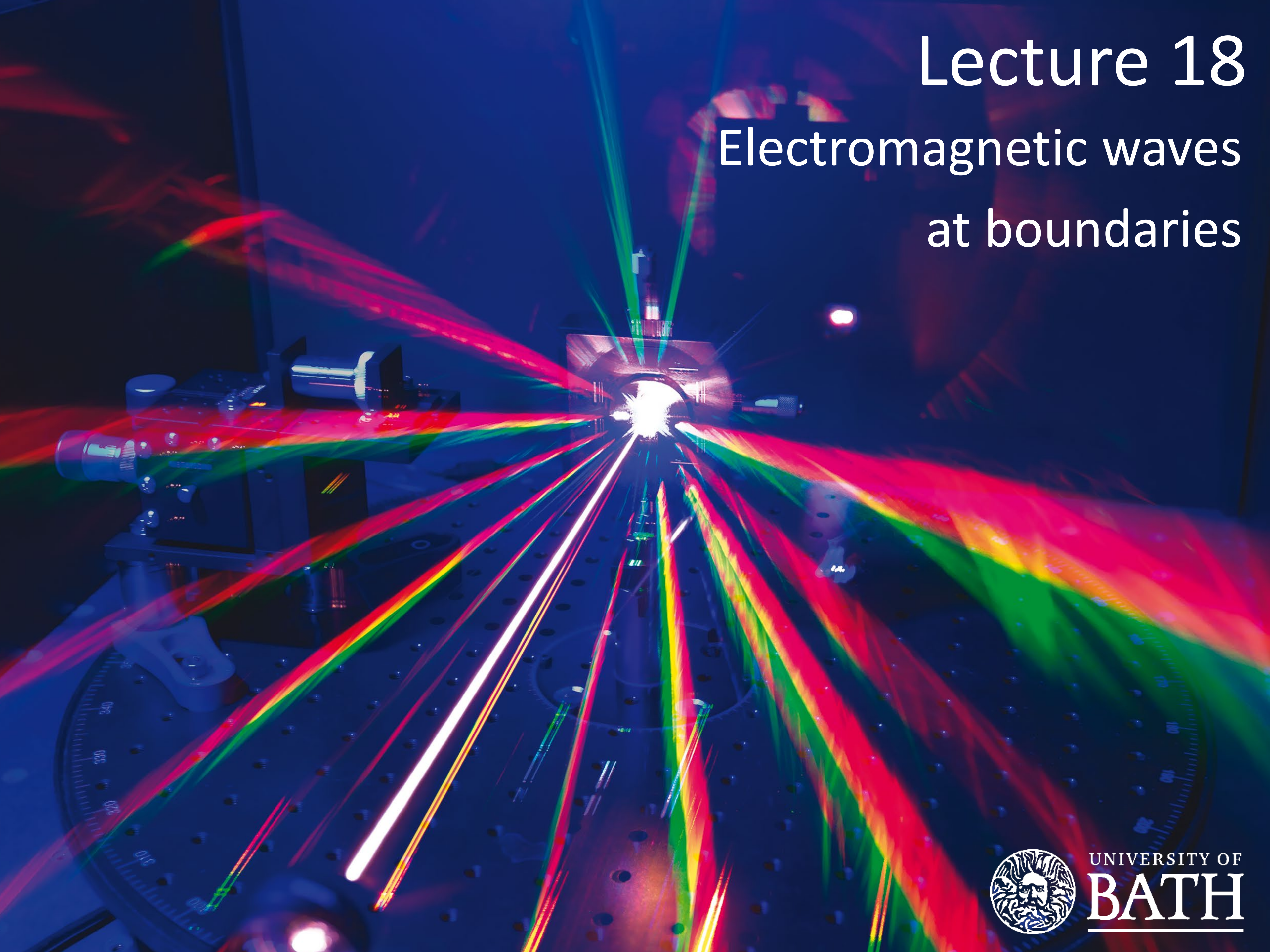
# Lecture 18

## Electromagnetic waves at boundaries



UNIVERSITY OF BATH

# Parallel and perpendicular in Relativity

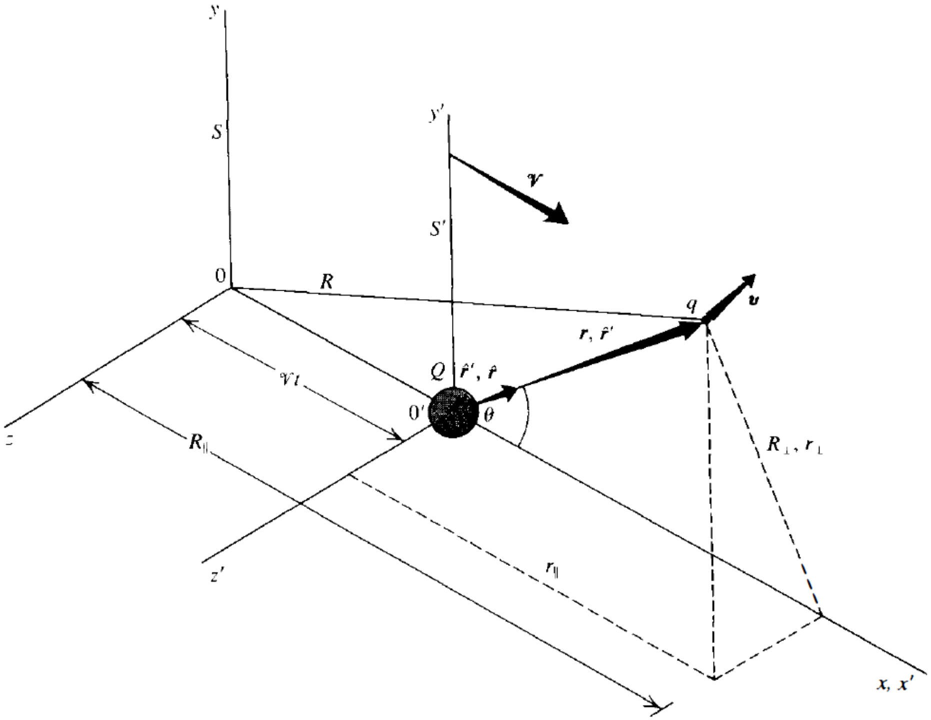



**Fig. 16-2.** The velocity of the charge $Q$ at the origin $O'$ of $S'$ is $\mathcal{V}\hat{\mathbf{x}}$ with respect to $S$. The velocity $\boldsymbol{v}$ of the charge $Q$ with respect to $S$ is arbitrary. All the unprimed variables shown are measured with respect to $S$.

The subscripts II and ⊥ refer, respectively, to the components that are either parallel or perpendicular to the motion of S' with respect to S. (Lorentz transformation)

In S’, the force exerted by Q on q is:

$$\boldsymbol{F}'_{Qq} = \frac{Qq\hat{\boldsymbol{r}}'}{4\pi\epsilon_0 r'^2} = \frac{Qq\boldsymbol{r}'}{4\pi\epsilon_0 r'^3},$$

In S’, the force exerted by Q on q is:

$$\boldsymbol{F}_{Qq} = \frac{Qq}{4\pi\epsilon_0} \frac{\boldsymbol{r} + \boldsymbol{v} \times (\boldsymbol{\mathcal{V}} \times \boldsymbol{r})/c^2}{\gamma^2 r^3 (1 - \beta^2 \sin^2\theta)^{3/2}}.$$

where $\gamma = (1 - \mathcal{V}^2/c^2)^{-1/2}$ and $\beta = \dfrac{\mathcal{V}}{c}$

Last lecture, we learned about **normal** and **transverse** (with respect to the interface). Today, we will discuss **parallel** and **perpendicular** (with respect to the plane of incidence). Let’s not confuse them.



Lorrain, Corson, Lorrain, Electromagnetic fields and waves, W. H. Freeman and Company, New York, 1988, p. 292-293.

# Last time we saw

At the boundary between two materials, EM wave is partially reflected and partially transmitted.

The plane of incidence is defined by the wave vector of the incident wave and a unit vector normal to the boundary.

On each side of the boundary, the electric and magnetic fields can be resolved into normal and tangential components.

For LIH materials, assuming no surface charges and no surface currents.

| | Electric fields | Magnetic fields |
|---|---|---|
| **Normal components** | $\vec{D}_{1n} = \vec{D}_{2n}$ | $\vec{B}_{1n} = \vec{B}_{2n}$ |
| **Tangential components** | $\vec{E}_{1t} = \vec{E}_{2t}$ | $\vec{H}_{1t} = \vec{H}_{2t}$ |

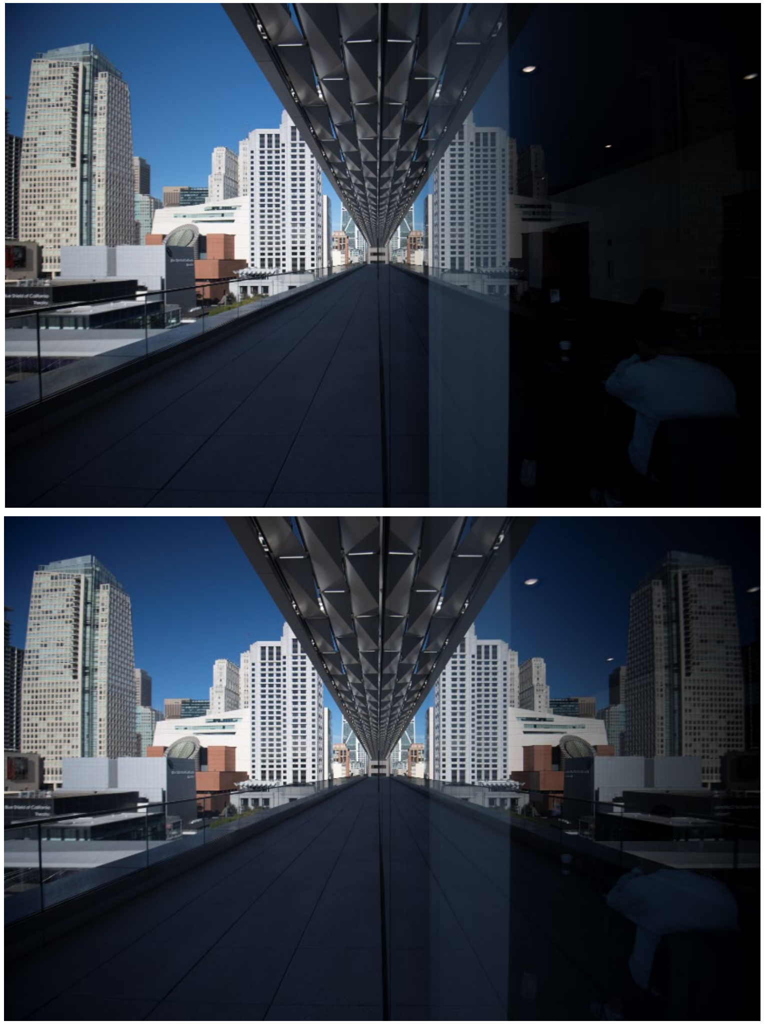
And today we will see…

# Overview

**In this Lecture we will look at:**

- Polarizing filters in photography
- Electromagnetic waves at normal incidence
- General incidence at the boundary and the law of reflection
- General incidence at the boundary and Snell's law
- Polarisation of an electromagnetic wave
- General incidence at the boundary for P-polarized light

# Polarizing filters in photography



Our experience shows us that at the boundary between two materials, some light is reflected and some light is transmitted. But how much?

We will consider a plane EM wave, where we have:

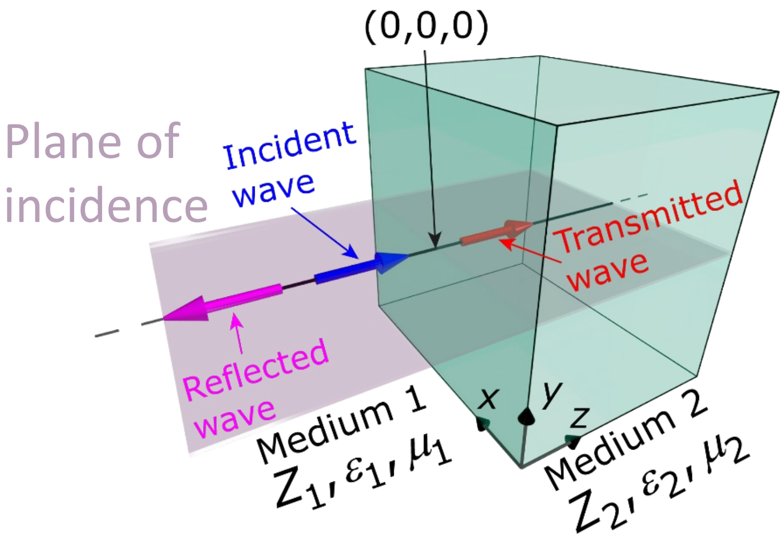
$$\vec{k} \times \vec{E} = \mu\omega\vec{H}$$

$$\vec{k} \cdot \vec{E} = 0$$

$$\vec{k} \cdot \vec{H} = 0$$

Consider light incident along the normal to the interface.

# Electromagnetic waves at normal incidence




EM arriving at the boundary between two materials, at normal incidence, is partially reflected and partially transmitted.

At the boundary, all three waves are associated with the motion of the same electron vibrations. So:

$$\omega_i = \omega_r = \omega_t = \omega = \frac{2\pi}{T}$$

Same angular frequency!

The incident E-field is only along $x$, so:

$$\vec{E}_i = E_{i0} e^{i(k_1 z - \omega t)} \hat{x}$$

The wave vector is along $z$, so from:

$$\vec{k} \times \vec{E} = \mu\omega\vec{H}$$

The H-field must be along $y$:

$$\vec{H}_i = H_{i0} e^{i(k_1 z - \omega t)} \hat{y}$$

Using the impedance in medium 1:

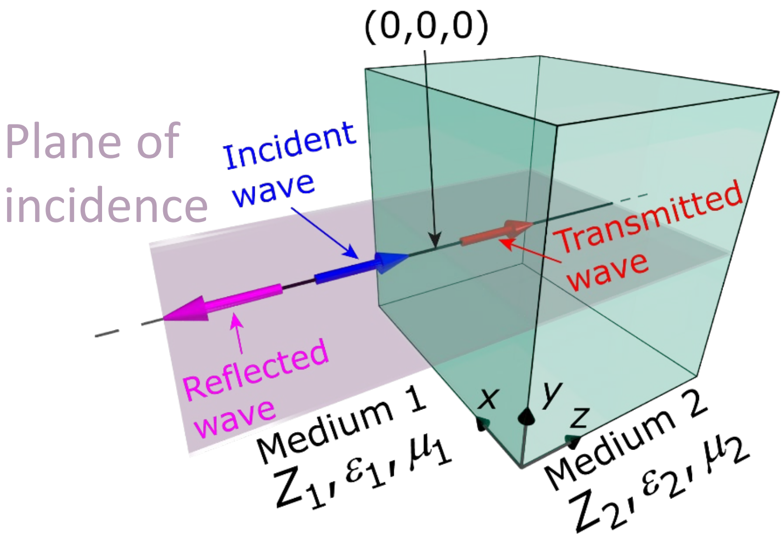
$$Z_1 = E_{i0} / H_{i0}$$

We get:

$$\vec{E}_i = E_{i0} e^{i(k_1 z - \omega t)} \hat{x} \quad \text{and} \quad \vec{H}_i = \frac{E_{i0}}{Z_1} e^{i(k_1 z - \omega t)} \hat{y}$$

How about the reflected wave?

# Electromagnetic waves at normal incidence




Incident wave:

$$\vec{E}_i = E_{i0} e^{i(k_1 z - \omega t)} \hat{x}$$

and

$$\vec{H}_i = \frac{E_{i0}}{Z_1} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected wave: [$k$ along negative $z$]

$$\vec{E}_r = E_{r0} e^{i(-k_1 z - \omega t)} \hat{x}$$

Because of:

$$\vec{k} \times \vec{E} = \mu\omega\vec{H}$$

The H-field must be along negative $y$:

$$\vec{H}_r = -H_{r0} e^{i(-k_1 z - \omega t)} \hat{y}$$

Using the impedance in medium 1:

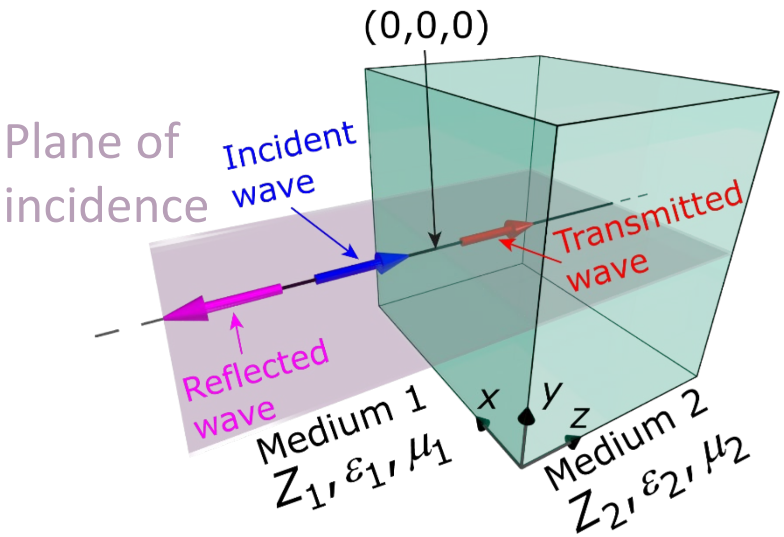
$$Z_1 = E_{r0} / H_{r0}$$

We get: $\vec{E}_r = E_{r0} e^{i(-k_1 z - \omega t)} \hat{x}$

and $\vec{H}_r = -\frac{E_{r0}}{Z_1} e^{i(-k_1 z - \omega t)} \hat{y}$

How about the transmitted wave?

# Electromagnetic waves at normal incidence




Incident wave:

$$\vec{E}_i = E_{i0} e^{i(k_1 z - \omega t)} \hat{x}$$

and

$$\vec{H}_i = \frac{E_{i0}}{Z_1} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected wave:

$$\vec{E}_r = E_{r0} e^{i(-k_1 z - \omega t)} \hat{x}$$

and

$$\vec{H}_r = -\frac{E_{r0}}{Z_1} e^{i(-k_1 z - \omega t)} \hat{y}$$

Transmitted wave: [using same logic]

$$\vec{E}_t = E_{t0} e^{i(k_2 z - \omega t)} \hat{x}$$

Then, using the impedance in medium 2:

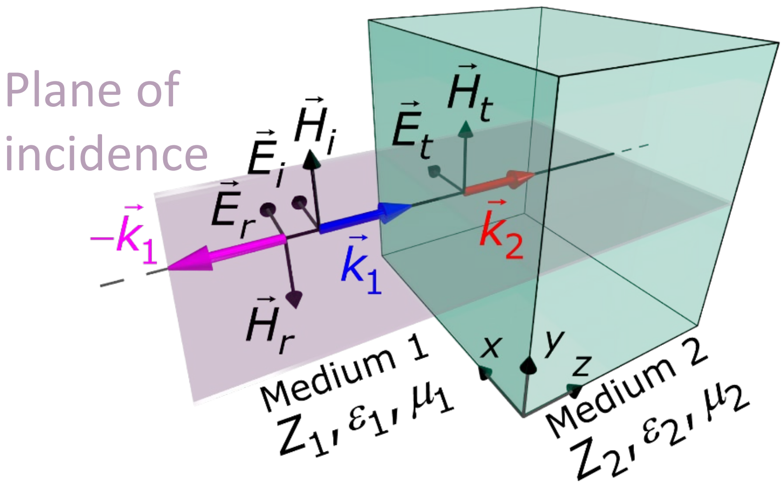
$$Z_2 = E_{t0} / H_{t0}$$

We get:

$$\vec{H}_t = \frac{E_{t0}}{Z_2} e^{i(k_2 z - \omega t)} \hat{y}$$

What are the total fields in media 1 and 2?

# Electromagnetic waves at normal incidence




Electric and magnetic field strength components, together with the wave vectors, for incident, reflected and transmitted electromagnetic waves.

All the vector components are tangential and tangential *E* and *H* components are continuous at the boundary.

The total fields are:

In medium 1:

$$\begin{cases} \vec{E}_1 = \vec{E}_i + \vec{E}_r \\ \vec{H}_1 = \vec{H}_i + \vec{H}_r \end{cases}$$

In medium 2:

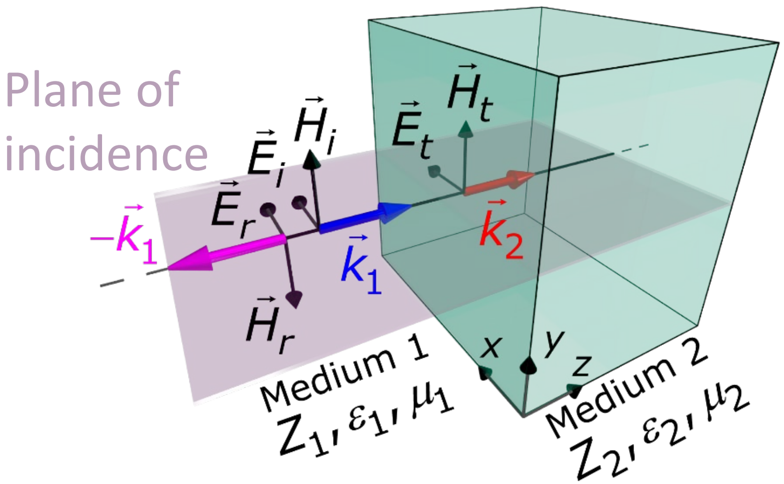
$$\begin{cases} \vec{E}_2 = \vec{E}_t \\ \vec{H}_2 = \vec{H}_t \end{cases}$$

At the boundary:

$$\begin{cases} \vec{E}_1 = \vec{E}_2 \\ \vec{H}_1 = \vec{H}_2 \end{cases}$$


How much is reflected or transmitted?

# Electromagnetic waves at normal incidence




Incident wave:

$$\vec{E}_i = E_{i0} e^{i(k_1 z - \omega t)} \hat{x}$$

and

$$\vec{H}_i = \frac{E_{i0}}{Z_1} e^{i(k_1 z - \omega t)} \hat{y}$$

Reflected wave:

$$\vec{E}_r = E_{r0} e^{i(-k_1 z - \omega t)} \hat{x}$$

and

$$\vec{H}_r = -\frac{E_{r0}}{Z_1} e^{i(-k_1 z - \omega t)} \hat{y}$$

Transmitted wave:

$$\vec{E}_t = E_{t0} e^{i(k_2 z - \omega t)} \hat{x}$$

and

$$\vec{H}_t = \frac{E_{t0}}{Z_2} e^{i(k_2 z - \omega t)} \hat{y}$$

We just saw that, at the boundary:

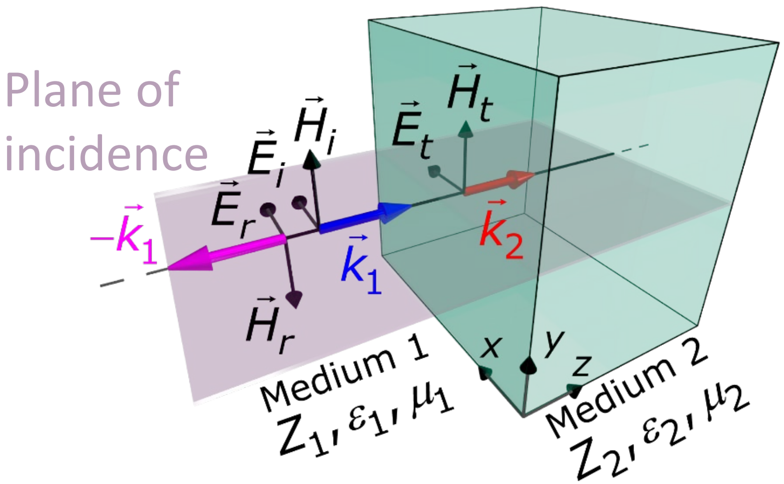
$$\begin{cases} \vec{E}_i + \vec{E}_r = \vec{E}_2 \\ \vec{H}_i + \vec{H}_r = \vec{H}_2 \end{cases}$$

Replacing:

$$\begin{cases} E_{i0} e^{i(k_1 z - \omega t)} \hat{x} + E_{r0} e^{i(-k_1 z - \omega t)} \hat{x} = E_{t0} e^{i(k_2 z - \omega t)} \hat{x} \\ \dfrac{E_{i0}}{Z_1} e^{i(k_1 z - \omega t)} \hat{y} - \dfrac{E_{r0}}{Z_1} e^{i(-k_1 z - \omega t)} \hat{y} = \dfrac{E_{t0}}{Z_2} e^{i(k_2 z - \omega t)} \hat{y} \end{cases}$$

We drop $e^{i(-\omega t)}$ as well as the unit vectors along $x$ and $y$

# Electromagnetic waves at normal incidence




We just saw that, at the boundary:

$$\begin{cases} \vec{E}_i + \vec{E}_r = \vec{E}_2 \\ \vec{H}_i + \vec{H}_r = \vec{H}_2 \end{cases}$$

We obtain:

$$\begin{cases} E_{i0} e^{i(k_1 z)} + E_{r0} e^{i(-k_1 z)} = E_{t0} e^{i(k_2 z)} \\ \dfrac{E_{i0}}{Z_1} e^{i(k_1 z)} - \dfrac{E_{r0}}{Z_1} e^{i(-k_1 z)} = \dfrac{E_{t0}}{Z_2} e^{i(k_2 z)} \end{cases}$$

At the boundary $z$=0, so

$$e^{i(k_1 z)} = e^{i(k_2 z)} = 1$$

And we get:

$$\begin{cases} E_{i0} + E_{r0} = E_{t0} \\ \dfrac{E_{i0}}{Z_1} - \dfrac{E_{r0}}{Z_1} = \dfrac{E_{t0}}{Z_2} \end{cases} \rightarrow \begin{cases} E_{i0} + E_{r0} = E_{t0} \\ E_{i0} - E_{r0} = \dfrac{Z_1}{Z_2} E_{t0} \end{cases}$$

We can add up the equations:

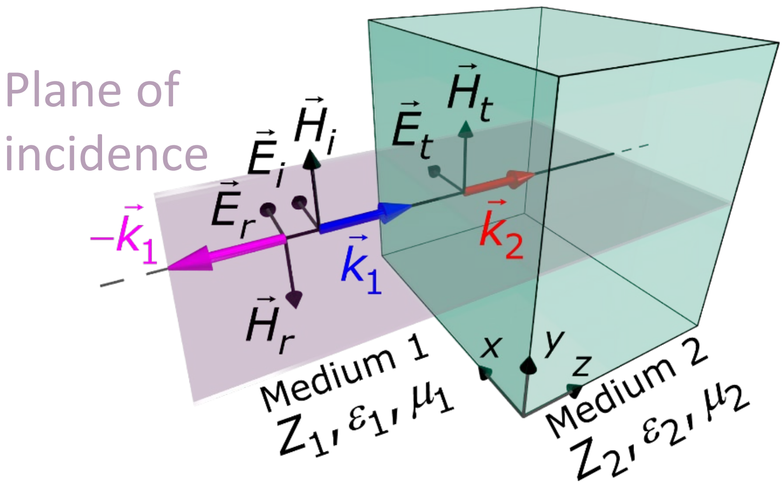
$$2E_{i0} = E_{t0} + \frac{Z_1}{Z_2} E_{t0} = \left(1 + \frac{Z_1}{Z_2}\right) E_{t0} =$$

$$= \frac{Z_2 + Z_1}{Z_2} E_{t0}$$

Which leads to:

$$E_{t0} = \frac{2Z_2}{Z_2 + Z_1} E_{i0}$$

# Electromagnetic waves at normal incidence




We just saw that, in transmission:

$$E_{t0} = \frac{2Z_2}{Z_2 + Z_1} E_{i0}$$

Then from:

$$E_{i0} + E_{r0} = E_{t0}$$

We have:

$$E_{r0} = E_{t0} - E_{i0} = \frac{2Z_2}{Z_1 + Z_2} E_{i0} - E_{i0} =$$

$$= \frac{2Z_2}{Z_1 + Z_2} E_{i0} - \frac{Z_1 + Z_2}{Z_1 + Z_2} E_{i0} =$$

$$= \frac{2Z_2 - Z_1 - Z_2}{Z_1 + Z_2} E_{i0}$$

So, we get: $E_{r0} = \dfrac{Z_2 - Z_1}{Z_1 + Z_2} E_{i0}$

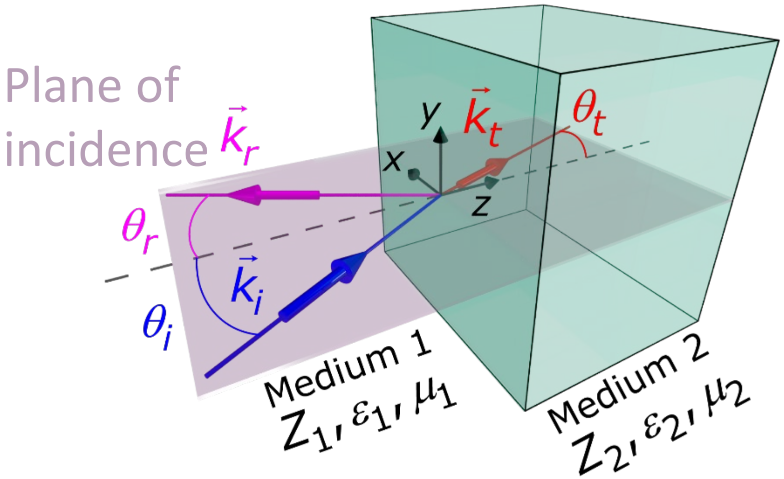
We define **reflection and transmission coefficients**:

$$\boxed{r_{\parallel/\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_1 + Z_2}} \qquad \boxed{t_{\parallel/\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2}{Z_1 + Z_2}}$$

Note: $r_{\parallel/\perp} \neq r$ [position] and $t_{\parallel/\perp} \neq t$ [time]

What if incidence is not normal?

# General incidence at the boundary and the law of reflection




The **plane of incidence** is defined by the wave vector of the incident wave and a unit vector normal to the boundary.

We saw that:

$$\omega_i = \omega_r = \omega_t = \omega$$

At the boundary, the phases of all three waves are identical (it is the same oscillations at the interface).

For the three waves of the form $e^{\left(\vec{k}\cdot\vec{r}-\omega t\right)}$

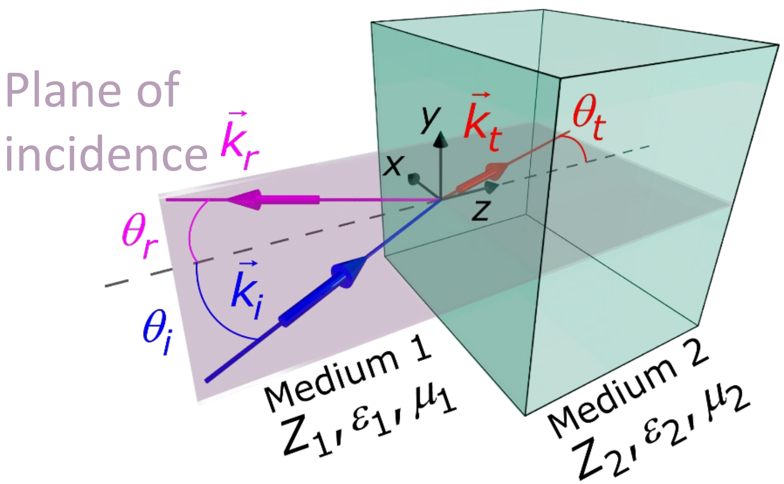
We can write:

$$\left(\vec{k}_i \cdot \vec{r}\right)_{\text{interface}} = \left(\vec{k}_r \cdot \vec{r}\right)_{\text{interface}} = \left(\vec{k}_t \cdot \vec{r}\right)_{\text{interface}}$$

$$\left(k_{ix}x + k_{iz}z\right) = \left(k_{rx}x + k_{rz}z\right) = \left(k_{tx}x + k_{tz}z\right)$$

$$\text{So} \begin{cases} k_{i,x} = k_{r,x} = k_{t,x} = k_x \\ k_{i,y} = k_{r,y} = k_{t,y} = k_y \end{cases}$$

# General incidence at the boundary and the law of reflection




The **plane of incidence** is defined by the wave vector of the incident wave and a unit vector normal to the boundary.

We just saw:

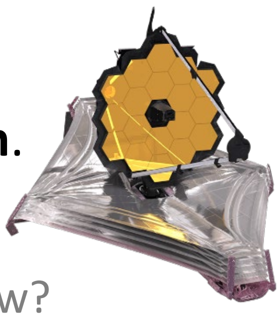
$$\begin{cases} k_{i,x} = k_{r,x} = k_{t,x} = k_x \\ k_{i,y} = k_{r,y} = k_{t,y} = k_y \end{cases}$$

Here: $k_{i,x} = k_{r,x} \quad k_{i,y} = k_{r,y} \quad k_{r,z} = -k_{i,z}$

Therefore, we must have:

$$\theta_i = \theta_r$$

which is the **law of reflection**.



How about Snell's law?

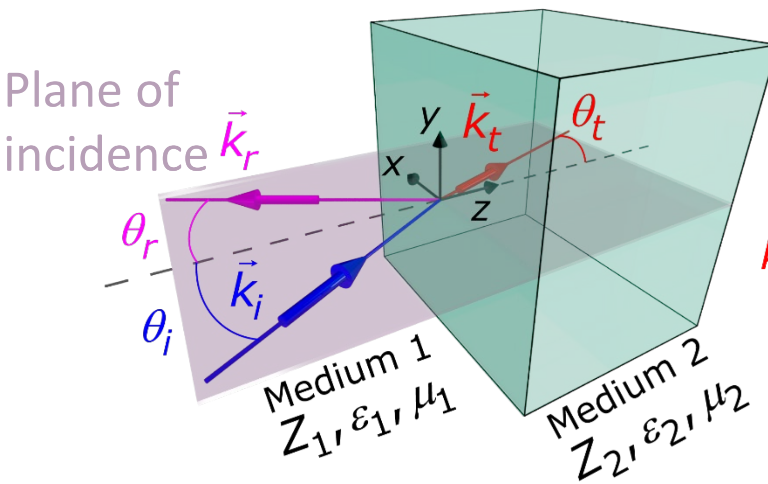
In vacuum, we have:

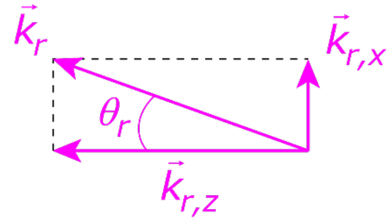
$$\left|\vec{k}\right| = \frac{2\pi}{\lambda} = \frac{\omega T}{\lambda} = \frac{\omega}{c} = \omega\sqrt{\varepsilon_0\mu_0}$$

In materials, we have:

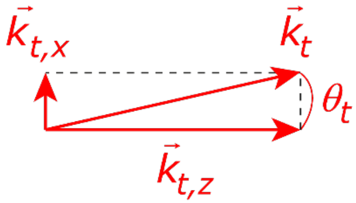
$$v_p = \frac{\omega}{\left|\vec{k}\right|} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{n}$$

So: $\left|\vec{k}\right| = \frac{\omega}{c} n$ (we will use on next slide)




$$\begin{cases} \left|\vec{k}_{r,x}\right| = \left|\vec{k}_r\right| \sin\theta_r = \left|\vec{k}_i\right| \sin\theta_i \\ \left|\vec{k}_{r,z}\right| = \left|\vec{k}_r\right| \cos\theta_r = \left|\vec{k}_i\right| \cos\theta_i \end{cases}$$



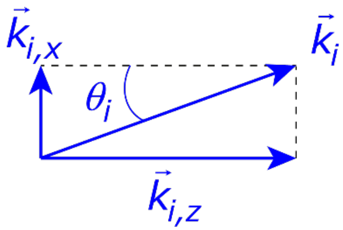

$$\begin{cases} \left|\vec{k}_{t,x}\right| = \left|\vec{k}_t\right| \sin\theta_t = \frac{\omega}{c}\sqrt{\varepsilon_{r2}\mu_{r2}} \sin\theta_t \\ \left|\vec{k}_{t,z}\right| = \left|\vec{k}_t\right| \cos\theta_t = \frac{\omega}{c}\sqrt{\varepsilon_{r2}\mu_{r2}} \cos\theta_t \end{cases}$$

We have: $k_i = \omega\underline{\underline{\sqrt{\mu_{r1}\varepsilon_{r1}}}}/c$

and: $k_t = \omega\underline{\underline{\sqrt{\mu_{r2}\varepsilon_{r2}}}}/c$

$$\boxed{\left|\vec{k}\right| = \frac{\omega}{c}\underline{\underline{n}}}$$

Resolving the wave vectors:




$$\begin{cases} \left|\vec{k}_{i,x}\right| = \left|\vec{k}_i\right| \sin\theta_i \\ \left|\vec{k}_{i,z}\right| = \left|\vec{k}_i\right| \cos\theta_i \end{cases}$$

From the conservation of angular momentum: $k_{i,x} = k_{r,x} = k_{t,x}$

So $\left|\vec{k}_i\right| \sin\theta_i = \left|\vec{k}_t\right| \sin\theta_t$, replacing:

$$\frac{\cancel{\omega}}{\cancel{c}}\sqrt{\mu_{r1}\varepsilon_{r1}} \sin\theta_i = \frac{\cancel{\omega}}{\cancel{c}}\sqrt{\mu_{r2}\varepsilon_{r2}} \sin\theta_t$$

Therefore: $\sqrt{\mu_{r1}\varepsilon_{r1}} \sin\theta_i = \sqrt{\mu_{r2}\varepsilon_{r2}} \sin\theta_t$

Snell's law: $\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$

# Polarisation of an electromagnetic wave

For an EM wave, the plane of oscillations of the electric field defines the direction of **light polarisation**.

> Note: Do not confuse the electric polarisation, which we discussed in dielectric materials (a material property!) and the light polarisation, which is a property of electromagnetic waves (a light property!)

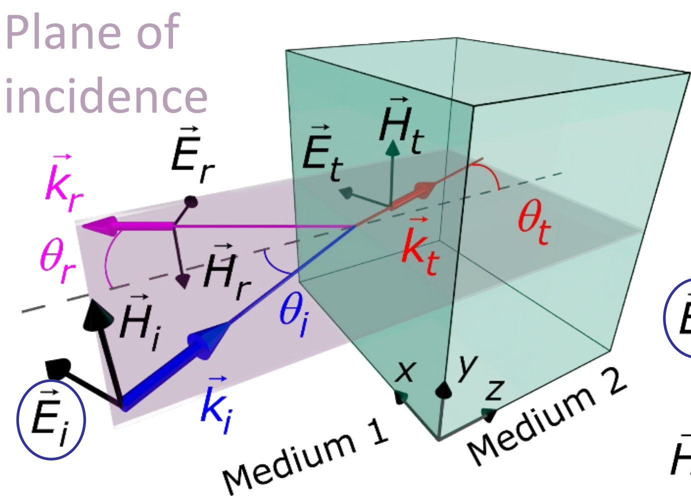
We can consider two cases of light polarisation:

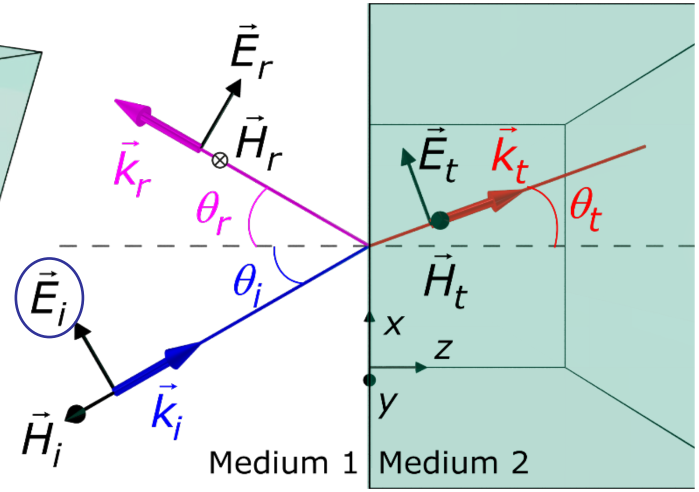
1. The polarisation of light is in the plane of incidence (the magnetic field oscillations are perpendicular to the plane of incidence). This polarisation state is referred to as **P-polarized** (from the German parallel). It can be indicated as ( $\parallel$ ).

2. The polarisation of light is perpendicular to the plane of incidence (the magnetic field oscillations are in the plane of incidence). This polarisation state is referred to as **S-polarized** (from the German senkrecht). It can be indicated as ( $\perp$ ).

Let's start with P-polarized light!

# General incidence at the boundary for P-polarized light



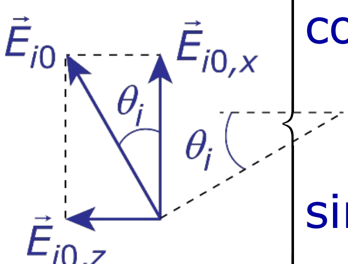




We can resolve all the vectors in the Cartesian coordinate system:

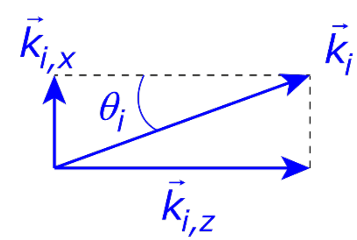
$$\vec{E}_i = \left(E_{i0,x} + E_{i0,z}\right) e^{i\left(k_{i,x}x + k_{i,z}z - \omega t\right)}$$

$$\vec{E}_i = E_{i0}\left(\cos\theta_i \hat{x} - \sin\theta_i \hat{z}\right) e^{i\left(k_1 \sin\theta_i x + k_1 \cos\theta_i z - \omega t\right)}$$

$$\vec{H}_i = \frac{E_{i0}}{Z_1} \hat{y} e^{i\left(k_1 \sin\theta_i x + k_1 \cos\theta_i z - \omega t\right)}$$

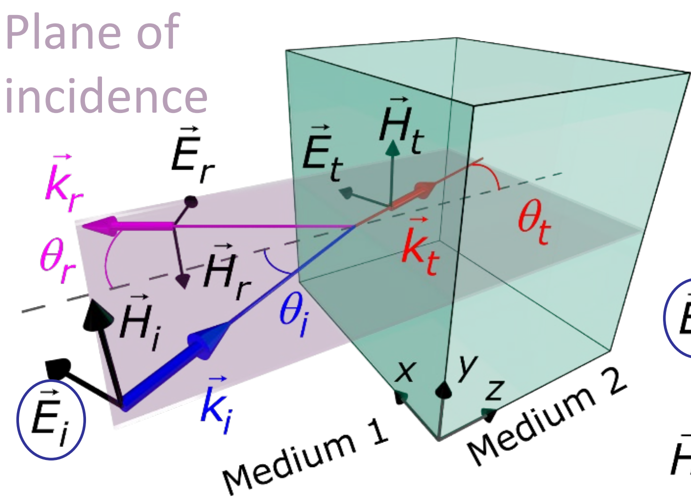


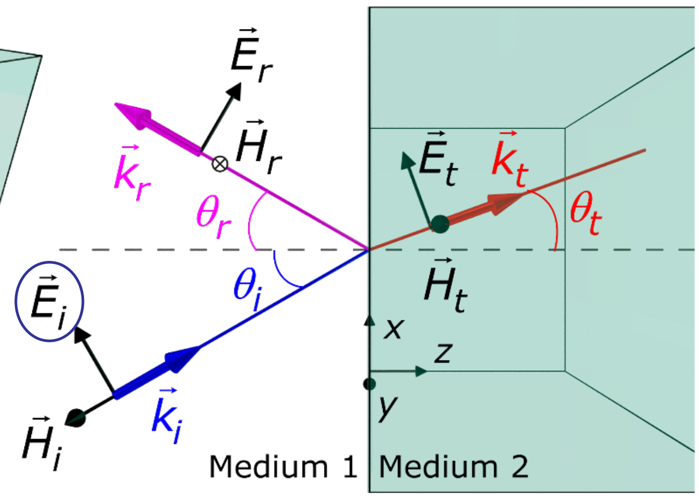
$$\begin{cases} \cos\theta_i = \dfrac{\left|\vec{E}_{i0,x}\right|}{\left|\vec{E}_{i0}\right|} \\ \sin\theta_i = \dfrac{\left|\vec{E}_{i0,z}\right|}{\left|\vec{E}_{i0}\right|} \end{cases} \rightarrow \begin{cases} \left|\vec{E}_{i0,x}\right| = \left|\vec{E}_{i0}\right| \cos\theta_i \\ \left|\vec{E}_{i0,z}\right| = \left|\vec{E}_{i0}\right| \sin\theta_i \end{cases}$$



$$\begin{cases} \left|\vec{k}_{i,x}\right| = \left|\vec{k}_i\right| \sin\theta_i \\ \left|\vec{k}_{i,z}\right| = \left|\vec{k}_i\right| \cos\theta_i \end{cases}$$

Repeat for all other vectors




We can resolve all the vectors in the Cartesian coordinate system:

$$\vec{E}_i = \left(E_{i0,x} + E_{i0,z}\right) e^{i\left(k_{i,x}x + k_{i,z}z - \omega t\right)}$$

$$\vec{E}_i = E_{i0}\left(\cos\theta_i \hat{x} - \sin\theta_i \hat{z}\right) e^{i\left(k_1 \sin\theta_i x + k_1 \cos\theta_i z - \omega t\right)}$$

$$\vec{E}_r = \left(E_{r0,x} + E_{r0,z}\right) e^{i\left(k_{r,x}x - k_{r,z}z - \omega t\right)}$$

$$\vec{E}_r = E_{r0}\left(\cos\theta_r \hat{x} + \sin\theta_r \hat{z}\right) e^{i\left(k_1 \sin\theta_r x - k_1 \cos\theta_r z - \omega t\right)}$$

$$\vec{E}_t = \left(E_{t0,x} + E_{t0,z}\right) e^{i\left(k_{t,x}x + k_{t,z}z - \omega t\right)}$$

$$\vec{E}_t = E_{t0}\left(\cos\theta_t \hat{x} - \sin\theta_t \hat{z}\right) e^{i\left(k_2 \sin\theta_t x + k_2 \cos\theta_t z - \omega t\right)}$$

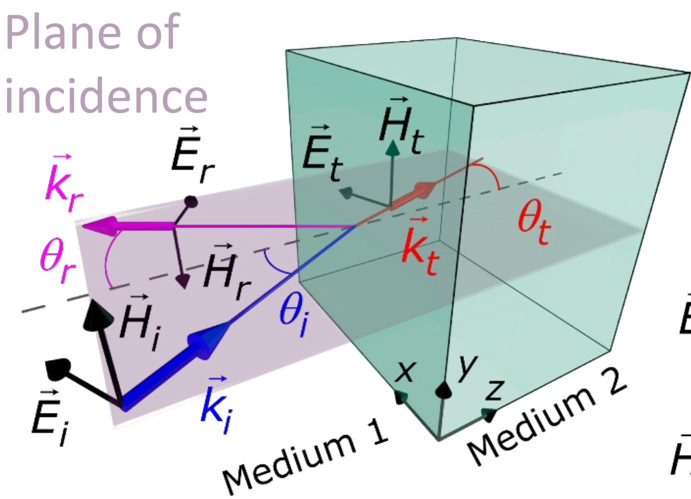
$$\vec{H}_i = \frac{E_{i0}}{Z_1} \hat{y} e^{i\left(k_1 \sin\theta_i x + k_1 \cos\theta_i z - \omega t\right)}$$

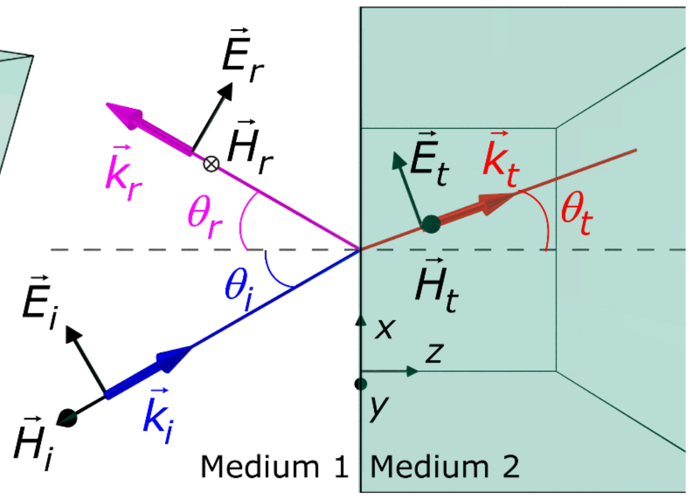
$$\vec{H}_r = \frac{E_{r0}}{Z_1}\left(-\hat{y}\right) e^{i\left(k_1 \sin\theta_r x - k_1 \cos\theta_r z - \omega t\right)}$$

$$\vec{H}_t = \frac{E_{t0}}{Z_1} \hat{y} e^{i\left(k_2 \sin\theta_t x + k_2 \cos\theta_t z - \omega t\right)}$$

Let's look at E.T.

# General incidence at the boundary for P-polarized light




We require:

$$\vec{E}_1\Big|_{\text{tangential}} = \vec{E}_2\Big|_{\text{tangential}}$$

Which is:

$$\vec{E}_1\Big|_x = \vec{E}_2\Big|_x$$

Therefore:

$$\left(\vec{E}_i + \vec{E}_r\right)\Big|_x = \vec{E}_t\Big|_x$$

$$\vec{E}_i = E_{i0}\left(\cos\theta_i\hat{x} - \sin\theta_i\hat{z}\right)e^{i\left(k_1\sin\theta_i x + k_1\cos\theta_i z - \omega t\right)}$$

$$\vec{E}_r = E_{r0}\left(\cos\theta_r\hat{x} + \sin\theta_r\hat{z}\right)e^{i\left(k_1\sin\theta_r x - k_1\cos\theta_r z - \omega t\right)}$$

$$\vec{E}_t = E_{t0}\left(\cos\theta_t\hat{x} - \sin\theta_t\hat{z}\right)e^{i\left(k_2\sin\theta_t x + k_2\cos\theta_t z - \omega t\right)}$$

We can replace: (knowing $\theta_i = \theta_r$)

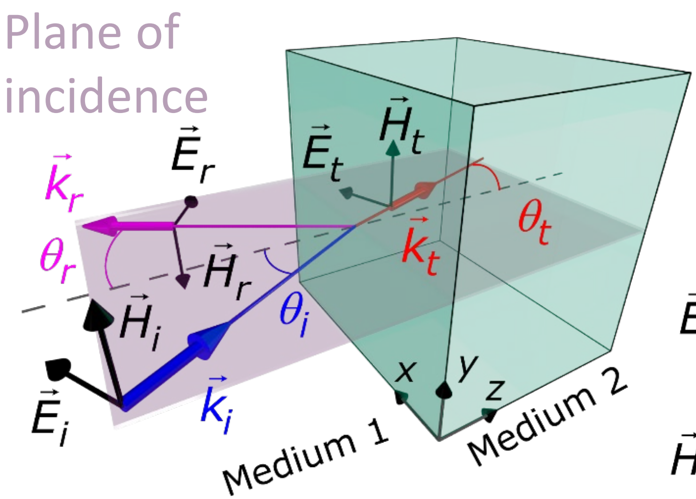
$$E_{i0}\cos\theta_i\hat{x} + E_{r0}\cos\theta_r\hat{x} =$$
$$= E_{t0}\cos\theta_t\hat{x}$$

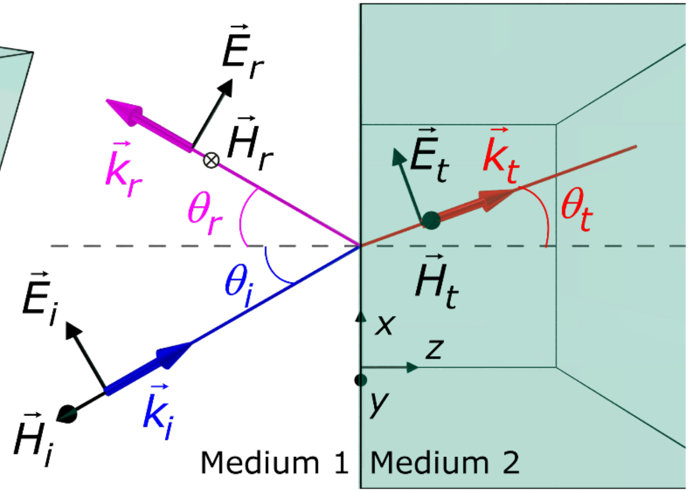
And we can drop the unit vectors:

$$E_{i0} + E_{r0} = E_{t0}\frac{\cos\theta_t}{\cos\theta_i}$$

Now the *H*-fields!

# General incidence at the boundary for P-polarized light




We just found:

$$\boxed{E_{i0} + E_{r0} = E_{t0}\frac{\cos\theta_t}{\cos\theta_i}}$$

Moreover:

$$\vec{H}_1\Big|_{\text{tangential}} = \vec{H}_2\Big|_{\text{tangential}}$$

Therefore:

$$\vec{H}_1\Big|_y = \vec{H}_2\Big|_y \rightarrow \left(\vec{H}_i + \vec{H}_r\right)\Big|_y = \vec{H}_t\Big|_y$$

Our *H*-fields are:

$$\vec{H}_i = \frac{E_{i0}}{Z_1}\hat{y}e^{i\left(k_1\sin\theta_i x + k_1\cos\theta_i z - \omega t\right)}$$

$$\vec{H}_r = -\frac{E_{r0}}{Z_1}\left(\hat{y}\right)e^{i\left(k_1\sin\theta_r x - k_1\cos\theta_r z - \omega t\right)}$$

$$\vec{H}_t = \frac{E_{t0}}{Z_1}\hat{y}e^{i\left(k_2\sin\theta_t x + k_2\cos\theta_t z - \omega t\right)}$$

We can now replace:

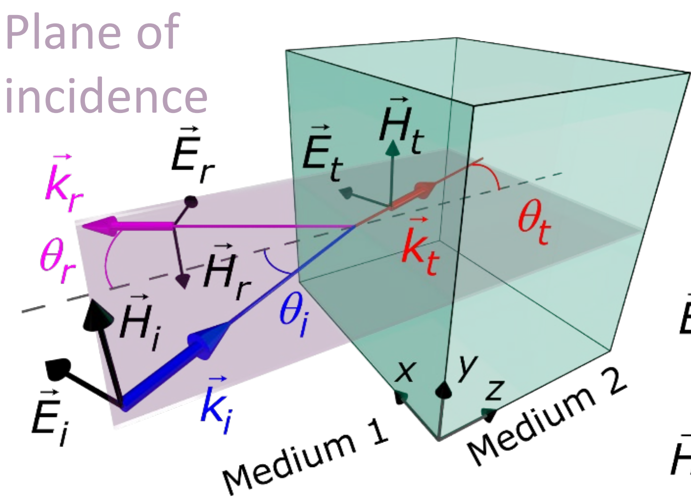
$$\left(\frac{E_{i0}}{Z_1} - \frac{E_{r0}}{Z_1}\right)\hat{y} = \frac{E_{t0}}{Z_2}\hat{y}$$

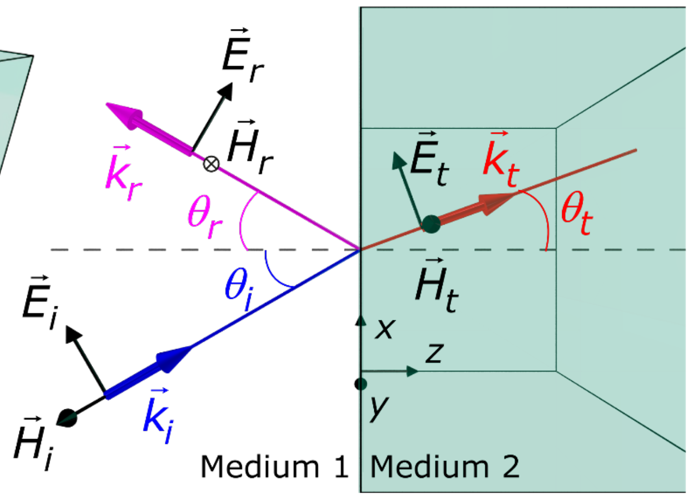
Dropping the unit vectors:

$$\boxed{E_{i0} - E_{r0} = \frac{Z_1}{Z_2}E_{t0}}$$

We add the two equations!

# General incidence at the boundary for P-polarized light




We just found:

$$\boxed{E_{i0} + E_{r0} = E_{t0}\frac{\cos\theta_t}{\cos\theta_i}}$$

Moreover:

$$\boxed{E_{i0} - E_{r0} = \frac{Z_1}{Z_2}E_{t0}}$$

Adding the two equations:

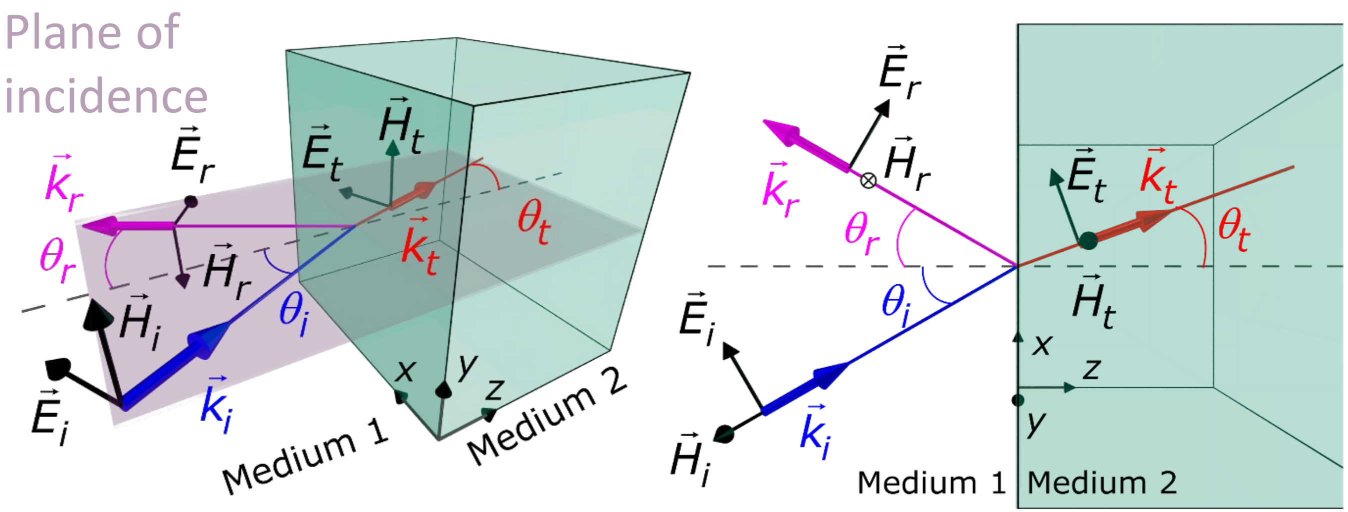
$$2E_{i0} = \left(\frac{\cos\theta_t}{\cos\theta_i} + \frac{Z_1}{Z_2}\right)E_{t0} = \left(\frac{Z_2\cos\theta_t + Z_1\cos\theta_i}{Z_2\cos\theta_i}\right)E_{t0} \rightarrow E_{t0} = \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0}$$

And replacing: $\underline{\underline{E_{i0}}} - E_{r0} = \frac{Z_1}{Z_2}E_{t0} = \left(\frac{Z_1}{Z_2}\right)\frac{2Z_2\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}E_{i0}$ [$Z_2$ cancels]

So, $E_{r0} = \left(\underline{\underline{1}} - \frac{2Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}\right)E_{i0}$

Just a few more steps!

# General incidence at the boundary for P-polarized light




We just found: $E_{r0} = \left(1 - \dfrac{2Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i}\right) E_{i0}$ and $E_{t0} = \dfrac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} E_{i0}$

So: $E_{r0} = \dfrac{Z_2 \cos\theta_t + Z_1 \cos\theta_i - 2Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} E_{i0} = \dfrac{Z_2 \cos\theta_t - Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} E_{i0}$

We can now define the **Fresnel coefficients for P-polarized light**:

$$\boxed{r_\parallel \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos\theta_t - Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i}} \qquad \boxed{t_\parallel \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i}}$$

Example question

# Example question 1

For non-magnetic materials, show that:

$$r_{\|} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \quad \text{and} \quad t_{\|} = \frac{4\sin\theta_t \cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}$$

To get you started:

# Example question 1

For non-magnetic materials, show that:

$$r_{\parallel} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \quad \text{and} \quad t_{\parallel} = \frac{4\sin\theta_t \cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}$$

Remembering that $Z = \sqrt{\frac{\mu_r}{\varepsilon_r}} Z_0$, for non magnetic materials ( $\mu_r = 1$) we have:

$Z = \frac{Z_0}{\sqrt{\varepsilon_r}} = \frac{Z_0}{n}$. Replace in Fresnel: $r_{\parallel} = \dfrac{\frac{Z_0}{n_2}\cos\theta_t - \frac{Z_0}{n_1}\cos\theta_i}{\frac{Z_0}{n_2}\cos\theta_t + \frac{Z_0}{n_1}\cos\theta_i} = \dfrac{\frac{\cos\theta_t}{n_2} - \frac{\cos\theta_i}{n_1}}{\frac{\cos\theta_t}{n_2} + \frac{\cos\theta_i}{n_1}}$

Because $\boxed{r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2\cos\theta_t - Z_1\cos\theta_i}{Z_2\cos\theta_t + Z_1\cos\theta_i}}$. Also: $n_1 \sin\theta_i = n_2 \sin\theta_t \rightarrow n_2 = \frac{\sin\theta_i}{\sin\theta_t} n_1$

We are going to need more space…

# Example question 1

For non-magnetic materials, show that:

$$r_{\parallel} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \quad \text{and} \quad t_{\parallel} = \frac{4\sin\theta_t \cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}$$

We just found that: $r_{\parallel} = \dfrac{\dfrac{\cos\theta_t}{n_2} - \dfrac{\cos\theta_i}{n_1}}{\dfrac{\cos\theta_t}{n_2} + \dfrac{\cos\theta_i}{n_1}}$ and $n_2 = \dfrac{\sin\theta_i}{\sin\theta_t} n_1$, now replacing:

$$r_{\parallel} = \frac{\dfrac{Z_0}{\dfrac{\sin\theta_i}{\sin\theta_t} n_1}\cos\theta_t - \dfrac{Z_0}{n_1}\cos\theta_i}{\dfrac{Z_0}{\dfrac{\sin\theta_i}{\sin\theta_t} n_1}\cos\theta_t + \dfrac{Z_0}{n_1}\cos\theta_i} = \frac{\dfrac{\sin\theta_t \cos\theta_t}{\sin\theta_i} - \cos\theta_i}{\dfrac{\sin\theta_t \cos\theta_t}{\sin\theta_i} + \cos\theta_i} = \frac{\sin\theta_t \cos\theta_t - \sin\theta_i \cos\theta_i}{\sin\theta_t \cos\theta_t + \sin\theta_i \cos\theta_i} =$$

$$= \frac{2\sin\theta_t \cos\theta_t - 2\sin\theta_i \cos\theta_i}{2\sin\theta_t \cos\theta_t + 2\sin\theta_i \cos\theta_i}$$

Now we use trigonometry.

# Example question 1

For non-magnetic materials, show that:

$$r_{\parallel} = \frac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i} \quad \text{and} \quad t_{\parallel} = \frac{4\sin\theta_t\cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}$$

We just found that: $r_{\parallel} = \dfrac{2\sin\theta_t\cos\theta_t - 2\sin\theta_i\cos\theta_i}{2\sin\theta_t\cos\theta_t + 2\sin\theta_i\cos\theta_i}$. We can now use the formula:

$\begin{cases} \sin(a+b) = \sin a\cos b + \cos a\sin b \\ \sin(2a) = 2\sin a\cos a \end{cases}$, and we obtain: $\boxed{r_{\parallel} = \dfrac{\sin 2\theta_t - \sin 2\theta_i}{\sin 2\theta_t + \sin 2\theta_i}}$. Next:

$$t_{\parallel} = \frac{2\dfrac{Z_0}{n_2}\cos\theta_i}{\dfrac{Z_0}{n_2}\cos\theta_t + \dfrac{Z_0}{n_1}\cos\theta_i} = \frac{\dfrac{2}{\dfrac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_i}{\dfrac{1}{\dfrac{\sin\theta_i}{\sin\theta_t}n_1}\cos\theta_t + \dfrac{1}{n_1}\cos\theta_i} = \frac{2\dfrac{\sin\theta_t\cos\theta_i}{\sin\theta_i}}{\dfrac{\sin\theta_t\cos\theta_t}{\sin\theta_i} + \cos\theta_i} =$$

$$\frac{2\sin\theta_t\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i} = \frac{4\sin\theta_t\cos\theta_i}{2\sin\theta_t\cos\theta_t + 2\sin\theta_i\cos\theta_i} \quad \text{and} \quad \boxed{t_{\parallel} = \frac{4\sin\theta_t\cos\theta_i}{\sin 2\theta_t + \sin 2\theta_i}}$$

Summary?

# Summary

At the boundary between two materials, EM wave is partially reflected and partially transmitted.

The plane of incidence is defined by the wave vector of the incident wave and a unit vector normal to the boundary.

On each side of the boundary, the electric and magnetic fields can be resolved into normal and tangential components.

For LIH materials, assuming no surface charges and no surface currents.

| | Electric fields | Magnetic fields |
|---|---|---|
| **Normal components** | $\vec{D}_{1n} = \vec{D}_{2n}$ | $\vec{B}_{1n} = \vec{B}_{2n}$ |
| **Tangential components** | $\vec{E}_{1t} = \vec{E}_{2t}$ | $\vec{H}_{1t} = \vec{H}_{2t}$ |

# Summary

At normal incidence, the reflection and transmission coefficients are:

$$r_{\parallel/\perp} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \qquad t_{\parallel/\perp} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2}{Z_1 + Z_2}$$

Snell’s law results from conservation of moment at the interface.

For a general angle of incidence, we distinguish two cases of light polarisation:

(i) in the plane of incidence (this is P-polarized light)

(ii) perpendicular to the plane of incidence (this is S-polarized light)

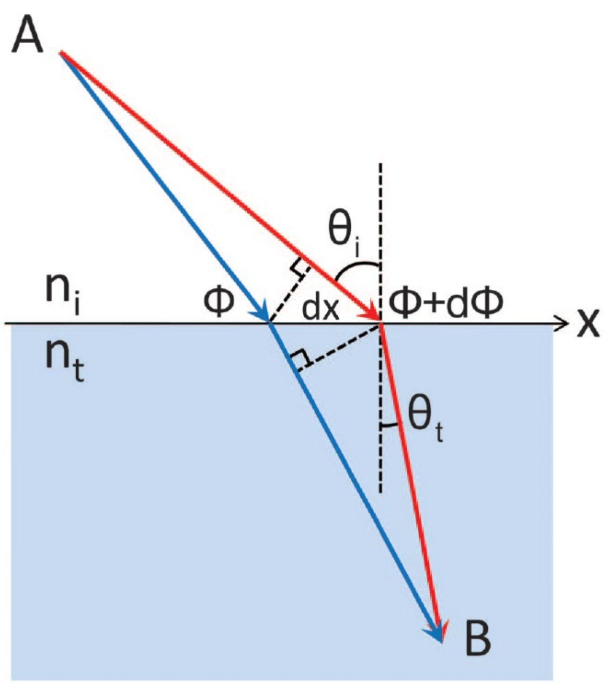
The Fresnel coefficients for P-polarized light are:

$$r_{\parallel} \equiv \frac{E_{r0}}{E_{i0}} = \frac{Z_2 \cos\theta_t - Z_1 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i} \qquad t_{\parallel} \equiv \frac{E_{t0}}{E_{i0}} = \frac{2Z_2 \cos\theta_i}{Z_2 \cos\theta_t + Z_1 \cos\theta_i}$$

Snell’s law?

# General Snell law

The introduction of an abrupt phase shift, denoted as a **phase discontinuity**, at the interface between two media allows us to revisit the laws of refraction.




$$n_t \sin\theta_t - n_i \sin\theta_i = \frac{\lambda_0}{2\pi}\frac{d\Phi}{dx}$$

$\theta_i$ : is the angle of incidence

$\theta_t$ : is the angle of refraction

$\Phi; \Phi + d\Phi$ : are, respectively, the phase discontinuities at the locations where the two paths cross the interface

$dx$ : is the distance between the crossing points

$n_i; n_t$ : are the refractive indices of the two media

$\lambda_0$ : is the wavelength in vacuum

What does such a material look like?



Yu N, Genevet P, Kats MA, Aieta F, Tetienne JP, Capasso F, Gaburro Z. Light propagation with phase discontinuities: generalized laws of reflection and refraction. Science. 2011 Oct 21;334(6054):333-7

# General Snell law

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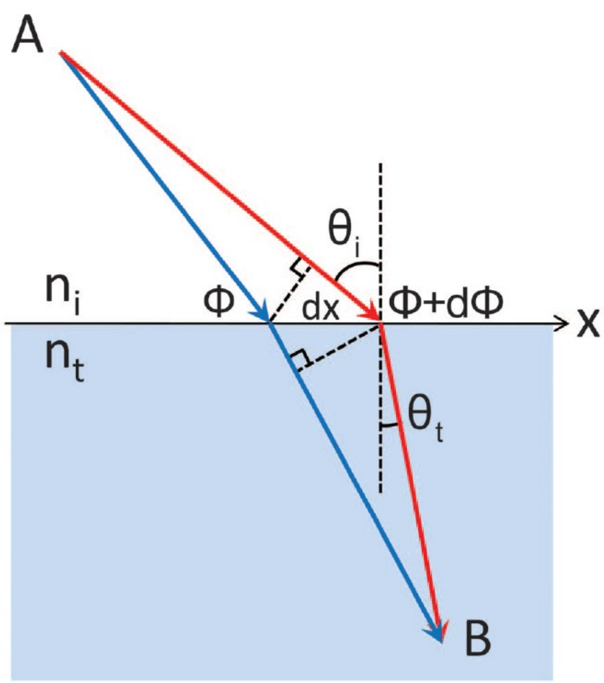
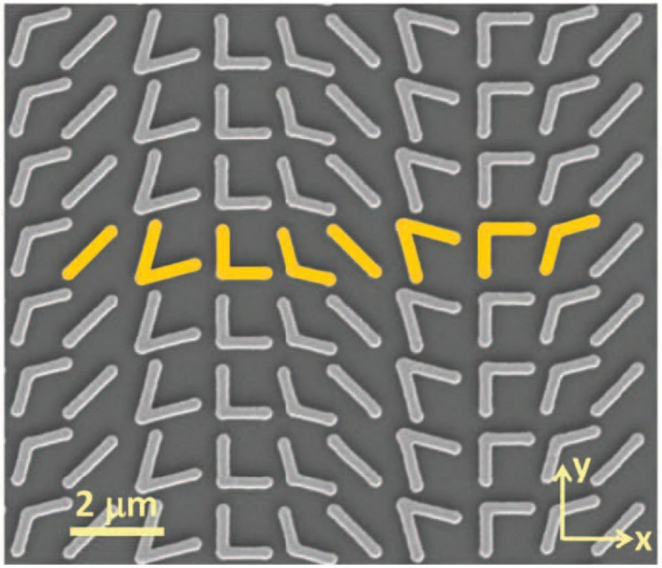



$$n_t \sin\theta_t - n_i \sin\theta_i = \frac{\lambda_0}{2\pi}\frac{d\Phi}{dx}$$




Au nanostructures on a Si substrate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