1 Example: Central potentials A4!

TISE $\underline{r} = (r, \theta, \phi)$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right] + V(r)u = Eu$$

1

We look for separable solut's of form:

2 $\quad u(r,\theta,\phi) = R(r) \times Y(\theta,\phi)$

2 => 1

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 Y\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, R\frac{\partial Y}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}R\frac{\partial^2 Y}{\partial\phi^2}\right] + VRY = ERY$$

$\downarrow \div RY = u$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{1}{Y}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{1}{Y}\frac{\partial^2 Y}{\partial\phi^2}\right] + V = E$$

$\downarrow \times r^2$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\sin\theta}\frac{1}{Y}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{1}{Y}\frac{\partial^2 Y}{\partial\phi^2}\right]+Vr^2=Er^2$$

$\downarrow$ $[\propto r]\ [\propto \theta\ \phi] = 0$

$$\left[-\frac{\hbar^2}{2\mu}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+Vr^2-Er^2\right]$$

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{\sin\theta}\frac{1}{Y}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{1}{Y}\frac{\partial^2 Y}{\partial\phi^2}\right]=0$$

Eqt° holds at every point in space $\Rightarrow$ each bracket cnst

Second bracket $= -\lambda$

$$\left[\frac{1}{\sin\theta}\frac{1}{Y}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{1}{Y}\frac{\partial^2 Y}{\partial\phi^2}\right]=-\lambda$$

$$\left[-\frac{\hbar^2}{2\mu}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+Vr^2-Er^2\right]+\frac{\hbar^2}{2\mu}\lambda=0$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+Vr^2-Er^2+\frac{\hbar^2}{2\mu}\lambda=0$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+Vr^2+\frac{\hbar^2}{2\mu}\lambda=Er^2$$

$\div r^2 \quad \times R$

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+VR+\frac{\hbar^2}{2\mu r^2}R\lambda=ER$$

(2) Spherical symmetric state require $4\pi\int_0^\infty dr\, r^2|u|^2=1$

$$u_{n00}(r)=A\frac{\sin\left(\frac{n\pi r}{a}\right)}{r}$$

$$\Downarrow$$

$$4\pi\int_0^\infty A^2\sin^2\left(\frac{n\pi r}{a}\right)dr=1$$

$$4\pi A^2\int_0^\infty \sin^2\left(\frac{n\pi r}{a}\right)dr=1$$

$$4\pi A^2\left(\frac{r}{2}-\frac{a}{4n\pi}\sin\left(\frac{2n\pi r}{a}\right)\right)=1$$

$$A=\frac{r}{8\pi}-\frac{a}{16n\pi^2}\sin\left(\frac{2n\pi r}{a}\right)$$

(3) a) $E_n=-\frac{1}{n^2}\frac{\mu e^4}{32\pi^2\epsilon_0^2\hbar^2}$

$= \cdots \mu$

if U replace μ by m_e $\Rightarrow$ Discrepancy error

$$DE: \frac{m_e - \mu}{m_e} = 1 - \frac{\mu}{m_e}$$

$$= 1 - \frac{m_p m_e}{(m_p + m_e) m_e}$$

$$= 0.0005444$$

$$\Downarrow \times \underline{100}$$

$$\underline{0.0544\ \%}$$

b) Wavelength btw red Balmer lines

$$\frac{1}{\lambda} = \frac{\mu}{m_e} \underset{\substack{\uparrow \\ \text{cnst}}}{R} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right)$$

$$R = \frac{m_e e^4}{32\pi^2 \epsilon_0^2 \hbar h c} = 1.097375 \times 10^7 m^{-1}$$

Hydrogen : $\mu = 9.104426 \times 10^{-31}$ kg

Deuterium : $\mu = 9.106904 \times 10^{-31}$ kg

c) Binding Energy $= \underset{\substack{\uparrow \\ \text{minimum} \\ \text{in potential well}}}{E_1}$

$$E_n = -\frac{1}{n^2} \frac{\mu e^4}{32\pi^2 \epsilon_0^2 \hbar^2} [eV]$$

$$32\pi\epsilon_0\hbar$$

$$E_1 = -\frac{m_e e^4}{64\pi^2\epsilon_0^2\hbar^2}$$

positron : $\mu = \frac{m_e}{2}$

$$= -6.8\,eV$$

④ Expectat° value

K.E ↳ $\langle\hat{T}\rangle = \iiint d^3r\, u(r,\theta,\phi)\left(-\frac{\hbar^2}{2\mu}\nabla^2\right)u(r,\theta,\phi)$

Potential E ↳ $\langle\hat{V}\rangle = \iiint d^3r\, u(r,\theta,\phi)V(r)u(r,\theta,\phi)$

$$\left(\frac{d}{dr}r^2\frac{d}{dr}\right)$$

$$u_{100}(r,\theta,\phi) = \sqrt{\frac{1}{\pi a_0^3}}\exp(-r/a_0)$$

$$e^{ax} = a e^{ax}$$

$$K.E = \langle\hat{T}\rangle = \frac{-\hbar}{2\pi a_0^3\mu}\iiint \exp(-r/a_0)\left(\frac{d}{dr}r^2\frac{d}{dr}\right)e^{-r/a_0}$$

$$= \frac{\hbar}{2\pi a_0^3\mu}\iiint\left(e^{-r/a_0}\right)\left(\frac{d}{dr}\underbrace{\frac{r^2}{a_0}e^{-r/a_0}}\right)$$

$$(u.v)' = u'v + uv'$$

$u = \frac{r^2}{a_0}$ $\quad u' = 2r$

$v = e^{-r/a_0}$ $\quad v' = -\frac{1}{a_0}e^{-r/a_0}$

$$2re^{-r/a_0} - \frac{r^2}{a_0^2}e^{-r/a_0}$$

$$= \frac{\hbar}{2\pi a_0^3\mu}\iiint\left(e^{-r/a_0}\right)\left(2re^{-r/a_0} - \frac{r^2}{a_0^2}e^{-r/a_0}\right)$$

$$\Big[= \frac{\hbar}{\ldots}\int \ldots e^{-2r/a_0} \ldots \iiint r^2 e^{-2r/a_0}\Big]$$

$$= \frac{\hbar}{2\pi a_0^3 \mu}\left[2 \iiint r e^{\cdots} \cdots - \frac{1}{a_0^2} \iiint r e^{\cdots} \cdots \right]$$

$$2 \iiint r e^{-r/a_0} \, dr \, d\theta \, d\phi = 2 a_0^2 \theta \phi$$

$$-\frac{1}{a_0^2} \iiint r^2 e^{-2r/a_0} \cdots = \frac{-1}{a_0^2} 2 a_0^3 \theta \phi$$

$$= -2 a_0 \theta \phi$$

$$= \frac{\hbar}{\cancel{2}\pi a_0^{\cancel{2}} \mu}\left[\cancel{2}\cancel{a_0} \theta \phi (a_0 - 1) \right]$$

$$= \frac{\hbar \theta \phi}{\pi a_0 \mu} (a_0 - 1)$$

3

$$A(1,2) \phi(1,2) = \alpha \phi(1,2)$$

$$\hat{P}_{12} \hat{A}(1,2) \phi(1,2) = \alpha \hat{P}_{12} \phi(1,2)$$

$\hat{P}_{12}$ is Hermitian if

$$\langle \phi(x_1, x_2) | \hat{P}_{12} \phi(x_1, x_2) \rangle = \langle \hat{P}_{12} \phi(x_1, x_2) | \phi(x_1, x_2) \rangle$$

$$\left\langle \phi(x_1,x_2) \middle| \hat{P}_{12}\phi(x_1,x_2) \right\rangle = \int dx_1 dx_2 \phi^*(x_1,x_2) \hat{P}_{12}\phi(x_1,x_2) = \int dx_1 dx_2 \phi^*(x_1,x_2)\phi(x_2,x_1).$$

Changing the dummy variables to $y_1 = x_2$ and $y_2 = x_1$ gives.

$$\int dx_1 dx_2 \phi^*(x_1,x_2)\phi(x_2,x_1) = \int dy_1 dy_2 \phi^*(y_2,y_1)\phi(y_1,y_2) = \int dy_1 dy_2 \left[\hat{P}_{12}\phi(y_1,y_2)\right]^* \phi(y_1,y_2)$$

Changing the dummy variables to $y_1 = x_1$ and $y_2 = x_2$ gives

$$\int dy_1 dy_2 \left[\hat{P}_{12}\phi(y_1,y_2)\right]^* \phi(y_1,y_2) = \int dx_1 dx_2 \left[\hat{P}_{12}\phi(x_1,x_2)\right]^* \phi(x_1,x_2) = \langle \hat{P}_{12}\phi(x_1,x_2) | \phi(x_1,x_2) \rangle$$

So,

$$\langle \phi(x_1,x_2) | \hat{P}_{12}\phi(x_1,x_2) \rangle = \langle \hat{P}_{12}\phi(x_1,x_2) | \phi(x_1,x_2) \rangle$$

as required.

(6) Bosons = symmetric wavef^{n}s
Ferm^{n}s = antisymmetric wvf^n

$$u_\pm = A\left[u_a u_b \pm u_a u_b\right]$$

$u_a = u_b$ & normalised

Note: orthogonality implies $\int d^3r\, u_a^* u_b = \delta_{ab}$
eigenf^{n}s

Find A

(1) Normalise

$$\int d^3r\, u_\pm^* u_\pm$$

$$= |A|^2 \int d^3r \left[u_a u_b \pm u_a u_b\right]^* \left[u_a u_b \pm u_a u_b\right]$$

$= 1$

multiplying out the integrand

$$\int d^3r\, u_\pm^* u_\pm = |A|^2\left[1 \pm 0 \pm 0 + 1\right]$$

$$= 2|A|^2 = 1$$

$$\therefore\ A = \frac{1}{\sqrt{2}}$$

Antisymmetric state: $u_- = A[u_a u_a - u_a u_a]$

$= 0$

state does not exist

Symmetric state: $u_+ = A[u_a u_a + u_a u_a]$

$= 4|A|^2 = 1$

$\therefore A = 1/2$

7 $V(x) = 0$

$0 \le x \le a$ Spatially dep part eigenf^{n}s

$V(x) = \infty$

$|x| > a$

with 8

A4 - non-interacting particles spatially $\propto$ eigenf^n eigenval

7. (a) From the lecture notes, the composite wavefunction for two distinguishable non-interacting particles in an external field with particle 1 in state a and particle 2 in state b is given by

distinguishable

$$u_{ab}(x_1,x_2) = u_a(x_1)u_b(x_2) \text{ with } E_{ab} = E_a + E_b = (n_a^2 + n_b^2)\frac{\hbar^2\pi^2}{2ma^2}.$$

The ground state corresponds to $a = b = 1$ with $n_a = n_b = 1$:-

$$u_{11}(x_1,x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right) \text{ with } E_{11} = 2\frac{\hbar^2\pi^2}{2ma^2}$$

and is non-degenerate. The first excited state corresponds to $n_a = 1, n_b = 2$ or $n_a = 2, n_b = 1$:-

$$u_{12}(x_1,x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) \text{ with } E_{12} = 5\frac{\hbar^2\pi^2}{2ma^2}$$

$$u_{21}(x_1,x_2) = \frac{2}{a}\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right) \text{ with } E_{21} = 5\frac{\hbar^2\pi^2}{2ma^2}$$

Interchanging the particle labels (e.g., 1→2 and 2→1) changes the energy eigenfunction but not change the energy eigenvalue, i.e., the first excited state is doubly degenerate.

bosons

(b) For bosons, $u_+(x_1,x_2) = A[u_a(x_1)u_b(x_2) + u_a(x_2)u_b(x_1)]$.

The ground state corresponds to $a = b = 1$ such that the normalisation constant $A = 1/2$ (see question 6). Then

$$u_{+,\text{ground}} = \frac{1}{2}[u_1(x_1)u_1(x_2) + u_1(x_2)u_1(x_1)] = u_1(x_1)u_1(x_2) = \frac{2}{a}\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right) \text{ with}$$

$E_{+,\text{ground}} = 2\frac{\hbar^2\pi^2}{2ma^2}$ and is non-degenerate.

For the first excited state the normalisation constant $A = 1/\sqrt{2}$ (see question 6) and

$$u_{+,\text{1st excited}} = \frac{1}{\sqrt{2}}[u_1(x_1)u_2(x_2) + u_1(x_2)u_2(x_1)] = \frac{1}{\sqrt{2}}\frac{2}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\right]$$

with $E_{\text{1st excited}} = 5\frac{\hbar^2\pi^2}{2ma^2}$. Interchanging the particle labels does not change $u_{+,\text{1st excited}}$ so the first excited state is non-degenerate.

fermions

(c) For fermions, $u_-(x_1,x_2) = \frac{1}{\sqrt{2}}[u_a(x_1)u_b(x_2) - u_a(x_2)u_b(x_1)]$.

For $a = b = 1$, $u_- = \frac{1}{\sqrt{2}}[u_1(x_1)u_1(x_2) - u_1(x_2)u_1(x_1)] = 0$ so the state does not exist (Pauli exclusion principle). The ground state is given by

$$u_{-,\text{ground}} = \frac{1}{\sqrt{2}}[u_1(x_1)u_2(x_2) - u_1(x_2)u_2(x_1)] = \frac{1}{\sqrt{2}}\frac{2}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\right]$$

with $E_{\text{ground}} = 5\frac{\hbar^2\pi^2}{2ma^2}$. In this case, interchanging the particle labels gives $-u_{-,\text{ground}}$ i.e. the negative of the original function. $|u_{-,\text{ground}}|^2 = |-u_{-,\text{ground}}|^2$, so the probability of locating the particles is unchanged, and the ground state is non-degenerate.

A4

8 Two non-interactive particles 1D ∞ square well

$$\hat{H} = \hat{T}_1 + \hat{T}_2$$

$$= \frac{\hbar^2}{2m}\frac{d^2}{dx_1^2} + \frac{\hbar^2}{2m}\frac{d^2}{dx_2^2}$$

energy eigenval : $E_n = n^2 \frac{\hbar^2\pi^2}{2ma^2}$

Apply hamiltonian ground states ex 7

a) For two distinguishable particles

$u_{\text{ground}}(x_1, x_2) = u_1(x_1)\, u_1(x_2)$

$2\ \hat{T}_1 u_1(x_1) = E_1 u_1(x_1)$:

$$\hat{H} u_{11} = (\hat{T}_1 + \hat{T}_2)\, u_1(x_1)\, u_1(x_2)$$

$$= u_1(x_2)\, \hat{T}_1 u_1(x_1) + u_1(x_1)\, \hat{T}_2 u_1(x_2)$$

$$= (E_1 + E_1)\, u_1(x_1)\, u_1(x_2)$$

$$= E_{ground}\, u_1(x_1)\, u_1(x_2)$$

$$\therefore E_{ground} = 2E_1 = 2 \frac{\hbar^2 \pi^2}{ma}$$

b) Bosons → also $u_{+,ground} = u_1(x_1)\, u_1(x_2)$

same result as a

c) Fermions

$$u_{-,ground} = \frac{1}{\sqrt{2}} \left[u_1(x_1)\, u_2(x_2) - u_1(x_2)\, u_2(x_1) \right]$$

$$\begin{cases} \hat{T} u_1(x_1) = E_1 u_1(x_1) \\ T u_2(x_1) = E_2 u_2(x_1) \\ \vdots \end{cases}$$

$$\hat{H} u_{-,ground}(x_1, x_2) = (\hat{T}_1 + \hat{T}_2)\, u_{-,ground}(x_1, x_2)$$

$$= \frac{u_2(x_2)}{\sqrt{2}}\, \hat{T}_1 u_1(x_1) + \frac{u_1(x_1)}{\sqrt{2}}\, \hat{T}_2 u_2(x_2)$$

$$- \left[\frac{u_1(x_2)}{} \hat{T} u \ldots (x \ldots) + \frac{u_2(x_1)}{} \hat{T} u (x \ldots \right]$$

$$= (E_1 + E_2)\frac{u_1(x_1)\,u_2(x_2)}{\sqrt{2}} - (E_2 + E_1)\frac{u_1(x_2)\,u_2(x_1)}{\sqrt{2}}$$

$$= (E_1 + E_2)\, u_{-,\text{ground}}$$

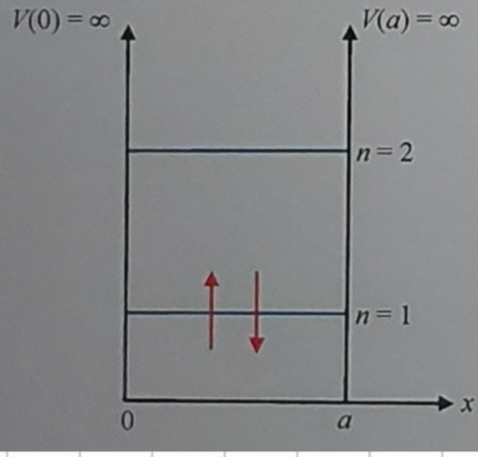
$$= E_{\text{ground}}\, u_{-,\text{ground}}$$

$$E_{\text{ground}} = E_1 + E_2 = 5\,\frac{\hbar^2\pi^2}{2ma^2}$$

9 Two non-interacting spin half fermions
1D ∞ square well

anti-parallel

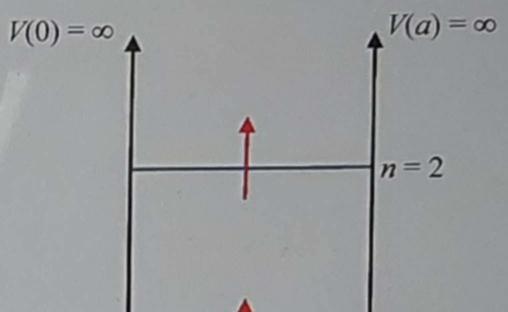
9. If the spins are anti-parallel, the total spin $S=0$. The overall eigenfunction $\phi(1,2)=u_+(1,2)\chi_{00}(1,2)$ has both a spatially dependent part $u_+(1,2)$ and a spin dependent part $\chi_{00}(1,2)=\frac{1}{\sqrt{2}}[\alpha_1\beta_2-\alpha_2\beta_1]$. The ground state corresponds to $a=b=1$, such that $u_+(1,2)=\frac{1}{2}[u_1(x_1)u_1(x_2)+u_1(x_2)u_1(x_1)]=u_1(x_1)u_1(x_2)$ which is finite (for the normalisation constant see question 6).

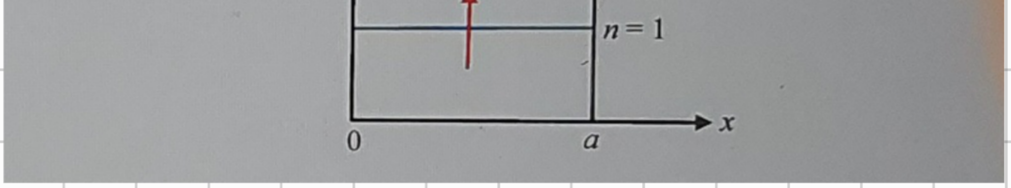


parallel

If the spins are parallel, the total spin $S=1$. The overall eigenfunction $\phi(1,2)=u_-(1,2)\chi_{1M_S}(1,2)$ has both a spatially dependent part $u_-(1,2)$ and a spin dependent part $\chi_{1M_S}(1,2)$ with $M_S=-1, 0$ or 1. The ground state corresponds to one electron in the state with $n=1$ and the other electron in the state with $n=2$ (see question 7).

$$u_{-,\text{ground}}=\frac{1}{\sqrt{2}}[u_1(x_1)u_2(x_2)-u_1(x_2)u_2(x_1)].$$





10 A4!

a) Distinguishable : $u(x_1, x_2) = u_a(x_1)\, u_b x_2$

orthonormal $\int dx_i\, u_a^*(x_i)\, u_b(x_i) = \delta_{ab}$

$$\langle x_1^2 \rangle = \iint u_a^* u_b^* x_1^2 u_a u_b \, dx_1 dx_2$$

$$= \int u_a^* x_1^2 u_a \, dx_1 \int |u_b|^2 dx_2$$

$$= \int u_a^* x_1^2 u_a \, dx_1 = \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \iint u_a^* u_b^* x_2^2 u_a u_b \, dx_1 dx_2$$

$$= \int u_b^* x_2^2 u_b \, dx_2 \int |u_a|^2 dx_1$$

$$= \int u_b^* x_2^2 u_b \, dx_2$$

$$= \langle x^2 \rangle_b$$

$$\langle x_1 x_2 \rangle = \int u_a^* x_1 u_a \, dx_1 \int u_b^* x_2 u_b \, dx_2$$

$$= \langle x \rangle_a \langle x \rangle_b$$

$$\langle (x_1 - x_2)^2 \rangle_{disting} = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2 \langle x_1 x_2 \rangle$$

$$= \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2 \langle x \rangle_a \langle x \rangle_b$$

b & c) Bosons / Fermions

$$u_\pm = \frac{1}{\sqrt{2}}\left[u_a u_b \pm u_a u_b\right]$$

$$\langle x_1^2 \rangle = \iint u_\pm^* \, x_1^2 \, u_\pm \, dx_1 \, dx_2$$

$$= \frac{1}{2}\begin{bmatrix} \int x_1^2 |u_a(x_1)|^2 dx_1 \int |u_b(x_2)|^2 dx_2 \\ + \int x_1^2 |u_b(x_1)|^2 dx_1 \int |u_a(x_2)|^2 dx_2 \\ \pm \int x_1^2 u_a(x_1)^* u_b(x_1) dx_1 \int u_b(x_2)^* u_a(x_2) dx_2 \\ \pm \int x_1^2 u_b(x_1)^* u_a(x_1) dx_1 \int u_a(x_2)^* u_b(x_2) dx_2 \end{bmatrix}$$

$$= \frac{1}{2}\left[\langle x^2 \rangle_a + \langle x_b^2 \rangle \pm 0 \pm 0\right]$$

$$= \frac{1}{2}\left[\langle x^2 \rangle_a + \langle x_b^2 \rangle\right]$$

Similarly $\langle x_2^2 \rangle = \frac{1}{2}\left[\langle x^2 \rangle_b + \langle x^2 \rangle_a\right]$

$$\langle x_1 x_2 \rangle = \iint u_\pm^* \, x_1 x_2 \, u_\pm \, dx_1 \, dx_2$$

$$= \langle x \rangle_a \langle x \rangle_b \pm \langle x \rangle_{ab} \langle x \rangle_{ba}$$

$$= \langle x \rangle_a \langle x \rangle_b \pm |\langle x \rangle_{ab}|^2$$

$$\Rightarrow \langle (x_1 - x_2)^2 \rangle = \langle (x_1 - x_2)^2 \rangle_{\text{distinguishable}} \mp 2 |\langle x \rangle_{ab}|^2$$

$\Downarrow$

identical bosons tend to be closer together

identical | bosons tend to | get closer together
| ferm°s | further apart

↓

system behaves as if there was a force of attract°/repuls° btw bosons/ferm°s

=

this force is not a force, but is often referred to as exchange force ← purely qm

Note that $\left\langle (x_1 - x_2)^2 \right\rangle_{\pm} = \left\langle (x_1 - x_2)^2 \right\rangle_{\text{distinguishable}}$ if $\langle x \rangle_{ab} = \int x u_a(x)^* u_b(x)\,dx = 0$ i.e. if there is no overlap between $u_a(x)$ and $u_b(x)$. So, if the particles are far enough apart, we can treat them as distinguishable: An electron in New York is distinguishable from an electron in London!

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$$\alpha_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \beta_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

AY!

$$\boxed{\hat{S}_{x_i} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_{y_i} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_{z_i} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$\hat{S}_{x_i}\alpha_i = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\beta_i \checkmark$$

$$\hat{S}_{x_i}\beta_i = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\alpha_i$$

$$\hat{S}_{zi}\,\beta_i = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{\hbar}{2}\beta_i$$

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$X = AA$

$\hat{S}X = S(S+1)AA \Rightarrow S =$ $\qquad \hat{S} = \hat{S}_1^2 + \hat{S}_2^2 + 2\left(\hat{S}_{x1}\hat{S}_{x2} + \hat{S}_{y1}\hat{S}_{y2} + \hat{S}_{z1}\hat{S}_{z2}\right)$

$\hat{S}_z X = \hbar M_S AA \Rightarrow M_S$

$\hat{S}_z = \hat{S}_{z1} + \hat{S}_{z2}$

$$\hat{S}_1^2 = S_{x1}^2 + \hat{S}_{y1}^2 + \hat{S}_{z1}^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\hat{S}_1^2\alpha_1 = \frac{3}{4}\hbar^2\alpha_1 \quad \& \quad \hat{S}_2\alpha_2 = \frac{3}{4}\hbar^2\alpha_2$$

A4!

1) $\hat{S}X_{11}(1,2)$ $\qquad + 2\hat{S}_{y1}\hat{S}_{y2}$

$$= \left(\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{x1}\hat{S}_{x2} + 2\hat{S}_{z1}\hat{S}_{z2}\right)\alpha_1\alpha_2$$

$$\hat{S}_1^2\alpha_1\alpha_2 = (\hat{S}_1^2\alpha_1)\alpha_2 = \frac{3}{4}\hbar^2\alpha_1\alpha_2$$

$$\hat{S}_2^2\alpha_1\alpha_2 = (\hat{S}_2^2\alpha_2)\alpha_1 = \frac{3}{4}\hbar^2\alpha_2\alpha_1$$

$$\hat{S}_{x1}\hat{S}_{x2}\alpha_1\alpha_2 = \frac{\hbar^2}{4}\beta_1\beta_2$$

$$\hat{S}_{y1}\hat{S}_{y2}\alpha_1\alpha_2 = -\frac{\hbar}{4}\beta_1\beta_2$$

$$\hat{S}_{z1}\hat{S}_{z2}\alpha_1\alpha_2 = \frac{\hbar^2}{4}\alpha_1\alpha_2$$

$$= \frac{6}{4}\hbar^2 \alpha_2 \alpha_1 + \frac{\hbar^2}{2}\alpha_1 \alpha_2$$

$$= 2\hbar^2 \alpha_1 \alpha_2 = S(S+1)\hbar^2 \alpha_1 \alpha_2$$

$$\therefore S = 1 \checkmark$$

$$\hat{S}_z \chi = M_s \hbar AA$$

$$\hat{S}_z \chi_{11} = \hat{S}_{z_1} \alpha_1 \alpha_2 + \hat{S}_{z_2} \alpha_1 \alpha_2$$

$$= \frac{\hbar}{2}\alpha_1 \alpha_2 + \frac{\hbar}{2}\alpha_2 \alpha_1$$

$$= \hbar \alpha_1 \alpha_2$$

$$= M_s \alpha_1 \alpha_2 \qquad \therefore M_s = \frac{\hbar}{\hbar} = 1 \checkmark$$

ii) $\chi_{1-1}(1,2) = \beta_1 \beta_2$

$$\hat{S}\chi_{1-1} = (\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{x_1}\hat{S}_{x_2} + 2\hat{S}_{y_1}\hat{S}_{y_2} + 2\hat{S}_{z_1}\hat{S}_{z_2})\beta_1\beta_2$$

$$\hat{S}_1 \beta_1 \beta_2 = \frac{3}{4}\hbar^2 \beta_1 \beta_2$$

$$\hat{S}_2 \beta_1 \beta_2 = -\frac{3}{4}\hbar^2 \beta_1 \beta_2$$

$$S_{x_1} S_{x_2} \beta_1 \beta_2 = \frac{\hbar^2}{4}\alpha_1 \alpha_2$$

$$S_{y_1} S_{y_2} \beta_1 \beta_2 = -\frac{\hbar^2}{4} \alpha_1 \alpha_2$$

$$S_{z_1} S_{z_2} \beta_1 \beta_2 = \frac{\hbar^2}{4} \beta_1 \beta_2$$

$$= \frac{6}{4} \hbar^2 \beta_2 \beta_1 + \frac{\hbar^2}{2} \beta_1 \beta_2$$

$$= 2\hbar^2 \beta_1 \beta_2 = S(S+1)\hbar^2 \beta_1 \beta_2$$

$$S = 1$$

$$\hat{S}_z \chi_{-1} = \hat{S}_{z_1} \beta_1 \beta_2 + \hat{S}_{z_2} \beta_1 \beta_2$$

$$= -\frac{\hbar}{2} \beta_1 \beta_2 - \frac{\hbar}{2} \beta_2 \beta_1$$

$$= -\hbar \beta_1 \beta_2 \Rightarrow M_s = -1$$

need all of these! $x, y, z, +, -$

$$\hat{S}_{x_i} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_{y_i} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(B)

$$\hat{S}_{\pm i} = \hat{S}_{x_i} \pm i \hat{S}_{y_i}$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right]$$

$$\Downarrow$$

$$\hat{S}_{+i} = \frac{\hbar}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad \hat{S}_{-i} = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$

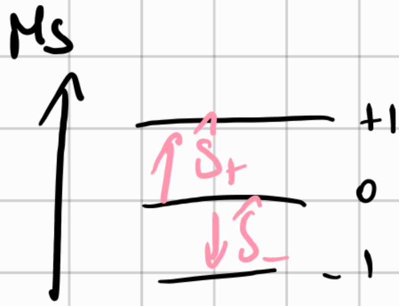
"Find the effect of $\hat{S}_{+i}$ in α_i" $\Longleftrightarrow$ $\hat{S}_{+i}\alpha_i = ?$

$$\hat{S}_{+i}\alpha_i = \frac{\hbar}{2}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}0\\ 0\end{pmatrix} = 0$$

$$\hat{S}_{-i}\alpha_i = \frac{\hbar}{2}\begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \begin{pmatrix}0\\ \hbar\end{pmatrix} \propto \beta_i$$



For the two-particle spin state $\chi_{11}(1,2)$

$$\hat{S}_+\chi_{11}(1,2) = \left(\hat{S}_{+1} + \hat{S}_{+2}\right)\alpha_1\alpha_2 = \left(\hat{S}_{+1}\alpha_1\right)\alpha_2 + \alpha_1\left(\hat{S}_{+2}\alpha_2\right) = 0, \text{ as required.}$$

$$\hat{S}_-\chi_{11}(1,2) = \left(\hat{S}_{-1} + \hat{S}_{-2}\right)\alpha_1\alpha_2 = \left(\hat{S}_{-1}\alpha_1\right)\alpha_2 + \alpha_1\left(\hat{S}_{-2}\alpha_2\right) = \hbar\left(\beta_1\alpha_2 + \alpha_1\beta_2\right) \propto \chi_{10}(1,2),$$

as required ($M_S = 0$ is the next rung on the ladder of M_S values when $S = 1$.)