

Chapter 1: Electric fields in materials

The previous part of this course dealt with electromagnetic fields in vacuum. Now, we consider what happens when materials are present. We begin with studying electric fields applied to **dielectric materials**. These materials are electric isolators – they typically lack free charges and do not conduct electricity. Instead, dielectrics have **bound charges** that can only slightly shift from their average equilibrium position, when subjected to an electric field.

Electric dipoles

The **electric dipole moment** is defined as the product of charge (q) and separation (\vec{r}), see Figure 1.1:

$$\vec{p} = q\vec{r}$$

Eq.1. 1

The charges themselves could be protons and electrons of an atom stretched in an external electric field, the poles of a polar molecule, or two oppositely charged metal spheres.

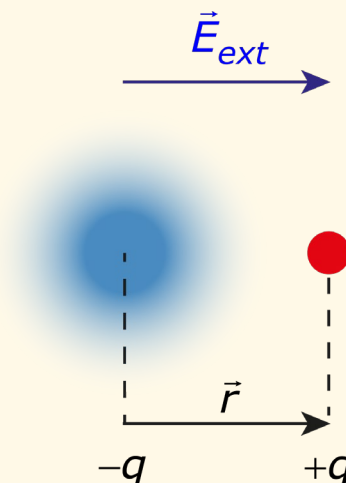


Figure 1.1. Charge separation in an atom due to an externally applied electric field.

Electric fields can induce dipole moments:

$$\vec{p} = \alpha \vec{E}_{ext},$$

Eq.1. 2

where \vec{E}_{ext} is the externally applied electric field and α is a proportionality term.

Consider an atom with a positively charged point nucleus ($+q$) and a negatively charged point electron ($-q$). This point electron can also represent the centre of the electron cloud surrounding the

nucleus. They are separated by distance of length r . Because we work with vectors here, we need some directions. So, we can choose a coordinate system centred on $-q$, where r is along the z -direction, and we can write: $\vec{r} = r\hat{z}$. The vector \vec{r} points from $-q$ to $+q$. In this representation, $+q$ feels two fields: the externally applied field (\vec{E}_{ext}) and the local electric field (\vec{E}_{local}) created by $-q$ (see Figure 1.2). At equilibrium, the effects of those two fields on $+q$ are balanced ($\vec{E}_{local} + \vec{E}_{ext} = 0$).

We can write an expression of \vec{E}_{local} from **Coulomb's law** (A-Level Physics):

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^3} r, \quad \text{Eq.1. 3}$$

but we need to adapt it to this coordinate system.

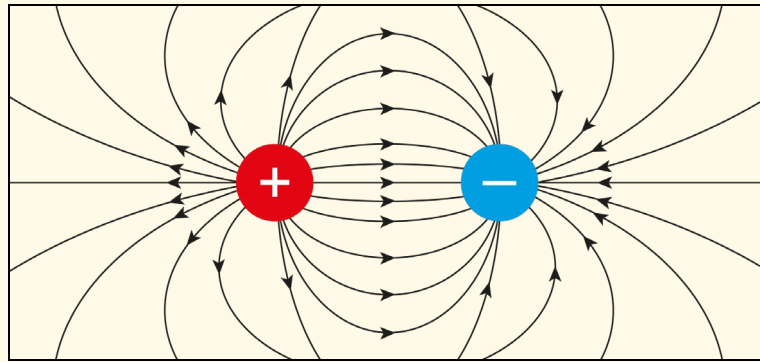


Figure 1.2. Electric field lines.

The local field created by $-q$, at the point where $+q$ is situated is:

$$\vec{E}_{local} = \frac{1}{4\pi\epsilon_0} \frac{-q}{r^3} \vec{r} = \frac{1}{4\pi\epsilon_0} \frac{-qr}{r^3} \hat{z} = \frac{-p}{4\pi\epsilon_0 r^3} \hat{z} = \frac{-\vec{p}}{4\pi\epsilon_0 r^3}. \quad \text{Eq.1. 4}$$

Therefore, we can write:

$$\vec{E}_{ext} = E_{ext} \hat{z} = -\vec{E}_{local} = \frac{p}{4\pi\epsilon_0 r^3} \hat{z}. \quad \text{Eq.1. 5}$$

It follows that:

$$\vec{p} = 4\pi\epsilon_0 r^3 \vec{E}_{ext}, \quad \text{Eq.1. 6}$$

and since we started with $\vec{p} = \alpha \vec{E}_{ext}$, we can state that:

$$\alpha = 4\pi\epsilon_0 r^3 = 3\epsilon_0 \left(\frac{4}{3} \pi r^3 \right) = 3\epsilon_0 V \quad \text{Eq.1. 7}$$

where V is the volume of a sphere with radius r . This is the volume of our atom with point nucleus and one point electron.

Therefore:

$$\vec{P} = 3\epsilon_0 V \vec{E}_{\text{ext}} \quad \text{Eq.1. 8}$$

Relative permittivity and electric susceptibility

We consider a dielectric cylinder of area A and thickness x , upon applying an electric field strength \vec{E} . The bound + and – charges are now slightly separated by the polarisation in every atom, see Figure 1.3. Within the material, the + and – charges cancel. However, at the surfaces, there is a net charge. Overall, the materials is now polarised, with **polarisation vector** \vec{P} . This polarisation must depend on the externally applied electric field and on the properties of the dielectric itself; we write this dependence as $\vec{P} = \epsilon_0 \chi_e \vec{E}$, where χ_e is the **electric susceptibility** of the material (i.e. it tells us how polarisable it is).

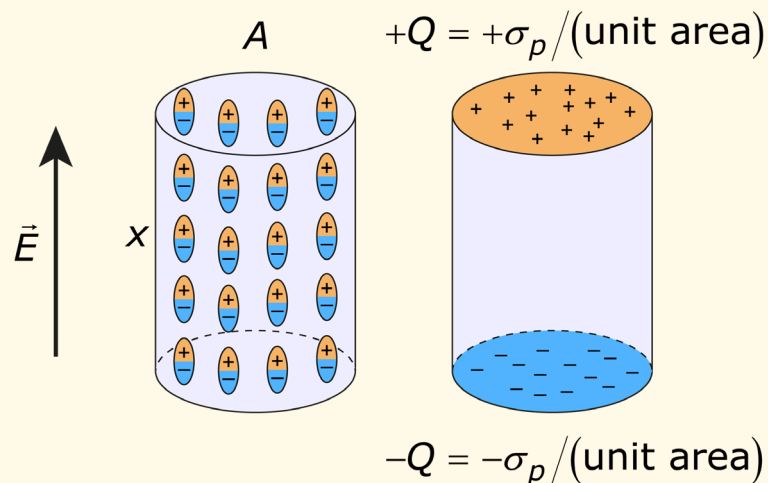


Figure 1.3. A dielectric cylinder of surface A and height x , upon applying an external electric field. Positive and negative charges Q are formed at the surfaces.

The resulting positive charge is $+Q = +\sigma_p / (\text{unit area})$, where σ_p is the **surface charge density**.

Similarly, the resulting negative charge is $-Q = -\sigma_p / (\text{unit area})$.

We can calculate the magnitude of the dipole moment of the whole block in two ways:

$$(\text{dipole moment}) = (\text{polarized charge}) \times (\text{separation}) = (\sigma_p A) x$$

or,

$$(\text{dipole moment}) = (\text{dipole moment/unit volume}) \times (\text{volume}) = P (Ax)$$

These must be the same, so that $\sigma_p = P$.

The magnitude of the polarisation tells us about the un-cancelled bound charge density on the surface of the dielectric.

Note: $\sigma_p = P$ also means that the unit of polarisation is the same as that of surface charge – coulombs per square meter.

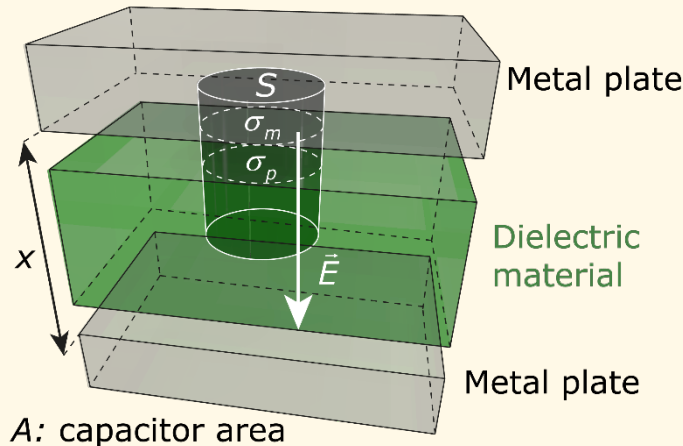


Figure 1.4. A dielectric materials between two metal plates.

In Figure 1.4, we define the following physical quantities:

- A : capacitor area
- x : thickness
- S : area of Gaussian cylinder
- ρ : the **charge density** enclosed in an arbitrary volume V .
- $\sigma_m = \frac{Q}{A}$: stored **surface charge density on the metal** surface.
- σ_p : polarisation charge density (it is opposite to σ_m)

We start with **Gauss' theorem**:

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_V \rho dv. \quad \text{Eq.1. 9}$$

Gauss' law states that the flux of the electric field out of an arbitrary closed surface is proportional to the electric charge enclosed by the surface, irrespective of how that charge is distributed.

\vec{E} leaves the cylinder on three sides: side, top and bottom.

The flux of \vec{E} leaving the cylinder is therefore $= 0 + 0 + \vec{E} \cdot \vec{S} = \vec{E} \cdot \vec{S}$ (side+top+bottom). The top is outside \vec{E} and along the side \vec{E} is not leaving the cylinder.

The charge enclosed is (metal and dielectric):

$$\sigma_m S + (-\sigma_p) S = (\sigma - \sigma_p) S. \quad \text{Eq.1. 10}$$

Therefore:

$$ES = \frac{(\sigma_m - \sigma_p) S}{\epsilon_0}, \quad \text{Eq.1. 11}$$

and

$$E = \frac{(\sigma_m - \sigma_p)}{\epsilon_0}. \quad \text{Eq.1. 12}$$

We can re-arrange using the magnitudes of the polarisation vector $\vec{P} = \epsilon_0 \chi_e \vec{E}$, where χ_e is the electric susceptibility:

$$\epsilon_0 E = \sigma_m - \sigma_p = \sigma_m - P = \sigma_m - \epsilon_0 \chi_e E. \quad \text{Eq.1. 13}$$

So that:

$$E = \frac{\sigma_m}{\epsilon_0 (1 + \chi_e)}, \quad \text{Eq.1. 14}$$

which give an expression for the electric field. Can we find another?

From A-Level Physics, the potential difference across the capacitor divided by the distance between the plates gives the field:

$$E = \frac{V}{x}. \quad \text{Eq.1. 15}$$

So,

$$\frac{V}{x} = \frac{\sigma_m}{\epsilon_0 (1 + \chi_e)} = \frac{Q/A}{\epsilon_0 (1 + \chi_e)}, \quad \text{Eq.1. 16}$$

which rearranges to

$$\frac{Q}{V} = \frac{\epsilon_0 (1 + \chi_e) A}{x}. \quad \text{Eq.1. 17}$$

Again, from A-Level Physics, we know something about Q and V :

$$C = \frac{Q}{V} \quad \text{Eq.1. 18}$$

Moreover, from A-Level Physics we have an expression for C in parallel plate capacitors:

$$C = \frac{A \epsilon_0 \epsilon_r}{x}. \quad \text{Eq.1. 19}$$

It follows that

$$\frac{A\varepsilon_0\varepsilon_r}{x} = \frac{\varepsilon_0(1 + \chi_e)A}{x}, \quad \text{Eq.1. 20}$$

i.e.

$$\varepsilon_r = 1 + \chi_e, \quad \text{Eq.1. 21}$$

where ε_r is the **relative permittivity** of the material.

Concepts of linearity, isotropy and homogeneity.

In this course, we will only discuss linear, isotropic and homogeneous (LIH) dielectric materials.

A dielectric material is **linear** if the **electric permittivity** ($\varepsilon = \varepsilon_0\varepsilon_r$) and the **electric susceptibility** (χ_e) do not change with the magnitude of the applied electric field.

A dielectric material is **isotropic** if the electric permittivity and the electric susceptibility do not change with the direction of the electric field in the material. In other words, all directions in the material are equivalent.

A dielectric material is **homogeneous** if the electric permittivity and the electric susceptibility do not change from point to point in the material. In other words, the permittivity is independent of the coordinates. We can say that all locations in the material are equivalent.

Not isotropic materials are called **anisotropic**. In anisotropic materials \vec{P} and \vec{E} are not parallel and are not simply related. For instance, this relationship may be nonlinear and could also depend on the past history of the sample, i.e. there can be a hysteresis.

Electric polarisation and dipole moment

Let us consider a **dielectric** medium. A dielectric is basically an electric insulator, like glass or water. We can represent such a material as a collection of positive and negative charges. In the simplest case, these are the nucleus and the electrons of an atom. Overall, the net charge of this atomic system is zero, but what if an electromagnetic wave comes along?

Under the influence of the electric field, the negative and positive charges will split. In turn, this is going to influence the electric field and that influence will show up in Maxwell's equations. We can now proceed and identify the new quantities.

As we have seen, the electric dipole moment is defined as:

$$\vec{p} = q_b \vec{r}. \quad \text{Eq.1. 22}$$

The two charges making up the individual dipole are now **bound charges**. This is often indicated with a small *b* written in subscript. The units are Coulomb meter ($C \cdot m$).

The **electric polarisation** of a region is defined as the dipole moment per unit volume (V). It is therefore a measure for the density of dipole moments:

$$\vec{P} = \frac{\sum \vec{p}}{V}. \quad \text{Eq.1. 23}$$

The electric polarisation is proportional to the electric field, depending on the dielectric material. Because the units of the dipole moment are $C \cdot m$, the units of polarisation are $C \cdot m^{-2}$ which is the same unit as a surface charge density, as we saw above.

If all the atoms have the same dipole moment, or if \bar{p} designates the average dipole moment, Eq. 1.23 becomes:

$$\vec{P} = \frac{N\bar{p}}{V} = n\bar{p}, \quad \text{Eq.1. 24}$$

where N is the number of molecules and n is the number of molecules per unit volume – the molecular density.

Electric polarisation at the surface

In Figure 1.5, we can see a unit volume inside a nonpolar dielectric, which encloses several atoms. At the maximum of the electric field (E) applied to this dielectric, the positive and negative charges are separated by a distance s . Note that when the electric field is applied to the right side of Figure 1.5, s is actually equivalent to the vector r in Eq. 1.22!

When E increases, N^+ positive charges and N^- negative charges cross the small element of area dA . The net charge that crosses dA is then:

$$dq_b = N^+ q_b - N^- (-q_b) = (N^+ + N^-) q_b. \quad \text{Eq.1. 25}$$

This is the sum of all charged crossing dA times the charge value. But, this is also the number of molecules within the unit volume:

$$V = \vec{s} \cdot d\vec{A}. \quad \text{Eq.1. 26}$$

We can easily check this: the volume of the rectangular box is the area times the height. Here,

$$\vec{s} \cdot d\vec{A} = |\vec{s}| |d\vec{A}| \cos \theta, \quad \text{Eq.1. 27}$$

with $\theta = 0$, so $\cos \theta = 1$. The magnitude of $d\vec{A}$ is the surface area and the magnitude of \vec{s} is the height.

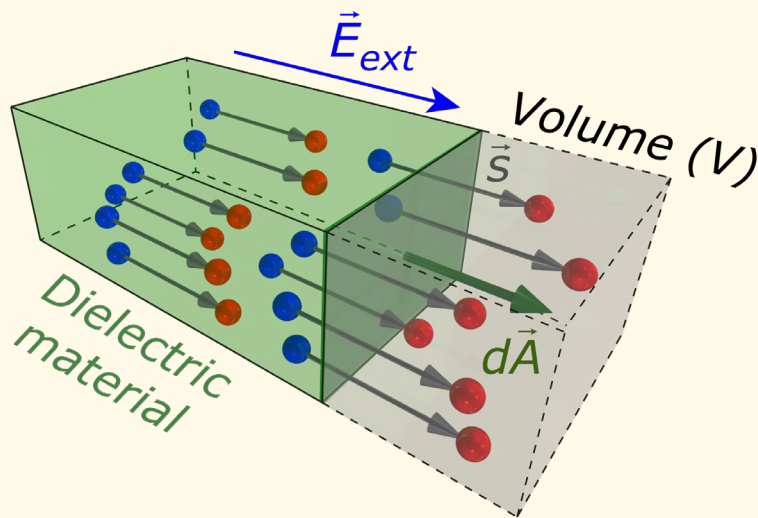


Figure 1.5. The unit volume V inside a nonpolar dielectric is the product of an element of area dA and of the maximum distance between the electrical charges (\vec{s}) under the influence on an external electric field E . The vector $d\vec{A}$ is perpendicular (i.e. normal) to the area dA .

Then remembering that n is the number of molecules per unit volume and that $\vec{s} = \vec{r}$,

$$dq_b = nVq_b = n(\vec{s} \cdot d\vec{A})q_b = nq_b\vec{s} \cdot d\vec{A} = n\vec{p} \cdot d\vec{A} = \vec{P} \cdot d\vec{A} \quad \text{Eq.1. 28}$$

Therefore, the net charge that crosses an element of surface $d\vec{A}$ in a dielectric, under the influence of an electric field, is proportional to the vector product of the polarisation by the vector $d\vec{A}$. This relationship is at the heart of the light-matter interaction, since its consequences (see below) will show up in Maxwell's equations. Note that it also means: $P = dq_b/dA = \sigma_p$, as we saw above.

The sources of electric polarisation

For instance, let us now consider the volume V enclosed behind dA . Eq. 1.28 tells us that the net charge that flows out of V , through the surface dA is $\vec{P} \cdot d\vec{A}$. We can rephrase this and say that the net charge that flows out of the closed surface A which encloses the volume V is:

$$Q_{out} = \int_A \vec{P} \cdot d\vec{A}, \quad \text{Eq.1. 29}$$

but then, it is clear that the charge that remains within V must be $-Q_{out}$.

Another way of looking at the charge within V is to consider the volume density of charge over the volume V . Let this volume **density of charge** be ρ_b . It follows then that:

$$\int_V \rho_b \cdot dv = -Q_{out} = -\int_A \vec{P} \cdot d\vec{A} = -\int_V (\nabla \cdot \vec{P}) dv, \quad \text{Eq.1. 30}$$

where the divergence theorem was used in order to obtain the last integral from the previous. Hence, we have:

$$\rho_b = -\nabla \cdot \vec{P} \quad \text{Eq.1. 31}$$

This means that there are sources of polarisation.

The total bound polarisation charge within a region is obtained by integrating over a volume:

$$Q_b = \int_V \rho_b \cdot dv = -\int_V (\nabla \cdot \vec{P}) dv = -\int_A \vec{P} \cdot d\vec{A}, \quad \text{Eq.1. 32}$$

which is Eq. 1.30 backwards.

Bound current density and the continuity equation

Under the influence of a time dependent electric field (such as the one in a light wave), by definition, the current flowing through a surface A is related to the **current density** \vec{J} by the formula:

$$\frac{dQ_b}{dt} = I = \int_A \vec{J} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{J}) dv, \quad \text{Eq.1. 33}$$

where we applied again the divergence theorem. Substituting for Q from Eq. 1.32 and then using Eq. 1.31 to substitute for the density of charge ρ_b , we obtain:

$$-\frac{d}{dt} \int_V \rho_b dv = -\int_V \left(\frac{\partial \rho_b}{\partial t} \right) dv = -\int_V \left(\frac{\partial (-\nabla \cdot \vec{P})}{\partial t} \right) dv = \int_V \left(\nabla \cdot \frac{\partial \vec{P}}{\partial t} \right) dv = \int_V (\nabla \cdot \vec{J}_b) dv \quad \text{Eq.1. 34}$$

And therefore:

$$\vec{J}_b = \frac{\partial \vec{P}}{\partial t}. \quad \text{Eq.1. 35}$$

This means that the motion of bound charges results in a polarisation or '**bound**' **current density** \vec{J}_b .

Note that from Eq. 1.34 we can also deduce that:

$$\nabla \cdot \vec{J}_b = -\frac{\partial \rho_b}{\partial t}. \quad \text{Eq.1. 36}$$

This expression is called the **Continuity Equation**.

Free charges and the electric flux density

The total charge in a dielectric is a sum of bound charges and of free charges:

$$\rho = \rho_b + \rho_f. \quad \text{Eq.1. 37}$$

The bound charges result from polarisation. The free charges do not.

We can now rewrite Maxwell's law, which mentions the total charge:

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f, \quad \text{Eq.1. 38}$$

and we obtain:

$$\nabla \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f. \quad \text{Eq.1. 39}$$

So, it looks like the free charges are the source of some field!

Let us then define this vector:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}. \quad \text{Eq.1. 40}$$

It is called the **electric flux density** (and sometimes the **electric displacement**).

Now, Gauss' law can be written as:

$$\nabla \cdot \vec{D} = \rho_f, \quad \text{Eq.1. 41}$$

with the benefit of having similar maths for both the free and the bound charge densities.

The total free charge is then:

$$Q_f = \int_V \rho_f \cdot dv = \int_V (\nabla \cdot \vec{D}) dv = \int_A \vec{D} \cdot d\vec{A}. \quad \text{Eq.1. 42}$$

The equation $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ is known as a **Constitutive Relation**. It can be rewritten as:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}. \quad \text{Eq.1. 43}$$

Replacing with the relative permittivity $\varepsilon_r = 1 + \chi_e$, we obtain:

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}. \quad \text{Eq.1. 44}$$

Note that in free space, the polarisation is zero, the relative permittivity is 1, so that $\vec{D} = \varepsilon_0 \vec{E}$ and we recover the forms of Gauss' law for free space.

Free charges versus bound charges

The presence of a **free volume charge density** (ρ_f) can polarise a dielectric and give rise to a **bound charge density** (ρ_b). If the dielectric is LIH, we can write:

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \left(1 - \frac{1}{\epsilon_r}\right) \vec{D}, \quad \text{Eq.1. 45}$$

where we can apply div to both sides:

$$\nabla \cdot \vec{P} = \left(1 - \frac{1}{\epsilon_r}\right) \nabla \cdot \vec{D}, \quad \text{Eq.1. 46}$$

which leads to:

$$\rho_b = -\left(1 - \frac{1}{\epsilon_r}\right) \rho_f. \quad \text{Eq.1. 47}$$

We now have an expression linking the bound and free charges. We can use this to calculate the total charge density:

$$\rho = \rho_b + \rho_f = -\left(1 - \frac{1}{\epsilon_r}\right) \rho_f + \rho_f = \frac{\rho_f}{\epsilon_r}. \quad \text{Eq.1. 48}$$

So, if $\rho_f \neq 0$, the total charge density ρ has the same sign as ρ_f , but it is smaller.

Moreover, at the interface between a conductor and a LIH dielectric, the conductor carries a **free surface charge density** σ_f and the dielectric carries a **bound surface charge density** σ_b .

Both \vec{D} and \vec{E} are normal to the interface.

Then

$$\sigma_f = D = \epsilon_0 \epsilon_r E = \epsilon_0 E + P = \epsilon_0 \epsilon_r E - \sigma_b, \quad \text{Eq.1. 49}$$

leading to:

$$E = \frac{\sigma_f + \sigma_b}{\epsilon_0}. \quad \text{Eq.1. 50}$$

Since $D = \epsilon_0 \epsilon_r E$,

$$\frac{D}{\epsilon_0 \epsilon_r} = \frac{\sigma_f}{\epsilon_0 \epsilon_r} = \frac{\sigma_f + \sigma_b}{\epsilon_0}, \quad \text{Eq.1. 51}$$

leading to:

$$\sigma_f = \epsilon_r (\sigma_f + \sigma_b). \quad \text{Eq.1. 52}$$

Rearranging:

$$\sigma_b = -\left(1 - \frac{1}{\epsilon_r}\right)\sigma_f \quad \text{Eq.1. 53}$$

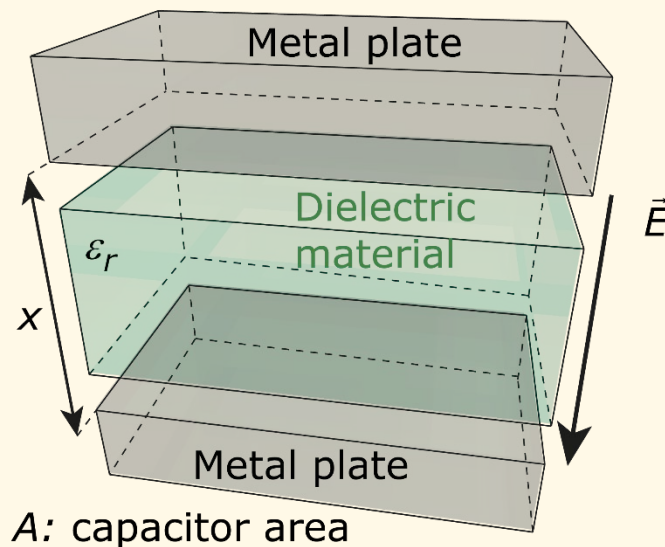
and the total charge is:

$$\sigma = \sigma_b + \sigma_f = -\left(1 - \frac{1}{\epsilon_r}\right)\sigma_f + \sigma_f = \frac{\sigma_f}{\epsilon_r}, \quad \text{Eq.1. 54}$$

which is similar to the expression for the volume charge density.

Energy stored in a dielectric

We consider a parallel plate capacitor with area A and plate separation x , that is filled with a dielectric with relative permittivity ϵ_r .



A : capacitor area

Figure 1.6. A parallel plate capacitor.

The work required to charge the capacitor is:

$$w = \frac{1}{2}CV^2, \quad \text{Eq.1. 55}$$

which is therefore also the stored energy W . From A-Level Physics, we know that $C = \frac{A\epsilon_0\epsilon_r}{x}$ and that $E = \frac{V}{x}$, i.e. $V = Ex$.

So, we can replace and rearrange (using $\vec{D} = \epsilon_0\epsilon_r\vec{E}$):

$$\frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{\epsilon_r\epsilon_0 A}{x}\right)(Ex)^2 = \frac{1}{2}(\epsilon_r\epsilon_0 E)E(Ax) = \frac{1}{2}DE \times \text{volume}.$$

Eq.1. 56

Therefore, the energy (w) stored per unit volume v is:

$$w = \frac{1}{2} \vec{D} \cdot \vec{E} \quad \text{Eq.1. 57}$$

and we have:

$$W = \int_V w dv. \quad \text{Eq.1. 58}$$

By integration over a path C in which the electric current flows (e.g. a length of wire), we obtain:

$$\vec{B} = \frac{\mu_0}{4\pi} \int_C \frac{Id\vec{L} \times \hat{r}}{r^2}. \quad \text{Eq.1. 59}$$

Summary

In a dielectric, microscopic dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.

Macroscopically, the combined effect of these dipole moments produces a polarisation $\vec{P} = \frac{\sum \vec{p}}{V}$, which is defined as the induced dipole moment per unit volume.

In LIH dielectrics, the polarisation originates from the surface charge density $\sigma_p = P$.

In LIH dielectrics: $\vec{P} = \epsilon_0 \chi_e \vec{E}$ with χ_e being the electric susceptibility of the material.

The electric flux density is $\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$.

The constitutive relation states that $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$.

We have a new Maxwell equation that states: $\nabla \cdot \vec{D} = \rho_f$

The relative permeability of the material ϵ_r is given by: $\epsilon_r = 1 + \chi_e$.

The polarisation has sources, these sources are the bound density of charge: $\rho_b = -\nabla \cdot \vec{P}$.

The bound density of charge is related to the free density of charge by: $\rho_b = -\left(1 - \frac{1}{\epsilon_r}\right)\rho_f$.

The surface bound density of charge is related to the surface free density of charge by: $\sigma_b = -\left(1 - \frac{1}{\epsilon_r}\right)\sigma_f$.

The energy stored in a dielectric is $W = \frac{1}{2} \int_V (\vec{D} \cdot \vec{E}) dv$.

Example question 1

A sample of diamond has a density of atoms of 3.5×10^{29} atoms per cubic meter. Diamond is a non-magnetic material and has a refractive index of 2.4. Carbon has a nucleus with a charge of $+6e$, surrounded by an electron cloud with charge $6e$.

If the diamond is given a polarization of 10^{-7} Cm^{-2} , what is the average separation between the centres of the positive and negative charges?

Answer: The magnitude of the polarization P is:

$$P = nq_b r,$$

where n is the number of atoms per unit volume, $\pm q_b$ is the bound charge of the dipole and r is the charge separation of the dipole. For a carbon atom, $q_b = 6e$. Therefore we find:

$$r = \frac{P}{nq_b} = \frac{P}{6ne} = \frac{10^{-7}}{6(3.5 \times 10^{29})(1.6 \times 10^{-19})}$$

$r \approx 3.0 \times 10^{-19} \text{ m}$,
which is a very small distance indeed.

Example question 2

[Old exam question] A material has an electrical susceptibility χ_e of 3.5. Calculate the magnitude of the electric dipole moment per unit volume, i.e. the polarization P , and the electric displacement, D , if the electric field E is 15 Vm^{-1} .(3)

Answer: We have $P = \varepsilon_0 \chi_e E$, $D = \varepsilon_0 \varepsilon_r E = \varepsilon_0 (1 + \chi_e) E$ and $D = \varepsilon_0 E + P$.

For the polarisation, we can use:

$$P = \varepsilon_0 \chi_e E = (8.85 \times 10^{-12})(3.5)(15) \approx 4.7 \times 10^{-10} \text{ Cm}^{-2}.$$

Then, for the electric flux density, we can use either

$$D = \varepsilon_0 \varepsilon_r E = \varepsilon_0 (1 + \chi_e) E = (8.85 \times 10^{-12})(4.5)(15) \approx 6.0 \times 10^{-10} \text{ Cm}^{-2}$$

or

$$D = \varepsilon_0 E + P = (8.85 \times 10^{-12})(15) + 4.7 \times 10^{-10} \approx 6.0 \times 10^{-10} \text{ Cm}^{-2}.$$

[One mark is awarded for correct equations and one mark each for the correct numerical answers.]

Example question 3

A parallel plate capacitor has a capacitance of 112 pF, a plate area of 96.5 cm^2 , and a mica dielectric ($\varepsilon_r = 5.4$). At a 55 V potential difference, calculate:

- (a) the electric field strength in the mica;
- (b) the magnitude of the free charge on the plates;
- (c) the magnitude of the induced surface charge;
- (d) the magnitude of the polarisation.

Answer: (a) We know that $C = \frac{A\epsilon_0\epsilon_r}{x}$ (so $d = \frac{A\epsilon_0\epsilon_r}{C}$) and that $E = \frac{V}{x}$, where x is the distance between the plates. Hence $E = \frac{VC}{A\epsilon_0\epsilon_r}$. Now

$$\text{we replace with values: } E = \frac{(55)(112 \times 10^{-12})}{(96.5 \times 10^{-4})(8.85 \times 10^{-12})(5.4)} \approx 13.4 \text{ kV}$$

(b) From the lecture, we know that $E = \frac{\sigma_m}{\epsilon_0\epsilon_r}$, so the density of free

charge on the plates is $\sigma_m = E\epsilon_0\epsilon_r$. To find the charge, we multiply this density by the area A and obtain

$$q_m = E\epsilon_0\epsilon_r A = (13.4 \times 10^3)(8.85 \times 10^{-12})(5.4)(96.5 \times 10^{-4}) \approx 6.18 \text{ nC}.$$

(c) From the lecture, the induced surface density of charge is

$\sigma_p = \sigma_m - E\epsilon_0$, so the surface charge is $\sigma_p A$. So,

$$q_p = A(E\epsilon_0\epsilon_r - E\epsilon_0) = AE\epsilon_0(\epsilon_r - 1). \text{ Numerically:}$$

$$q_p = (96.5 \times 10^{-4})(13.4 \times 10^3)(8.85 \times 10^{-12})(4.4) \approx 5.03 \text{ nC}.$$

(d) For the polarisation we have $\sigma_p = P$, so $P = \sigma_m - E\epsilon_0 = E\epsilon_0(\epsilon_r - 1)$

$$\text{Numerically: } P = (13.4 \times 10^3)(8.85 \times 10^{-12})(4.4) \approx 522 \text{ nC/m}^2.$$

Example question 4

[2016 Exam Question] Describe what it means that, with respect to electric and magnetic fields, a medium is

(a) linear (1)

(b) isotropic (1)

(c) homogeneous (1)

Answer:

(a) For the electric fields, the polarization density \vec{P} (and the displacement field \vec{D}) is proportional to the external electric field,

$$\vec{P} = \epsilon_0\chi_e\vec{E} \text{ (and } \vec{D} = \epsilon_0\epsilon_r\vec{E}). \text{ For the magnetic fields, the}$$

magnetization \vec{M} (magnetic field \vec{B}) is proportional to the magnetic field intensity \vec{H} , $\vec{M} = \mu_0\chi_m\vec{H}$ (and $\vec{B} = \mu_0\mu_r\vec{H}$).

[Equations expressing linear relations between the vector fields (for example, $\vec{P} = \epsilon_0\chi_e\vec{E}$ and $\vec{M} = \mu_0\chi_m\vec{H}$ are necessary for a full mark.]

(b) At a set point in the medium, ϵ_r (or χ_e) and μ_r (or χ_m) do not depend on the direction of the applied external field \vec{E} and \vec{H} , respectively.

[It is enough to mention only one of the two proportionality constants relevant for electric/magnetic fields (for example, ϵ_r and μ_r but not χ_e or χ_m).]

(c) The relative permittivity ϵ_r (electric susceptibility χ_e) and relative permeability μ_r (magnetic susceptibility χ_m) do not depend on the position within the medium.

[It is enough to mention only one of the two proportionality constants relevant for electric/magnetic fields (for example, ϵ_r and μ_r but not χ_e or χ_m).]

Example question 5

[2005 Exam Question] A charge of uniform charge density ρ is distributed throughout a medium with $\epsilon_r = 1$ throughout a sphere of radius R . The electric displacement (\vec{D}) inside the sphere is given by, $\vec{D} = \frac{\rho r}{3} \hat{r}$, for $r < R$.

Find the energy stored inside the charge distribution (i.e. for $r \leq R$). (3)

Answer:

We are given \vec{D} and we know that $\vec{D} = \epsilon \vec{E} = \epsilon_0 \epsilon_r \vec{E}$. Since we have

$\epsilon_r = 1$, we can write that $\vec{E} = \frac{\rho r}{3\epsilon_0} \hat{r}$.

[1 mark definition and/or finding E]

The energy density is given by

$$w = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1}{2} \left(\frac{\rho r}{3} \hat{r} \right) \cdot \left(\frac{\rho r}{3\epsilon_0} \hat{r} \right) = \frac{\rho^2 r^2}{18\epsilon_0}.$$

[1 mark for finding the energy density; 1/2 mark for definition but wrong density]

Integrate over the volume of the sphere to find the stored energy.

$$U = \int_0^R w dV = \int_0^R \frac{\rho^2 r^2}{18\epsilon_0} (4\pi r^2 dr)$$

$$U = \frac{4\pi\rho^2}{18\epsilon_0} \int_0^R r^4 dr = \frac{4\pi\rho^2}{18\epsilon_0} \left[\frac{r^5}{5} \right]_0^R = \frac{2\pi\rho^2 R^5}{45\epsilon_0} \text{ J.}$$

[1 mark for setting up integral correctly]