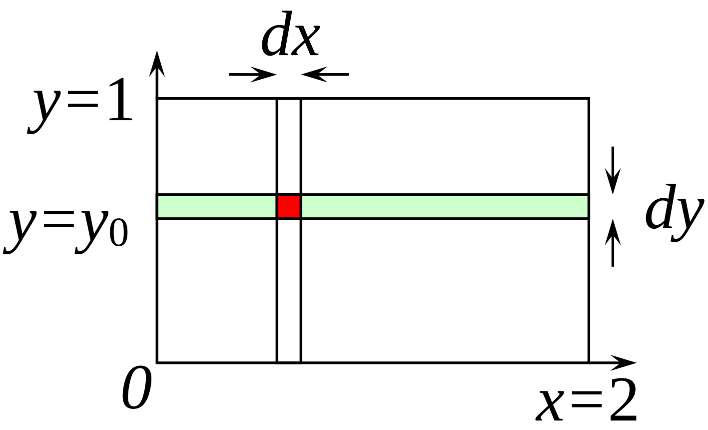
Case 1 – integration limits are constants:



Patches $dS = dxdy$ must cover whole range $0 \le x \le 2$ and $0 \le y \le 1$.

We essentially do a “partial integration” with respect to x (regarding y as constant) and another with respect to y (x constant).

- amounts to integrating first along a vertical or horizontal strip, then through rest of S.

eg: at a given $y = y_0$, charge in horizontal strip is

$$dy \times \int_{x=0}^{x=2} \sigma(x, y_0)\, dx = f(y_0) dy.$$

Then total charge

$$Q = \int_{y=0}^{y=1} f(y) dy = \int_{y=0}^{y=1} \left[\int_{x=0}^{x=2} \sigma(x, y)\, dx \right] dy.$$

Taking vertical strips we get the same answer via

$$Q = \int_{x=0}^{x=2} \left[\int_{y=0}^{y=1} \sigma(x, y) dy \right] dx.$$

Examples

1. $\sigma = x^2 + y$. Perform 2 "partial" integrals, starting with the innermost one:

$$Q = \int_0^2 \left[\int_0^1 dy\,(x^2 + y) \right] dx = \int_0^2 \left[x^2 y + \tfrac{1}{2} y^2 \right]_0^1 dx$$

$$= \int_0^2 \left(x^2 + \tfrac{1}{2} \right) dx = \left[\tfrac{1}{3} x^3 + \tfrac{1}{2} x \right]_0^2 = \tfrac{1}{3} 2^3 + \tfrac{1}{2} 2 = \tfrac{11}{3}$$

[Check you get the same if x integral taken 1st]

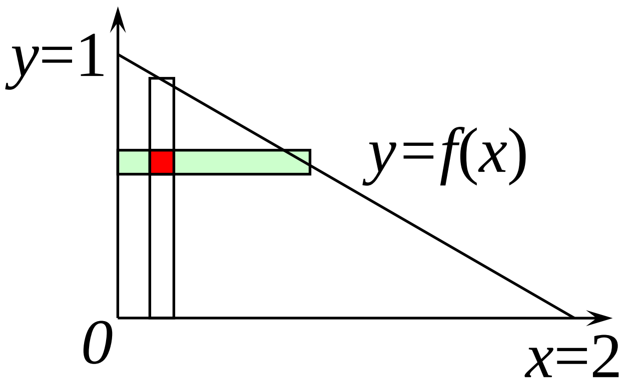
2. $\sigma = x^2 y$, so $Q = \int_{x=0}^{x=2} \left[\int_{y=0}^{y=1} x^2 y\, dy \right] dx$.

Safest approach – start from innermost integral.

BUT, as σ can be written as product $X(x)Y(y)$, and limits are constants, can evaluate Q as product of 2 integrals:

$$Q = \int_{x=0}^{x=2} x^2 dx \int_{y=0}^{y=1} y\, dy = \frac{2^3}{3} \frac{1^2}{2} = \frac{4}{3}.$$

Case 2 – integration limits are NOT constants:



Use equation of lines at boundary to set limits. Choose order of partial integrations to suit.

Example:

$\sigma = x^2 y$; S area shown, with $f(x) = 1 - \frac{1}{2}x$:

Horizontal strips:

$$Q = \int_{y=0}^{y=1} \left[\int_{x=0}^{x=2(1-y)} x^2 y \, dx \right] dy$$

- must do x integral 1st, as it depends on y.

Vertical strips:

$$Q = \int_{x=0}^{x=2} \left[\int_{y=0}^{y=1-x/2} x^2 y \, dy \right] dx$$

- must do y integral 1st, as it depends on x.
- This is easier:

$$Q = \int_{x=0}^{x=2} \left[\tfrac{1}{2} x^2 (1 - \tfrac{x}{2})^2 \right] dx = \ldots = \tfrac{2}{15}.$$

3.2.1 Flux & flux integrals

Flux is a measure of the “flow” of some quantity passing through a given area S.

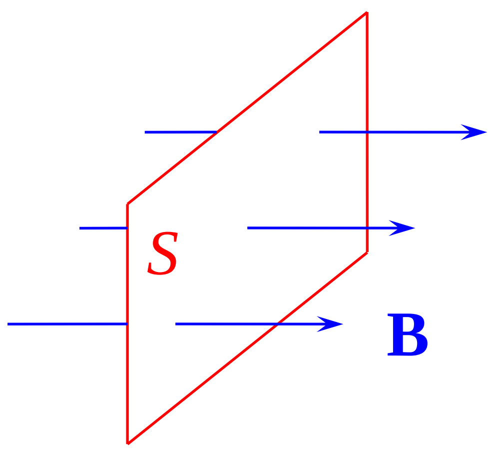
Examples:
Mass flux – rate of mass flow across unit area
Faraday’s Law $\varepsilon = -\dfrac{\partial \Phi}{\partial t}$; Φ is “magnetic flux”.

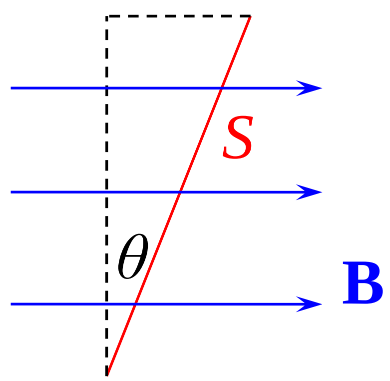
Can think of flux as a measure of the number of field (or flux) lines passing through S.

$\int_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S}$ and $\oint_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S}$ are “flux integrals” of the vector field $\mathbf{F}(\mathbf{r})$ over surface S. How do they arise?

Magnetic flux:

Constant magnetic field $\mathbf{B}$, flat aperture area S:





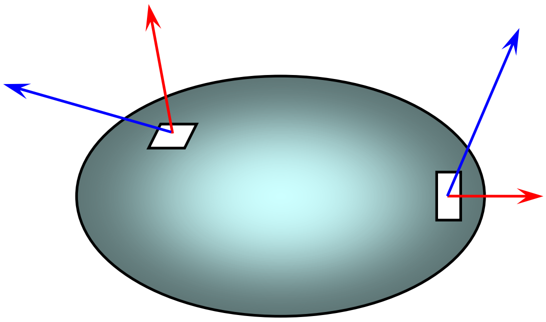
θ:	0	$\pi/2$	θ
Φ:	$\lvert\mathbf{B}\rvert S$	0	$\lvert\mathbf{B}\rvert S\cos\theta$

Define vector $\mathbf{S} = S\hat{\mathbf{n}}$ with $\hat{\mathbf{n}}$ normal to aperture.

Then for any θ, $\Phi = \mathbf{B} \cdot \mathbf{S}$ (a scalar)

But what if S is not a simple flat shape, or
$\mathbf{B}$ varies over S (or both)?

Use calculus – split S into small patches $d\mathbf{S} = dS\hat{\mathbf{n}}$



Flux through infinitesimal patch is

$$d\Phi = \mathbf{B}(\mathbf{r}) \cdot d\mathbf{S}$$

Add up all contributions from surface S to find total flux through S for ANY $\mathbf{B}(\mathbf{r})$ and ANY "open" S:

$$\int_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{S}$$

Note: If S is closed:

1. Write $\oint_S \mathbf{B} \cdot d\mathbf{S}$
2. Choose OUTWARD direction for $d\mathbf{S}$. Then $\oint_S \mathbf{B} \cdot d\mathbf{S}$ measures net outward flux of $\mathbf{B}$ across closed surface S.

3.2.2 Evaluation of flux integrals

1. Split integral into single-face integrals $\int_S \mathbf{F}\cdot d\mathbf{S}$
2. For each face
   (a) Write down $d\mathbf{S}$ - eg $d\mathbf{S} = -dxdy\,\mathbf{k}$
   (only 2 coordinates will vary; $d\mathbf{S}$ parallel to 3rd)
   (b) Evaluate dot product $\mathbf{F}\cdot d\mathbf{S}$
   (c) Evaluate the (scalar) surface integral

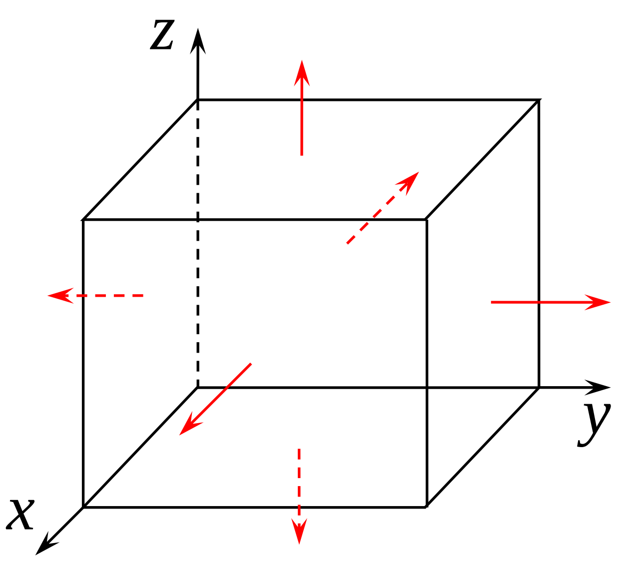
3. Add up integrals from each face

Worked example

Evaluate the net flux out of the box $0 \le x, y, z \le L$ of the vector field $\mathbf{F} = 2xyz\mathbf{i} - y^2 z\,\mathbf{j}$.

Split into 6 integrals, each over 1 of the square faces.

On each face, $d\mathbf{S}$ is **outward** normal



$$I = \oint_S \mathbf{F}\cdot d\mathbf{S} = I_{x=0} + I_{x=L} + I_{y=0} + I_{y=L} + I_{z=0} + I_{z=L}$$

On $x=0,L$ faces, $d\mathbf{S}=\mp dydz\,\mathbf{i}$, $\mathbf{F}\cdot d\mathbf{S}=\mp 2xyz\,dydz$

On $x=0$ face $\mathbf{F}\cdot d\mathbf{S}=0$ so $I_{x=0}=0$.

On $x=L$ face $\mathbf{F}\cdot d\mathbf{S}=+2L\,yz\,dydz$, so

$$I_{x=L}=2L\int_0^L ydy\int_0^L zdz=2L\cdot\frac{L^2}{2}\cdot\frac{L^2}{2}=\frac{L^5}{2}$$

On $y=0,L$ faces, $d\mathbf{S}=\mp dxdz\,\mathbf{j}$, $\mathbf{F}\cdot d\mathbf{S}=\pm y^2 z\,dxdz$

On $y=0$ face $\mathbf{F}\cdot d\mathbf{S}=0$ so $I_{y=0}=0$.

On $y=L$ face $\mathbf{F}\cdot d\mathbf{S}=-L^2\,z\,dxdz$, so

$$I_{y=L}=-L^2\int_0^L dx\int_0^L zdz=-L^2\cdot L\cdot\frac{L^2}{2}=-\frac{L^5}{2}$$

On $z=0,L$ faces, $d\mathbf{S}=\mp dxdy\,\mathbf{k}$, $\mathbf{F}\cdot d\mathbf{S}=0$, so

$I_{z=0}=I_{z=L}=0$.

Adding up all contributions we get

$$I=0+\frac{L^5}{2}+0-\frac{L^5}{2}+0+0=0$$

[Not a surprise; $\nabla\cdot\mathbf{F}=2yz-2yz=0$ everywhere & flux and divergence are closely related.]