**PH20014: Electromagnetism 1**

**Problem Sheet 3**

**Question 1.**

A plane wave in air is **incident normally** on an infinite lossless dielectric material having $\varepsilon_r = 3$, $\mu_r = 1$. If the incident wave is $\vec{E}_j = 10\cos(\omega t - x)\hat{j}$ V/m, find
(a) the wavelength $\lambda_i$ and the value of $\omega$ of the incident wave; (6.28 m, $3\times10^8$ rad/s)
(b) the wavelength $\lambda_t$ of the transmitted wave; (3.63 m)
(c) the amplitude of the incident magnetic field; (0.026 $\mathrm{Am^{-1}}$)
(d) the amplitude of the transmitted electric field; (7.32 $\mathrm{Vm^{-1}}$)
(e) the time-averaged transmitted power. (0.123 $\mathrm{Wm^{-2}}$)

**Answer:** The wave is given as $\vec{E}_j = 10\cos(\omega t - x)\hat{j}$.

(a) We can compare to the standard form $E = A(\omega t - kx)$ and determine that $k = 1$.

We know that $\lambda = 2\pi/k$, so here $\lambda = 2\pi \approx 6.28$ m. Moreover, $v_p = \dfrac{\omega}{k} = \dfrac{c}{n}$ and $n_{\text{air}} = 1$, so $\omega = ck = 299792458$ rad/s.

(b) For the transmitted wave in the dielectric medium, the phase velocity is given by: $v_p = \dfrac{c}{\sqrt{\varepsilon_r \mu_r}} = \dfrac{c}{\sqrt{3}}$. We also know that $v_p = \dfrac{\omega}{k} = \dfrac{\omega}{2\pi/\lambda} = \dfrac{\omega}{\omega T/\lambda} = \dfrac{\lambda}{T} = \lambda f$ and the frequency $f = \dfrac{v_p}{\lambda}$ is the same in both materials: $f = \dfrac{c}{\lambda_{\text{air}}} = \dfrac{c}{\lambda_{\text{material}}\sqrt{3}}$. Therefore

$$\lambda_{\text{material}} = \frac{\lambda_{\text{air}}}{\sqrt{3}} \approx \frac{6.28}{\sqrt{3}} = 3.63 \text{ m.}$$

(c) From the definition of the impedance, we know that $\dfrac{|E_{\text{air}}|}{|H_{\text{air}}|} = Z = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377$ Ω, which we assume is negligibly different from vacuum. So, $|H_{\text{air}}| = \dfrac{|E_{\text{air}}|}{377} = \dfrac{10}{377} = 0.026$ A/m.

(d) For the transmitted field, we need to find the transmission coefficient at the interface. We have $Z_{\text{air}} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 377$ Ω and $Z_{\text{material}} = \sqrt{\dfrac{\mu}{\varepsilon}} = \sqrt{\dfrac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\dfrac{\mu_0}{\varepsilon_0 3}} = \dfrac{Z_{\text{air}}}{\sqrt{3}} = 218$ Ω.

The plane wave in air is incident normally, so for transmission, according to the lectures (Eq. 4. 70): $E_{t0} = \dfrac{2Z_2}{Z_2 + Z_1}E_{i0}$. We can rearrange this to $E_{t0} = \dfrac{2}{1 + \dfrac{Z_1}{Z_2}}E_{i0}$.

Here, we can write $E_{t0} = \dfrac{2}{1 + \dfrac{Z_{\text{air}}}{Z_{\text{material}}}}E_{i0} = \dfrac{2}{1+\sqrt{3}}E_{i0} = \dfrac{2}{1+\sqrt{3}}10 \approx 7.32$ V/m.

(e) We use the Poynting vector $\langle \vec{S} \rangle = \frac{1}{2}\vec{E}\times\vec{H}$. There is no need to find the H-field in the medium as we can use the impedance, which we already know. $H_{\text{material}} = \frac{E_{\text{material}}}{Z_{\text{material}}}$ leads to $\langle \vec{S} \rangle = \frac{1}{2}\frac{E_{t0}^2}{Z_{\text{material}}} = \frac{\sqrt{3}}{2Z_{\text{air}}}E_{t0}^2 = \frac{1}{2}\frac{\sqrt{3}}{377}(7.32)^2 \approx 0.123$ Wm$^{-2}$.

**Question 2.**

A plane wave **propagates through a dielectric medium** in the region $z \le 0$ with $\varepsilon_r = 9$ and $\mu_r = 1$ with $\vec{H}_j = -0.2\cos\left(10^9 t - Kx - K\sqrt{8}z\right)\hat{j}$ A/m. The wave is **incident on a boundary with air**, which forms the *xy* plane, at *z* = 0. *[Note that $K = k\sin\theta_i$ and that $\sqrt{8}K = k\cos\theta_i$ in the usual notation, where k is the wavevector in the dielectric material]* Find:

(a) $\theta_i$ and $\theta_t$; (19.47º; 90º - incident wave at critical angle)

(b) the values of *K* and *k*; (3.33 m-1; 10 m-1)

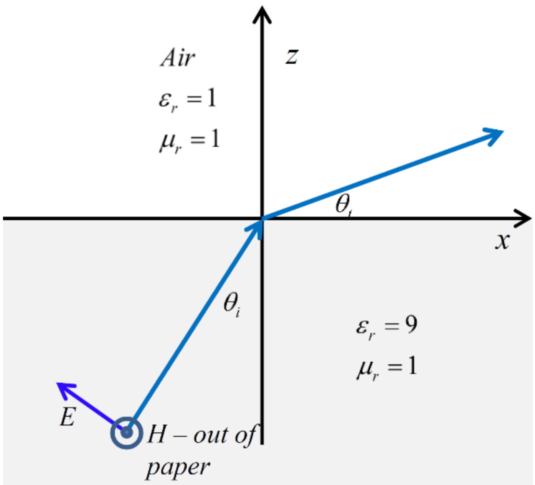
(c) the wavelengths in the dielectric and in the air; (0.628m, 1.88 m)

(d) the incident electric field; $\left(-23.7\hat{i} + 8.38\hat{k}\right)\cos\left(10^9 t - Kx - K\sqrt{8}z\right)$ Vm-1

(e) the transmitted and reflected electric fields;
$150.7\cos\left(10^9 t - 3.333x\right)\hat{k}$ V/m and $\left(23.7\hat{i} + 8.38\hat{k}\right)\cos\left(10^9 t - Kx + K\sqrt{8}z\right)\hat{k}$ V/m

(f) the Brewster angle. (18.43 º)

**Answer:** The wave is given as $\vec{H}_j = -0.2\cos\left(10^9 t - Kx - K\sqrt{8}z\right)\hat{j}$.



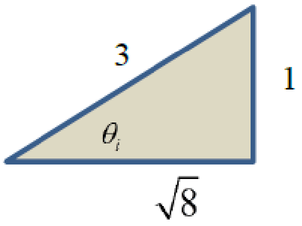

At the interface, we apply Snell's law:
$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_{\text{air}}}{n_{\text{medium}}}.$$

We can then replace:
$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{1}{\sqrt{\varepsilon_r\mu_r}} = \frac{1}{\sqrt{9}} = \frac{1}{3}.$$

(a) Snell's law leads to: $\sin\theta_t = 3\sin\theta_i$. We need to find $\theta_i$ from the description of the incoming wave. We know that for the incident wave:
$\vec{H}_j = -0.2\cos\left(10^9 t - Kx - K\sqrt{8}z\right)\hat{j}$, which we can compare with a general wave:
$\vec{H}_j = -A\cos\left(\omega t - kx\sin\theta_i - kz\cos\theta_i\right)\hat{j}$ where $k = 2\pi/\lambda$. We can immediately see that $\omega = 10^9$ rad/s. From taking the ratio of the magnitude of the x and z components of the wave vector, we can see that $\frac{k\sin\theta_i}{k\cos\theta_i} = \tan\theta_i = \frac{K}{K\sqrt{8}} = \frac{1}{\sqrt{8}}$.

- We can use a calculator to find this angle, which is 19.47°. We can then calculate its cos and sin.

- We can also use geometry: $\tan\theta_i = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{1}{\sqrt{8}}$, so the two sides of the associated right angle triangle are 1 and $\sqrt{8}$. The hypotenuse is given by Pythagoras. $a^2 + b^2 = c^2 \rightarrow 1^2 + \sqrt{8}^2 = 9 = 3^2$.




So, we have $\sin\theta_i = \frac{1}{3}$ and therefore $\sin\theta_t = 3\sin\theta_i = 3\frac{1}{3} = 1$. From this, it follows that $\theta_i = 19.47°$ and $\theta_t = 90°$. The angle of incidence is the critical angle, for which the transmitted wave is propagating along the interface.

(b) In the medium, $v_p = \frac{c}{n_{\text{medium}}} = \frac{c}{\sqrt{\varepsilon_r\mu_r}} = \frac{c}{\sqrt{9}} = \frac{c}{3}$ and $v_p = \frac{\omega}{k}$, so

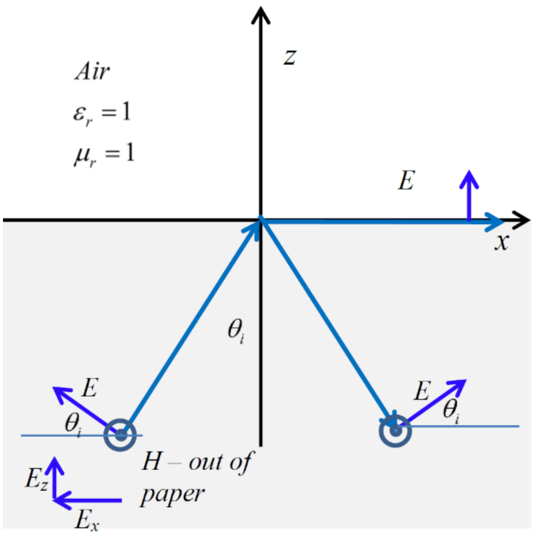
$k = \frac{\omega}{v_p} = \frac{10^9}{\frac{1}{3}(299792458)} \approx 10 \text{ m}^{-1}$. And since we have $k\sin\theta_i = K$, $K = \frac{1}{3}k \approx 3.33 \text{ m}^{-1}$.

(c) In the dielectric: $\lambda_{\text{medium}} = \frac{2\pi}{k} = \frac{2\pi}{10} \approx 0.628 \text{ m}$.

In the air: $\lambda_{\text{air}} = \frac{c}{f} = \frac{2\pi c}{\omega} = \frac{2\pi(299792458)}{10^9} \approx 1.88 \text{ m}$.

[Alternatively, we can use the fact that the frequency $f = v_p/\lambda$ is the same in both materials, so that we can establish a relationship between the wavelengths in both materials. We then just need to evaluate the phase velocities $v_p = c/n$. We obtain: $\frac{c/n_{\text{medium}}}{\lambda_{\text{medium}}} = \frac{c/n_{\text{air}}}{\lambda_{\text{air}}}$, which simplifies to $\frac{1}{n_{\text{medium}}\lambda_{\text{medium}}} = \frac{1}{n_{\text{air}}\lambda_{\text{air}}}$ and, because $n_{\text{air}} = 1$, we have $\lambda_{\text{air}} = \lambda_{\text{medium}} n_{\text{medium}}$. Note that this relationship is general: at the interface the wavelength changes by a factor of $n_{\text{medium}}$. So here, $\lambda_{\text{air}} = 0.628 \times 3 \approx 1.88 \text{ m}$.

(d) We are now looking at the incident wave.




So far, we have examined the projections of the wave vector on the $x$ and $z$ axis. Now, we are going to do the same for the amplitude vector.

For an amplitude $A$, we clearly have: $E_x = -A\cos\theta_i$ and $E_z = A\sin\theta_i$.

We then use the impedance: $\frac{A}{|\vec{H}|} = Z_{\text{material}}$ and we know that:

$$Z_{\text{material}} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0 9}} = \frac{Z_{\text{air}}}{3}.$$

Hence: $A = \frac{Z_{\text{air}}}{3}\left|\vec{H}\right| = \frac{377}{3}0.2 \approx 25.13$ V/m.

Therefore: $E_x = -(25.13)(0.943) \approx 23.7$ V/m and $E_z = (25.13)\times(0.33) \approx 8.38$ V/m.

In vector notations:
$$\vec{E}_i = \left(-23.7\hat{i} + 8.38\hat{k}\right)\cos\left(10^9 t - 3.33x - 3.33\sqrt{8}z\right) = \\ = \left(-23.7\hat{i} + 8.38\hat{k}\right)\cos\left(10^9 t - 3.33x - 9.42z\right).$$

(e) The reflected wave has the same component of wave vector in the $x$ direction but the component in the $z$ direction has an opposite sign. Therefore, the phase term of the reflected wave is simply $\cos\left(10^9 t - 3.33x + 9.42z\right)$.

In order to find the amplitude of the reflected wave, we use Fresnel's coefficients. We have to select the Fresnel coefficient for the correct polarization.
The electric field is in the plane of incidence, that is to say, we have P-polarization.
From the lectures (section 'General case of incidence at the boundary: P-polarized light', Eq. 4. 111), we have $\boxed{r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}}$.

Here, the angle of incidence is the critical angle and we have $\sin\theta_t = 1$, which means that $\cos\theta_t = 0$, so we can write: $r_{\parallel} = \frac{-Z_1\cos\theta_i}{Z_1\cos\theta_i} = -1$.

[We have $n_1\sin\theta_i = n_2\sin\theta_t$ and at the critical angle we have $\sin\theta_i = n_2/n_1$, so $\sin\theta_t = 1$]

As we expect, the wave undergoes total internal reflection. The amplitude reflected is the same as the incident amplitude but the direction of the x-component is changed, see diagram. [In general, materials do have loss; for instance, the refractive indices of metals are mostly imaginary. This means that in general, the Fresnel coefficients are imaginary numbers (see question 5 below). The fact that $r_{\parallel} < 0$ shows that the wave has experienced a phase change of $\pi$ radians upon reflection at the boundary. This is because $-1 = e^{i\pi}$, so $\tilde{r}_{\parallel} = -\frac{\vec{E}_{r0}}{\vec{E}_{i0}} = -\frac{\vec{A}_{r0}e^{i\phi_r}}{\vec{A}_{i0}e^{i\phi_i}} = -\frac{\vec{A}_{r0}}{\vec{A}_{i0}}e^{i(\phi_r-\phi_i)} = e^{i\pi}\frac{\vec{A}_{r0}}{\vec{A}_{i0}}e^{i(\phi_r-\phi_i)} = \frac{\vec{A}_{r0}}{\vec{A}_{i0}}e^{i(\phi_r-\phi_i+\pi)}$ means that the wave has reflected with same amplitude ($A$=1) but with a phase shift of $\pi$.]

In vector notations: $\vec{E}_r = \left(23.7\hat{i} + 8.38\hat{k}\right)\cos\left(10^9 t - 3.33x - 9.42z\right)$ V/m.

For the transmitted wave, the Fresnel coefficient is given by $\boxed{t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}}$, which is Eq. 4. 112 from the lecture notes. Again, we use the fact that $\cos\theta_t = 0$ and we obtain:

$$\frac{A_t}{A_i} = \frac{2Z_2\cos\theta_i}{Z_1\cos\theta_i} = \frac{2Z_2}{Z_1} = \frac{2Z_{\text{air}}}{Z_{\text{material}}} = \frac{2Z_{\text{air}}}{Z_{\text{air}}/3} = 6.$$

Therefore: $A_t = 6A_i = (6)(25.13) = 150.7$ V/m.

We know that the wave vector of the transmitted wave is along the interface, therefore the E-field of the transmitted wave only has a component in the x-direction.

In the air, $c = f\lambda = \frac{\omega}{k}$. Thus $k = \frac{\omega}{c} = \frac{10^9}{299792458} = 0.33$ m$^{-1}$.

In vector notations we have: $\vec{E}_t = \left(150.7\hat{k}\right)\cos\left(10^9 t - 3.33x\right)$ V/m.

(f) For the Brewster angle: $\tan\theta_B = \frac{n_2}{n_1} = \sqrt{\frac{1}{\varepsilon_r}} = \frac{1}{3}$. Therefore: $\theta_B \approx 18.4°$.

## Question 3.

Consider the case of a plane wave polarised with its E-field **perpendicular** to the plane of incidence. The wave propagates in vacuum and reflects from a **non-magnetic** material with a **relative permittivity > 1**. Show that the reflected wave is $\pi$ radians out of phase with the incident wave for any angle of incidence. [Hint: You may wish to use Snell's Law]

This question is about the reflectivity for light polarized with its E-field perpendicular to the plane of incidence, i.e**. S-polarized light**. In this case, the H-field is in the place of incidence. From the lecture notes (section 'General case of incidence at the boundary: S-polarized light', Eq. 4. 144) $\boxed{r_\perp \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}}$.

In this case light is incident from vacuum, so $Z_1 = Z_{\text{vacuum}} = Z_0$. We also have:

$Z_2 = Z_{\text{material}} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \frac{Z_0}{n}$ because $n = \sqrt{\varepsilon_r\mu_r} = \sqrt{\varepsilon_r}$ with $\mu_r = 1$ in a non-magnetic material. We can now replace: $r_\perp = \frac{\frac{Z_0}{n}\cos\theta_i - Z_0\cos\theta_t}{\frac{Z_0}{n}\cos\theta_i + Z_0\cos\theta_t} = \frac{\frac{\cos\theta_i}{n} - \cos\theta_t}{\frac{\cos\theta_i}{n} + \cos\theta_t}$.

Next, we use Snell's law: $\sin\theta_i = n\sin\theta_t$, so $\frac{1}{n} = \frac{\sin\theta_t}{\sin\theta_i}$ and we replace:

$$r_\perp = \frac{\frac{\sin\theta_t}{\sin\theta_i}\cos\theta_i - \cos\theta_t}{\frac{\sin\theta_t}{\sin\theta_i}\cos\theta_i + \cos\theta_t} = \frac{\sin\theta_t\cos\theta_i - \sin\theta_i\cos\theta_t}{\sin\theta_t\cos\theta_i + \sin\theta_i\cos\theta_t}.$$

We can simplify this equation using a standard trigonometry relation.

Eq. 4. 151 in the lecture notes is: $\begin{cases}\sin(a+b) = \sin a\cos b + \cos a\sin b \\ \sin(a-b) = \sin a\cos b - \cos a\sin b\end{cases}$

Leading to: $r_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$. (This is a general relation for this polarization and non-magnetic materials).

At the interface from vacuum to the material, the relative permittivity is increasing and the refractive index is also increasing. Thus, from Snell's law ($\sin\theta_i = n\sin\theta_t$), we know that, for any angle of incidence $\theta_i > \theta_t$. Therefore $\theta_t - \theta_i < 0$.

The sin of a negative number is also negative, therefore $r_\perp = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$ is always negative. [note that $180 > \theta_t + \theta_i > 0$ always].

When $r_\perp < 0$, the physical meaning is that there is a phase change of $\pi$ radians upon reflection from the material.

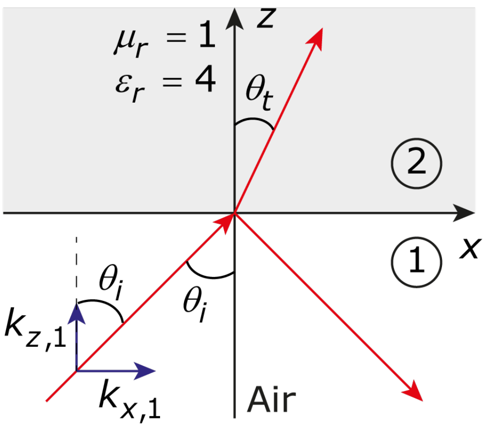
## Question 4.

*[2016 Exam question]* A uniform plane wave is **propagating in air** with an electric field given by $\vec{E} = 10\cos\left[\left(19.21\times10^8\right)t - 4x - 5z\right]\hat{y}$ [V/m], with $x$ and $z$ measured in metres and $t$ in seconds. It hits the surface of a lossless dielectric slab with $\varepsilon_r = 4$, $\mu_r = 1$ occupying the half-space $z \geq 0$. Find

(a) the **polarisation** of the incident wave with respect to the plane of incidence, (2)
(b) the **angle of incide**nce, (1)
(c) the **reflected electric field** $\vec{E}_r$ (6)

(a) Inspecting the mathematical expression for $\vec{E}$, we conclude that $\vec{k} = (4,0,5)$. The unit vector normal to the surface is $\hat{n} = \hat{z}$. Both the vectors are contained in the *xz*-plane which means the *xz*-plane is the plane of incidence.
[*Comment:* The form of the wave in this problem is written as $\cos\left(\omega t - \vec{k}\cdot\vec{r}\right)$ - this is a convention. Notice that, the cosine function is symmetric, $\cos(-x) = \cos(x)$, so that the wave can be written as: $\vec{E} = 10\cos\left[4x + 5z - \left(19.21\times 10^8\right)t\right]\hat{y}$, which corresponds to $\cos\left(\vec{k}\cdot\vec{r} - \omega t\right)$.]

Inspecting the form of the electric field, we see that $\vec{E} \propto \hat{y}$. Hence, the electric field oscillates along the *y*-axis. The polarization of the wave is (electric field) perpendicular to the plane of incidence. In other words, the wave is S-polarized.




(b) we can write that $\tan\theta_i = \dfrac{k_{x,1}}{k_{z,1}} = \dfrac{4}{5}$, thus

$$\theta_i = \arctan\frac{4}{5} \approx 38.66\,°.$$

(c) The angle of reflection is equal to the angle of incidence, so that

$$k_{x,1}^{reflected} = k_{x,1}^{incident} \text{ and } k_{z,1}^{reflected} = -k_{z,1}^{incident}.$$

Also, the frequency of the incident and reflected waves are the same, so the reflected electric field must have the form

$$\vec{E}_r = \vec{E}_{r0}\cos\left[\left(19.21\times 10^8\right)t - 4x - 5z\right]\hat{y}.$$

At the same time, Fresnel's equations give us

$r_\perp \equiv \dfrac{E_{r0}}{E_{i0}} = \dfrac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}$, with $Z_N = \sqrt{\dfrac{\mu_0\mu_{rN}}{\varepsilon_0\varepsilon_{rN}}}$, where $N = 1,2$ is an integer that designates the material.

Therefore:
$$E_{r0} = \frac{\sqrt{\dfrac{\mu_0\mu_{r2}}{\varepsilon_0\varepsilon_{r2}}}\cos\theta_i - \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\cos\theta_t}{\sqrt{\dfrac{\mu_0\mu_{r2}}{\varepsilon_0\varepsilon_{r2}}}\cos\theta_i + \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\cos\theta_t}E_{i0} = \frac{\sqrt{\dfrac{\mu_{r2}}{\varepsilon_{r2}}}\cos\theta_i - \cos\theta_t}{\sqrt{\dfrac{\mu_{r2}}{\varepsilon_{r2}}}\cos\theta_i + \cos\theta_t}E_{i0}.$$

To obtain the amplitude $E_{r0}$, we need to know the angle of transmission, $\theta_t$. This value can be obtained from Snell's law: $n_2\sin\theta_t = n_1\sin\theta_i$, which we can rewrite as:
$$\theta_t = \arcsin\left(\frac{n_1\sin\theta_i}{n_2}\right) = \arcsin\left(\frac{n_1\sin\theta_i}{\sqrt{\mu_{r2}\varepsilon_{r2}}}\right) = \arcsin\left(\frac{\sin 38.66}{\sqrt{1}\sqrt{4}}\right) \approx 18.2\,°.$$

Having a value for $\theta_t$, we find
$$E_{r0} = \frac{\frac{1}{2}\cos(38.66) - \cos(18.2)}{\frac{1}{2}\cos(38.66) + \cos(18.2)}10 \approx -4.17 \text{ V/m}.$$

Hence, the reflected electric field is $\vec{E}_r = -4.18\cos\left[\left(19.21\times 10^8\right)t - 4x + 5z\right]\hat{y}$ V/m.

**Question 5.**

> *[HARDER]* This question is about refraction and reflection of light at the surface of a good conductor. You may assume that the refractive index is >>1 for a good conductor *(More on this in electromagnetism 2)*. [the material is non-magnetic]
> (a) Summarise how light transmitted into a good conductor is **refracted**.
> (b) Using the expressions for the impedance of free space and the impedance of a good conductor find a good approximation for the **reflectivity (*R*)** of a metal surface. You can **assume normal incidence** to the surface to simplify the calculation.

(a) For a metal, we know that $\sigma \gg \varepsilon\omega$ and that the wave number of the propagating wave is $\gamma = -\alpha + i\beta$.

For a good conductor, $\alpha = \beta = \sqrt{\frac{\sigma\mu\omega}{2}} = \sqrt{\frac{\sigma\mu(2\pi f)}{2}} = \sqrt{\sigma\mu_r\mu_0\pi f}$ .

In general, $k = \frac{2\pi}{\lambda}$ and in a material with refractive index $n$, for a wave with wavelength in free space $\lambda_0$, we have: $k = \frac{2\pi n}{\lambda_0}$. Also in free space, $c = \frac{\lambda_0}{T} = f\lambda_0 \rightarrow \lambda_0 = \frac{c}{f}$, so we can write $k = \frac{2\pi f n}{c} = \frac{\omega n}{c}$ .

In a material, $\beta$ is the component of the complex wave vector that is linked to the phase term of wave propagation. Hence, we can write: $k = \frac{\omega n}{c} = \beta = \sqrt{\frac{\sigma\mu\omega}{2}}$. Thus, it follows that $\frac{\omega n}{c} = \sqrt{\frac{\sigma\mu\omega}{2}}$ and $n = \frac{c}{\omega}\sqrt{\frac{\sigma\mu\omega}{2}}$.

As we have $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$, we can substitute: $n = \frac{1}{\omega\sqrt{\varepsilon_0\mu_0}}\sqrt{\frac{\sigma\mu_0\mu_r\omega}{2}} = \sqrt{\frac{\sigma\mu_0\mu_r\omega}{\omega^2\varepsilon_0\mu_0 2}} = \sqrt{\frac{\sigma\mu_r}{\omega\varepsilon_0 2}}$.

For a non-magnetic material, $\mu_r = 1$, so $n = \sqrt{\frac{\sigma}{2\omega\varepsilon_0}}$.

We know that for a good conductor $\sigma \gg \varepsilon\omega$, so we can conclude that $n$ is large. If $n$ is large, then the light is strongly refracted towards the normal when transmitted into the material.

(b) We will use the results for normal incidence to look at the general features of reflectivity from a good conductor.

At normal incidence, we have $\theta_i = \theta_t = 0$, so $\cos\theta_i = \cos\theta_t = 1$. The Fresnel coefficients are then (regardless of the polarization): $r_{\parallel/\perp} = \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_2 + Z_1}$.

For incidence from free space onto a good conductor, $Z_1 = Z_0$.

For the material, from the lectures, we have (Eq. 3.118):

$$Z_2 \approx \sqrt{\frac{\mu\omega}{\sigma}}e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{\sigma\delta}e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{\sigma\delta}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \frac{\sqrt{2}}{\sigma\delta}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = \frac{1+i}{\sigma\delta}$$, so $Z_2 = \frac{1+i}{\sigma\delta}$,

where $\delta = \sqrt{\frac{2}{\sigma\mu\omega}}$.

Thus, $r_{\parallel/\perp} = \dfrac{\dfrac{(1+i)}{\sigma\delta} - Z_0}{\dfrac{(1+i)}{\sigma\delta} + Z_0} = \dfrac{(1+i) - Z_0\sigma\delta}{(1+i) + Z_0\sigma\delta} = \dfrac{(1 - Z_0\sigma\delta) + i}{(1 + Z_0\sigma\delta) + i}$.

For the reflectivity, $R = r_{\parallel/\perp} r^*_{\parallel/\perp} = \left[\dfrac{(1 - Z_0\sigma\delta) + i}{(1 + Z_0\sigma\delta) + i}\right]\left[\dfrac{(1 - Z_0\sigma\delta) - i}{(1 + Z_0\sigma\delta) - i}\right] = \dfrac{(1 - Z_0\sigma\delta)^2 + 1}{(1 + Z_0\sigma\delta)^2 + 1}$.

This is a correct expression for the reflectivity but it does not give us great physical insights. To make it more meaningful, we can rearrange, using $X = Z_0\sigma\delta$ for clarity:

$$R = \frac{(1-X)^2 + 1}{(1+X)^2 + 1} = \frac{1 - 2X + X^2 + 1}{1 + 2X + X^2 + 1} = \frac{2 - 2X + X^2 + [4X - 4X]}{2 + 2X + X^2} = 1 - \frac{4X}{2 + 2X + X^2}.$$

We also have $X = Z_0\sigma\delta = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\sigma\sqrt{\dfrac{2}{\sigma\mu_0\mu_r\omega}} = \sqrt{\dfrac{2\sigma}{\varepsilon_0\omega}}$.

For a good conductor, $X$ is large, since $\sigma > \varepsilon\omega$. In addition, if $X$ is large, we can approximate the denominator: $2 + 2X + X^2 \approx X^2$. Therefore, we can write:

$R \approx 1 - \dfrac{4X}{X^2} = 1 - \dfrac{4}{X}$ and $R \approx 1 - 4\sqrt{\dfrac{\varepsilon_0\omega}{2\sigma}}$. Here again, we use the fact that $\sigma > \varepsilon\omega$ in a good conductor, therefore $\sqrt{\varepsilon_0\omega/2\sigma}$ is small and the reflectivity is $R \approx 1$.

It follows that metals are very reflective. The light that is transmitted into the metal decays over a few oscillations.