Homogeneous : all terms involve dependent var &/or its derivs → Linear

$\frac{\partial T^2}{\partial x^2} - \frac{1}{k}\frac{\partial T}{\partial t} = 0$ ✓ $R\frac{\partial Q}{\partial t} + \frac{Q}{c} = V$ ✗ $\frac{d^2y}{dx^2} + \frac{dy}{dt} - \sin x = 0$ ✗

$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$ | $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ ∴ L

$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + k|\psi|^2\psi$ | $u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 - u^2$ ∴ NL

can be

PDE linear if dep var & derivs occur only first pwr² & no products dependen on var & it's derivs

$L(a\psi + b\Phi) = aL(\psi) + bL(\Phi)$ — add to def

ψ & Φ are arbitrary f° & a & b are arbitrary non-zero consts!

- commutative Σ $L+M = M+L$
- associative $(L+M)+N = L+(M+N)$ & $(LM)N = L(MN)$
- distributive $L(c_1M + c_2N) = c_1LM + c_2LN$
- commutative × (const coeff) $LM = ML$

Div th $\int_S \vec{A}\,dS = \int_V \nabla\cdot\vec{A}\,dV$

PDES

- elliptic $B^2 - 4AC < 0$ steady-state
- hyperbolic $B^2 - 4AC > 0$ vibrat°
- parabolic $B^2 - 4AC = 0$ transit°

- Poisson $\nabla^2\Phi = -S(\vec{r})$
- Laplace $\nabla^2\Phi = 0$
- Diffus° $\nabla^2\Phi - \frac{1}{k}\frac{\partial\Phi}{\partial t} = -S(\vec{r},t)$
- wave $\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -S(\vec{r},t)$
- Helmholtz $\nabla^2\Phi + \lambda\Phi = 0$
- Schrod $-\frac{\hbar^2}{2m}\nabla^2\Phi + V(\vec{r})\Phi = i\hbar\frac{\partial\Phi}{\partial t} = E\Phi$

$$L(u) = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

$L(u) = G(x,y)$ $= 0$ ∴ homog; $\neq 0$ ∴ inhom

Solving ODEs

Homogeneous $0 = \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y$

2 solut°s → comb $y(x) = c_1y_1(x) + c_2y_2(x)$

(ex) • $P(x) = 0$, $Q(x) = \omega^2$ $\xrightarrow{GS}$ $\frac{d^2y}{dx^2} + \omega^2y = 0$

• $P(x) = \frac{1}{x}$, $Q(x) = 1 - \frac{n^2}{x^2}$ $\xrightarrow{GS}$ $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$

Inhomog $F(x) = \cdots$

y_p = sol ∴ General sol → $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

Pwr series ~needs more work $\frac{d^2y}{dx^2} + \omega^2y = 0$

GS → $y(x) = \sum_{s=0}^{\infty} a_s x^{s+q}$

recurrence relat°: $a_{t+2} = \frac{-\omega^2}{(q+t+2)(q+t+1)}a_t$

Boundary condit°s

(ex) wave eqt° 1D: $\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c}\frac{\partial^2\phi}{\partial t^2} = 0$

$L(\phi) = 0$, $L = \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

General solut°: $\phi(x,t) = \sum_k [A_k e^{i(kx-kct)} + B_k e^{i(kx+kct)}]$

BC: $\phi(x,0) = \sin 3x$ & $\frac{\partial\phi(x,0)}{\partial t} = 0$

⇒ $\phi(x,0) = \sum_k [-ikc A_k e^{ikx} + ikc B_k e^{ikx}]$

$= \sum_k -ikc(A_k - B_k)e^{ikx}$

$\phi(x,t) = \sum_k A_k e^{ikx}(e^{-ikct} + e^{ikct})$

$= \sum_k 2A_k e^{ikx}\cos(kct)$

BC ⇓ $\phi(x,0) = \sum_k 2A_k e^{ikx} = \sin 3x = \frac{e^{i3x}}{2i} - \frac{e^{-i3x}}{2i}$

$2A_3 = 1/2i$; $2A_{-3} = -1/2i$; other $A_n = 0$

$\phi(x,t) = \sum_k 2A_k e^{ikx}\cos(kct) = \sin(3x)\cos(3ct)$

Separat° of var

1D wave eqt°: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

$u(x,t) = X(x)T(t)$

$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

$\frac{\partial^2 u}{\partial x^2} = \frac{d^2X}{dx^2}T$, $\frac{\partial^2 u}{\partial t^2} = X\frac{d^2T}{dt^2}$ } $\frac{d^2X}{dx^2} - \frac{1}{c^2}X\frac{d^2T}{dt^2} = 0$

$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2} = -k^2$

Possible solut°s:

$\frac{d^2X}{dx^2} = -k^2X$ $\frac{d^2T}{dt^2} = -k^2c^2T$

General solut°s:

$$\begin{cases} X(x) = a_k e^{ikx} + b_k e^{-ikx} \\ T(t) = c_k e^{ikt} + d_k e^{-ikt} \end{cases}$$

Diffus° chem species

Change tot amount species: $\Delta M_1 = \delta t\int_V Q(\vec{r},t)\,dV$

flux vector: $\vec{J}$

Chge due diffus° $\Delta M_2 = \delta t\int -\vec{J}\,dS = D\delta t\int_S \nabla p\,dS$

tot change: $\Delta M = M_1 + M_2$

we know $\Delta M = \delta t\int_V \frac{\partial p}{\partial t}dV$ div th

∴ $\int_V \frac{\partial p}{\partial t}dV = \int_V Q(\vec{r},t)dV + D\int_V \nabla^2 p\,dV$

$\frac{\partial p}{\partial t} = Q(\vec{r},t) + D\nabla^2 p$

$\nabla^2 p - \frac{1}{D}\frac{\partial p}{\partial t} = -\frac{Q(\vec{r},t)}{D}$

Temp distribut°

$\nabla^2 T - \frac{\rho c}{k}\frac{\partial T}{\partial t} = -\frac{P(\vec{r})}{k}$

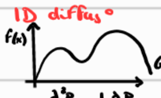
steady state: $\frac{\partial p}{\partial t} = 0$, $\frac{\partial T}{\partial t} = 0$

∴ $\nabla^2 p = -\frac{Q(\vec{r})}{D}$

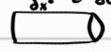
$\nabla^2 T = -P(\vec{r})/k$

Poisson!

1D diffus°



$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c}\frac{\partial p}{\partial t}$



$p(x,t) = X(x)T(t)$

$T\frac{d^2X}{}$

G.S: $p(x,t) = X(x)T(t)$

$= \sum_a (A_a\sin ax + B_a\cos ax)e^{-a^2ct}$

1D-diffus° eqt°: $\frac{\partial^2 p}{\partial x^2} = \frac{1}{c}\frac{\partial p}{\partial t}$

$\mathcal{F}[p(x,t)] = P(k,t) = \int_{-\infty}^{\infty} p(x,t)e^{-ikx}dx$

$\mathcal{F}\left[\frac{\partial^2 p(x,t)}{\partial x^2}\right] = -k^2P(k,t)$

$\mathcal{F}\left[\frac{1}{c}\frac{\partial p}{\partial t}\right] = \frac{1}{c}\frac{\partial P(k,t)}{\partial t}$

$-k^2P(k,t) = \frac{1}{c}\frac{\partial P(k,t)}{\partial t}$ . . .

* Divergent: $s_i \to \infty$ as $i \to \infty$
* Convergent: $S = \lim_{i\to\infty} s_i$
  - → Absolutely $\sum_n u_n$ & $\sum_n |u_n|$ both converge
  - → Conditionally div / but con
* Oscillatory: partial sums oscillate but never converge

Tsts

- $\lim_{n\to\infty} u_n \neq 0$ ∴ div $\lim_{n\to\infty}\left(\frac{(n+1)^a}{n^a}\right) = 1$
- ratio tst: $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)$ $\begin{cases} <1 \to \text{conv} \\ >1 \to \text{div} \\ =1 \to \text{undet} \end{cases}$
- bigger series converge ⇒ so does smaller; smaller series div ⇒ does bigger

Fourier Transform

$$\mathcal{F}[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$\mathcal{F}^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

Point → ordinary: $P(x)$ & $Q(x)$ finite at $x = x_0$

↓ singular: P&Q div as $x \to x_0$

- ↘ regular: $(x-x_0)P$ & $(x-x_0)^2Q$ remain finite as $x \to x_0$
- ↘ irregular ← expans° point may fail

N° sols: a indicial eq → q_1, q_2

- $q_1 = q_2$ → 1 sol
- $q_1 - q_2$ = int → larger q will give a sol, smaller may not
- $q_1 - q_2 \neq$ int ↓ 2 sols

Linear superposition:

You should know, and be able to justify, that the solution to a linear PDE given by $L(u) = G(x,y)$ can be written as $u = v + \sum_k c_k w_k$,

where $c_1, c_2, c_3, \ldots$ are constant coefficients, L is a differential operator, v is a particular solution of the inhomogeneous linear equation (i.e. when $G \neq 0$), and w is a particular solution of the associated homogeneous equation (i.e. when $G = 0$).

Solutions to common ODEs:

You should be able to write down (immediately!) the solutions to the following ODEs:

$\frac{dy}{dx} = \pm ky \Rightarrow y = A\exp(\pm kx)$

$\frac{d^2y}{dx^2} = k^2y \Rightarrow y = A\exp(kx) + B\exp(-kx)$ *OR* $y = C\cosh(kx) + D\sinh(kx)$

$\frac{d^2y}{dx^2} = -k^2y \Rightarrow y = A\exp(ikx) + B\exp(-ikx)$ *OR* $y = C\cos(kx) + D\sin(kx)$

where A, B, C, D and k are constants.

Displacement profile t=0

$u(x,0) = \sum_n A_n\sin(nx) = \frac{2}{\pi} \times \begin{cases} Px & \text{for } 0 \le x \le \pi/2 \\ P(\pi - x) & \text{for } \pi/2 \le x \le \pi \end{cases}$

$A_n = \frac{2}{\pi} \times \int_0^{\pi}\sin(nx)\underbrace{f(x)}\,dx$ profile of spring t=0

symmetry f(x): integral will be non-zero for odd values

$A_n = \frac{2}{\pi}\cdot 2\cdot\int_0^{\pi/2}\sin(nx)\left[\frac{2Px}{\pi}\right]dx$ — integral restricted to $0 \le x \le \pi/2$; n odd only

$I = \int_0^{\pi/2} x\sin(nx)\,dx \xrightarrow{\text{integ by parts}} I = \frac{1}{n^2}[\sin(nx)]_0^{\pi/2}$

Now $[\sin(nx)]_0^{\pi/2} = 1$ for $n = 1, 5, 9, \ldots$

$[\sin(nx)]_0^{\pi/2} = -1$ for $n = 3, 7, 11, 15, \ldots$

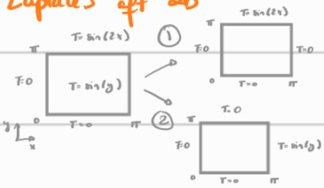
$[\sin(nx)]_0^{\pi/2} = 0$ for even values of n

⇓ general express°: $[\sin(nx)]_0^{\pi/2} = (-1)^m$, $m = \frac{n-1}{2}$ n always odd

coeffs of n:

∴ $I = \frac{(-1)^m}{n^2} \to A_n = \frac{4}{\pi}\cdot 2\frac{2P}{\pi}\cdot\frac{(-1)^m}{n^2} = \frac{8P}{\pi^2}\cdot\frac{(-1)^m}{(2m+1)^2}$

(ex) Laplace's eqt° ads



① $T(x,y) = \sum_n A_n\sin(nx)\sinh(ny)$

We know $T(x,\pi) = \sin(2x) = \sum A_n\sin(nx)\sinh(n\pi)$

∴ $A_n = 0$ for $n \neq 2$, when $n = 2$: $1 = A_2\sinh(2\pi)$, $A_2 = 1/\sinh(2\pi)$

$T(x,y) = \sum_n A_n\sin(nx)\sinh(ny) = \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}$

② by analogy x ↔ y & noting sin(2x) as top boundary conditi°

$T(x,y) = \frac{\sin(y)\sinh(x)}{\sinh(\pi)}$

Full solut°: $T(x,y) = \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)} + \frac{\sin(y)\sinh(x)}{\sinh(\pi)}$

9.1.3. Definition of a vector space

A set of vectors **a**, **b** and **c** is said to form a linear vector space satisfies the following:

i. **Commutative & associative addition:**

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$$
$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$$

ii. **Associative multiplication:**

$$\lambda(\mu)\mathbf{a} = (\lambda\mu)\mathbf{a}$$

iii. **Distributive addition and multiplication:**

$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}$$
$$(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$$

(where λ and μ are arbitrary scalars).

iv. There exists a null vector **0** such that $\mathbf{a}+\mathbf{0} = \mathbf{a}$ for all **a**

v. Multiplication by unity leaves any vector unchanged, i. $1\times\mathbf{a} = \mathbf{a}$.

vi. All vectors have a corresponding negative vector such $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$. It follows from this that $-\mathbf{a} = (-1)\times\mathbf{a}$.

9.9. Hermitian operators

An operator L is defined as being Hermitian, if the following condit holds over the interval $[a,b]$:

$$\int_a^b f^*(x)[Lg(x)]dx = \int_a^b [Lf(x)]^* g(x)dx \quad (9.12$$

We've already shown that the eigenvalues of Hermitian matrices are real, and that their eigenfunctions are orthogonal (for distinct eigenvalues).

In a similar way, the eigenvalues of Hermitian operators are real, an their eigenfunctions can be chosen to be orthogonal.

Because of this, Hermitian operators are frequently used in quantum mechanics. Their eigenvalues give the only possible measured valu an observable quantity (such as energy or angular momentum). The observable quantities must be real – this fits with the eigenvalues of Hermitian operators being real.

The infinite set of eigenfunctions of a Hermitian operator form a complete basis set over the interval in question, so that we can expa any function $y(x)$ as an infinite series of eigenfunctions over this interval:

$$y(x) = \sum_{n=0}^{\infty} c_n y_n(x) \quad (9.12$$

Again, this fits with the formulation of quantum mechanics, where t wavefunction for a particle in an infinite potential well (for example be written a linear combination of the infinite set of eigenfunctions, each weighted by a coefficient c_n.

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

Hermitian conjugate

$A^\dagger = (A^*)^T = (A^T)^*$

Properties

- eigenvals are real
- anti-H matrix eigvs are imaginary
- H & a-H = normal matrices

$a^\dagger = (a_1^*, a_2^*, \ldots a_N^*)$

$a^\dagger b = \sum_{i=1}^{N} a_i^* b_i$

$\langle a|b\rangle = a^*\cdot b = \underset{\text{orthonormal basis}}{a^\dagger b}$

* orthogonal $\langle a|b\rangle = 0$
* norm vector $||a|| = \langle a|a\rangle^{1/2}$
* inner product cplx vector $\langle a|b\rangle = a^*\cdot b = \sum_{i=1} a_i^* b_i$

$\langle\hat{e}_i|\hat{e}_j\rangle = \delta_{ij}$

Inner product

$\langle a|b\rangle = \langle b|a\rangle^*$

→ $\langle a+b|c\rangle = \langle a|c\rangle + \langle b|c\rangle$

c → vctr → $\langle a|b+c\rangle = \langle a|b\rangle + \langle a|c\rangle$

→ const → $\langle a|cb\rangle = c\langle a|b\rangle$

↳ $\langle ca|b\rangle = c^*\langle a|b\rangle$

Determinants

$|A^T| = |A|$

$|A^*| = |A|^*$

$|A^\dagger| = |A|^*$

Inverse matrices

$(A^{-1})^{-1} = A$

$(A^T)^{-1} = (A^{-1})^T$

$(A^\dagger)^{-1} = (A^{-1})^\dagger$

Product of matrics

$|AB| = |A||B| = |BA|$

$(AB)^* = A^*B^*$

$(AB)^T = B^TA^T$

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^\dagger = B^\dagger A^\dagger$

Linear operator $\vec{y} = A\vec{x}$

$(A+B)\underline{x} = A\underline{x} + B\underline{x}$

$(\lambda A)\underline{x} = \lambda(A\underline{x})$

$(AB)\underline{x} = A(B\underline{x})$

$AB\underline{x} \neq BA\underline{x}$ (generally)

two operators = if $A\underline{x} = B\underline{x}$ ∀

$Ix = x$

$AA^{-1} = A^{-1}A = I$

Normalised funct° $\hat{f} = \frac{f}{||f||}$

Spherical Harmonics

solut°: $\Psi = R\,\Theta\,\Phi$ $= R(r)\,Y(\theta,\phi)$

$Y_l^m = (-1)^m\left[\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}P_l^m\cos(\theta)\exp(im\phi)$ Legendre Polynomials

Orthogonal f&g if $\langle f|g\rangle = \int_a^b f^*g\,dx = 0$

in addit°: $\langle f|f\rangle = \int_a^b f^2dx = 1$, $\langle g|g\rangle = \int_a^b g^2dx = 1$

(ex) sin & cos:

$\int_0^{2\pi}\sin(mx)\sin(nx)dx = \pi\delta_{mn}$

$\int_0^{2\pi}\sin(mx)\cos(nx)dx = \pi\delta_{mn}$

orthogonal Legendre Polynomials $\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$

form a set of orthogonal f° [-1, 1]

$\int_{-1}^{1} P_l^m P_{l'}^m dx = \frac{2(l+m)!}{(2l+1)(l-m)!}\delta_{ll'}$

$\int_{-1}^{1}\frac{P_l^m P_l^{m'}}{1-x^2}dx = \frac{(l+m)!}{m(l-m)!}\delta_{mm'}$

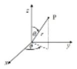
7. SPHERICAL HARMONICS

7.1. Laplace's Equation in Spherical Polars

Reminder of Laplace's Equation:

$$\nabla^2\Psi = 0$$

Let's now solve this in spherical polar co-ordinates. Reminder from the vector calculus section of PH20107.



Remember that in this co-ordinate system point P is defined by its radial distance from the origin (r), its polar angle (or latitude) (θ) measured from the z-axis, and its azimuth angle (or longitude) (φ).

The ranges for these variables are:

$0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$.

Writing Laplace's Equation in spherical polar co-ordinates gives:

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} = 0 \quad (7.1)$$

As usual, let's try separation of variables, with

$$\Psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \quad (7.2)$$
$$\Psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \quad (7.3)$$

Substituting this into the Laplace Equation, dividing through by $\Psi = RY$, and multiplying through by r^2 gives:

[illegible]

Since the left hand side of the equation is a function of r only, while the right hand side is a function of θ and φ only, they must each be equal to a separation constant, which we'll call k.

So:

[illegible]

Let's start with the R(r) equation, expanding it out:

[illegible]

Rewrite this using the substitution $r = \exp(t)$ and $R(r) = S(t)$:

[illegible]

This is now a constant coefficient 2nd order ODE, and so is easy to solve:

[illegible]

By inspection, we can therefore see that

$\lambda_1 + \lambda_2 = -1$ (7.13)

$\lambda_1\lambda_2 = -k$ (7.14)

So if we call one of the roots l, the other root will be $-(l+1)$.

From equations (7.13) and (7.14), we therefore also now know that

$k = l(l+1)$ (7.15)

Note that we don't know more than this for now, though. For example, we don't yet know whether or not l might have to be an integer.

[illegible]

As if by magic! This is exactly the form of the associated Legendre Equation in equation (6.98)!!

Since we already know that the condition for obtaining convergent solutions to the Legendre Equation for all finite values of x is that l is a non-negative integer. So this must also be the case for our solution to the Laplace Equation in spherical polar co-ordinates.

We can therefore write the solution to this equation as

$y(x) = P_l^m(x)$ (7.29)

where $P_l^m(x)$ is the associated Legendre function for the particular values of m and l.

However, we were originally trying to solve the Θ(θ) equation.

[illegible]

Note that, similarly to Bessel functions, there are two kinds of Legendre functions – Legendre functions of the first kind are conventionally labelled $P_l^m(x)$, and Legendre functions of the second kind are conventionally labelled $Q_l^m(x)$.

[illegible]

Power Series

Homogeneous

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx}\,Q(x)y = 0$

two solut°s: $y(x) = c_1y_1(x) + c_2y_2(x)$

Inhomogeneous

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$

y_p solut°: $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

Example 1:

If $P(x) = 0$ and $Q(x) = \omega^2$, then equation (6.1) becomes:

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \quad (6.3)$$

with two distinct solutions: $y_1 = A\sin(\omega x)$ and $y_2 = B\cos(\omega x)$.

Example 2:

If $P(x) = \frac{1}{x}$ and $Q(x) = \left(1 - \frac{n^2}{x^2}\right)$, then equation (6.1) becomes:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0. \quad (6.4)$$

This is Bessel's equation (compare with equation (4.66)), and so the two distinct solutions are: $y_1 = J_n(x)$ and $y_2 = Y_n(x)$.

Solut°s of ODEs:

$\frac{d^2y}{dx^2} + \omega^2y = 0$ ↓ sol $y(x) = \sum_{s=0}^{\infty} a_s x^{s+q}$

Matrices

any N×N matrix = Σ symm & anti-symm

$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

= Σ Hermitian & a-H

$= \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger)$

Det orthogonal matrix $|A| = \pm 1$

Unitary matrix $\underbrace{\text{non-singular}}_{|A| \neq 0}$ $AA^\dagger = I$ & $A^\dagger = A^{-1}$

eigenvals unitary matrix have unit modulus $\lambda^*\lambda = |\lambda^2| = 1$

Normal matrix: $AA^\dagger = A^\dagger A$

$A^{-1}(A^{-1})^\dagger = A^{-1}(A^\dagger)^{-1} = (A^\dagger A)^{-1}$

- If two matrices commute all their eigvecs con
- If two matrices have same eigenvecs, they commute

FOURIER SIN & COS

10. APPENDIX: FOURIER SINE AND COSINE SERIES FOR FINITE-RANGE FUNCTIONS

Remember from the Fourier part of unit PH20107 that if an infinite periodic function $f(t)$ has a period T and $\omega_0 = 2\pi/T$, then its Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t), \quad (10.1)$$

where

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)dt \qquad n = 0, 1, 2, 3\ldots.. \quad (10.2)$$

and

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)dt \qquad n = 1, 2, 3\ldots.. \quad (10.3)$$

Remember also that for an odd function, $a_n = 0$, so that:

$$f(t) = \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t) \quad (10.4)$$

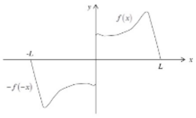
Similarly, for an even function, $b_n = 0$, so that:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) \quad (10.5)$$

When solving PDEs we will often need to represent a function that is defined over a finite interval as a Fourier series containing only either sine or cosine terms.

Suppose that we want to represent a function $f(x)$ defined within a finite interval $0 \le x \le L$ as a Fourier series that contains only sine terms.

As outlined in Appendix C of your lecture notes for the Fourier part of PH20107, we can do this by imagining that we artificially extend $f(x)$ to negative x as an odd function, as shown in the diagram below:



We can then imagine repeating this unit (i.e. the odd function which now extends over the interval $-L \le x \le L$) with a period $T = 2L$, to give an infinite periodic odd function.

By comparison with equations (10.4) and (10.3), we can therefore write the Fourier series for $f(x)$ in the finite interval $0 \le x \le L$ as:

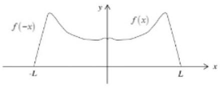
$$f(x) = \sum_{n=1}^{\infty} b_n\sin(n\omega_0 x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right), \quad (10.6)$$

where

$$b_n = \frac{2}{T}\int_{-L}^{L} f(x)\sin(n\omega_0 x)dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (10.7)$$

Similarly, we might instead want to represent the same function $f(x)$, defined over the same finite interval as a Fourier series that contains only cosine terms.

In this case, we now imagine artificially extending $f(x)$ to negative x as an even function, as shown in the diagram below:



Again, we then imagine repeating this unit with a period $T = 2L$, to give an infinite periodic even function.

By comparison with equations (10.5) and (10.2), we can therefore write the Fourier series for $f(x)$ in the finite interval $0 \le x \le L$ as:

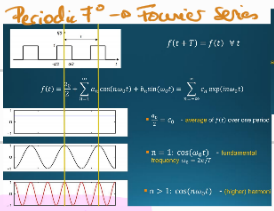
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right), \quad (10.8)$$

where

$$a_n = \frac{2}{T}\int_{-L}^{L} f(x)\cos(n\omega_0 x)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

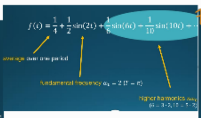
must converge for sol to be right!

Periodic f° → Fourier Series


$f(t+T) = f(t)\ \forall t$

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + b_n\sin(\omega_0 t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_0 t)$

- $\frac{a_0}{2} = c_0$ - average of $f(t)$ over one period
- n = 1: $\cos(\omega_0 t)$ - fundamental frequency $\omega_0 = 2\pi/T$
- n > 1: $\cos(n\omega_0 t)$ - (higher) harmonics

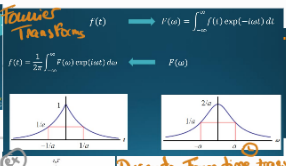

$f(t) = \frac{1}{4} + \frac{1}{2}\sin(2t) + \frac{1}{6}\sin(6t) - \frac{1}{10}\sin(10t) - \cdots$

odd $f(-t) = -f(t) \to a_n = 0$

even $f(-t) = f(t) \to b_n = 0$

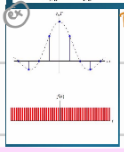
reflect° only around t = 0

Fourier Transforms


$f(t) \Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$

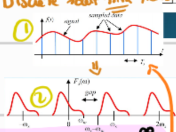
$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega t)\,d\omega \Leftarrow F(\omega)$

ex

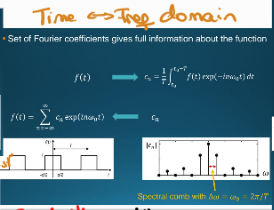


Discrete Four time transf

①



Time ↔ Freq domain

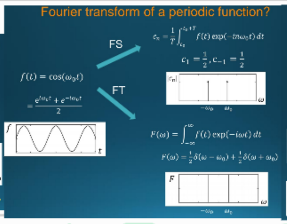

- Set of Fourier coefficients gives full information about the function

$f(t) \Rightarrow c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\exp(-in\omega_0 t)\,dt$

$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_0 t) \Leftarrow c_n$

Spectral comb with $\Delta\omega = \omega_0 = 2\pi/T$

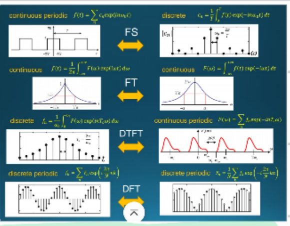
Fourier transform of a periodic function?


$f(t) = \cos(\omega_0 t) = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}$

FS: $c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\exp(-in\omega_0 t)\,dt$, $c_1 = \frac{1}{2}, c_{-1} = \frac{1}{2}$

FT: $F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)\,dt$

$F(\omega) = \frac{1}{2}\delta(\omega - \omega_0) + \frac{1}{2}\delta(\omega + \omega_0)$



Integrals 101

$\int (a+bt)\sin(\omega t)\,dt = \frac{b\sin(t\omega) - \omega(a+bt)\cos(t\omega)}{\omega^2} + c$

$\int (a+bt)\cos(\omega t)\,dt = \frac{\omega(a+bt)\sin(t\omega) + b\cos(t\omega)}{\omega^2} + c$

$\int (a+bt)e^{i\omega t}\,dt =$

$\int_{r=0}^{r=\infty} r^3 e^{-r/a_0}\,dr = \left[-a_0 \underset{=0}{r^3} e^{-r/a_0} - 3a_0^2 \underset{=0}{r^2} e^{-r/a_0} - 6a_0^3 \underset{=0}{r}\, e^{-r/a_0} - 6a_0^4 \underbrace{e^{-r/a_0}}_{=1}\right]_0^{\infty}$

$= 6a_0^4$

$\frac{d}{dr}$	$\int dr$
r^3	$+\ e^{-r/a_0}$
$3r^2$	$-\ a_0 e^{-r/a_0}$
$6r$	$+\ a_0^2 e^{-r/a_0}$
6	$-\ a_0^3 e^{-r/a_0}$
0	$+\ a_0^4 e^{-r/a_0}$

Coeffs ↳ $\begin{cases} a_n = 0 \\ b_n = \frac{1}{\pi n} \Rightarrow f(t) = \sum_{n=1}^{+\infty} \frac{1}{\pi n}\sin(2nt) \\ \omega_0 = 2 \end{cases}$

↳ c_n corresponds freq $\omega = n\omega_0$

$F(\omega) = e^{-|\omega|}$

$= \frac{1}{2\pi}\left\{\int_{-\infty}^{0} e^{\omega} e^{i\omega t}\,d\omega + \int_0^{\infty} e^{-\omega} e^{i\omega t}\,d\omega\right\}$

TRIG STUFF

$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

$\cos(x) = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\cos(2A) = 1 - \sin^2(A)$

$\text{sinc}(x) = \frac{e^{ix} - e^{-ix}}{2x}$

Sampling thm recorded data propert f°: ①

$f_s(t) = f(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

$F(\omega) = \mathcal{F}[f(t)]$: original data

FT of recorded data ⇓

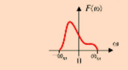
$F_s(\omega) = \mathcal{F}[f_s(t)] = \mathcal{F}\left[f(t) \times \sum_{n=-\infty}^{\infty}\delta(t - nT_s)\right]$

frequency Convolut° thm ↳ $= \frac{1}{2\pi} F(\omega) * \mathcal{F}\left[\sum_{n=-\infty}^{\infty}\delta(t - nT_s)\right]$

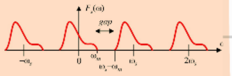
FT of Dirac comb ↳ $= \frac{\omega_s}{2\pi} F(\omega) * \delta(\omega - m\omega_s)$

convolut° with δ funct° ↳ $= \frac{1}{T_s}\sum_{m=-\infty}^{\infty} F(\omega - m\omega_s)$ ②

ie, an infinite train of copies of $F(\omega)$ separated by ω_s.
If the transform of the original signal looks like



where ω_m is the highest significant frequency component in the signal, then the transform of the recorded data looks like



So $F(\omega)$ (and hence the original signal $f(t)$ by IFT) can be completely recovered from the data by filtering out frequency components $|\omega| > \omega_m$ in the data, provided the copies of $F(\omega)$ do not overlap.

To avoid this overlap, the marked gap must be greater than zero:

$$\omega_s - \omega_m > \omega_m \Rightarrow \omega_s > 2\omega_m$$

In terms of "hertz" frequency, $f = \omega/2\pi$:

> The sampling rate (f_s) must be at least twice the highest frequency in the signal (f_m):
>
> $f_s \geq 2f_m$

Nyquist sampling theorem / criterion

Aliasing : prevent by ↑ sampling rate till $f_s > 2f_m$ / filter signal before sampling: remove high freq components till $2f_m < f_s$

ex · wagon wheels appearing to rotate backwards in old westerns
· spurious squeals in over-compressed audio files

Reminder: Frequency ↙ $f = \frac{1}{\Delta t}$ [Hz] ↘ angular freq $\omega = \frac{2\pi}{\Delta t} = 2\pi f$ [rad/s]

Bandwidth theorem: width $F(\omega), \Delta\omega \propto \frac{1}{\text{width of } f(t), \Delta t}$

Find FT ⋯ :

1. *Evaluate the integral*

Fine in principle but can be tedious and the answer you get may not be in the neatest form. Do this if you have no alternative or if the integral looks quick and easy.

2. *Use the table on p. 12 of the formula book*

Obviously the quickest method, but there aren't many functions in the table. Familiarise yourself with the ones that are, to save time in an exam!

The commonly-encountered "top hat" function is called *rect* in the table. Note that the formulae to the right of the curly bracket are two equivalent definitions, not one compound definition!

$$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} H(t+\frac{\tau}{2}) - H(t-\frac{\tau}{2}) & \text{[one definition]} \\ 1, \quad |t| < \tau/2 \\ 0, \quad |t| > \tau/2 \end{cases} \Big\}\ \text{[another definition]}$$

3. *Use the properties on p. 11 of the formula book*

Do this if the function you want to transform is suitably related to another whose transform you already know.

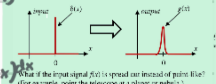
Don't forget about the trivial-looking linearity property, which is not in the formula book.

Int by parts

$\int uv\,dx = u\int v\,dx - \int u'\left(\int v\,dx\right)dx$

Convolut°

The impulse response or "point spread function" ($g(x)$, say) of an imperfect instrument is what the instrument records when the input is an ideal point source, represented by $\delta(x)$. For example, diffraction in a telescope may make it image a point like star $\delta(x)$ as a finite spot $g(x)$.



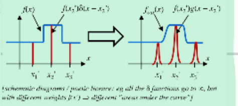
What if the input signal $f(x)$ is spread out instead of point like? (For example, point the telescope at a planet or nebula.)

Express $f(x)$ as a continuous distribution of point sources $\delta(x - x')$, each centred at $x = x'$ with "weight" $f(x')$:

$f(x) = \int_{-\infty}^{\infty} f(x')\delta(x - x')dx'$ [sift property of δ]

The instrument then distorts each of the distribution's point sources $\delta(x - x')$ into the point spread function $g(x - x')$, so the recorded output is

$f_{out}(x) = \int_{-\infty}^{\infty} f(x')g(x - x')dx'$



(schematic diagrams / poetic licence: eg all the δ functions go to ∞, but with different weights $f(x')$ ⇒ different "areas under the curve")

This operation on functions f and g is called their *convolution* and is written:

find it in the formula book

$$\boxed{(f * g)(x) = \int_{-\infty}^{\infty} f(x')g(x - x')dx'}$$

dummy variable x'

- The result of convolution is a *function* whose *name* is $f*g$, which here takes x as its argument. When performing the integral, x is treated as a constant.
- ie, in this context, * does not represent multiplication or complex conjugate
- It doesn't matter which way round f and g go:

$$(f * g)(x) = (g * f)(x)$$

4.2. Convolution Theorem

If $F(\omega) = \mathcal{F}[f(t)]$, $G(\omega) = \mathcal{F}[g(t)]$

it can be shown that

find it in the formula book $\boxed{\mathcal{F}[(f*g)(t)] = F(\omega)\times G(\omega)}$ *convolution theorem*

The transform of the convolution = the product of the transforms.

Similarly (in the other direction) it can be shown that

find it in the formula book $\boxed{\mathcal{F}[f(t)\times g(t)] = \frac{1}{2\pi}(F*G)(\omega)}$ *frequency convolution theorem*

The transform of the product = 1/(2π) × the convolution of the transforms.

The frequency convolution theorem is a kind of "product rule" for FTs, while the convolution theorem does the same for IFTs.

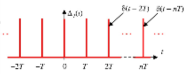
Dirac comb Shah Δ_T ← Special f°s → Heavy side (Unit step) H

↓

δ Dirac Delta (Pulse)

Definition: $\boxed{\Delta_T(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT)}$

- An infinite train of δ functions separated by T:



- It represents "something impulsive that happens repeatedly"
- $\Delta_T(t)$ is a periodic function. You can show (problem sheet) that its FT is proportional to another Dirac comb, with separation $\omega_0 = 2\pi/T$:

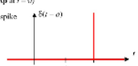
$\mathcal{F}[\Delta_T(t)] = \omega_0\Delta_{\omega_0}(\omega)$

$= \omega_0\sum_{m=-\infty}^{\infty}\delta(\omega - m\omega_0)$

Definition (2 parts): $\boxed{\begin{matrix}\delta(t) = 0 \quad t \neq 0 \\ \int_{-\infty}^{\infty}\delta(t)dt = 1\end{matrix}}$ [NB the value of δ(0) is not defined!]

- An infinitely-thin infinitely-high "spike" at $t = 0$, with unit area underneath:

- It represents "something that happens all at once" (eg an impulsive force $\mathbf{F} = \Delta\mathbf{p}\,\delta(t)$ causes change of momentum $\Delta\mathbf{p}$ at $t = 0$)
- $\delta(t - a)$ is a spike at $t = a$:

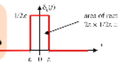


- Mathematicians are not comfortable about calling δ(t) a function - but physicists (like Dirac) can be less scrupulous
- δ(t) can be thought of as the derivative of the step function

$$\delta(t) = \frac{dH(t)}{dt}$$

(proof: integrate both sides from −∞ to t)

- There are numerous ways to define δ(t) as the limit of a sequence of well-behaved functions. eg consider the top-hat function $\delta_\varepsilon(t)$, with width 2ε and height 1/2ε.



As ε gets smaller the function becomes taller and narrower while its area stays equal to 1. Hence $\delta(t) = \lim_{\varepsilon\to 0}\delta_\varepsilon(t)$

Sift property of δ(t)

Consider the integral of a delta function multiplied by an arbitrary other function $f(t)$:

$\int_{-\infty}^{\infty} f(t)\delta(t - a)dt = \int_{-\infty}^{\infty} f(a)\delta(t - a)dt$

[Since $\delta(t - a)$ is zero except at $t = a$, $f(t)$ in the integral can be replaced by $f(a)$]

$= f(a)\int_{-\infty}^{\infty}\delta(t - a)dt$ [f(a) is a constant]

$= f(a)$ [integral of δ(t) is 1]

⇒ Sift property: $\boxed{\int_{-\infty}^{\infty} f(t)\delta(t - a)dt = f(a)}$

Integration with δ picks out the value of f at the spike of the δ.

The sift property is not in the formula book, but you are expected to know it.

Fourier Transform of δ(t)

$F(\omega) = \mathcal{F}[\delta(t - a)]$

$= \int_{-\infty}^{\infty}\delta(t - a)e^{-i\omega t}dt$

$= e^{-i\omega a}$ [sift property of δ]

⇒ $\boxed{\mathcal{F}[\delta(t - a)] = e^{-i\omega a}}$ NOT in the formula book: use time shift on the next result, or do the integral

In particular: $\mathcal{F}[\delta(t)] = 1$ find it in the formula book

The spectrum of a delta function is independent of ω: an impulse δ has equal amounts of every frequency.

Inverse Fourier Transform of δ(ω)

Repeating the exercise with the IFT integral gives

$\boxed{\mathcal{F}[e^{i\omega_0 t}] = 2\pi\delta(\omega - \omega_0)}$ find it in the formula book

[Intuitively this is hardly surprising - it says that the spectrum of a single-frequency wave contains only one frequency ...]

In particular: $\mathcal{F}[1] = 2\pi\delta(\omega)$

Now we can Fourier transform constants, something you may not have guessed from looking at the integral definition of FT.

The FT and IFT of δ are given in the formula book, so no need to memorise them. Nevertheless they are very easy to re-derive, given the sift property of δ.

Convolution with δ function

If $g(x) = \delta(x - a)$

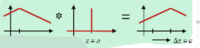
then $(f * g)(x) = \int_{-\infty}^{\infty} f(x')g(x - x')dx'$ [defn]

$= \int_{-\infty}^{\infty} f(x')\delta(x - a - x')dx'$ [spike where $x' = x - a$]

$= f(x - a)$ [sift property]

That is, $\boxed{f(x) * \delta(x - a) = f(x - a)}$

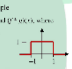
- convolution with $\delta(x - a)$ reproduces the original function but with the argument x replaced by $x - a$;
- equivalently, it replaces the δ symbol with f;
- equivalently, it shifts the function to the right by $\Delta x = a$.



Example

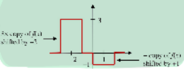
Q. Find $(f * g)(x)$, where

$f(x) =$



$g(x) = 3\delta(x - 2) - \delta(x - 1)$

A. $(f * g)(x) = 3f(x - 2) - f(x - 1)$ [convolution with δ]



6.4. Power Series Solution of ODEs

Consider again the equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0, \quad (6.7)$$

and let's try the solution

$$y(x) = x^q\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots\right) \quad (6.8)$$
$$= \sum_{s=0}^{\infty} a_s x^{s+q},$$

where q is chosen so that $a_0 \neq 0$.

We need to determine q and the coefficients a_s.

From equation (6.8), we have:

$$\frac{dy}{dx} = \sum_{s=0}^{\infty} a_s(s+q)x^{s+q-1} \quad (6.9)$$

$$\frac{d^2y}{dx^2} = \sum_{s=0}^{\infty} a_s(s+q)(s+q-1)x^{s+q-2} \quad (6.10)$$

For the example equation (6.5), this gives:

$$\sum_{s=0}^{\infty} a_s(s+q)(s+q-1)x^{s+q-2} + \omega^2\sum_{s=0}^{\infty} a_s x^{s+q} = 0 \quad (6.11)$$

Now equation (6.11) must be true for all x. This means that **the coefficients of each and every power of x have to be equal to zero.**

So now look in turn at each of the powers of x in equation (6.11):

6.4.1. Lowest power: indicial equation

The lowest power in equation (6.11) occurs when $s = 0$ in the first series in this equation. So the lowest power is x^{q-2}.

But we know that the coefficients of every power of x have to be equal to zero. So let's set the coefficient of this lowest power of x (i.e. x^{q-2}) to zero:

$$a_0 q(q-1) = 0 \quad (6.12)$$

But we already know that $a_0 \neq 0$.

So:

$$q(q-1) = 0 \quad (6.13)$$

Therefore $q = 0$ or $q = 1$.

Equation (6.13) is known as the ***indicial equation***, whose roots determine the possible values of q.

It is always formed by considering the coefficient of the lowest power of the independent variable (here x) in the original equation.

6.4.2. Second lowest power

The next lowest power in equation (6.11) occurs when $s = 1$ in the first series in this equation. So the next lowest power is x^{q-1}.

So the coefficient for the x^{q-1} term is:

$$a_1(q+1)q = 0 \quad (6.14)$$

But we already know that $q = 0$ or $q = 1$.

If $q = 0$, then a_1 is undetermined.

If $q = 1$, then $a_1 = 0$.

add the box on the transpose of a product - see notes .

An example on using DFT/FFT

Suppose, you recorded an input signal by taking $N = 100$ data points at regular time intervals $\Delta t = 1s$ (i.e. your sampling frequency $f_s = 1Hz$, which is one measurement per second, and the corresponding angular frequency is $\omega_s = 2\pi$ rad/s). If you use a standard FFT function to analyse its spectrum, you should expect:

- The resulting spectrum to be within the interval $\omega \in (-\pi, \pi)$ rad/s, i.e. $f \in (-0.5, 0.5)$ Hz. [cf. Nyquist criterion: maximal signal frequency is smaller or equal half the sampling frequency!]
- The spectral resolution to be $\Delta\omega = \frac{2\pi}{100} = \frac{\pi}{50}$ rad/s (or $\Delta f = 0.01\ Hz$)
- The first 51 data points of an FFT output correspond to spectral amplitudes at frequencies

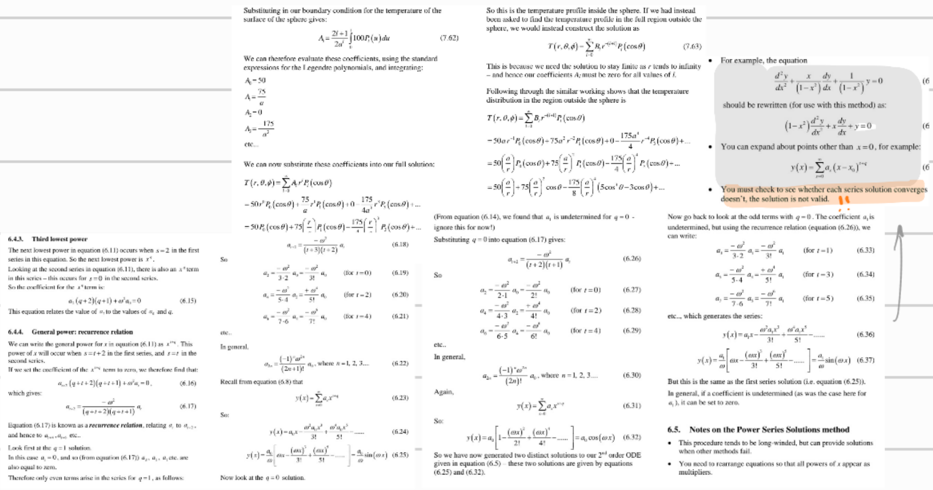
$$\omega_j = 0, \Delta\omega, 2\Delta\omega, \ldots, 49\Delta\omega, 50\Delta\omega = \pi$$

[Note that for the 51st data point j=50=N/2, and therefore starting from the 52nd data point we should shift all frequencies by $-\frac{2\pi}{\Delta t} = -100\Delta\omega = -2\pi$ rad/s]

- The remaining 49 data points correspond to spectral amplitudes at frequencies:

$$\omega_j = 51\Delta\omega - 2\pi = -49\Delta\omega, 52\Delta\omega - 2\pi = -48\Delta\omega, -47\Delta\omega, -46\Delta\omega, \ldots, -2\Delta\omega, -\Delta\omega$$

i.e. the negative half of the $(-\pi, \pi)$ frequency range



Substituting in our boundary condition for the temperature of the surface of the sphere gives:

$$A_l = \frac{2l+1}{2a^l}\int_0^1 100 P_l(u)\,du \qquad (7.62)$$

We can therefore evaluate these coefficients, using the standard expressions for the Legendre polynomials, and integrating:

$A_0 = 50$

$A_1 = \frac{75}{a}$

$A_2 = 0$

$A_3 = \frac{-175}{a^3}$

etc...

We can now substitute these coefficients into our full solution:

$$T(r,\theta,\phi) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$= 50 r^0 P_0(\cos\theta) + \frac{75}{a} r^1 P_1(\cos\theta) + 0 - \frac{175}{4a^3} r^3 P_3(\cos\theta) + \ldots$$

$$= 50 P_0(\cos\theta) + 75\left(\frac{r}{a}\right) P_1(\cos\theta) - \frac{175}{4}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \ldots$$

So this is the temperature profile inside the sphere. If we had instead been asked to find the temperature profile in the full region outside the sphere, we would instead construct the solution as

$$T(r,\theta,\phi) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta) \qquad (7.63)$$

This is because we need the solution to stay finite as r tends to infinity – and hence our coefficients A_l must be zero for all values of l.

Following through the similar working shows that the temperature distribution in the region outside the sphere is

$$T(r,\theta,\phi) = \sum_{l=0}^{\infty} B_l r^{-(l+1)} P_l(\cos\theta)$$

$$= 50 a r^{-1} P_0(\cos\theta) + 75 a^2 r^{-2} P_1(\cos\theta) + 0 - \frac{175a^4}{4} r^{-4} P_3(\cos\theta) + \ldots$$

$$= 50\left(\frac{a}{r}\right) P_0(\cos\theta) + 75\left(\frac{a}{r}\right)^2 P_1(\cos\theta) - \frac{175}{4}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \ldots$$

$$= 50\left(\frac{a}{r}\right) + 75\left(\frac{a}{r}\right)^2 \cos\theta - \frac{175}{8}\left(\frac{a}{r}\right)^4 \left(5\cos^3\theta - 3\cos\theta\right) + \ldots$$

- For example, the equation

$$\frac{d^2y}{dx^2} + \frac{x}{(1-x^2)}\frac{dy}{dx} + \frac{1}{(1-x^2)}y = 0 \qquad (6$$

should be rewritten (for use with this method) as:

$$(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0 \qquad (6$$

- You can expand about points other than $x=0$, for example:

$$y(x) = \sum_{s=0}^{\infty} a_s (x-x_0)^{s+q} \qquad (6$$

- You must check to see whether each series solution converges doesn't, the solution is not valid. !!

6.4.3. Third lowest power

The next lowest power in equation (6.11) occurs when $s=2$ in the first series in this equation. So the next lowest power is x^q.

Looking at the second series in equation (6.11), there is also an x^q term in this series – this occurs for $s=0$ in the second series.

So the coefficient for the x^q term is:

$$a_2(q+2)(q+1) + \omega^2 a_0 = 0 \qquad (6.15)$$

This equation relates the value of a_2 to the values of a_0 and q.

6.4.4. General power: recurrence relation

We can write the general power for x in equation (6.11) as x^{t+q}. This power of x will occur when $s=t+2$ in the first series, and $s=t$ in the second series.

If we set the coefficient of the x^{t+q} term to zero, we therefore find that:

$$a_{t+2}(q+t+2)(q+t+1) + \omega^2 a_t = 0, \qquad (6.16)$$

which gives:

$$a_{t+2} = \frac{-\omega^2}{(q+t+2)(q+t+1)} a_t \qquad (6.17)$$

Equation (6.17) is known as a ***recurrence relation***, relating a_t to a_{t-2}, and hence to a_{t-4}, a_{t-6} etc..

Look first at the $q=1$ solution.

In this case $a_1 = 0$, and so (from equation (6.17)) a_3, a_5, a_7 etc. are also equal to zero.

Therefore only even terms arise in the series for $q=1$, as follows:

$$a_{t+2} = \frac{-\omega^2}{(t+3)(t+2)} a_t \qquad (6.18)$$

So

$$a_2 = \frac{-\omega^2}{3\cdot 2} a_0 = \frac{-\omega^2}{3!} a_0 \qquad (\text{for } t=0) \qquad (6.19)$$

$$a_4 = \frac{-\omega^2}{5\cdot 4} a_2 = \frac{+\omega^4}{5!} a_0 \qquad (\text{for } t=2) \qquad (6.20)$$

$$a_6 = \frac{-\omega^2}{7\cdot 6} a_4 = \frac{-\omega^6}{7!} a_0 \qquad (\text{for } t=4) \qquad (6.21)$$

etc..

In general,

$$a_{2n} = \frac{(-1)^n \omega^{2n}}{(2n+1)!} a_0, \text{ where } n = 1, 2, 3\ldots \qquad (6.22)$$

Recall from equation (6.8) that

$$y(x) = \sum_{t=0}^{\infty} a_t x^{t+q} \qquad (6.23)$$

So:

$$y(x) = a_0 x - \frac{\omega^2 a_0 x^3}{3!} + \frac{\omega^4 a_0 x^5}{5!} - \ldots\ldots \qquad (6.24)$$

$$y(x) = \frac{a_0}{\omega}\left[\omega x - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \ldots\ldots\right] = \frac{a_0}{\omega}\sin(\omega x) \qquad (6.25)$$

Now look at the $q=0$ solution.

(From equation (6.14), we found that a_1 is undetermined for $q=0$ - ignore this for now!)

Substituting $q=0$ into equation (6.17) gives:

$$a_{t+2} = \frac{-\omega^2}{(t+2)(t+1)} a_t \qquad (6.26)$$

So

$$a_2 = \frac{-\omega^2}{2\cdot 1} a_0 = \frac{-\omega^2}{2!} a_0 \qquad (\text{for } t=0) \qquad (6.27)$$

$$a_4 = \frac{-\omega^2}{4\cdot 3} a_2 = \frac{+\omega^4}{4!} a_0 \qquad (\text{for } t=2) \qquad (6.28)$$

$$a_6 = \frac{-\omega^2}{6\cdot 5} a_4 = \frac{-\omega^6}{6!} a_0 \qquad (\text{for } t=4) \qquad (6.29)$$

etc..

In general,

$$a_{2n} = \frac{(-1)^n \omega^{2n}}{(2n)!} a_0, \text{ where } n = 1, 2, 3\ldots \qquad (6.30)$$

Again,

$$y(x) = \sum_{t=0}^{\infty} a_t x^{t+q} \qquad (6.31)$$

So:

$$y(x) = a_0\left[1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \ldots\ldots\right] = a_0\cos(\omega x) \qquad (6.32)$$

So we have now generated two distinct solutions to our 2nd order ODE given in equation (6.5) – these two solutions are given by equations (6.25) and (6.32).

Now go back to look at the odd terms with $q=0$. The coefficient a_1 is undetermined, but using the recurrence relation (equation (6.26)), we can write:

$$a_3 = \frac{-\omega^2}{3\cdot 2} a_1 = \frac{-\omega^2}{3!} a_1 \qquad (\text{for } t=1) \qquad (6.33)$$

$$a_5 = \frac{-\omega^2}{5\cdot 4} a_3 = \frac{+\omega^4}{5!} a_1 \qquad (\text{for } t=3) \qquad (6.34)$$

$$a_7 = \frac{-\omega^2}{7\cdot 6} a_5 = \frac{-\omega^6}{7!} a_1 \qquad (\text{for } t=5) \qquad (6.35)$$

etc.., which generates the series:

$$y(x) = a_1 x - \frac{\omega^2 a_1 x^3}{3!} + \frac{\omega^4 a_1 x^5}{5!} - \ldots\ldots \qquad (6.36)$$

$$y(x) = \frac{a_1}{\omega}\left[\omega x - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \ldots\ldots\right] = \frac{a_1}{\omega}\sin(\omega x) \qquad (6.37)$$

But this is the same as the first series solution (i.e. equation (6.25)).

In general, if a coefficient is undetermined (as was the case here for a_1), it can be set to zero.

6.5. Notes on the Power Series Solutions method

- This procedure tends to be long-winded, but can provide solutions when other methods fail.
- You need to rearrange equations so that all powers of x appear as multipliers.

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