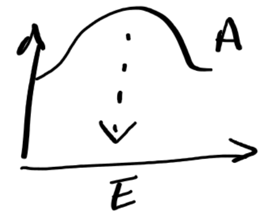
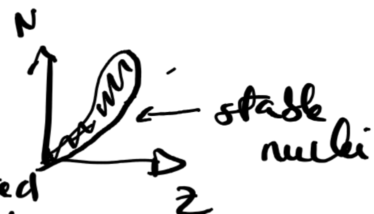
3. NUCLEAR MODELS

Aims of this section: to explain

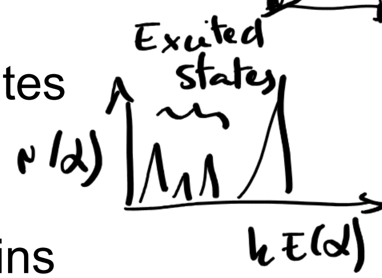
- the binding energy curve $\dfrac{B(Z,N)}{A}$



- $\dfrac{N}{Z}$ for heavy nuclei



- excited states



- nuclear spins

- "magic" nuclei

combinat°s protons & nucleons that give extra stability ← extra binding energy

Two main models:

1. Liquid drop model (classical)

2. Shell model (simplified QM)

3.1. Liquid Drop Model (classical)

The nuclear force works at short ranges.

∴ expect binding energy to be $\propto A$ ← mass $\propto$ n° nucleons there are

(as for a liquid drop)

But there are other contributions to the Binding Energy:

e.g. the surface (cf. liquid drop)

the charge (protons $\propto$ C interact°)

QM effects ...

quantum mechanical

need to be able to ... why does this eq^{n} ...

answer

box like it does?

SEMI-EMPIRICAL MASS FORMULA

$\pm, \wedge, \div, \dots ?$

$$B(Z,N) = a\,A - b\,A^{\frac{2}{3}} - s\,\frac{(N-Z)^2}{A} - d\,\frac{Z^2}{A^{\frac{1}{3}}} - \frac{\delta}{A^{\frac{1}{2}}}$$

A good fit is obtained with the following values:

not to be known

"Bulk" parameter a = 15.835 MeV

"Surface" parameter b = 18.33 MeV

"Symmetry" parameter s = 23.20 MeV

"Coulomb" parameter d = 0.714 MeV

"Pairing" parameter

$$\delta = \begin{cases} +11.12 \text{ MeV for odd N, odd Z} \\ -11.12 \text{ MeV for even N, even Z} \\ 0 \text{ MeV for even-odd N, Z} \end{cases}$$

Excellent agreement for A > 20

(i) Bulk term a A

a : basic Binding Energy per nucleon

$\cong$ bulk cohesive energy of liquid

(ii) Surface term – b $A^{2/3}$

Nucleons near the surface are less well bound (as they have fewer neighbours)

$\cong$ surface tension in liquids

Negative term: less Binding Energy for these nucleons

$\Rightarrow$ spherical nuclei

Contribution $\propto$ no. of nucleons near the surface

$\propto$ surface area

$\propto R_{nuc}^2$

But $R_{nuc} \propto A^{1/3}$

$\therefore$ surface term $\propto A^{2/3}$

instead of n st

continued by p

build diag

together

(iii) Symmetry term $-s\dfrac{(N-Z)^2}{A}$ ← ↑ larger as $N \neq Z$

Favours equal numbers of protons and neutrons, i.e. N = Z

desired in this symmetry logic

QM explanation: see Shell Model

Bulk term ∝ A, hence extra A in denominator

(iv) Coulomb term $-d\dfrac{Z^2}{A^{\frac{1}{3}}}$ ← reason why U bnd to more n rich nuclei
$\underbrace{n}_{\text{nucleon}}$

Nucleus: charge $Z\,e$ in small volume

Energetically unfavourable

Energy of uniformly charged sphere $= \dfrac{3}{5}\dfrac{Z^2e^2}{4\pi\varepsilon_0 R_{nuc}}$

$$\propto \frac{Z^2}{R_{nuc}}$$

factor of unity ⟶ R_{nuc}

$$\propto \frac{Z^2}{A^{\frac{1}{3}}}$$

This term forces $Z < N$ for large values of A.

(v) Pairing term δ

$-\frac{\delta}{A^{1/2}}$

Purely empirical.

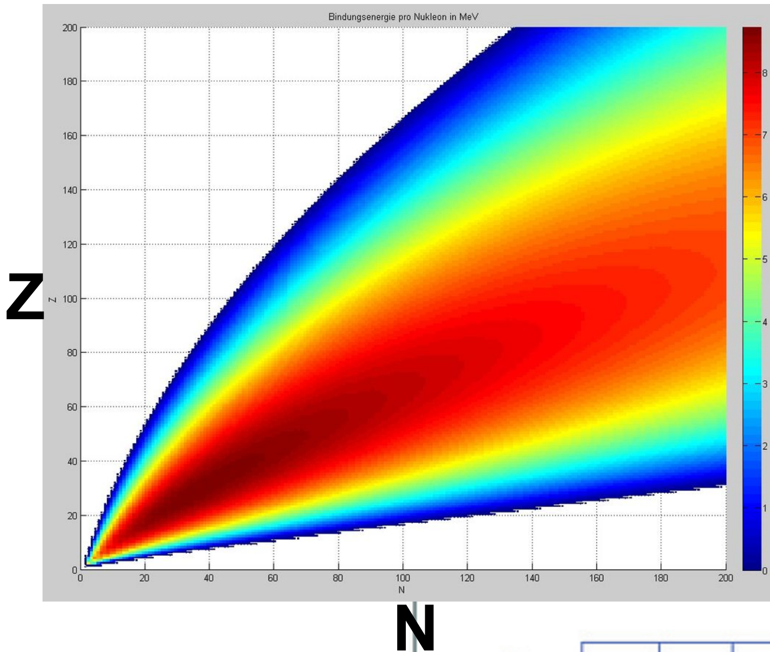
Favours even-even nuclei ← most stable (eventually converge to this)
over odd-odd and odd-even nuclei

least stable

Note: only 4 stable odd-odd nuclei

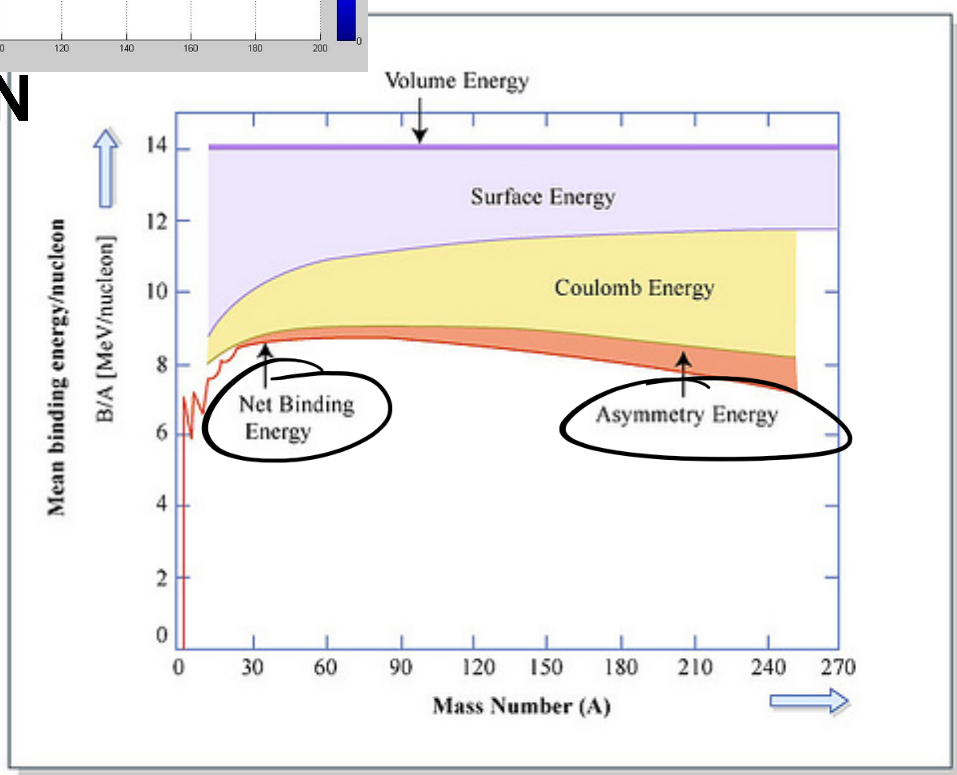
$$ {}^{2}_{1}H \quad {}^{6}_{3}Li \quad {}^{10}_{5}B \quad {}^{14}_{7}N $$

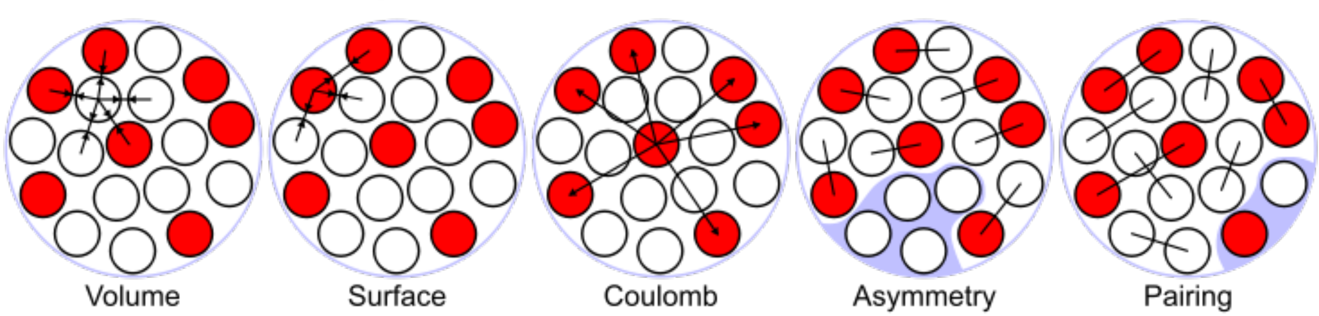
$$B(Z,N) = a\,A - b\,A^{\frac{2}{3}} - s\,\frac{(N-Z)^2}{A} - d\,\frac{Z^2}{A^{\frac{1}{3}}} - \frac{\delta}{A^{\frac{1}{2}}}$$



label axis when representing
↓
<u>Valley of Stability</u>
(Color coding: B/A)

<u>Binding energy per nucleon</u>





β-stability

β-decay : A remains constant

Let us look at the atomic masses of a set of isobars

$$m_a(Z,N)c^2 = \{N\,m_n + Z\,(m_p + m_e)\}c^2 - B(Z,N)$$

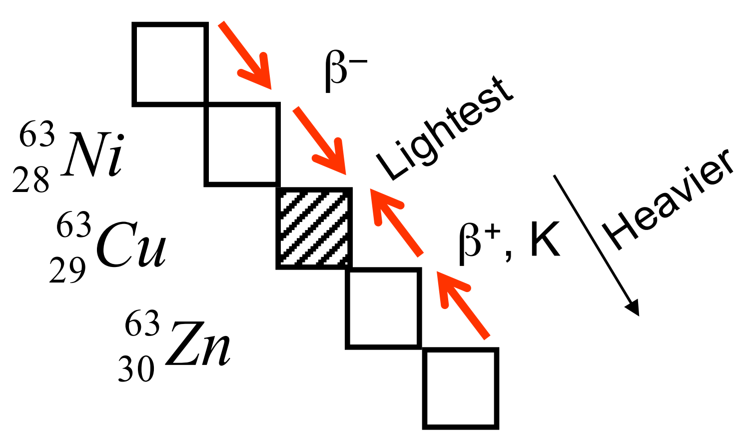
Writing N = A – Z, with A constant:

$$m_a\,c^2 = \alpha - \beta\,Z + \gamma\,Z^2$$ *(parabolic function of Z)*

where

$$\begin{cases} \alpha = A\,m_n\,c^2 - a\,A + b\,A^{2/3} + s\,A + \delta\,A^{-1/2} \\ \beta = 4\,s + (m_n - m_p - m_e)c^2 \\ \gamma = \dfrac{4\,s}{A} + \dfrac{d}{A^{\frac{1}{3}}} \end{cases}$$

β-decay tends toward the lightest atom in a set of isobars



So the atomic mass is minimum when:

$$\frac{dm_a}{dZ} = 0 \Leftrightarrow -\beta + 2\,\gamma\, Z = 0$$

$$\Rightarrow Z_{\min} = \frac{\beta}{2\,\gamma}$$

$$\boxed{Z_{\min} = \left(\frac{4\,s + (m_n - m_p - m_e)c^2}{4\,s + d\,A^{\frac{2}{3}}} \right) \times \frac{A}{2}}$$

$4\,s$ = 92.8 MeV

$(m_n - m_p - m_e)\,c^2 \approx$ 0.8 MeV

d = 0.714 MeV

Z_{min} can therefore be approximated as

$$\boxed{Z_{\min} \approx \frac{A}{1.98 + 0.015\,A^{\frac{2}{3}}}}$$

← relat° btw ideal n° of protons & mass n° = gives U most stable config → how many N & P should U need to build X?

This gives the Z of the most stable nucleus

s-term → favours $N = Z$
d-term → favours $N > Z$

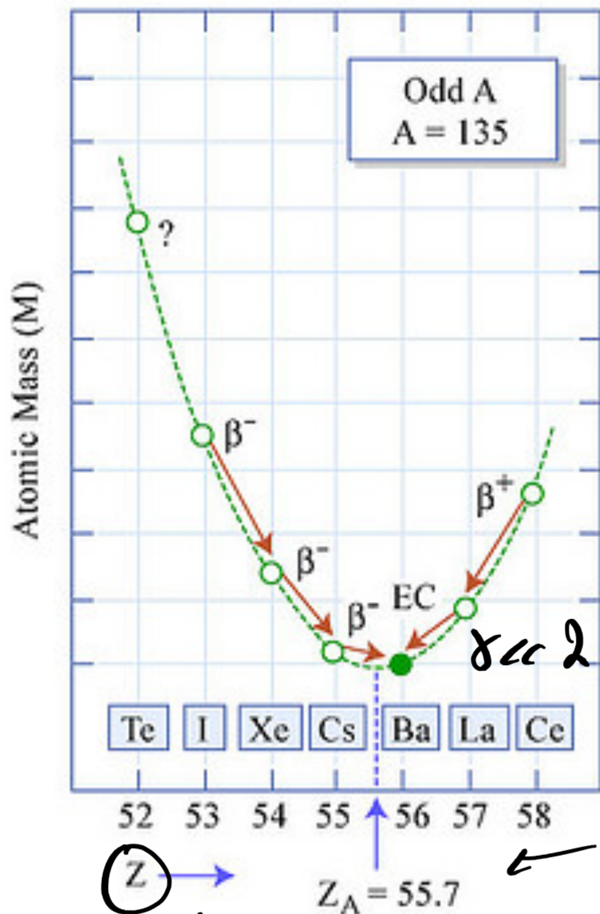
But A and Z are integers. Let us look at the two cases:

A odd a even

(a) A odd

$\therefore\ \delta = 0 \Longrightarrow m_a$ gives single parabola

<u>Example:</u> $A = 135 \Longrightarrow Z_{min} = 55.7$



$\delta \ll 2m_e$ ← difference

α

← emit positrons/capture fast

emit β per side

K capture always possible
β^+ needs $\Delta m_a > 2\ m_e$

Some form of β-decay down to nucleus closest to Z_{min}.

↳ there are conditions of positions

For odd A, one can predict:

- 1 stable isobar
- Which one it is
- Decay modes of unstable nuclei

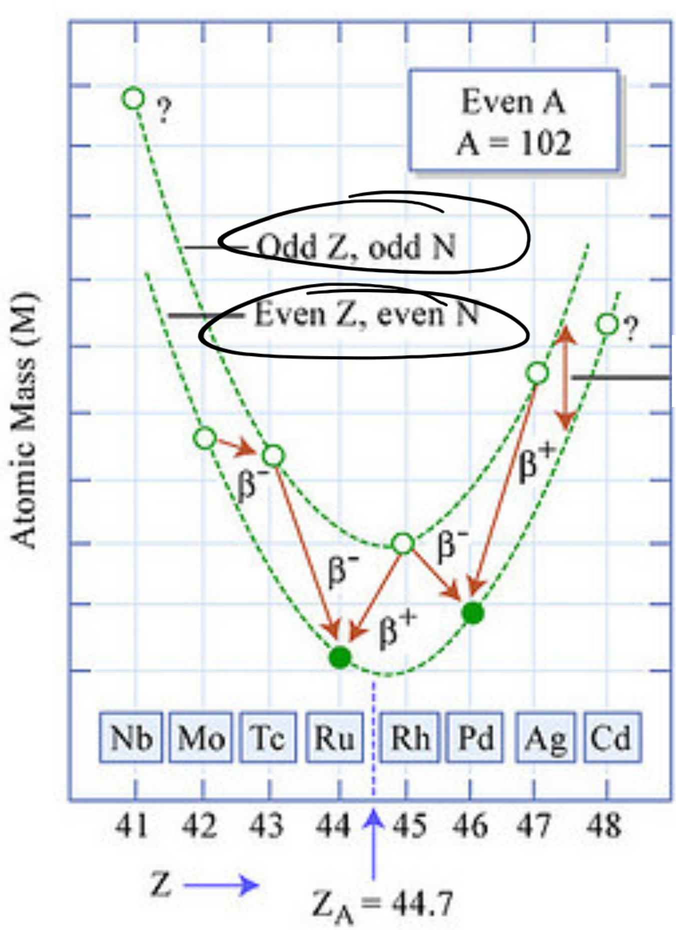
(b) A even

$\delta A^{-1/2}$ term becomes important.

Even-even: $\delta = -11.2$ MeV
Odd-odd: $\delta = +11.2$ MeV

$\Rightarrow$ TWO parabolae, separated by $\dfrac{22.4}{A^{\frac{1}{2}}}$ MeV

Example: $A = 102$, $Z_{min} = 44.7$



In this case, the liquid drop model predicts:

- 2 stable nuclei
- the decay modes
- 1 odd-odd nucleus which can decay both ways

(for A even, often 2 stable isobars ...)

The liquid drop model also predicts:

$$\begin{cases} \dfrac{N}{Z} & \text{curve} \\ \dfrac{B}{A} & \text{curve} \end{cases}$$

$$Z_{\min} \approx \left(\frac{4s}{4s + d\,A^{\frac{2}{3}}} \right) \times \frac{A}{2}$$

dropping the $m_n - m_p - m_e$ term

$$\left.\begin{aligned} \frac{N}{Z_{\min}} &= \frac{A - Z_{\min}}{Z_{\min}} = \frac{A}{Z_{\min}} - 1 = 1 + \frac{d\,A^{\frac{2}{3}}}{2s} \\ \frac{B}{A} &\cong a - \frac{b}{A^{\frac{1}{3}}} - \frac{s\,d\,A^{\frac{2}{3}}}{4s + d\,A^{\frac{2}{3}}} \end{aligned}\right\} \text{for stable nuclei}$$

But the liquid drop model does not cover:

- light nuclei
- magic numbers
- spin
- excited states ...

3.2. THE SHELL MODEL

Liquid drop model works poorly when N and/or Z close to:

2, 8, 20, 28, 50, 82, 126 $\leftarrow$ "magic numbers"

These are more stable than predicted by the liquid drop model

$^{4}_{2}He \quad ^{16}_{8}O \quad ^{208}_{82}Pb$ are doubly magic

Tin (Z = 50) even has 10 stable isobars.

The shell model is a simplified QM model.

<u>Main aim:</u> predict nuclear energy levels
(shells, cf. atomic levels)

To do that, we need to know the potential well due to the
strong/Coulomb force that each nucleon feels

We use the idea of "self-consistent fields".

Each neutron moves in the <u>average</u> potential well caused by the other neutrons and protons.

Each proton moves in the <u>average</u> potential well of the protons and neutrons (strong force) <u>AND</u> the average Coulomb potential of the other protons.

Guess potential well

↓

Calculate energy levels and wavefunctions

↓

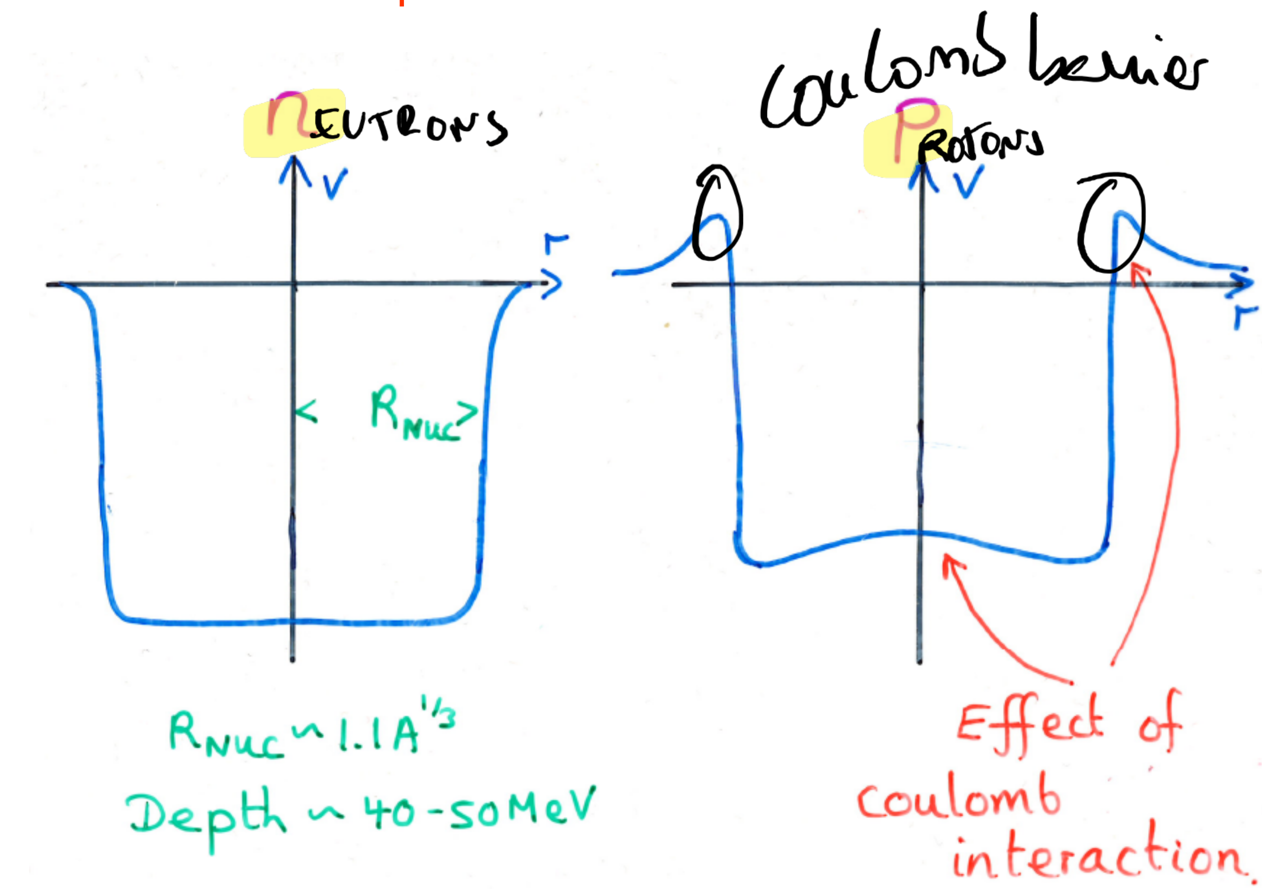
Fill up with protons and neutrons

↓

From nucleon density, calculate potential well

What do these potential wells look like?

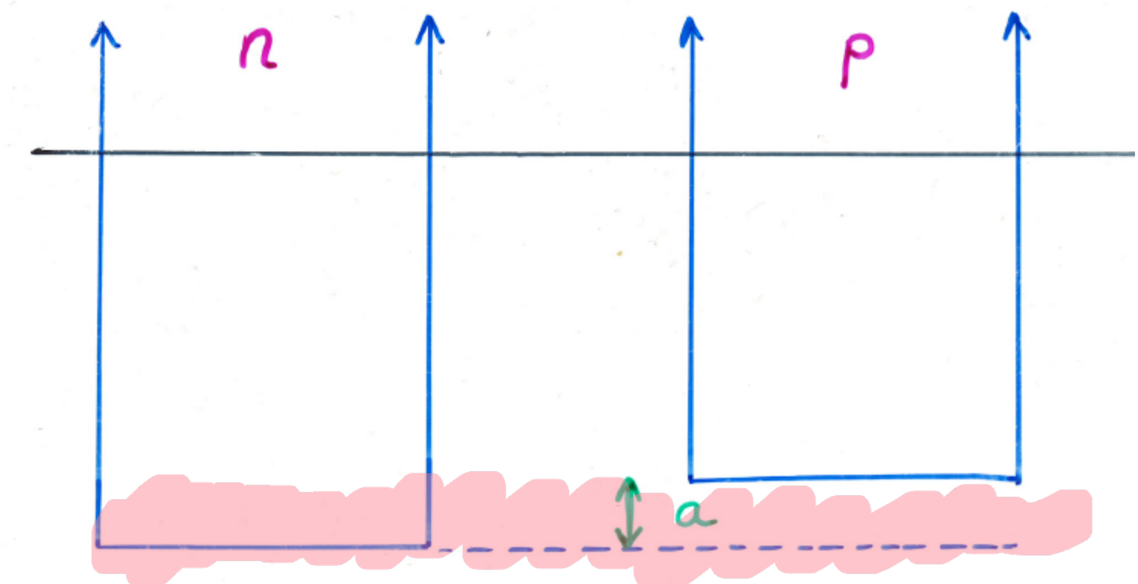
What does the potential look like?



- Shape of well is roughly constant for all nuclei
- Order of energy levels same for all nuclei
- Depth roughly constant
- Energy levels closer as R_{nuc} increases
- Energy levels filled to within about 8 MeV from top of well

(average energy to remove nucleon ~ 8 MeV – B.E. per nucleon)

This can be approximated with square wells:



a = average Coulomb energy – increases with *Z*

a ~ 10 MeV for medium *A*

<u>Process</u>

(1) Calculate energy levels

(2) Fill up with neutrons and protons
(obeying exclusion principle)

(1) Energy levels

nuclei are

Spherically symmetric, 3-D potential well.

We can use the radial part of the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right)+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}R+V(r)R=E\,R$$

potential well

energy

eigenvalue

often written as $\Psi(r)$

$R(r)$ wavefunction

l orbital angular momentum quantum number

l	0	1	2	3	4	5
	s	p	d	f	g	h

spectroscopic notation

For each l value, we can obtain a set of values of energy (eigenvalues):

$$E_n \qquad n = 1, 2, 3, \ldots$$

schröd eqtº abstract form:

$$\hat{O}\Psi = E\,\Psi$$

eigenvalue

n: principal quantum nº

p: orbital ang. momentum q nº

magnetic nº

as eigenvalue

m_ℓ:

∴ write levels as $E_{n,l}$

For a square well, the order is:

n	1	1	1	2	1	2	1
l	0	1	2	0	3	1	4
	1s	1p	1d	2s	1f	2p	1g

(2) Fill up levels

Degeneracy:
Expect $(2\,l + 1)$ states for each level

$$m_l = -\,l,\ -\,l + 1,\ \ldots,\ 0,\ \ldots,\ l - 1,\ l$$

E.g. for $l = 1$, $m_l = -\,1, 0, 1$

Also 2 particles for each state: spin up ↑ and spin down ↓

⟹ $2 \times (2\,l + 1)$ particles in each shell.

	1s	1p	1d	2s	1f	2p	1g
2 (2 *l* +*1)*	2	6	10	2	14	6	18
Running Total	2	8	18	20	34	40	58
Magic #?	√	√	X	√	X	X	X

THIS DOES NOT WORK …

What is wrong? What is missing?

In fact, the potential V(r) has an extra term:

$$f(r)L \cdot S$$

orbital and spin angular momentum operators

(first proposed in 1948 …)

⟹ spin-orbit coupling

l and s couple to form a total angular momentum quantum number j

$$j = \begin{cases} l + \dfrac{1}{2} \\ l - \dfrac{1}{2} \end{cases}$$

The shells are labelled by j as well as l and n

Each level with total angular momentum quantum number j contains $(2j + 1)$ nucleons with different m_j

Example: $l = 2$

We have $j = 5/2$ or $j = 3/2$

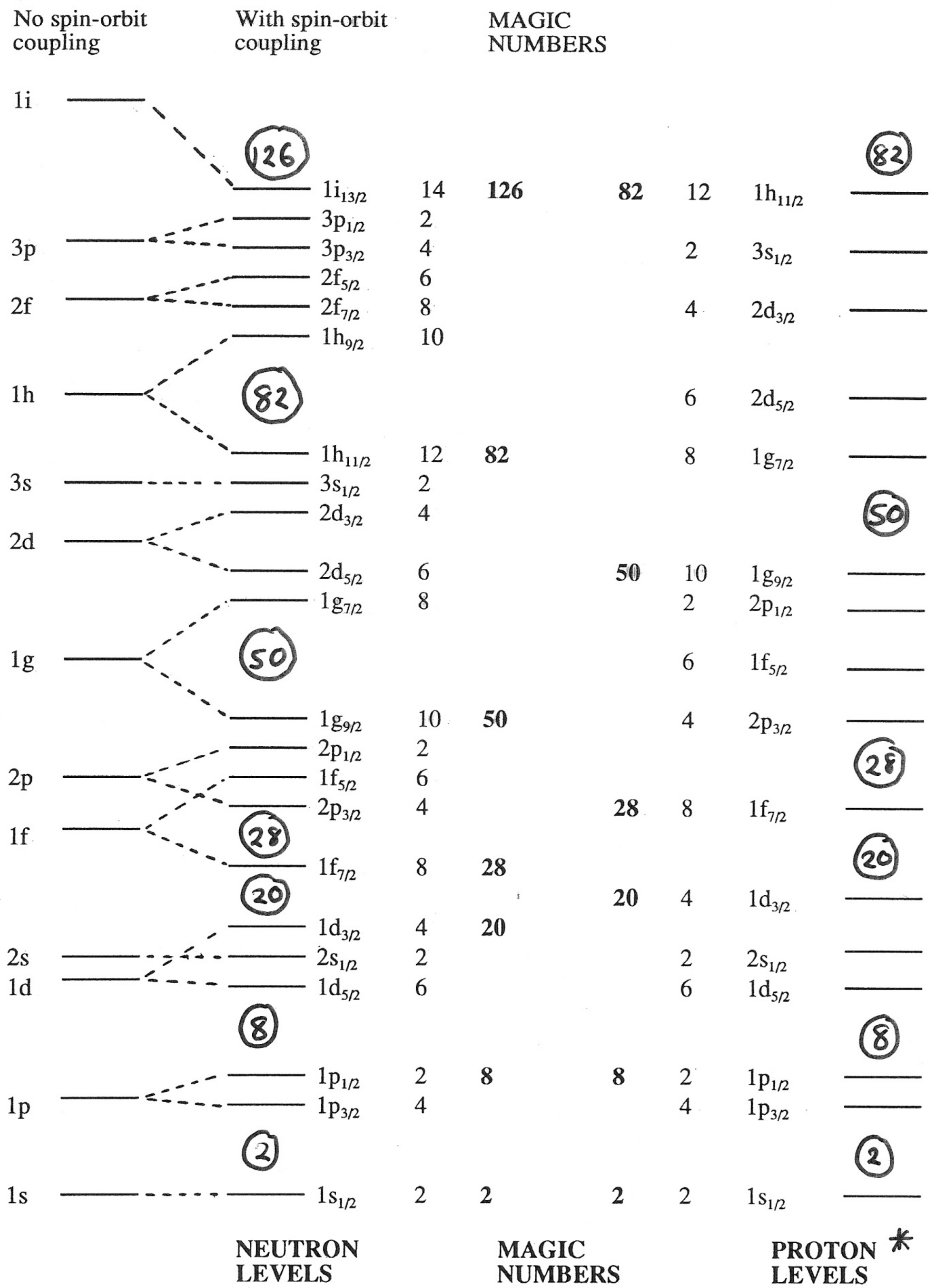
m_j	5/2	
	3/2	3/2
	1/2	1/2
	-1/2	-1/2
	-3/2	-3/2
	-5/2	
	6 states	4 states

Total = 10 (agrees with 2 (2 l + 1))

However: $\left\{\begin{array}{l} l + ½ \\ l - ½ \end{array}\right.$ levels are split in energy

Denotes state: nl_j

$$radial \cdot orbital_{total} \qquad \text{e.g.} \quad 2d_{5/2}$$



* Shows top occupied p-level at correct level relative to n-levels. Other p-levels will bunch up underneath.

Nuclear spin-orbit coupling is similar to the one in atomic physics, but …

(i) is not electromagnetic in origin;

(ii) produces large splits which increase with *l*;

(iii) $l + ½$ level is lower for nuclear coupling

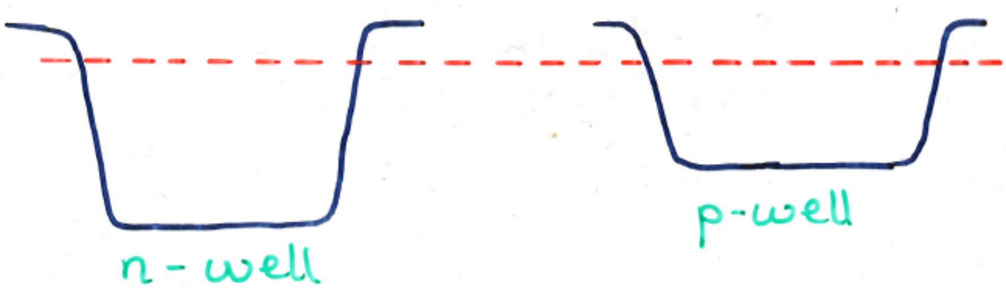
And the consequence is … shells overlap

So $2s_{½}$ comes before $1d_{3/2}$

Magic numbers correspond to the filling up of levels with a larger than average gap to the next level:
more stable …

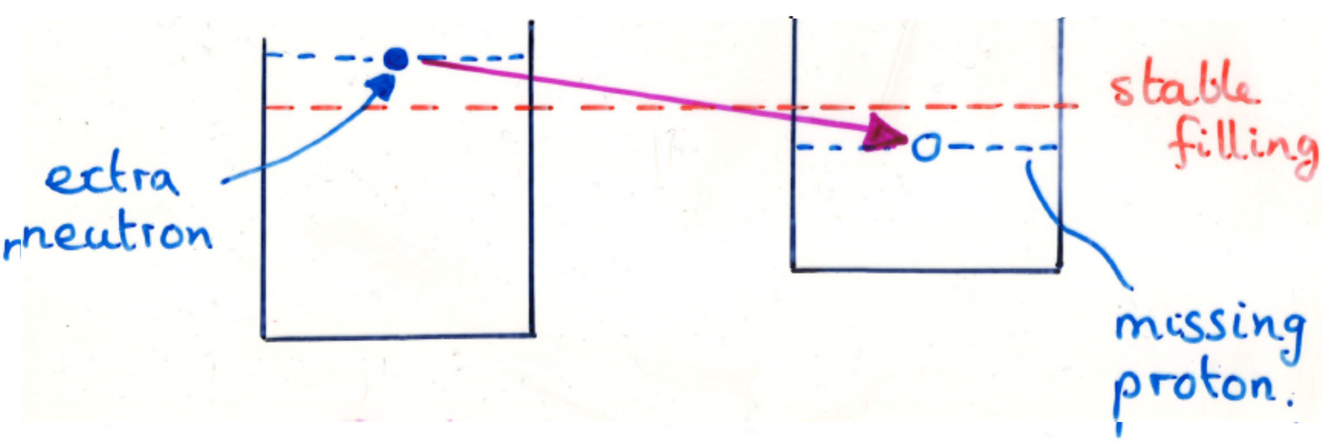
Level filling

Fill with protons and neutrons to same absolute level:



As Z increases, Coulomb shift increases …

To avoid different energy level diagram for each Z, show topmost occupied proton level at correct level relative to neutron levels



β decay: $n \rightarrow p^+ + e^- + \overline{\nu}_e$

Nuclear spin J: (also noted I)

The nucleus is composite – but it has an overall intrinsic angular momentum (i.e. spin)

⟹ Nuclear magnetic moment

The overall spin is made up of the individual *j*'s of each nucleon.

Rules

(a) Filled levels have a total angular momentum of 0

(b) Successive protons and neutrons pair off to give a total angular momentum of 0

- All even-even nuclei have $J = 0$
- Even-odd nuclei have $J = j$ of unpaired nucleon

Odd-odd nuclei: no general rule …

Examples for some light nuclei:

	N	Z	J
${}^{3}_{2}He$	1	2	$\frac{1}{2}$
${}^{4}_{2}He$	2	2	0
${}^{5}_{2}He$	3	2	$\frac{3}{2}$
${}^{16}_{8}O$	8	8	0
${}^{17}_{8}O$	9	8	$\frac{5}{2}$
${}^{17}_{8}O^{*}$			$\frac{1}{2}$

lone neutron is excited to next level $2s_{1/2}$