

Lecture 12

Electric fields in materials

Last time we saw that

In a dielectric, microscopic dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.

Macroscopically, the combined effect of these dipole moments produces a polarisation, defined as the induced dipole moment per unit volume: $\vec{P} = \frac{\sum \vec{p}}{V}$

In LIH dielectrics, the polarisation originates from the surface charge density $\sigma_p = P$.

In LIH dielectrics: $\vec{P} = \epsilon_0 \chi_e \vec{E}$ with χ_e being the electric susceptibility of the material.

The relative permeability of the material is given by: $\epsilon_r = 1 + \chi_e$

Overview

In this Lecture we will look at:

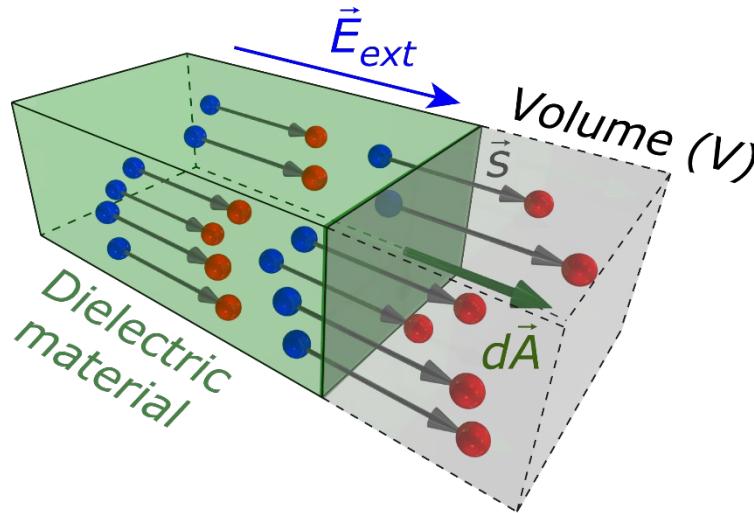
- Electric polarisation at the surface
- The sources of electric polarization
- Bound current density and the continuity equation
- Free charges and the electric flux density
- Free charges versus bound charges
- The energy stored in a dielectric

First, the electric polarisation at the surface!

Electric polarisation at the surface

We saw that: $\sigma_p = P$

Now, consider a unit volume V near a surface dA of a dielectric:



An external E -field splits the charges by a distance s , near an element of area dA , inside a volume $V=sdA$.

When E increases, $N+$ positive charges and $N-$ negative charges cross the small element of area dA . The net charge that crosses dA is then

$$dq_b = N^+ q_b - N^- (-q_b) = (N^+ N^-) q_b$$

This is the sum of all charged crossing dA times the charge value. But, this is also the number of molecules within the unit volume: $V = \vec{s} \cdot d\vec{A}$

Then remembering that n is the number of molecules per unit volume and $\vec{s} = \vec{r}$:

$$\begin{aligned} dq_b &= nVq_b = n(\vec{s} \cdot d\vec{A})q_b \\ &= nq_b \vec{s} \cdot d\vec{A} = n\vec{p} \cdot d\vec{A} = P d\vec{A} \end{aligned}$$

And also:

$$P = \frac{dq_b}{dA} / dA = \sigma_p$$

How is this important?

The sources of electric polarisation

We just saw that:

$$dq_b = \vec{P} \cdot d\vec{A}$$

Inside the dielectric, the net charge that flows out of the closed surface A which encloses the volume V is

$$Q_{out} = \int_A \vec{P} \cdot d\vec{A}$$

but then, it is clear that the charge that remains within V must be $-Q_{out}$

We can also use the **volume density of charge ρ_b** to find the charge inside the dielectric:

$$\int_V \rho_b \cdot dV = -Q_{out}$$

So, using the divergence theorem:

$$\int_V \rho_b \cdot dV = - \int_A \vec{P} \cdot d\vec{A}$$
$$= - \int_V (\nabla \cdot \vec{P}) dV$$

Therefore: $\vec{P}_b = -\nabla \vec{P}$

This means that there are sources of polarization.

The total bound polarisation charge within a region is obtained by integrating over a volume

$$Q_b = \int_V \rho_b dV$$
$$= - \int_V (\nabla \cdot \vec{P}) dV = - \int_A \vec{P} \cdot d\vec{A}$$

How does this bound charge change with time?

Bound current density and the continuity equation

We just saw that:

$$Q_b = \int_V P_b dV = - \int_V (\mathbf{D} \cdot \vec{P}) dV$$

Under the influence of a **time dependent** electric field (such as the one in a light wave), by definition, the **current flowing** through a surface A is **related** to the **current density** \vec{J} by :

$$\frac{dQ_p}{dt} = I = \int_A \vec{J} \cdot d\vec{A} = \int_V (\mathbf{D} \cdot \vec{J}) dV$$

Substituting:

$$-\frac{d}{dt} \int_V P_b dV = \int_V \left(\frac{dP_b}{dt} \right) dV = - \int_V \left(\frac{\partial (-\nabla \cdot \vec{P})}{\partial t} \right) dV = \dots \\ = \int_V (\mathbf{D} \cdot \vec{J}_b) dV$$

We discussed bound charges; how about free charges?

Therefore:

$$\vec{J}_b = \frac{\partial \vec{P}}{\partial t}$$

This means that the **motion of bound charges** results in a **polarisation** or **'bound' current density** \vec{J}_b .

We also deduce that:

$$\mathbf{D} \text{ — mixed}$$

which is called the **Continuity Equation**.

Free charges and the electric flux density

The total charge in a dielectric is:

$$\rho = \rho_b + \rho_f$$

The bound charges result from polarisation. The free charges do not.

From Maxwell's law:

$$\epsilon_0 \nabla \cdot \vec{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \vec{P} + \rho_f$$

Therefore:

It looks like the free charges are the source of some field!

Let's define: $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

This is the **electric flux density** (aka **the electric displacement**)

Note: The equation $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ is known as a **Constitutive Equation**.

Now we can write Maxwell's law as:

$$\vec{D} \cdot \nabla = \rho_f \quad \text{Compare to}$$

The total free charge is now:

We can also write:

Replacing with:

$$\text{We obtain: } \vec{D} = \epsilon_0 \epsilon_f \vec{E} = \epsilon \vec{E}$$

In free space $\epsilon_f = 1$ so $\vec{D} = \epsilon_0 \vec{E}$.

In practice?



Example question

[Old exam question] A material has an electrical susceptibility χ_e of 3.5. Calculate the magnitude of the electric dipole moment per unit volume, i.e. the polarization P , and the electric displacement, D , if the electric field E is 15 Vm^{-1} .

Example question

[Old exam question] A material has an electrical susceptibility χ_e of 3.5. Calculate the magnitude of the electric dipole moment per unit volume, i.e. the polarization P , and the electric displacement, D , if the electric field E is 15 Vm^{-1} .

Here, we have: $P = \epsilon_0 \chi_e E$, $D = \epsilon_0 \epsilon_r E = \epsilon_0 (1 + \chi_e) E$ and $D = \epsilon_0 E + P$

For the polarisation:

$$P = \epsilon_0 \chi_e E = (8.85 \times 10^{-12})(3.5)(15) \approx 4.7 \times 10^{-10} \text{ Cm}^{-2}$$

Then, for the electric flux density, we can use either

$$D = \epsilon_0 \epsilon_r E = \epsilon_0 (1 + \chi_e) E = (8.85 \times 10^{-12})(4.5)(15) \approx 6.0 \times 10^{-10} \text{ Cm}^{-2}$$

or,

$$D = \epsilon_0 E + P = (8.85 \times 10^{-12})(15) + 4.7 \times 10^{-10} \approx 6.0 \times 10^{-10} \text{ Cm}^{-2}$$

[One mark is awarded for correct equations and one mark each for the correct numerical answers.]

How are free and bound charges related?

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Free charges versus bound charges

In LIH dielectrics, we have a **free volume charge density** (ρ_f) and a **bound volume charge density** (ρ_b).

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \left(1 - \frac{1}{\epsilon_r}\right) \vec{D}$$

Apply div to both sides:

Which leads to:

$$\rho_b = - \left(1 - \frac{1}{\epsilon_r}\right) \rho_f$$

And we can calculate the total ρ :

We also have a **free surface charge density** (σ_f) and a **bound surface charge density** (σ_b).

Leading to:

$$\text{Since } \vec{D} = \epsilon_0 \epsilon_r \vec{E}$$

Therefore:

$$\text{Rearranging } \sigma_b = - \left(1 - \frac{1}{\epsilon_r}\right) \sigma_f$$

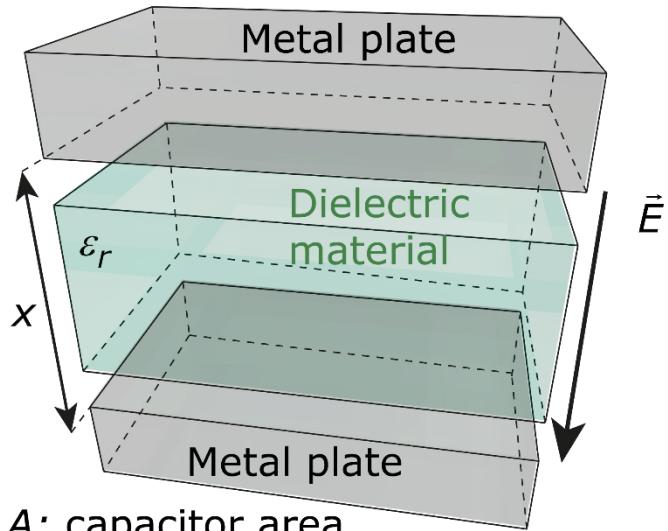
And we can calculate the total σ :

What is the energy stored in a dielectric?



The energy stored in a dielectric

We consider a parallel plate capacitor:



Work required to charge the capacitor:

$$w = \frac{1}{2} C_U^2$$

We also have: $C = \frac{A\epsilon_0\epsilon_r}{x}$ and $E = \frac{V}{x}$

Then, using: $\vec{D} = \epsilon_r \epsilon_0 \vec{E}$ and $V = E x$

$$\begin{aligned} \frac{1}{2} C V &= \frac{1}{2} \left(\frac{\epsilon_r \epsilon_0 A}{x} \right) (E x^2) \\ &= \frac{1}{2} (\epsilon_r \epsilon_0 E) E (A x) = \frac{1}{2} D E \times \text{vol} \end{aligned}$$

A: capacitor area

Therefore, the energy (w) stored per unit volume v is: $w = \frac{1}{2} \vec{D} \cdot \vec{E}$

And we have the **energy stored**: $w = \int_V w dV$

What about nonlinear materials?

Example question

[2005 Exam Question] A charge of uniform charge density ρ is distributed throughout a medium with $\epsilon_r = 1$ throughout a sphere of radius R. The electric displacement (\vec{D}) inside the sphere is given by $\vec{D} = \frac{\rho r}{3} \hat{r}$ for $r < R$.

Find the energy stored inside the charge distribution (i.e. for $r \leq R$). (3)

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Find the energy stored inside the charge distribution (i.e. for $r \leq R$). (3)

We have $\vec{D} = \frac{\rho r}{3} \hat{r}$ and we know that $\vec{D} = \epsilon \vec{E} = \epsilon_0 \epsilon_r \vec{E}$. [1 point]

So, we have $\vec{E} = \frac{\rho r}{3\epsilon_0 \epsilon_r} \hat{r}$ and since $\epsilon_r = 1$, we can write $\vec{E} = \frac{\rho r}{3\epsilon_0} \hat{r}$.

The energy density is $w = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1}{2} \left(\frac{\rho r}{3} \hat{r} \right) \cdot \left(\frac{\rho r}{3\epsilon_0} \hat{r} \right) = \frac{\rho^2 r^2}{18\epsilon_0}$. [1 point]

Integrate over the volume of the sphere to find the stored energy. [1 point]

$$W = \int_0^R w dV = \int_0^R \frac{\rho^2 r^2}{18\epsilon_0} (4\pi r^2 dr) = \frac{4\pi\rho^2}{18\epsilon_0} \int_0^R r^4 dr = \frac{4\pi\rho^2}{18\epsilon_0} \left[\frac{r^5}{5} \right]_0^R = \frac{2\pi\rho^2 R^5}{45\epsilon_0}$$

Example question

[2016 Exam Question] Describe what it means that, with respect to electric and magnetic fields, a medium is:

- (a) linear (1)
- (b) isotropic (1)
- (c) homogeneous (1)

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(a) For the electric fields, the polarization density \bar{P} (and the displacement field \bar{D}) is proportional to the external electric field $\bar{P} = \epsilon_0 \chi_e \bar{E}$ (and $\bar{D} = \epsilon_0 \epsilon_r \bar{E}$). For the magnetic fields, the magnetization \bar{M} (magnetic field \bar{B}) is proportional to the magnetic field intensity $\bar{M} = \mu_0 \chi_m \bar{H}$ (and $\bar{B} = \mu_0 \mu_r \bar{H}$).

(b) At a set point in the medium ϵ_r (or χ_e) and μ_r (or χ_m) do not depend on the direction of the applied external field \bar{E} and \bar{H} , respectively.

(c) The relative permittivity ϵ_r (electric susceptibility χ_e) and relative permeability μ_r (magnetic susceptibility χ_m) do not depend on the position within the medium.



Summary

In a dielectric, microscopically dipole moments $\vec{p} = q\vec{r}$ are induced by externally applied electric fields.

Macroscopically, the combined effect of these dipole moments produces a polarization, defined as the induced dipole moment per unit volume: $\vec{P} = \frac{\sum \vec{p}}{V}$.

In LIH dielectrics, the polarization originates from the surface charge density $\sigma_p = P_s$.

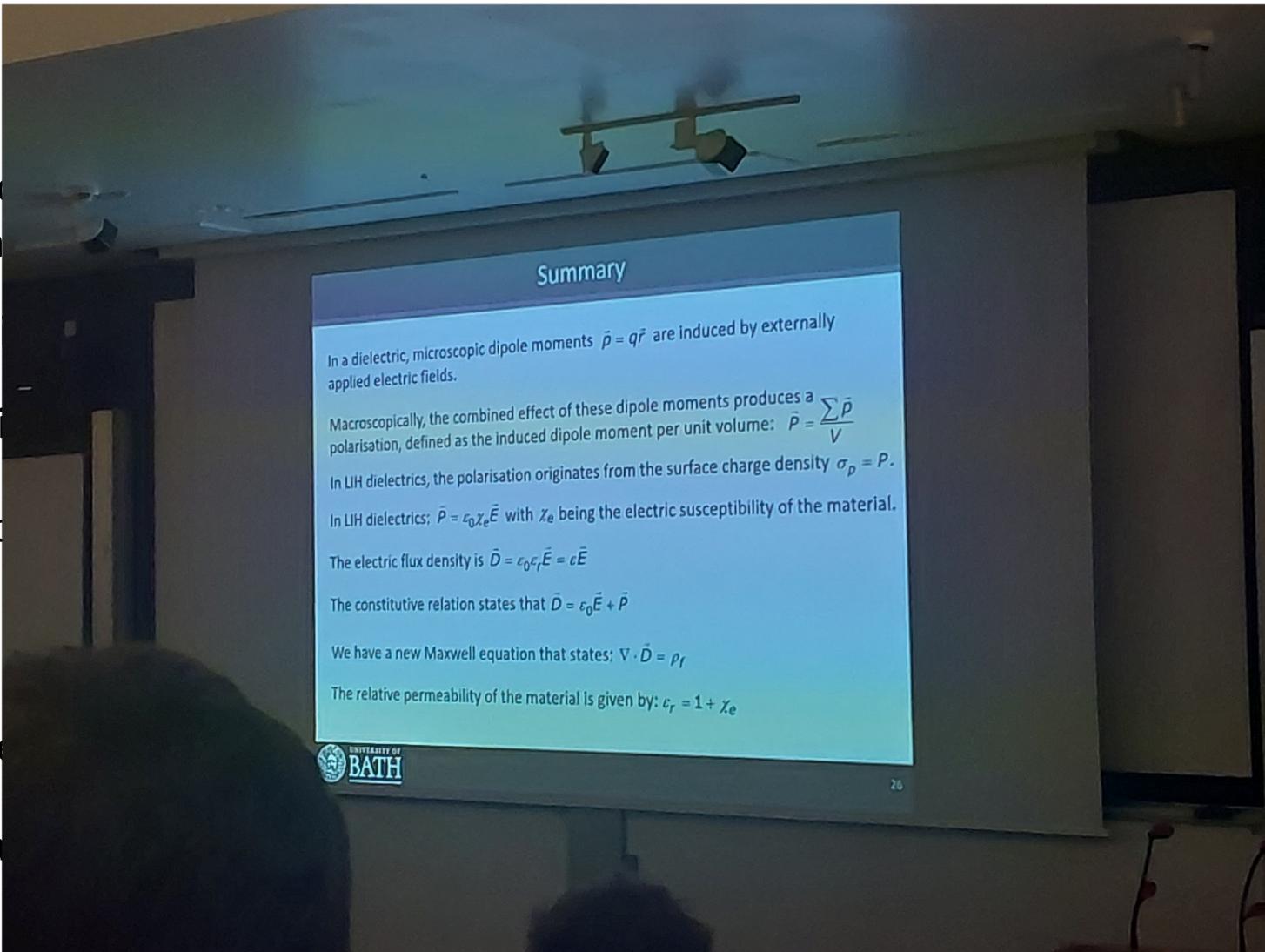
In LIH dielectrics: $\vec{P} = \epsilon_0 \chi_e \vec{E}$ with χ_e being the electric susceptibility of the material.

The electric flux density is $\vec{D} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E}$

The constitutive relation states that $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

We have a new Maxwell equation that states: $\nabla \cdot \vec{D} = \rho_f$

The relative permeability of the material is given by: $\epsilon_r = 1 + \chi_e$



Summary

The polarisation has sources, these sources are the bound density of charge:

The bound density of charge is related to the free density of charge by:

The surface bound density of charge is related to the surface free density of charge by:

The energy per unit volume stored in a dielectric is:

How about magnetic materials?