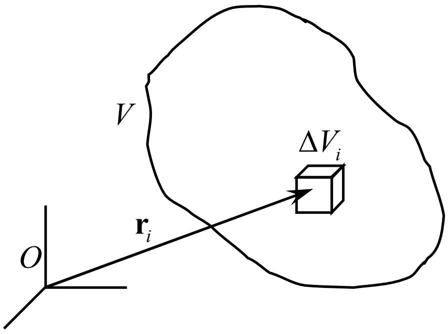
MATHEMATICAL OPERATIONS ON FIELD QUANTITIES

A. *INTEGRATION*

A1. Volume integral of a scalar field



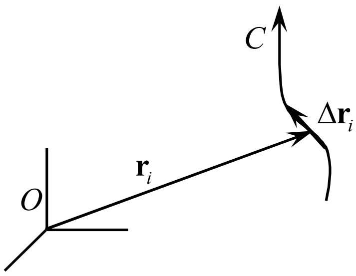
Consider a volume V in a region in which a scalar field $\psi(\mathbf{r})$ is defined.

Divide V into N small volume elements ΔV_i, centred around points with position vectors $\mathbf{r}_i$, $i = 1, \ldots, N$.

Then form the summation $\sum_{i=1}^{N} \psi(\mathbf{r}_i)\, \Delta V_i$.

Take the limit as $N \to \infty$, $\Delta V_i \to 0$ to obtain the **volume integral** $\int_V \psi\, dV \equiv \lim_{\substack{N\to\infty \\ \Delta V_i \to 0}} \sum_{i=1}^{N} \psi(\mathbf{r}_i) \Delta V_i$.

A2. Tangential line integral of a vector field



We consider a path C in a vector field $\mathbf{F}(\mathbf{r})$.

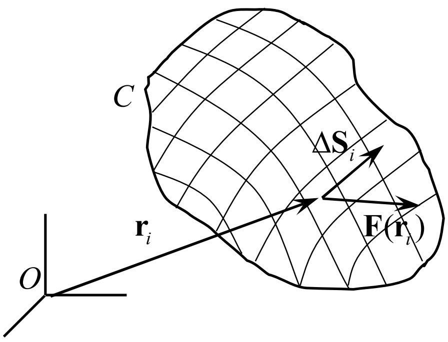
Divide C into N small tangential elements $\Delta\mathbf{r}_i$, also written $\Delta s_i \hat{\mathbf{T}}_i$, at position vectors $\mathbf{r}_i$, $i = 1, \ldots, N$.

Then form the summation $\sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i$.

Take the limit as $N \to \infty$, $\Delta s_i \to 0$ to obtain the **tangential line integral** $\int_C \mathbf{F} \cdot d\mathbf{r} \equiv \lim_{\substack{N\to\infty \\ \Delta s_i \to 0}} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{r}_i$.

If C forms a closed loop, usually write $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

A3. Flux integral of a vector field



Consider a surface S bounded by the closed curve C in a vector field $\mathbf{F}(\mathbf{r})$.

Divide the surface into N small patches. Let $\Delta\mathbf{S}_i$ be the vector area of patch i, equal to the area of the patch × the unit vector normal to the patch: $\Delta\mathbf{S}_i = \Delta S_i \hat{\mathbf{n}}_i$, at position vectors $\mathbf{r}_i$, $i = 1, \ldots, N$.

Then form the summation $\sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{S}_i$.

The required integral is then $\int_S \mathbf{F} \cdot d\mathbf{S} \equiv \lim_{\substack{N\to\infty \\ \Delta S_i \to 0}} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \Delta\mathbf{S}_i$. This integral is often called the **flux** of $\mathbf{F}$ across S. If S forms a closed surface, write $\oint_S \mathbf{F} \cdot d\mathbf{S}$; $d\mathbf{S}$ then points **outwards**.

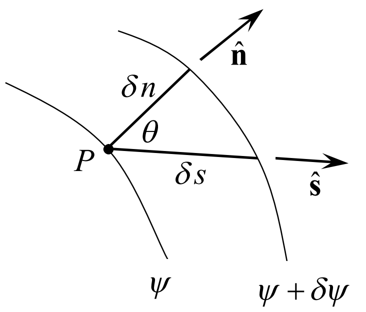
B. *DIFFERENTIATION*

B1. Differentiation of a vector by a scalar

$\mathbf{F}(\mathbf{r},t)$ is a vector quantity. Its time derivative, for example, may be defined by $\mathbf{F}(\mathbf{r},t+\delta t)=\mathbf{F}(\mathbf{r},t)+\left(\dfrac{\partial \mathbf{F}}{\partial t}\right)\delta t$ in the limit $\delta t \to 0$.

B2. The gradient of a scalar field

Here we consider the rate of change of a **scalar** field $\psi(\mathbf{r})$ with distance.



Take two neighbouring contours, with values ψ and $\psi+\delta\psi$.

Along a general direction $\hat{\mathbf{s}}$, the **directional derivative** is defined by

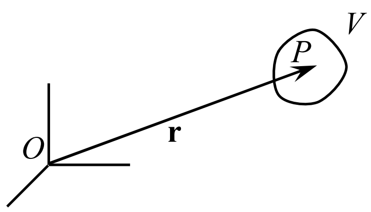
$$\frac{\partial \psi}{\partial s}=\lim_{\delta s\to 0}\frac{\delta\psi}{\delta s}.$$

The *maximum* value this derivative can take occurs when δs is taken along the normal direction $\hat{\mathbf{n}}$ between the contours. We define the **gradient** of ψ at point P using the vector

$$\mathrm{grad}\,\psi \equiv \frac{\partial \psi}{\partial n}\hat{\mathbf{n}}.$$

The modulus of this vector gives the maximum spatial derivative of ψ at point P ; the direction in which the maximum derivative occurs is the direction of the vector $\mathrm{grad}\,\psi$. Note that $\mathrm{grad}\,\psi$ is a **vector field** derived from a **scalar field.**

B3. The divergence of a vector field



Let P be a point in a **vector** field $\mathbf{F}(\mathbf{r})$. Around P construct the small volume V having (closed) surface S , and calculate the net **flux** of $\mathbf{F}(\mathbf{r})$ across S .

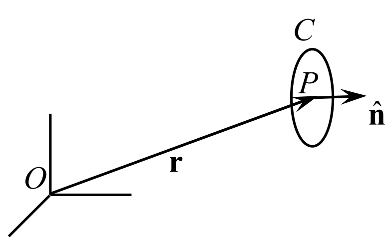
The **divergence** of $\mathbf{F}(\mathbf{r})$ at P is then defined as

$$\mathrm{div}\mathbf{F}=\lim_{V\to 0}\left\{\frac{1}{V}\oint_S \mathbf{F}\cdot d\mathbf{S}\right\}.$$

$\mathrm{div}\mathbf{F}$ is a scalar field derived from a vector field, and measures the net flux originating or disappearing at a point.

For the vector fields of interest in this unit, a non-zero divergence implies a **source** (if positive) or **sink** (if negative) of field at that point.

B4. The curl of a vector field



Again, let P be a point in a **vector** field $\mathbf{F}(\mathbf{r})$. Around P construct a small loop C enclosing (open) surface S . Calculate the **tangential line integral** of $\mathbf{F}(\mathbf{r})$ around C and ROTATE the loop until the integral has a maximum value. In this orientation, let $\hat{\mathbf{n}}$ be the direction normal to the loop.

The **curl** of $\mathbf{F}(\mathbf{r})$ at P is then defined as

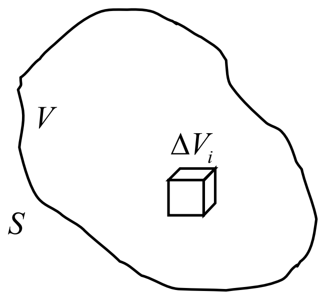
$$\mathrm{curl}\mathbf{F}=\hat{\mathbf{n}}\lim_{S\to 0}\left\{\frac{1}{S}\oint_C \mathbf{F}\cdot d\mathbf{r}\right\}.$$

$\mathrm{curl}\mathbf{F}$ is a vector field derived from a vector field. A non-zero curl at a point indicates a rotational element to the field at that point. The direction of $\mathrm{curl}\mathbf{F}$ is along the axis of rotation, and the modulus is a measure of local rotation rate.

C. *TWO IMPORTANT INTEGRAL THEOREMS*

We will use each of these integral theorems a lot…

C1. The divergence theorem



Consider a volume V enclosed by surface S in a vector field $\mathbf{F}(\mathbf{r})$.

Let γ denote the flux of $\mathbf{F}$ out of S: $\gamma = \oint_S \mathbf{F}\cdot d\mathbf{S}$.

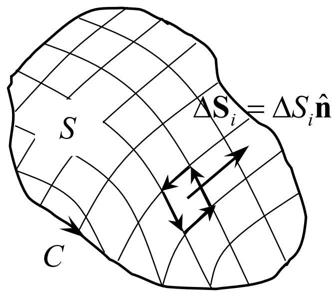
Divide V into N small volume elements ΔV_i, each enclosed by surface S_i. The flux of $\mathbf{F}$ from a single element is $\oint_{S_i} \mathbf{F}\cdot d\mathbf{S}_i$. The sum of all N terms in V is equal to γ, since contributions from internal surfaces cancel. Thus

$$\gamma = \sum_{i=1}^{N} \oint_{S_i} \mathbf{F}\cdot d\mathbf{S}_i = \sum_{i=1}^{N} \left\{ \frac{1}{\Delta V_i} \oint_{S_i} \mathbf{F}\cdot d\mathbf{S}_i \right\} \Delta V_i.$$

Now take the limit as $N \to \infty$, $\Delta V_i \to 0$. As $\Delta V_i \to 0$, the term in brackets becomes $\mathrm{div}\mathbf{F}$. As $N \to \infty$, the sum on the right becomes a volume integral. Thus

$$\oint_S \mathbf{F}\cdot d\mathbf{S} = \int_V \mathrm{div}\mathbf{F}\, dV. \qquad \text{(DIVERGENCE THEOREM)}$$

C2. Stokes' theorem



Let C be a closed curve around the edge of an arbitrary (but not closed) surface S in a vector field $\mathbf{F}(\mathbf{r})$.

Divide the surface into N small patches and calculate the tangential line integral of $\mathbf{F}$ around the perimeter of each patch. The sum of all N of these integrals is equal to the tangential line integral of $\mathbf{F}$ around C since contributions on internal edges cancel. Thus

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \sum_{i=1}^{N} \oint_{C_i} \mathbf{F}\cdot d\mathbf{r}_i = \sum_{i=1}^{N} \left\{ \frac{1}{\Delta S_i} \oint_{C_i} \mathbf{F}\cdot d\mathbf{r}_i \right\} \Delta S_i \\ &= \sum_{i=1}^{N} \{\text{component of curl}\mathbf{F}\text{ along normal to }\Delta S_i\}\, \Delta S_i \\ &= \sum_{i=1}^{N} \mathrm{curl}\mathbf{F}\cdot d\mathbf{S}_i.\end{aligned}$$

In the limit as $N \to \infty$, $\Delta S_i \to 0$,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S \mathrm{curl}\mathbf{F}\cdot d\mathbf{S}. \qquad \text{(STOKES' THEOREM)}$$

Each of these theorems helps us to relate physics inside a region to physics on the boundary of that region. This is extremely useful in any field theory. In this sense the theorems are just extensions of the basic result of integral calculus

$$\int_a^b d\phi = \int_a^b \frac{d\phi}{dx} dx = \phi(b) - \phi(a).$$

D. ***EVALUATION OF DERIVATIVES IN CARTESIAN COORDINATES***

All of the maths in this handout has so far been independent of any coordinate system. This is one of the major advantages of using vector calculus; general results and proofs, valid for all shapes and (well-behaved) fields, in all (orthonormal) coordinate systems, may be obtained in one go.

However, to **evaluate** fields, their derivatives and integrals we need to adopt a coordinate system. In this unit we shall nearly always use Cartesian coordinates (x, y, z) with basis vectors $\hat{\mathbf{e}}_x = \hat{\mathbf{i}}$, $\hat{\mathbf{e}}_y = \hat{\mathbf{j}}$ and $\hat{\mathbf{e}}_z = \hat{\mathbf{k}}$. Here we look at the way to evaluate various derivatives in Cartesian coordinates.

D1. Differentiation of a vector by a scalar

In this unit, the scalar is usually time, t. If in component form, $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_x\hat{\mathbf{k}}$, then

$$\frac{\partial \mathbf{F}}{\partial t} = \frac{\partial F_x}{\partial t}\hat{\mathbf{i}} + \frac{\partial F_y}{\partial t}\hat{\mathbf{j}} + \frac{\partial F_z}{\partial t}\hat{\mathbf{k}}.$$

N.B. It is important to remember that F_x is not necessarily a **function** of x alone; it is the x-**component** of $\mathbf{F}$. In general, F_x is a function of (x, y, z, t).

D2. Spatial derivatives of fields

The 3 key spatial derivatives can all be written in terms of the **vector operator** ∇**:**

$$\text{grad}\,\psi = \nabla\psi \qquad \text{div}\mathbf{F} = \nabla\cdot\mathbf{F} \qquad \text{curl}\mathbf{F} = \nabla\times\mathbf{F}\,.$$

This is the notation that will be used in this unit. In Cartesian coordinates we have

$$\nabla \equiv \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\,,$$

so that the derivatives can be evaluated in Cartesian coordinates as follows:

$$\nabla\psi = \hat{\mathbf{i}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\psi}{\partial z}$$

gradient of a scalar field ψ

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

divergence of a vector field $\mathbf{F}$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

curl of a vector field $\mathbf{F}$

D3. Triple products

We will often apply ∇ twice to form triple products. An important example is the **Laplacian** operator $\nabla\cdot\nabla\psi = \nabla^2\psi = \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} + \dfrac{\partial^2\psi}{\partial z^2}$. In addition (and a bit confusingly) we **define** ∇^2 acting on a vector field component by component: $\nabla^2\mathbf{F} \equiv \hat{\mathbf{i}}\nabla^2 F_x + \hat{\mathbf{j}}\nabla^2 F_y + \hat{\mathbf{k}}\nabla^2 F_z = \hat{\mathbf{i}}\left(\dfrac{\partial^2 F_x}{\partial x^2} + \dfrac{\partial^2 F_x}{\partial y^2} + \dfrac{\partial^2 F_x}{\partial z^2}\right) + \cdots$

Other useful triple products are $\nabla\times\nabla\psi$ which is zero for any scalar field ψ, $\nabla\cdot(\nabla\times\mathbf{F})$ which is zero for any vector field $\mathbf{F}$, and $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$, which is useful when deriving wave equations. AN EXCELLENT SOURCE OF INFORMATION FOR THE VECTOR CALCULUS NEEDED ON THIS UNIT IS p18 OF “Basic Formulae and Statistical Tables”.