Problem Sheet 1 – Differentiation of Vectors & Space Curves

The general idea of this sheet is to get you going with differentiating vector quantities, then to start exploring some properties of space curves.

1. Let $\mathbf{a}$ and $\mathbf{b}$ be time-dependent vectors: $\mathbf{a} = (2t)\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = (2t)^2\mathbf{j}$. Calculate

$$\frac{d\mathbf{a}}{dt}; \qquad \frac{d\mathbf{b}}{dt}; \qquad \mathbf{a} \times \mathbf{b}; \qquad \frac{d}{dt}(\mathbf{a} \times \mathbf{b}); \qquad \mathbf{a} \times \frac{d\mathbf{b}}{dt}; \qquad \frac{d\mathbf{a}}{dt} \times \mathbf{b}$$

   and hence verify that, for this particular choice of $\mathbf{a}$ and $\mathbf{b}$,

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \times \mathbf{b}.$$

2. Express the following curves in the parametric form $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.
   (a) $y = 4x^2$
   (b) $3x + 6y - z = 2$, $x + y = 0$
   (c) the circle of radius 2 centred at (1,2) in the (x, y) plane

3. Sketch the curve $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$.

4. Find the unit tangent vector $\hat{\mathbf{T}}$ for the following curves, both for a general point t and at the given points P.
   (a) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ at $P = (1, 1, 0)$.
   (b) $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j} + 4t\mathbf{k}$ at $P = (0, 3, 2\pi)$.

5. For the helix given by $\mathbf{r}(t) = \cos(2\pi t)\mathbf{i} + \sin(2\pi t)\mathbf{j} + (2\pi t)\mathbf{k}$, find
   (a) $d\mathbf{r}/dt$, $d^2\mathbf{r}/dt^2$ and $|d\mathbf{r}/dt|$.
   (b) the distance travelled s as a function of t (assume $s = 0$ when $t = 0$)
   (c) the curve expressed in terms of s, ie $\mathbf{r}(s)$
   (d) the unit tangent vector $\hat{\mathbf{T}}$ expressed both in terms of s and t, ie $\hat{\mathbf{T}}(t)$ and $\hat{\mathbf{T}}(s)$.

6. In Cartesian coordinates Newton's law of motion can be expressed as

$$\mathbf{F} = m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k})$$

   where $\mathbf{F}$ is a force and m is the mass of a particle. Solve these (three) differential equations for the case where $\mathbf{F} = qE\mathbf{j}$ (q and E are constants), given the initial conditions

$$\text{at} \quad t = 0: \quad x = 0, \quad \dot{x} = v_0, \quad y = 0, \quad \dot{y} = 0, \quad z = 0, \quad \dot{z} = 0.$$

   Your solution $\mathbf{r}(t)$ describes the trajectory of the particle - in this case the trajectory of a particle of charge q in a constant electric field $E\mathbf{j}$.