Chapter 3: Electromagnetic waves in linear, isotropic and homogeneous materials

How do EM waves propagate through materials? It seems very complicated. There are many types of materials and they do not seem to have a lot in common. How can we understand an EM wave propagating through air (a mostly lossless dielectric), or thick glass (a more lossy dielectric), or a thin layer of gold (a good conductor)?

What is the difference between dielectric and conducting materials for EM propagation?

Throughout these lectures, the term $\sigma/\omega\varepsilon$ is key. It describes conductivity per wavelength, with respect to the dielectric properties of the material.

Reminder: EM waves in vacuum

What do we know about EM waves in vacuum?

Maxwell's equations in vacuum:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \text{(ME1}_\text{V}\text{)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{(ME2}_\text{V}\text{)}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(ME3}_\text{V}\text{)}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(ME4}_\text{V}\text{)}$$

From the first part of Electromagnetism 1, we know how to:

- Solve Maxwell's equations to find a wave equation whose phase velocity is the speed of light.
- The electric and magnetic field are linked and the **impedance** is the ratio of the amplitudes of these fields.
- The wave carries energy as shown by the Poynting vector.

Electromagnetic waves in materials

Maxwell's equations in materials:

$$\boxed{\nabla \cdot \vec{D} = \rho_f} \qquad \text{(ME1}_\text{M}\text{)}$$

$$\boxed{\nabla \cdot \vec{B} = 0} \qquad \text{(ME2}_\text{M}\text{)}$$

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \qquad (\mathrm{ME3_M})$$

$$\boxed{\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}} \qquad (\mathrm{ME4_M})$$

In LIH materials:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \qquad \text{Eq. 3. 1}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} \qquad \text{Eq. 3. 2}$$

$$\vec{H} = \frac{1}{\mu_0 \mu_r} \vec{B} = \frac{1}{\mu} \vec{B} \qquad \text{Eq. 3. 3}$$

$$\vec{M} = \chi_m \vec{H} \qquad \text{Eq. 3. 4}$$

The key point here is that, for materials, we now have Maxwell's equations that are mathematically very similar to those in vacuum. We can therefore manipulate them in a similar way and extract meaningful physical information without having to do very different maths. In other words, we can transpose our pre-existing knowledge onto a new problem.

The wave equation in an ideal LIH dielectric (insulator)

What happens to Maxwell's equations in the absence of free charges and current?

Taking the curl of $\mathrm{ME3_M}$, we obtain:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right). \qquad \text{Eq. 3. 5}$$

On the left hand side, we can apply:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}. \qquad \text{Eq. 3. 6}$$

But without free charges, $\rho_f = 0$, so

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 0. \qquad \text{Eq. 3. 7}$$

It follows that

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{E}. \qquad \text{Eq. 3. 8}$$

On the right hand side,

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \nabla \times \left(-\frac{\partial \left(\mu \vec{H}\right)}{\partial t}\right) = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) = -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right) \qquad \text{Eq. 3. 9}$$

but without free currents, $\vec{J}_f = 0$, so

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\mu \frac{\partial^2 \vec{D}}{\partial t^2} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} . \qquad \text{Eq. 3. 10}$$

Combining the left and right hand sides:

$$\boxed{\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}}, \qquad \text{Eq. 3. 11}$$

which looks like a wave equation.

Phase velocity in an ideal LIH insulator

The **phase velocity** is:

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0 \mu_r \varepsilon_0 \varepsilon_r}} \text{, so } \boxed{v_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}}} . \qquad \text{Eq. 3. 12}$$

This means that in an ideal LIH dielectric, the electromagnetic wave propagates similarly to the way it does in free space – it remains a plane wave but its phase velocity is lowered.

We now introduce a new quantity: the **refractive index** of a material, which is defined as:

$$n = \frac{c}{v_p} = \sqrt{\mu_r \varepsilon_r} . \qquad \text{Eq. 3. 13}$$

Also note that

$$v_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}\sqrt{\mu_r \varepsilon_r}} = \frac{1}{\sqrt{\mu\varepsilon}} . \qquad \text{Eq. 3. 14}$$

Note: The **Maxwell relation** ($n = \sqrt{\mu_r \varepsilon_r} \approx \sqrt{\varepsilon_r}$) states that, for common materials, the refractive index is proportional to the relative permittivity, since the relative permeability for most materials is close to one, $\mu_r \approx 1$.

Intrinsic impedance in an ideal LIH insulator

How does a linearly polarized plane wave travel inside an ideal insulator?

We consider a plane wave travelling along the *x*-direction:

$$\vec{E} = E_0 e^{i(kx - \omega t)} \hat{y} . \qquad \text{Eq. 3. 15}$$

We then look at the curl of this vector:

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_0 e^{i(kx-\omega t)} & 0 \end{vmatrix} = \frac{\partial}{\partial x}\left[E_0 e^{i(kx-\omega t)}\right]\hat{z} = ikE_0 e^{i(kx-\omega t)}\hat{z}\,. \quad \text{Eq. 3. 16}$$

Remembering ME3$_M$, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we can write:

$$ikE_0 e^{i(kx-\omega t)}\hat{z} = -\frac{\partial \vec{B}}{\partial t}\,. \quad \text{Eq. 3. 17}$$

We can integrate this with respect to *t*:

$$\int\left(-\frac{\partial \vec{B}}{\partial t}\right)dt = \int\left(ikE_0 e^{i(kx-\omega t)}\hat{z}\right)dt = \frac{ik}{-i\omega}E_0 e^{i(kx-\omega t)}\hat{z} + \mathrm{C}\,. \quad \text{Eq. 3. 18}$$

Where the constant is zero, because we are only interested in the oscillations of the field. Therefore, we can write:

$$-\vec{B} = \frac{ik}{-i\omega}E_0 e^{i(kx-\omega t)}\hat{z} \quad \text{Eq. 3. 19}$$

and

$$\vec{B} = \frac{k}{\omega}E_0 e^{i(kx-\omega t)}\hat{z}\,. \quad \text{Eq. 3. 20}$$

We also know that $\vec{B} = \mu \vec{H}$, so

$$\vec{H} = \frac{k}{\mu\omega}E_0 e^{i(kx-\omega t)}\hat{z} = H_0 e^{i(kx-\omega t)}\hat{z}\,. \quad \text{Eq. 3. 21}$$

We remember that the **intrinsic impedance** *Z* is the ratio of (complex) amplitudes:

$$Z = \frac{E_0}{H_0} = \frac{E_0}{\frac{k}{\mu\omega}E_0} = \frac{\mu\omega}{k} = \mu v_p = \mu\frac{1}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{\mu_r}{\varepsilon_r}}Z_0 \quad \text{Eq. 3. 22}$$

where *Z* is the intrinsic impedance of the material and Z_0 is the wave impedance in vacuum: $Z_0 \approx 377\ \Omega$.

In a perfect dielectric, an EM wave propagates in the same way as in vacuum but with scaled phase velocity:

$$v_p = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = \frac{c}{n} \quad \text{Eq. 3. 23}$$

and with scaled impedance:

$$Z = \sqrt{\frac{\mu_r}{\varepsilon_r}}Z_0\,. \quad \text{Eq. 3. 24}$$

So far, we discussed dielectrics. How about conducting materials?

Ohm's law and the current density

Now we look at conducting materials. A conducting material obeys Ohm's law: $V = R \cdot I$, so $I = V/R$.

From A-Levels, we also know **Pouillet's law** which states that:

$$R = \rho \frac{L}{A}, \qquad \text{Eq. 3. 25}$$

where R is the electrical resistance, L is the length of the material, A is the cross-sectional area of the material and ρ is the **electric resistivity**.

Together with $E = V/L$, we can substitute:

$$I = \frac{L \cdot E}{\rho \frac{L}{A}} = \frac{1}{\rho} A \cdot E, \qquad \text{Eq. 3. 26}$$

which leads to:

$$\frac{I}{A} = \frac{1}{\rho} E = J = \sigma E \qquad \text{Eq. 3. 27}$$

and generally:

$$\boxed{\vec{J} = \sigma \vec{E}}. \qquad \text{Eq. 3. 28}$$

Here we used the fact that J is the magnitude of the **current density** (the amount of charge per unit time that flows through a unit area of a chosen cross section) and σ is the **conductivity** (the inverse of the resistivity).

Modified wave equation in a conducting medium

How does a linearly polarized plane wave travel inside a conducting material?

Remembering ME3M, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we look at the curl of this equation:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right). \qquad \text{Eq. 3. 29}$$

On the left hand side, we can apply:

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}. \qquad \text{Eq. 3. 30}$$

But without free charges, $\rho_f = 0$. There is not bulk density in charge in the conductor. This is because all the free charges inside the material repel and try to stay away from each other. So, the bulk density of free charges is zero, but there are free charges at the surface and the surface density of charges is not zero. There will be a conductivity at the surface ($\sigma \neq 0$). So,

$$\nabla \cdot \vec{D} = \varepsilon \nabla \cdot \vec{E} = 0. \qquad \text{Eq. 3. 31}$$

It follows that

$$\nabla \times \left(\nabla \times \vec{E}\right) = -\nabla^2 \vec{E}. \qquad \text{Eq. 3. 32}$$

On the right hand side,

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = \nabla \times \left(-\frac{\partial\left(\mu \vec{H}\right)}{\partial t}\right) = -\mu \frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right) = -\mu \frac{\partial}{\partial t}\left(\vec{J}_f + \frac{\partial \vec{D}}{\partial t}\right). \qquad \text{Eq. 3. 33}$$

Now we have

$$\vec{J}_f = \sigma \vec{E}. \qquad \text{Eq. 3. 34}$$

So,

$$\nabla \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu \frac{\partial^2 \vec{D}}{\partial t^2} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}, \qquad \text{Eq. 3. 35}$$

where we used our knowledge of $\vec{D} = \varepsilon \vec{E}$.
Combining both sides of the equation, we obtain a **modified wave equation**:

$$\boxed{\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}}. \qquad \text{Eq. 3. 36}$$

Solving the modified wave equation

Here, a plane wave solution does not quite work. But we can use very similar maths:

$$\boxed{\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}}. \qquad \text{Eq. 3. 37}$$

For the plane wave we discussed before, we simply have:

$$\gamma = ik. \qquad \text{Eq. 3. 38}$$

On the left hand side of the modified wave equation, we have:

$$\nabla^2 \vec{E} \to \frac{\partial^2 \vec{E}}{\partial x^2} = \frac{\partial^2\left(\vec{E}_0 e^{(\gamma x - i\omega t)}\right)}{\partial x^2} = \gamma^2 \vec{E}. \qquad \text{Eq. 3. 39}$$

On the right hand side of the modified wave equation, we have:

$$\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} = \mu\varepsilon \left(-i\omega\right)^2 \vec{E} + \mu\sigma\left(-i\omega\right)\vec{E}. \qquad \text{Eq. 3. 40}$$

Combining both sides, we obtain:

$$\gamma^2\vec{E} = \mu\varepsilon\left(-i\omega\right)^2\vec{E} + \mu\sigma\left(-i\omega\right)\vec{E} = -\mu\varepsilon\omega^2\vec{E} - \mu\sigma\omega\vec{E}. \qquad \text{Eq. 3. 41}$$

We take γ to be complex, so we can write:

$$\boxed{\gamma = -\alpha + i\beta}, \qquad \text{Eq. 3. 42}$$

(the minus sign in front of α is for convenience, this will become clear later). We then obtain:

$$\gamma^2 = \left(-\alpha + i\beta\right)^2 = \alpha^2 - \beta^2 - i2\alpha\beta = -\mu\varepsilon\omega^2 - i\mu\sigma\omega, \qquad \text{Eq. 3. 43}$$

where we can identify the real and imaginary parts:

$$\left.\begin{aligned}\alpha^2 - \beta^2 &= -\mu\varepsilon\omega^2\\ -2\alpha\beta &= -\mu\sigma\omega\end{aligned}\right\} \rightarrow \frac{-2\alpha\beta}{\alpha^2-\beta^2} = \frac{2\alpha\beta}{\beta^2-\alpha^2} = \frac{\cancel{\mu}\sigma\cancel{\omega}}{\cancel{\mu}\varepsilon\omega^{\cancel{2}}} = \frac{\sigma}{\varepsilon\omega}. \qquad \text{Eq. 3. 44}$$

This ratio is dimensionless:

$$\frac{\sigma}{\varepsilon\omega} = \frac{\left[s\cdot \cancel{m}^{-1}\right]}{\left[F\cdot \cancel{m}^{-1}\right]\left[s^{-1}\right]} = \frac{\left[\dfrac{s^3\cdot A^2}{m^2\cdot kg}\right]}{\left[\dfrac{s^4\cdot A^2}{m^2\cdot kg}\right]\left[s^{-1}\right]}. \qquad \text{Eq. 3. 45}$$

The dielectric function

Next, we are going to use $\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$ in equation ME4$_M$, which is $\nabla\times\vec{H} = \vec{J}_f + \dfrac{\partial\vec{D}}{\partial t}$. Here, we can use

$$\frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}, \qquad \text{Eq. 3. 46}$$

so

$$\vec{E} = -\frac{1}{i\omega}\frac{\partial\vec{E}}{\partial t}. \qquad \text{Eq. 3. 47}$$

So,

$$\nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} = \sigma\left(-\frac{1}{i\omega}\frac{\partial\vec{E}}{\partial t}\right) + \varepsilon\frac{\partial\vec{E}}{\partial t} = \left(i\frac{\sigma}{\omega} + \varepsilon\right)\frac{\partial\vec{E}}{\partial t} \qquad \text{Eq. 3. 48}$$

$$\nabla\times\vec{H} = \left(i\frac{\sigma}{\omega} + \varepsilon_0\varepsilon_r\right)\frac{\partial\vec{E}}{\partial t} = \varepsilon_0\left(\varepsilon_r + i\frac{\sigma}{\varepsilon_0\omega}\right)\frac{\partial\vec{E}}{\partial t} = \varepsilon_0\varepsilon\left(\omega\right)\frac{\partial\vec{E}}{\partial t}. \qquad \text{Eq. 3. 49}$$

Note that

$$\varepsilon_r + i\frac{\sigma}{\varepsilon_0\omega} = \varepsilon_r\left(1 + i\frac{\sigma}{\varepsilon\omega}\right), \qquad \text{Eq. 3. 50}$$

where the ratio we discussed above appears.

The equation

$$\nabla \times \vec{H} = \varepsilon_0 \varepsilon(\omega) \frac{\partial \vec{E}}{\partial t} \qquad \text{Eq. 3. 51}$$

has a familiar mathematical shape and we have introduced a new quantity – the **dielectric function**:

$$\boxed{\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}}, \qquad \text{Eq. 3. 52}$$

which is often written as:

$$\varepsilon(\omega) = \varepsilon' + i\varepsilon''. \qquad \text{Eq. 3. 53}$$

The dielectric function links time-varying external electric fields to the magnetic field intensity created in response inside the material. The dielectric function completely describes the response of a material to electric fields oscillating with angular frequency ω. In particular, it captures both the 'insulating aspect', expressed by the dielectric "constant" ε_r, and the conduction of a material, through the appearance of conductivity σ in the dimensionless ratio $\frac{\sigma}{\varepsilon\omega}$.

Intrinsic impedance in a conducting medium

Now, let's have a look at the intrinsic impedance of the material, when both the dielectric constant and the conductivity are taken into account.

We saw before that

$$Z = \frac{\mu\omega}{k} = \frac{\mu\omega}{\text{wave vector}}. \qquad \text{Eq. 3. 54}$$

The field we are working with here is $\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$. For a plane wave, $\gamma = ik$, so

$$k = \frac{\gamma}{i} = -i\gamma. \qquad \text{Eq. 3. 55}$$

Therefore,

$$Z = \frac{\mu\omega}{-i\gamma}. \qquad \text{Eq. 3. 56}$$

We also found that

$$\gamma^2 = \mu\omega(-\varepsilon\omega - i\sigma). \qquad \text{Eq. 3. 57}$$

So, $k = -i\sqrt{\mu\omega(-\varepsilon\omega - i\sigma)} = -i\sqrt{-(\mu\varepsilon\omega^2 + i\mu\sigma\omega)}$ and we can write:

$$Z = \frac{\mu\omega}{-i\sqrt{-\left(\mu\varepsilon\omega^2 + i\,\mu\sigma\omega\right)}} = \frac{\mu\omega}{-i\sqrt{i^2\left(\mu\varepsilon\omega^2 + i\,\mu\sigma\omega\right)}} = \frac{\sqrt{\mu^2\omega^2}}{-ii\sqrt{\mu\varepsilon\omega^2 + i\,\mu\sigma\omega}}.$$

$$Z = \sqrt{\frac{\mu^2\omega^2}{\mu\varepsilon\omega^2\left(1 + i\,\frac{\mu\sigma\omega}{\mu\varepsilon\omega^2}\right)}} = \sqrt{\frac{\mu}{\varepsilon\left(1 + i\,\frac{\sigma}{\varepsilon\omega}\right)}}$$

Eq. 3. 58

The intrinsic impedance is now a complex number, where the imaginary part is present because of the dimensionless ratio $\frac{\sigma}{\varepsilon\omega}$.

This imaginary part can be neglected if $\frac{\sigma}{\varepsilon\omega} << 1$, which is to say, if the conductivity is such that $\sigma << \varepsilon\omega$, i.e. if the conductive properties are much poorer in comparison to the dielectric properties, at a given frequency.
Note that if we neglect the imaginary part, we obtain

$$Z \approx \sqrt{\frac{\mu}{\varepsilon}} \qquad \text{Eq. 3. 59}$$

which is the expression for a lossless dielectric material.

Therefore, $\frac{\sigma}{\varepsilon\omega}$ quantifies how good a conductor a material is at a given frequency. (which also makes sense, since the ratio is conductivity per frequency!)

The complex refractive index in a conductive medium

We have established that γ can be complex in

$$\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}. \qquad \text{Eq. 3. 60}$$

We can therefore write:

$$\vec{E} = \vec{E}_0 e^{(-\alpha x + i\beta x - i\omega t)} = \vec{E}_0 e^{(-\alpha x)} e^{i(\beta x - \omega t)}. \qquad \text{Eq. 3. 61}$$

In this form, the term $e^{-\alpha x}$ represents an exponential decay with distance x and decay constant α.

Our wave vector is complex because it describes both propagation and decay of the wave:

$$k \to \gamma = -\alpha + i\beta, \qquad \text{Eq. 3. 62}$$

since we have

$$\vec{E} = \vec{E}_0 e^{(-\alpha x + i\beta x - i\omega t)} = \vec{E}_0 e^{i\left(-\frac{\alpha x}{i} + \beta x - \omega t\right)} = \vec{E}_0 e^{i(i\alpha x + \beta x - \omega t)} = \vec{E}_0 e^{i[(\beta + i\alpha)x - \omega t]}$$

Eq. 3. 63

Because we have a complex wave vector, we now also have a **complex refractive index**:

$$\boxed{\tilde{n} = n + i\eta = \sqrt{\varepsilon(\omega)} = \sqrt{\varepsilon' + i\varepsilon''}},$$ Eq. 3. 64

where n is the refractive index (describing the change in phase velocity) and η is the **extinction coefficient** (describing the attenuation/absorption of the EM wave).

The electric and magnetic fields in a conductive medium

Let's take another look at our complex intrinsic impedance:

$$Z = \frac{\mu\omega}{k} = \frac{\mu\omega}{\text{wave vector}} = \frac{\mu\omega}{\beta + i\alpha}\frac{\beta - i\alpha}{\beta - i\alpha} = \frac{\mu\omega}{\alpha^2 + \beta^2}(\beta - i\alpha).$$ Eq. 3. 65

So,

$$\text{Re}[Z] = \frac{\mu\omega\beta}{\alpha^2 + \beta^2}$$ Eq. 3. 66

and

$$\text{Im}[Z] = -\frac{\mu\omega\alpha}{\alpha^2 + \beta^2}.$$ Eq. 3. 67

We then consider the polar form of complex numbers:

$$a + ib = re^{i\phi} = r(\cos\phi + i\sin\phi),$$ Eq. 3. 68

so that

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{b}{a}.$$ Eq. 3. 69

In our case,

$$\tan\phi = -\frac{\alpha}{\beta}$$ Eq. 3. 70

and

$$Z = |Z|e^{i\phi}.$$ Eq. 3. 71

But we can also define

$$\tan\varphi = \tan(-\phi) = -\tan\phi = \frac{\alpha}{\beta}.$$ Eq. 3. 72

In that case,

$$Z = |Z|e^{i\phi} = |Z|e^{-i\varphi}$$ Eq. 3. 73

We can write the electric and magnetic fields as:

$$\vec{E} = E(\vec{r})e^{-i\omega t}\hat{n}_1 \qquad \text{Eq. 3. 74}$$

and

$$\vec{H} = H(\vec{r})e^{-i\omega t}\hat{n}_2. \qquad \text{Eq. 3. 75}$$

where $\hat{n}_1$ and $\hat{n}_2$ are unit vectors in the directions of the respective fields.

The ratio of complex amplitudes is then:

$$Z = \frac{E(\vec{r})}{H(\vec{r})} \qquad \text{Eq. 3. 76}$$

With our chosen expression $\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$ there is only a dependence on x. Therefore, we can write:

$$E(x) = H(x)Z = H(x)|Z|e^{-i\varphi} \rightarrow E(x) = |Z|H(x)e^{-i\varphi}. \qquad \text{Eq. 3. 77}$$

But we recall that

$$E(x) = E_0 e^{-\alpha x} e^{i(\beta x - \omega t)}. \qquad \text{Eq. 3. 78}$$

So,

$$H(x) = \frac{E_0 e^{-\alpha x} e^{i(\beta x - \omega t)}}{\frac{E_0}{H_0} e^{-i\varphi}} = H_0 e^{-\alpha x} e^{i(\beta x - \omega t)} e^{i\varphi} =$$
$$H(x) = H_0 e^{-\alpha x} e^{i(\beta x - \omega t + \varphi)} \qquad \text{Eq. 3. 79}$$

From

$$\begin{cases} E(x) = \dfrac{E_0}{H_0} H_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ H(x) = H_0 e^{-\alpha x} e^{i(\beta x - \omega t + \varphi)} \end{cases}, \qquad \text{Eq. 3. 80}$$

we get

$$\boxed{\begin{cases} \vec{E}(x) = |Z|\vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ \vec{H}(x) = \vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t + \varphi)} \end{cases}}. \qquad \text{Eq. 3. 81}$$

where we see that the electric and magnetic fields of the EM wave are shifted in phase.

The loss tangent in a conductive medium

In '**lossy dielectrics**' there is a power loss of the electromagnetic field. Lossy dielectrics are conductive media, but they are **poor conductors**. So far, we have learned that in a lossy dielectric:

1. The EM wave decays (described by the decay constant α).
2. The electric and magnetic fields oscillate, but no longer in phase. The phase shift between them is described by the angle φ, which is given by $\tan\varphi = \frac{\alpha}{\beta}$.

Then using trigonometry and substituting for $\tan\varphi$:

$$\tan 2\varphi = \frac{2\tan\varphi}{1-(\tan\varphi)^2} = \frac{2\frac{\alpha}{\beta}}{1-\left(\frac{\alpha}{\beta}\right)^2} = \frac{2\frac{\alpha}{\beta}}{\frac{\beta^2}{\beta^2}-\frac{\alpha^2}{\beta^2}} = \frac{2\frac{\alpha}{\beta}\beta^2}{\beta^2-\alpha^2} = \frac{2\alpha\beta}{\beta^2-\alpha^2} = \frac{\sigma}{\omega\varepsilon}.$$

Eq. 3. 82

For convenience, we can introduce another angle, such as:

$$\tan\theta = \tan 2\varphi = \frac{\sigma}{\omega\varepsilon},$$ Eq. 3. 83

which we shall call the **loss tangent**. It quantifies how good a conductor a material is, at a given (angular) frequency. Because $\theta = 2\varphi$, we can also use the loss tangent to find the phase difference between the electric and magnetic fields in the conducting material.

We can therefore write:

$$\boxed{\tan\theta = \tan 2\varphi = \frac{\sigma}{\omega\varepsilon} = \left|\frac{\text{ohmic current}}{\text{displacement current}}\right|}.$$ Eq. 3. 84

The wave parameters in a conductive medium

These are the two currents that make up the total current in the material, as we see from Maxwell's equation:

$$\nabla\times\vec{H} = \vec{J}_f + \frac{\partial\vec{D}}{\partial t} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} = \sigma\vec{E} - i\omega\varepsilon\vec{E},$$ Eq. 3. 85

where we used $\frac{\partial\vec{E}}{\partial t} = -i\omega\vec{E}$, which comes from $\vec{E} \propto e^{-i\omega t}$.

Next, we need to find exact expressions for the wave **propagation parameters** α and β, in order to discuss the two limiting cases, corresponding to 'poor' and 'good' conductors. We started with $\gamma = -\alpha + i\beta$. What are α and β?

We have already established that

$$\gamma^2 = -\mu\varepsilon\omega^2 - i\mu\sigma\omega \quad \text{Eq. 3. 86}$$

and that

$$\alpha^2 - \beta^2 = -\mu\varepsilon\omega^2 . \quad \text{Eq. 3. 87}$$

We can therefore calculate $|\gamma|^2$:

$$|\gamma|^2 = \gamma\gamma^* = \sqrt{\gamma^2}\left(\sqrt{\gamma^2}\right)^* = \sqrt{-\mu\varepsilon\omega^2 - i\mu\sigma\omega}\sqrt{-\mu\varepsilon\omega^2 + i\mu\sigma\omega} . \quad \text{Eq. 3. 88}$$

Combining:

$$\begin{aligned} |\gamma|^2 &= \sqrt{-\left(\mu\varepsilon\omega^2 + i\mu\sigma\omega\right)}\sqrt{-\left(\mu\varepsilon\omega^2 - i\mu\sigma\omega\right)} = \\ |\gamma|^2 &= \sqrt{\left(\mu\varepsilon\omega^2 + i\mu\sigma\omega\right)\left(\mu\varepsilon\omega^2 - i\mu\sigma\omega\right)} \end{aligned} . \quad \text{Eq. 3. 89}$$

where we can factorise, so that

$$|\gamma|^2 = \sqrt{\mu^2\omega^2\left(\varepsilon\omega + i\sigma\right)\left(\varepsilon\omega - i\sigma\right)} . \quad \text{Eq. 3. 90}$$

Now, we can use

$$(a+b)(a-b) = a^2 - b^2 \quad \text{Eq. 3. 91}$$

to obtain

$$|\gamma|^2 = \mu\omega\sqrt{(\varepsilon\omega)^2 - (i\sigma)^2} = \mu\omega\sqrt{\varepsilon^2\omega^2 + \sigma^2} = \mu\omega\sqrt{\varepsilon^2\omega^2\left(1 + \frac{\sigma^2}{\varepsilon^2\omega^2}\right)} , \quad \text{Eq. 3. 92}$$

where we recognise the loss tangent

$$|\gamma|^2 = \mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} . \quad \text{Eq. 3. 93}$$

But we can also write

$$\mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} = |\gamma|^2 = \gamma\gamma^* = (-\alpha + i\beta)(-\alpha - i\beta) = \alpha^2 + \beta^2 \quad \text{Eq. 3. 94}$$

Therefore, we now have

$$\left.\begin{aligned} \beta^2 + \alpha^2 &= \mu\varepsilon\omega^2\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} \\ \beta^2 - \alpha^2 &= \mu\varepsilon\omega^2 \end{aligned}\right\} \rightarrow 2\beta^2 = \mu\varepsilon\omega^2\left[1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right] \quad \text{Eq. 3. 95}$$

we added the two equations.

So, now we have an expression for β:

$$\beta = \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[1 + \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} . \quad \text{Eq. 3. 96}$$

We can do the same thing for α:

$$\left.\begin{aligned}\beta^2+\alpha^2&=\mu\varepsilon\omega^2\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\\ \beta^2-\alpha^2&=\mu\varepsilon\omega^2\end{aligned}\right\}\rightarrow 2\alpha^2=\mu\varepsilon\omega^2\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right] \qquad \text{Eq. 3. 97}$$

where we subtracted the two equations. This leads to:

$$\alpha=\sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}. \qquad \text{Eq. 3. 98}$$

The expressions for α and β look complicated and we do not need to remember them. But they serve to understand the two limiting cases of 'poor' and 'good' conductors:

1. A 'poor' conductor is one where $\frac{\sigma}{\varepsilon\omega}\ll 1$ and the Ohmic current is much smaller than the displacement current.
2. A 'good' conductor is one where $\frac{\sigma}{\varepsilon\omega}\gg 1$ and the Ohmic current is much larger than the displacement current.

Special case I: Poor conductors (lossy dielectrics)

We consider the case of poor conductors.

1. Loss tangent: $\tan\theta=\frac{\sigma}{\varepsilon\omega}\ll 1\rightarrow\theta\approx 0$.
2. Phase angle: $\varphi=\frac{1}{2}\theta\approx 0$.
3. Calculating the **propagation parameters** α and β.

For

$$\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}=\left[1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]^{\frac{1}{2}}=(1+x)^{\frac{1}{2}}, \qquad \text{Eq. 3. 99}$$

we can use the **binomial theorem**, which states that:,

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+nx^{n-1}+x^n, \qquad \text{Eq. 3. 100}$$

so here, for $\frac{\sigma}{\varepsilon\omega}\ll 1$, we have:

$$\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \approx 1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2, \qquad \text{Eq. 3. 101}$$

since the other terms within the series are much too small to consider. We can then write:

$$\alpha \approx \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[-1+1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]} = \omega\sqrt{\frac{\mu\varepsilon}{4}\left(\frac{\sigma}{\varepsilon\omega}\right)^2}. \qquad \text{Eq. 3. 102}$$

$$\alpha \approx \omega\sqrt{\frac{\mu\varepsilon}{4}\frac{\sigma^2}{\varepsilon^2\omega^2}} = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\varepsilon}}$$

For a perfect dielectric $\sigma = 0$, and hence $\alpha = 0$, which agrees with our expectation that there should be no exponential decay for the EM wave in the material.

We can now do the same analysis for β.

$$\beta \approx \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[+1+1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]} = \omega\sqrt{\frac{\mu\varepsilon}{2}2\left[1+\frac{1}{4}\left(\frac{\sigma}{\varepsilon\omega}\right)^2\right]} \approx \qquad \text{Eq. 3. 103}$$

$$\beta \approx \omega\sqrt{\mu\varepsilon}$$

4. The phase velocity $v_p = \dfrac{\omega}{\text{wave vector}}$:

We have seen above that our wave vector is

$$k + i\kappa = \beta + i\alpha. \qquad \text{Eq. 3. 104}$$

Earlier, we established that

$$\vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)}, \qquad \text{Eq. 3. 105}$$

where it is clear that the decay constant α does not influence the phase. Therefore, to find the phase velocity we only take β

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \text{Eq. 3. 106}$$

which is the same as in a perfect dielectric.

Since the phase velocity is that of a perfect dielectric, the only real effect we can expect from propagation in a lossy dielectric is that the wave will be attenuated (amplitude falling as indicated by $e^{-\alpha x}$).

5. The intrinsic impedance of the medium:

$$Z = \sqrt{\frac{\mu}{\varepsilon\left(1+i\dfrac{\sigma}{\varepsilon\omega}\right)}} \approx \sqrt{\frac{\mu}{\varepsilon}}. \qquad \text{Eq. 3. 107}$$

6. The electric and magnetic fields:

We have seen that $\varphi = \frac{1}{2}\theta \approx 0$, so that means the electric and magnetic fields are in phase. We can write:

$$\begin{cases} \vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ \vec{H} = \vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \end{cases}. \qquad \text{Eq. 3. 108}$$

Special case II: Good conductors

We consider the case of good conductors.

1. Loss tangent: $\tan\theta = \frac{\sigma}{\varepsilon\omega} \gg 1 \rightarrow \theta \approx \frac{\pi}{2}$.
2. Phase angle: $\varphi = \frac{1}{2}\theta \approx \frac{\pi}{4}$.
3. Calculating the propagation parameters α and β:

For $\frac{\sigma}{\varepsilon\omega} \gg 1$, we have $\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \approx \frac{\sigma}{\varepsilon\omega}$, so that

$$\alpha \approx \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[-1+\frac{\sigma}{\varepsilon\omega}\right]} \approx \omega\sqrt{\frac{\mu\cancel{\varepsilon}}{2}\frac{\sigma}{\cancel{\varepsilon}\omega}} = \sqrt{\frac{\mu\sigma\omega}{2}} \qquad \text{Eq. 3. 109}$$

and

$$\beta \approx \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[+1+\left(\frac{\sigma}{\varepsilon\omega}\right)\right]} = \approx \omega\sqrt{\frac{\mu\cancel{\varepsilon}}{2}\frac{\sigma}{\cancel{\varepsilon}\omega}} = \sqrt{\frac{\mu\sigma\omega}{2}}. \qquad \text{Eq. 3. 110}$$

Therefore, for good conductors:

$$\alpha \approx \beta \approx \sqrt{\frac{\mu\sigma\omega}{2}}. \qquad \text{Eq. 3. 111}$$

4. The phase velocity $v_p = \frac{\omega}{\text{wave vector}}$:

$$v_p = \frac{\omega}{\beta} = \frac{\sqrt{\omega^2}}{\sqrt{\frac{\mu\sigma\omega}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}} = \omega\delta, \qquad \text{Eq. 3. 112}$$

where

$$\boxed{\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \alpha^{-1}} \qquad \text{Eq. 3. 113}$$

is the **skin depth**.

In a good conductor, there is **dispersion**, as the phase velocity depends on frequency.

Note: by definition, **dispersion** is the phenomenon in which the phase velocity of a wave depends on its frequency. In practice, this often means that light is separated by individual colours as it travels through a material.

5. The intrinsic impedance of the medium:

Using the phase angle, we have:

$$Z = |Z|e^{i\phi} = |Z|e^{-i\varphi} = |Z|e^{-i\frac{\pi}{4}}. \qquad \text{Eq. 3. 114}$$

We also have:

$$|Z|^2 = ZZ^*, \qquad \text{Eq. 3. 115}$$

so

$$|Z| = \left[ZZ^*\right]^{\frac{1}{2}}. \qquad \text{Eq. 3. 116}$$

Therefore,

$$|Z| = \left[\sqrt{\frac{\mu}{\varepsilon\left(1+i\frac{\sigma}{\varepsilon\omega}\right)}}\sqrt{\frac{\mu}{\varepsilon\left(1-i\frac{\sigma}{\varepsilon\omega}\right)}}\right]^{\frac{1}{2}} \approx \left[\sqrt{\frac{\mu}{\cancel{\varepsilon}\left(i\frac{\sigma}{\cancel{\varepsilon}\omega}\right)}}\sqrt{\frac{\mu}{\cancel{\varepsilon}\left(-i\frac{\sigma}{\cancel{\varepsilon}\omega}\right)}}\right]^{\frac{1}{2}} \qquad \text{Eq. 3. 117}$$

$$|Z| \approx \sqrt{\frac{\mu\omega}{\sigma}}$$

Combining:

$$Z \approx \sqrt{\frac{\mu\omega}{\sigma}}e^{-i\frac{\pi}{4}} = \frac{\sqrt{2}}{\sigma\delta}e^{-i\frac{\pi}{4}}, \qquad \text{Eq. 3. 118}$$

because

$$\sqrt{\mu\omega} = \frac{1}{\delta}\sqrt{\frac{2}{\sigma}}. \qquad \text{Eq. 3. 119}$$

6. The electric and magnetic fields:

Because $\varphi = \frac{1}{2}\theta \approx \frac{\pi}{4}$, the electric and magnetic fields are 45° out of phase. Moreover, the wave attenuates very quickly, as $e^{-\alpha x} = e^{-x/\delta}$. After a distance of one wavelength, the amplitude drops by

$$e^{-\lambda/\delta} = e^{-2\pi/(\beta\delta)} = e^{-2\pi} \approx 1/535. \qquad \text{Eq. 3. 120}$$

Therefore:

$$\begin{cases} \vec{E} = \vec{E}_0 e^{-\frac{x}{\delta}} e^{i\left(\frac{x}{\delta}-\omega t\right)} \\ \vec{H} = \vec{H}_0 e^{-\frac{x}{\delta}} e^{i\left(\frac{x}{\delta}-\omega t+\frac{\pi}{4}\right)} \end{cases}. \qquad \text{Eq. 3. 121}$$

Note: The wave parameter β also describes the wavelength of the wave, since

$$\lambda = \frac{2\pi}{\text{wave vector}} = \frac{2\pi}{\beta}. \qquad \text{Eq. 3. 122}$$

Summary

In LIH materials, Maxwell's equations become: $\nabla \cdot \vec{D} = \rho_f$, $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$.

In an ideal LIH dielectric, the electromagnetic wave propagates similarly to the way it does in free space – it remains a plane wave but its phase velocity is lowered: $v_p = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$.

The wave equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$ changes to a modified wave equation $\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t}$ in conductive medium.

A solution to the modified wave equation is $\vec{E} = \vec{E}_0 e^{(\gamma x - i\omega t)}$ with $\gamma = -\alpha + i\beta$, so $\vec{E} = \vec{E}_0 e^{-\alpha x} e^{i(\beta x - \omega t)}$. Upon solving an important ratio appears: $\frac{\sigma}{\varepsilon\omega}$.

This ratio is part of the dielectric function of the material: $\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0 \omega}$ and of its complex refractive index: $\tilde{n} = n + i\eta = \sqrt{\varepsilon(\omega)} = \sqrt{\varepsilon' + i\varepsilon''}$.

The ratio is defined as the loss tangent:

$$\tan\theta = \tan 2\varphi = \frac{\sigma}{\omega\varepsilon} = \left|\frac{\text{ohmic current}}{\text{displacement current}}\right|.$$

When an EM wave propagates in a lossy dielectric, its amplitude decays at a rate α. The electric and magnetic fields oscillate with a phase shift described by the angle φ in $\tan\varphi = \alpha/\beta$. Altogether we have:

$$\begin{cases} \vec{E}(x) = |Z|\vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t)} \\ \vec{H}(x) = \vec{H}_0 e^{-\alpha x} e^{i(\beta x - \omega t + \varphi)} \end{cases}.$$

In poor conductors, we have $\frac{\sigma}{\varepsilon\omega} \ll 1$. The EM plane wave decays slowly. The electric and magnetic field are in phase.

In good conductors, we have $\frac{\sigma}{\varepsilon\omega} \gg 1$. The EM wave decays rapidly and we defined the skin depth as $\delta = \sqrt{\frac{2}{\mu\sigma\omega}} = \alpha^{-1}$. The electric and magnetic fields no longer oscillate in phase.

Example question 1

[from Sadiku] In free space $\vec{H} = 0.1\cos\left(2\times10^8 t + kx\right)\vec{a}_y\ \text{Am}^{-1}$.

Determine:
(a) The direction of propagation.
(b) The period.
(c) The wavelength (λ).
(d) The wave vector magnitude.

Answer:
(a) The wave propagates along the negative *x* direction.
(b) $T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi\times10^8} = \underline{\underline{31.42}}$ ns.
(c) $\lambda = uT = 3\times10^8 \times 31.42\times10^{-9} = \underline{\underline{9.425}}$ m.
(d) $k = \beta = \frac{2\pi}{\lambda} = \underline{\underline{0.67}}\ \text{rad}\cdot\text{m}^{-1}$.

Example question 2

[from Sadiku] An EM wave propagating in a certain medium is described by: $\vec{E} = 25\sin\left(2\pi\times 10^6 t - 6x\right)\vec{a}_z$ Vm^{-1}.

(a) Determine the direction of propagation.
(b) Compute the period (T).
(c) Calculate the wavelength (λ).
(d) Establish the velocity (u). [hint $u = \omega/\beta$]

Answer:

(a) The wave propagates along the positive x direction.

(b) $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2\pi\times 10^6} = \underline{\underline{1\ \mu\text{s}}}$.

(c) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6} = \underline{\underline{1.047}}$ m.

(d) $u = \dfrac{\omega}{\beta} = \dfrac{2\pi\times 10^6}{6} = \underline{\underline{1.047\times 10^6}}$ ms^{-1}.

Example question 3

[from Sadiku] At 50 MHz, a lossy dielectric material is characterized by $\mu = 2.1\mu_0$, $\varepsilon = 3.6\varepsilon_0$ and $\sigma = 0.08$ Sm^{-1}. If $\vec{E}_s = 6e^{-\gamma x}\vec{a}_z$ Vm^{-1}, compute:

(a) γ
(b) The wavelength (λ).
(c) The wave velocity.

Answer:

(a) We know that $\varepsilon_0 = \dfrac{1}{\mu_0 c^2}$ and that $\mu_0 = 4\pi 10^{-7}$ so,

$$\varepsilon_0 = \frac{1}{4\pi 10^{-7}c^2} = \frac{1}{4\pi 10^{-7}c^2} = \frac{1}{4\pi 10^{-7}\times 9\times 10^{16}} = \frac{10^{-9}}{36\pi}.$$

We also know $\omega = 2\pi f$, with $f = 50$ MHz $= 5\times 10^7$ Hz. So, $\omega = 10\pi 10^7 = \pi 10^8$.

Hence,

$$\frac{\sigma}{\omega\varepsilon} = \frac{0.08}{\left(\pi 10^8\right)(3.6)\left(\dfrac{10^{-9}}{36\pi}\right)} = \frac{0.08}{(3.6)\left(\dfrac{10^{-1}}{36}\right)} = \frac{0.08}{\left(36\times 10^{-1}\right)\left(\dfrac{10^{-1}}{36}\right)} = 8.$$

(b) Next, $\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{\omega}{c}\sqrt{\frac{2.1\times 3.6}{2}\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}.$

So, $\alpha = \frac{\left(\pi 10^8\right)}{3\times 10^8}\sqrt{\frac{2.1\times 3.6}{2}\left[-1+\sqrt{1+64}\right]} \approx 5.41.$

Similarly,

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{\left(\pi 10^8\right)}{3\times 10^8}\sqrt{\frac{2.1\times 3.6}{2}\left[1+\sqrt{1+64}\right]} \approx 6.13.$$

Hence, $\gamma = -\alpha + i\beta \approx \underline{\underline{-5.41 + i6.13}}\ \text{m}^{-1}.$

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6.13} \approx \underline{\underline{1.025}}\ \text{m}.$

(c) The wave velocity is $u = \frac{\omega}{\beta} = \frac{2\pi\times 50\times 10^6}{6.13} = \underline{\underline{5.10\times 10^7}}\ \text{ms}^{-1}$

Example question 4

[from Sadiku] A lossy material has $\mu = 5\mu_0$ and $\varepsilon = 2\varepsilon_0$. If at 5 MHz, the phase constant is 10 rad/m, calculate:
(a) The loss tangent.
(b) The conductivity of the material.
(c) The complex permittivity.
(d) The attenuation constant.

Answer:
(a) From the lectures:

$$\beta = \sqrt{\frac{1}{2}\mu\varepsilon\omega^2\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}$$

Hence, $\beta = \omega\sqrt{\frac{10\mu_0\varepsilon_0}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{\omega}{c}\sqrt{5\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}.$

We also know $\omega = 2\pi f$, with $f = 5$ MHz $= 5\times10^6$ Hz. So, $\omega = 10\pi10^6 = \pi10^7$. Hence,

$$10 = \frac{\pi\times10^7\times\sqrt{5}}{3\times10^8}\sqrt{\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{\pi\sqrt{5}}{30}\sqrt{\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}$$

$$\left(\frac{300}{\pi\sqrt{5}}\right)^2 = \left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right] \rightarrow \left(\frac{300}{\pi\sqrt{5}}\right)^2 - 1 = \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}$$

$$1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2 = \left[\left(\frac{300}{\pi\sqrt{5}}\right)^2 - 1\right]^2 \rightarrow \left(\frac{\sigma}{\varepsilon\omega}\right)^2 = \left[\left(\frac{300}{\pi\sqrt{5}}\right)^2 - 1\right]^2 - 1$$

and $\frac{\sigma}{\varepsilon\omega} = \sqrt{\left[\left(\frac{300}{\pi\sqrt{5}}\right)^2 - 1\right]^2 - 1} \approx \underline{\underline{1823}}$.

(b) We know that $\sigma = \varepsilon\omega\tan(\theta) = 2\varepsilon_0\omega\tan(\theta)$. We also know that $\varepsilon_0 = \frac{1}{\mu_0 c^2}$ and that $\mu_0 = 4\pi10^{-7}$ so, $\varepsilon_0 = \frac{1}{4\pi10^{-7}c^2}$. We take $c = 3\times10^{-8}$, so $\varepsilon_0 = \frac{1}{4\pi10^{-7}\times9\times10^{16}} = \frac{10^{-9}}{36\pi}$. Therefore,

$$\sigma = 2\left(\frac{10^{-9}}{36\pi}\right)\left(\pi\times10^7\right)(1823) = \frac{10^{-2}}{18}(1823) \approx \underline{\underline{1.013}}\ \mathrm{Sm}^{-1}.$$

(c) From the lectures:

$$\varepsilon(\omega) = \varepsilon_r(\omega) + i\frac{\sigma(\omega)}{\varepsilon_0\omega} = 2\times\left(\frac{10^{-9}}{36\pi}\right) + i\frac{1.013}{\pi\times10^7} \approx \underline{\underline{1.77\times10^{-11} + i3.22\times10^{-8}}}\ \mathrm{Fm}^{-1}$$

(d) From the lectures $\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}$ and

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}, \text{ so}$$

$$\frac{\alpha}{\beta} = \frac{\sqrt{\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}}{\sqrt{\left[+1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]}} = \sqrt{\frac{-1+\sqrt{1+1823^2}}{+1+\sqrt{1+1823^2}}} \approx \sqrt{\frac{1822}{1824}}$$ and since $\beta = 10$,

we have $\alpha \approx 10\sqrt{\frac{1822}{1824}} \approx \underline{\underline{9.99}}$ m^{-1}.

Example question 5

[from Sadiku] Calculate the skin depth and the velocity of propagation for a uniform plane wave at frequency 6 MHz travelling in polyvinylchloride ($\mu_r = 1$ and $\varepsilon_r = 4$, $\tan\theta = 7\times10^{-2}$).

Answer:

From the lectures:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[-1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{2\pi f}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[-1+\sqrt{1+49\times10^{-4}}\right]}$$

$$\alpha = \frac{2\pi\times6\times10^6}{c}\sqrt{\frac{1\times4}{2}\left[-1+\sqrt{1+49\times10^{-4}}\right]} \approx 8.8\times10^{-3}$$

Therefore the skin depth is $\delta = \alpha^{-1} \approx \underline{\underline{113.75}}$ m.

Also from the lectures:

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]} = \frac{2\pi f}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[1+\sqrt{1+49\times10^{-4}}\right]} \approx 0.25$$

The wave velocity is then $u = \frac{\omega}{\beta} = \frac{2\pi\times6\times10^6}{0.25} = \underline{\underline{1.5\times10^8}}$ ms^{-1}