Problem Sheet 3 – Integration of Scalar and Vector Fields

The idea of this sheet is to get you used to integrating scalar and vector fields, using Cartesian coordinates. There will be more chances to integrate later in the unit.

1. If $\mathbf{F}$ is the vector field $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j}$ evaluate the line integral

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} \qquad \text{where} \quad A = (0,0,0) \quad \text{and} \quad B = (1,1,0)$$

   along the paths

   (a) $(0,0,0) \longrightarrow (0,1,0) \longrightarrow (1,1,0)$ in straight lines [needs 2 integrals]
   (b) $(0,0,0) \longrightarrow (1,1,0)$ in a straight line
   (c) along the short arc of the circle $x^2 + (y-1)^2 = 1$.

2. For the vector field $\mathbf{A} = (2xy+1)\mathbf{i} + (x^2+4y)\mathbf{j}$ evaluate the tangential line integral $\int_C \mathbf{A} \cdot d\mathbf{r}$ along the curves

   (a) $y = x$, from (0,0) to (1,1)
   (b) $y = x^2$, from (0,0) to (1,1)

   What property of $\mathbf{A}$ ensures that these results are the same? Find the potential function associated with $\mathbf{A}$ and hence confirm your result for (a) and (b).

3. Determine which of the following fields are *conservative*. For those which are, find the corresponding potential function.

   (a) $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$
   (b) $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$
   (c) $\mathbf{F} = y\sin(z)\mathbf{i} + x\sin(z)\mathbf{j} + xy\cos(z)\mathbf{k}$

4. Use a surface integral to find the area between the curves $y = x^2$ and $y = x^3$. You are advised to sketch the curves first.

5. Evaluate the flux integral of $\mathbf{A} = 3y\mathbf{i} - 12\mathbf{j} + 8x\mathbf{k}$ over the triangle in the yz plane with vertices at $(0,0,0)$, $(0,1,0)$ and $(0,0,3)$.

6. Let $\phi = 45x^2y$ and let V denote the closed region bounded by the planes $4x+2y+z=8$, $x=0$, $y=0$ and $z=0$. Evaluate the volume integral of ϕ over this volume.