PH30030: Quantum Mechanics   Problems Sheet 3

This problems sheet covers section 3 of the course, on solutions of the time-independent Schrodinger equation (TISE) for many particle systems.

1. Work carefully through the separation of variables in spherical polar coordinates for the case where the potential energy depends only on the radial coordinate. Make sure that you can understand where the equation for the radial part of the wavefunction, i.e.,

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(V(r)+\frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right)R(r)=E\,R(r)$$

   comes from.

2. Consider a particle in a 3D spherical infinite well defined by $V(r)=0$ for $r \le a$, $V(r)=\infty$ for $r > a$, where a is the radius of the well. The spherically symmetric eigenfunctions have the form $u_{n00}(\underline{r}) = A\dfrac{\sin\left(\frac{n\pi r}{a}\right)}{r}$, where A is a normalisation constant. Normalise this wavefunction.

3. The motion of the nucleus in the hydrogen atom can be accounted for by using the centre of mass coordinate system. The reduced mass μ, and not the electron mass m_e, then appears in the energy eigenvalues of the TISE.
   (a) Find the percentage discrepancy in the energy eigenvalues E_n if m_e is used instead of μ.
   (b) Find the separation in wavelength between the red Balmer lines (transitions from n = 3 to n = 2) for both hydrogen and deuterium.
   (c) Find the binding energy of positronium, where the proton in hydrogen is replaced by a positron. The positron has the same mass as an electron but opposite charge.
   (d) For muonic hydrogen, the electron is replaced by a muon, which has the same charge and mass 206.77m_e. Find the wavelength for the Lyman-α line (transitions from n = 2 to n = 1).

4. The normalised $1s$ eigenfunction of the hydrogen atom is $u_{100}(r,\theta,\phi)=\sqrt{\dfrac{1}{\pi a_0^3}}\exp\left(-r/a_0\right)$ where $a_0=\dfrac{4\pi\epsilon_0}{e^2}\dfrac{\hbar^2}{\mu}$. Calculate the expectation value of the kinetic energy and the potential energy for this eigenfunction. Show that the sum of these expectation values is the energy of the $1s$ state. Hint: the useful integrals on problems sheet 1 will be useful here!

5. If $\phi(x_1,x_2)$ is the eigenfunction describing the positions of two particles at coordinates x_1 and x_2, show that the exchange operator $\hat{P}_{12}$ is Hermitian.
   Hint: You can treat x_1 and x_2 as dummy variables.

6. If the states for two particles $u_a(\underline{r}_i)$ and $u_b(\underline{r}_i)$ are orthogonal, and both are normalised, find the normalisation constant A in the expression
$u_\pm(\underline{r}_1,\underline{r}_2) = A\left[u_a(\underline{r}_1)u_b(\underline{r}_2) \pm u_a(\underline{r}_2)u_b(\underline{r}_1)\right]$.

   If $u_a(\underline{r}_i) = u_b(\underline{r}_i)$ and is normalised, find A.

   Note: Orthonormality of the eigenfunctions implies that $\int d^3r_i\, u_a^*(\underline{r}_i)\, u_b(\underline{r}_i) = \delta_{ab}$.

7. Consider two non-interacting particles of mass m in a one-dimensional infinite potential well described by $V(x) = 0$ for $0 \le x \le a$, $V(x) = \infty$ for $|x| > a$. The one-particle states are given by $u_n(x) = \sqrt{\dfrac{2}{a}}\sin\left(\dfrac{n\pi x}{a}\right)$ with $E_n = n^2\dfrac{\hbar^2\pi^2}{2ma^2}$. Find the spatially dependent part of the energy eigenfunctions and corresponding energy eigenvalues for: -

   (a) The ground state and first excited state of two distinguishable particles. Find the degeneracy of these states.
   (b) The ground state and first excited state of two indistinguishable bosons. Find the degeneracy of these states.
   (c) The ground state of two indistinguishable fermions. Find the degeneracy of this state.

8. Consider two non-interacting particles of mass m in a 1D infinite square well. Construct the Hamiltonian. Apply this Hamiltonian to each of the ground state wavefunctions of question 7 to find the energy eigenvalues.

9. Consider two non-interacting spin-half fermions in a 1D infinite square well. Find the ground state wavefunction if the spins of the fermions are either (i) anti-parallel or (ii) parallel.

10. Consider a two-particle system in 1D. Suppose that one particle is in the state $u_a(x)$ and the other is in the state $u_b(x)$ where these states are orthonormal. Find the expectation value of the square of the separation distance between the two particles
$\left\langle (x_1 - x_2)^2 \right\rangle = \left\langle x_1^2 \right\rangle + \left\langle x_2^2 \right\rangle - 2\left\langle x_1 x_2 \right\rangle$ if the particles are (a) distinguishable, (b) indistinguishable bosons, and (c) indistinguishable fermions.

    Note: Orthonormality of the states implies that $\int dx_i\, u_a^*(x_i)\, u_b(x_i) = \delta_{ab}$

11. For the single-particle spin states $\alpha_i = \begin{pmatrix}1\\0\end{pmatrix}$ and $\beta_i = \begin{pmatrix}0\\1\end{pmatrix}$, use the Pauli matrices to show that

    $\hat{S}_{xi}\alpha_i = \frac{\hbar}{2}\beta_i$, $\hat{S}_{xi}\beta_i = \frac{\hbar}{2}\alpha_i$, $\hat{S}_{yi}\alpha_i = i\frac{\hbar}{2}\beta_i$, $\hat{S}_{yi}\beta_i = -i\frac{\hbar}{2}\alpha_i$,
    $\hat{S}_{zi}\alpha_i = \frac{\hbar}{2}\alpha_i$ and $\hat{S}_{zi}\beta_i = -\frac{\hbar}{2}\beta_i$.

12. Consider a two-particle system of identical spin ½ fermions. By using the operators $\hat{S}^2$ and $\hat{S}_z$, and the relations derived in question 11, find the quantum numbers *S* and M_S corresponding to the spin states (i) $\chi_{11}(1,2) = \alpha_1\alpha_2$, (ii) $\chi_{1-1}(1,2) = \beta_1\beta_2$, (iii) $\chi_{10}(1,2) = \frac{1}{\sqrt{2}}[\alpha_1\beta_2 + \alpha_2\beta_1]$, and (iv) $\chi_{00}(1,2) = \frac{1}{\sqrt{2}}[\alpha_1\beta_2 - \alpha_2\beta_1]$.

13. Use the Pauli matrices to construct the ladder operators $\hat{S}_{\pm i} = \hat{S}_{xi} \pm i\hat{S}_{yi}$ for a spin ½ fermion in state *i*. Find the effect of $\hat{S}_{+i}$ and $\hat{S}_{-i}$ on the single particle spin state α_i. Hence, for a two-particle system of identical spin ½ fermions, find the effect of $\hat{S}_{+} = \hat{S}_{+1} + \hat{S}_{+2}$ and $\hat{S}_{-} = \hat{S}_{-1} + \hat{S}_{-2}$ on the two-particle spin state $\chi_{11}(1,2) = \alpha_1\alpha_2$.