin\left\{2\,\frac{2\pi}{T}\,\frac{T}{2}\right\}}{2\omega_0}\right] = \frac{1}{2} \qquad [\sin(2\pi) \equiv 0]$$

- $$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)dt$$
$$= \frac{2}{T}\int_0^{T/2} \sin(\omega_0 t)\cos(n\omega_0 t)dt$$

didn't bother writing the zero term ...

[Derivation is like b_n (trig identity, division by zero when n = 1, substitutions) but the answer for $n \neq 1$ is nonzero]

$$= \frac{1}{T}\int_0^{T/2}\left[\sin\{\omega_0(n+1)t\} - \sin\{\omega_0(n-1)t\}\right]dt \qquad \dagger$$

$$= \frac{1}{T}\left[\frac{-\cos\{\omega_0(n+1)t\}}{\omega_0(n+1)} + \frac{\cos\{\omega_0(n-1)t\}}{\omega_0(n-1)}\right]_0^{T/2} \qquad \text{if } n \neq 1$$

$$= \frac{1}{T\omega_0}\left[\frac{-\cos\left\{(n+1)\frac{2\pi}{T}\frac{T}{2}\right\}+1}{(n+1)} + \frac{\cos\left\{(n-1)\frac{2\pi}{T}\frac{T}{2}\right\}-1}{(n-1)}\right]$$

$$= \frac{1}{2\pi}\left[1+(-1)^n\right]\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

get it? work it out / revise!

[Used: $\omega_0 = 2\pi/T$, $\cos(0) \equiv 1$, $\cos[(n \pm 1)\pi] \equiv -(-1)^n$]

$$= -\frac{[1+(-1)^n]}{\pi(n^2-1)} \qquad \text{if } n \neq 1 \qquad \text{[common denominator]}$$

[Now do n = 1, from the step† before the division by 0]

$$a_1 = \frac{1}{T}\int_0^{T/2} \sin\{2\omega_0 t\}\,dt \qquad [\dagger \text{ with } n = 1]$$

$$= \frac{1}{T}\left[\frac{-\cos\{2\omega_0 t\}}{2\omega_0}\right]_0^{T/2}$$

$$= \frac{1}{2T\omega_0}\left[-\cos\left\{2\,\tfrac{2\pi}{T}\,\tfrac{T}{2}\right\}+1\right] = 0 \qquad [\cos(2\pi) \equiv 1]$$

- So
$$\begin{cases} a_1 = 0 & \\ a_n = -\dfrac{[1+(-1)^n]}{\pi(n^2-1)} & n = 0, 2, 3, 4, \ldots \\ b_1 = 1/2 & \\ b_n = 0 & n = 2, 3, 4, \ldots \end{cases}$$

all n except n = 1

- FS: $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin(\omega_0 t) - \frac{1}{\pi}\sum_{n=2}^{\infty}\frac{[1+(-1)^n]}{(n^2-1)}\cos(n\omega_0 t)$$

$a_0/2$ *b_1* *a_n for $n > 1$*

$$= \frac{1}{\pi} + \frac{1}{2}\sin(\omega_0 t) - \frac{2}{\pi}\left\{\frac{1}{3}\cos(2\omega_0 t) + \frac{1}{15}\cos(4\omega_0 t) + \frac{1}{35}\cos(6\omega_0 t) + \ldots\right\}$$

That example was full of complications, including: lack of symmetry, a variable as the period T, special-case values of n, and use of trig formulae like $\cos(n\pi) = (-1)^n$.

Some of these complications may appear in exam questions.

This is where the material in Appendix C (page 33) fits in.

1.3. Complex Fourier Series

Replace sin and cos in the FS with complex exponentials

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{(e^{in\omega_0 t} + e^{-in\omega_0 t})}{2} + b_n \frac{(e^{in\omega_0 t} - e^{-in\omega_0 t})}{2i}$$

[if you don't follow the line above, you need to <u>revise complex numbers NOW</u> because you're seriously going to need it]

ie we can write

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}}$$ [complex FS]

find it in the formula book

where the coefficients c_n are to be determined for all integers n (including negative n).

The component $e^{in\omega_0 t}$ has frequency $\omega = n\omega_0$, which is negative for negative n.

[-ve frequency? just the other half of, eg, $\cos(\omega t) = \frac{(e^{i\omega t} + e^{-i\omega t})}{2}$]

<u>Derivation of c_n</u> (similar to that for a_n and b_n)

We will need the following integral for *non-zero* integer N:

$$\int_0^T e^{iN\omega_0 t} dt = \left[\frac{e^{iN\omega_0 t}}{iN\omega_0} \right]_0^T$$ [only for $N \neq 0$, else $\div 0$]

$$= \frac{e^{iN\frac{2\pi}{T}T} - 1}{iN\omega_0}$$ [subst $\omega_0 = 2\pi/T$]

$$= 0$$ [$e^{iN2\pi} = \cos(2\pi N) + i \sin(2\pi N)$
$= 1$ for integer N]

Now the derivation - start with the complex FS:

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$$

Multiply both sides by $e^{im\omega_0 t}$, where m is an arbitrary integer, then integrate both sides over one period:

$$\int_{t_0}^{t_0+T} f(t)e^{-im\omega_0 t}dt=\sum_{n=-\infty}^{\infty} c_n\int_{t_0}^{t_0+T} e^{in\omega_0 t}e^{-im\omega_0 t}dt$$

$$=\sum_{n=-\infty}^{\infty} c_n\int_{0}^{T} e^{i(n-m)\omega_0 t}dt \qquad \text{[choose } t_0 = 0\text{]}$$

Now we showed above that the integral on the RHS is zero if $N = n - m \neq 0$, so in the infinite sum only $n = m$ survives:

$$=c_m\int_0^T dt$$

$$=c_m T$$

Hence

$$\boxed{c_n=\frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-in\omega_0 t}dt} \qquad [\text{rename } n\to m]$$

find it in the formula book

In general the c_n are complex, but there's only one integral to work out. Symmetry is not very useful for complex FS because $e^{-in\omega_0 t}$ (and hence the integrand) is neither even or odd.

A plot of c_n against frequency $\omega = n\omega_0$ is call the *spectrum* of $f(t)$. If c_n is complex this requires two plots:

amplitude spectrum: $|c_n|$ against ω

phase spectrum: $\phi_n = \arg(c_n)$ against ω

remember what these mean? if not, revise complex numbers NOW!!!

<u>Example</u>

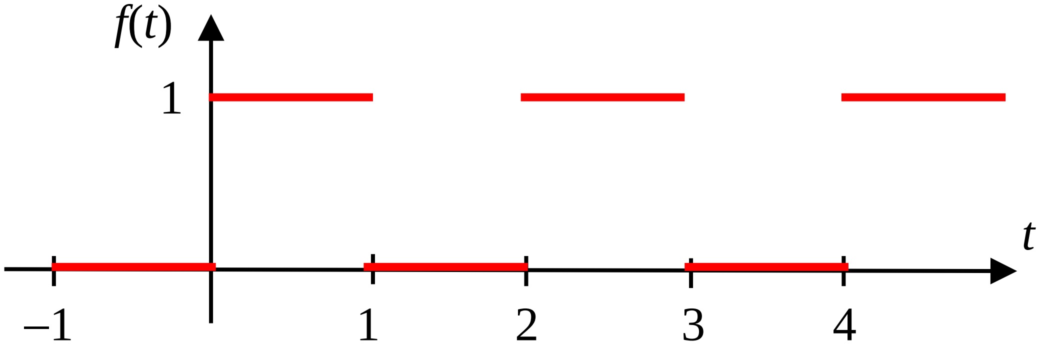
Q. Find the complex FS for

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$$

$$f(t+2) = f(t) \quad \forall t$$

and plot its amplitude and phase spectra.

A. Sketch $f(t)$: it's a square wave, though not the same as the one before



$$T = 2 \quad \Rightarrow \quad \omega_0 = 2\pi/T = \pi$$

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-in\omega_0 t}\,dt$$ [from formula book]

$$= \frac{1}{2}\int_0^1 1.e^{-in\pi t}\,dt \qquad *$$ [subst T, ω_0 and $f(t)$, and the piece from 1 to 2 is zero]

$$= \frac{1}{2}\left[\frac{e^{-in\pi t}}{-in\pi}\right]_0^1$$ if $n \neq 0$ [else ÷0]

$$= \frac{1}{-2in\pi}(e^{-in\pi} - 1)$$

$$= \frac{-i}{2n\pi}[1-(-1)^n]$$

[$e^{-in\pi} = \cos(-n\pi) + i\sin(-n\pi)$
$= \cos(n\pi) = (-1)^n$]

[also $1/i = -i$]

Now go back to * to do special case $n = 0$:

$$c_0 = \frac{1}{2}\int_0^1 1.dt = \frac{1}{2}$$

So the complex FS is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} = \frac{1}{2} - \frac{i}{2\pi}\sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{[1-(-1)^n]}{n} e^{in\pi t}$$

c_0 term separate because it doesn't fit the pattern

The Q also wants the amplitude and phase spectra. It's worth being methodical. Start with a table for small-ish values of n:

n	ω	c_n	$\lvert c_n \rvert$	$\arg(c_n)$
0	0	1/2	1/2	0
1	π	$-i/\pi$	$1/\pi$	$-\pi/2$
–1	$-\pi$	i/π	$1/\pi$	$\pi/2$
2	2π	0	0	undefined
–2	-2π	0	0	undefined
3	3π	$-i/3\pi$	$1/3\pi$	$-\pi/2$
–3	-3π	$i/3\pi$	$1/3\pi$	$\pi/2$
4	4π	0	0	undefined
–4	-4π	0	0	undefined

arg(0)

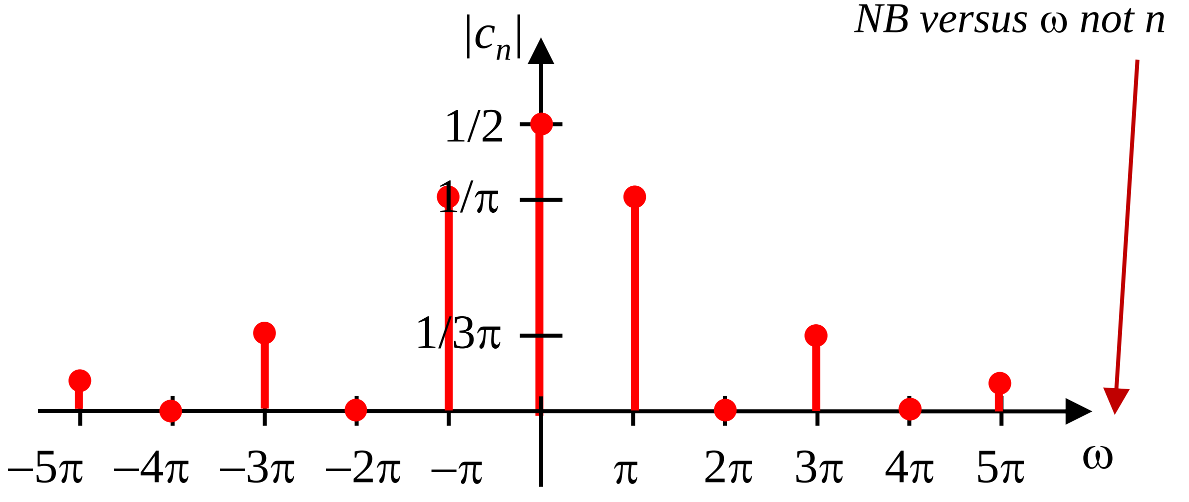
$\omega = n\omega_0$

modulus is always real and non-negative

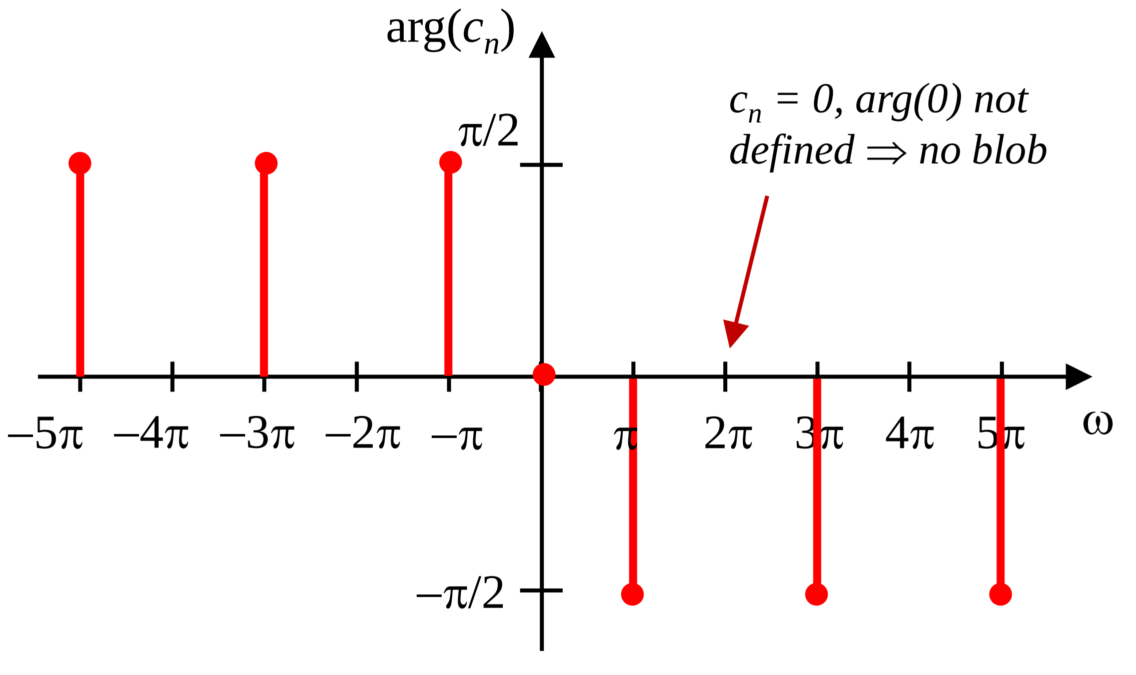
arg (if it exists) is always between $-\pi$ and π

Plot spectra using "blobs on sticks"

Amplitude spectrum: $|c_n|$ against ω



Phase spectrum: $\arg(c_n)$ against ω

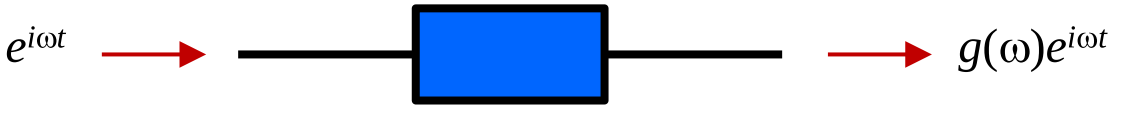


Example

Q. A high-pass filter circuit has a *transfer function*

$$g(\omega) = \frac{i\omega}{1 + i\omega}$$

meaning that if the input signal is a single-frequency sinusoidal wave $e^{i\omega t}$ then the output signal is $g(\omega)e^{i\omega t}$:



What is the output signal if the *non-sinusoidal* square wave $f(t)$ from the previous example is the input signal?

A. The complex FS from the previous example

$$f_{in}(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t}$$

expresses the signal in terms of single-frequency components $e^{in\pi t}$ with frequency $\omega = n\pi$. The n-th component therefore becomes $g(n\pi)e^{in\pi t}$ after passing through the circuit. Hence the output summing all the components is

$$f_{out}(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi t} g(n\pi)$$

$$= -\frac{i}{2\pi} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{[1-(-1)^n]}{n} \left(\frac{in\pi}{1+in\pi} \right) e^{in\pi t}$$

[Substituted c_n from before and $g(\omega)$ from the Q. The c_0 term doesn't survive because $g(0) = 0$.]

Suitably-many terms of this series can be summed to show what happens if you pass a square wave into an a.c. coupled oscilloscope: see Appendix D on page 35.

Time and frequency domains

The high-pass filter example typifies the use of Fourier techniques. The input signal $f(t)$ has a simple form as a function of t, "in the time domain". On the other hand the response of the circuit has a simple form as a function of ω, "in the frequency domain". Its response to a single-frequency input is simple, but *different for different frequencies*. We need Fourier techniques to transform our description of the signal between the two domains - to look at the signal from the time or frequency point of view.

$f(t)$ tells us how much of the signal exists at time t while c_n tells us how much of the signal exists at frequency $\omega = n\omega_0$. Each description of the signal is complete. If you know it in the time domain as $f(t)$ then you can find it in the frequency domain as c_n by performing the integral. If you know it in the frequency domain as c_n then you can find it in the time domain as $f(t)$ by summing the FS.

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-in\omega_0 t}\,dt$$

time domain | *frequency domain*

Fourier analysis

$f(t)$ | c_n (with $\omega = n\omega_0$)

Fourier synthesis

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$$

2 Fourier Transforms

2.1. Development from Fourier Series

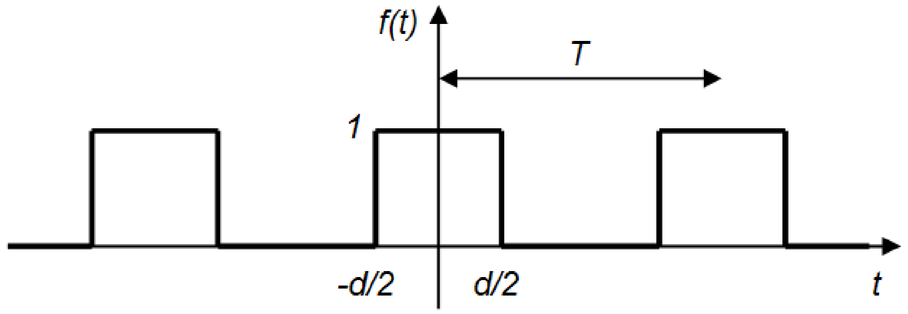
Firstly, a specific example:

Q. Find the complex FS of

$$f(t) = \begin{cases} 1 & |t| < d/2 \\ 0 & d/2 < |t| < T/2 \end{cases}$$

$$f(t+T) = f(t) \quad \forall t$$

A. Sketch the function:



It's a train of "top hat" pulses, width d, separated by T.

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-in\omega_0 t}\,dt \qquad [\omega_0 = 2\pi/T]$$

$$= \frac{1}{T}\int_{-d/2}^{d/2} e^{-in\omega_0 t}\,dt = \frac{1}{T}\left[\frac{e^{-in\omega_0 t}}{-in\omega_0}\right]_{-d/2}^{d/2} \qquad \text{if } n \neq 0 \qquad [\text{else} \div 0]$$

$$= \frac{1}{T}\left[\frac{e^{-in\omega_0 d/2} - e^{in\omega_0 d/2}}{-in\omega_0}\right] = \frac{1}{T}\frac{2}{n\omega_0}\left[\frac{e^{in\omega_0 d/2} - e^{-in\omega_0 d/2}}{2i}\right]$$

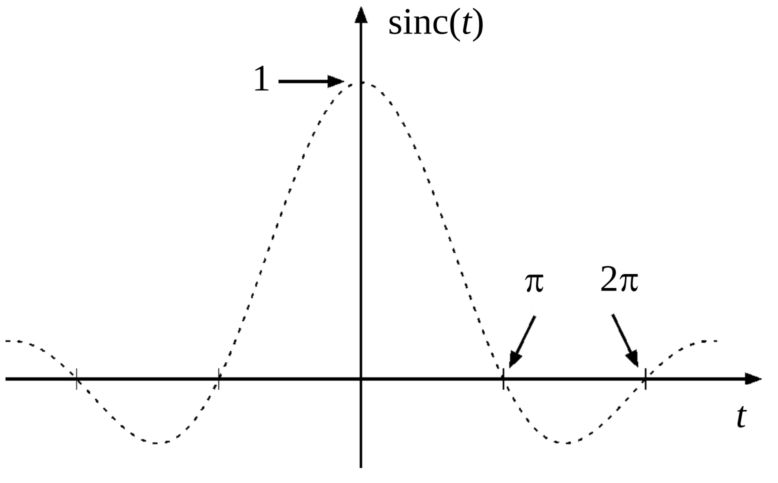
$$= \frac{1}{T}\frac{2}{n\omega_0}\sin\left(\frac{n\omega_0 d}{2}\right) \qquad [\sin\theta \equiv \frac{e^{i\theta} - e^{-i\theta}}{2i}]$$

$$= \frac{d}{T}\frac{\sin(n\omega_0 d/2)}{(n\omega_0 d/2)}$$

$$= \frac{d}{T}\operatorname{sinc}\left(\frac{n\omega_0 d}{2}\right)$$

where "sinc" represents the very-useful continuous function

$$\boxed{\operatorname{sinc}(t) = \begin{cases} \dfrac{\sin(t)}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}}$$



Now special case n = 0: $\quad c_0 = \dfrac{1}{T}\displaystyle\int_{-d/2}^{d/2} dt = \dfrac{d}{T}$

so $\quad c_n = \dfrac{d}{T}\operatorname{sinc}\left(\dfrac{n\omega_0 d}{2}\right)$ *happens to be right* even for n = 0.

Hence the FS for $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{d}{T}\operatorname{sinc}\left(\frac{n\omega_0 d}{2}\right) e^{in\omega_0 t}$$

<u>The spectrum of $f(t)$</u>

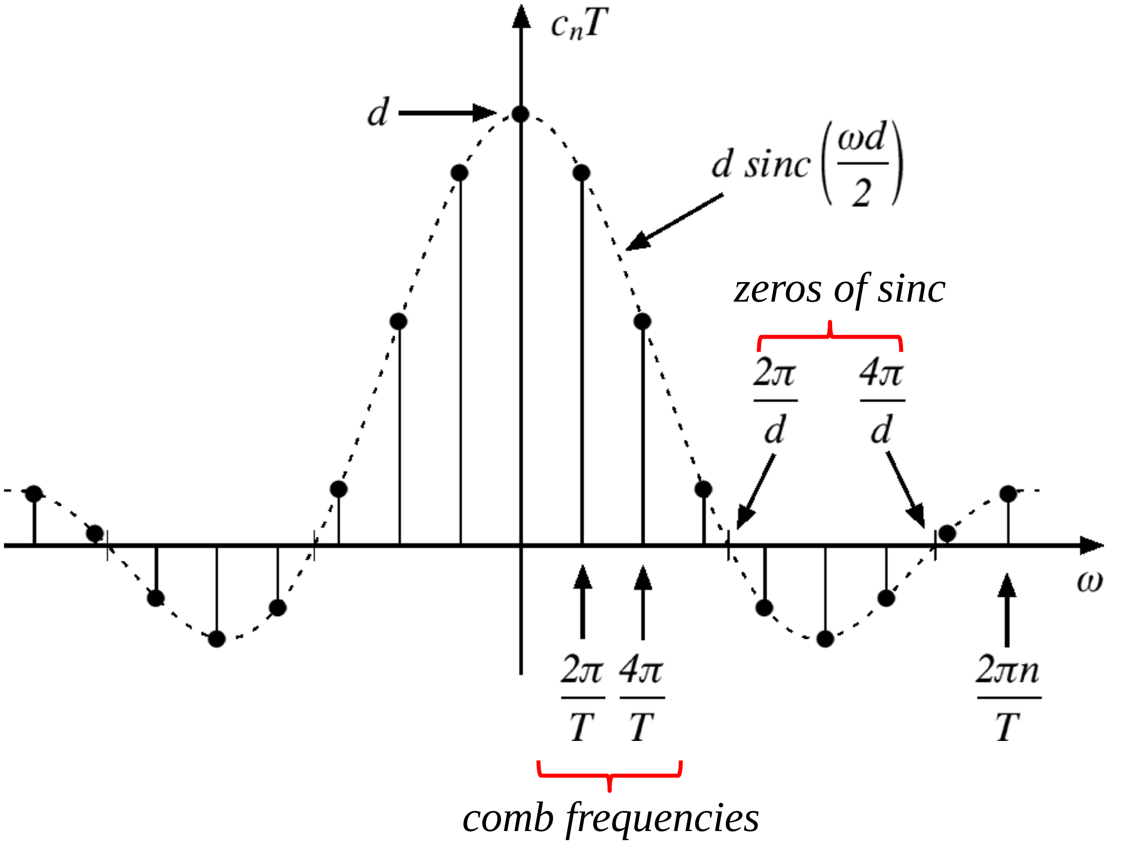
The spectrum is the variation of c_n with frequency ω, where

$$\omega \equiv n\omega_0 = \frac{2\pi n}{T} \qquad (1)$$

c_n is real so we can plot its spectrum directly - no need to plot amplitude and phase spectra separately. We will multiply c_n from the previous example by constant T before plotting the spectrum:

$$c_n T = d \ \mathrm{sinc}\left(\frac{\omega d}{2}\right) \qquad (2)$$

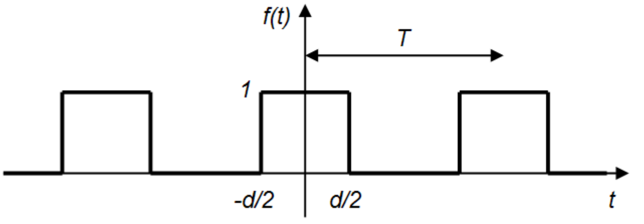
The spectrum is this function of ω, sampled at a comb of discrete values separated by $\Delta\omega = 2\pi/T$ as given by (1). For example if $T = 3.4\,d$:



So, referring back to the original function $f(t)$, the shape of the spectrum (2) depends only on the width of each pulse d while the discrete sampling of it (1) depends only on the separation of the pulses T.

A non-periodic version of $f(t)$

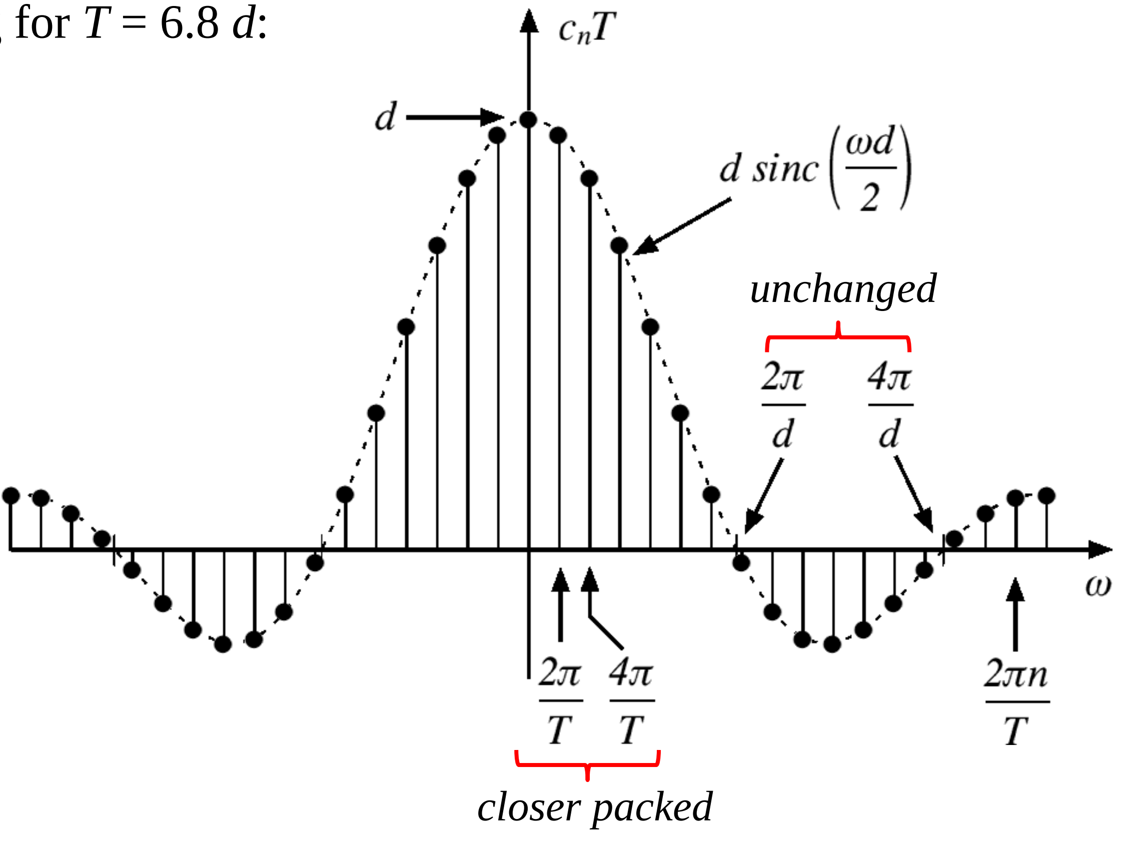
If the pulse separation T increases while the pulse width d stays the same, then



•the comb of frequencies (1) becomes *more densely packed*,

•but the sinc "envelope" (2) of the spectrum *doesn't change*.

eg for $T = 6.8\,d$:



As $T \to \infty$, the pulses become infinitely spaced: the central pulse remains but the others are "banished" to infinity. Simultaneously the comb of frequencies in the spectrum becomes zero-spaced:

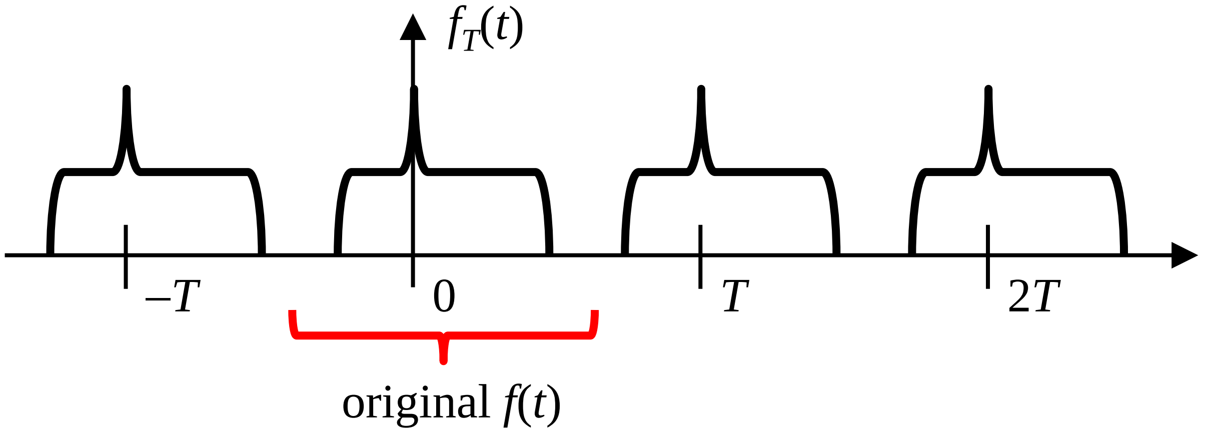
•$f(t)$ becomes *non-periodic* (a single top-hat pulse)

□ω in the spectrum (2) becomes a *continuous variable*

This spectrum is called the *Fourier transform* of $f(t)$.

Now, the general case:

To define the Fourier transform of an arbitrary non-periodic function $f(t)$, consider the complex FS of the periodic function $f_T(t)$ formed by repeating $f(t)$ with period T



then recover $f(t)$ by letting $T \to \infty$.

Fourier transform: Consider the function

$$F_T(\omega) = c_n T \tag{3}$$

$$= \int_{-T/2}^{T/2} f_T(t) e^{-i\omega t} dt \qquad [\text{choose } t_0 = -T/2]$$

where "allowed" frequencies are $\omega = 2\pi n/T$ so their spacing is

$$\delta\omega = \frac{2\pi}{T} \tag{4}$$

As $T \to \infty$:

$$F_T(\omega) \to F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \tag{5}$$

In (4), $\delta\omega \to 0$ and ω becomes a continuous variable. The continuous function $F(\omega)$ replacing the discrete coefficients c_n is the *Fourier transform* of $f(t)$ and is defined by (5).

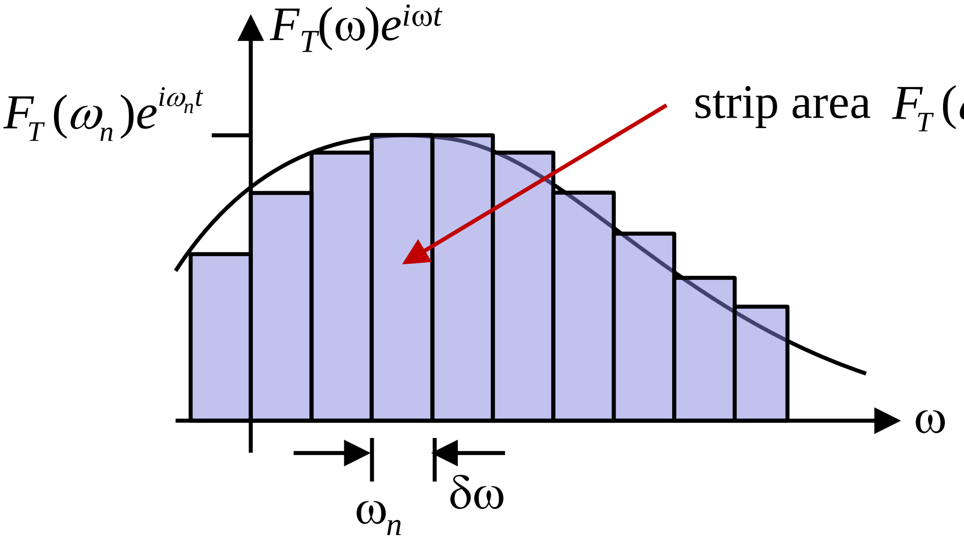
<u>Inverse Fourier transform</u>: Meanwhile, the Fourier series itself becomes

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega t} \qquad \text{[FS with implicit } \omega = n\omega_0 \text{ from (1)]}$$

$$= \sum_{n=-\infty}^{\infty} \frac{F_T(\omega)}{T} e^{i\omega t} \qquad [c_n \text{ from (3)}]$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F_T(\omega) e^{i\omega t} \delta\omega \qquad [T \text{ from (4)}]$$

The sum is the area of an infinite set of rectangular strips:



As $T \to \infty$ and $\delta\omega \to 0$, the strips narrow to zero and the sum becomes the area under the smooth curve, ie an *integral*:

$$f_T(t) \to f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \qquad (6)$$

This integral recovers $f(t)$ from its Fourier transform $F(\omega)$, and so is the *inverse Fourier transform* of $F(\omega)$. Generalising the Fourier series, it expresses non-periodic $f(t)$ as a *continuous* distribution of sinusoids $e^{i\omega t}$, with amplitudes $\propto F(\omega)$.

2.2. Fourier Transforms by Integration

In summary, the *Fourier transform* (FT) from (5) is a definite integral performed on a function of one variable *f*(*t*) that returns a new function of a different variable *F*(ω):

find it in the formula book

$$\boxed{F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt} \qquad FT$$

In this integral, *t* is a dummy variable of integration and ω is treated as a constant.

An equally-simple *inverse Fourier transform* (IFT) integral (6) recovers *f*(*t*) from *F*(ω):

find it in the formula book

$$\boxed{f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega} \qquad IFT$$

Now ω is the variable and *t* is treated like a constant.

The *Fourier transform operator* $\mathcal{F}$ represents the act of taking the FT, and $\mathcal{F}^{-1}$ is the inverse:

$$F(\omega) = \mathcal{F}[f(t)]$$

$$f(t) = \mathcal{F}^{-1}[F(\omega)]$$

The integrals are similar but the variables are swapped, the sign of the exponent changes, and the IFT has a factor of 1/2π.

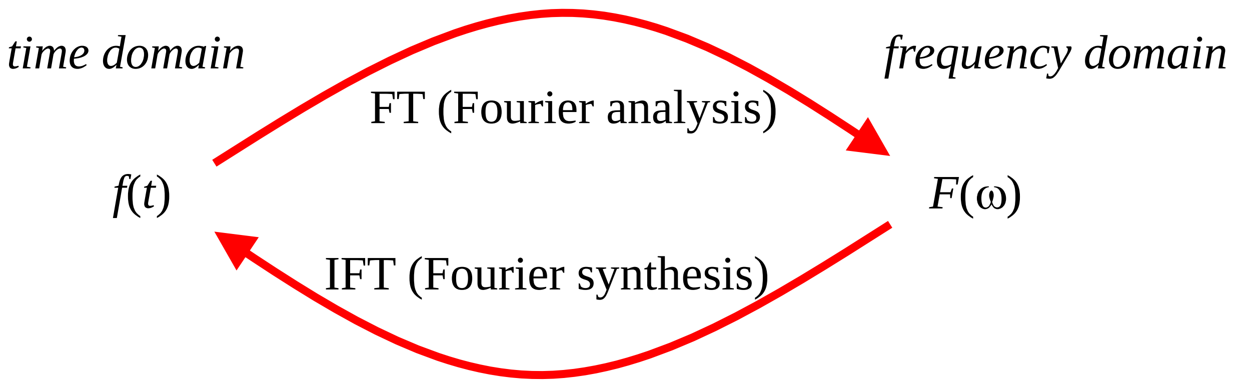
Other definitions of FT and IFT handle the 2π and/or the signs differently. We're following the definition in the formula book. You may therefore find expressions for FTs in the wider world that differ from ours by multiplying factors and/or sign changes. However, all that matters is that $\mathcal{F}^{-1}\mathcal{F}[f(t)] \equiv f(t)$.

The integral definitions appear on p. 11 of the formula book and so do not need to be memorised. Note the useful information in the formula book, including the short list of transforms on p. 12 that can save you some effort.

Time and frequency domains (again)

Fourier transforms are the generalisation of the Fourier series, and give the frequency content (or spectrum) of a time dependence that need not be periodic. $f(t)$ represents something (like a wave) in the time domain, and its Fourier transform $F(\omega)$ represents that thing in the frequency domain.

The functions $f(t)$ and $F(\omega)$ in the *Fourier transform pair* contain the same information: given either, you can deduce the other through the appropriate integral. They are just different ways of looking at the same thing.



Spatial Fourier transforms

The two variables t and ω are described as being *conjugate* to each other. If t represents time then ω represents ordinary temporal frequency (in angular "rad/s" units of course: "hertz" frequency is $\omega/2\pi$).

$$\text{time} \quad t \leftrightarrow \omega \quad \text{"temporal" frequency}$$

Other conjugate pairs of variables include position x and wave constant or *spatial frequency* k.

$$\text{space} \quad x \leftrightarrow k \quad \text{"spatial" frequency}$$

Hence $F(k) = \mathcal{F}[f(x)]$ etc, where the integrals have x and k instead of t and ω but otherwise work in exactly the same way.

However, unlike time, we can have more than one dimension of space and hence *2-D and 3-D Fourier transforms*. Indeed in abstract spaces we can have arbitrarily many dimensions.

eg, 2-D:

2-D ⇒ double integral

$$F(k_x, k_y) = \mathcal{F}[f(x, y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) e^{-ik_x x} e^{-ik_y y} dx dy \qquad FT$$

$$f(x, y) = \mathcal{F}^{-1}[F(k_x, k_y)] = \left(\tfrac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x, k_y) e^{ik_x x} e^{ik_y y} dk_x dk_y \qquad IFT$$

one factor of 2π per dimension

In general, in n-D where position $\mathbf{r} = (x, y, z, \ldots)$ and spatial frequency (or wave-vector) $\mathbf{k} = (k_x, k_y, k_z, \ldots)$:

$$F(\mathbf{k}) = \mathcal{F}[f(\mathbf{r})] = \int_{\mathbf{r}} f(\mathbf{r}) e^{-i\mathbf{k}.\mathbf{r}} d\mathbf{r}$$

$$f(\mathbf{r}) = \mathcal{F}^{-1}[F(\mathbf{k})] = \left(\tfrac{1}{2\pi}\right)^n \int_{\mathbf{k}} F(\mathbf{k}) e^{i\mathbf{k}.\mathbf{r}} d\mathbf{k}$$

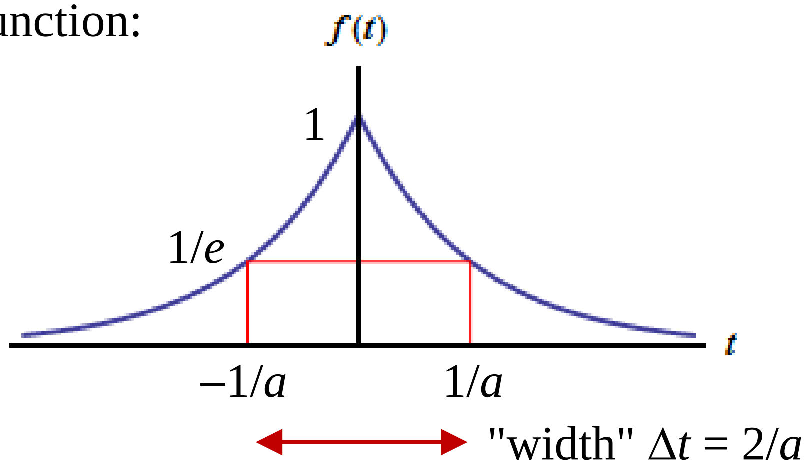
The mathematical connection between *real space* and the *reciprocal space* of solid-state physics (a.k.a. k-space or momentum-space, in which Brillouin zones exist) is a FT. Also 2-D FTs are important in the theory of diffraction.

Example

Q. Find the FT of $f(t) = e^{-a|t|}$, where $a > 0$.

A. Actually this one is in the table in the formula book (check before ploughing ahead!), but here's how to do it by integration:

Sketch the function:



$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad \text{[defn of FT]}$$

$$= \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t} dt \quad \text{[subst } f(t)\text{]}$$

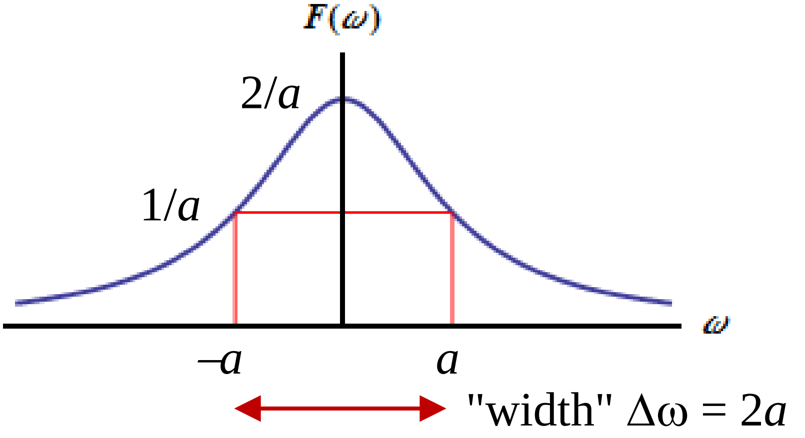
$$= \int_{-\infty}^{0} e^{at} e^{-i\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-i\omega t} dt \quad \text{[to handle } |t|\text{]}$$

$$= \int_{-\infty}^{0} e^{(a-i\omega)t} dt + \int_{0}^{\infty} e^{-(a+i\omega)t} dt \quad \text{[combine exponentials]}$$

$$= \left[\frac{e^{(a-i\omega)t}}{(a-i\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_{0}^{\infty}$$

$$= \left[\frac{e^{at} e^{-i\omega t}}{(a-i\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-at} e^{-i\omega t}}{-(a+i\omega)}\right]_{0}^{\infty} \quad \text{[split exponentials]}$$

$$= \frac{1}{(a-i\omega)} + \frac{1}{(a+i\omega)} = \frac{2a}{a^2+\omega^2}$$



<u>Example</u>

Q. Find the IFT of $F(\omega)=\begin{cases}\omega & 0<\omega<1\\ 0 & otherwise\end{cases}$

A.

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega \qquad \text{[defn of IFT]}$$

$$=\frac{1}{2\pi}\int_{0}^{1}\omega e^{i\omega t}d\omega \qquad \text{[subst } F(\omega)\text{]}$$

$$=\frac{1}{2\pi}\left[\omega\frac{e^{i\omega t}}{it}\right]_0^1-\frac{1}{2\pi}\int_0^1\frac{e^{i\omega t}}{it}d\omega \qquad \text{[by parts]}$$

[Careful here! In an IFT make sure you integrate with respect to ω not *t*. Likewise, when you substitute the limits, make sure you replace ω not *t*. There should be no ω in the answer!]

$$=\frac{1}{2\pi}\frac{e^{it}}{it}-\frac{1}{2\pi}\left[\frac{e^{i\omega t}}{(it)^2}\right]_0^1$$

$$=\frac{e^{it}}{2\pi it}+\frac{1}{2\pi t^2}(e^{it}-1)$$

[scruffy - can't simplify further]

The Bandwidth Theorem

In the first of the previous two examples, note how the width of $f(t)$ and the width of $F(\omega)$ depend on the parameter a in reciprocal ways, so that:

$$\boxed{\text{width of } F(\omega),\ \Delta\omega \propto \frac{1}{\text{width of } f(t),\ \Delta t}}$$

ie a narrow $f(t)$ gives a wide $F(\omega)$ and vice versa. This relationship holds in general for all functions and their FTs.

Aside (not exam material): The constant of proportionality depends on the functional form of $f(t)$ but has a lower bound. A rigorous treatment replaces the ad hoc definitions of width we used in the example: if Δt is the standard deviation of $|f(t)|^2$ and $\Delta\omega$ is the standard deviation of $|F(\omega)|^2$, then

$$\Delta\omega\Delta t \geq 1/2$$

A wave packet that extends over a time Δt has a spread of (angular) frequencies of at least $1/(2\Delta t)$.

Applying the bandwidth theorem to a quantum particle with energy $E = \hbar\omega$ (Einstein) and momentum $p = \hbar k$ (de Broglie):

$$\Delta E \Delta t \geq \hbar / 2$$

$$\Delta p \Delta x \geq \hbar / 2$$

$\Rightarrow$ *Heisenberg's uncertainty principle* comes directly from the theory of Fourier transforms, given the wave-particle duality of quantum mechanics.

2.3. Properties of Fourier Transforms

There are several properties listed on p. 11 of the formula book that can help you to find new FTs and IFTs from old ones without evaluating any more integrals.

In the following, let $F(\omega) = \mathrm{F}\,[f(t)]$

$$G(\omega) = \mathrm{F}\,[g(t)]$$

1 . Linearity

$$\boxed{\begin{aligned}\mathrm{F}\,[Af(t)+Bg(t)] &= A\mathrm{F}\,[f(t)]+B\mathrm{F}\,[g(t)]\\ &= AF(\omega)+BG(\omega)\end{aligned}}$$

where A and B are constants. This most basic of properties is actually not listed in the formula book, but you are expected to know about it.

2. Time shift

$$\boxed{\mathrm{F}\,[g(t+\tau)] = e^{i\omega\tau}G(\omega)}$$

find it in the formula book

where τ is a constant

Proof:

$$\begin{aligned}
LHS &= \int_{-\infty}^{\infty} g(t+\tau)e^{-i\omega t}dt && \text{[defn of FT]}\\
&= \int_{-\infty}^{\infty} g(t')e^{-i\omega(t'-\tau)}dt' && [t' = t+\tau]\\
&= \int_{-\infty}^{\infty} g(t')e^{-i\omega t'}e^{i\omega\tau}dt' && \text{[factorise exp]}\\
&= e^{i\omega\tau}\int_{-\infty}^{\infty} g(t')e^{-i\omega t'}dt' && \text{[constant factor]}\\
&= e^{i\omega\tau}G(\omega) = RHS && \text{[defn of FT]}
\end{aligned}$$

3 . Reflection

$$\boxed{\mathrm{F}\,[g(-t)] = G(-\omega)}$$

find it in the formula book

4. Frequency shift

$$\boxed{\mathrm{F}\,[e^{i\omega_0 t} g(t)] = G(\omega - \omega_0)}$$

find it in the formula book

where ω_0 is a constant

5. Scale change

$$\boxed{\mathrm{F}\,[g(t/T)] = TG(\omega T)}$$

find it in the formula book

where T is a positive constant

6. Conjugate

$$\boxed{\mathrm{F}\,[g^*(t)] = G^*(-\omega)}$$

find it in the formula book

where * denotes the complex conjugate

7. Derivative

$$\mathrm{F}\left[\frac{dg(t)}{dt}\right] = i\omega G(\omega)$$

Repeating:

$$\mathrm{F}\left[\frac{d^{(n)}g(t)}{dt^{(n)}}\right] = (i\omega)^n G(\omega)$$

find it in the formula book

In other words, FT converts differential equations into algebraic equations, often making them easier to solve.

Another version of this property applies to derivatives of the function of ω instead:

$$\boxed{\mathrm{F}\,[tg(t)] = i\frac{dG(\omega)}{d\omega}}$$

find it in the formula book

8 . Symmetry (also known as Inversion)

$$\boxed{\mathcal{F}\,[G(t)] = 2\pi g(-\omega)}$$

find it in the formula book

Look carefully at what's going on here. The transform function $G(\omega) = \mathcal{F}\,[g(t)]$ now acts on t instead of ω - the roles of the two variables are reversed. We can use this property to find the FT of a function $f(t)$ which we notice has the same form as a known transform function $G(\omega)$.

NB common student mistake! Note that $g(-\omega)$ is $g(t)$ with <u>every</u> t replaced by $-\omega$.

For example, if $$g(t) = \begin{cases} 1+t & 0<t<1 \\ 0 & otherwise \end{cases}$$

[including the t in the inequality!]

then $$g(-\omega) = \begin{cases} 1-\omega & 0<-\omega<1 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} 1-\omega & -1<\omega<0 \\ 0 & otherwise \end{cases}$$

[*then* tidy up the inequality]

NB, all these properties are written as Fourier transforms. They can of course equivalently be written as *inverse* Fourier transforms by applying $\mathcal{F}^{-1}$ to both sides.

eg frequency shift: $$\mathcal{F}^{-1}[G(\omega-\omega_0)] = e^{i\omega_0 t} g(t)$$

<u>Example</u>

Q. Find the FTs of

(a) $f_1(t) = e^{-9|t|}$

(b) $f_2(t) = e^{-|t-2|}$

(c) $f_3(t) = \dfrac{1}{1+t^2}$

given that the FT of $f(t) = e^{-|t|}$ is $F(\omega) = \mathcal{F}[f(t)] = \dfrac{2}{1+\omega^2}$

A. (a)

[First spot that $f_1(t)$ looks like $f(t)$ with a scale change. Start each answer by writing the function you *want* in terms of the function you've *got*:]

$$f_1(t) = e^{-9|t|} = e^{-|9t|} = f(9t)$$

So

$$F_1(\omega) = \frac{1}{9} F\left(\frac{\omega}{9}\right)$$

[scale change property with $T = 1/9$]

$$= \frac{1}{9} \times \frac{2}{1+(\omega/9)^2}$$

[F from the question]

$$= \frac{18}{81+\omega^2}$$

Important point! In an exam question you are of course expected to show how you obtain your answer. So if you use a FT property you need to say so, eg like this.

(b) $$f_2(t) = e^{-|t-2|} = f(t-2)$$

$$F_2(\omega) = e^{-2\omega i} F(\omega) = \frac{2e^{-2\omega i}}{1+\omega^2}$$

[time shift property with τ = –2]

(c) $$f_3(t) = \frac{1}{1+t^2} = \frac{1}{2}\frac{2}{(1+t^2)} = \frac{1}{2}F(t)$$

$$F_3(\omega) = \frac{1}{2} \times 2\pi f(-\omega) = \pi e^{-|-\omega|} = \pi e^{-|\omega|}$$

[symmetry property]

<u>Example</u>

Q. Use the formula book to find the IFTs of

(a) $$F_1(\omega) = e^{-\omega^2}$$

(b) $$F_2(\omega) = e^{-(\omega+3)^2}$$

A. (a) formula book p. 12: if $f(t) = e^{-at^2} = e^{-t^2/4}$ [let a = 1/4] then

$$F(\omega) = \mathrm{F}\,[\,f(t)] = (\pi / a)^{1/2} e^{-\omega^2/4a} = 2\sqrt{\pi} e^{-\omega^2}$$

$$F_1(\omega) = F(\omega) / 2\sqrt{\pi}$$

["want" in terms of "got"]

$$f_1(t) = \frac{1}{2\sqrt{\pi}} e^{-t^2/4}$$

(b) $$F_2(\omega) = e^{-(\omega+3)^2} = F_1(\omega+3)$$

$$f_2(t) = f_1(t) e^{-3it} = \frac{1}{2\sqrt{\pi}} e^{-t^2/4} e^{-3it}$$

2.4. Summary: Strategy for finding FTs

You now have three methods for finding Fourier transforms and inverse transforms. You will need to recognise which is the most profitable for a given problem.

1. *Evaluate the integral*

Fine in principle but can be tedious and the answer you get may not be in the neatest form. Do this if you have no alternative or if the integral looks quick and easy.

2. *Use the table on p. 12 of the formula book*

Obviously the quickest method, but there aren't many functions in the table. Familiarise yourself with the ones that are, to save time in an exam!

The commonly-encountered "top hat" function is called *rect* in the table. Note that the formulae to the right of the curly bracket are two equivalent definitions, not one compound definition!

$$\mathrm{rect}\left(\tfrac{t}{\tau}\right) = \begin{cases} H(t+\tfrac{\tau}{2}) - H(t-\tfrac{\tau}{2}) & \text{[one definition]} \\ \left.\begin{array}{ll} 1, & |t| < \tau/2 \\ 0, & |t| > \tau/2 \end{array}\right\} & \text{[another definition]} \end{cases}$$

3. *Use the properties on p. 11 of the formula book*

Do this if the function you want to transform is suitably related to another whose transform you already know.

Don't forget about the trivial-looking linearity property, which is not in the formula book.

3. Special Functions

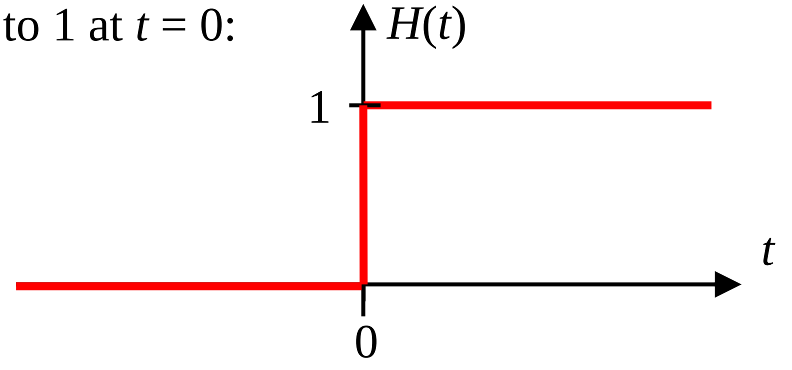
It is useful to define special functions to represent various kinds of discontinuity that can be encoutered in physical problems.

3.1. The Heaviside (Unit Step) Function $H(t)$

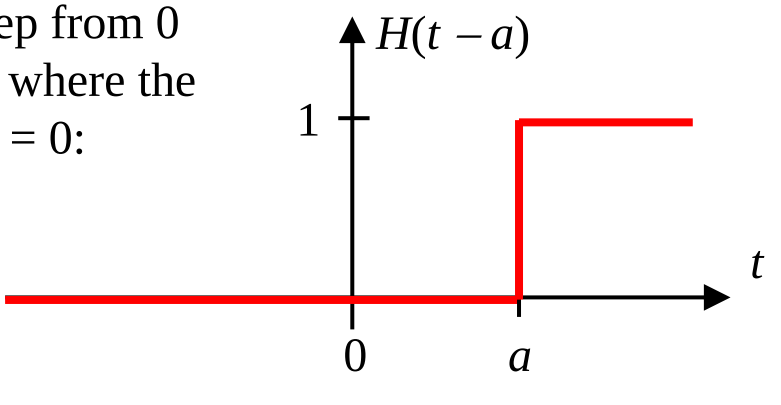
Definition:

$$\boxed{H(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}}$$

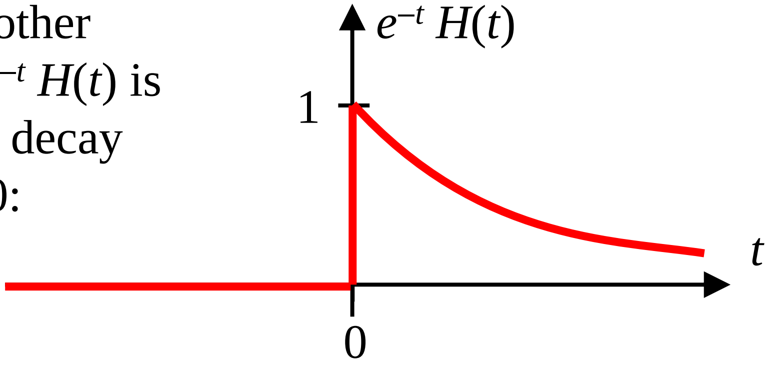
- A step from 0 to 1 at $t = 0$:



- It represents "something switching on or off"
- $H(t - a)$ is a step from 0 to 1 at $t = a$, ie where the argument $t - a = 0$:



- combine with other functions, eg $e^{-t} H(t)$ is an exponential decay starting at $t = 0$:



(The first definition of *rect* in the formula book is another example.)

3.2. ***The Dirac Delta (Impulse) Function*** $\boldsymbol{\delta(t)}$

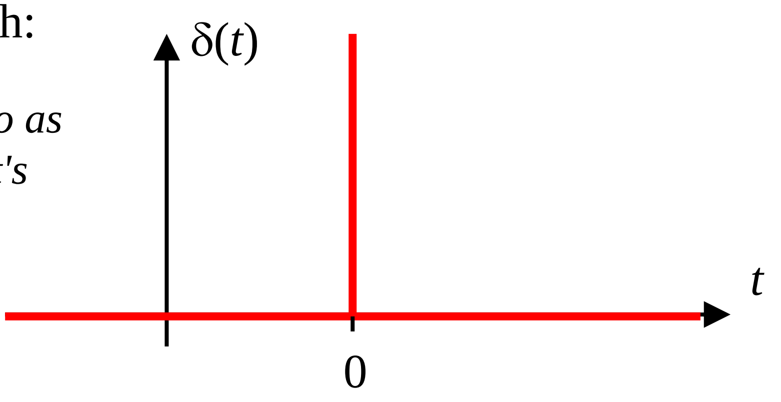
Definition (2 parts):

$$\boxed{\begin{aligned} &\delta(t) = 0 \qquad t \neq 0 \\ &\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \end{aligned}}$$

[NB the value of δ(0) is not defined!]

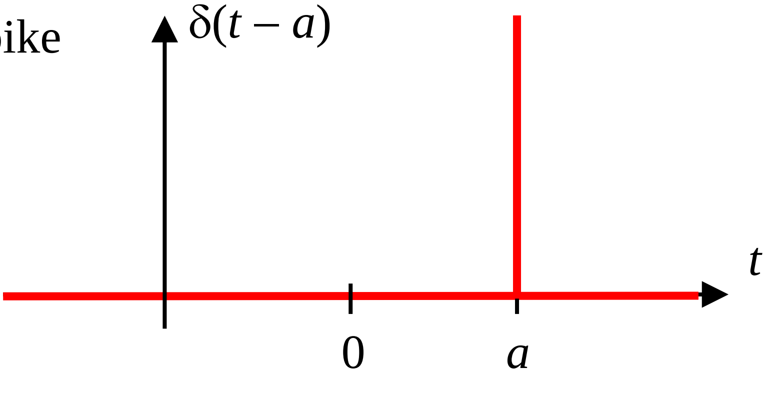
- An infinitely-thin infinitely-high "spike" at t = 0, with unit area underneath:

y axis shifted left, so as not to obscure what's happening at t = 0



- It represents "something that happens all at once" (eg an impulsive force **F** = **Δp** δ(t) causes change of momentum **Δp** at t = 0)

- δ(t – a) is a spike at t = a:



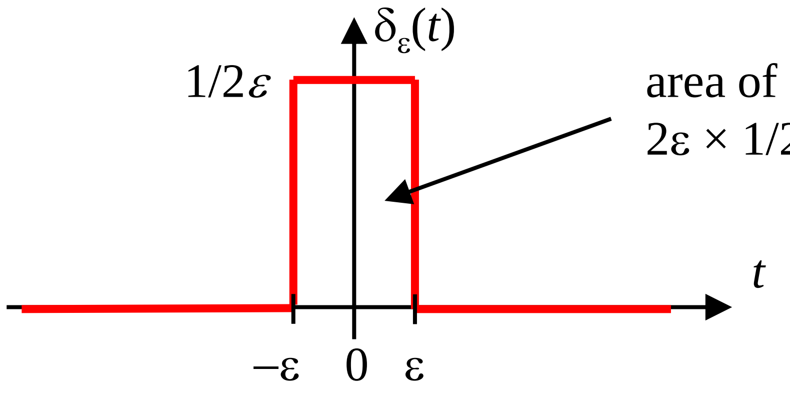
- Mathematicians are not comfortable about calling δ(t) a function - but physicists (like Dirac) can be less scrupulous!

- δ(t) can be thought of as the derivative of the step function:

$$\delta(t) = \frac{dH(t)}{dt}$$

(proof: integrate both sides from –∞ to t)

- There are numerous ways to define $\delta(t)$ as the limit of a sequence of well-behaved functions. eg consider the top-hat function $\delta_\varepsilon(t)$, with width 2ε and height $1/2\varepsilon$.



As ε gets smaller the function becomes taller and narrower while its area stays equal to 1. Hence $\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$

Sift property of $\delta(t)$

Consider the integral of a delta function multiplied by an arbitrary other function $f(t)$:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = \int_{-\infty}^{\infty} f(a)\delta(t-a)dt$$

[Since $\delta(t-a)$ is zero except at $t = a$, $f(t)$ in the integral can be replaced by $f(a)$]

$$= f(a)\int_{-\infty}^{\infty} \delta(t-a)dt \qquad [f(a) \text{ is a constant}]$$

$$= f(a) \qquad [\text{integral of } \delta(t) \text{ is } 1]$$

$\Rightarrow$ Sift property: $\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)}$

Integration with δ picks out the value of f at the spike of the δ.

The sift property is not in the formula book, but you are expected to know it.

Fourier Transform of δ(*t*)

$$F(\omega) = \mathrm{F}\,[\delta(t-a)]$$
$$= \int_{-\infty}^{\infty} \delta(t-a) e^{-i\omega t} dt$$
$$= e^{-i\omega a} \qquad \text{[sift property of } \delta\text{]}$$

$$\Rightarrow \qquad \boxed{\mathrm{F}\,[\delta(t-a)] = e^{-i\omega a}}$$

NOT in the formula book: use time-shift on the next result, or do the integral

In particular: $\mathrm{F}\,[\delta(t)] = 1$

find it in the formula book

The spectrum of a delta function is independent of ω: an impulse δ has equal amounts of every frequency.

Inverse Fourier Transform of δ(ω)

Repeating the exercise with the IFT integral gives

$$\boxed{\mathrm{F}\,[e^{i\omega_0 t}] = 2\pi\delta(\omega-\omega_0)}$$

find it in the formula book

[Intuitively this is hardly surprising - it says that the spectrum of a single-frequency wave contains only one frequency ...]

In particular: $\mathrm{F}\,[1] = 2\pi\delta(\omega)$

Now we can Fourier-transform constants, something you may not have guessed from looking at the integral definition of FT.

The FT and IFT of δ are given in the formula book, so no need to memorise them. Nevertheless they are very easy to re-derive, given the sift property of δ.

Example

Q. Find a solution of the differential equation

$$-\frac{d^2 f}{dt^2} + f(t) = \delta(t)$$

[This could be the equation of motion of a pendulum struck by a hammer (an impulsive force) at $t = 0$]

A. FT both sides, writing $\mathcal{F}[f(t)] = F(\omega)$:

$$-(i\omega)^2 F(\omega) + F(\omega) = 1$$ [derivative property on LHS, FT of δ on RHS]

$$F(\omega) = \frac{1}{1+\omega^2}$$ [rearrange for $F(\omega)$]

Now IFT both sides: $f(t) = \frac{1}{2} e^{-|t|}$ [using tables]

Transform techniques are a powerful way of solving differential equations, even with "strange" inhomogeneous parts like $\delta(t)$.

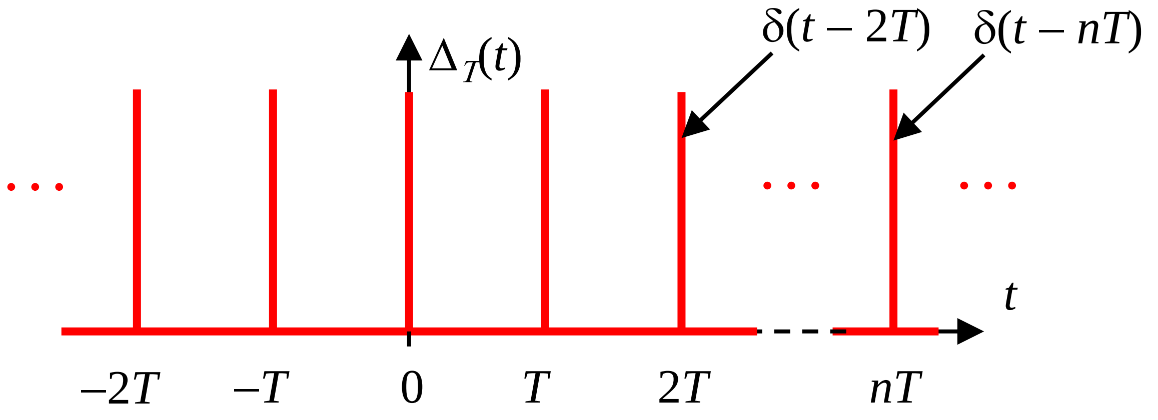
Using the FT here, we have only found a particular integral. To obtain the general solution we need to add the complementary function found by solving LHS = 0 by other methods (see YR1 differential equations).

3.3. ***The Dirac Comb (Shah) Function*** $\Delta_T(t)$

Definition:

$$\boxed{\Delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)}$$

- An infinite train of δ functions separated by T:



- It represents "something impulsive that happens repeatedly"

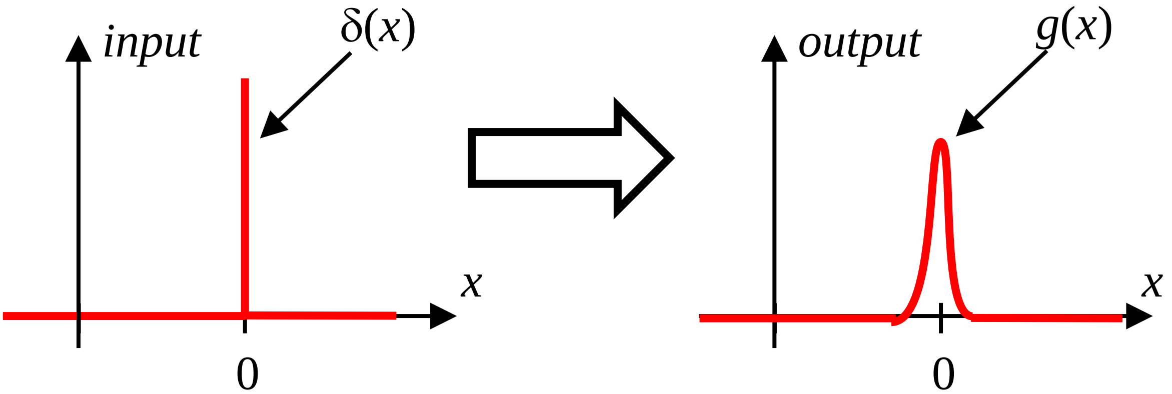
- $\Delta_T(t)$ is a periodic function. You can show (problem sheet) that its FT is proportional to another Dirac comb, with separation $\omega_0 = 2\pi/T$:

$$\begin{aligned} F\,[\Delta_T(t)] &= \omega_0 \Delta_{\omega_0}(\omega) \\ &= \omega_0 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_0) \end{aligned}$$

4. Convolution and Signal Processing

4.1. Convolution

The impulse response or "point spread function" ($g(x)$, say) of an imperfect instrument is what the instrument records when the input is an ideal point source, represented by $\delta(x)$. For example, diffraction in a telescope may make it image a point-like star $\delta(x)$ as a finite spot $g(x)$.



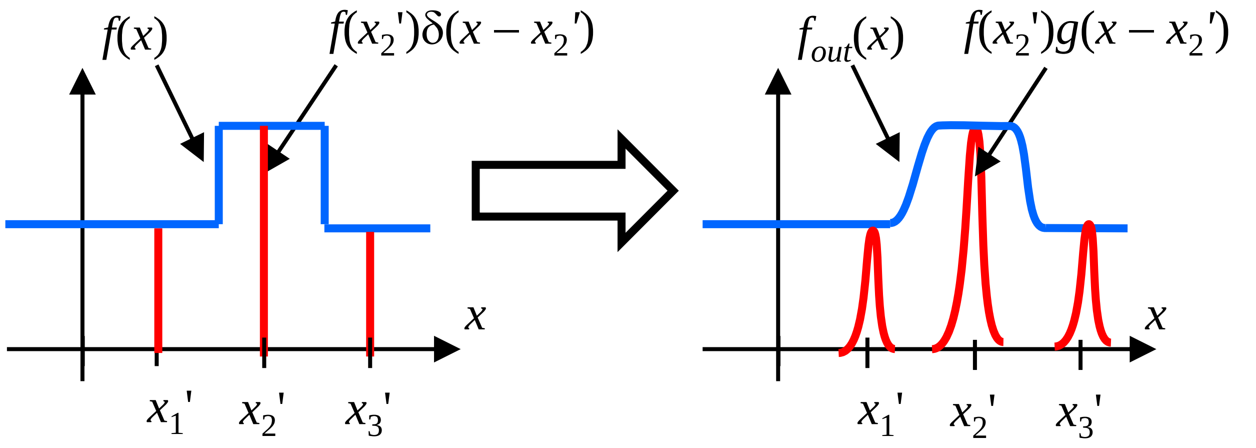
What if the input signal $f(x)$ is spread out instead of point-like? (For example, point the telescope at a planet or nebula.)

Express $f(x)$ as a continuous distribution of point sources $\delta(x - x')$, each centred at $x = x'$ with "weight" $f(x')$:

$$f(x) \equiv \int_{-\infty}^{\infty} f(x')\delta(x-x')dx' \qquad \text{[sift property of } \delta]$$

The instrument then distorts each of the distribution's point sources $\delta(x - x')$ into the point spread function $g(x - x')$, so the recorded output is

$$f_{out}(x) = \int_{-\infty}^{\infty} f(x')g(x-x')dx'$$



[schematic diagrams / poetic licence: eg all the δ functions go to ∞, but with different weights f(x') ⇒ different "areas under the curve"]

This operation on functions *f* and *g* is called their *convolution* and is written:

find it in the formula book

$$\boxed{(f*g)(x)=\int_{-\infty}^{\infty} f(x')g(x-x')dx'}$$

dummy variable *x*'

- The result of convolution is a *function* whose *name* is *f* **g*, which here takes *x* as its argument. When performing the integral, *x* is treated as a constant.
- ie, in this context, * does not represent multiplication or complex conjugate
- It doesn't matter which way round *f* and *g* go:

$$(f*g)(x)\equiv(g*f)(x)$$

- The verb for convolution is "convolve"; we say *f* and *g* are convolved together
- It can be awkward working out the convolution integral, but it's very simple when one of the functions is a delta function:

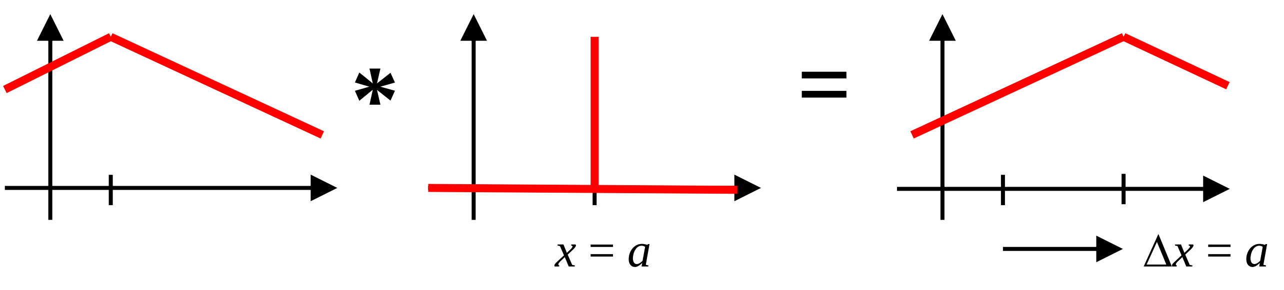
Convolution with δ function

If $$g(x) = \delta(x-a)$$

then
$$(f*g)(x) = \int_{-\infty}^{\infty} f(x')g(x-x')dx' \quad \text{[defn]}$$
$$= \int_{-\infty}^{\infty} f(x')\delta(x-a-x')dx' \quad \text{[spike where } x' = x-a\text{]}$$
$$= f(x-a) \quad \text{[sift property]}$$

That is, $$\boxed{f(x)*\delta(x-a) = f(x-a)}$$

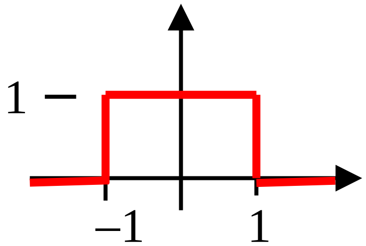
- convolution with $\delta(x-a)$ reproduces the original function but with the argument x replaced by $x-a$;
- equivalently, it replaces the δ symbol with f;
- equivalently, it shifts the function to the right by $\Delta x = a$.



Example

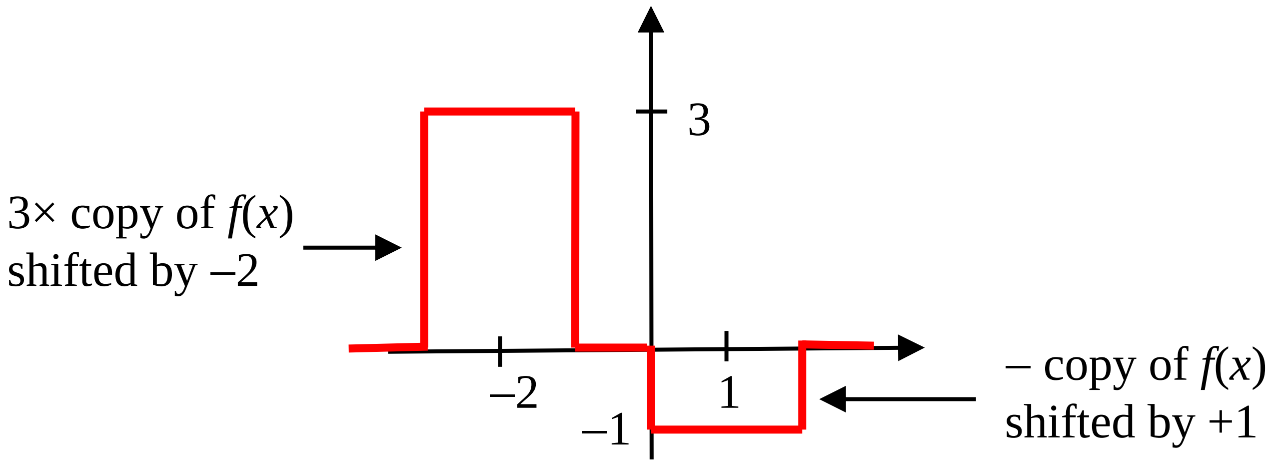
Q. Find $(f*g)(x)$, where

$f(x) =$



$g(x) = 3\delta(x+2) - \delta(x-1)$

A. $(f * g)(x) = 3f(x + 2) - f(x - 1)$ [convolution with δ]



4.2. Convolution Theorem

If
$$F(\omega) = \mathrm{F}\,[f(t)]$$
$$G(\omega) = \mathrm{F}\,[g(t)]$$

it can be shown that

find it in the formula book

$$\boxed{\mathrm{F}\,[(f * g)(t)] = F(\omega) \times G(\omega)}$$

convolution theorem

The transform of the convolution = the product of the transforms

Similarly (in the other direction) it can be shown that

find it in the formula book

$$\boxed{\mathrm{F}\,[f(t) \times g(t)] = \frac{1}{2\pi}(F * G)(\omega)}$$

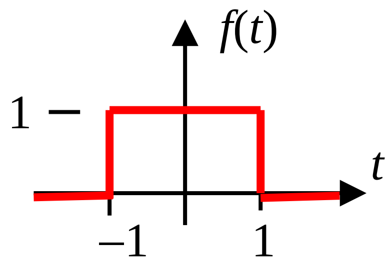
frequency convolution theorem

The transform of the product = 1/(2π) × the convolution of the transforms

The frequency convolution theorem is a kind of "product rule" for FTs, while the convolution theorem does the same for IFTs.

<u>Example</u>

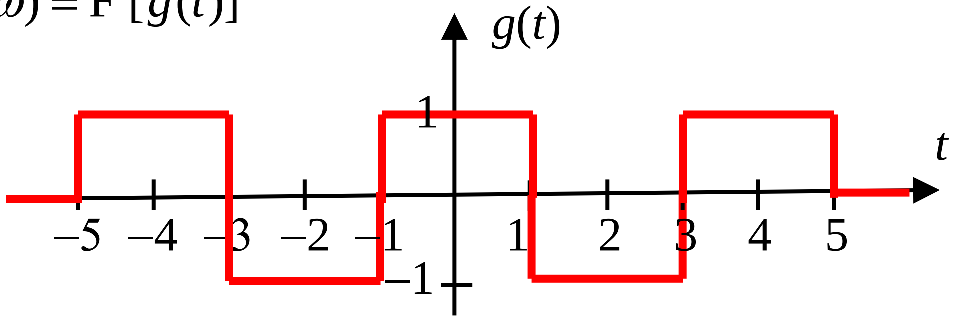
Q. $f(t)$ =



Given that $F(\omega) = \mathrm{F}\,[f(t)] = 2\ \mathrm{sinc}(\omega)$

what is $G(\omega) = \mathrm{F}\,[g(t)]$

where $g(t)$ =



A. Write $g(t)$ as the convolution of $f(t)$ with some δ functions:

$$
\begin{aligned}
g(t) &= f(t+4) - f(t+2) + f(t) - f(t-2) + f(t-4) \\
&= f(t) * \left[\delta(t+4) - \delta(t+2) + \delta(t) - \delta(t-2) + \delta(t-4)\right]
\end{aligned}
$$

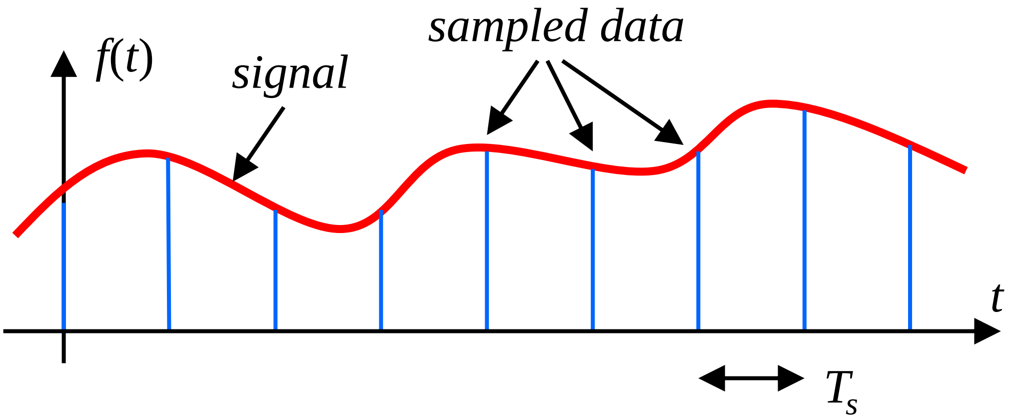
Apply the convolution theorem:

$$
\begin{aligned}
G(\omega) &= F(\omega) \times \mathrm{F}\,[\delta(t+4) - \delta(t+2) + \delta(t) - \delta(t-2) + \delta(t-4)] \\
&= F(\omega) \times \left[e^{i4\omega} - e^{i2\omega} + 1 - e^{-i2\omega} + e^{-i4\omega}\right] \qquad \text{[FT of } \delta\text{]} \\
&= 2\ \mathrm{sinc}(\omega) \times \left[1 - 2\cos(2\omega) + 2\cos(4\omega)\right]
\end{aligned}
$$

Of course this question could also be answered using the time shift property...

4.3. Sampling Theorem

A digital instrument samples a continuous signal $f(t)$ with a time interval T_s, ie at (angular) frequency $\omega_s = 2\pi / T_s$:



The *recorded* data is proportional to a function $f_s(t)$:

$$f_s(t) = f(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

which represents the sampling process by multiplication with a Dirac comb of spacing T_s.

Although the signal between the samples is ignored, under certain circumstances the entire original signal can be reconstructed from the sampled data!

Let $F(\omega) = \mathrm{F}\,[\,f(t)]$ be the FT of the original signal.

The FT of the recorded data is

$$F_s(\omega) = \mathrm{F}\,[\,f_s(t)]$$

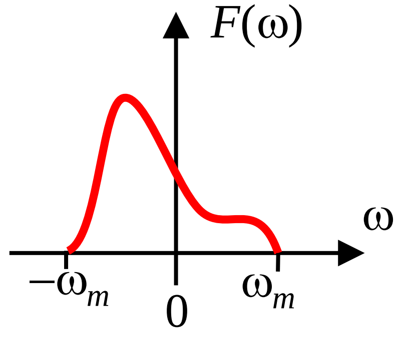
$$= \mathrm{F}\left[\,f(t)] \times \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] \qquad \text{[subst } f_s(t)\text{]}$$

$$= \frac{1}{2\pi} F(\omega) * \mathrm{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] \qquad \text{[frequency convolution theorem]}$$

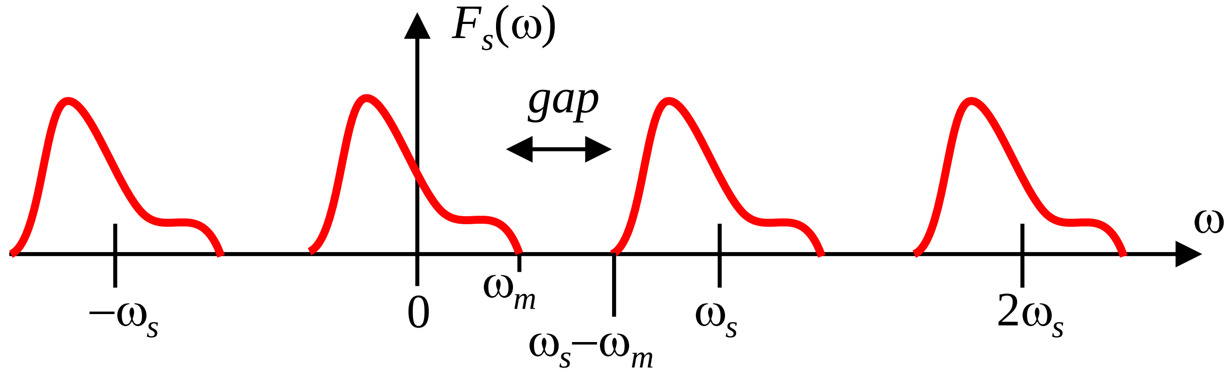
$$= \frac{\omega_s}{2\pi} F(\omega) * \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_s) \qquad \text{[FT of Dirac comb]}$$

$$= \frac{1}{T_s} \sum_{m=-\infty}^{\infty} F(\omega - m\omega_s) \qquad \text{[convolution with } \delta \text{ function]}$$

ie, an infinite train of copies of $F(\omega)$ separated by ω_s.
If the transform of the original signal looks like



where ω_m is the highest significant frequency component in the signal, then the transform of the recorded data looks like



So $F(\omega)$ (and hence the original signal $f(t)$ by IFT) can be completely recovered from the data by filtering out frequency components $|\omega| > \omega_m$ in the data, <u>provided the copies of $F(\omega)$ do not overlap</u>.

To avoid this overlap, the marked gap must be greater than zero:

$$\omega_s - \omega_m \geq \omega_m \Rightarrow \omega_s \geq 2\omega_m$$

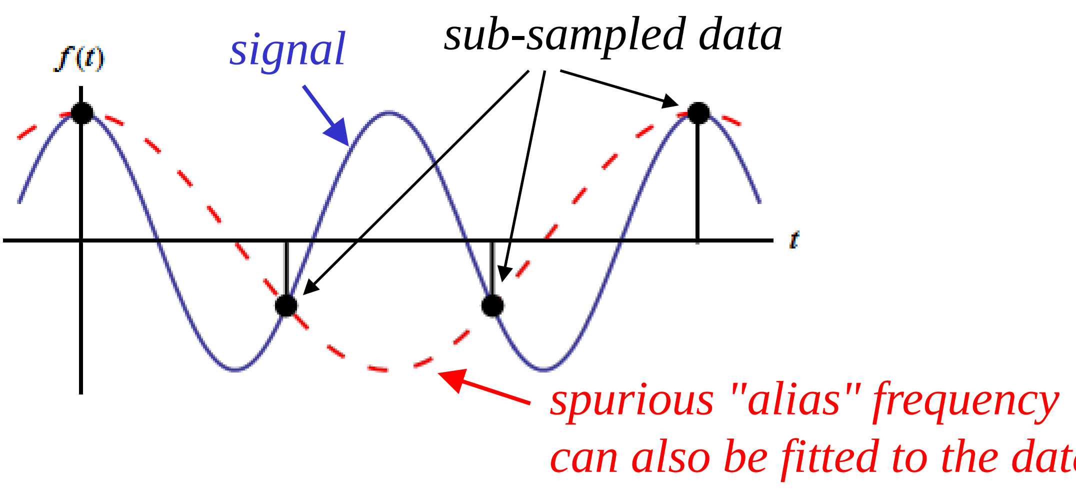
In terms of "hertz" frequency, $f = \omega/2\pi$:

The sampling rate (f_s) must be at least twice the highest frequency in the signal (f_m): $$f_s \geq 2 f_m$$

Nyquist sampling theorem / criterion

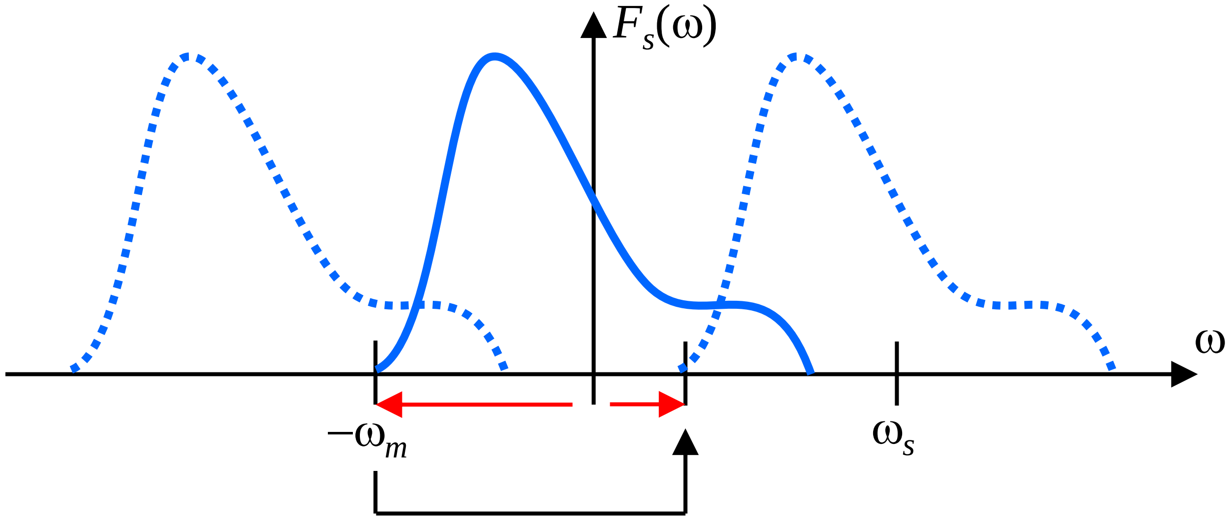
Aliasing

If the Nyquist criterion is not satisfied, adjacent copies of $F(\omega)$ in $F_s(\omega)$ overlap and the original $F(\omega)$ cannot be recovered. Spurious low-frequency components are introduced within $-\omega_m > \omega > \omega_m$, a degradation called *aliasing*:



In the above diagram: the Nyquist criterion indicates that the high-frequency signal should be sampled at least twice per period. Because it is not sampled that often, it is possible to fit a lower-frequency cos function to the same data points. This is the alias, and represents a low frequency that is completely absent from the original signal.

In the frequency domain:



high-f component shows up as a low-f alias

Examples of aliases:

- wagon wheels appearing to rotate backwards in old westerns
- water appearing to slow down or drip upwards in strobe light
- spurious squeals in over-compressed audio files
- artifacts in over-compressed image or movie files

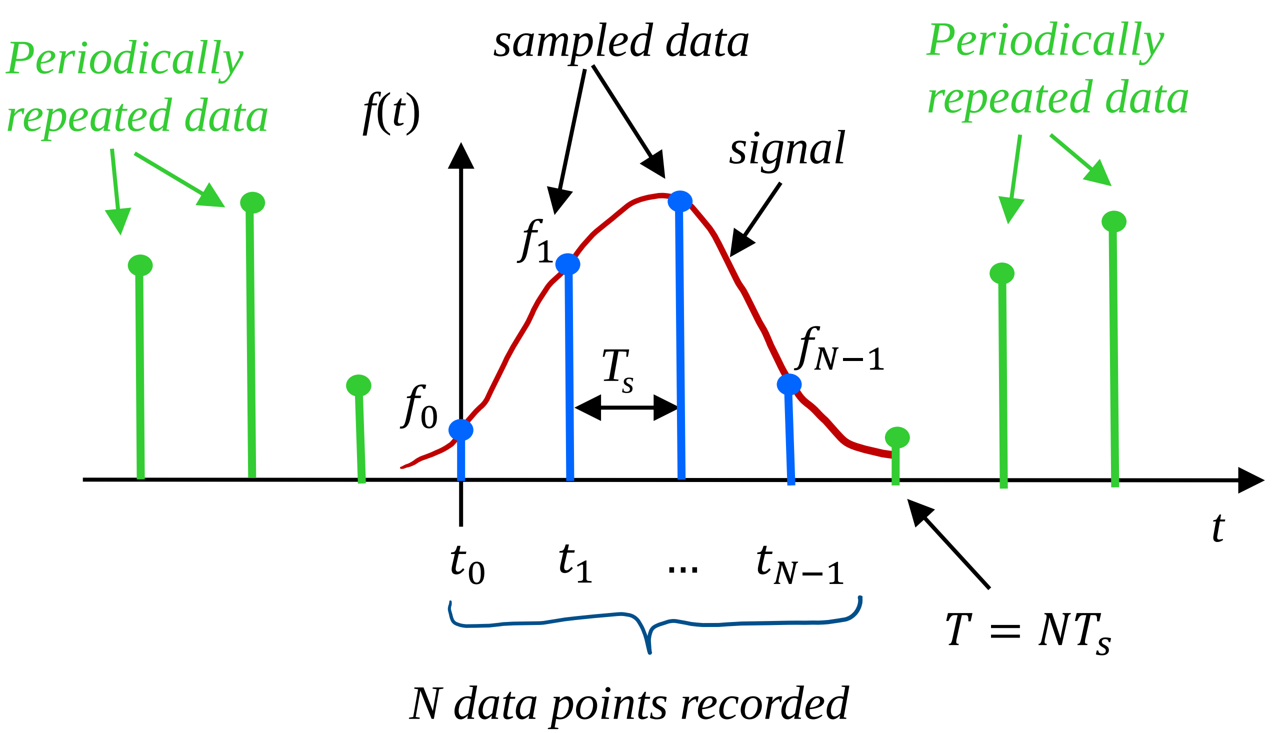
Two ways to prevent aliasing:

1. (obviously) increase the sampling rate until $f_s > 2\ f_m$; or

2. filter the signal *before sampling*, to remove high-frequency components until $2\ f_m < f_s$. This blurs or *smooths* the data, but at least it doesn't introduce non-existent components.

4.4. Discrete and Fast Fourier Transforms

In any practical application, a sampled continuous signal will be a **finite set** of data points. E.g. consider a signal which has been recorded using a fixed sampling interval T_s with N data points in total. We will label the corresponding time instants as t_j with $j = 0,1,\dots,N-1$, and the corresponding values of the signal amplitude as $f_j = f\left(t_j\right)$. It is convenient to set the first data point at $t = 0$, such that $t_j = jT_s$.



To obtain the spectrum, we will use a technique similar to what was used for finite range functions (see Appendix C of part 1). We introduce a periodic version of the continuous signal function $f_T(t)$ by periodically repeating the pattern of recorded data points. The period of this function will be $T = NT_s$, i.e. $f_T(t + NT_s) = f_T(t)\ \forall\ t$.

We can then calculate Complex Fourier Series coefficients of this newly introduced periodic function $f_T(t)$:

$$c_n = \frac{1}{NT_s}\int_0^{T_s} f_T(t)\exp\left[-in\left(\frac{2\pi}{NT_s}\right)t\right]dt$$

We only know values of the function at the discrete set of data points $t_j = jT_s$ within the one period window $0 < t < T = NT_s$. We can use the trapezoid rule to approximate the above integral:

$$c_n \approx \frac{1}{NT_s}\sum_{j=0}^{N-1} f_j\ \exp\left[-in\left(\frac{2\pi}{NT_s}\right)jT_s\right]\cdot T_s$$

Recorded values of the function

$$f_T(t_j) = f(t_j) = f_j$$

Time points

$$t_j = jT_s$$

Width of each trapezoid $\Delta t = T_s$

Observe that the sampling interval T_s cancels out everywhere, and the result depends only on the number of points N:

n is the index of the complex Fourier series coefficient (amplitude of the n-th harmonic $\omega_n = 2\pi n/(NT_s)$)

$$c_n \approx \frac{1}{N}\sum_{j=0}^{N-1} f_j\ \exp\left[-i\left(\frac{2\pi}{N}\right)jn\right]$$

j is the summation index

The obtained coefficients c_n appear to be periodic: $c_{n+N} = c_n$.
This is easy to see:

$$c_{n+N} = \frac{1}{N}\sum_{j=0}^{N-1} f_j \, \exp\left[-i\left(\frac{2\pi}{N}\right)j(n+N)\right] =$$

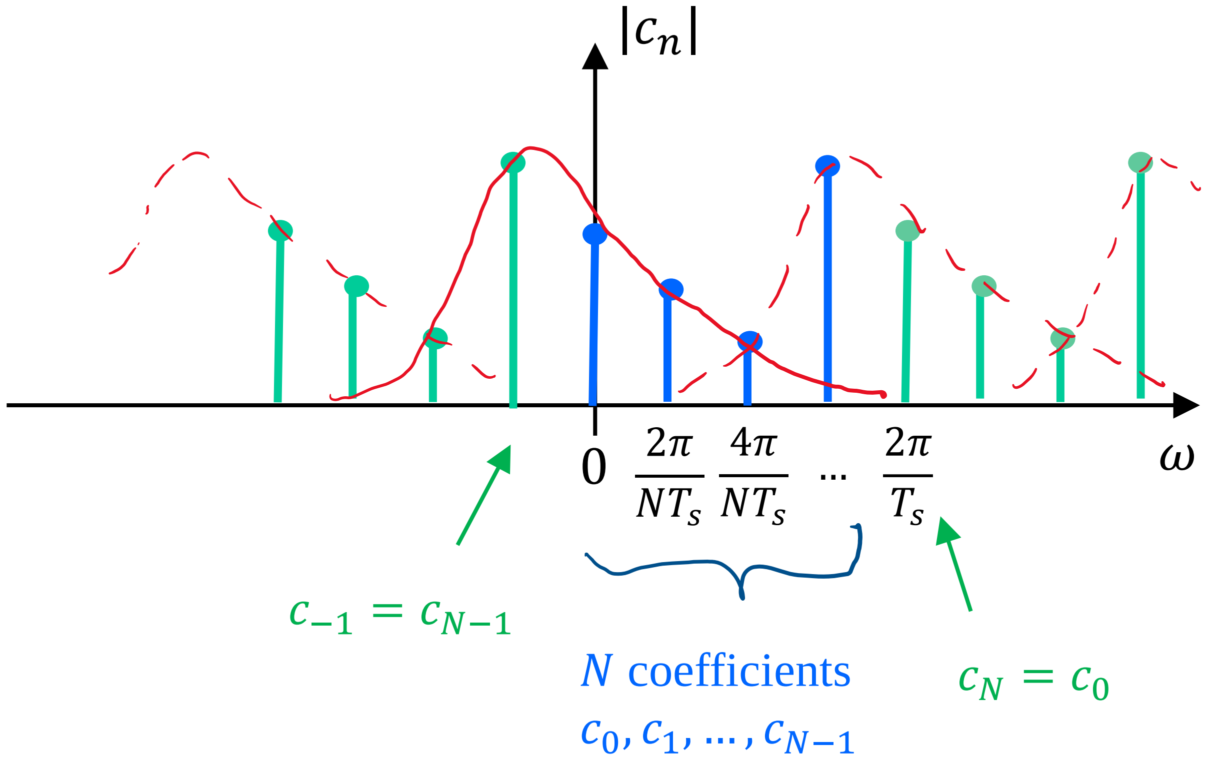
$$= \frac{1}{N}\sum_{j=0}^{N-1} f_j \, \exp\left[-i\left(\frac{2\pi}{N}\right)jn - i\left(\frac{2\pi}{N}\right)jN\right] =$$

$$= \frac{1}{N}\sum_{j=0}^{N-1} f_j \, \exp\left[-i\left(\frac{2\pi}{N}\right)jn\right]\underbrace{\exp[-i2\pi j]}_{=\,1,\ \textit{since } j \textit{ is an integer!}} =$$

$$= \frac{1}{N}\sum_{j=0}^{N-1} f_j \, \exp\left[-i\left(\frac{2\pi}{N}\right)jn\right] = c_n$$

Hence there are only N independent c_n coefficients!

The periodicity of c_n coefficients is the **direct result of aliasing**. The initial data was sampled with the sampling period T_s. Hence the spectrum is periodic with the period $\omega_s = 2\pi/T_s$



Discrete input	$\Longrightarrow$	Periodic spectrum
Periodic input	$\Longrightarrow$	Discrete spectrum
Discrete + periodic input	$\Longrightarrow$	Discrete + periodic spectrum

Sampling period T_s determines the maximal frequency $\omega_m = \frac{\pi}{T_s}$

Number of points N determine the frequency resolution $\Delta\omega = \frac{2\pi}{NT_s}$

Inverse transform (complex Fourier series):

$$f_T(t) = \sum_{j=-\infty}^{+\infty} c_j \exp\left[i\left(\frac{2\pi n}{NT_s}\right)t\right]$$

Formally require an infinite series. However, it is possible to prove rigorously, that to obtain the original discrete set of data points $f_j = f(jT_s) = f_T(jT_s)$ you only need N coefficients:

$$f_j = \sum_{n=0}^{N-1} c_n \exp\left[i\left(\frac{2\pi}{N}\right)jn\right]$$

Discrete Fourier Transform:

Transforms a discrete set of data points (not a function!) $\{x_j\}, j = 0,1, \ldots N-1$ according to:

$$X_n = DFT[\{x_j\}] = \sum_{j=0}^{N-1} x_j \exp\left[-i\left(\frac{2\pi}{N}\right)jn\right], n = 0,1, \ldots, N-1$$

and the inverse operation:

$$x_j = IDFT[\{X_n\}] = \frac{1}{N}\sum_{n=0}^{N-1} X_n \exp\left[i\left(\frac{2\pi}{N}\right)jn\right], j = 0,1, \ldots, N-1$$

When implemented on a computer, the above summations require $\sim N^2$ operations per transform – this is slow.

However, the algorithm can run much faster if N is a power of two (e.g. N=1024). This is known as **Fast Fourier Transform (FFT)**.

Calculating Fourier Transforms on a computer

If you need to compute

A Fourier Series $c_n = \frac{1}{T}\int_0^T f(t)\exp\left[in\left(\frac{2\pi}{T}\right)t\right]dt$,

or a Fourier Transform $F(\omega) = \int_{-\infty}^{\infty} f(t)\exp[i\omega t]\,dt$

with N data points $f(t_j) = f_j$ sampled at regular time steps with the interval T_s (such that $T = N\mathrm{T}_s$), you should:

1) Calculate Discrete Fourier Transform $F_n = DFT[\{f_j\}]$ using an FFT routine *(for fastest calculation use N as a power of 2, e.g. $N = 256$ will work much faster than $N = 250$)*;

2) The Fourier Series coefficients can be obtained from the Discrete Fourier Transform as:

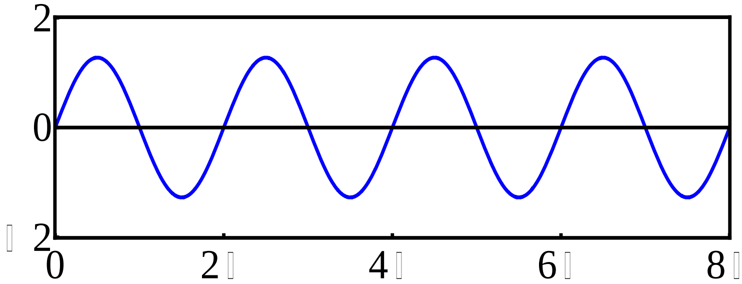
$$\mathrm{c_n} \approx \begin{cases} (1/N)F_n, & n = 0,1,2, \dots \frac{N}{2} \\ (1/N)F_{n+N}, & n = -1,-2, \dots -\frac{N}{2}+1 \end{cases}$$

3) The Fourier Transform function can be obtained from the Discrete Fourier Transform as:

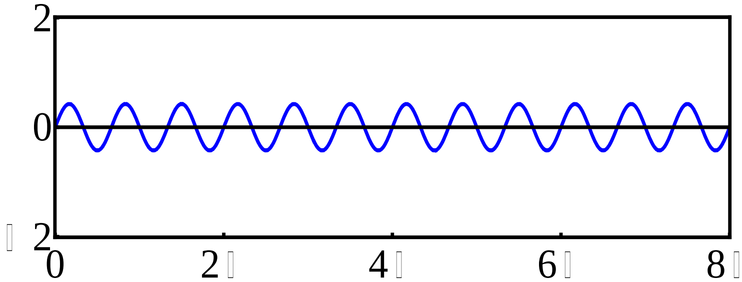
$$F(\omega = 2\pi n/(NT_s)) \approx \begin{cases} T_s \cdot F_n, & n = 0,1,2, \dots \frac{N}{2} \\ T_s \cdot F_{n+N}, & n = -1,-2, \dots -\frac{N}{2}+1 \end{cases}$$

Appendix A: Fourier series of a square wave

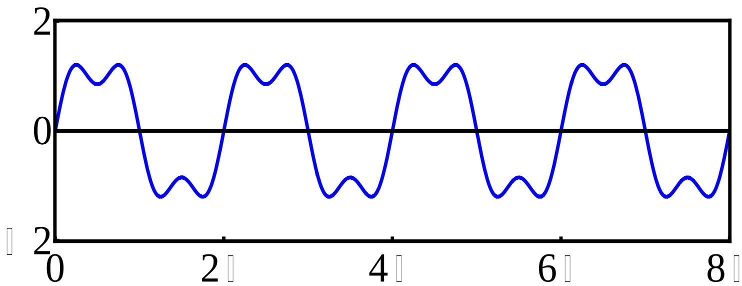
$$\frac{4}{\pi}\sin(t)$$



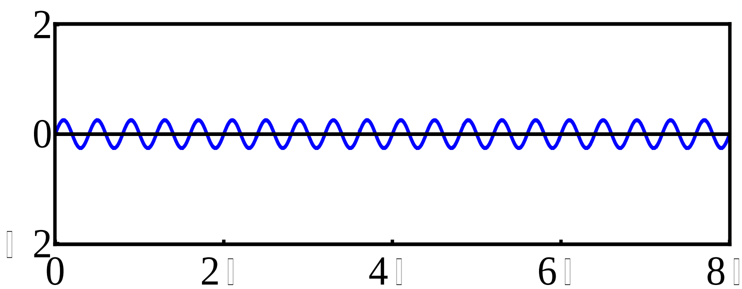
$$+\frac{4}{\pi}\sin(3t)/3$$



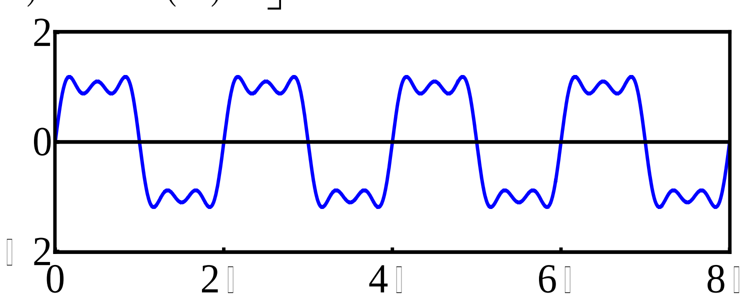
$$=\frac{4}{\pi}\left[\sin(t)+\sin(3t)/3\right]$$



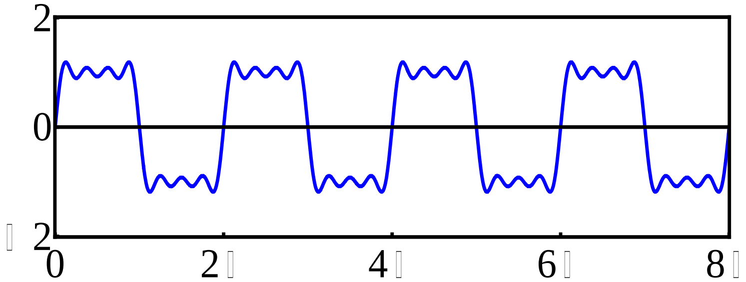
$$+\frac{4}{\pi}\sin(5t)/5$$



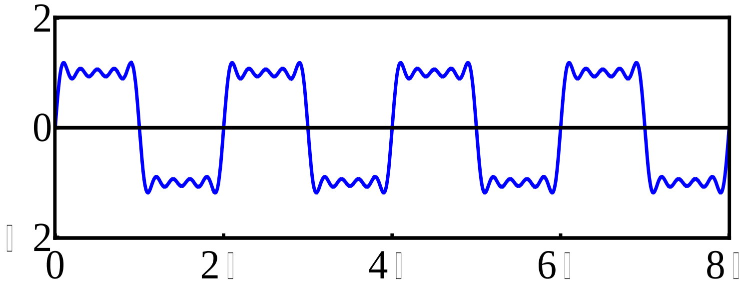
$$=\frac{4}{\pi}\left[\sin(t)+\sin(3t)/3+\sin(5t)/5\right]$$



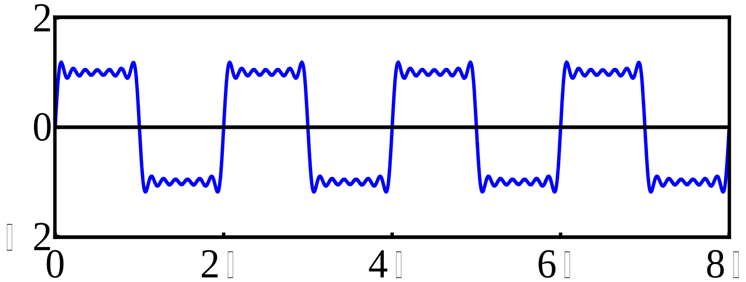
$$\frac{4}{\pi}\left[\sin(t)+\sin(3t)/3+\sin(5t)/5+\sin(7t)/7\right]$$



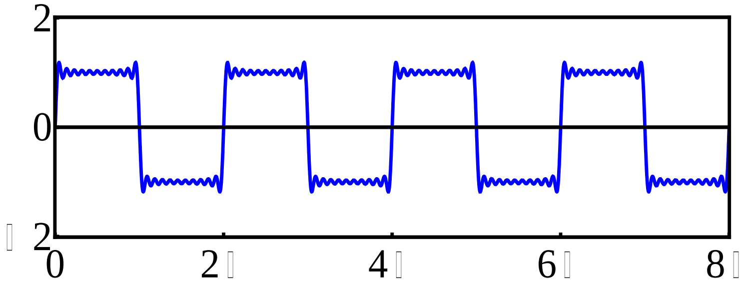
$$\frac{4}{\pi}\left[\sin(t)+\ldots+\sin(9t)/9\right]$$



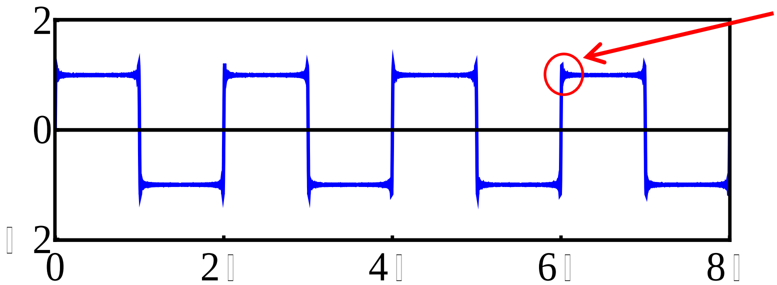
$$\frac{4}{\pi}\left[\sin(t)+\ldots+\sin(13t)/13\right]$$

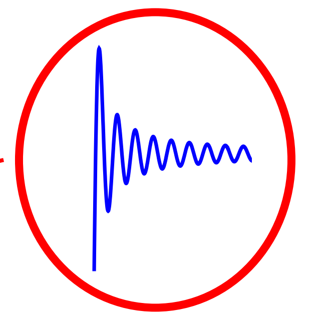


$$\frac{4}{\pi}\left[\sin(t)+\ldots+\sin(21t)/21\right]$$



$$\frac{4}{\pi}\left[\sin(t)+\ldots+\sin(99t)/99\right]$$

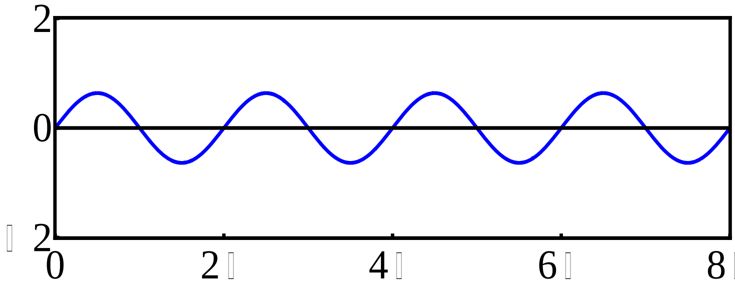




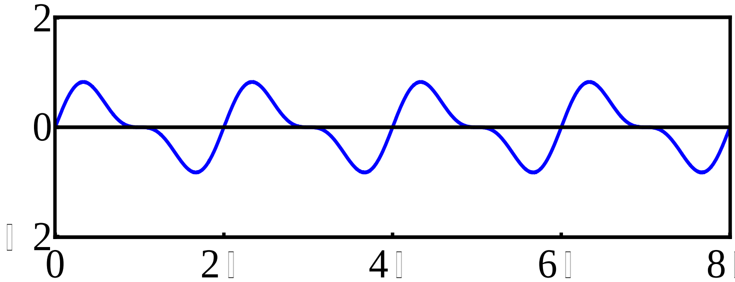
Gibbs phenomenon: ~9% overshoot at step discontinuites. Adding more terms to the series reduces the width of the oscillations but their height remains ~9% of the step.

Appendix B: <u>Fourier series of a sawtooth wave</u>

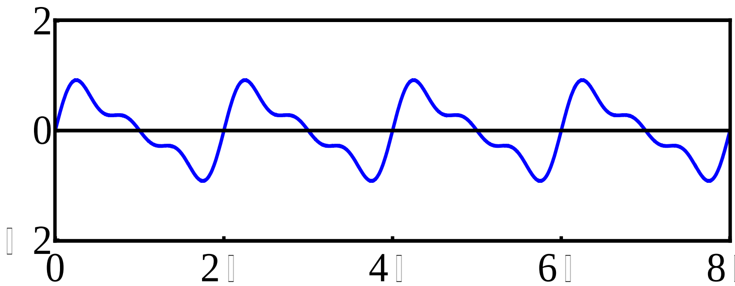
$$\frac{2}{\pi}\sin(t)$$



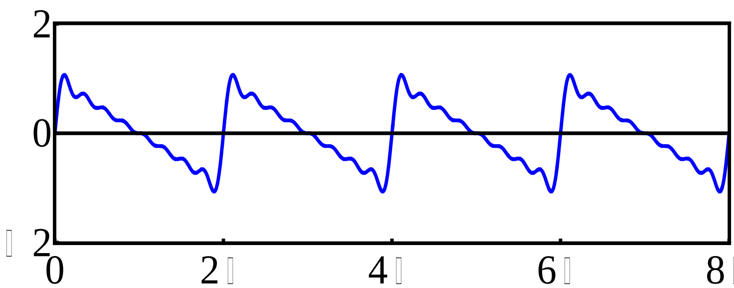
$$\frac{2}{\pi}\left[\sin(t)+\sin(2t)/2\right]$$



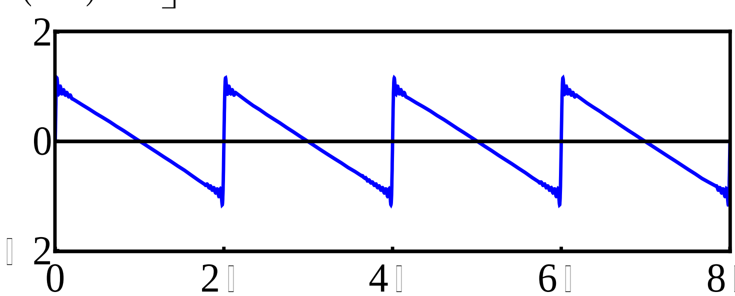
$$\frac{2}{\pi}\left[\sin(t)+\sin(2t)/2+\sin(3t)/3\right]$$



$$\frac{2}{\pi}\left[\sin(t)+...+\sin(8t)/8\right]$$



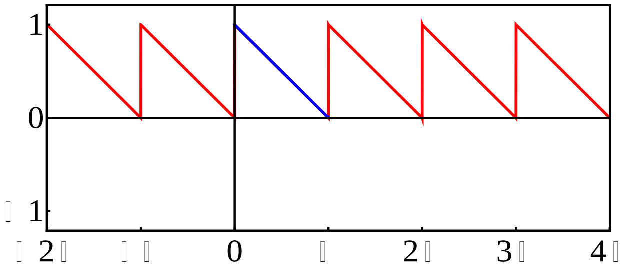
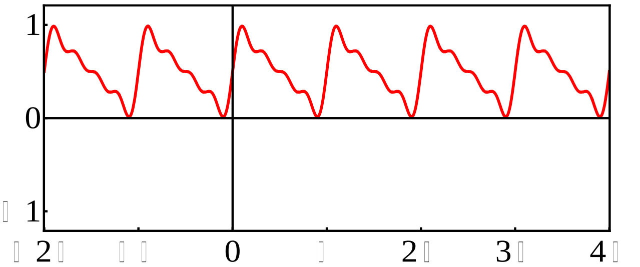
$$\frac{2}{\pi}\left[\sin(t)+...+\sin(50t)/50\right]$$



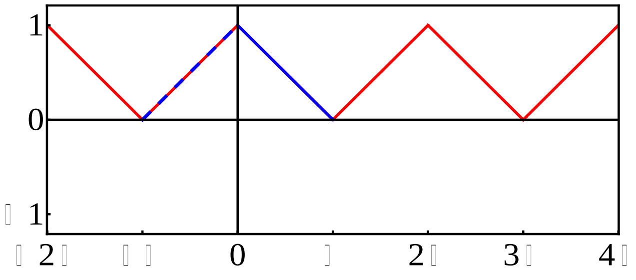
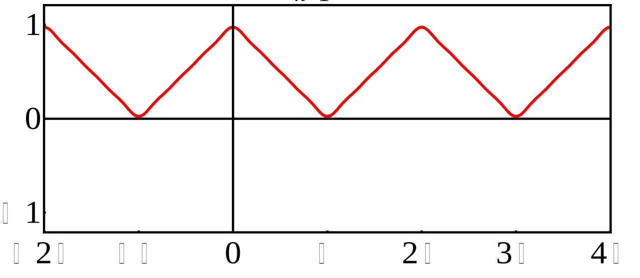
Appendix C: Finite-range functions

Fourier series represent infinite periodic functions as sums of sinusoidal components. Consider these three functions, each shown with its FS summed over the first 5 terms:

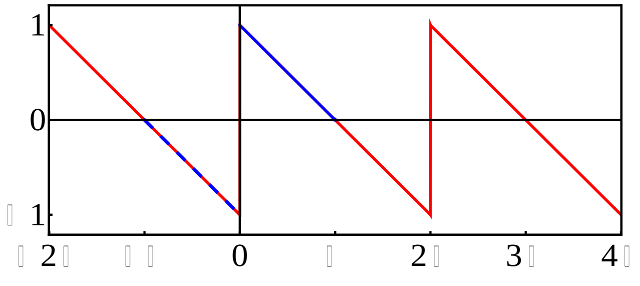
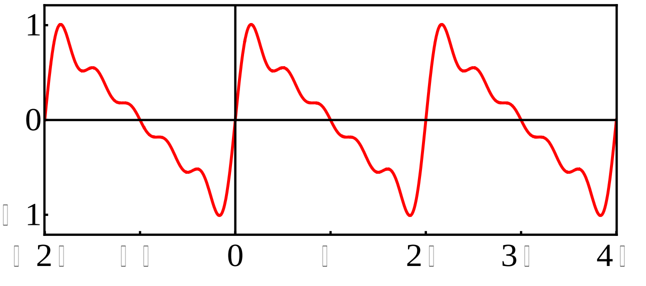
$$g_1(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(2nt)$$

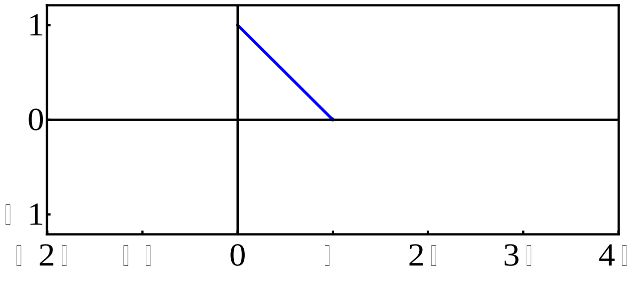
$$g_2(t) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos(nt)$$

$$g_3(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(nt)$$

Although the three functions (and their FS) are quite different, they do coincide when $0 < t < \pi$. Each FS can therefore represent the finite range function $g(t)$, which is only defined for $0 < t < \pi$:



$$g(t) = 1 - t/\pi \quad 0 < t < \pi$$

Such finite-range functions can arise as solutions of differential equations with boundary conditions, where values outside the range are meaningless or irrelevant. (One example might be a solution of the heat conduction equation applied to an insulated metal bar whose ends are held at fixed temperatures.)

There's an infinite number of ways to represent such a finite-range function as a FS, depending on what values are taken outside the range of interest. The functions on the previous page represent the three simplest ways:

$g_1(t)$: Just repeat the function $g(t)$ with period equal to the range of interest (ie, $T = \pi$). This yields the infinite periodic function $g_1(t)$. Because $g_1(t)$ is neither even nor odd it is necessary to evaluate both a_n and b_n integrals to find the FS (although, as you can see, in this case a_0 is the only non-zero a_n).

$g_2(t)$: Artificially extend $g(t)$ to negative t as an even function (shown as a broken line in the $g_2(t)$ plot) then repeat this unit with period equal to twice the range of interest (ie, $T = 2\pi$). Because the resulting infinite periodic function $g_2(t)$ is now even, it is enough to evaluate only the a_n integral. This gives what is called a *Fourier cosine series*.

$g_3(t)$: Artificially extend $g(t)$ to negative t as an odd function (shown as a broken line in the $g_3(t)$ plot) then repeat this unit with period equal to twice the range of interest (ie, $T = 2\pi$). Because the resulting infinite periodic function $g_3(t)$ is now odd, it is enough to evaluate only the b_n integral. This gives a *Fourier sine series*.

Which method is best? It depends on the function ... Two things to consider:

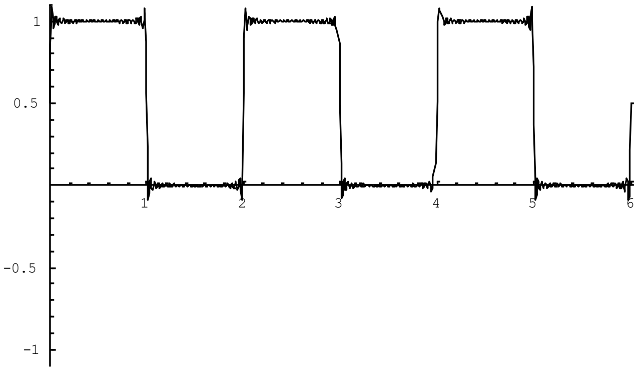
1. How much effort is involved? $g_1(t)$ requires the least thought but more effort, because we must work out both integrals a_n and b_n. However, in some cases the repeated function is even or odd anyway.

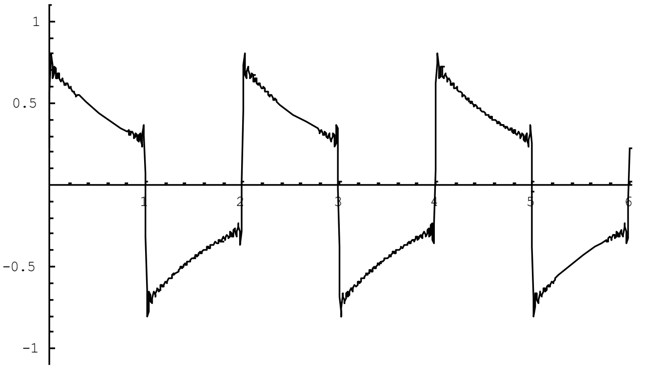
2. How good is the FS for a given number of terms? In the example above, the FS for $g_2(t)$ looks much better than the others for the same number of terms. This is because the other two functions have step discontinuities, so their FS suffer from the Gibbs phenomenon. Therefore it's usually best to choose an extended function without step discontinuities.

Appendix D: High-pass filter

A high-pass filter removes low frequencies from a signal and transmits just the high frequencies. Physical devices can filter real waves (eg an electronic high-pass filter can be built from resistors and capacitors), or computers can perform the task on digital signals by calculations based on Fourier analysis.

Here are the sums to $n = \pm 50$ of two complex Fourier series. The one on the left is is an input square wave, and the one on the right is the output after the lowest-frequency components have been removed:





The flat tops of the square wave have become slanted, like exponential decays. Notice also that the "d.c." component has gone - the output function averages zero whereas the input averages 0.5.

An oscilloscope will do this sort of thing to a square wave from a signal generator if "ac coupling" (a kind of electronic high-pass filter) is selected: