Section 2. Angular momentum

- Important example of the use of operators in quantum mechanics

- Leads to analysis of spin

- Lays the foundation for solutions of the Schrödinger equation in 3D

2.1 Definitions, operators and commutators

- In classical mechanics, the angular momentum of a particle at position $\underline{r}$ and with momentum $\underline{p}$ is:

$$\boxed{\underline{L} = \underline{r} \times \underline{p}}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \begin{array}{l} \underline{i}\left(yp_z - zp_y\right) \\ -\underline{j}\left(xp_z - zp_x\right) \\ +\underline{k}\left(xp_y - yp_x\right) \end{array}$$

In quantum mechanics we replace $\underline{r}$ and $\underline{p}$ with the corresponding operators (see section 1.2), so the angular momentum operator becomes:

$$\boxed{\hat{\underline{L}} = \hat{\underline{r}} \times \hat{\underline{p}}}$$

Cartesian components are:

$$\boxed{\begin{aligned}\hat{L}_x &= \hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y \\ \hat{L}_y &= \hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z \\ \hat{L}_z &= \hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x\end{aligned}}$$

What is the commutator of $\hat{L}_x$ and $\hat{L}_y$?

$$\left[\hat{L}_x, \hat{L}_y\right] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x$$

$$= \left(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right)\left(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z\right) - \left(\hat{z}\hat{p}_x - \hat{x}\hat{p}_z\right)\left(\hat{y}\hat{p}_z - \hat{z}\hat{p}_y\right)$$

The only non-commuting operators here are $\hat{z}$ and $\hat{p}_z$, so

see section 1.6

$$\left[\hat{L}_x, \hat{L}_y\right] = \hat{y}\hat{p}_x\left(\hat{p}_z\hat{z} - \hat{z}\hat{p}_z\right) + \hat{x}\hat{p}_y\left(\hat{z}\hat{p}_z - \hat{p}_z\hat{z}\right)$$

$$= \left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right)\underbrace{\left(\hat{z}\hat{p}_z - \hat{p}_z\hat{z}\right)}_{=\left[\hat{z},\hat{p}_z\right]=i\hbar}$$

$$= i\hbar\underbrace{\left(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x\right)}_{\hat{L}_z} = i\hbar\hat{L}_z$$

see problems sheet for details

We find

$$\boxed{\begin{aligned}\left[\hat{L}_x, \hat{L}_y\right] &= i\hbar\hat{L}_z\\ \left[\hat{L}_y, \hat{L}_z\right] &= i\hbar\hat{L}_x\\ \left[\hat{L}_z, \hat{L}_x\right] &= i\hbar\hat{L}_y\end{aligned}}$$

note the cyclic permutation

- We can also define the square magnitude of the total angular momentum as

$$\boxed{\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2}$$

classically: -
$\underline{L}^2 = \underline{L} \bullet \underline{L} = L_x^2 + L_y^2 + L_z^2$

- What are the commutators of $\hat{L}^2$ with $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$?

e.g $$\left[\hat{L}^2, \hat{L}_z\right] = \left[\hat{L}_x^2, \hat{L}_z\right] + \left[\hat{L}_y^2, \hat{L}_z\right] + \left[\hat{L}_z^2, \hat{L}_z\right]$$

We find that the first two commutators on the rhs cancel, and the last commutator is zero, so

$$\boxed{\left[\hat{L}^2, \hat{L}_x\right] = \left[\hat{L}^2, \hat{L}_y\right] = \left[\hat{L}^2, \hat{L}_z\right] = 0}$$

see problems sheet

- The separate components of angular momentum are not compatible and cannot be measured simultaneously

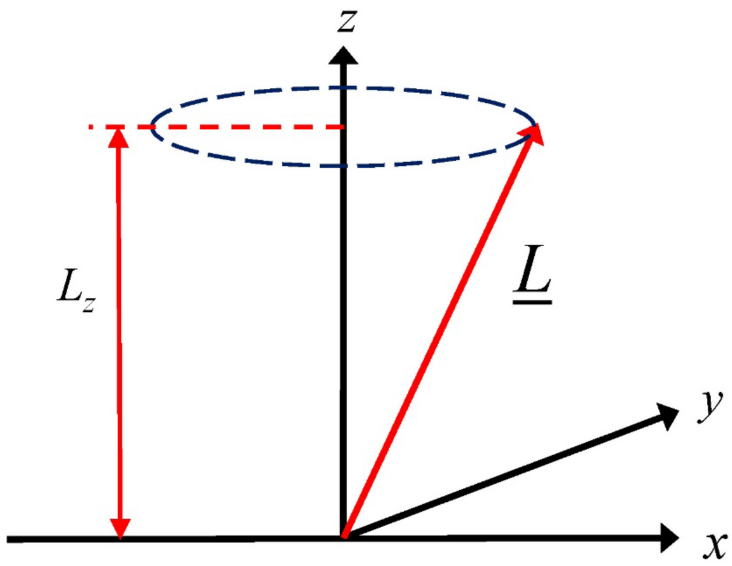
- Each component is compatible with the total angular momentum

We can therefore look for solutions which are common eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$

see section 1.6

We could choose $\hat{L}_x$ or $\hat{L}_y$, but $\hat{L}_z$ is conventional

Aside: Spinning top



Spinning top with spin angular momentum $\underline{L}_{\text{spin}} = I\underline{\omega}$ precesses about the *z* axis with orbital angular momentum $\underline{L}_{\text{orbital}}$.

Total angular momentum $\underline{L} = \underline{L}_{\text{spin}} + \underline{L}_{\text{orbital}}$ changes direction, but $|\underline{L}|$ is constant in the absence of an external field.

Observables L_x and L_y are variables, but L^2 and L_z are constants

Uncertainty:

$$\Delta A\,\Delta B \geq \frac{1}{2}\left|\int \psi^* \left[\hat{A}, \hat{B}\right] \psi\right|$$

see section 1.6

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$

Commutation relation – see above

So

$$\Delta L_x\,\Delta L_y \geq \frac{1}{2}\left|\int \psi^* i\hbar\hat{L}_z\psi\right|$$

$$\geq \frac{\hbar}{2}\left|\int \psi^* \hat{L}_z\psi\right|$$

$|i| = 1$

$$\geq \frac{\hbar}{2}\left|\underbrace{\left\langle \hat{L}_z \right\rangle}_{}\right|$$

expectation value of $\hat{L}_z$

Back to the problem: eigenvalues of $\hat{L}^2$ and $\hat{L}_z$

- To help in the analysis, we define two more operators

$$\boxed{\begin{aligned}\hat{L}_+ &= \hat{L}_x + i\hat{L}_y \\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y\end{aligned}}$$

- Properties of $\hat{L}_+$ and $\hat{L}_-$

$$\begin{aligned}\hat{L}_+\hat{L}_- &= \left(\hat{L}_x + i\hat{L}_y\right)\left(\hat{L}_x - i\hat{L}_y\right) \qquad (1) \\ &= \hat{L}_x^2 + \hat{L}_y^2 - i\hat{L}_x\hat{L}_y + i\hat{L}_y\hat{L}_x \\ &= \hat{L}^2 - L_z^2 - i\underbrace{\left[\hat{L}_x, \hat{L}_y\right]}_{= i\hbar\hat{L}_z} \\ &= \hat{L}^2 - L_z^2 + \hbar\hat{L}_z\end{aligned}$$

Similarly

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$$

So

$$\boxed{\left[\hat{L}_+, \hat{L}_-\right] = 2\hbar\hat{L}_z}$$

$$\left[\hat{L}_z,\hat{L}_+\right]=\underbrace{\left[\hat{L}_z,\hat{L}_x\right]}_{=i\hbar\hat{L}_y}+i\underbrace{\left[\hat{L}_z,\hat{L}_y\right]}_{=-i\hbar\hat{L}_x}$$

$$=i\hbar\left(\hat{L}_y-i\hat{L}_x\right) \qquad (2)$$

So

$$\boxed{\left[\hat{L}_z,\hat{L}_+\right]=\hbar\hat{L}_+}$$

Similarly

$$\boxed{\left[\hat{L}_z,\hat{L}_-\right]=-\hbar\hat{L}_-}$$

see problems sheet for details

2.2 Eigenvalues of $\hat{L}^2$ and $\hat{L}_z$

Because $\hat{L}^2$ and $\hat{L}_z$ commute, they must have a common set of eigenfunctions. We can write

$$\hat{L}^2 |\phi_n\rangle = \alpha_n |\phi_n\rangle \quad \text{and} \quad \hat{L}_z |\phi_n\rangle = \beta_n |\phi_n\rangle$$

The eigenvalues α_n and β_n can be determined just using operator expressions, together with the condition

$$\boxed{\alpha_n \geq \beta_n^2}$$

Measured value for L_z^2 cannot be larger than measured value for L^2

The algebra….

$$\hat{L}_z\left|\phi_n\right\rangle = \beta_n\left|\phi_n\right\rangle$$

$$\hat{L}_+\hat{L}_z\left|\phi_n\right\rangle = \beta_n\,\hat{L}_+\left|\phi_n\right\rangle$$

$$\overbrace{\left[\hat{L}_z,\hat{L}_+\right]}^{\boxed{\text{from above}}} = \hbar\hat{L}_+ \quad \text{so} \quad \hat{L}_+\hat{L}_z = \hat{L}_z\hat{L}_+ - \hbar\hat{L}_+$$

$$\therefore \hat{L}_+\hat{L}_z\left|\phi_n\right\rangle = \left(\hat{L}_z\hat{L}_+ - \hbar\hat{L}_+\right)\left|\phi_n\right\rangle$$

$$= \hat{L}_z\left(\hat{L}_+\left|\phi_n\right\rangle\right) - \hbar\left(\hat{L}_+\left|\phi_n\right\rangle\right)$$

$$= \beta_n\left(\hat{L}_+\left|\phi_n\right\rangle\right)$$

$$\Rightarrow \quad \boxed{\hat{L}_z\left(\hat{L}_+\left|\phi_n\right\rangle\right) = \left(\beta_n + \hbar\right)\left(\hat{L}_+\left|\phi_n\right\rangle\right)}$$

Similarly, we can show

$$\boxed{\hat{L}_z\left(\hat{L}_-\left|\phi_n\right\rangle\right) = \left(\beta_n - \hbar\right)\left(\hat{L}_-\left|\phi_n\right\rangle\right)}$$

$\hat{L}_+|\phi_n\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $\beta_n + \hbar$ (provided $\hat{L}_+|\phi_n\rangle \neq 0$)

Eigenvalue increased by $\hbar$

$\hat{L}_-|\phi_n\rangle$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $\beta_n - \hbar$ (provided $\hat{L}_-|\phi_n\rangle \neq 0$)

Eigenvalue decreased by $\hbar$

More algebra….

$$\hat{L}^2|\phi_n\rangle = \alpha_n|\phi_n\rangle$$

$$\Rightarrow \quad \hat{L}_+\hat{L}^2|\phi_n\rangle = \alpha_n\ \hat{L}_+|\phi_n\rangle$$

$$\hat{L}_-\hat{L}^2|\phi_n\rangle = \alpha_n\ \hat{L}_-|\phi_n\rangle$$

$\hat{L}^2$ commutes with $\hat{L}_x$ and $\hat{L}_y$, so it must commute with $\hat{L}_+$ and $\hat{L}_-$

can swop order of operators

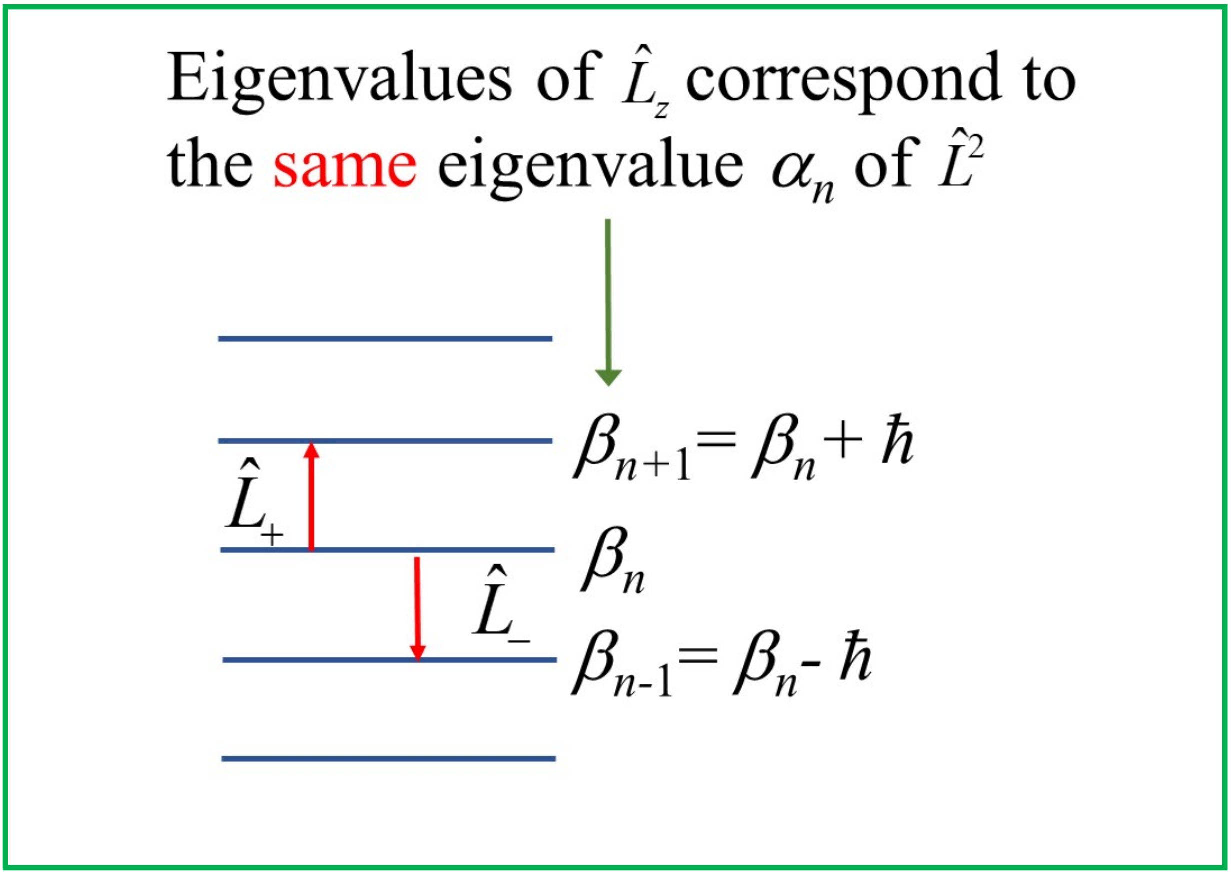
$$\Rightarrow \quad \hat{L}^2\left(\hat{L}_+|\phi_n\rangle\right) = \alpha_n\left(\hat{L}_+|\phi_n\rangle\right)$$

$$\hat{L}^2\left(\hat{L}_-|\phi_n\rangle\right) = \alpha_n\left(\hat{L}_-|\phi_n\rangle\right)$$

$\hat{L}_+|\phi_n\rangle$ and $\hat{L}_-|\phi_n\rangle$ are eigenfunctions of $\hat{L}^2$ with eigenvalue α_n

- For each eigenvalue of $\hat{L}^2$ there are a set of eigenfunctions with different $\hat{L}_z$ eigenvalues. The $\hat{L}_+$ and $\hat{L}_-$ operators "raise" or "lower" the eigenfunctions within this set.

- $\hat{L}_+$ and $\hat{L}_-$ are called "ladder operators" or "creation" and "annihilation" operators



- We now use the condition $\alpha_n \geq \beta_n^2$. This implies there is a maximum and minimum value of β. Call these $\beta_{\max}$ and $\beta_{\min}$, with corresponding eigenfunctions $|\phi_{\max}\rangle$ and $|\phi_{\min}\rangle$

Must be true! The state $\hat{L}_+|\phi_{max}\rangle$ of $\hat{L}_z$ is zero because it does not exist

Yet more algebra….

$$\boxed{\hat{L}_+|\phi_{max}\rangle = 0} \quad \text{so} \quad \hat{L}_-\hat{L}_+|\phi_{max}\rangle = 0$$

From above

$$\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$$

so

$$\left(\hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z\right)|\phi_{max}\rangle = 0$$

i.e.

$$\left(\alpha - \beta_{max}^2 - \hbar\beta_{max}\right)|\phi_{max}\rangle = 0$$

$$\therefore \quad \boxed{\alpha = \beta_{max}\left(\beta_{max} + \hbar\right)}$$

In a similar way, starting from $\boxed{\hat{L}_-|\phi_{min}\rangle = 0}$ we find

$$\boxed{\alpha = \beta_{min}\left(\beta_{min} - \hbar\right)}$$

The state $\hat{L}_-|\phi_{min}\rangle$ of $\hat{L}_z$ is zero because it does not exist

It follows that

$$\boxed{\beta_{\min} = -\beta_{\max}}$$

such that

$$\begin{aligned}\alpha &= \beta_{\min}\left(\beta_{\min} - \hbar\right) \\ &= -\beta_{\max}\left(-\beta_{\max} - \hbar\right) \\ &= \beta_{\max}\left(\beta_{\max} + \hbar\right)\end{aligned}$$

as required

Neighbouring values of β on the $\hat{L}_z$ “ladder” differ by $\hbar$, so

$$\beta_{\max} - \beta_{\min} = n\hbar \quad (n \text{ integer})$$

$$\text{So } \beta_{\max} - (-\beta_{\max}) = 2\beta_{\max} = n\hbar$$

$$\text{or } \beta_{\max} = n\hbar/2$$

$$\Rightarrow \boxed{\beta_{\max} = -\beta_{\min} = \frac{n}{2}\hbar \equiv \ell\,\hbar}$$

So

$$\alpha = \beta_{\max}\left(\beta_{\max} + \hbar\right) = \ell\hbar\left(\ell\hbar + \hbar\right) = \ell\left(\ell + 1\right)\hbar^2$$

$$\alpha = \beta_{\min}\left(\beta_{\min} - \hbar\right) = \ell\hbar\left(\ell\hbar + \hbar\right) = \ell\left(\ell + 1\right)\hbar^2$$

Then

$$\hat{L}^2\left|\phi\right\rangle = \alpha\left|\phi\right\rangle \text{ with } \alpha = \ell\left(\ell + 1\right)\hbar^2$$

$\ell = n/2$ is integer or half integer

Also

$$\hat{L}_z\left|\phi\right\rangle = \beta\left|\phi\right\rangle \text{ with } \beta_{\min} = -\ell\hbar \text{ and } \beta_{\max} = +\ell\hbar$$

$$\hat{L}_z\left(\hat{L}_+\left|\phi\right\rangle\right) = \left(\beta + \hbar\right)\left(\hat{L}_+\left|\phi\right\rangle\right)$$

$$\hat{L}_z\left(\hat{L}_-\left|\phi\right\rangle\right) = \left(\beta - \hbar\right)\left(\hat{L}_-\left|\phi\right\rangle\right)$$

Ladder of eigenvalues of $\hat{L}_z$ separated by integer units of $\hbar$, $\Delta\beta = \pm\hbar$

Finally, we have

Eigenvalues of $\hat{L}^2$ are $\alpha = \ell(\ell+1)\hbar^2$, with ℓ an integer or "half integer"

For each value of ℓ, eigenvalues of $\hat{L}_z$ can be written as $\beta = m\hbar$, where m varies in integer steps between $-\ell$ and $+\ell$

- For orbital angular momentum (i.e. what we've been talking about so far) only integer values of ℓ matter.

- However, for more general angular momentum, the half integer solutions are also relevant.

ℓ	Magnitude of L	Possible L_z values
0	0	0
1	$\sqrt{2}\hbar$	$-\hbar, 0, \hbar$
2	$\sqrt{6}\hbar$	$-2\hbar, -\hbar, 0, \hbar, 2\hbar$
$\vdots$	$\vdots$	$\vdots$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}\hbar$	$-\frac{\hbar}{2}, \frac{\hbar}{2}$

$$L^2 = \ell(\ell+1)\hbar^2 \Rightarrow L = \sqrt{\ell(\ell+1)}\hbar$$

$$L_z = m\hbar, \ \ m = -\ell, -\ell+1, \ldots, +\ell$$

For $\ell = 1/2$ see later

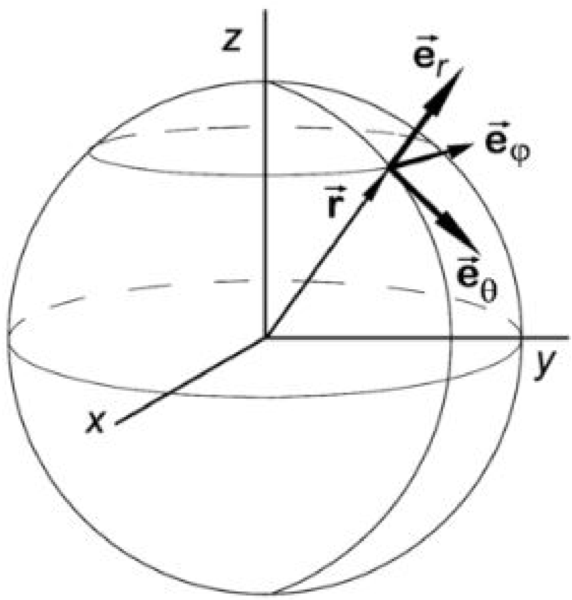
2.3 Eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$

These cannot be obtained just from operator algebra – we have to solve the equations. It is most convenient to use spherical polar coordinates.

Start from

$$\underline{\hat{L}} = \underline{\hat{r}} \times \underline{\hat{p}} = -i\hbar\, \underline{r} \times \underline{\nabla}$$

Use spherical polars: -



$\underline{r}$: intercepts sphere at point P

Unit vectors: -

$\underline{e}_r$: in radial direction at point P

$\underline{e}_\theta$: tangent to sphere through point P with constant longitude

$\underline{e}_\phi$: tangent to sphere through point P with constant latitude

In spherical polars

$$\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \underline{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

So

$$\underline{\hat{L}} = -i\hbar \begin{vmatrix} \underline{e}_r & \underline{e}_\theta & \underline{e}_\phi \\ r & 0 & 0 \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{vmatrix}$$

$$= -i\hbar\left(\underline{e}_\phi \frac{\partial}{\partial \theta} - \underline{e}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\right)$$

The relation between Cartesian and spherical polar unit vectors is: -

$$\begin{pmatrix} \underline{e}_x \\ \underline{e}_y \\ \underline{e}_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_r \\ \underline{e}_\theta \\ \underline{e}_\phi \end{pmatrix}$$

- Unit vector in z direction is

$$\underline{e}_z = \underline{e}_r \cos\theta - \underline{e}_\theta \sin\theta$$

and

$$\hat{L}_z = \underline{e}_z \bullet \hat{\underline{L}} \Rightarrow \boxed{\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}}$$

where we have used $\underline{e}_r \bullet \underline{e}_\phi = 0,\ \underline{e}_\theta \bullet \underline{e}_\theta = 1$ etc.

- For $\hat{L}^2$ we find

$$\boxed{\hat{L}^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right)}$$

- We want to solve

ℓ integer

$$\boxed{\begin{aligned} \hat{L}^2 \left|Y_{\ell m}(\theta,\phi)\right\rangle &= \ell(\ell+1)\,\hbar^2 \left|Y_{\ell m}(\theta,\phi)\right\rangle \\ \hat{L}_z \left|Y_{\ell m}(\theta,\phi)\right\rangle &= m\hbar \left|Y_{\ell m}(\theta,\phi)\right\rangle \end{aligned}}$$

m integer from $-\ell$ to $+\ell$

$Y_{\ell m}(\theta,\phi)$ are the common eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ with eigenvalues $\ell(\ell+1)\hbar^2$ and $m\hbar$ respectively

- The solutions $Y_{\ell m}(\theta,\phi)$ are called spherical harmonics. Derivations of them can be found in textbooks and in other units.

- The lowest few spherical harmonics are

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

s orbital

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \; \exp(\pm i\phi)$$

p orbitals

$$Y_{20} = \sqrt{\frac{5}{16\pi}} \left(3\cos^2\theta - 1\right)$$

$$Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}} \cos\theta \sin\theta \exp(\pm i\phi)$$

$$Y_{2\pm2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \; \exp(\pm 2i\phi)$$

d orbitals

Normalisation: $\displaystyle\int_0^{2\pi}\int_0^{\pi} \left|Y_{\ell m}(\theta,\phi)\right|^2 \sin\theta \; d\theta d\phi = 1$

2.4 Spin angular momentum

- The Stern-Gerlach experiment allows us to measure the z component of angular momentum. (see additional notes)

- Orbital angular momentum is insufficient to explain results for atoms

- We therefore postulate that quantum particles have an intrinsic angular momentum, or spin

- We postulate spin operators $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ with the same commutation properties as $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, i.e.

$$\boxed{\begin{aligned}\left[\hat{S}_x, \hat{S}_y\right] &= i\hbar\, \hat{S}_z \\ \left[\hat{S}_y, \hat{S}_z\right] &= i\hbar\, \hat{S}_x \\ \left[\hat{S}_z, \hat{S}_x\right] &= i\hbar\, \hat{S}_y\end{aligned}}$$

- We can also define

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$$

All of the previous analysis for angular momentum operators follows through, and we can conclude that:

- Eigenvalues of $\hat{S}^2$ are $s(s+1)\hbar^2$, with s an integer or "half integer"

- Eigenvalues of $\hat{S}_z$ are $m_s\hbar$, where m_s varies in integer steps between $-s$ and s

For spin, the half integer solutions matter. Atoms with no orbital angular momentum and a single unpaired electron split into two beams in a Stern-Gerlach experiment.

$\therefore$ $\boxed{\text{Electrons are spin-half particles, i.e. } s = \frac{1}{2}}$

$\boxed{m_s = -1/2 \text{ or } 1/2}$

2.5 Pauli spin matrices

What are the eigenfunctions of $\hat{S}^2$ and $\hat{S}_z$ for a spin-half particle?

spin is an intrinsic property of a particle

These cannot be functions of the particle's position, and so we need a representation that doesn't depend on spatial coordinates

In section 1.11 we discussed the matrix representation of quantum mechanics. This turns out to be a natural representation for spin angular momentum

Operators become matrices

Bra/Kets become row/column vectors

- The Pauli spin matrices are defined as

$$\boxed{\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

- The spin components are

$$\boxed{\hat{S}_x = \frac{\hbar}{2}\sigma_x \quad \hat{S}_y = \frac{\hbar}{2}\sigma_y \quad \hat{S}_z = \frac{\hbar}{2}\sigma_z}$$

These obey all the commutation relations, e.g.

$$\left[\hat{S}_x, \hat{S}_y\right] = \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

$$= \frac{\hbar^2}{4}\left(\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right)$$

$$= i\hbar\,\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\,\hat{S}_z$$

see additional notes

- It is simple to calculate the eigenvalues and eigenfunctions (i.e. eigenvectors) of $\hat{S}_x, \hat{S}_y, \hat{S}_z$:

Component	Eigenvalue	Eigenvector
$\hat{S}_x$	$+\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$
$\hat{S}_x$	$-\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$
$\hat{S}_y$	$+\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$
$\hat{S}_y$	$-\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$
$\hat{S}_z$	$+\frac{\hbar}{2}$ $\boxed{m_s = 1/2}$	$\begin{pmatrix}1\\0\end{pmatrix}$ “spin up”
$\hat{S}_z$	$-\frac{\hbar}{2}$ $\boxed{m_s = -1/2}$	$\begin{pmatrix}0\\1\end{pmatrix}$ “spin down”

most commonly used

- What about $\hat{S}^2$? We find

$$\hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_z^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ becomes

$$\boxed{\hat{S}^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

- As expected, $\hat{S}^2$ commutes with $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$

- All of the above eigenvectors are also eigenvectors of $\hat{S}^2$ with eigenvalue $\underbrace{\frac{3}{4}}_{\boxed{= s\,(s+1)\,\hbar^2}}\hbar^2$, as expected for $s = \frac{1}{2}$