PH20014: Electromagnetism 1

Problem Sheet 1: Answers

Electric Fields

Question 1.

At a particular temperature and pressure, a helium gas contains 5×10^{25} atoms per cubic metre. If an electric field of 10 kV m^{-1} applied to the gas causes an average electron cloud shift of 10^{-18} m, find the relative permittivity of the helium.

[ε_r=1.00018]

Let *P* be the magnitude of the induced dipole moment per unit volume.

[Note $\vec{P}$ is generally a vector but we only need to examine the magnitude of the vector here.]

From the lecture notes (section on 'Electric polarisation'), $\vec{P} = \frac{N\vec{p}}{V} = n\vec{p}$ where *n* is the number of dipoles per unit volume and $\vec{p}$ is the individual dipole on one atom.

$\vec{p} = q_b\vec{r}$ where q_b is the charge on the dipole and $\vec{r}$ is the separation of the charges.

From the question, the centre of the electron cloud of the helium has moved by $r = 10^{-18}$ m compared to the centre of the nucleus.

Helium has $Z = 2$ and so the charge involved is 2*e* and there are 5×10^{25} atoms per cubic meter.

$$P = np = nq_br = nq_br = n2ea = \left(5 \times 10^{25}\right)2\left(1.6 \times 10^{-19}\right)\left(10^{-18}\right) = 1.6 \times 10^{-11} \text{ Cm}^{-2} .$$

[Note: That is a polarisation in units of Coulombs per square meter coming from the units of N [m^{-3}], *e*[*C*] and a[*m*]. This is the same as the unit of electric displacement $\vec{D}$, which you can deduce from Gauss's Law for electric displacement. This is also consistent with the definition that $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$.]

But, by definition: $\vec{P} = \chi_e\varepsilon_0\vec{E}$. Looking just at the magnitude of the vector, $P = \chi_e\varepsilon_0 E$. So $\chi_e = P/\varepsilon_0 E$ where *E* is 10^4 Vm^{-1}.

This gives $\chi_e = \dfrac{1.6 \times 10^{-11}}{\left(8.85 \times 10^{-12}\right)\left(10^4\right)} = 1.8 \times 10^{-4}$ [Note – no units the susceptibility is just a number]

Finally, $\varepsilon_r = 1 + \chi_e = 1.00018$. [Relative permittivity also has no units as it's a number]

Question 2.

A material has an electric susceptibility of 3.5. Calculate the electric dipole moment, P, and the D-field if the electric field E is 15 Vm^{-1}.

[P = 4.6 x 10^{-10} Cm^{-2}, D = 6.0 x 10^{-10} Cm^{-2}]

Definitions from the course:

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_r \varepsilon_0 \vec{E}$

$\vec{P} = \chi_e \varepsilon_0 \vec{E}$

$\varepsilon_r = 1 + \chi_e$

Substituting in values from the question:

$P = 3.5\varepsilon_0 E = (3.5)(15)\varepsilon_0$

$P = 52.5\varepsilon_0$ Cm^{-2}

$P = 4.6 \times 10^{-10}$ Cm^{-2}

$D = \varepsilon_r \varepsilon_0 E = (1 + \chi_e)\varepsilon_0 E$

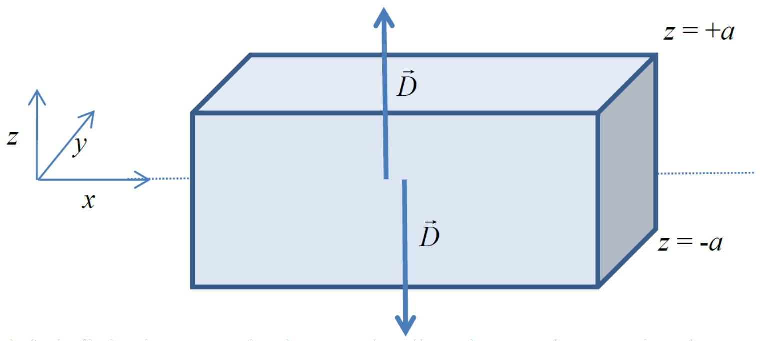
$D = (4.5)(15)\varepsilon_0 = 67.5\varepsilon_0$

$D = 6.0 \times 10^{-10}$ Cm^{-2}.

Question 3.

A slab of dielectric of relative permittivity ε_r fills the space between $z = \pm a$ in the x-y plane and contains a uniform density of free charges ρ_f per unit volume. Using Gauss' Law, find $\vec{E}$, $\vec{D}$ and $\vec{P}$ as functions of z. What is the surface density of polarisation charge on the surface of the dielectric?

[$\sigma = P = \pm(\varepsilon_r - 1)\rho_f\, a/\varepsilon_r$]



The slab is infinite in extent in the x and y directions as it occupies the x-y plane between $z = \pm a$.

If there is an electric field then it will **only** have a component in the z-direction as any field component in the x or y direction will break the symmetry and make one x-y direction special. As there is no one special direction in an infinite x-y plane, there is no x-y component of the electric field.

Using the vector version of Gauss' law, we know that, $\nabla \cdot \vec{D} = \rho_f$, assuming that $D_x = D_y = 0$.

We have, $\frac{\partial \vec{D}}{\partial z} = \rho_f$ which gives $D_z = \int \rho_f dz$.

So, $D_z = \rho_f z + C$ and $C = 0$ as $D_z = 0$ if $\rho_f = 0$.

Therefore, $D_z = \rho_f z$.

We also have $\vec{D} = \varepsilon\vec{E}$, so $E_z = \frac{D_z}{\varepsilon} = \frac{\rho_f z}{\varepsilon_r \varepsilon_0}$.

To find *P*, we will look at a general result first.

$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ so $\vec{P} = \vec{D} - \varepsilon_0\vec{E} = \varepsilon_r\varepsilon_0\vec{E} - \varepsilon_0\vec{E} = (\varepsilon_r - 1)\varepsilon_0\vec{E}$.

As $\vec{D} = \varepsilon_r\varepsilon_0\vec{E}$, we can combine these as

$\boxed{\vec{P} = \frac{\varepsilon_r - 1}{\varepsilon_r}\vec{D}}$ (This last result is a general one)

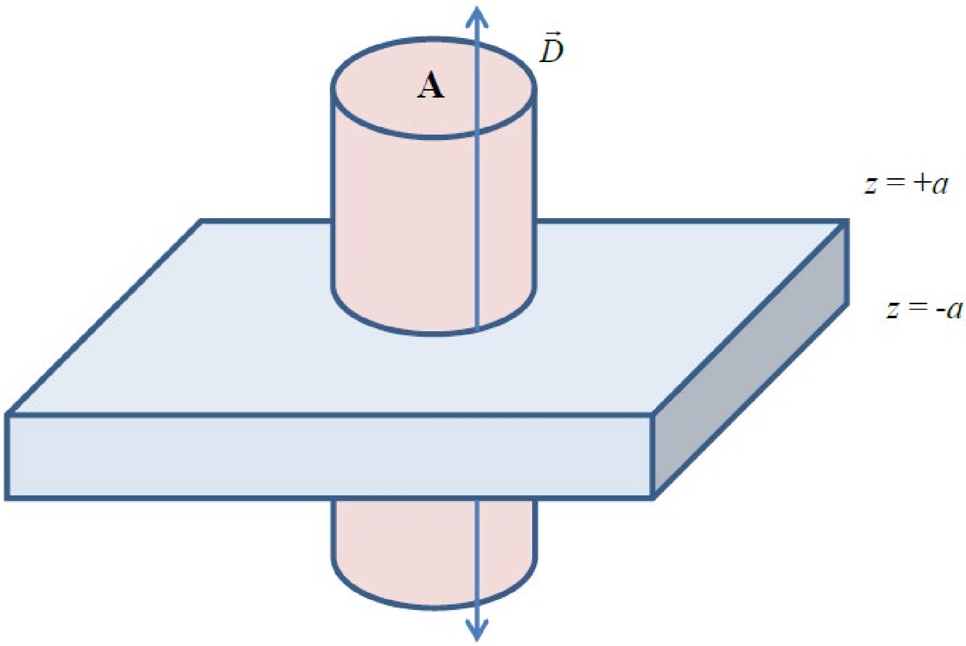
So, here, $P = \frac{\varepsilon_r - 1}{\varepsilon_r}\rho_f z$.

The solution so far has been for $-a < z < +a$. That is for the field inside the material.

At the top of the material, $\sigma_b = |P| = \frac{\varepsilon_r - 1}{\varepsilon_r}\rho_f a$ [See lecture note section on 'Relative permittivity and electric susceptibility']

At the bottom of the material, $\sigma_b = -\frac{\varepsilon_r - 1}{\varepsilon_r}\rho_f a$.

<u>Outside the material</u>, for $z > a$ and $z < -a$, we solve the problem as we would for a distance from a sheet of charge.



Using Gauss' Law for the Gaussian surface shown – the cylinder or pillbox shape $\oint_S \vec{D}\cdot d\vec{S} = \rho_f V$ with *V* the volume.

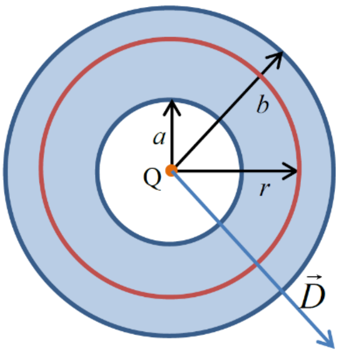
Here, $2DA = \rho_f[2aA]$, which simplifies to: $D = \rho_f a$.

So, the magnitude of electric displacement, *D*, outside the material is the same as the value it has reached at the surface of the material using $z = a$ in the result from inside the material.

Question 4.

A charge Q is placed at the centre of a spherical shell of LIH dielectric material with relative permittivity ε_r. The shell has an inner radius a, and an outer radius b, and the rest of space is a vacuum. Using Gauss' Law, find (a) the displacement field $\vec{D}$ at any distance, r, from the charge, (b) Find the electric field for (i) $r < a$, (ii) $a < r < b$ and (iii) $r > b$ and (c) find the electrostatic energy stored in the dielectric.
[(a) $D=Q/(4\pi R^2)$, (b) (i),(iii) $E = Q/(4\pi\varepsilon_0 R^2)$, (ii) $E = Q/(4\pi\varepsilon_r\varepsilon_0 R^2)$, (c) $U=Q^2/(8\pi\varepsilon_r\varepsilon_0)\,(1/a - 1/b)$]

This question has to be solved using Gauss' law for materials ($\nabla \cdot \vec{D} = \rho_f$) as the version of Gauss' law you have previously used in vacuum ($\nabla \cdot \vec{E} = \rho/\varepsilon_0$) **only** works if ***all*** charges in the situation are used. "All" means both the free charge and the bound polarisation charge giving a surface charge all contribute to ρ. Thus, we can solve this easily if we use $\nabla \cdot \vec{D} = \rho_f$ and *only* look at the free charge.



The free charge Q is at the centre of a spherical shell of dielectric material.

There is no special direction in the question. So, thinking in terms of a spherical coordinates, we expect every angle θ and ϕ to be equivalent. The field can only depend on the coordinate r.

We use a spherical Gaussian surface with radius r centred on the charge Q. The D-field has the same value at every point on the Gaussian surface as every point on the surface has the same value of r and the radial field is perpendicular to the surface of the Gaussian surface.

(a) From lecture notes (section on 'Free charges and the electric flux density'),

$$Q_f = \int_V \rho_f \cdot dv = \int_V \left(\nabla \cdot \vec{D}\right) dv = \int_A \vec{D} \cdot d\vec{A}, \text{ so } Q_f = \vec{D} \cdot \vec{A}.$$

Hence, $Q_f = \vec{D}(r)\left[4\pi r^2\right] \cdot \hat{r}$ and $\vec{D}(r) = \dfrac{Q_f}{4\pi r^2}\hat{r}$, where $\hat{r}$ is a unit vector in the r-direction.

This is true for all values of r, as the same free charge Q_f is enclosed for all values of r.

(b) The general relationship is that $\vec{D} = \varepsilon\vec{E} = \varepsilon_r\varepsilon_0\vec{E}$. Hence $\vec{E} = \dfrac{\vec{D}}{\varepsilon_r\varepsilon_0}$, where ε_r must be specified for each medium.

Thus:

(i) Vacuum. $\varepsilon_r = 1$. $\quad \vec{E} = \dfrac{\vec{D}}{\varepsilon_0} = \dfrac{Q_f}{4\pi\varepsilon_0 r^2}$

(ii) Dielectric with ε_r. $\quad \vec{E} = \dfrac{\vec{D}}{\varepsilon_0\varepsilon_r} = \dfrac{Q_f}{4\pi\varepsilon_0\varepsilon_r r^2}$

(iii) Same as (i).

(c) We note from the lecture notes (section on 'Energy stored in a dielectric') that $w = \frac{1}{2}\vec{D} \cdot \vec{E}$. This expression is for *w* the scalar energy density. This varies with *r* so has to be integrated over the volume of the dielectric in order to include all the contributions to the stored energy.

$w = \frac{1}{2}\vec{D} \cdot \vec{E} = \frac{1}{2}\left(\frac{Q_f}{4\pi r^2}\hat{r}\right) \cdot \left(\frac{Q_f}{4\pi\varepsilon_0\varepsilon_r r^2}\right) = \frac{Q_f^2}{32\pi^2\varepsilon_0\varepsilon_r r^4}$. In the dielectric with ε_r.

Thus, $U = \int_{r=a}^{r=b} \frac{Q_f^2}{32\pi^2\varepsilon_0\varepsilon_r r^4} dv$.

To integrate *dv*, we could use $dv = r^2 \sin\theta dr d\theta d\phi$ but there is no θ or ϕ dependence in the expression, so we can just use:
$dv = 4\pi r^2 dr$ (volume = area x thickness).

[Note: we can find this second expression for *dv* by integrating the first version over θ and ϕ with the appropriate limits or by considering it as an onion skin type layer of area *A* of a sphere with thickness *dr*, so that $dv = Adr = 4\pi r^2 dr$.]

Hence, $U = \int_{r=a}^{r=b} \frac{Q_f^2}{32\pi^2\varepsilon_0\varepsilon_r r^4} dv = \int_a^b \frac{Q_f^2}{8\pi\varepsilon_0\varepsilon_r r^2} dr = \frac{Q_f^2}{8\pi\varepsilon_0\varepsilon_r}\left[-\frac{1}{r}\right]_a^b = \frac{Q_f^2}{8\pi\varepsilon_0\varepsilon_r}\left(\frac{a-b}{ab}\right)$.

This stored energy is a measured in joules. So $U = \frac{Q^2}{8\pi\varepsilon_0\varepsilon_r}\left(\frac{a-b}{ab}\right)$ J.

Magnetic Fields

Question 5.

A material has a magnetic susceptibility of 0.01. Calculate the magnetic dipole moment per unit volume *M* and the *B*-field if the *H*-field is 10^3 A m^{-1}.

[M=10 Am^{-1}, B=1.3 x 10^{-3} T]

Basic relationships for magnetism:

$\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$

$\vec{M} = \chi_m \vec{H}$

$\vec{B} = \mu_0\left(1 + \chi_m\right)\vec{H} = \mu_0\mu_r\vec{H}$

Here, we look only at the magnitude of the vectors. Given that $\chi_m = 0.01$,

$M = \chi_m H = \left(10^{-2}\right)\left(10^3\right) = 10$ Am^{-1}.

Moreover,

$B = \mu_0\left(1.01\right)H = \mu_0\left(1.01\right)10^3 = 1010\mu_0$

$\mu_0 = 4\pi \times 10^{-7}$ Hm^{-1}.

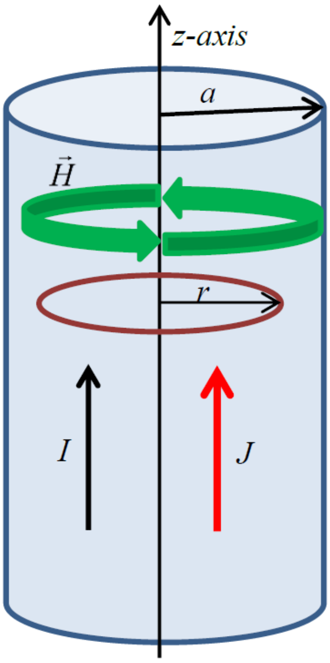
$B = 1.3 \times 10^{-3}$ T.

Question 6.

An infinitely long cylindrical conductor of radius *a* and permeability $\mu_0\mu_r$ is placed along the *z*-axis. The conductor carries a uniformly distributed current *I* along *z*. Use Ampère's Law to find *H* and then *M* for 0 < r < a.

[M= $(\mu_r\text{-}1)\ I\ R\ /(2\pi a^2)$]

This question requires use of the rules of magnetism in a material. We do not really want to perform a calculation for $\vec{B}$ that depends on the free current flowing, the effective surface current and the magnetisation current density; that would be too complicated. Instead, we work with $\vec{H}$ and find it by considering only the real currents flowing.



The (free) current flowing down the cylindrical conductor is *I*. It is uniformly distributed, which means that the current density is the same throughout the wire.

The area of the wire is πa^2.

Thus, the uniform current density is $J_f = \dfrac{I}{A} = \dfrac{I}{\pi a^2}$.

Using Ampère's law, $\nabla \times \vec{H} = \vec{J}_f + \dfrac{\partial \vec{D}}{\partial t}$, in integral form for a situation with no electric field – or a static *E*-field, $\oint_L \vec{H} \cdot d\vec{L} = \iint_S \vec{J}_f \cdot d\vec{S}$ applied to the circular Ampèrian path at radius *r*.

We expect that loops of *H*-field are being created by the current. As all points a distance *r* from the centre of the cylinder are equivalent, i.e. there is no angular variation in the situation for the uniformly distributed current, we expect the field to be constant along the Ampèrian path.

Thus, $\oint_L \vec{H} \cdot d\vec{L} = \iint_S \vec{J}_f \cdot d\vec{S}$ leads to:

$H(2\pi r) = J_f\left(\pi r^2\right) = \dfrac{I}{\pi a^2}\pi r^2$, which simplifies to $H = \dfrac{Ir}{2\pi a^2}$. Using vector notations:

$\vec{H} = \dfrac{Ir}{2\pi a^2}\hat{\theta}$, where $\hat{\theta}$ is a tangential unit vector (perpendicular to $\hat{r}$).

The question requires $\vec{M}$ where $\vec{M} = \chi_m \vec{H}$, so $\vec{M} = \dfrac{\chi_m I r}{2\pi a^2}\hat{\theta}$.

Alternatively, we could use $\mu_r = 1 + \chi_m$ in the form $\chi_m = \mu_r - 1$ to substitute into $\vec{M} = \chi_m \vec{H}$.

Possible extensions to the question.

a) Find $\vec{B}$. (To solve, we find $\vec{H}$ then use $\vec{B} = \mu \vec{H}$)

b) Find the field inside and outside the cylinder/wire. (When the Ampèrian loop is outside the cylinder the enclosed current just become *I*.)

c) Find the magnetostatic energy, *U*, stored in the wire (We use $w = \dfrac{1}{2}\vec{B} \cdot \vec{H}$ and integrate to find $U = \int w dv$.)

d) Find the self-inductance, L, of the wire from the stored energy (we could use $U = \frac{1}{2}LI^2$).

Question 7.

In a certain homogeneous isotropic medium for which $\mu_r = 4$, the magnetic field (B-field) is given in mT by $\vec{B} = 2\hat{i} - 5\hat{j} + 4\hat{k}$, where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors. Calculate (a) the magnetic susceptibility χ_m, (b) the H-field – the magnetic field intensity, (c) the magnetisation $\vec{M}$ and (d) the magnetic energy stored per unit volume.
[(a) χ=3, (b) H=(2, -5, 4)/4μ_0 mAm^{-1}, (c) M= 3(2, -5, 4)/4μ_0 mAm^{-1}, (d) W = 4.5 J m^{-3}]

From the question, we know that $\mu_r = 4$ and $\vec{B} = 2\hat{i} - 5\hat{j} + 4\hat{k}$ mT.
[Note: The fact that the initial field is in millitesla is recognised in the subsequent units.]
(a) We know that $\mu_r = 1 + \chi_m$ and we have $\mu_r = 4$, so $\chi_m = 3$.
(b) We know that $\vec{B} = \mu\vec{H} = \mu_0\mu_r\vec{H}$, so $\vec{H} = \dfrac{\vec{B}}{\mu_0\mu_r}$ and we can replace:

$$\vec{H} = \frac{1}{4\mu_0}\left(2\hat{i} - 5\hat{j} + 4\hat{k}\right) \text{ mAm}^{-1}.$$

(c) We know that $\vec{M} = \chi_m\vec{H}$, so we can replace:

$$\vec{M} = \frac{\chi_m}{4\mu_0}\left(2\hat{i} - 5\hat{j} + 4\hat{k}\right) = \frac{3}{4\mu_0}\left(2\hat{i} - 5\hat{j} + 4\hat{k}\right) \text{ mAm}^{-1}$$

(d) We know that $w = \frac{1}{2}\vec{B}\cdot\vec{H}$, so we can substitute:

$$w = \frac{1}{2}\left(2\hat{i} - 5\hat{j} + 4\hat{k}\right)\times 10^{-3} \times \left[\frac{1}{4\mu_0}\left(2\hat{i} - 5\hat{j} + 4\hat{k}\right)\right]\times 10^{-3} = \frac{1}{8\mu_0}(4 + 25 + 16)\times 10^{-6}$$

and $w = \dfrac{45\times 10^{-6}}{8\left(4\pi\times 10^{-7}\right)} = 4.5$ J m^{-3}.

Note that this energy density had no dependence on spatial coordinates as the B-field and hence the H-field are not spatially varying. In this special case, if asked to find the energy stored in a volume V then $U = wV$.

Question 8.

Use your new definition of $\vec{B}$ and a Maxwell equation to show that $\nabla\cdot\vec{H} = -\nabla\cdot\vec{M}$.
Using your understanding of divergence, explain the meaning of this result.
What does this tell you about the fields around a bar magnet?

It is the general case that $\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right)$.

It is also universally true that $\nabla\cdot\vec{B} = 0$.
Hence, $\nabla\cdot\left[\mu_0\left(\vec{H} + \vec{M}\right)\right] = \mu_0\nabla\cdot\left(\vec{H} + \vec{M}\right) = \mu_0\left(\nabla\cdot\vec{H} + \nabla\cdot\vec{M}\right) = 0$.

Because $\mu_0 \neq 0$, this requires that $\nabla\cdot\vec{H} + \nabla\cdot\vec{M} = 0$ and therefore: $\nabla\cdot\vec{H} = -\nabla\cdot\vec{M}$.

A bar magnet is a magnetised bar of material. The vacuum surrounding it contains a magnetic field ($\vec{B}$) that originates from an *H*-field ($\vec{B} = \mu_0 \vec{H}$) as there can be no magnetisation of the vacuum. Where does this *H*-field come from? There are no real external currents flowing in a bar magnet and we start by associating the *H*-field with currents.

The equation above shows that where there is a divergence of the magnetisation there is a (negative) divergence of the *H*-field. This means that a sink of magnetisation becomes a source of *H*-field. Thus, the boundaries of the magnet where there is spatial variation in magnetisation produce the *H*-field that gives the magnetic field around the bar magnet.

We noted in the lectures (section 'Macroscopic view: the magnetisation') that for a magnetised material there is a **magnetisation-induced volume current density** $\vec{J}_m = \nabla \times \vec{M}$ and a **magnetisation-induced surface current density** $\vec{k}_m = \vec{M} \times \hat{n}$, where $\hat{n}$ is the unit vector normal to the surface. The other way to understand the field around the magnet is that it is created by these currents related to the spatial configuration of the magnetisation.

Question 9.

The Dielectric Relaxation Time. Use the following equations,

$$\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t} \qquad \text{(The continuity Equation)},$$

$$\vec{J}_f = \sigma \vec{E} \qquad \text{(Ohm's law) and}$$

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\varepsilon_0} \qquad \text{(Gauss' Law)},$$

to find a differential equation which has the solution $\rho_f = \rho_{f0} e^{-t/\tau}$, where $\tau = \frac{\varepsilon_0}{\sigma}$. ρ_{f0} is the initial charge density at $t = 0$ and τ is the relaxation or rearrangement time.

Using the equations in the question: $\nabla \cdot \vec{J}_f = \nabla \cdot \left(\sigma \vec{E}\right) = \sigma \nabla \cdot \vec{E} = \sigma \frac{\rho_f}{\varepsilon_0}$.

But we also have: $\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$.

So, $\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon_0} \rho_f$, which we can rewrite as $\frac{\partial \rho_f}{\rho_f} = -\frac{\sigma}{\varepsilon_0} \partial t$ and integrate

$$\int \frac{\partial \rho_f}{\rho_f} = -\frac{\sigma}{\varepsilon_0} \int \partial t$$

to obtain $\ln \rho_f = -\frac{\sigma}{\varepsilon_0} t + C$. Taking the exponential of both sides we see that:

$e^{\ln \rho_f} = e^{-\frac{\sigma}{\varepsilon_0} t + C} = e^C e^{-\frac{\sigma}{\varepsilon_0} t}$, which simplifies to $\rho_f = \rho_{f0} e^{-\frac{t}{\tau}}$, where $\tau = \frac{\varepsilon_0}{\sigma}$.

ρ_{f0} is the initial charge density at $t = 0$ and τ is the relaxation or rearrangement time.

[Note: the dielectric relaxation time is short for good conductors and long for good dielectrics. For copper, $\tau = 1.5 \times 10^{-19}$ s whilst for fused quartz $\tau = 51.2$ days.]