Divergence $\nabla \cdot \mathbf{F}$ in non-Cartesian coordinates

This is an "optional extra" for those who like to see where things come from.

Coordinate-free definition of divergence

When trying to find an interpretation of $\nabla \cdot \mathbf{F}$ in Cartesian coordinates, we effectively took a flux integral over a small volume $\delta V = \delta x \delta y \delta z$, and showed that the divergence measures local sources of flux. In fact we were using the definition of divergence in terms of a flux integral:

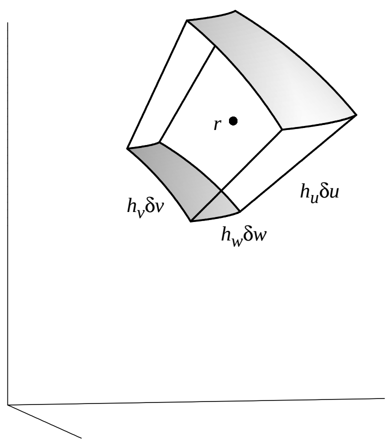
$$\boxed{\text{divergence of } \mathbf{F} \text{ at } \mathbf{r} = \text{div}\mathbf{F}(\mathbf{r}) = \nabla \cdot \mathbf{F}(\mathbf{r}) = \lim_{V\to 0} \frac{1}{V} \oint_S \mathbf{F} \cdot d\mathbf{S}}$$

The surface S encloses the volume V, which as $V \to 0$ converges around the point $\mathbf{r}$. The integral $\oint_S \mathbf{F} \cdot d\mathbf{S}$ gives the total flux emanating from the enclosed volume V (remember, the direction of surface element $d\mathbf{S}$ is the *outward* normal).

Divergence in a general orthogonal coordinate system.

We know how to evaluate the divergence in Cartesian coordinates. How do we evaluate $\nabla \cdot \mathbf{F}$ when $\mathbf{F}$ is a vector field expressed in any other orthogonal coordinate system? To derive the answer we repeat the process of evaluating a flux integral over the surface S of a small volume V surrounding the point $\mathbf{r} = (u, v, w)$, using what we now know about surface elements in a general coordinate system. The range of the *coordinates* in V is chosen to be

$$\begin{aligned} u &\pm \delta u/2 \\ v &\pm \delta v/2 \\ w &\pm \delta w/2 \end{aligned}$$



but of course the size of the volume depends on the scale factors, and is $V = (h_u\delta u)(h_v\delta v)(h_w\delta w)$ as shown in the figure (though pedants should note that, strictly speaking, these dimensions pass through $\mathbf{r}$ and not along the edges of the box).

There are 6 faces we need to consider. First look at the shaded faces, on which u is a constant $(u \pm \delta u/2)$ and v and w vary.

To keep things a bit easier, we will assume that V is so small that $\mathbf{F}$ can be taken as a constant across any given face, but may take different values on each face. The flux crossing the upper shaded face is then given by

$$\Phi_{u+} = \mathbf{F}(u + \delta u/2, v, w) \cdot \delta\mathbf{S}_{u+}$$

the subscript $u+$ denoting which face we are on. The surface element for this face is

$$\begin{aligned} \delta\mathbf{S}_{u+} &= +(\hat{\mathbf{e}}_v \times \hat{\mathbf{e}}_w)\, h_v h_w \delta v \delta w \\ &= +\hat{\mathbf{e}}_u\ h_v(u + \delta u/2, v, w)\ h_w(u + \delta u/2, v, w)\ \delta v\ \delta w. \end{aligned}$$

In this expression we note the key point that in a general coordinate system the scale factors h_u, h_v and h_w vary with position. Next we write the component of the vector $\mathbf{F}$ in the direction $\hat{\mathbf{e}}_u$ as F_u, so

$$\mathbf{F}(u, v, w) = F_u(u, v, w)\hat{\mathbf{e}}_u + F_v(u, v, w)\hat{\mathbf{e}}_v + F_w(u, v, w)\hat{\mathbf{e}}_w.$$

Then the flux is

$$\Phi_{u+} = F_u(u+\delta u/2, v, w) h_v(u+\delta u/2, v, w) h_w(u+\delta u/2, v, w)\ \delta v\ \delta w$$

This is similar to the Cartesian case, except for the dependence of the scale factors h_v and h_w on the coordinates. If we write $G_u = F_u h_v h_w$ (with all 3 terms evaluated at the same coordinates) we are back on more familiar territory:

$$\begin{aligned}\Phi_{u+} &= +G_u(u+\delta u/2, v, w) \times \delta v \delta w \\ &\approx \left(G_u(u,v,w) + \frac{\delta u}{2}\frac{\partial G_u}{\partial u}(u,v,w)\right)\ \delta v\ \delta w\end{aligned}$$

where the second expression is a Taylor expansion of the first and becomes exact as $\delta u \to 0$. On the lower shaded face, the outward normal is $-\hat{\mathbf{e}}_u$ and F_u, h_v and h_w must be evaluated at $(u - \delta u/2, v, w)$. Then

$$\begin{aligned}\Phi_{u-} &= -G_u(u-\delta u/2, v, w) \times \delta v \delta w \\ &\approx \left(-G_u(u,v,w) + \frac{\delta u}{2}\frac{\partial G_u}{\partial u}(u,v,w)\right)\ \delta v\ \delta w\end{aligned}$$

and the total flux across the two shaded faces is

$$\Phi_{u+} + \Phi_{u-} = \frac{\partial G_u}{\partial u}\delta u\ \delta v\ \delta w = \frac{\partial}{\partial u}(h_v h_w F_u)\delta u\ \delta v\ \delta w.$$

This whole process is repeated for the other pairs of faces at $v \pm \delta v/2$ and $w \pm \delta w/2$. Adding, we get

$$\oint_S \mathbf{F}\cdot d\mathbf{S} = \delta u \delta v \delta w \left[\frac{\partial}{\partial u}(F_u\ h_v\ h_w) + \frac{\partial}{\partial v}(F_v\ h_w\ h_u) + \frac{\partial}{\partial w}(F_w\ h_u\ h_v)\right]_{(u,v,w)}.$$

Finally, dividing by the volume $V = h_u h_v h_w \delta u \delta v \delta w$ enclosed by S and taking the limit we find our answer:

$$\boxed{\lim_{V\to 0}\frac{1}{V}\oint_S \mathbf{F}\cdot d\mathbf{S} = \nabla\cdot\mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(F_u\ h_v\ h_w) + \frac{\partial}{\partial v}(F_v\ h_w\ h_u) + \frac{\partial}{\partial w}(F_w\ h_u\ h_v)\right]}$$

Explicit expressions.

In Cartesians (x, y, z), $h_x = h_y = h_z = 1$ and

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

In cylindrical polars (ρ, ϕ, z), $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$ and

$$\nabla\cdot\mathbf{F} = \frac{1}{\rho}\left[\frac{\partial (F_\rho \rho)}{\partial \rho} + \frac{\partial F_\phi}{\partial \phi} + \frac{\partial (F_z \rho)}{\partial z}\right] = \frac{1}{\rho}\frac{\partial (\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

In spherical polars (r, θ, ϕ), $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$ and

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial (F_r r^2 \sin\theta)}{\partial r} + \frac{\partial (F_\theta r\sin\theta)}{\partial \theta} + \frac{\partial (F_\phi r)}{\partial \phi}\right] \\ &= \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (F_\phi)}{\partial \phi}\end{aligned}$$