Lecture 12

Electric fields in materials

Last time we saw that

In a dielectric, microscopic dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.

Macroscopically, the combined effect of these dipole moments produces a polarisation, defined as the induced dipole moment per unit volume: $\vec{P} = \dfrac{\sum \vec{p}}{V}$

In LIH dielectrics, the polarisation originates from the surface charge density $\sigma_p = P$.

In LIH dielectrics: $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ with χ_e being the electric susceptibility of the material.

The relative permeability of the material is given by: $\varepsilon_r = 1 + \chi_e$

Overview

In this Lecture we will look at:

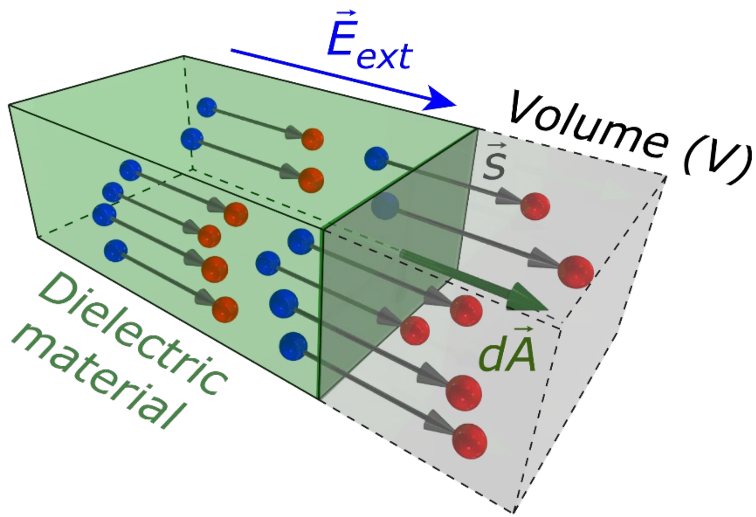
- ❑ Electric polarisation at the surface
- ❑ The sources of electric polarization
- ❑ Bound current density and the continuity equation
- ❑ Free charges and the electric flux density
- ❑ Free charges versus bound charges
- ❑ The energy stored in a dielectric

First, the electric polarisation at the surface!

Electric polarisation at the surface

We saw that: $\sigma_p = P$

Now, consider a unit volume *V* near a surface *dA* of a dielectric:



An external *E*-field splits the charges by a distance *s*, near an element of area *dA*, inside a volume *V=sdA.*

When *E* increases, *N+* positive charges and *N–* negative charges cross the small element of area *dA*. The net charge that crosses *dA* is then

$$dq_b = N^+ q_b - N^-(-q_b) = (N^+ + N^-)q_b$$

This is the sum of all charged crossing *dA* times the charge value. But, this is also the number of molecules within the unit volume: $V = \vec{s} \cdot d\vec{A}$

Then remembering that *n* is the number of molecules per unit volume and $\vec{s} = \vec{r}$:

$$dq_b = nVq_b = n(\vec{s} \cdot d\vec{A})q_b$$
$$= nq_b \vec{s} \cdot d\vec{A} = n\vec{p} \cdot d\vec{A} = \vec{P} \cdot d\vec{A}$$

And also: $P = dq_b / dA = \sigma_p$

polarised

How is this important?

The sources of electric polarisation

We just saw that:

$$dq_b = \vec{P} \cdot d\vec{A}$$

Inside the dielectric, the net charge that flows out of the closed surface A which encloses the volume V is

$$Q_{out} = \int_A \vec{P} \cdot d\vec{A}$$

but then, it is clear that the charge that remains within V must be $-Q_{out}$

We can also use the **volume density of charge** ρ_b to find the charge inside the dielectric:

$$\int_V \rho_b \, dV = -Q_{out}$$

So, using the divergence theorem:

$$\int_V \rho_b \, dV = -\int_A \vec{P} \cdot d\vec{A}$$
$$= -\int_V (\nabla \cdot \vec{P}) \, dV$$

Therefore: $\rho_b = -\nabla \vec{P}$

This means that there are sources of polarization.

The total bound polarisation charge within a region is obtained by integrating over a volume

$$Q_b = \int \rho_b \, dV$$

$$= -\int_V (\nabla \cdot \vec{P}) \, dV = -\int_A \vec{P} \cdot d\vec{A}$$

How does this bound charge change with time?

Bound current density and the continuity equation

We just saw that:

$$Q_b = \int_V \rho_b \, dV = -\int_V (\nabla \cdot \vec{P}) \, dV$$

Under the influence of a time dependent electric field (such as the one in a light wave), by definition, the current flowing through a surface *A* is related to the **current density** $\vec{J}$ by :

$$\frac{dQ_b}{dt} = I = \int_A \vec{J} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{J}) \, dV$$

Substituting:

$$-\frac{d}{dt} \int_V \rho_b \, dV = \int_V \left(\frac{\partial \rho_b}{\partial t} \right) dV = -\int_V \left(\frac{\partial (-\nabla \cdot \vec{P})}{\partial t} \right) dV = \ldots$$

$$= \int_V (\nabla \cdot \vec{J}_b) \, dV$$

Therefore:

$$\vec{J}_b = \frac{\partial \vec{P}}{\partial t}$$

This means that the motion of bound charges results in a polarisation or **'bound' current density** $\vec{J}_b$.

We also deduce that:

∇ —— missed

which is called the **Continuity Equation**.

We discussed bound charges; how about free charges?

Free charges and the electric flux density

The total charge in a dielectric is:

$$\rho = \rho_b + \rho_f$$

The bound charges result from polarisation. The free charges do not.

From Maxwell's law:

$$\varepsilon_0 \nabla \cdot \vec{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f$$

Therefore:

It looks like the free charges are the source of some field!

Let's define: $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

This is the **electric flux density** (aka **the electric displacement**)

Note: The equation $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ is known as a **Constitutive Equation**.

Now we can write Maxwell's law as:

$\nabla \cdot \vec{D} = \rho_f$ Compare to

The total free charge is now:

We can also write:

Replacing with:

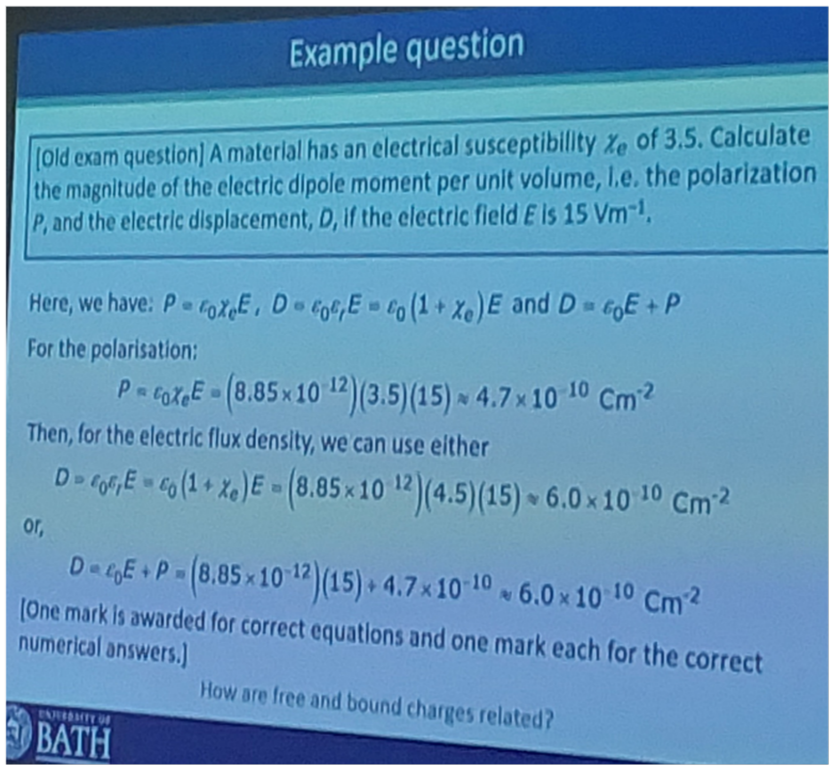
We obtain: $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$

In free space $\vec{P} = 0$; $\varepsilon_r = 1$ so $\vec{D} = \varepsilon_0 \vec{E}$.

In practice?

Example question

[Old exam question] A material has an electrical susceptibility χ_e of 3.5. Calculate the magnitude of the electric dipole moment per unit volume, i.e. the polarization P, and the electric displacement, D, if the electric field E is 15 $\mathrm{Vm^{-1}}$.



Free charges versus bound charges

In LIH dielectrics, we have a **free volume charge density** (ρ_f) and a **bound volume charge density** (ρ_b).

$$\vec{P} = \vec{D} - \varepsilon_0 \vec{E} = \left(1 - \frac{1}{\varepsilon_r}\right)\vec{D}$$

Apply div to both sides:

Which leads to:

$$\rho_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\rho_f$$

And we can calculate the total ρ :

We also have a **free surface charge density (** σ_f) and a **bound surface charge density** (σ_b).

Leading to:

Since $D = \varepsilon_0 \varepsilon_r E$

Therefore:

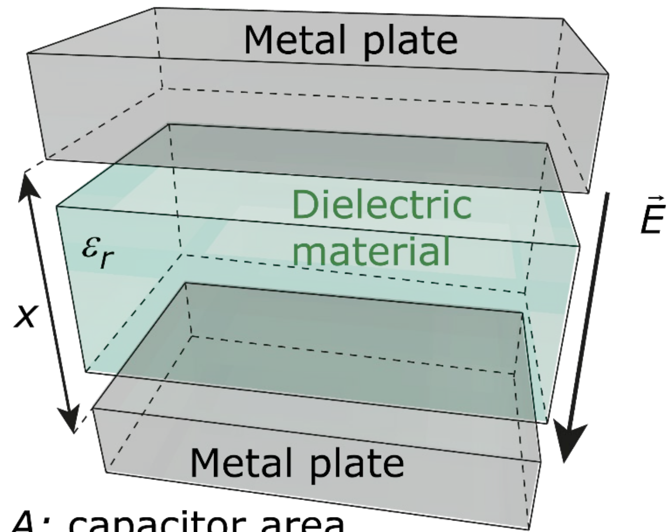
Rearranging $\sigma_b = -\left(1 - \frac{1}{\varepsilon_r}\right)\sigma_f$

And we can calculate the total σ :

What is the energy stored in a dielectric?

The energy stored in a dielectric

We consider a parallel plate capacitor:



A: capacitor area

Work required to charge the capacitor:

$$W = \frac{1}{2} C V^2$$

We also have: $C = \frac{A\varepsilon_0\varepsilon_r}{x}$ and $E = \frac{V}{x}$

Then, using: $\vec{D} = \varepsilon_0\varepsilon_r \vec{E}$ and $V = E x$

$$\frac{1}{2} C \vec{V} = \frac{1}{2}\left(\frac{\varepsilon_r \varepsilon_0 A}{x}\right)(E x^2)$$

$$= \frac{1}{2}(\varepsilon_r \varepsilon_0 E) E (A x) = \frac{1}{2} D E \times vol$$

Therefore, the energy (w) stored per unit volume v is: $w = \frac{1}{2}\vec{D}\cdot\vec{E}$

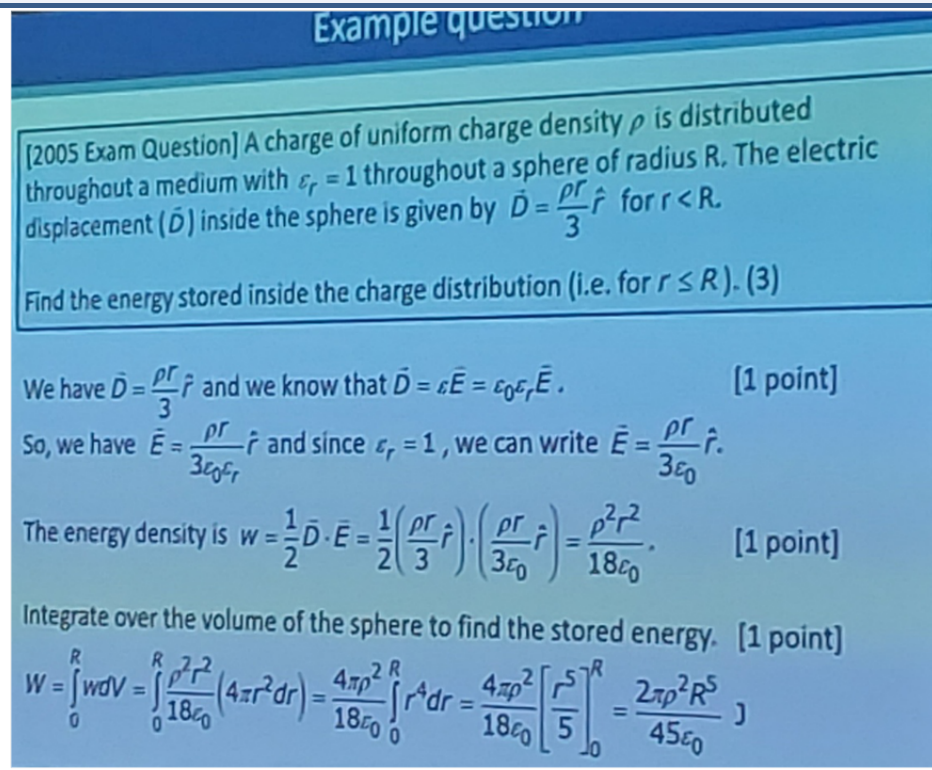
And we have the **energy stored**: $W = \int_v w \, dv$

What about nonlinear materials?

Example question

[2005 Exam Question] A charge of uniform charge density ρ is distributed throughout a medium with $\varepsilon_r = 1$ throughout a sphere of radius R. The electric displacement ($\vec{D}$) inside the sphere is given by $\vec{D} = \frac{\rho r}{3}\hat{r}$ for r < R.

Find the energy stored inside the charge distribution (i.e. for $r \leq R$). (3)



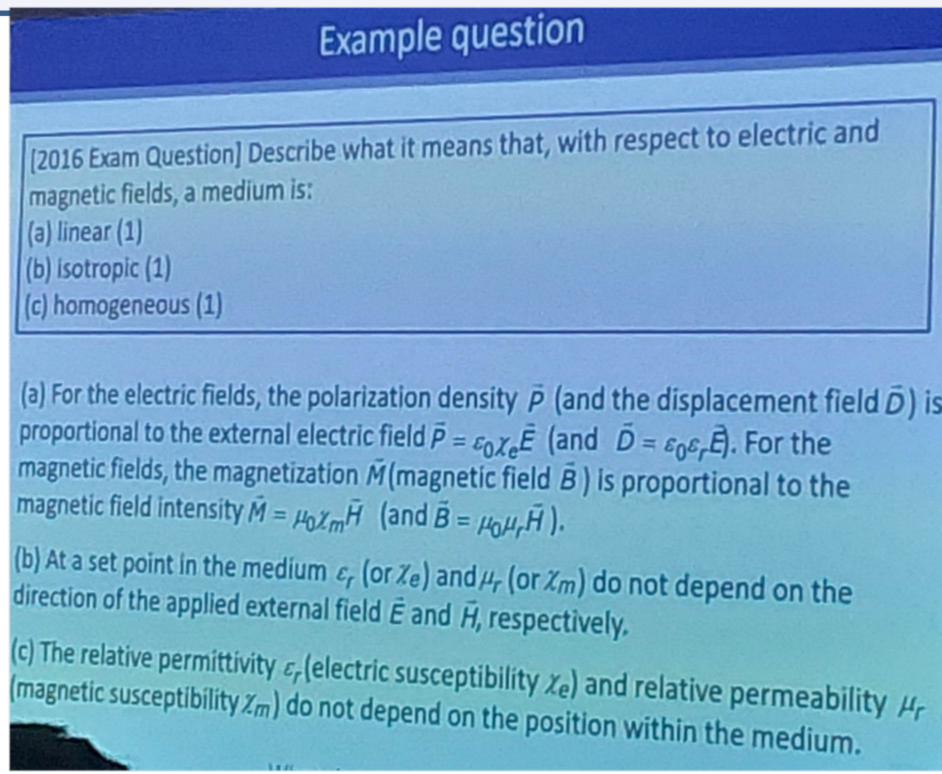
Example question

[2016 Exam Question] Describe what it means that, with respect to electric and magnetic fields, a medium is:

(a) linear (1)

(b) isotropic (1)

(c) homogeneous (1)



Summary

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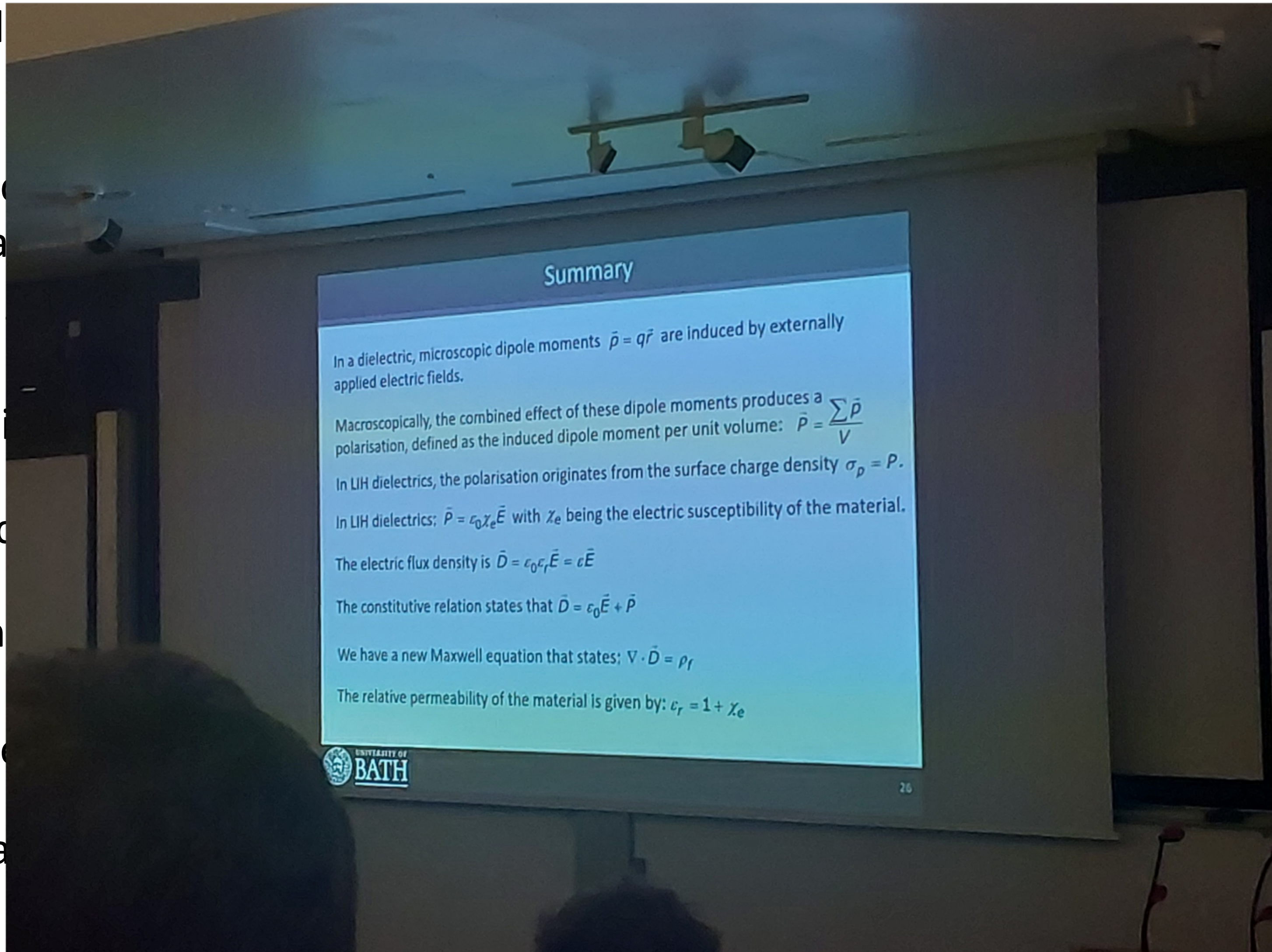

Summary

The polarisation has sources, these sources are the bound density of charge:

The bound density of charge is related to the free density of charge by:

The surface bound density of charge is related to the surface free density of charge by:

The energy per unit volume stored in a dielectric is:

How about magnetic materials?