Problem Sheet 1 - De-dimensionalisation

Some practice doing de-dimensionalisation & thinking about differential equations.

1. In the lectures I de-dimensionalised the governing equation for a train on level ground. Repeat the process, but leaving the dimensionless product in front of (a) the rolling friction term, and (b) the tractive force term. Show that if $A \to 2A$, $B \to 3B$ and $T \to \frac{4}{3}T$, each of the possible dimensionless products remains unchanged.

2. The equation for a falling stone is
$$m\frac{dv}{dt} + Bv^2 = mg.$$
What are the dimensions of each term? How many variables are there, and how many parameters? How many parameters would you expect to reduce this to by de-dimensionalisation? What sign should B have, and why?

3. De-dimensionalise the falling stone equation, keeping as many dimensionless products as possible $= 1$. Explain how you would calculate the true value of distance moved by the stone, its momentum and its kinetic energy in terms of your dimensionless variables.

4. An "inverted pendulum" has a mass m at the top of a rod of length ℓ free to pivot at its base. If a trolley (mass M) is attached to the pivot, it may be moved horizontally (along the x axis) to keep the pendulum upright (think of someone trying to balance a ruler on the end of her finger). The equations of motion for the pendulum are
$$(M+m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta = 0 \qquad \text{and} \qquad m\ell(-g\sin\theta + \ddot{x}\cos\theta + \ell\ddot{\theta}) = 0,$$
where the dots denote differentiation with respect to time t, g is acceleration due to gravity, and θ is the angle of the rod to the vertical.

   There are 3 variables in this problem. Identify them, and state their dimensions. Then de-dimensionalise the coupled equations, keeping as many terms as possible of order 1, and deriving expressions for any dimensionless products.

5. Fluid mechanics is governed by the Navier-Stokes equations
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P - g\mathbf{k} + \frac{\eta}{\rho}\nabla^2\mathbf{v}$$
for the fluid velocity $\mathbf{v}$. η is the fluid viscosity, ρ its density, P the pressure and g the acceleration due to gravity.

   What are the dimensions of each term? De-dimensionalise the equation by employing characteristic scales L, v_o, P_o and T for length, speed, pressure and time respectively. Show that the generic behaviour of fluids depends on the 3 dimensionless products
$$R = \frac{\rho v_o L}{\eta} \qquad F = \frac{v_o^2}{gL} \qquad S = \frac{L}{Tv_o}.$$
Under what conditions would you expect to be able to ignore the effects of (a) gravity and (b) viscosity?