6.3 Curl of a vector field, $\nabla \times \mathbf{F}$

When trying to find an interpretation of $\nabla \times \mathbf{F}$ in Cartesian coordinates, we effectively evaluated a tangential line integral around a small loop. We were actually making use of the definition of the curl in terms of a line integral:

$$\hat{\mathbf{n}} \cdot \mathrm{curl}\mathbf{F} = \hat{\mathbf{n}} \cdot \left(\nabla \times \mathbf{F}(\mathbf{r}) \right) = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{F} \cdot d\mathbf{r}$$

The path C encloses area A. $\hat{\mathbf{n}}$ is normal to the area. As $A \to 0$, the path encloses point $\mathbf{r}$. The shape of A is arbitrary.

Curl in a general orthogonal coordinate system

Again, this coordinate-free definition can be used to derive a formula for $\nabla \times \mathbf{F}$ in other orthogonal coordinate systems – see h/o on moodle if you wish.

As with grad and div, the derivation is mostly about keeping tabs on the effect of scale factors.

Result: $$\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w} \begin{vmatrix} h_u \hat{\mathbf{e}}_u & h_v \hat{\mathbf{e}}_v & h_w \hat{\mathbf{e}}_w \\ \dfrac{\partial}{\partial u} & \dfrac{\partial}{\partial v} & \dfrac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{vmatrix}$$

Explicit expressions for $\nabla \times \mathbf{F}$

system	$\nabla \times \mathbf{F}$
Cartesian	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$
Cylindrical polar	$\dfrac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$
Spherical polar	$\dfrac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\,\hat{\mathbf{e}}_\theta & r\sin\theta\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$

You should remember that these formulae exist.

All on p19 of “Basic Formulae and Statistical Tables”

Example:

Find curl of the (magnetic) field

$$\mathbf{H} = \left\{ \begin{array}{ll} \dfrac{I\rho}{2\pi a^2}\hat{\mathbf{e}}_\phi & \text{for } \rho \le a \\ \dfrac{I}{2\pi\rho}\hat{\mathbf{e}}_\phi & \text{for } \rho > a \end{array} \right\} = H(\rho)\hat{\mathbf{e}}_\phi$$

Cylindrical polar coordinates are the natural choice:

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ 0 & \rho H(\rho) & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left\{\hat{\mathbf{e}}_\rho\left[-\frac{\partial}{\partial z}\big(\rho H(\rho)\big)\right] - \rho\hat{\mathbf{e}}_\phi[0] + \hat{\mathbf{e}}_z\left[\frac{\partial}{\partial\rho}\big(\rho H(\rho)\big)\right]\right\}$$

$$= \frac{1}{\rho}\hat{\mathbf{e}}_z\frac{\partial}{\partial\rho}\big(\rho H(\rho)\big).$$

So $$\nabla \times \mathbf{H} = \left\{ \begin{array}{ll} \dfrac{I}{\pi a^2}\hat{\mathbf{e}}_z & \text{for } \rho \le a \\ \mathbf{0} & \text{for } \rho > a \end{array} \right.$$

6.4 The Laplacian and the equations of science

We can now derive $\nabla^2\Phi = \nabla\cdot\nabla\Phi$ when $\Phi = \Phi(u,v,w)$:

$$\nabla^2\Phi = \nabla\cdot\nabla\Phi = \nabla\cdot\left(\frac{1}{h_u}\frac{\partial\Phi}{\partial u}\hat{\mathbf{e}}_u + \frac{1}{h_v}\frac{\partial\Phi}{\partial v}\hat{\mathbf{e}}_v + \frac{1}{h_w}\frac{\partial\Phi}{\partial w}\hat{\mathbf{e}}_w\right)$$

So

$$\nabla^2\Phi = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial\Phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial\Phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial\Phi}{\partial w}\right)\right]$$

eg Cartesians (x,y,z); $h_x = h_y = h_z = 1$ and

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

as before.

Explicit expressions for $\nabla^2\Phi$

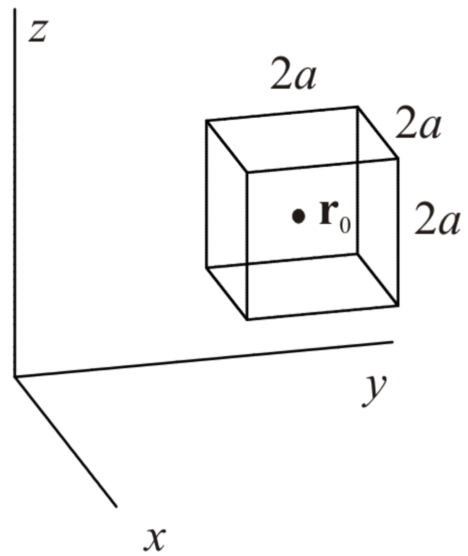
Cyl. polar	$\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2\Phi}{\partial\phi^2} + \rho\frac{\partial^2\Phi}{\partial z^2}\right]$
Sph. polar	$\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right]$

These appear (more-or-less) on p19 of “Basic Formulae and Statistical Tables”

$\nabla^2\Phi$ and the Equations of Science

What does $\nabla^2\Phi$ measure?

Take a cube, side $2a$, centre $\mathbf{r}_0 = (x_0, y_0, z_0)$.



The average value of $\Phi(\mathbf{r})$ in the cube is

$$\langle\Phi\rangle = \frac{1}{V}\int_V \Phi(\mathbf{r})\,dV.$$

Now expand $\Phi(\mathbf{r})$ about $\mathbf{r}_0$ in a Taylor expansion:

$$\begin{aligned}
\Phi(x,y,z) = \Phi(x_0,y_0,z_0) &+ (x-x_0)\left.\frac{\partial\Phi}{\partial x}\right|_{\mathbf{r}_0} + (y-y_0)\left.\frac{\partial\Phi}{\partial y}\right|_{\mathbf{r}_0} + (z-z_0)\left.\frac{\partial\Phi}{\partial z}\right|_{\mathbf{r}_0} \\
&+ \frac{1}{2}\left\{(x-x_0)^2\left.\frac{\partial^2\Phi}{\partial x^2}\right|_{\mathbf{r}_0} + (y-y_0)^2\left.\frac{\partial^2\Phi}{\partial y^2}\right|_{\mathbf{r}_0} + (z-z_0)^2\left.\frac{\partial^2\Phi}{\partial z^2}\right|_{\mathbf{r}_0}\right. \\
&\qquad + (x-x_0)(y-y_0)\left.\frac{\partial^2\Phi}{\partial x\partial y}\right|_{\mathbf{r}_0} + (y-y_0)(z-z_0)\left.\frac{\partial^2\Phi}{\partial y\partial z}\right|_{\mathbf{r}_0} \\
&\qquad \left. + (z-z_0)(x-x_0)\left.\frac{\partial^2\Phi}{\partial z\partial x}\right|_{\mathbf{r}_0}\right\} + \ldots
\end{aligned}$$

Substituting into the integral, and using $V = 8a^3$ gives (try it!)

$$\langle \Phi \rangle = \Phi(x_0, y_0, z_0) + \frac{a^2}{6}\left[\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}\right]_{\mathbf{r}_0} + O(a^4).$$

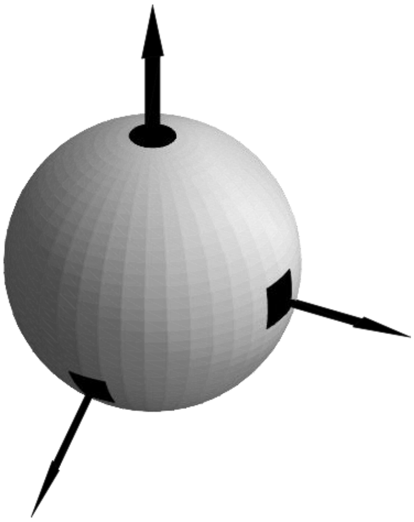
Therefore

$$\frac{a^2}{6}\nabla^2 \Phi \approx \langle \Phi \rangle - \Phi$$

$\nabla^2 \Phi$ is a measure of the difference between the value of the scalar Φ and its average value in the immediate neighbourhood.

Alternatively:

From the integral definition of the divergence



$$\nabla^2\Phi = \nabla\cdot\nabla\Phi = \lim_{V\to 0}\frac{1}{V}\oint_S \nabla\Phi\cdot d\mathbf{S}.$$

Let S be surface enclosing small volume V surrounding point $\mathbf{r}$.

$\nabla\Phi$ points up the slope in Φ. So if $\mathbf{r}$ is at a

(i) local **maximum,** $\nabla\Phi$ points inwards everywhere on $S \Rightarrow \nabla^2\Phi < 0$

(ii) local **minimum** $\rightarrow$ $\nabla\Phi$ points outwards everywhere on $S \Rightarrow \nabla^2\Phi > 0$

(iii) general point, $\nabla\Phi$ will point inwards or outwards at different parts of S, $\nabla\Phi\cdot d\mathbf{S}$ will be variously +ve and -ve, and the surface integral will be the net result of contributions. Thus

$\nabla^2\Phi$ measures the net gradient at $\mathbf{r}$

Equations of Science involving $\nabla^2\Phi$

$\nabla^2\Phi$ provides a measure of how $\Phi(\mathbf{r})$ relates to nearby values; $\therefore$ no surprise that it turns up in many important differential equations of science.

$$\nabla^2\Phi \begin{matrix} > \\ < \end{matrix} 0 \quad \Rightarrow \quad \text{net local} \begin{matrix} \text{minimum} \\ \text{maximum} \end{matrix} \text{in } \Phi$$

How might this cause a physical effect?

- "A locally high value of Φ will cause it to decrease with time"

$$\rightarrow \quad -\nabla^2\Phi \propto -\frac{\partial\Phi}{\partial t} \qquad \text{or} \qquad \nabla^2\Phi = \kappa\frac{\partial\Phi}{\partial t}.$$

This is the DIFFUSION EQUATION which arises in problems involving heat conduction, chemicals in suspension and aspects of materials science.

- "Φ will tend to become as smooth as possible"

$$\rightarrow \quad \nabla^2\Phi = 0$$

- LAPLACE'S EQUATION, which describes the electrostatic and gravitational potentials in vacuum (i.e. most of the Universe).

• "A local maximum in Φ results from some source"

$$\rightarrow \quad -\nabla^2\Phi = \rho(\mathbf{r})$$

This is the POISSON EQUATION, which e.g. gives the electrostatic potential due to a distribution of charges.

• "A local minimum causes Φ to spring back in time"

$$\rightarrow \quad \nabla^2\Phi \propto \frac{\partial^2\Phi}{\partial t^2} \qquad \text{or} \qquad \nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}.$$

This is the WAVE EQUATION, used in many areas of science.

These equations, along with combinations such as the wave equation plus source term, represent most of the fundamental equations of science involving scalar fields; and all involve ∇^2, the Laplacian.

Finally, ∇^2 also appears in a slightly peculiar expression, namely

$$\nabla^2\Phi \propto i\frac{\partial\Phi}{\partial t},$$

with $i = \sqrt{-1}$. This is in the SCHRÖDINGER EQUATION, which is a central part of quantum theory.