Homogeneous: all terms involve dependent var &/or its derivs → **Linear** $L(a\psi + b\Phi) = aL(\psi) + bL(\Phi)$

$\frac{\partial T^2}{\partial x^2} - \frac{1}{k}\frac{\partial T}{\partial t} = 0$ ✓ $\quad R\frac{\partial Q}{\partial t} + \frac{Q}{C} = V$ ✗ $\quad \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$ | $\frac{\partial \Delta u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ✓

$\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial t} - \sin x = 0$ ✗ $\quad i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + V|\psi|^2\psi$ | $\omega\frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial y}\right)^2 - u^2 = 0$ ✗

- commutative Σ $L+M = M+L$
- associative $(L+M)+N = L+(M+N)$ $\quad 2(LM)N = L(MN)$
- distributive $L(c_1M + c_2N) = c_1LM + c_2LN$
- commutative w (const coeff) $LM = ML$

can be

PDE linear if
- dep var & derivs occur only first pwr
- no products dependent on var & it's derivs

$L(u) = A\frac{\partial^2}{\partial x^2} + B\frac{\partial}{\partial x\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$

$L(u) = G(x,y) = 0$ ∴ homog
$\neq 0$ ∴ inhom

Div th $\int_S \vec{A}\,ds = \int_V \nabla\cdot\vec{A}\,dV$

PDEs
- elliptic $B^2 - 4AC < 0$ steady-state
- hyperbolic $B^2 - 4AC > 0$ vibrat°
- parabolic $B^2 - 4AC = 0$ transit°

- Poisson $\nabla^2\Phi = -\rho(\vec{r})$
- Laplace $\nabla^2\Phi = 0$
- Diffus° $\nabla^2\Phi - \frac{1}{k}\frac{\partial\Phi}{\partial t} = -\delta(\vec{r},t)$
- wave $\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\delta(\vec{r},t)$
- Helmholtz $\nabla^2\Phi + \lambda\Phi = 0$
- Schröd $-\frac{\hbar^2}{2m}\nabla^2\Phi + V(\vec{r})\Phi = i\hbar\frac{\partial\Phi}{\partial t} = E\Phi$

Solving ODEs

Homogeneous: $0 = \frac{d^2y}{dx^2} + P(x)\frac{dy}{d} + Q(x)y$

2 solut's $\xrightarrow{\text{comb}}$ $y(x) = c_1y_1(x) + c_2y_2(x)$

ex
- $P(x) = 0$, $Q(x) = \omega^2$ $\xrightarrow{GS}$ $\frac{d^2y}{dx^2} + \omega^2 y = 0$
- $P(x) = \frac{1}{x}$, $Q(x) = 1 - \frac{n^2}{x^2}$ $\xrightarrow{GS}$ $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$

Inhomog $F(x) = \cdots$

y_p = sol ∴ General sol $\rightarrow y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

Pwr series $\frac{d^2y}{dx^2} + \omega^2 y = 0$

GS → $y(x) = \sum_{s=0}^{\infty} a_s x^{s+q}$

needs more work

recurrence relat°: $a_{s+2} = \frac{-\omega^2}{(q+s+2)(q+s+1)}a_s$

Boundary condit's

ex wave eqt° 1D $\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c}\frac{\partial^2\phi}{\partial t^2} = 0$

$L(\phi) = 0$

$L = \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

General sol^n: $\phi(x,t) = \sum_k [A_k e^{ik(x-ct)} + B_k e^{ik(x+ct)}]$

BC: $\phi(x,0) = \sin 3x$ & $\frac{\partial\phi(x,0)}{\partial t} = 0$

$\Rightarrow \phi(x,0) = \sum_k [-ikc\,A_k e^{ikx} + ikc\,B_k e^{-ikx}]$

$= \sum_k -ikc\,(A_k - B_k)e^{ikx}$

$\phi(x,t) = \sum_k A_k e^{ikx}(e^{-ikct} + e^{ikct})$

BC $= \sum_k 2A_k e^{ikx}\cos(kct)$

$\phi(x,0) = \sum_k 2A_k e^{ikx} = \sin 3x = \frac{e^{i3x}}{2i} - \frac{e^{-i3x}}{2i}$

$2A_3 = 1/2i$
$2A_{-3} = -1/2i$
other $A_n = 0$

$\phi(x,t) = \sum_k 2A_k e^{ikx}\cos(kct) = \sin(3x)\cos(3ct)$

Separat° of var

$u(x,t) = X(x)T(t)$

1D wave eqt°: $\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$

sol form: $\phi(x,t) = X(x)T(t)$

$\frac{\partial^2\phi}{\partial x^2} = \frac{d^2X}{dx^2}T$ $\quad \frac{d^2X}{dx^2}T - \frac{1}{c^2}X\frac{d^2T}{dt^2} = 0$

$\frac{\partial^2\phi}{\partial t^2} = X\frac{d^2T}{dt^2}$

$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -k^2$

$\frac{d^2X}{dx^2} = -k^2X$ $\quad \frac{d^2T}{dt^2} = -k^2c^2T$

$X(x) = ae^{ikx} + be^{-ikx}$

$T(t) = c_ke^{ickt} + d_ke^{-ickt}$

$k = \frac{2\pi}{\lambda}$

$\omega = ck$

$\phi(x,t) = X(x)T(t)$

$\phi(x,t) = e^{i(kx - \omega t)}$

$X_k(x) = a_ke^{ikx} + b_ke^{-ikx}$

$T_k(t) = c_ke^{ickt} + d_ke^{-ickt}$

$\phi(x,t) = \sum_k X_k(x)T_k(t)$

$= X(x)Y(y)Z(z)T(t)$

$= R(\vec{r})\Theta(\theta)\Phi(\phi)T(t)$

Diffus° chem species

Change tot amount species: $\Delta M_1 = \delta t\int_V Q(\vec{r},t)\,dV$

flow vector: $\vec{J}$

Change due diffus°: $\Delta M_2 = \delta t\int_S -\vec{J}\cdot d\vec{s} = \delta t\int_V \nabla\cdot\vec{J}\,dV$

tot change: $\Delta M = M_1 + M_2$

we know $\Delta M = \delta t\int_V \frac{\partial\rho}{\partial t}dV$

$\therefore \int_V \frac{\partial\rho}{\partial t}dV = \int_V Q(\vec{r},t)\,dV + \delta t\int_V \nabla^2\rho\,dV$

$\frac{\partial\rho}{\partial t} = Q(\vec{r},t) + D\nabla^2\rho$

$\nabla^2\rho - \frac{1}{D}\frac{\partial\rho}{\partial t} = -\frac{Q(\vec{r},t)}{D}$

Temp distribut°

$\nabla^2 T - \frac{\rho c}{k}\frac{\partial T}{\partial t} = -\frac{P(\vec{r})}{k}$

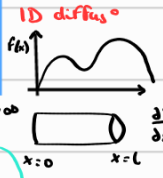
steady state: $\frac{\partial\rho}{\partial t} = 0$, $\frac{\partial T}{\partial t} = 0$

$\therefore \nabla^2\rho = -\frac{Q(\vec{r})}{D}$

$\nabla^2 T = -P(\vec{r})/k$

Poisson!

1D diffus°



$\frac{\partial^2\rho}{\partial x^2} = \frac{1}{c}\frac{\partial\rho}{\partial t}$

$p(x,t) = X(x)T(t)$

$T\,d^2X$

G.S: $p(x,t) = X(x)T(t) = \sum_a (A_a\sin ax + B_a\cos ax)e^{-a^2ct}$

1D diffus° eqt°: $\frac{\partial^2\rho}{\partial x^2} = \frac{1}{c}\frac{\partial\rho}{\partial t}$

$\mathcal{F}[\rho(x,t)] = P(u,t) = \int_{-\infty}^{\infty}\rho(x,t)e^{-iux}dx$

$\mathcal{F}\left[\frac{\partial^2\rho(x,t)}{\partial x^2}\right] = -u^2P(u,t)$

$\mathcal{F}\left[\frac{1}{c}\frac{\partial\rho}{\partial t}\right] = \frac{1}{c}\frac{\partial P(u,t)}{\partial t}$

$-u^2P(u,t) = \frac{1}{c}\frac{\partial P(u,t)}{\partial t}$ ...

- **Divergent**: $S_i \to \infty$ as $i \to \infty$
- **Convergent**: $S = \lim_{i\to\infty} S_i$
- **Oscillatory**: partial sums oscillate but never converge

Tsts
- $\lim_{n\to\infty} u_n \neq 0$ ∴ div
- ratio tst: $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)$ $\begin{cases} <1 \to \text{conv} \\ >1 \to \text{div} \\ =1 \to \text{undet} \end{cases}$
- bigger series converges ⇒ so does smaller; smaller series div ⇒ does bigger

Fourier Transform

$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$

- Absolutely $\sum_n u_n$ & $\sum_n |u_n|$ both converge
- Conditionally $\sum_n u_n$ conv but $\sum_n |u_n|$ div

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

Points → ordinary: $P(x)$ & $Q(x)$ finite

→ singular: P & Q div at $x = x_0$
- regular: $(x-x_0)P$ & $(x-x_0)^2Q$ remain finite as $x \to x_0$
- irregular: ...approach infinity...

expansion may fail

ν° sols
- indicial eq^n: q_1, q_2
- $q_1 - q_2 \neq$ int: q_1, q_2 → 2 sols
- $q_1 - q_2 =$ int → larger q will give a sol, smaller may not