# 5. NUCLEAR REACTIONS

*The aim of this section is to introduce:*

- *the basic features of reactions;*
- *the concept of cross-section*
- *the centre of mass frame*
- *how to use this in fission and fusion reactions*

A typical reaction can be written as:

$$a + X \rightarrow Y + b$$

Examples:

$${}^{1}_{0}n + {}^{10}_{5}B \rightarrow {}^{4}_{2}He + {}^{7}_{3}Li + 2.8\ \mathrm{MeV}$$

*B.E. 64.75 28.3 39.25*

*(used in reactor control, or in detecting neutrons …)*

$${}^{2}_{1}H + {}^{16}_{8}O \rightarrow {}^{17}_{8}O^{*} + p$$

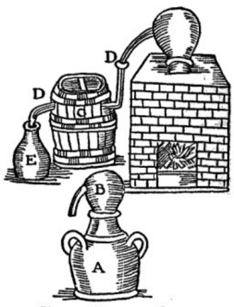
*(deuteron stripping, creating excited states)*

$${}^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{141}_{56}Ba^{*} + {}^{92}_{36}Kr^{*} + 3\,n + energy!!!$$

*(an infamous reaction)*

$${}^{1}_{1}p + {}^{204}_{82}Pb \rightarrow {}^{197}_{79}Au + {}^{8}_{4}Be$$
$$\rightarrow 2\,\alpha$$

*(another famous reaction …)*



## Typical features:

(i) Usually, light nucleons (*n* or *p*) or light nucleus, incident on a heavier one

(ii) Energy released Q is:

Q = total binding energy of products
– total binding energy of reactants

If Q > 0, the reaction is EXOTHERMIC
If Q < 0, the reaction is ENDOTHERMIC
(it requires energy to take place)
*(the incident particle must have a threshold K.E. for the reaction to occur)*

(iii) Strong interaction involved:
Nuclei must approach within ~ 1 fm to react …

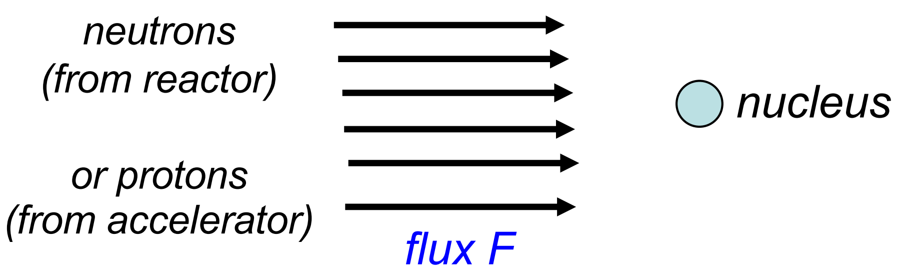
(iv) The strength of the reaction is measured by its cross-section $\sigma$

(v) We need to work in the "Centre of Mass" frame

# 5.1. CROSS-SECTION

The flux of incident particles is
the number of incident particles per unit area per unit time
*(units: $m^{-2}$ $s^{-1}$)*

Let us consider a single nucleus




The number of particles interacting per unit time is proportional to the flux F.

We can write it as: $\sigma \times F$

$\sigma$ is the cross-section (unit: $m^2$)

$\sigma \times F$ is the number of particles incident on an area $\sigma$ per unit time: $\sigma$ is the effective cross-section of the nucleus.

$\sigma$ is not simply related to the cross-sectional area of the nucleus: it gives an idea of the magnitude …

The cross-section area of an average nucleus is:

$$\approx \pi \, R_{nuc}^2$$

$$\approx \pi \times (1.1)^2 \times A^{\frac{2}{3}} \times 10^{-30} \text{ m}^2$$

For ${}^{238}_{92}U$ : $\approx 1.45 \times 10^{-28} \text{ m}^2$

For ${}^{16}_{8}O$ : $\approx 0.25 \times 10^{-28} \text{ m}^2$

Units: barns (1 barn = $10^{-28}$ m$^2$)

The cross-sections vary in range between

$\sigma \sim 0$ to $10^4$ barns

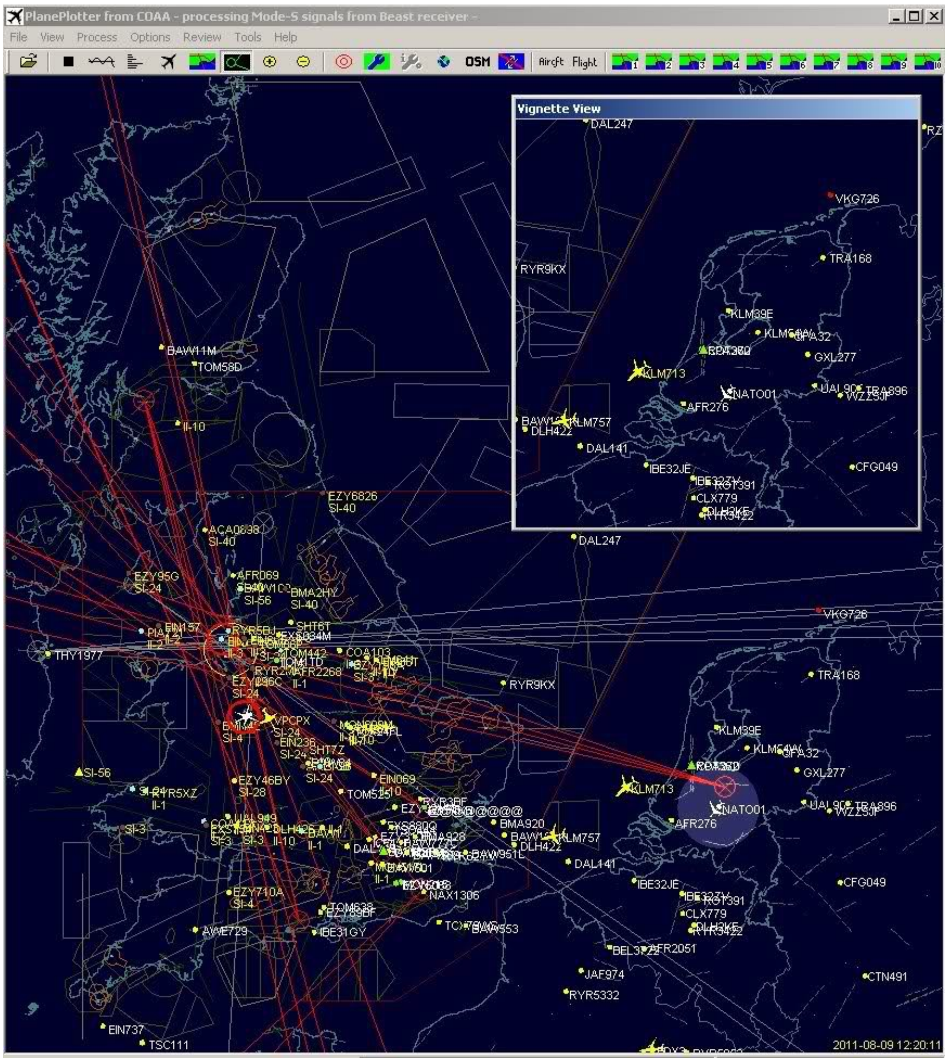
(this is <u>much larger than the “physical” cross-section</u>)

<u>Analogy:</u>
The radar cross-section of an object is related to how easily it interacts with electromagnetic waves:

| | |
|---|---|
| typical bird | 0.01 m$^2$ |
| human | 1 m$^2$ |
| average car | 100 m$^2$ |
| B-2 stealth bomber | 0.75 m$^2$ |

*(all rather different from the physical cross-section)*


USAF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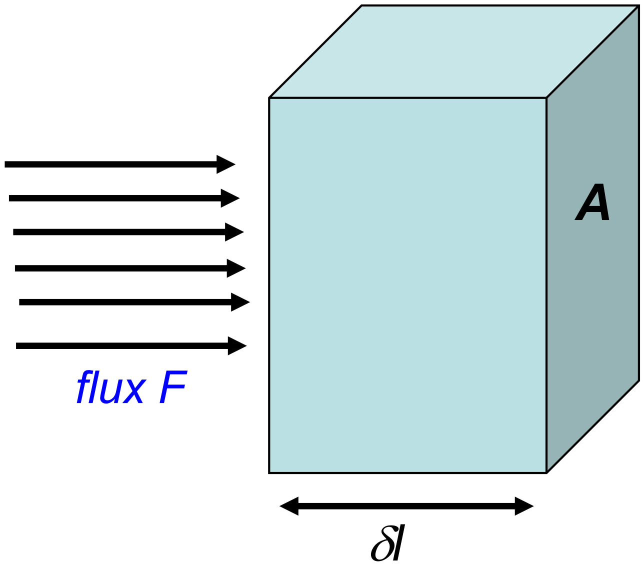
For a target with $\rho_{nuc}$ nuclei per unit volume:




*(assuming flux is not reduced along distance $\delta l$)*

The number of reactions per unit time is:

$$\underbrace{F\,\sigma}_{\text{nucleus}} \times \underbrace{\rho_{nuc}\,A\,\delta l}_{\text{no. of nuclei}}$$

*(assuming single scattering)*

The number of particles incident per unit time is:

$$F\ A$$

Therefore the probability of reacting in thickness $\delta l$ is:

$$\rho_{nuc}\ \sigma\ \delta l = \frac{\delta l}{l_0}$$

$l_0$ is the mean free path of incident particles

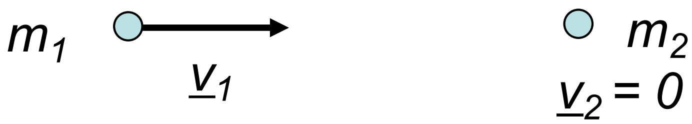
For $\rho_{nuc}$ = $10^{29}$ $m^{-3}$

| | | |
|---|---|---|
| | $\sigma$ = 1 barn | 100 barns |
| | $l_0$ = 10 cm | 1 mm |

# 5.2. CENTRE OF MASS FRAME

We shall consider non-relativistic cases, with stationary targets.

## Lab frame:




$$KE_{lab} = \frac{1}{2} m_1 \, v_1^2$$

## Centre of Mass (CM) frame:

It moves with velocity $v_{CM}$ so that:

$$m_1 \left( v_1 - v_{CM} \right) = m_2 \, v_{CM}$$

i.e. the net momentum is 0


Hence:

$$v_{CM} = \frac{m_1}{m_1 + m_2} v_1$$

$$KE_{CM} = \frac{1}{2} m_1 \left( v_1 - v_{CM} \right)^2 + \frac{1}{2} m_2 \, v_{CM}^2$$

$$KE_{CM} = \frac{1}{2} m_1 \left(v_1 - v_{CM}\right)^2 + \frac{1}{2} m_2 v_{CM}^2$$

Substituting for $v_{CM}$ and rearranging, one gets:

$$\boxed{\begin{aligned} KE_{CM} &= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v_1^2 \\ KE_{CM} &= \frac{m_2}{m_1 + m_2} KE_{lab} \end{aligned}}$$

In the CM frame, there is <u>no net momentum.</u>
Thus, all KE can go into the reaction.

In the LAB frame, there is a net momentum, which must be conserved. All KE is <u>not</u> available.

Only the KE in the CM frame is available for the reaction (e.g. to excite the target nucleus).

If Q is required to make an endothermic reaction occur, in the LAB frame, one must supply:

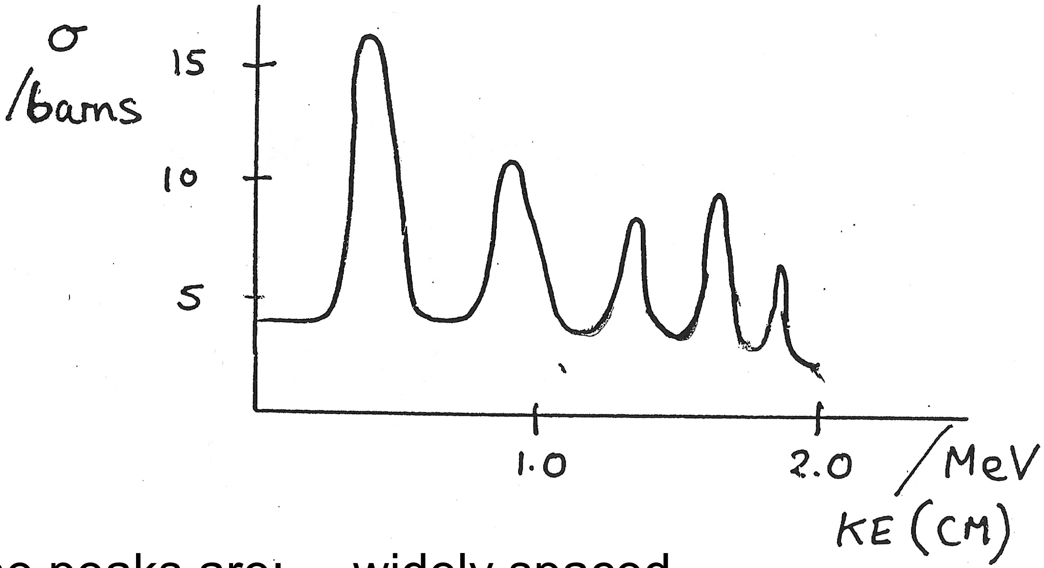
$$\frac{m_1 + m_2}{m_2} Q$$

*This is why we fire light particles at heavy ones …*

# 5.3. NEUTRON REACTIONS

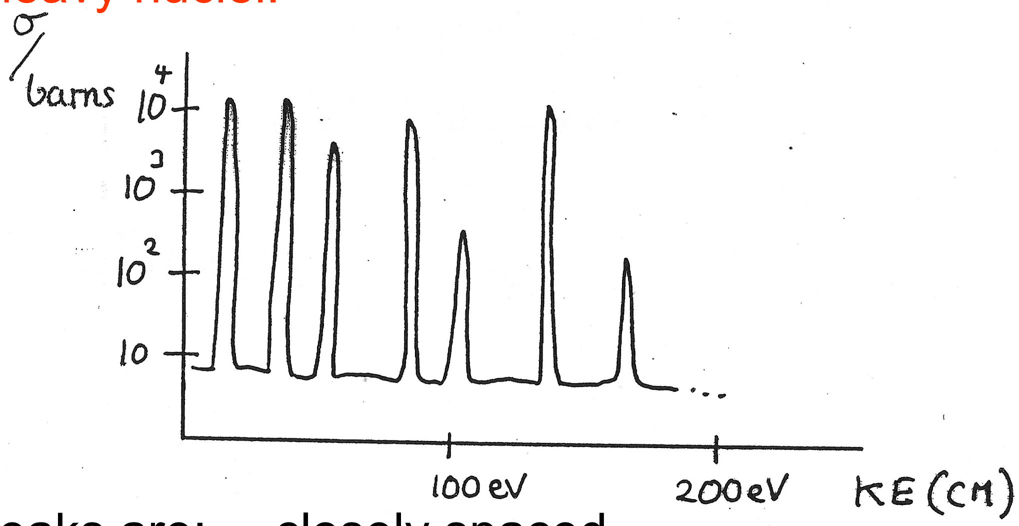
$$n + {}^{A}_{Z}X \rightarrow products$$

With light nuclei: (e.g. $n + {}^{16}_{8}O \rightarrow \cdots$)




The peaks are: widely spaced
broad
low (a few barns)

With heavy nuclei:




The peaks are: closely spaced
narrow
high ($10^3$ – $10^4$ barns)

Why do we have resonant peaks?

There is formation of a compound nucleus:

$$n + {}^{A}_{Z}X \rightarrow {}^{A+1}_{Z}X^{*} \rightarrow products$$

The neutron is absorbed into the nucleus.
Its energy is shared between nucleons.

∴ a while before any nucleon(s) have enough energy to escape

*This is a two-stage process.*

*The decay mode carries little information about the formation process.*

Resonance occurs when the initial KE (in the CM frame) matches the energy of the excited state of the compound nucleus ${}^{A+1}_{Z}X$

OFF RESONANCE

- energies do not match
- n doesn't enter nucleus (low probability)
- low $\sigma$

ON RESONANCE

- energies match (within $\Delta$E)
- n enters target -> "long-lived" compound nucleus
- decay by breakup or $\gamma$-decay

*Note: the binding energy of the neutron in the compound nucleus is available for the excitation of the target nucleus.*

*Typical values: 4 – 8 MeV …*

Let us look at a typical example:

$$n + {}^{16}_{8}O \rightarrow {}^{17}_{8}O^{*}$$

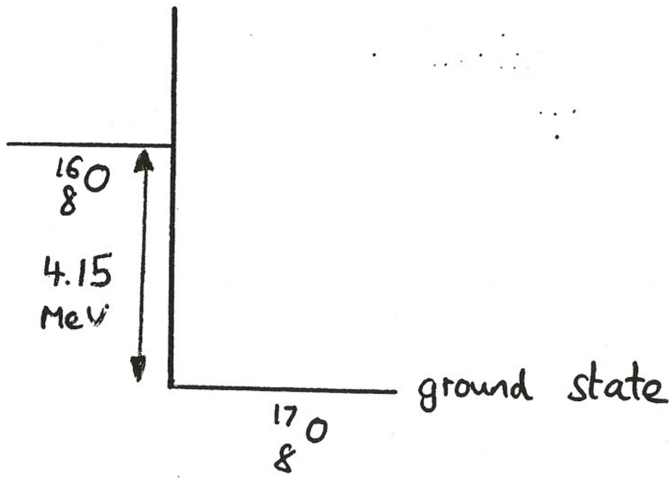
We need the binding energies of each …

$^{16}_{8}O$ B.E. = 127.62 MeV

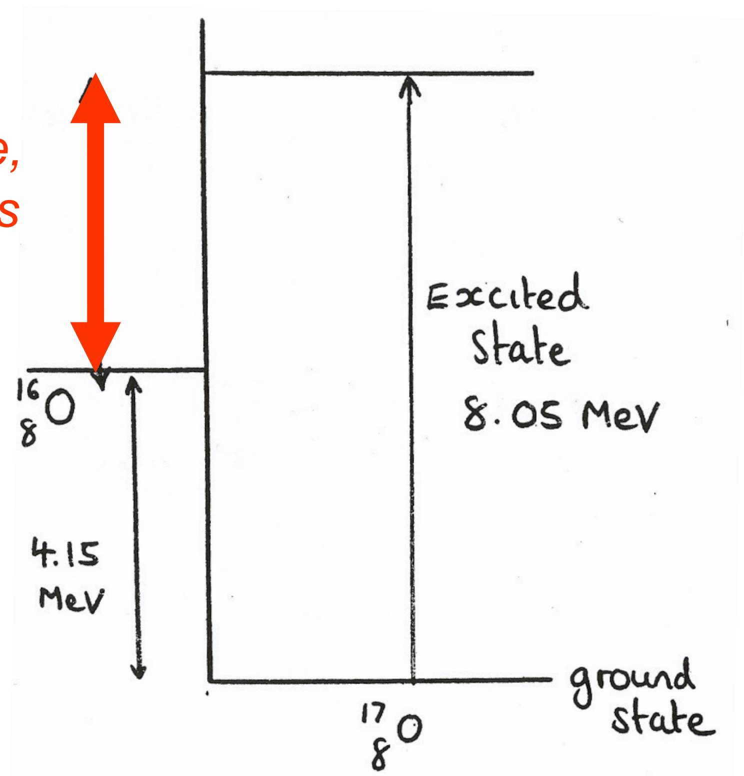
$^{17}_{8}O$ B.E. = 131.77 MeV *(ground state)*

$$\therefore {}^{17}_{8}O \text{ is more stable than } {}^{16}_{8}O$$

The difference is: 131.77 – 127.62 = 4.15 MeV
We can use this value to draw the energy level diagram:




*To be at resonance, we must supply this energy (as KE)*



For: $n+{}^{16}_{8}O \rightarrow {}^{17}_{8}O^{*}$

The resonance occurs when:

4.15 MeV + KE(CM) of n = excited energy level of ${}^{17}_{8}O$

binding energy of last neutron in ${}^{17}_{8}O$

So, for 8.05 MeV excited state, we must supply:

KE(CM) = 8.05 – 4.15 = 3.9 MeV

In the LAB frame, $KE = 3.9 \times \frac{M + m_n}{M}$

$$= 3.9 \times \frac{17}{16} = \underline{\underline{4.14 \text{ MeV}}}$$

*(here it is adequate to use A for mass …)*

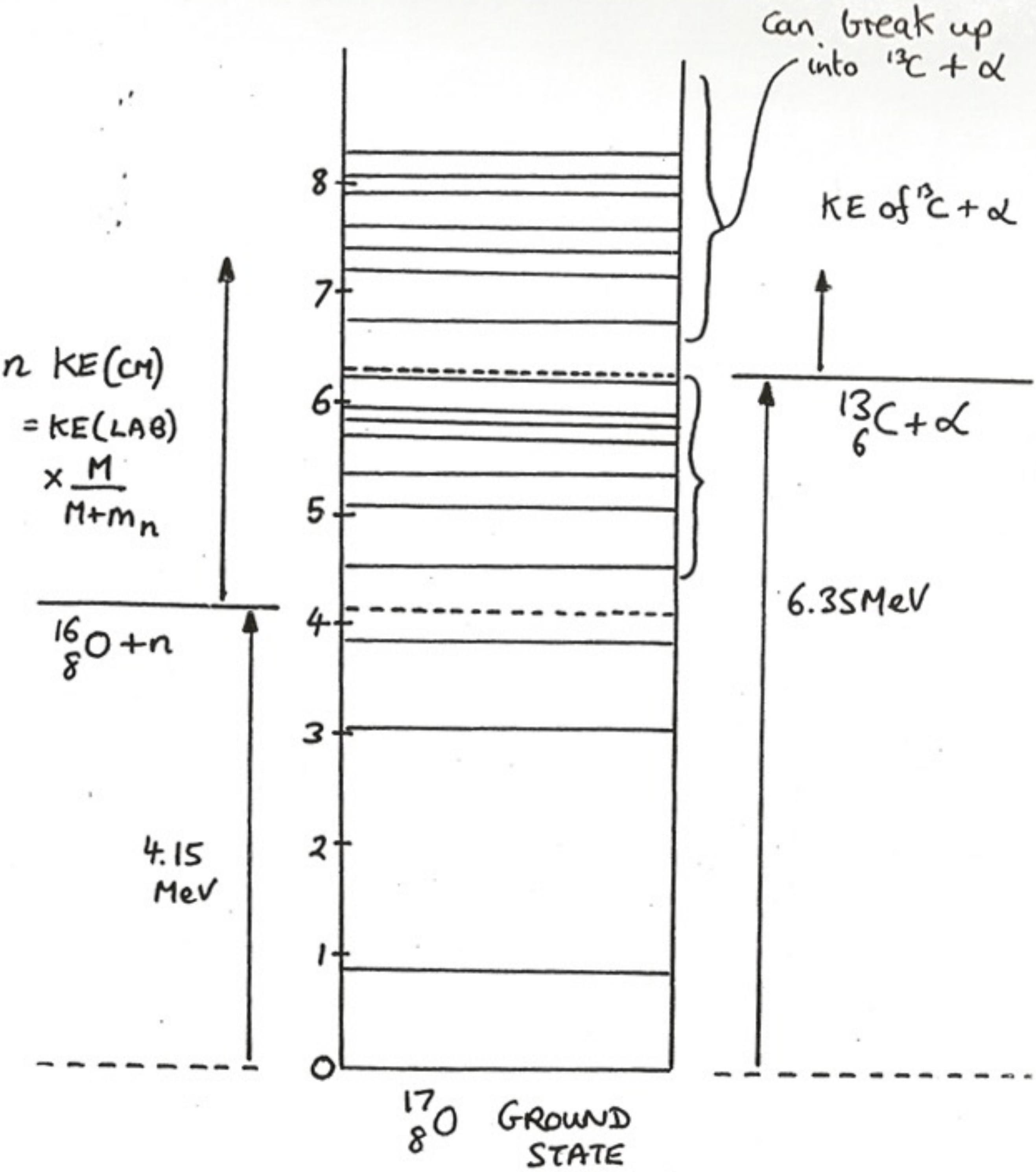
$$n+{}^{16}_{8}O \rightarrow {}^{17}_{8}O^{*} \rightarrow {}^{13}C + \alpha$$

Q = -2.2 MeV (endothermic reaction)

Hence, if n has 4.14 MeV of KE(LAB), products will have:

4.14 – 2.2 = 1.94 MeV of KE(LAB)

# $n + {}^{16}_{8}O$ Reaction




Stationary (no KE) in CM frame.

## Reaction products:

They are determined by the decay of intermediate ${}^{A+1}_{Z}X^{*}$

e.g. $n + {}^{16}_{8}O \rightarrow {}^{17}_{8}O^{*}$

$\rightarrow n + {}^{16}_{8}O$ Resonant elastic scattering

$\rightarrow n + {}^{16}_{8}O^{*}$ inelastic

$\rightarrow {}^{13}_{6}C + \alpha$ reaction

$\rightarrow {}^{17}_{8}O + \gamma$ Neutron capture.

Neutron capture is weak for low values of *A* (they break up faster) but it becomes important for high values of *A*.

# 5.4. CHARGED PARTICLE REACTIONS

Very similar to neutron reactions, except for the
Coulomb barrier (*cf. $\alpha$-decay)*

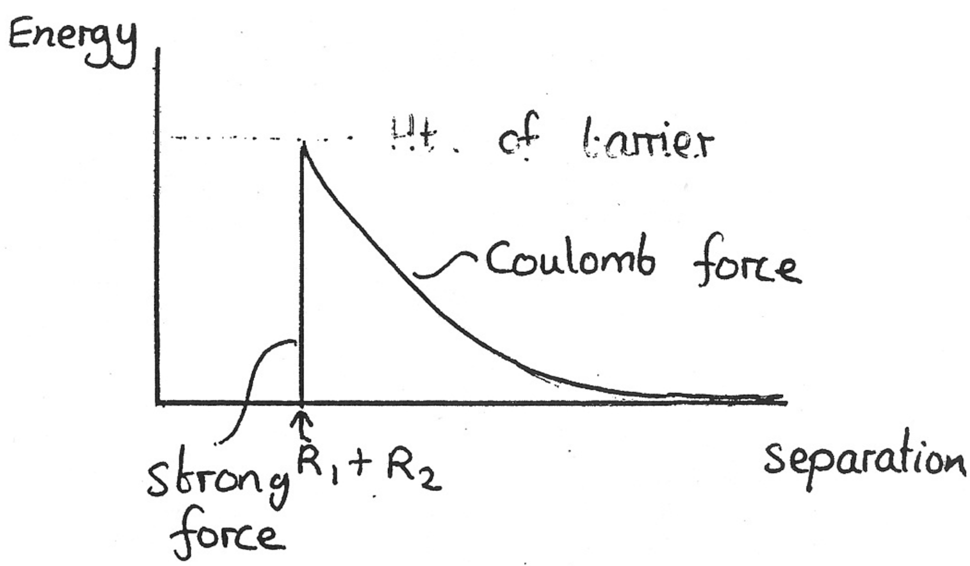
To experience the strong force, classically, nuclei must touch.

Energy supplied must be:

$$\frac{Z_1 \, Z_2 \, e^2}{4\pi \, \varepsilon_0 \left(R_1 + R_2\right)} \approx \frac{Z_1 \, Z_2}{\left(A_1^{1/3} + A_2^{1/3}\right)} \times 1.3 \text{ MeV}$$

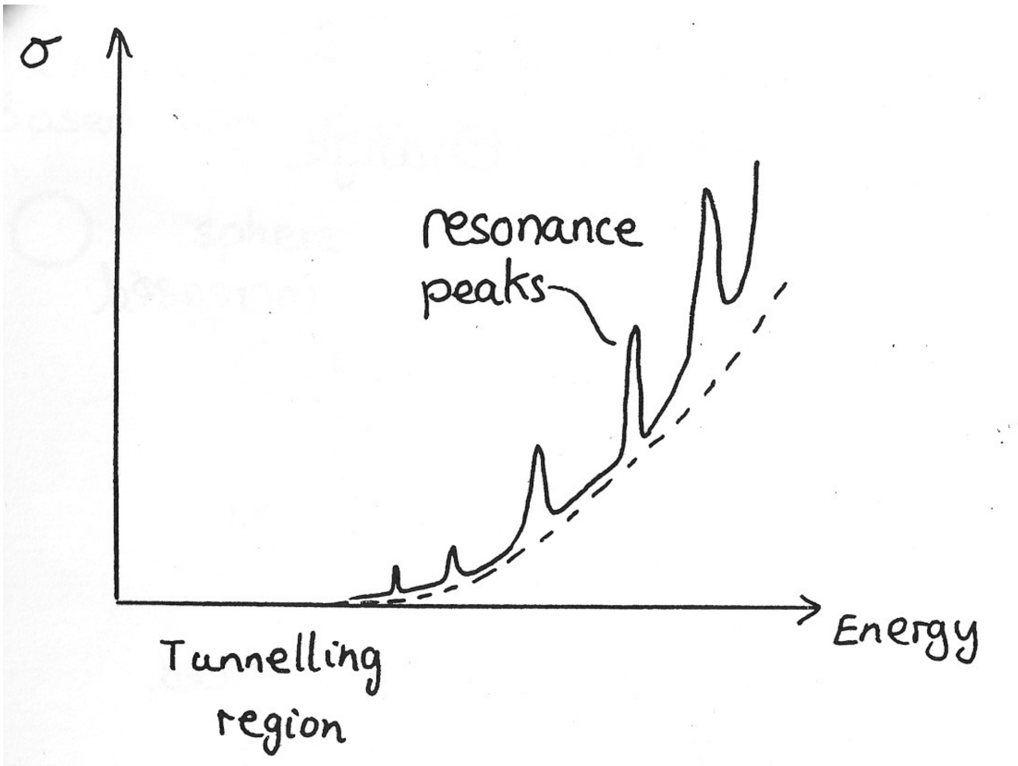
So, for example, for: ${}^{13}_{6}C + \alpha \rightarrow {}^{16}O + n$

the barrier is around 4 MeV



- **If KE « Barrier Height**
  - must tunnel to interact
  - very small $\sigma$
  - only light particles react

- **If KE » Barrier Height**
  - same as neutron reactions




missed lecture

# 5.5. NUCLEAR FISSION

$$\text{HEAVY NUCLEUS} \xrightarrow{\textit{splits}} \text{2 light nuclei} + \text{(neutrons)} + \textit{energy}$$

Two cases:

1. spontaneous fission

   some heavy nuclei are unstable

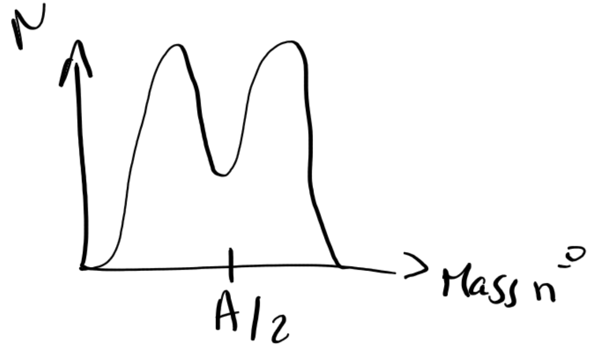
   (form of radioactive decay)

2. induced fission

   follows a nuclear reaction

   (usually neutron + heavy nucleus)

$$n + {}^{A}_{Z}X \quad {}^{A+1}_{Z}X^{*} \rightarrow \text{PRODUCTS}$$




## Spontaneous Fission

Let us consider the symmetric case (rare, but easier):

$$ {}^{A}_{Z}X \rightarrow 2\ {}^{A/2}_{Z/2}Y $$

The energy released is:

$$ Q = 2B(A/2, Z/2) - B(A,Z) $$

The Liquid Drop Model gives the binding energy of a nucleus of mass number *A*, containing *Z* protons and *N* neutrons as:

$$ B(Z,N) = a\ A - b\ A^{\frac{2}{3}} - s\frac{(N-Z)^2}{A} - d\frac{Z^2}{A^{\frac{1}{3}}} + \delta $$

where *a* = 15.84 MeV, *b* = 18.33 MeV, *s* = 23.20 MeV, *d* = 0.714 MeV.

Using the liquid drop model (and ignoring the pairing term δ)

no influence

→ bulk term $\quad a\ A$

→ symmetry term $\quad s\frac{(N-Z)^2}{A}$

$\left.\begin{matrix} a\ A \\ s\frac{(N-Z)^2}{A} \end{matrix}\right\}$ are unaffected

$\alpha A = 2a\left(\frac{A}{2}\right)$

$\frac{(N-Z)^2}{A}s = 2\frac{s\left(\frac{N}{2}-\frac{Z}{2}\right)^2}{A/2}$

└ don't promote or inhibit

Only the surface and Coulomb terms change.

- the surface energy is increased:

$$b\,A^{2/3} \Rightarrow 2\,b\left(\frac{A}{2}\right)^{\frac{2}{3}} = 1.26\,b\,A^{\frac{2}{3}}$$

↳ This decreases B
i.e makes fiss° less likely

- the Coulomb energy is reduced:

↳ B ↑ => f° + likely

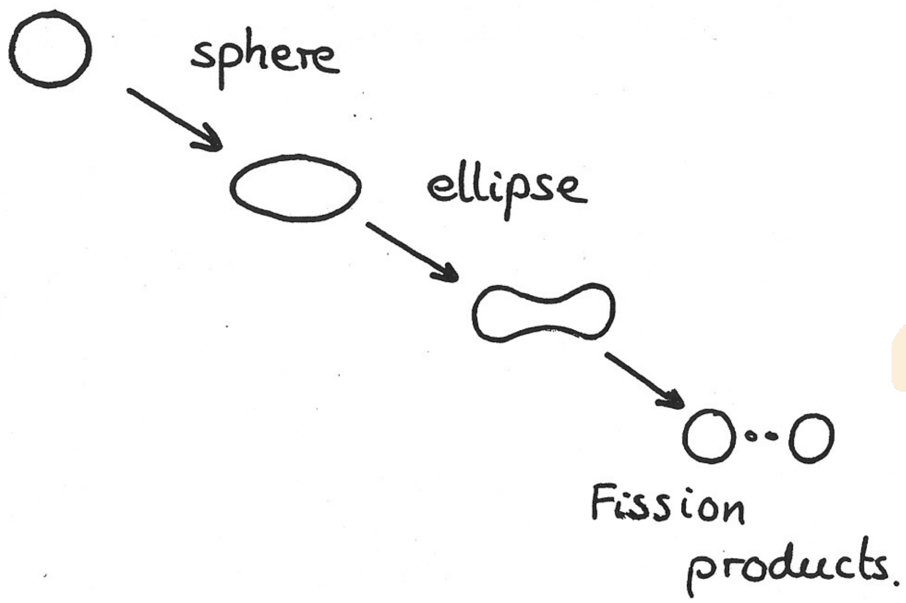
$$\frac{d\,Z^2}{A^{1/3}} \Rightarrow \frac{2\,d\left(\frac{Z^2}{4}\right)}{\left(\frac{A}{2}\right)^{\frac{1}{3}}} \Rightarrow 0.62\,\frac{d\,Z^2}{A^{\frac{1}{3}}}$$

Works out when Q ≥ 0 (i.e. for Z ≥ 45)

For larger values of Z, Q can be large (~100-200 MeV)

## Fission Model

Based on the idea of a liquid drop

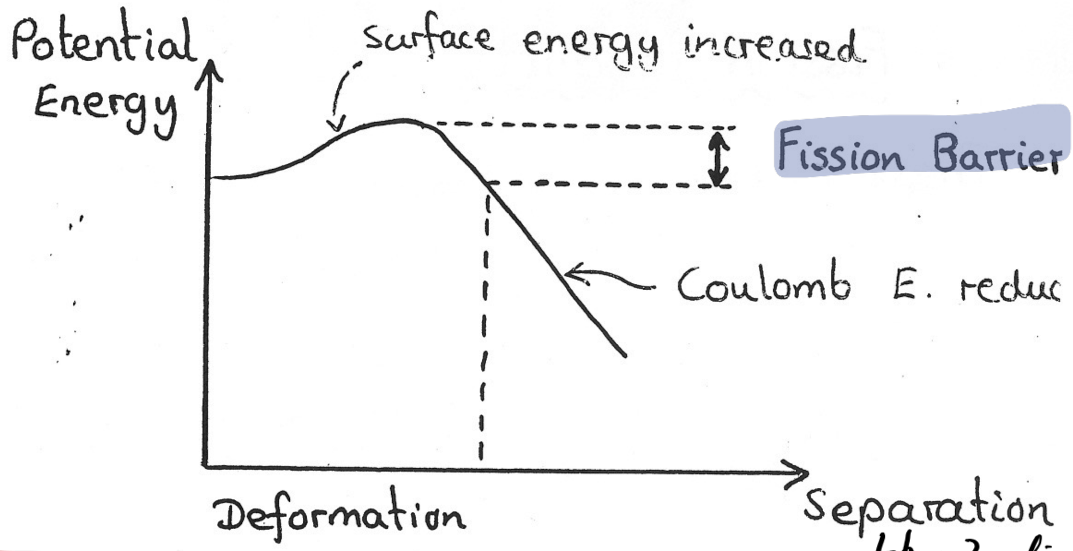



Fission yields:

an increase in surface energy

a decrease in Coulomb energy

But only gain Coulomb energy when the products are separated from each other …




btw 2 fiss products (that U are trying to separate)

The fission barrier is ~ 5-6 MeV for A ~ 240

log $t_{1/2}$

Fission is a tunnelling process and lifetimes are long

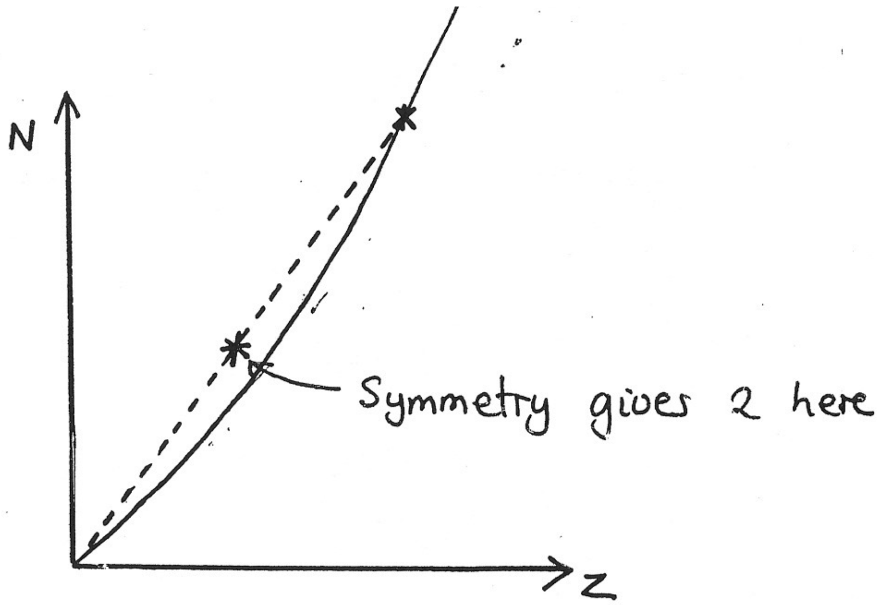
| | | |
|---|---|---|
| $^{238}_{92}U$ | $\alpha$-decay<br>~$10^9$ years | fission<br>~$10^{16}$ years |

For Z ≥ 144, the barrier tends to 0.

For A ≥ 260, fission becomes the dominant decay mode.

## Fission products - random jets

neutron rich highly excited




Symmetric fission releases maximum energy.
Fragments usually asymmetric.

Following fission:

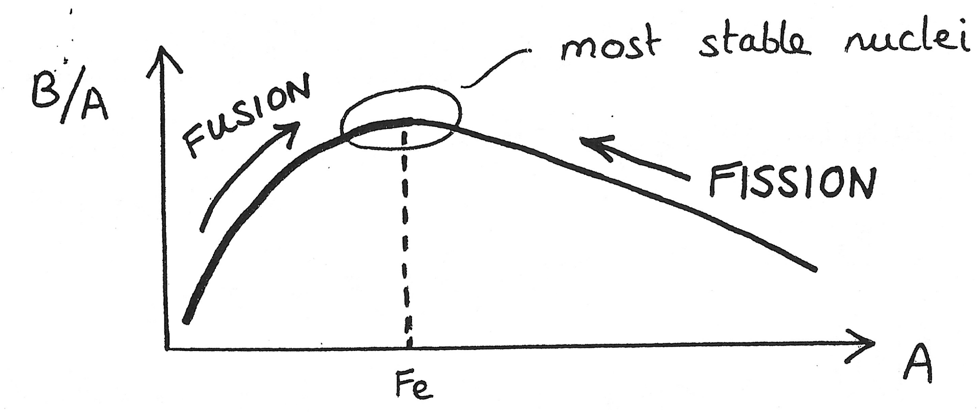
- “prompt” neutrons “boil off”
  *(fast: ~ $10^{-20}$ s)*
- $\gamma$-decay to ground states
  *(fast: ~ $10^{-16}$ s)*
- products are still n-rich: $\beta$-decay back to $\beta$-stability line
  *(slow: minutes/years)*
- more ‘delayed neutrons’ sometimes emitted after $\beta$-decay
  *(e.g. important for control of nuclear power plants)*

$$^{87}_{35}Br \xrightarrow{\beta^-} {}^{87}_{36}Kr$$
$$\xrightarrow{\beta^-} {}^{87}_{36}Kr^* \rightarrow {}^{86}_{36}Kr + n$$

# 5.6. NUCLEAR FUSION

BIG ENERGY

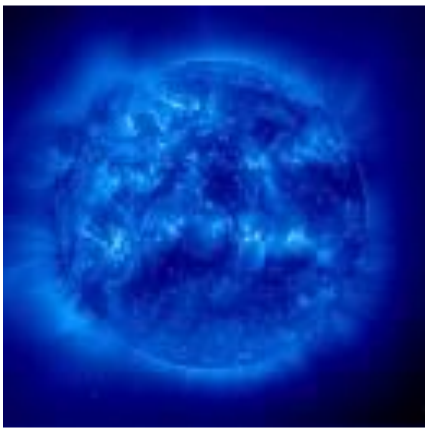
Let us consider the binding energy/nucleon curve:




Two light nuclei fusing can increase the total B.E.
∴ releases energy

This is important in:

stars (like the Sun)
fusion reactors



The solar energy flux at the surface of the Earth is 1.4 $kW/m^2$
Total output = $4 \times 10^{26}$ W

Where does this energy come from?

inputs

2 protons → 1 goes through β decay

## Basic reactions

waste products

2 x (1) $p + p \rightarrow {}^2_1H + e^+ + \nu_e$

$\left({}^1_1H\right)$

Q = 0.4 MeV

(+1 MeV from $e^+ + e^- \rightarrow 2\gamma$)

weak interaction – slow

2 x (2) $p + {}^2_1H \rightarrow {}^3_2He + \gamma$

Q = 5.5 MeV

1x (3) ${}^3_2He + {}^3_2He \rightarrow {}^4_2He + 2p$

Q = 12.9 MeV

Overall: $\boxed{4p \rightarrow {}^4_2He}$

Q = 24.7 MeV
(+2 MeV)

This is the main hydrogen burning cycle in the Sun.
It is called the PPI chain
*(there is also a PP-II, PP-III etc.)*

But the protons/nuclei are charged: there is a Coulomb barrier to fusion. How much?

The barrier height is:

$$\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 \left(R_1 + R_2\right)} \approx \frac{1.44\, Z_1 Z_2}{\left(R_1 + R_2\right)}\,\mathrm{MeV} \geq 1\mathrm{MeV}$$

What energies do the particles have?

High temperature $\Longrightarrow$ nuclei are ionised

Helium

Gas $\Longrightarrow$ plasma (mix of nuclei and electrons)

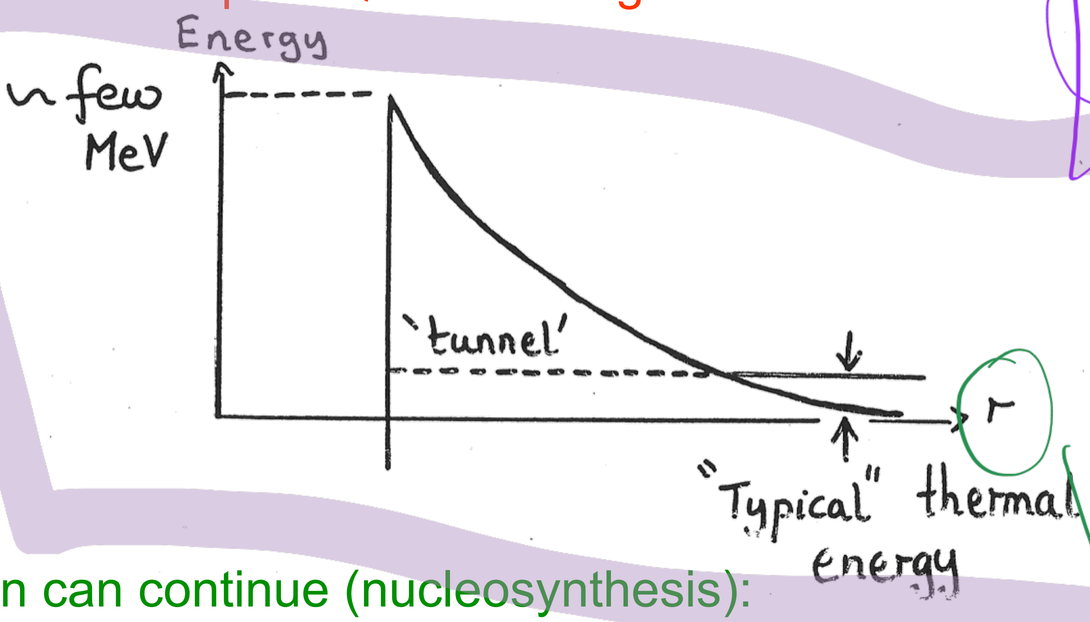
Distribution of energies = Maxwell-Boltzmann distribution

The average kinetic energy is: $\frac{3}{2}kT$

For the temperature of the Sun ($> 10^7$ K), this means that

typical KE << barrier height

➢ reactions require QM tunnelling




distance apart

Fusion can continue (nucleosynthesis):

$$^{4}_{2}He \rightarrow {}^{12}_{6}C \rightarrow {}^{16}_{8}O \rightarrow {}^{24}_{12}Mg \cdots \rightarrow {}^{56}_{26}Fe$$

But barriers are higher, requiring significantly higher T

*(that's why heaviest elements are only created in supernovae)*

## FUSION REACTORS

Can we generate fusion energy on Earth?

The PPI chain is not suitable (too slow)

One possibility: D-D reaction (D = Deuterium)

$$ {}^2_1H + {}^2_1H \rightarrow {}^3_2He + n \qquad 3.3\,\mathrm{MeV} $$

$$ \rightarrow {}^3_1H + p \qquad 4.0\,\mathrm{MeV} $$

Even better: D-T reaction (T = Tritium)

$$ {}^2_1H + {}^3_1H \rightarrow {}^4_2He + n \qquad 17.6\,\mathrm{MeV!} $$

Energy out of react'

This is much more suitable:

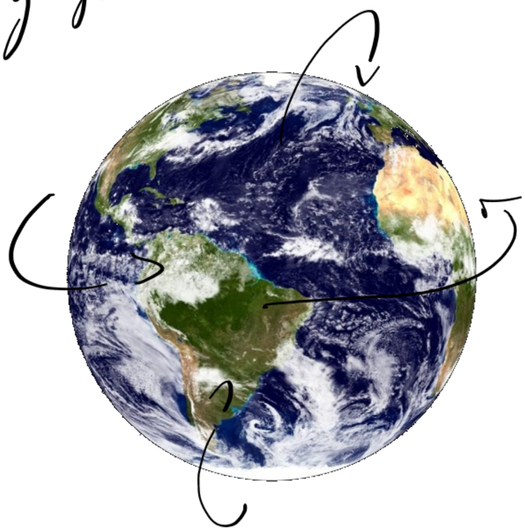
- high Q
- lower barrier

$\Big\}$ ↑ likely fusi'

Coulomb

Where do we get D from?

0.015% D in water



Where do we get T from?

$$ n + {}^6_3Li \rightarrow {}^3_1H + \alpha $$

↳ more fusion

## Requirements

"Breakeven" energy out $\cong$ energy in
"Ignition" reaction self-sustaining
↳ needed for energy producto

(1) Plasma temperature must be high ($>10^8$ K)

(2) Plasma density $n$ (ions/m$^3$) must be high

(3) Must hold plasma for sufficient time $\tau$ (confinement time)

For T > $10^8$ K, i.e. k T ~ 10 keV, we need for breakeven

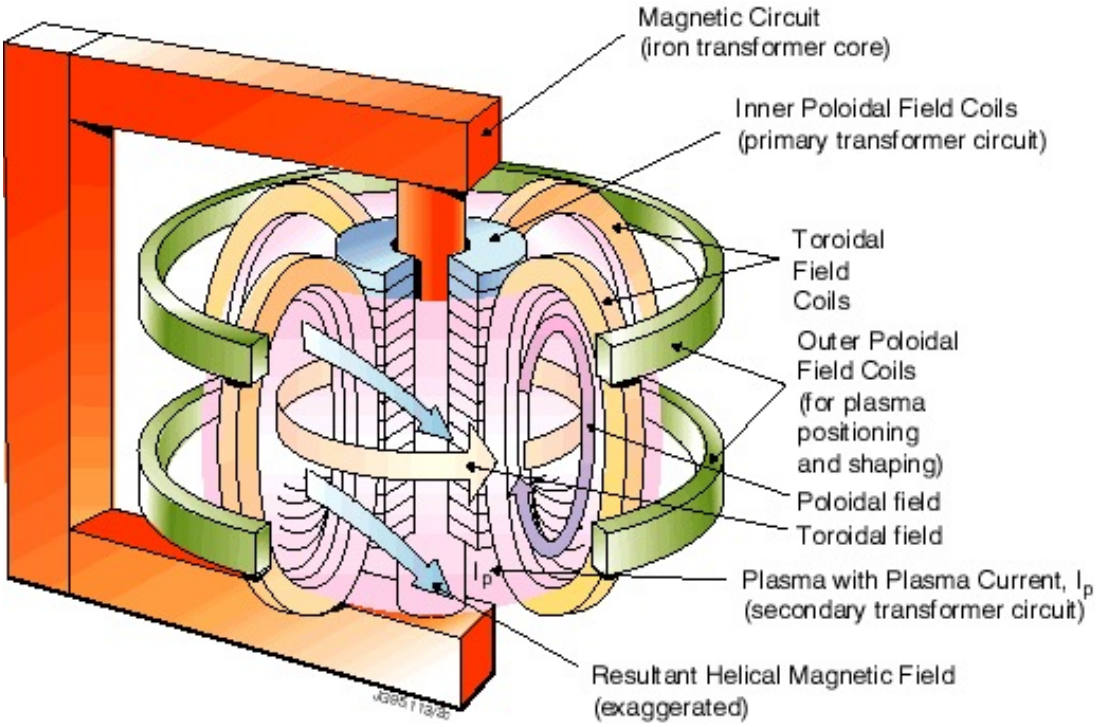
$$n\,\tau > 10^{20} \text{ s m}^{-3} \qquad \text{("Lawson criterion")}$$

This is higher still for ignition.

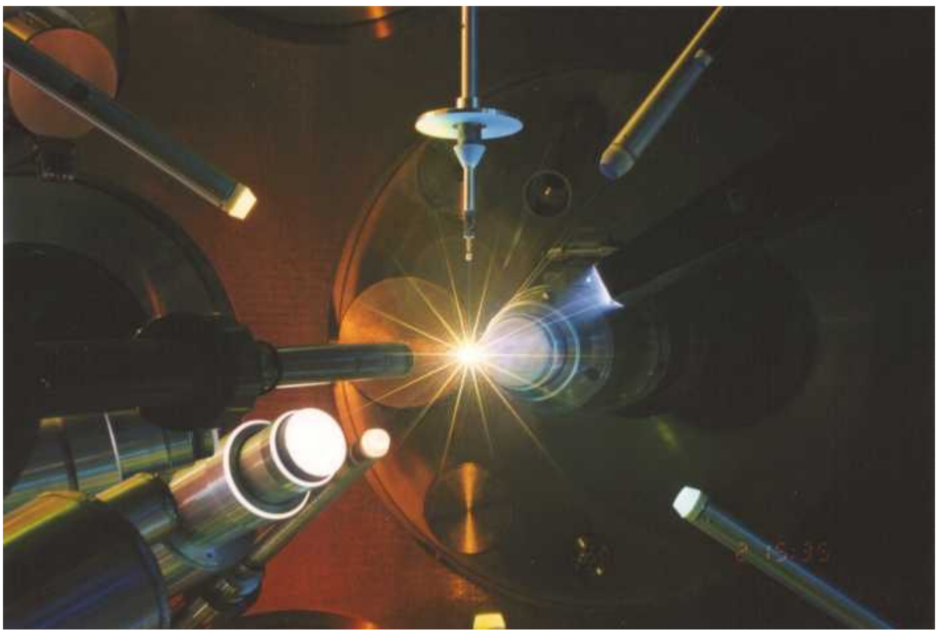
## Confinement methods

- *Magnetic Confinement Fusion (MCF)*
  Plasma confined in magnetic field (e.g. Tokamak).
  It is held in a torus (e.g. Joint European Torus, 2.8 m ∅)
  Maximum output: 16 MW
  power out/power in = 65%

- *Inertial Confinement Fusion (ICF)*
  DT mixture in plastic spheres (1 mm ∅)
  Vaporised by intense energy burst, applied symmetrically (with lasers or ion beams)
  Pellet implosion generates T ~ $10^8$ K and very high $n$
  Confinement time ~ $10^{-9}$ s




Typical Tokamak design



Typical ICF chamber

## REACTION RATE

Rate for 1 nucleus = F $\sigma$

*flux of incident nuclei*

*cross-section*

Reaction rate per unit volume = F $\sigma$ $\rho_1$

number density of target nuclei

For colliding beams of particles: F = $\rho_2$ *v*

number density of incident nuclei

velocity of incident nuclei

$\therefore$ reaction rate per unit volume = $\rho_1$ $\rho_2$ $\sigma$ v

We need to average over all possible relative velocities.

Hence reaction rate per unit volume = $\rho_1 \rho_2 \overline{\sigma v}$

$$\overline{\sigma v} = \int_0^{+\infty} P(v)\,\sigma(v)\,v\;dv$$

Where P(v)dv is the probability of having a relative velocity between *v* and *v* + *dv* (Maxwell-Boltzmann distribution)

$\rho_1 \rho_2 \overline{\sigma v}$ for different nuclei

$\frac{1}{2}\rho_1^2 \overline{\sigma v}$ for identical nuclei (division by 2 in order not to count pairs twice)

$$\overline{\sigma\, v} = \int_0^{+\infty} \underbrace{P(v)}\, \underbrace{\sigma(v)\, v}\, dv$$

$$\sim e^{-\frac{mv^2}{2kT}}$$

*exponentially decreases at large v*

*tunnelling: exponentially decreases at low v*