3. Integration of Scalar and Vector Fields

Recall the definite integral of a function of 1 variable

$$I = \int_a^b f(x)dx$$

This expression comes from finding the area of rectangles of width δx and height $f(x)$ in the limit $\delta x \to 0$.

dx is infinitesimal piece of the x-axis.

You already know lots of rules for evaluating definite integrals…

eg: $$\int_a^b x\,dx = \left[\tfrac{1}{2}x^2\right]_a^b = \tfrac{1}{2}(b^2 - a^2)$$

We will use these rules to help us evaluate 3 other types of integral, useful when the integrand is a scalar field $\phi(\mathbf{r})$ or a vector field $\mathbf{a}(\mathbf{r})$.

Generic name	Where we integrate	element
Line integral	Along a line or curve	dr, $d\mathbf{r}$
Surface integral	Over a surface	dS, $d\mathbf{S}$
Volume integral	Over a volume	dV

For each type of integral we will consider

1. How is the integral constructed?
2. Why are such integrals useful to a scientist?
3. How do we evaluate them?

You then get a chance to practise number 3!

3.1 Tangential line integrals

3.1.1 Construction

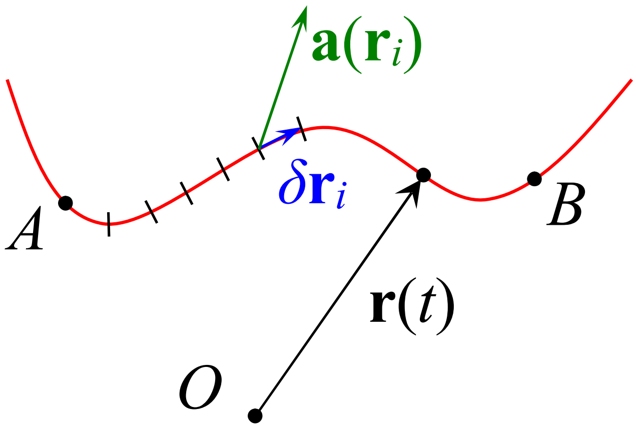
Consider a **curve** $\mathbf{r}(t)$ through a region of space which contains a **vector field** $\mathbf{a}(\mathbf{r})$.

Break $\mathbf{r}(t)$ into small tangential segments

$$\delta\mathbf{r}_1, \quad \delta\mathbf{r}_2, \quad \delta\mathbf{r}_3, \quad \ldots \quad \delta\mathbf{r}_i, \quad \ldots \quad \delta\mathbf{r}_N$$

Magnitude of $\delta\mathbf{r}_i$ is δs_i; direction is along local tangent to the curve.

For each segment we can calculate $\mathbf{a}(\mathbf{r}_i)\cdot\delta\mathbf{r}_i$



We add up all the dot products from each of the N segments between points A and B:

$$\sum_{i=1}^{N}\mathbf{a}(\mathbf{r}_i)\cdot\delta\mathbf{r}_i$$

The **tangential line integral** of $\mathbf{a}(\mathbf{r})$ between A and B on the curve is this sum, in the limit that all the $\delta s_i \to 0$.

So tangential line integral $= \int_A^B \mathbf{a}(\mathbf{r})\cdot d\mathbf{r}$

If line is expressed as curve C, we write $\int_C \mathbf{a}(\mathbf{r})\cdot d\mathbf{r}$

If C is a closed loop, we write $\oint_C \mathbf{a}(\mathbf{r})\cdot d\mathbf{r}$

3.1.2. Why do tangential line integrals arise?

Suppose we want to know the “work done” W in moving an object along a curve C from A to B. We know that

Work done = force x distance

1. C is a straight line, constant force acts along line.

2. As 1., but constant force and line NOT aligned

3. Force is general $\mathbf{F}(\mathbf{r})$ and C an arbitrary curve

This is general case. For infinitesimal segment, $\mathbf{F}(\mathbf{r})$ is constant, $d\mathbf{r}$ is straight, so $dW = \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$.

Adding up all the bits of work done gives

$$W = \int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$

In this unit we will evaluate fairly simple tangential line integrals.

Note, though, that the work done moving an object from A to B can **always** be written $W = \int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$, whether or not we know how to do the integral.

This can be very useful.

3.1.3 Evaluation of tangential line integrals

First do the dot product.

Write $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$, so

$$\int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_A^B \left[F_x(x,y,z)dx + F_y(\mathbf{r})dy + F_z(\mathbf{r})dz \right].$$

KEY POINT:

Do **NOT** now integrate the first part of this from x_A to x_B etc. Instead, we must integrate along the curve between A and B. To do this, we **parameterise** the curve, linking all the terms in the integral to the single parameter (see section 1.3).

Examples

A particle moves through a (2d) force field $\mathbf{F} = xy\mathbf{i} - y^2\mathbf{j}$. Calculate the work done in moving from $A = (0,0)$ to $B = (2,1)$

(i) Along a straight line

(ii) Along the curve $y = \frac{1}{4}x^2$

Work

$$W = \int_A^B \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_A^B \left[F_x dx + F_y dy \right] = \int_A^B \left[xydx - y^2 dy \right]$$

(i) Straight line can be parameterised as $x = t, \quad y = \frac{1}{2}t$ with t from 0 to 2. With this parameterisation, $dx = dt, \quad dy = \frac{1}{2}dt$ and

$$W_{(i)} = \int_0^2 \left[t \times \tfrac{1}{2}t \times dt - \tfrac{1}{4}t^2 \times \tfrac{1}{2}dt \right] = \int_0^2 \tfrac{3}{8}t^2 dt = 1$$

(ii) Curve $y = \frac{1}{4}x^2$ can be parameterised as $x = 2s, \quad y = s^2$ with s from 0 to 1. With this parameterisation, $dx = 2\,ds, \quad dy = 2s\,ds$ and

$$W_{(ii)} = \int_0^1 \left[2s \times s^2 \times 2\,ds - s^4 \times 2s\,ds \right] = \int_0^1 \left(4s^3 - 2s^5 \right) ds$$

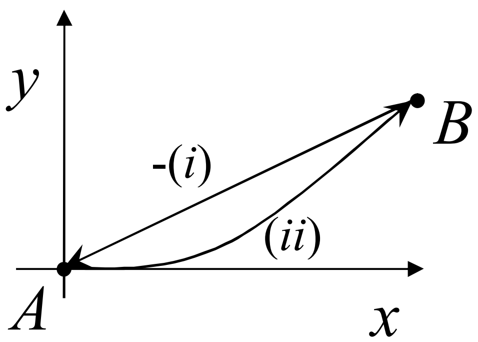
$$= \left[s^4 - \tfrac{1}{3}s^6 \right]_0^1 = \frac{2}{3}$$

NOTE – **result depends on path taken:**

$$W_{(i)} = 1; \quad W_{(ii)} = \frac{2}{3}.$$

So in this case, the tangential line integral around a closed path is non-zero:

Since $\int_{-C} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ (all the $d\mathbf{r} \to -d\mathbf{r}$)



$$\oint \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -\frac{1}{3}$$

in this case.