Lecture 1:

Some basics of mathematical modelling

The need for computer modelling & simulation

Analytical methods are very powerful, but they cannot begin to cover all phenomena we wish to understand.

There is a need to deal with

- Equations with no analytical solution
- Manageable equations with difficult auxiliary conditions
- Situations where modelling via an equation is inappropriate

Analytic approximation techniques (perturbation, variation, separation of scales…) sometimes help. More often we will need to find numerical solutions.

This unit:
an introduction to some techniques of computational modelling

Setting up the model

- Models based on first principles (*"ab initio calculations"*)

  e.g. Newton's laws, Maxwell equations

  - The governing equations are well-known
  - Often resource-demanding (many degrees of freedom, large system sizes)

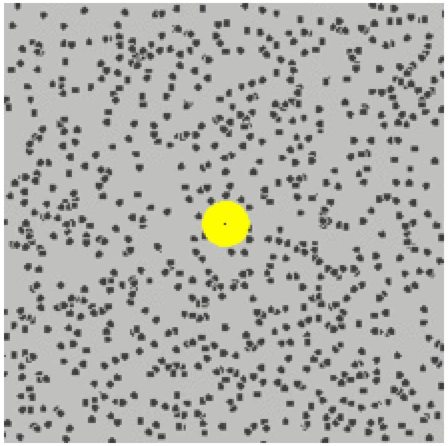
- Reduced problem-specific models

  - derived from first-principles under certain approximations

  - Empirical or semi-empirical

  - Easy to handle, require minimum resources
  - Often linked to well-studied systems/equations
  - Require careful considerations of validity of approximations, empirical terms

Setting up the model



An example: *Brownian motion*

'ab initio':

$$m_1 \frac{d^2\vec{r}_{\mathbf{1}}}{dt^2} = \sum_j \vec{F}_{1j}$$

$$m_2 \frac{d^2\vec{r}_{\mathbf{2}}}{dt^2} = \sum_j \vec{F}_{2j}$$

$$m_3 \frac{d^2\vec{r}_{\mathbf{3}}}{dt^2} = \sum_j \vec{F}_{3j}$$

...

Langevin equation:

$$m \frac{d^2\vec{r}}{dt^2} = -\gamma\vec{v} + \vec{S}(t)$$

Empirical terms:
were not derived, but simply postulated

Another example: supercontinuum generation in an optical fibre

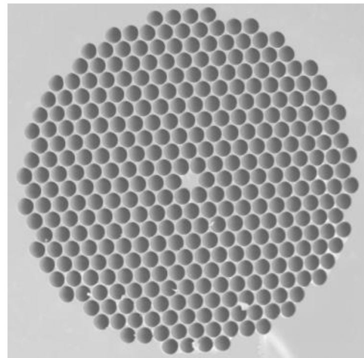
'ab initio':

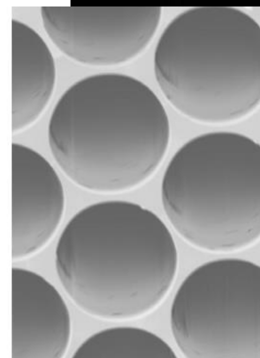
$$\nabla \times \vec{E} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{\boldsymbol{D}}}{\partial t}$$

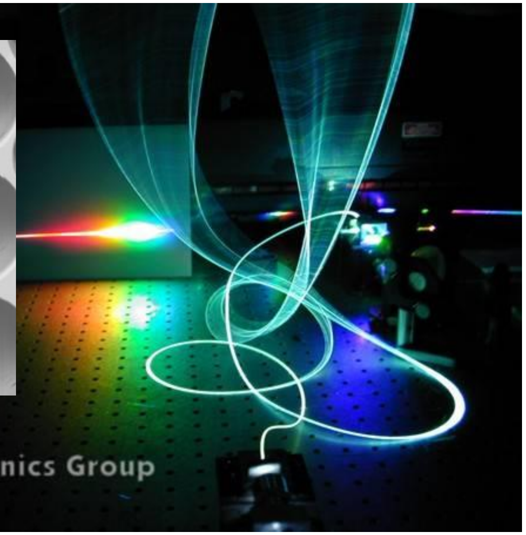
$$\vec{D} = \epsilon_0 \boxed{\epsilon(\vec{r})} \vec{E} + \overrightarrow{P_{nl}}(\vec{E})$$

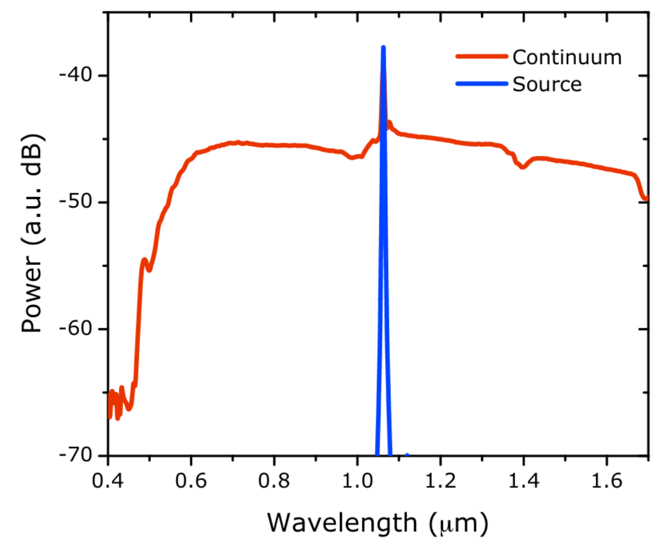
$$\vec{B} = \mu_0 \vec{H}$$





complex fibre geometry





Reduced model (derived from Maxwell's eqs):

$$i\frac{\partial \Psi}{\partial z} = -\boxed{\frac{\beta_2}{2}}\frac{\partial^2 \Psi}{\partial t^2} - \boxed{\frac{\beta_3}{6}}\frac{\partial^3 \Psi}{\partial t^3} + \boxed{\gamma}|\Psi|^2\Psi$$

Generalized Nonlinear Schrödinger equation

https://en.wikipedia.org/wiki/Supercontinuum

The Major Equations of Science

- often a useful starting point.

Broadly, can organise in terms of time dependence:

Time:	none	$\frac{\partial}{\partial t}$	$\frac{\partial^2}{\partial t^2}$
Single object or variable	Algebraic or transcendental equation	ODE(1)	ODE(2)
Coupled objects or variables	Simultaneous A/T Eqns	Simultaneous ODE(1)	Simultaneous ODE(2)
Continuum or field	Elliptic PDE(2)	Parabolic PDE(2)	Hyperbolic PDE(2)

Most major (linear) eqns of science; all contain ∇^2 …

+ exceptions !

Some well-known equations

Elliptic:

$$\nabla^2\Phi = 0$$ Laplace

$$\nabla^2\Phi = f(\underline{r})$$ Poisson

$$(\nabla^2 + k)\Phi = 0$$ Helmholtz

Parabolic:

$$\nabla^2\Phi + g(\underline{r},t) = \frac{1}{D}\frac{\partial\Phi}{\partial t}$$ Diffusion/Heat

$$-\frac{\hbar^2}{2m}\nabla^2\Phi + V(\underline{r},t)\Phi = i\hbar\frac{\partial\Phi}{\partial t}$$ Schrödinger

Hyperbolic:

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$$ Wave

A lot is known about these, so use them to guide you.

eg: if you see terms in $\partial^2 u / \partial t^2$ and u , expect oscillation (or exponential decay).

Linear vs Nonlinear equations

$$i\frac{\partial\Psi}{\partial z} = -\frac{\beta_2}{2}\frac{\partial^2\Psi}{\partial t^2} - \frac{\beta_3}{6}\frac{\partial^3\Psi}{\partial t^3} + \gamma|\Psi|^2\Psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\Phi + V(\underline{r},t)\Phi = i\hbar\frac{\partial\Phi}{\partial t}$$ Schrödinger

+ a nonlinear term

+ higher order derivative(s)

Check: substitute $\psi \to A\psi$, *equation becomes different!*

Generalized Nonlinear Schrödinger equation

Nonlinear equations often require special methods in computational physics!

- **Establishing connections** with the known in literature models is important for gaining insight into properties of the system and confidence in your modelling tools

- Use freedom in choosing values of model parameter/setting to zero some as the limiting case

- May be useful to introduce artificial parameters (equation terms) to establish a “bridge” between your model and a well-studied system

Another example: "bridging" two systems

- System to study:

$$\Phi^2 \cdot \nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

- Known in literature system (wave equation):

$$\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

- Modified system

$$(\mathbf{1} - \boldsymbol{\epsilon} + \boldsymbol{\epsilon}\Phi^2) \cdot \nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

$\epsilon = 0$ - standard wave equation

$\epsilon = 1$ - your system

Auxiliary conditions (ACs)

- No time dependence + at least one spatial coordinate

"Boundary Value Problems"(BVP):

(i) Dirichlet condition: specify $\Phi(s)$ on boundary

(ii) Neumann condition: specify $\left.\dfrac{\partial \Phi}{\partial n}\right|_s$

(iii) Mixed condition (Robin condition): specify $\Phi(s) + a\left.\dfrac{\partial \Phi}{\partial n}\right|_s$

(iv) Periodic boundary condition: $\Phi(\vec{r} + \overrightarrow{K_j}) = \Phi(\vec{r})$

Auxiliary conditions (ACs)

- First-order time derivative, constant coefficients, stationary problem

$$\Phi(t) \sim \exp(-i\omega t)$$

$$\frac{\partial}{\partial t} \rightarrow (-i\omega)$$

Eigen-value problem $\lambda\Phi = \hat{L}\Phi$

Example: Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\vec{\mathrm{r}})\Psi$$

$$\Psi(t) \sim \exp\left(-\frac{iE}{\hbar}t\right) \quad \Rightarrow \quad E\Psi = \left(-\frac{\hbar^2}{2m}\nabla^2\Psi + V(\vec{\mathrm{r}})\right)\Psi$$

Auxiliary conditions (ACs)

- Time-dependent, "Initial Value Problem" (IVP)

First-order, $\frac{\partial}{\partial t}$: specify Φ at $t = t_0$

Second-order, $\frac{\partial^2}{\partial t^2}$: specify Φ and $\frac{\partial \Phi}{\partial t}$ at $t = t_0$

Higher-orders, $\frac{\partial^m}{\partial t^m}$: specify $\Phi, \frac{\partial \Phi}{\partial t}, \ldots \frac{\partial^{m-1}\Phi}{\partial t^{m-1}}$ at $t = t_0$

- Correct ACs are vital in any mathematical or computational model

- When dealing with PDEs, often a combination of BVP (for spatial coordinates) and IVP (for time) is required

- Auxiliary conditions are set:
  - Explicitly (while setting up the model)
  - Implicitly (via using specific numerical techniques such as FFT)

Summary:

- **First-principles** vs **reduced** (approximate) models

  - Consider required resources, limitations of the model

- Establishing connections with known in literature models

  - Use freedom in setting parameter values

- Auxiliary conditions:

  - Boundary Value Problem
  - Eigenvalue Problem
  - Initial Value Problem

Lecture 2:

De-dimensionalization

(simplifying your model)

Dealing with physical units

- Each physical quantity has units (dimensions). Your starting model is likely to have variables and parameters measured in physical units.

Example (ODE): A train moving in x-direction on level ground.

$$m\frac{d^2x}{dt^2} = -A\frac{dx}{dt} - B\left(\frac{dx}{dt}\right)^2 + T$$

mass*acceleration   rolling friction   air drag   tractive force

(semi-empirical model. Can you identify empirical terms?)

Dealing with physical units

$$m\frac{d^2x}{dt^2} = -A\frac{dx}{dt} - B\left(\frac{dx}{dt}\right)^2 + T$$

- **Each term in this equation must have the same units** of $\mathrm{kg{\cdot}m{\cdot}s}^{-2} = \mathrm{N}$
- Have a look at each variable and parameter separately:

$\boldsymbol{m}$ - Mass in kg

$\boldsymbol{x}$ - Position in m

$\boldsymbol{t}$ - Time in s

$\boldsymbol{T}$ - Force in N

$\boldsymbol{\frac{dx}{dt}}$ - First derivative (velocity) in …

$\boldsymbol{\frac{d^2x}{dt^2}}$ - Second derivative (acceleration) in …

$\boldsymbol{A}$ - Rolling friction coefficient in …

$\boldsymbol{B}$ - Air drag coefficient in …

De-dimensionalisation (non-dimensionalisation)

- Systematically reduce number of parameters by making each term in equation dimensionless
- Also makes generic behaviour easier to spot.

$$m\frac{d^2x}{dt^2} = -A\frac{dx}{dt} - B\left(\frac{dx}{dt}\right)^2 + T$$

Note: only derivatives of x appear in this equation, so simplify: $v = \dot{x}$, and

$$m\frac{dv}{dt} + Av + Bv^2 = T \qquad (1)$$

$$m\frac{dv}{dt} + Av + Bv^2 = T \qquad ①$$

Goal is $v(t)$: t is the independent VARIABLE
v is the dependent VARIABLE

T, m, A, B are PARAMETERS

1. Define dimensionless versions of the variables (we have 2).

$$v = D_1 u \qquad t = D_2 \tau$$

Here τ is dimensionless time, so D_2 is a time. Treat it as a timescale that we are free to choose when it suits us.

Similarly, u is dimensionless velocity, so D_1 is a velocity

NOTE: You only need to do it for variables which have dimensions! E.g. if the original equation has variable angle ϕ (measured in rads – i.e. dimensionless!) you should not touch it. Introducing another dimensionless angle would not help you in any way!

$$m\frac{dv}{dt}+Av+Bv^2=T \qquad ①$$

2. Substitute into ①

$$m\frac{D_1}{D_2}\frac{du}{d\tau}+AD_1u+B(D_1)^2u^2=T$$

3. Make the equation dimensionless. There are choices! eg

$$\div\, m\frac{D_1}{D_2} \quad \rightarrow \quad \frac{du}{d\tau}+\frac{AD_2}{m}u+\frac{BD_1D_2}{m}u^2=\frac{TD_2}{D_1m}. \qquad ②$$

this term is dimensionless, so all are.

Moreover, since u is dimensionless (just a number), so are all the "dimensionless products" $\frac{AD_2}{m}$, $\frac{BD_1D_2}{m}$ & $\frac{TD_2}{D_1m}$. Now choose D_1 & D_2 to make ② as simple as possible…

$$\frac{du}{d\tau} + \frac{AD_2}{m}u + \frac{BD_1D_2}{m}u^2 = \frac{TD_2}{D_1 m}. \qquad ②$$

4. Fix scales – usually some choice…

eg choose $\frac{AD_2}{m} = 1$; this sets timescale $D_2 = \frac{m}{A}$.

then choose $\frac{TD_2}{D_1 m} = 1$; this sets speed scale $D_1 = \frac{T}{A}$.

We've made all our choices, so any remaining products are fixed:

Here, $\frac{BD_1D_2}{m} = \frac{BT}{A^2} \equiv \boldsymbol{b}$ is the single parameter left.

Our (1-parameter) governing equation is now

$$\frac{du}{d\tau} + u + bu^2 = 1$$

- a dimensionless equation with a dimensionless parameter b

$$\frac{du}{d\tau} + u + bu^2 = 1$$

5. Consider dimensionless product(s)

$$\frac{BT}{A^2} \equiv \boldsymbol{b}$$

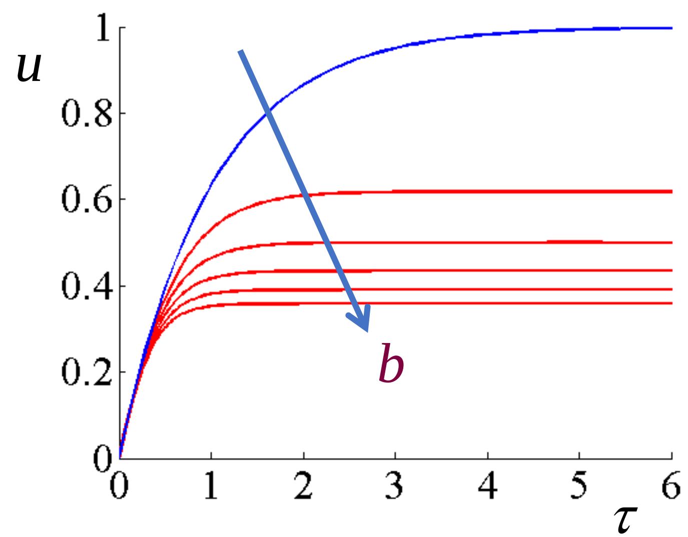
- They reveal generic behaviour

  eg if $A \to A/2$ only need $T/4$ tractive force for same b
- If $b \ll 1$ may be able to neglect bu^2 term.
- Parameter reduction => easier to explore all solutions

6. Solve dimensionless equation

eg with

$u(0) = 0$



terminal speeds

$u_T(b)$

Note: u_T can be obtained analytically!

7. Take care to convert back to $v(t)$ for interpretation

$$v = D_1 u, \quad t = D_2 \tau$$

$D_1 = \frac{T}{A}$ - units of speed

$D_2 = \frac{m}{A}$ - units of time

$$v_T = D_1 u_T = \frac{T}{A} u_T$$

$$\frac{BT}{A^2} \equiv \boldsymbol{b}$$

eg keep A & B the same; double T

$b \to 2b$ so u_T DEcreases.

(u_T is decreasing for larger b – check the plot on the previous slide!)

seems odd, but real terminal speed $v_T = \frac{T}{A} u_T$ Increases!

De-dimensionalisation of higher-order derivatives:

Take $v = D_1 u, \quad t = D_2 \tau$ & compute $\dfrac{d^2 v}{dt^2}$:

$$\frac{d^2 v}{dt^2} = \frac{d^2 (D_1 u)}{dt^2}$$

$$= D_1 \frac{d}{dt}\left(\frac{du}{d\tau} \frac{d\tau}{dt} \right)$$

$$\frac{1}{D_2}$$

$$= \frac{D_1}{D_2} \frac{d^2 u}{d\tau^2} \frac{d\tau}{dt}$$

$$= \frac{D_1}{\left(D_2\right)^2} \frac{d^2 u}{d\tau^2},$$

& in general $\dfrac{d^n v}{dt^n} = \dfrac{D_1}{\left(D_2\right)^n} \dfrac{d^n u}{d\tau^n}$

Alternative method of de-dimensionalisation:

Choose scales (suitable for the problem) at the outset.

So for our train model, perhaps $v = Vu, \quad t = \theta\tau$

with $V = 125\text{mph}$ and $\theta = 1\text{h}$.

We cannot now minimise the number of dimensionless products

BUT we can ensure our dimensionless variables are $O(1)$

- useful when computing.

Summary:

- De-dimensionalising your model is useful for:

  - reducing the number of parameters;

  - identifying generic behaviour;

  - working with convenient variables of the order of $O(1)$.

- Remember to convert back to physical units for correct interpretation of your results!

Lecture 3:

Some further examples

An example: an optical pulse propagating in a fibre

$$i\frac{\partial \mathrm{A}}{\partial z} = -\frac{\beta_2}{2}\frac{\partial^2 \mathrm{A}}{\partial t^2} + \gamma|\mathrm{A}|^2\mathrm{A}$$

z - Length along the fibre (in m)

$\boldsymbol{t}$ - time (in s)

$\Big\}$ **Independent variables**

$\mathbf{A(t,z)}$ - Amplitude of the pulse (**in ???**) **Dependent variable (field)**

$\boldsymbol{\beta_2} = \mathbf{2}\ \mathrm{ps^2 m^{-1}}$ - Dispersion parameter of the fibre

$\boldsymbol{\gamma} = \mathbf{0.01}\ \mathrm{W^{-1} m^{-1}}$ - Nonlinear parameter of the fibre

$\Big\}$ **Parameters**

An example: an optical pulse propagating in a fibre

$$i\frac{\partial \mathrm{A}}{\partial z} = -\frac{\beta_2}{2}\frac{\partial^2 \mathrm{A}}{\partial t^2} + \gamma|\mathrm{A}|^2\mathrm{A}$$

Geometry Parameters

$$\boldsymbol{\beta_2} = \mathbf{2}\ \mathrm{ps}^2\mathrm{m}^{-1}$$

$$\boldsymbol{\gamma} = \mathbf{0.01}\ \mathrm{W}^{-1}\mathrm{m}^{-1}$$

Input Parameters

- Fibre length
- Pulse duration
- Pulse peak amplitude

Potentially, a large parameter space to explore!

Reduce parameters by de-dimensionalization

$$i\frac{\partial \mathrm{A}}{\partial z} = -\frac{\beta_2}{2}\frac{\partial^2 \mathrm{A}}{\partial t^2} + \gamma|\mathrm{A}|^2\mathrm{A}$$

Introduce dimensionless variables

$$z = L\zeta \qquad t = T\tau \qquad A = U\psi$$

$$\Rightarrow \quad i\frac{U}{L}\frac{\partial \psi}{\partial \zeta} = -\frac{1}{2}\frac{\beta_2 U}{T^2}\frac{\partial^2 \psi}{\partial \tau^2} + \gamma U^3|\psi|^2\psi$$

$$\Rightarrow \quad i\frac{\partial \psi}{\partial \zeta} = -\frac{1}{2}\frac{\beta_2 L}{T^2}\frac{\partial^2 \psi}{\partial \tau^2} + \gamma L U^2|\psi|^2\psi$$

Reduce parameters by de-dimensionalization

$$i\frac{\partial\psi}{\partial\zeta} = -\frac{1}{2}\frac{\beta_2 L}{T^2}\frac{\partial^2\psi}{\partial\tau^2} + \gamma L U^2|\psi|^2\psi$$

<u>Can select L, T, and U to set all coefficients to 1</u>

E.g.: 1) fix L to be the length of the fibre (such that $0 < \zeta < 1$)

2) Set $\frac{|\beta_2|L}{T^2} = 1 \quad \Rightarrow \quad T = \sqrt{|\beta_2|L}$

3) Set $|\gamma|LU^2 = 1 \quad \Rightarrow \quad U = \sqrt{1/(|\gamma|L)}$

OR: 1) fix T to be the input pulse width (such that τ is O(1))

2) Set $\frac{|\beta_2|L}{T^2} = 1 \quad \Rightarrow \quad L = T^2/|\beta_2|$

3) Set $|\gamma|LU^2 = 1 \quad \Rightarrow U = T\sqrt{|\beta_2/\gamma|}$

$$i\frac{\partial\psi}{\partial\zeta} = -\frac{\mathrm{sgn}(\beta_2)}{2}\frac{\partial^2\psi}{\partial\tau^2} + \mathrm{sgn}(\gamma)\,|\psi|^2\psi$$

Note some important tendencies

$$\frac{|\beta_2|L}{T^2} = 1 \qquad |\gamma|LU^2 = 1$$

- Changing fibre dispersion $|\beta_2| \rightarrow 2|\beta_2|$ is equivalent to:

  - (whilst keeping the pulse duration fixed) reducing the fibre length L by half and increasing pulse peak amplitude by $\sqrt{2}$

or

  - (whilst keeping the fibre length fixed) stretching the pulse duration by $\sqrt{2}$

- Changing fibre nonlinearity $|\gamma| \rightarrow 2|\gamma|$ is equivalent to reducing the input pulse amplitude by $\sqrt{2}$

Theory – Experiment Phrasebook

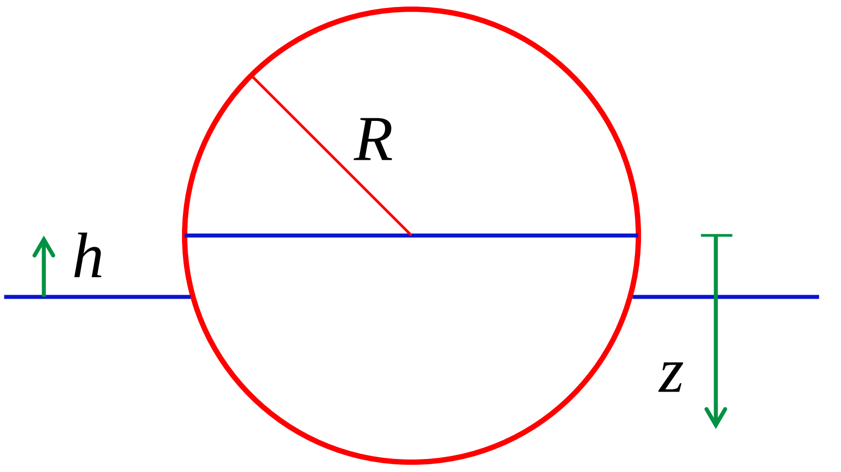
"Set $T = 100fs = 0.1ps$;
Set input pulse duration =1;
Set $L = \frac{T^2}{|\beta_2|} = \frac{0.1^2 ps^2}{1\, ps^2/m} = 0.01m$;
Set propagation length $\xi_m = \frac{1m}{L} = 100$;
Set $U^2 = T^2 \left|\frac{\beta_2}{\gamma}\right| = 0.1^2 \left|\frac{1}{0.01}\right| = 1W$
Set pulse amplitude $\Psi_0 = \sqrt{\frac{1kW}{U^2}} = \sqrt{10^3}$ *"*
(note: amplitude^2 gives power)

"My fibre has $\beta_2 = 1\frac{ps^2}{m}$, $\gamma = 0.01\ (Wm)^{-1}$*, and it is 1m long. My input pulse is* $100\ fs$ *long with a peak power of 1kW. What will I observe at the output?"*

" I observe some interesting effects when I set pulse amplitude $\Psi_0 = 100$ *and propagate over distance* $\xi_m = 10$*"*

"You will observe some interesting effects if you set pulse peak power $P_0 = 100^2 \cdot T^2 \left|\frac{\beta_2}{\gamma}\right|$ *and use fibre length* $L = 10 \cdot T^2/|\beta_2|$*, where* β_2 *and* γ *are your fibre parameters, and* T *is the duration of your input pulse"*

Case study: model the motion of a buoy half-filled with water



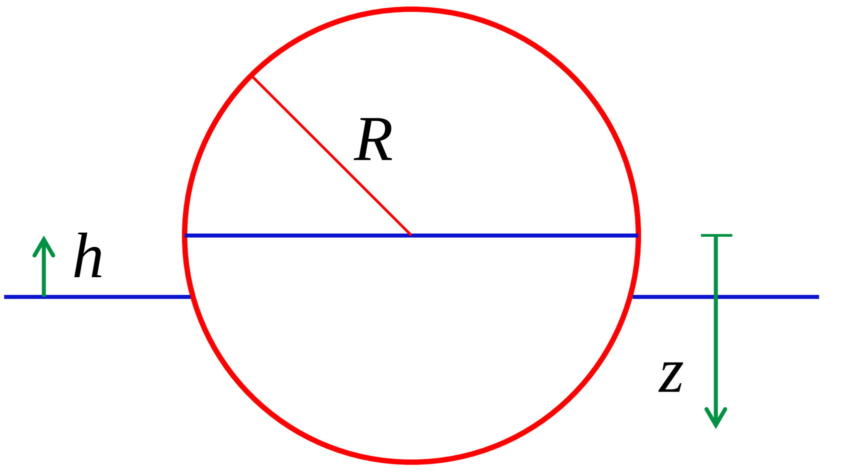
- "Ab initio" modelling must involve a combination of Newton's laws (for the buoy) and hydrodynamics - very resource demanding and complicated!

- Focus on vertical oscillations of the buoy and derive a simple model

- Expect damped oscillations – hence an oscillator-type model ($d^2x/dt^2 = -\omega_0^2 x + damping$)

Case study: model the motion of a buoy half-filled with water

- Forcing term: water waves? wind? Assume not
- Consider vertical motion only
- 2nd law of Newton:



$$m\frac{d^2h}{dt^2} = -mg + \rho g V_{disp} + drag\ forces$$

gravity

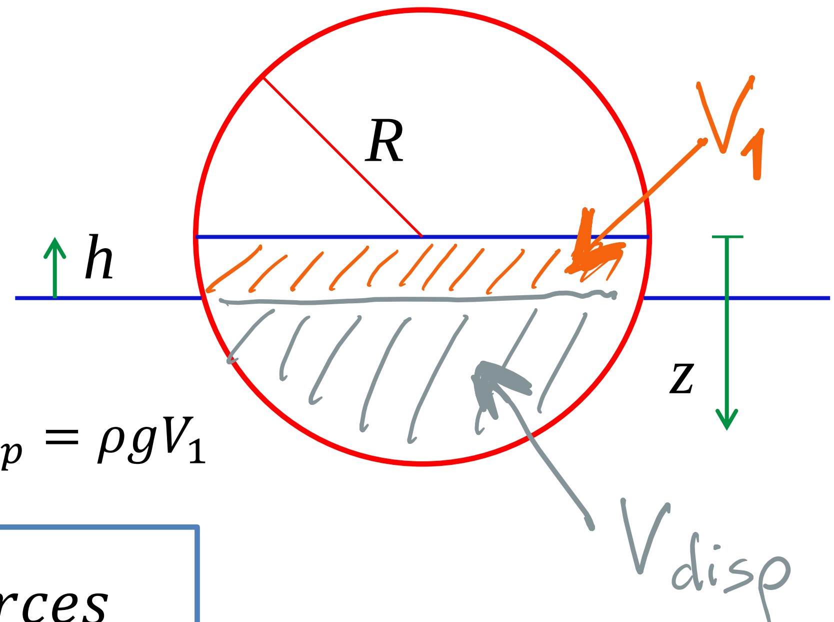
Restoring force
(=weight of displaced water)

$$m\frac{d^2h}{dt^2} = -mg + \rho g V_{disp} + drag\ forces$$

Neglect plastic shell, assume water only

$$\rightarrow m = \rho \frac{V_{sphere}}{2}$$

$$mg - \rho g V_{disp} = \rho \frac{V_{sphere}}{2} g - \rho g V_{disp} = \rho g V_1$$



$$\boxed{m\frac{d^2h}{dt^2} = -\rho g V_1 + drag\ forces}$$

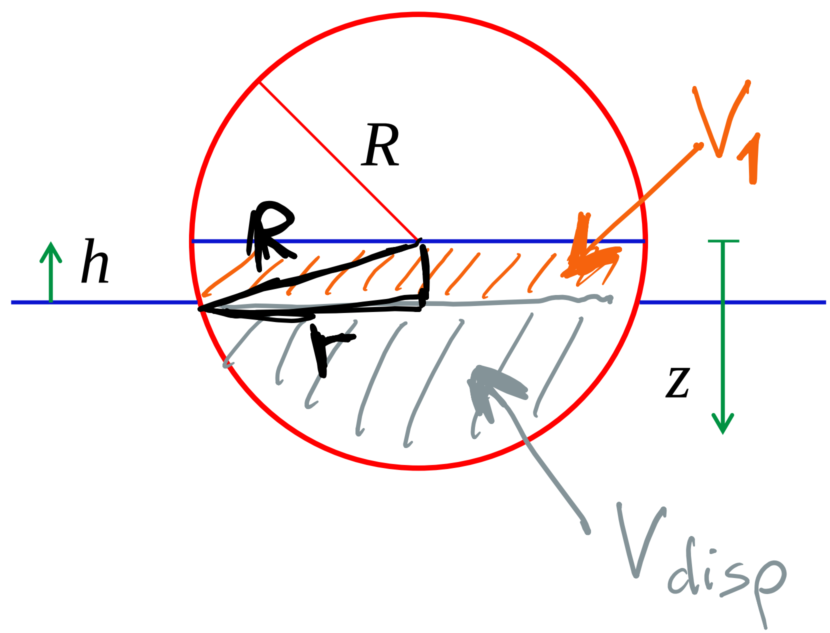
- Expect an oscillator-type model ($d^2x/dt^2 = -\omega_0^2 x + damping$)

=> Need to express V_1 (and drag forces) via h

$$\boxed{m\frac{d^2h}{dt^2} = -\rho g V_1 + drag\ forces}$$

$$V_1 = \int_0^h \pi r^2 dz$$

$$= \pi\int_0^h (R^2 - z^2)dz = \pi\left(R^2 h - \frac{1}{3}h^3\right)$$



$$\boxed{m\frac{d^2h}{dt^2} = -\rho g\pi\left(R^2h - \frac{1}{3}h^3\right) + drag\ forces}$$

- Damping (energy loss) forces

Relevant science is fluid dynamics.
First-principles forces difficult. Semi-empirically, expect

"viscous drag" $\propto$ speed $= \dfrac{dh}{dt}$ (eg Stokes: $F = 6\pi\eta vr$)

"inertial drag" $\propto \left(\dfrac{dh}{dt}\right)^2$ (eg cars: $F = \frac{1}{2}Cv^2A$)

- Damping via wave generation? Ignore.

Warning: careful with empirical terms!

- Drag forces should oppose motion

- ”viscous drag” ~ speed $= \frac{dh}{dt}$

$$m\frac{d^2h}{dt^2} = -\alpha\frac{dh}{dt}$$

negative for positive velocity $\left(\frac{dh}{dt} > 0\right)$
and positive for negative velocity $\left(\frac{dh}{dt} < 0\right)$ ✓

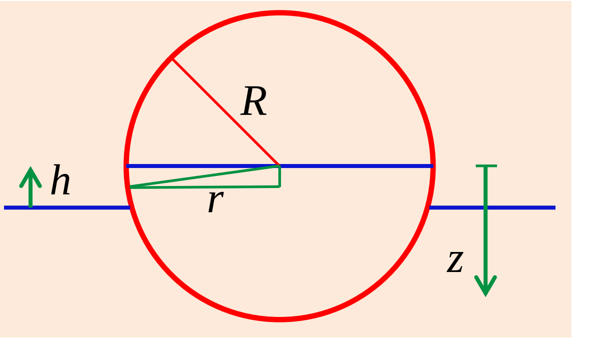
- ”inertial drag” ~ speed^2 $= \left(\frac{dh}{dt}\right)^2$

$$m\frac{d^2h}{dt^2} = -\beta\left(\frac{dh}{dt}\right)^2$$

negative for positive velocity $\left(\frac{dh}{dt} > 0\right)$
negative for negative velocity $\left(\frac{dh}{dt} < 0\right)$ ✗

Fix: use $v|v|$ instead of v^2!

$$m\frac{d^2h}{dt^2} = -\beta\left|\frac{dh}{dt}\right|\frac{dh}{dt}$$

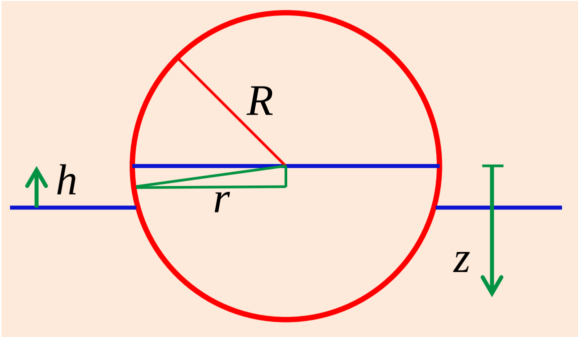


Our governing ODE is

$$\frac{2}{3}\pi R^3\rho\frac{d^2h}{dt^2}+\alpha\frac{dh}{dt}+\beta\left|\frac{dh}{dt}\right|\frac{dh}{dt}+\rho g\pi R^2 h-\frac{1}{3}\rho g\pi h^3=0$$

- A damped oscillator equation
- … with some nonlinear terms

Let’s make it look neater now!



Our governing ODE is

$$\frac{2}{3}\pi R^3\rho\frac{d^2h}{dt^2}+\alpha\frac{dh}{dt}+\beta\left|\frac{dh}{dt}\right|\frac{dh}{dt}+\rho g\pi R^2 h-\frac{1}{3}\rho g\pi h^3=0$$

Variables: h, t

Parameters: $R, \rho, \alpha, \beta \quad (g)$

De-dimensionalise: Let $h = Hy, \quad t = Tx$, so

$$\frac{2}{3}\pi R^3\rho\frac{H}{T^2}\frac{d^2y}{dx^2}+\alpha\frac{H}{T}\frac{dy}{dx}+\beta\frac{H^2}{T^2}\left|\frac{dy}{dx}\right|\frac{dy}{dx}+\rho g\pi R^2 Hy-\frac{1}{3}\rho g\pi H^3y^3=0$$

$$\frac{d^2y}{dx^2}+\frac{3\alpha T}{2\pi R^3\rho}\frac{dy}{dx}+\frac{3\beta H}{2\pi R^3\rho}\left|\frac{dy}{dx}\right|\frac{dy}{dx}+\frac{3gT^2}{2R}y-\frac{gH^2T^2}{2R^3}y^3=0$$

$$\frac{d^2y}{dx^2}+\frac{3\alpha T}{2\pi R^3\rho}\frac{dy}{dx}+\frac{3\beta H}{2\pi R^3\rho}\left|\frac{dy}{dx}\right|\frac{dy}{dx}+\frac{3gT^2}{2R}y-\frac{gH^2T^2}{2R^3}y^3=0$$

Now simplify by choice of scales (not free choice this time).

eg: leave dimensionless products in damping terms by setting

$$\frac{3gT^2}{2R}=1 \quad \rightarrow \text{ timescale } \quad T=\sqrt{\frac{2R}{3g}} \quad \text{, then set}$$

$$\frac{gH^2T^2}{2R^3}=1 \quad \rightarrow \text{ length scale } \quad H=\sqrt{\frac{2R^3}{g}\frac{3g}{2R}}=\sqrt{3}R$$

Dimensionless equation is then

$$\boxed{\frac{d^2y}{dx^2}+a\frac{dy}{dx}+b\left|\frac{dy}{dx}\right|\frac{dy}{dx}+y-y^3=0}$$

with dimensionless products $a=\dfrac{3\alpha\sqrt{2R}}{2\pi R^3\rho\sqrt{3g}}$, $\qquad b=\dfrac{3\sqrt{3}\beta}{2\pi R^2\rho}$.

Solving the ODE

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + b\left|\frac{dy}{dx}\right|\frac{dy}{dx} + y - y^3 = 0$$

- cannot by done analytically

However, as a useful first step, solve those parts that can:

If $a \ll 1,\ b \ll 1$ & $y \ll 1\ \rightarrow\ y^3 \ll y$ so ignore y^3,

approx equation is $\frac{d^2y}{dx^2} + y = 0$ with solution $y = P\cos x + Q\sin x$

Should then expect oscillation at angular frequency

1 (dimensionless) or T^{-1} rad/s as $t = Tx$.

$$T = \sqrt{\frac{2R}{3g}}$$

For general values of a, b, y we need to solve the equation numerically

Summary:

- Considered two examples

- De-dimensionalizing helps to simplify the model and understand some important properties (before we set off with the numerical studies)