Examples involving grad

(1) If $\phi = \sin(x^2 - xy)$ find $\operatorname{grad}\phi$.

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

The partial derivatives are

$$\frac{\partial\phi}{\partial x} = \cos(x^2 - xy) \times (2x - y)$$

$$\frac{\partial\phi}{\partial y} = \cos(x^2 - xy) \times (-x)$$

$$\frac{\partial\phi}{\partial z} = 0.$$

So $\operatorname{grad}\phi$ is given by

$$\operatorname{grad}\phi = (2x - y)\cos(x^2 - xy)\mathbf{i} - x\cos(x^2 - xy)\mathbf{j}$$

NOTE: See handout for plots of ϕ and $\operatorname{grad}\phi$.

(2) In electrostatics, the electric field $\mathbf{E}$ and the electric potential V are related by $\mathbf{E} = -\text{grad}\,V$.

(i) If $V = cz$, where c is a constant:

$$\text{grad}\,V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + c\mathbf{k}$$

So $\mathbf{E} = -c\mathbf{k}$, representing a uniform electric field in the z direction.

(ii) If $V = \dfrac{q}{4\pi\varepsilon_o r}$, where $r = \sqrt{(x^2 + y^2 + z^2)}$.

See if you can show that

$$\mathbf{E} = -\text{grad}\,V = \frac{q}{4\pi\varepsilon_o}\frac{\mathbf{r}}{r^3}$$

NOTE: This represents the potential and field from a stationary point charge q.

(3) If $\phi = x^2 - y^2 + 2yz + 2z^2$, find the gradient of ϕ at $(2,-1,1)$ in the $(1,1,1)$ direction.

From the definition of $\text{grad}\,\phi$ we have

$$\text{grad}\,\phi = 2x\mathbf{i} + (-2y+2z)\mathbf{j} + (2y+4z)\mathbf{k}.$$

At $(2,-1,1)$, this becomes

$$\text{grad}\,\phi = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

The directional derivative in the $(1,1,1)$ direction is defined by

$$\frac{d\phi}{ds} = \text{grad}\,\phi \cdot \hat{\mathbf{s}} \qquad \text{with} \qquad \hat{\mathbf{s}} = \frac{1}{\sqrt{3}}(1,1,1).$$

The directional derivative then becomes

$$\frac{d\phi}{ds} = (4,4,2) \cdot \frac{1}{\sqrt{3}}(1,1,1) = \frac{10}{\sqrt{3}}$$

NOTE: This value of 5.77 is less than the *maximum* gradient at $(2,-1,1)$ which is given by $|\,\text{grad}\,\phi\,| = 6$.

2.2 Divergence and Curl of a vector field

We have seen how the gradient of a scalar field can be analysed in terms of

$$\mathrm{grad}\,\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

Now define a **vector operator** (“del” or “nabla”)

$$\nabla = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \quad \text{or}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

In terms of ∇, $\mathrm{grad}\,\phi$ can be written as

$$\mathrm{grad}\,\phi \equiv \nabla \phi$$

What do we get if we apply ∇ to **vector** fields?

This can be done 2 ways – through the

dot product or the

cross product.

A general vector field can be written as

$$\mathbf{a}(x,y,z) = a_x(x,y,z)\mathbf{i} + a_y(x,y,z)\mathbf{j} + a_z(x,y,z)\mathbf{k}$$

1. Dot product

$$\nabla \cdot \mathbf{a} \equiv \mathrm{div}\,\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$\nabla \cdot \mathbf{a}$ is a **scalar field**.

2. Cross product

$$\nabla \times \mathbf{a} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \times (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})$$

so

$$\nabla \times \mathbf{a} \equiv \mathrm{curl}\;\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right)\mathbf{i} - \left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \right)\mathbf{j} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right)\mathbf{k}$$

$\nabla \times \mathbf{a}$ is a **vector field**.

Examples:

(a) The position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
What are $\nabla \cdot \mathbf{r}$ and $\nabla \times \mathbf{r}$?

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

(b) If $\mathbf{a} = xy\mathbf{i} + z^2\mathbf{j} + 3xyz\mathbf{k}$ find $\nabla \cdot \mathbf{a}$ and $\nabla \times \mathbf{a}$.

$$\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = y + 0 + 3xy = y(1+3x)$$

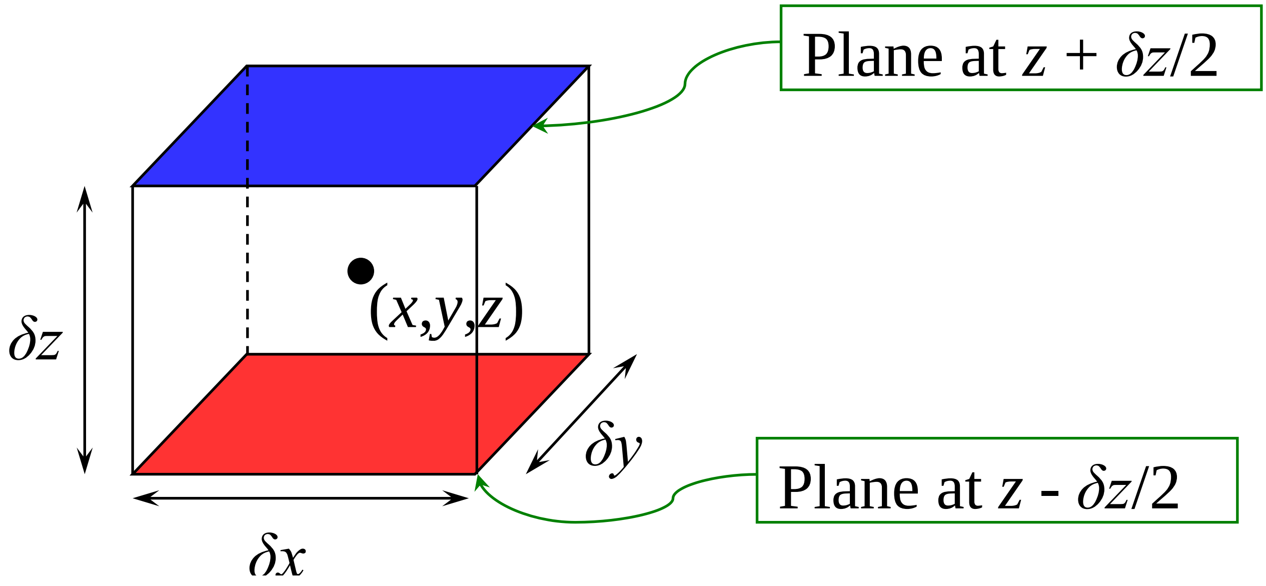
This is a scalar field (the value depends on position)

$$\nabla \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & z^2 & 3xyz \end{vmatrix} = (3xz - 2z)\mathbf{i} - 3yz\mathbf{j} + -x\mathbf{k}$$

The answer is a vector field.

2.2.1 Interpretation of divergence

Consider the small box shown below of volume $\delta V = \delta x\,\delta y\,\delta z$, and centred at (x, y, z).



The *flux* of a vector field $\mathbf{a}$ across any face is defined as

Flux=Outwards normal component
of $\mathbf{a}$ x Area of face

Flux of $\mathbf{a}$ from the red face at $z - \delta z/2$ is therefore

$$\begin{aligned}\text{Flux} &= \left[-a_z(x, y, z - \frac{\delta z}{2})\right] \times \delta x \delta y \\ &\approx \left[-a_z(x, y, z) + \frac{\delta z}{2}\frac{\partial a_z}{\partial z}(x, y, z)\right] \times \delta x \delta y.\end{aligned} \quad (1)$$