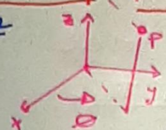
CARTESIAN

x, y, z



CYLINDRICAL POLAR

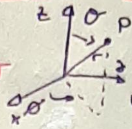
$0 \le \rho < \infty$

$0 \le \phi < 2\pi$

$-\infty < z < \infty$

← Polar axis

SPHERICAL POLAR



$0 < r < \infty$

$0 \le \theta < \pi$

$0 \le \phi < 2\pi$

Translat°:

$x = \rho\cos(\phi) = r\sin(\theta)\cos(\phi)$

$y = \rho\sin(\phi) = r\sin(\theta)\sin(\phi)$

$z = z = r\cos(\theta)$

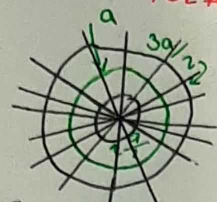
$\rho = \sqrt{x^2+y^2}$ $\quad r = \sqrt{x^2+y^2+z^2}$

$\phi = \tan(y/x)$ $\quad \theta = \cos^{-1}\left(z/\sqrt{x^2+y^2+z^2}\right)$

$z = z$ $\quad \phi = \tan^{-1}(y/x)$

PLANE POLAR

$\div 8$



$r = a \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}(b\theta)$

If r/a = NaN ↓ DO NOT ADD!

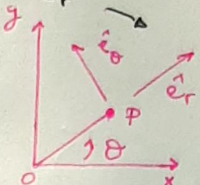
Steps: ① θ | r/a

0, $\pi/8$, $2\pi/8$, ⋮, $15\pi/8$

② Delimit all $\frac{r}{a} = 0$ "no" area

③ unite points (neatly)

Posit° vector $r = r\hat{e}_r$



$\hat{e}_r = \cos(\theta)\vec{i} + \sin(\theta)\vec{j}$

$\hat{e}_\theta = -\sin(\theta)\vec{i} + \cos(\theta)\vec{j}$

$\frac{d\hat{e}_r}{dt} = \dot{\theta}\hat{e}_\theta$ $\quad \frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\hat{e}_r$

$\hat{e}_\rho = \cos(\phi)\vec{i} + \sin(\phi)$

$\hat{e}_\phi = -\sin(\phi)\vec{i} + \cos(\phi)\vec{j}$

$\hat{e}_z = u \to u_z = 1$

Special cases:

- $R = a\cos\theta$ $\quad a>0 \quad a<0$
- $R = a\sin\theta$:
- $R = a \pm b\cos\theta$ (remember ±)
  - $a>0$, $b>0$: $|a-b|$, $a+b$
  - $\frac{a}{b} = 1$: lies in axis opposite cos/sin element
  - $1 < \frac{a}{2} < 2$
  - $\frac{a}{b} \ge 2$

or $\hat{S}$ Scale factor $\hat{e}_u = \frac{\partial r}{\partial u} / \left|\frac{\partial r}{\partial u}\right|$

Unit vector $\quad h_u = \left|\frac{\partial r}{\partial u}\right|$ $\quad \hat{e}_u = \frac{1}{h_u}\frac{\partial r}{\partial u}$

Posit° vector $\quad \vec{r} = r\hat{e}_r$

Velocity vector $\quad \vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ ≠ Velocity $\frac{dr}{dt} = v$ $\quad$ Speed $\left|\frac{dr}{dt}\right| = |V|$

Accelerat° $\quad a = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$

Magnitude or δS ⊛ $\quad a^2$ ← max gradient $|\nabla\phi(a,b,c)|$

Unit Tangent vector $\quad \hat{T} = \frac{dr}{dt} / \left|\frac{dr}{dt}\right|$

Tangent in direct° velocity $\quad \hat{T} = \frac{v}{|v|} = \hat{v}$

$\delta\phi = \phi(x+\delta x, y+\delta y, z+\delta z) - \phi(x,y,z)$ $\quad$ SCALAR ← FIELDS → VECTOR a

$\frac{\delta\phi}{\delta x} = \lim_{\delta x \to 0}\left[\frac{\phi(x+\delta x, y, z) - \phi(x,y,z)}{\delta x}\right] \Rightarrow$ direct° $\phi(x+\delta x, y, z) - \phi(x,y,z)$ magnitude

$\frac{\partial\phi}{\partial x}\delta x$ $\quad$ direct° are steepest $\nabla\phi$ $\quad \frac{d\phi}{ds} = 0$

Finding potential

$\frac{\partial}{\partial x} F = \dots + c$

$\frac{\partial}{\partial y} F = \dots + c$ $\Rightarrow \phi = \dots + c$

Equat° of a plane $\quad r \cdot n = a \cdot n$

Tangential to the equipotential surface of ϕ at a $\quad n = \nabla\phi$ at a ← posit° $(\cdot,\cdot,\cdot)$

$r \cdot n = \nabla\phi(a) \cdot a$

potential $F = \nabla\phi$ $\quad$ Direct°

$\nabla \times F = 0 \Rightarrow F(r)$ Conservative

$\int_B^A F(r)\,dr = \int_A^B \nabla\phi\,dr \Leftarrow \oint_C F(r)\,dr = 0$

$= \phi_B - \phi_A$ ⊛

$\frac{d\phi}{ds} = \nabla\phi \cdot \hat{S} = |\nabla\phi|\cos(\theta)$

$\delta\phi \simeq \nabla\phi\,\delta r \simeq \nabla\phi\,\hat{S}\,\delta S \simeq |\nabla\phi|\cos(\theta)\,\delta S$

Parametrising

- Circle: $x^2+y^2=r^2 \Rightarrow x = r\cos(t)$, $y = \pm r\sin(t)$
- ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \Rightarrow x = a\cos(t)$, $y = \pm b\sin(t)$

replace when needed: $(y-1)^2 \Rightarrow \alpha^2$, $\alpha = \sin(\theta)$, $y = 1+\sin(\theta)$

- $A \to C$: $A \to B \to C$, $(0,0,0)$ $(0,1,0)$ $(1,1,0)$

$F = a\vec{i} + b\vec{j} + c\vec{j}$

$\int_A^C F\,dr = \int_A^B F\,dr + \int_B^C F\,dr$

$r = x\vec{i} + y\vec{j} + z\vec{k} \to dr = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$

Grad → $\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

DIV → $\nabla a = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$ = SCALAR FIELD

Curl → $\nabla \times a = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a_x & a_y & a_z \end{vmatrix}$: VECTOR FIELD ← Cartesian

1st \ 2nd	grad	div	curl
grad	×	ok	×
div	∇^2	×	0
curl	0	×	ok

A→B: $0 \le t \le 0 \to x = 0$, $dx = 0$; $0 \le y \le 1 \to y = t$, $dy = dt$; $0 \le z \le 0 \to z = 0$, $dz = 0$ $\quad$ $F \xrightarrow{param} F_p$ $\quad \int_0^1 F_p\,dy$ ← $F_p\,dx = 0$, $F_p\,dz = 0$

B→C: same… $0 \le t \le 1$; $0 \le x \le 1 \to x = t$; $1 \le y \le 1 \to y = 1$; $0 \le z \le 0$ …

- $F = y\,i + 2x\,j \xrightarrow{param} x = t,\ y = t,\ z = 0$

Average value Φ

$\langle\Phi\rangle = \frac{1}{V}\int_V \Phi(r)\,dV$

$a\nabla^2\Phi = \langle\Phi\rangle - \Phi$ measures net spread at r

$\hat{n} \cdot \text{curl } F = \hat{n}(\nabla \times F(r))$

$= \lim_{A \to 0}\frac{1}{A}\oint F\,dr$ enclosing r

Laplacean

$\nabla^2\Phi = \nabla \cdot \left(\frac{1}{h_u}\frac{\partial\Phi}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial\Phi}{\partial v}\hat{e}_v + \frac{1}{h_w}\frac{\partial\Phi}{\partial w}\hat{e}_w\right)$

$= \frac{1}{h_u h_w h_v}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial\Phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial\Phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial\Phi}{\partial w}\right)\right]$

- $h_x = h_y = h_z = 1$
- $\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2\Phi}{\partial\phi^2} + \rho\frac{\partial^2\Phi}{\partial z^2}\right]$
- $\frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial r}\left(r^2\sin(\theta)\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right]$

Curl $\frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \rho\hat{e}_\phi & \hat{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$

$\frac{1}{r^2\sin(\theta)}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin(\theta)\hat{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin(\theta)F_\phi \end{vmatrix}$

Diffusion eqt°

$\nabla^2\Phi = k\frac{\partial\Phi}{\partial t}$

$\nabla^2\Phi$ provides good measure of how $\Phi(r)$ relates to nearby values

Smooth: $\nabla^2\Phi = 0$

local max Φ from some source: $-\nabla^2\Phi = \rho(r)$ Poisson eqt°

$\nabla^2\Phi > 0 \Rightarrow$ net local min in Φ; < 0 max

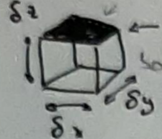
local min causes Φ to spring back in time: $\nabla^2\Phi \propto \partial^2\Phi/\partial t^2$ or $\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$

Schröd eqt°: $\nabla^2\Phi \propto \frac{\partial\Phi}{\partial t}$

Rate change 2 — Max rate change ↓

Flux = outwards of a normal component × Area of a face $\int_S F(r)\,ds$

Int: $\oint_S F(r)\,ds$



— plane: $z + \delta z/2$

bottom

Flux bottom plane: $[-a(x,y,z-\frac{\delta z}{2})] \times \delta x \, \delta y$

Net flux bottom & top plane: $\frac{\delta a_z}{\delta z} \times \delta V$ $\quad \frac{d}{dx} e^{2x} = 2e^{2x}$

Net flux whole box: $\nabla \cdot a \times \delta V$

Magnetic Flux: Φ

Faraday's Law: $\mathcal{E} = \frac{\partial \Phi}{\partial t}$

Evaluat° Flux int

1) Split into faces → $\int_S F\,ds$
2) For each face:
 a) write ds → $ds = -dx\,dy\,\hat{k}$ (only 2 coord vary)
 b) eval $F \cdot ds$
 c) eval (scalar) surface int
3) Σ all int

Newton's Laws

$F_x = m\frac{d^2x}{dt^2}$, $F_y = m\frac{d^2y}{dt^2}$, $F_z = m\frac{d^2z}{dt^2}$

3D: $m\frac{d^2r}{dt} = qE + q\frac{dr}{dt} \times B$: Eqt° of mot°

$F = q(E + v \times B)$: Electromag & mag fields

$F = m\frac{d^2r}{dt^2}$: Mass × Accelerat°

Eqt° of mot°: $F = -\frac{GMm}{r}\hat{e}_r$ $\quad F = \frac{Q_1 Q_2}{4\pi\varepsilon_0} r^2$

$F = ma \Rightarrow -\frac{c}{r^2}\hat{e}_r = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{e}_\theta$

Mot° central F: $\dot{\theta} = h/r^2$

$-\frac{c}{r} = \ddot{r} - \frac{h^2}{r^3}$ with $\ddot{r} = -\frac{c}{r^2} + \frac{h^2}{r^3}$

Replace $u = \frac{1}{r} \Rightarrow \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{h}{r^2}$ & $\frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = \frac{1}{r^2}\frac{dr}{d\theta}$

2nd order diff eqt°: $-\frac{h^2}{r^2}\frac{d^2u}{d\theta} = -\frac{c}{r^2} + \frac{h^2}{r^3} \Leftrightarrow \frac{d^2u}{d\theta^2} + u = \frac{c}{h^2}$

General solut°: $u = A\cos(\theta) + B\sin(\theta) + c/h^2$

$\frac{du}{d\theta}$ when $\theta = 0 \Rightarrow B = 0$ & $r(\theta) = r(-\theta)$

$B = 0 \Rightarrow u = \frac{1}{r} = A\cos(\theta) + \frac{c}{h^2} \Leftrightarrow r = \frac{h^2}{a}\frac{1}{1 + \frac{Ah^2}{c}\cos(\theta)}$

is the curve $r = \frac{a}{1 + e\cos(\theta)}$ with $a = \frac{h^2}{c}$ $\quad 2+ = Ah^2/c$

Vol int — Int — Surface int

dV

$Q = \int_S \theta\,ds$

ds has 1 Cart coord fixed $\Rightarrow ds = \begin{cases} dy\,dz \\ dy\,dx \\ dx\,dy \end{cases}$

Intg Limits

Line int

$\int_A^B a(r)\,dr$

How to find?

I. Show $F = \vec{i} - 2\vec{j} - y\vec{k}$

II. $F = \nabla\phi$: find

III. Eval $\int_A^B F(r)\,ds$

conservative? (check)

I. Show $\nabla \times F = 0$

II. $F = \nabla\phi$

$F_x = \partial\phi/\partial x$
$F_y = \partial\phi/\partial y$
$F_z = \partial\phi/\partial z$

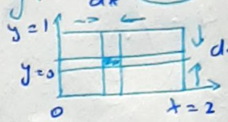
Integrate these eqt°

Inspect $\phi(r) = \dots$

III. F consv

$\int_A^B F(r)\,dr = \phi_B - \phi_A$

↳ CNST



$ds = dx\,dy$

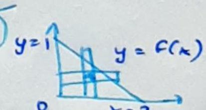
Given $y = y_0$: change in horizontal strip

$dy \times \int_{x=0}^{x=2} \theta(x, y_0)\,dt = f(y_0)\,dy$

Tot change:

$Q = \int_{y=0}^{y=1} f(y)\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=2} \theta(x,y)\,dx\,dy$

↳ CNST



Gauss div thm: $\boxed{\int_V \operatorname{div} F\,dV = \oint_S F\,ds}$

$\operatorname{div} F \simeq \frac{1}{\delta V_i}\oint_{S_i} F\,ds$

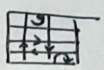
$\Sigma \operatorname{div} F\,\delta V_i \xrightarrow[\delta V_i \to 0]{} \int_V \operatorname{div} F\,dv$

= Tot outward flux of F across all surface δV_i

$= \Sigma_i \oint_{S_i} F\,ds$

- Flux through any internal face is = & opposite to the flux through the shared face of the adjacent block ⇒ Adding all the fluxes: $\Sigma \oint_{S_i} F\,ds$; gauss div thm
- Tot div = Tot outward flux

Stoke's thm: $\boxed{\int_S \operatorname{curl} F \cdot ds = \oint F\,dx}$



- Horizontal strips:

$Q = \int_{y=0}^{y=1}\int_{x=0}^{x=2(1-y)} x^2 y\,dx\,dy$

- Vertical strips:

$Q = \int_{x=0}^{x=2}\int_{y=0}^{y=1-x/2} x^2 y\,dy\,dx$

TOT: $Q = \int_{x=0}^{x=2} \frac{1}{2}x^2(1 - \frac{x}{2})\,dx$

Green's thm: → $\int_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dv = \oint_S \phi\nabla\psi\,ds$

Proof: ↳ $\int_V [\phi\nabla^2\psi - \psi\nabla^2\phi]\,dV = \oint_S (\phi\nabla\psi - \psi\nabla\phi)\,ds$

$\nabla(\phi\nabla\psi) = (\nabla\phi)\cdot(\nabla\psi) + \phi\nabla^2\psi$: vector identity

Integrate both sides over vol: RHS: $\int_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dv$

LHS: $\int_V \nabla(\phi\nabla\psi)\,dV = \oint_S \phi\nabla\psi\,ds$ ← div thm

Max rate change $|\vec{\nabla}T|_p = \|a\vec{i} + b\vec{j} + c\vec{k}\| = $ value

~~Rate change in direct° Q(d, e, f) at distance P~~

Rate change of T with distance at P in direct° at point Q with coord (d, e, f)?

$\frac{dT}{ds} = \vec{\nabla}T \cdot \vec{s}$

$\vec{s} = \vec{Q} - \vec{P}$

$\frac{dT}{ds} = \vec{\nabla}T \cdot \vec{s}$ $\quad \vec{s} = \vec{Q} - \vec{P} = (h, i, l)$ $\quad \hat{s} = \frac{\vec{s}}{|\vec{s}|}$ $\quad [\vec{\nabla} \cdot \vec{T} \cdot \hat{s} = n^2]$ Answer

Specific Heat Capacity

$[JK^{-1}kg^{-1}] \leftarrow [JK^{-1}] \div$ mass

$C_V = \left(\frac{đQ}{dT}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V$

$C_p = C_V + p\left(\frac{dV}{T}\right) = nRT$, $C_p = \left(\frac{\partial U}{dT}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p$

Enthalpy $H = U + pdV$

$dH = dU + pdV + Vdp$
$= đQ + Vdp$
$= C_V dT + pdV$

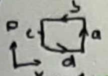
Heat $dQ = dU - dW$
$= dU + pdV$

Heat = E exchanged directly btw microscopic dof of freedom (microscopic work)

\# Temperature – thermometer
= Measuring – pyrometer
– thermocouple

Exact = integrable $\frac{d}{dT}$
∝ phase diff
∝ ~~path~~
$\oint dr = 0$

Work = E exchanged by transfer btw direct macroscopic observables W



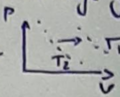
a: $\Delta W = -\int_{V_1}^{V_2} p_1 dV = -p_1 \Delta V$
c: $\Delta W = p_1 \Delta V$
b/d: $\Delta W = 0$

$\Rightarrow \Delta W = (p_2 - p_1)(V_2 - V_1)$
$= \int_i^f x\,dx$

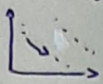
→ Gas: $-pdV$
→ Magnetic: Bdm
→ General: $x\,dx$ (Force, conjugate displacement)

Way to change gas

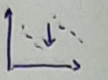
Isobar $P = cnst$



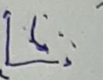
Isotherm $T = cns$



Isochore $V = cnst$



Adiabatic $\Delta Q = 0$ no heat exchange



$pV^\gamma = cnst$

IDEAL GAS: $pV = nRT$

$\underbrace{\left(\frac{\partial U}{\partial T}\right)_V}_{C_V} = \underbrace{\left(\frac{\partial U}{\partial T}\right)_p}_{C_p}$

$C_p = C_V + p\left(\frac{\partial V}{T}\right)$
$C_p > C_V$
$đQ = dU + đW$
$= C_V dT + pdV$
$= C_p T - dH$

if $đQ = 0$
$pdV = -C_V dT$
$\ln(p) = -\gamma \ln V + cnst$
$\gamma = C_p / C_V$

$F = pA$, $đW = -Fdx = -pAdx$
$dU = đQ - pdV$

[JK⁻¹]

Entropy S
$\Delta S = S_f - S_i$
$= \int_i^f \frac{đQ_{rev}}{T}$
$\oint dS = 0$
$dS = đQ_{rev}/T$
$\int_{abd} dS = \int_{acd} dS$
$\Delta W = -\int p\,dV$
$\Delta W = \frac{1}{2} \kappa_T \bar{V} (p_2^2 - p_1^2)$

Linear coeff of thermal expans°
$\alpha = \frac{1}{\ell}\frac{d\ell}{dT}$
$\alpha\,dT = \frac{1}{\ell}d\ell$
path indp $\alpha(T - T_0) = \ln(\ell/\ell_0)$
$\ell = \ell_0 e^{\alpha \Delta T}$

Hydrostatic pressure
$V = V(p, T)$
$dV = \left(\frac{\partial V}{\partial p}\right)_T dp + \left(\frac{\partial V}{\partial T}\right)_p dT$
$= -V\kappa_T\,dp + V\beta_p\,dT$

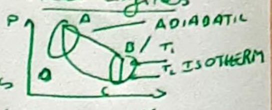
isothermal compressibility; thermal coeff expans°

$df\ x\,dx + y\,dy$: cont everywhere
$\left(\frac{\partial x}{\partial y}\right)_x = \left(\frac{\partial y}{\partial x}\right)_y$

Isothermal compress° (ideal gas)
$\Delta W = -\int p\,dV$
$pV = nRT$, $V = nRT$
$= -nRT\int_{V_1}^{V_2}\frac{dV}{V} = -nRT\ln\left(\frac{V_2}{V_1}\right)$
$= -nRT\ln\left(\frac{p_1}{p_2}\right) = nRT\ln\left(\frac{p_2}{p_1}\right)$

Isobaric change (solid)
$\beta_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$
$\Delta W = -\int p\,dV = \int_{p_1}^{p_2} p\kappa_T V dp - \int_{T_1}^{T_2} p\beta_p V dT$
$= -\int_{T_1}^{T_2} p\beta_p V dT = -p\bar{V}\beta_p(T_2 - T_1)$

Heat Engines



$\boxed{T_1} \xrightarrow{Q_1} (C_1) \xrightarrow{Q_2} \boxed{T_2} \xrightarrow{Q_2} (C_2) \xrightarrow{Q_3} \boxed{T_3}$, with W_1 from C_1, W_2 from C_2

$\frac{Q_1}{Q_2} = f_1(T_1, T_2)$
$\frac{Q_2}{Q_3} = f_2(T_2, T_3)$

$\boxed{T_1} \to (C_1 + C_2) \to \boxed{T_3}$

$\frac{Q_1}{Q_3} = f_3(T_1, T_3) \Rightarrow \frac{Q_1}{Q_3} = \frac{T_1}{T_3}$

Carnot engine = cycle isotherms & adiabatics
(reversible) quasistatic & no hysteresis, with fixed temp, taking heat

$\boxed{T_1 C_1 | T_2 C_2}$, $W = Q_1 - Q_2$, $\frac{T_1}{T_2} = \frac{Q_1}{Q_2}$

$\oint dU = Q_1 - Q_2 + W = 0$
heat supplied; work done on

work done by: $-W = Q_1 - |Q_2|$

Efficiency: $\eta = \frac{\text{work out}}{\text{heat in}}$
$\eta_c > \eta_{other}$, $\eta = 1 - \frac{|Q_2|}{Q_1}$

G	M	k	d	c	m	μ	n	p
10^9	10^6	10^3	10^{-1}	10^{-2}	10^{-3}	10^{-6}	10^{-9}	10^{-12}

→ 0th: If 2 syst are separately in thermal equilibrium they are also in therm eq° with each other

→ 1st: E is conservd. Heat & Work are both forms of E
infinitesimal $dU = đW + đQ$; finite $\Delta U = \Delta W + \Delta Q$

→ 2nd: no process is possible whose sole result is
↳ complete convers° of heat into work (Kelvin)
↳ Transfer of heat from a colder body to a hotter (Clausius)

always check for violat°: $\Delta Q = -\Delta U - \Delta W$ Clausius thm

Proof:
$W = Q_{V_1} - Q_{V_2} = Q_1 - Q_{C_2}$
$\frac{W}{Q_{V_1}} > \frac{W}{Q_{C_1}} \Rightarrow \eta_V > \eta_C$, $Q_{V_1} < Q_{C_1}$ violets 2nd law
$Q_{C_2} - Q_{V_2} = Q_1 - Q_{V_1}$

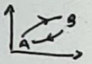
$\boxed{C, T_1}$ → O → $\boxed{E_1 T_2}$

$đQ_1 = C\,dT$
$đQ_2 = \frac{T_2}{T_1} đQ_1 = T_2\,dS_1$
$= T_2 \frac{C}{T} dT$
$đW = đQ_1 - đQ_2$
$= C|T_2 - T_1| - |C T_2 \ln\left(\frac{T_2}{T_1}\right)|$
$\Delta U = C(T_2 - T_1) \Rightarrow \Delta U = C(T_2 - 1)$
$C = \lim_{\Delta T \to 0}\left(\frac{\Delta Q}{\Delta T}\right) = \frac{đQ}{dT}$ ← extensive
$\Rightarrow dS_1 = \frac{C}{T}dT$

quasistatic; no hysteresis
Reversible engine (η_c)

Cycles
→ reversible: $\eta_c \geq \eta_{other} \Rightarrow 1 - \frac{Q_2^c}{Q_1^c} \geq 1 - \frac{Q_2^o}{Q_1^o} \Rightarrow \frac{Q_2^c}{Q_1^c} \leq \frac{Q_2^o}{Q_1^o}$
→ irreversible:
$\frac{Q_1^o}{T_1} - \frac{Q_2^o}{T_2} \leq 0 \to \Sigma\frac{Q^o}{T} \leq 0$

$\oint_{cycle}\frac{đQ}{T} \leq 0$

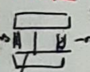


$\int_A^B \frac{đQ}{dT} \leq \int_B^A \frac{đQ_{rev}}{T} \leq 0$
$\int_A^B \frac{đQ}{dT} \leq \int_A^B \frac{đQ_{rev}}{T}$
$\frac{đQ}{T} \leq \frac{đQ_{rev}}{T} = dS$

$\oint_{cycle}\frac{đQ}{T} \leq 0$

Thermally isolated syst
$đQ = 0 \to dS \geq 0$
assumpt° universe ⇒ HEAT DEATH

Joule-Kelvin



porous plug

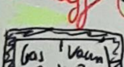
Push gas slowly: Adiabatic process ← no heat exchange
Isenthalpic process: $H_1 = H_2$

$\Delta W_1 = \int_{V_1}^0 dV p_1 = p_1 V_1$
$\Delta W_2 = -p_2 V_2$
$\Delta W = p_1 V_1 - p_2 V_2$
$đQ = 0 \Rightarrow \Delta U = \Delta W$
$U_2 - U_1 = p_1 V_1 - p_2 V_2$

JK effect: compressed gas in a big space reduced } inversion temp; warm up ⇒ INFLAT°

soon ↓ as gas cools

JK coeff $\alpha_J = \left(\frac{\partial T}{\partial V}\right)_U = -\frac{\left(\frac{\partial U}{\partial V}\right)}{\left(\frac{\partial U}{\partial T}\right)_V} = \frac{-1}{C_V}\left[T\left(\frac{\partial p}{\partial T}\right)_V - p\right]$



$dU = đQ + đW = 0$

$pV = RT\left(1 + \frac{B_2(T)}{V} + \frac{B_3(T)}{V^3} + \dots\right)$

Real gas: $p = \frac{RT}{V} + \frac{RT}{V^2}B_2(T)$

Perfect gas – no interact° btw particle
– $U = U(T)$ only
– distance btw particle does not a U
$T \not\propto \Delta V$

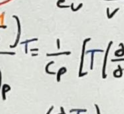
$\alpha_J = \frac{-1}{C_V}\frac{RT^2}{V^2}\frac{dB_2(T)}{T}$

$dH = 0$
$\alpha_J \equiv \left(\frac{\partial T}{\partial p}\right)_H = \frac{-\left(\frac{\partial H}{\partial p}\right)}{\left(\frac{\partial H}{\partial T}\right)_p} = \frac{1}{C_p}\left[T\left(\frac{\partial H}{\partial T}\right)_p - V\right]$

Entropy ideal gas: $\alpha_J = 0$
$\Delta S = \int_{V_A}^{V_A + V_B}\frac{p\,dV}{T} = nR\ln 2$
$= nR\ln\left(\frac{V_f}{V_i}\right)$

$\alpha_{JW} \equiv \left(\frac{\partial T}{\partial p}\right)_H$

H cnst



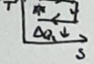
$(V_1, \Delta r, 0)$
Inside: $\alpha_{JW} > 0$, $T p \alpha T T$
inversion curve
Outside: gas compress ↓T (He, H₂)

missing last 2 pages thermal + pb2 notes (end)
^ see marking

LAWS OF ENTROPY

$\boxed{T_1} \to \boxed{T_2}$
$đQ = 0$: from outside isolated
$T_1 > T_2 \to đQ > 0$
0th: 1 & 2 not eq
1st: $-đQ_1 = đQ_2$
2nd: irrev & spont heat flow: 1→2 impossible

by Clausius:
$dS_1 = -đQ/T_1$
$dS_2 = +đQ/T_2$
$dS_{tot} = đQ\left(\frac{1}{T_2} - \frac{1}{T_1}\right) > 0$



$\Delta Q_1 = T_1\int_A^B dS = T_1(S_B - S_A)$
$\Delta Q_2 = T_2(S_D - S_C)$
$\oint dU = 0 = \Delta Q + \Delta W$
$\oint dS = \Delta Q$ ← area enclosed
$W' = -\Delta W = 0$

$dU = TdS - pdV$
$dH = TdS + Vdp$
$dF = -SdT - pdV$
$dG = -SdT + Vdp$

$dU = đQ + đW$: 1st
$đW = -pdV$: reversible change V
$đQ = TdS$: rev chg / 2nd law
} $dU = TdS - pdV$

all var F of S (path indp)
• closed syst
• work done by ΔV only

Maxwell Relat°
• Internal E: $U \to \left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$
• Enthalpy $\to \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p$
• Helmotz Free E: $F \to \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$
• Gibbs Free E: $G \to \left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$

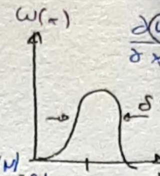
Statistical weight ω_i : tot n° accessible microstates

$P_i = \omega_i / \text{Tot n° microstates}$

$\omega(n) = {}^{N}C_n = \frac{N!}{n!(N-n)!}$

$\frac{1}{k_B T} = \frac{\partial \ln \omega}{\partial U}$

$\ln \omega \simeq N \ln(N) - n\ln(n) - (N-n)\ln(N-n)$ = tot n° arrangements

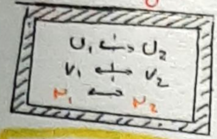


$\frac{\partial \omega}{\partial x} = 0 \Rightarrow \frac{\partial \ln \omega}{\partial x} = \frac{1}{\omega}\frac{\partial \omega}{\partial x} = 0$

extensive & maximised closed syst

$\boxed{\frac{\partial \cosh(x)}{\partial x} = \frac{e^{x} - e^{-x}}{2} = \sinh(x)}$

Closed syst in ϵ:



$U_0 = U_1 + U_2$
$V_0 = V_1 + V_2$
$N_0 = N_1 + N_2$ } fixed

$\boxed{x! \simeq x^x e^{-x} \qquad \ln x! \simeq x\ln x - x}$

Entropy $dS = \frac{\partial Q}{T}$ (therm)

$S_0 = S_1 + S_2$

$\omega_0 = \omega_1 + \omega_2$

$\ln \omega_0 = \ln \omega_1 + \ln \omega_2$

$dS = \frac{\partial S}{\partial U} dU + \frac{\partial S}{\partial V} dV + \frac{\partial}{\partial N} dN$

$= \frac{dU}{T} + p\frac{dV}{T} - \frac{\mu}{T} dN \Rightarrow$

$S = k_B \ln \omega$

$\Delta S = \Delta Q / T \leftarrow \Delta Q = mgh$

1st Law: $dU = TdS - pdV + \mu dN$

2nd Law: $\left(\frac{1}{T_2} - \frac{1}{T_2}\right)\delta U_2 \geq 0$

Chemical potential

note: $n = N_x$

$\mu = -T\left(\frac{-\partial S}{\partial N}\right)_{U,V}$

$= -k_B T \left(\frac{\partial \ln \omega}{\partial N}\right)_{U,V}$

$N_1 = N_2 \Rightarrow$ diffusive eq^m

$T \propto U$

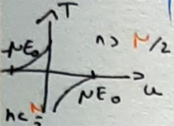
$\frac{1}{k_B T} = \frac{\partial \ln \omega}{\partial U}$

$r(U)$

$n(U) = \frac{N}{2} + \frac{U}{2\epsilon_0}$

Temp

$T = \frac{2\epsilon_0}{k_B} \frac{1}{\ln\left(\frac{N\epsilon_0 - U}{N\epsilon_0 + U}\right)}$



$n > N/2$

$n < \frac{N}{2}$

Two lvl syst

$dU = TdS - Fdl$

↳ work done

$\uparrow$ spin ↑ ϵ_0 $\quad$ $\downarrow$ spin ↓ $-\epsilon_0$

$U = (2n - N)\epsilon_0 \Rightarrow$

$N - n = \frac{N}{2} - \frac{U}{2\epsilon_0}$

$F = T\left(\frac{\partial S}{\partial l}\right)_U = k_B T \frac{\partial \ln \omega}{\partial l} = \frac{k_B T}{2d} \ln\left(\frac{Nd - l}{Nd + l}\right)$

$l(n) = [n - (N - n)]d = [2n - N]d$

$n(l) = \frac{N}{2} + \frac{l}{2d}$

$\omega(U) = \frac{N!}{\left(\frac{N}{2} + \frac{U}{2\epsilon_0}\right)!\left(\frac{N}{2} - \frac{U}{2\epsilon_0}\right)!}$

Attempt 1 stat mech

limit $l \ll Nd$

$F(l) = \frac{-k_B T}{N d^2}$

$\boxed{l(1+x) \simeq x + O(x^2)}$

Maxwell relat° for therm: $\left(\frac{\partial S}{\partial l}\right)_T = \left(\frac{\partial F}{\partial T}\right)_l$

Helmotz Free energy $F = -k_B T \ln(Z)$: const T

$F = U - TS$

$U = F + TS = k_B \frac{\partial \ln Z}{\partial T}$

$p = \left(\frac{-\partial F}{\partial V}\right)_{T,N} = k_B T\left(\frac{\partial \ln Z}{\partial V}\right)_{T,N}$

$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -k_B T \left(\frac{\partial \ln Z}{\partial N}\right)_{T,V}$

Boltzmann distribut° $P(\epsilon) = \frac{1}{Z} e^{-\epsilon / k_B T}$

$Z = \sum_i e^{-\epsilon_i / k_B T}$

Two lvl syst

$Z = \sum_i e^{-\epsilon_i / k_B T} = e^{\epsilon_0 / k_B T} + e^{-\epsilon_0 / k_B T}$ (up, down)

$U = k_B T^2 \left(\frac{\partial \ln Z}{\partial T}\right)_{V,N} = k_B T^2 \frac{\partial}{\partial T} \ln 2\cosh\left(\frac{\epsilon_0}{k_B T}\right)$

$= \epsilon_0 \tanh\left(\frac{\epsilon_0}{k_B T}\right)$

$C = \left(\frac{\partial U}{\partial T}\right)_{V,N} = \frac{\epsilon_0^2}{k_B T^2} \operatorname{sech}^2 \frac{\epsilon_0}{k_B T}$

Mechanical eq^m: no unbalanced force, no pressure grad

approaching ⇓

$[\delta S = \delta S_1 + \delta S_2 \geq 0]$: 2nd Law

Volume syst changes:

$\frac{\delta S_1}{\delta V_1}\delta V_1 + \frac{\delta S_2}{\delta V_2}\delta V_2 \geq 0$

$\delta V_1 = -\delta V_2 \quad \hookrightarrow \left(\frac{\delta S_2}{\delta V_2} - \frac{\delta S_1}{\delta V_1}\right)\delta V_2 \geq 0$

$p = T\frac{\partial S}{\partial V} \quad \hookrightarrow \left(\frac{p_2}{T_2} - \frac{p_1}{T_2}\right)\delta V_2 \geq 0$

- temp equalised
- $\delta V_2 > 0$ iff $p_2 > p_1$
- subsyst at higher p gains vol until pressure is equalised

E levels $P_g(\epsilon_i) = g\, P_m(\epsilon_i) = g\, A e^{-\epsilon_i / k_B T}$ (A " Z^{-1}) : Boltz dist

$\epsilon_l = l(l+1)\frac{\hbar^2}{2I} \qquad \hbar = \frac{h}{2\pi}$

$G(\epsilon)$ **states with** $\epsilon \leq \epsilon \rightarrow G(\epsilon + \delta) - G(\epsilon) \simeq \frac{dG}{d\epsilon}\delta\epsilon$ (sts btw ϵ & $\epsilon + \delta\epsilon$)

$\frac{dG}{d\omega} = \frac{dG}{d\epsilon}\frac{d\epsilon}{d\omega} = \hbar \frac{dG}{d\epsilon}$

density states per unit ϵ

Density per freq ω

$Z = \sum_{\text{states}} e^{-\epsilon / k_B T} = \sum_\epsilon g_\epsilon e^{-\epsilon / k_B T}$

$\sum_\epsilon \rightarrow \int$

$Z = \int_0^\infty d\epsilon\, e^{-\epsilon / k_B T} \frac{dG}{d\epsilon}$

$\delta N_2 > 0$ iff $\mu_1 > \mu_2$.

syst with lower chem potential gains $+$ 'ill eq^m

Diffusive eq^m

reach diffus° i.e gravitational eq^m

$\frac{\delta S_1}{\delta N_1}\delta N_1 + \frac{\delta S_2}{\delta N_2}\delta N_2 \geq 0 \quad \} \ \delta N_1 = -\delta N_2$

$\left(\frac{\delta S_1}{\delta N_2} - \frac{\delta S_1}{\delta N_1}\right)\delta N_2 \geq 0 \quad \} \ \mu = -T\, \partial S / \partial N$

$\left(\frac{\mu_1}{T_1} - \frac{\mu_2}{T_2}\right)\delta \geq 0$

– Particle in box

Conservat° laws:
- Electric charg
- Momentum (angular)
- E (tot) $E_{TOT} = KE + E_0$; $KE = mg\Delta h$
- Quark flavour (not weak)
- Baryon n° $B = \frac{1}{3}(N(q) - N(\bar{q}))$
- Indiv lepton n° $L_e = N(e^-) + N(e^+) + N(\nu_e) - N(\bar{\nu}_e)$; $L_\mu = \dots$

uncertain: $\Delta t \simeq \frac{\hbar}{2\Delta E}$

Rest mass not matter

$\Delta E = m_0 c^2 = E_0$

can exist for $\Delta t = \frac{\hbar}{2 m_0 c^2}$

max range: $= c\Delta t = \frac{\hbar c}{2E_0}$

$m_0 = 0 \Rightarrow$ virtual photon $\Rightarrow$ Long range Force ($m = 0$)

virtual wave can travel 2×10^{-10} m, short range

$\hookrightarrow$ weak F appears very weak

KE: $\gamma \geq E_0$; $\leq E_0$ weak int $\to$ very weak when ~90 GeV

large % weak int: $\frac{\hbar c}{KE} \leq \frac{\hbar c}{E_0}$ p/$\bar{p}$ lepton photon

range $\propto$ Forces $\leftarrow$ Intrinsic strength

- Gravity (graviton) $\rightarrow$ acts on all mass/E $\leftarrow \propto 1/r^2$
- Electromag (photon: γ) $\rightarrow$ electrically charged particle ; $<10^{-16}$ s
- Weak (Boson, Meson) $\leftarrow$ experienced by hadrons & leptons ; spin life $>10^{-12}$ s ; quark flavour conserved
- Strong (Gluon) $\leftarrow$ acts on quarks, not leptons ; $\sim 10^{-21}$ s ; colour charge: r, g, b

E.M F = Strong F

Atomic v.s Nuclear
- Length: 10^{-10} m ; $1 fm = 10^{-15}$ m
- Time: $\sim10^{-18}$ s ; $10^{-17} - 10^{-23}$ s
- Mass: $m_e = 0.511$ MeV/c^2 ; $m_p = 938.5$, $m_n = 939.6$
- KE: 10–1000 eV ; 1–10 MeV

CP transforms: charge conjugation $\to$ particle – antiparticle ; parity: $r \to -r$ mirror

Interact°:
- WEAK – change quark flavour ; neutrino – antineutrino involved } no CP symmetry
- EM – real photon emitted/absorbed, lifetime $10^{-16} - 10^{-10}$ s
- STRONG – quark flav conserved, lifetime $< 10^{-20}$ s

CP symm $\Rightarrow$ process & anti-process eqly likely to occur

Rest mass $\leftrightarrow$ KE

React°: $A + X \rightarrow B + C$; $\underbrace{M_A + M_X \leq M_B + M_C}$ KE converted into mass

Decay $\alpha t \alpha N$: $A \rightarrow B + C$; $\underbrace{M_A \geq M_B + M_C \text{ (rest mass)}}$ no violat° E conserv

Quantum gravity: $\hbar$: ML^2T^{-1} ; G: $M^{-1}L^3T^{-2}$; c: LT^{-1}

Planck { length: $\left(\frac{G\hbar}{c^3}\right)^{1/2} \sim 10^{-35}$ m ; energy: $\left(\frac{\hbar c^5}{G}\right)^{1/2} \sim 10^{19}$ GeV

Photon decay (high E) $\simeq \geq 10^{32}$ years

Fermion ($\frac{1}{2}$ spin, Pauli excl respected) ; Boson

Hadron: Baryon (3q or 3$\bar{q}$) $\to$ DECAY ; Meson (q – $\bar{q}$ pair) DECAY $\to$ lepton, photon

Lepton (point like) ; Quark (up, down, …) composite ; Gauge Boson

e^- neutrino, μ^- muon

- Half life: $t_{1/2} = \frac{\ln(2)}{\lambda} \Leftarrow e^{-\lambda t} = 1/2$
- Mean life or lifetime $= 1/\lambda$
- Activity source $= \lambda N$ Bq
- Initial activity / Decay rate $= \lambda N_0$ Bq
- <Activity 1^{st} $\frac{1}{2}$ life>: $A = \frac{N_0/2}{\ln(2)/\lambda} = \frac{N_0\lambda}{2\ln(2)}$ Bq

$\lambda = \frac{\delta N}{N \delta t} \Leftarrow$ decay a δt

Particle mot° $\psi = e^{ikx}$: wavefield mot° only "sees" obj size $\lambda < a$

dB wave $\lambda_B = \frac{h}{p} \Rightarrow \lambda_B = \frac{hc}{KE}$ & $p = \frac{\hbar c}{E_0}$ range F by exchange particle

$k = \frac{p}{\hbar}$

special relat: $E^2 = (pc)^2 + (m_0c^2)^2$; $= KE + m_0c^2$; only in extreme cases ($KE \gg m_0c^2$!)

Mass nucleus $= A m$ / mass nucleon

$=$ mass protons + mass neutrons – $\frac{\text{binding } E}{c^2}$ $\leftarrow$ ~8.78 MeV

- $V_{nucleus} = \frac{4}{3}\pi R_0^3 A$ fm^3 (vol)
- vitesse $\to v = \sqrt{\frac{2T}{m}}$

$e^- + \bar{\nu}_e = 0 \to$ Balance!

$\mu^- = \nu_\mu \leftarrow$ Muons int°

may have implicat° ${}^{A}_{Z}X \leftarrow \Sigma A$ product $= A_{initial} - 1$ if e^+

If ${}^{a}_{b}X + {}^{c}_{d}Y = {}^{a+c}_{b+d}Z + ?$

γ photon: mass defect $\to E$

What decay is possible? $\to \Delta m$
- $< 2m_e$: K-capture or β^+
- $\geq 2m_e$: β^+ decay possible

${}^{A}_{Z}X$: A·u kg

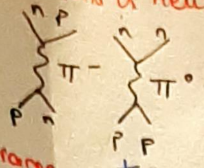
$Q =$ (mass nucleus – mass products) $\times c^2 > 0$

$\Updownarrow$

$Q = BE_{products} - BE_{nucleus}$

Particle

Photons & neutrons



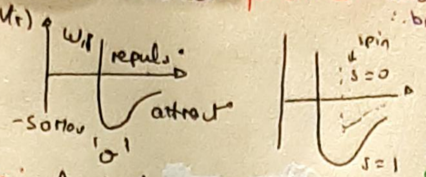
] mesons

range $\sim \hbar / 2 m_0 c$
$\sim c\Delta t$
$\sim \hbar c / 2E_0$

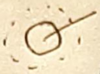
p-p
n-n
n-p
deuteron : binding E ~ 2.2 MeV

not emit : Pauli exclus° ~ cannot have 2 identical fermions in 1 quantum state

Nucleon-nucleon potential



Nuclei spherical



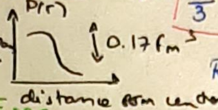
size scattering exp high E/n $E^2 = p^2c^2 + m_0^2c^4$
λ_B ≈ nuclear size
$\lambda = 2\pi\hbar / p$

Heavy Nuclei (A > 12)

- reasonably well defined rad
- narrow surface reg° (≥2.5 fm)
- cnst density near center

$\lambda \leq 1$ fm ← nucleon E ≥ 100 MeV

$N°_{nucleon} = A = \boxed{\frac{4\pi}{3} R_{nuc}^3} \times$ density $\times A = V_{nucleus}$



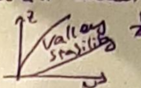
nucleon / fm³

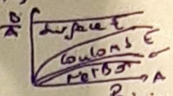
$R_{nuc} = \frac{1}{A^{1/3}}$ fm $= 1.1 A^{1/2}$? ≈ correct
} force has a short range

LIQUID DROP MODEL

classical $\xrightarrow{\text{does not cover}}$ • magic n° • spin • excited state
• light nuclei
(~A > 20) A = Z + N

nuclear force → short ranges
∴ binding E ∝ A ← mass ∝ n° nucleons
∝ surface
∝ charge
∝ QM effects





Asymmetry E

$B(Z,N) = \underbrace{aA}_{} - \underbrace{bA^{2/3}}_{} - s\underbrace{\frac{(N-Z)^2}{A}}_{} - \underbrace{\frac{dZ^2}{A^{1/3}}}_{} - \frac{\delta}{A^{1/2}}$] Pairing term: favors even-even over odd-odd nuclei

Bulk cohesive E per nucleon
surface term
favour = n° neutrons & protons
Favours bounding to more N rich nuclei
Z^2 / Factor of unity

α Bind E
∝ n° nucleon surface
∝ surface area
$\propto R_{nuc}^2 \rightarrow R_{nuc} \propto A^{1/3}$
Bulk term

^x_8A ^x_3B
heavier
stable

β stability

β decay - remains

$m_a(Z,N)c^2 = \{Nm_n + Z(m_p + m_e)\}c^2 - B(Z,N)$

$m_a c^2 = \alpha - \beta Z + \gamma Z^2$

β decay tends towards lightest atom in set of isobars.

β^- β^+ heavier

Atomic mass is min when: $\frac{dm_a}{dZ} = 0 \Leftrightarrow -\beta + 2\gamma Z = 0$

$\Rightarrow Z_{min} = \beta / 2\gamma \Rightarrow Z_{min} = \left(\frac{4s + (m_n - m_p - m_e)c^2}{4s + dA^{2/3}}\right) \times \frac{A}{2}$

$\simeq A / (1.98 + 0.015 A^{2/3})$: most stable config

$Z_{min} \simeq \left(\frac{4s}{4s + dA^{2/3}}\right) \times \frac{A}{2}$

stable nuclei $\begin{cases} \frac{N}{Z_{min}} = 1 + \frac{dA^{2/3}}{2s} \\ \frac{B}{A} \simeq a - \frac{b}{A^{1/3}} - \frac{s\,dA^{2/3}}{4s + dA^{1/3}} \end{cases}$

M $\delta \ll 2m_e$
$\boxed{A \text{ odd}}$ ∴ δ = 0
Z_A m_a single parabola
1 stable isobar

$\boxed{A \text{ even}}$ Two parabola separated by $22.4/A^{1/2}$ MeV
2 stable nuclei
1 odd-odd nucleus: can decay both ways

SHELL MODEL

← simple QM model

certain n° Z & N are poorly explained by liquid drop model (too stable): 2, 8, 20, 28 : magic n°

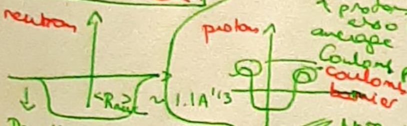
• Spin-orbit coupling is also way diff drop : V(r) has an extra term
↳ no electromag origin
↳ produces large splits
↳ $j = l + 1/2$ lvl lower for nuclear coupling

fills levels with bigger E gaps

shell overlaps

Predict nuclear E levels { neutrons / protons : average potential 2 cases

Potential well:



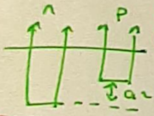
~ 1.1 $A^{1/3}$

Depth ~ 40-50 MeV

↑ protons also have average Coulomb pot°

Note: • shape cnst all nuclei
• order E levels same all nuclei
• depth roughly cnst
• E levels filled to ~8 MeV top well
• E levels closer as R_{nuc} ↑

1) Guess potential well
2) Calc E levels & wavef°
3) Fill up with protons & neutrons
4) From nucleon density calc potential well



Principal quantum n° : n
Orbital angular momentum quantum n° : l
Magnetic quantum n° : $m_l = -l, -l+1, \ldots, 0, l-1, l$

Degeneracy (2l+1) states per level

stable fill:

spin ↑↓ ⇒ $2 \times (2l+1)$ particles per shell

Filled level

Schrö'd eq°:

$-\frac{\hbar}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}R + \underbrace{V(r)}_{\text{potential well}}R = \underbrace{E}_{\text{E eignvl}}R$

Excited states: unstable

LIGHT: : HEAVY

I = J : nuclear spin

• Tot angular momentum ~ 0
• Successive protons & neutrons pair off = angular mom = 0

β decay : $n \rightarrow p^+ + e^- + \bar{\nu}_e$

• even-even nuclei J = 0
even-odd = J = j of unpaired nucleons

lifetime ≥ 10^{-16} s
ΔE ≤ few eV width

Unstable
γ decay (to lower E lvls
fission to light nuclei } lifetime $10^{-18/20}$ s, ΔE ≤ 0.1 MeV

Nuclei not ↓ lowest E levels ≃ Grand st

↑ E level ∝ ↓ B : ^A_ZX

: formed, decays after ~ nuclear reactions

Nuclei II

2

Nuclear react$^\circ$: $a + X \to Y + b$

└ light nucleon or nucleus incident to heavier one

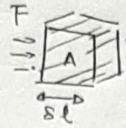
└ E released = Q = tot binding E of products − tot binding E of reactants

$Q > 0$ react$^\circ$ → Exothermic

< 0 Endothermic : require E to take place

incident particle must have a K.E threshold

Φ Flux $= \frac{N}{t}$ = n$^\circ$ particles incident per unit time



$F\sigma \times P_{nuc}$ / $A\delta t$

$P_{nuc}\,\sigma\,\delta L = \frac{\delta L}{L_0}$ mean free path

Pros reacting with thickness δL incident

effective cross sect$^\circ$ nuclei $\simeq \sigma$ → give an idea of magnitude

why we fire light particle towards heavy one

must supply: $\frac{m_1 + m_2}{m_2} Q$ (endothermic)

{ LAB frame → net mom ✓
{ CM frame → net momentum = 0
└ KE available for react$^\circ$

see next

Nuclei 2

m_1 O → v_1 m_2 O $v_2 = 0$

$KE_{lab} = \frac{1}{2} m_1 v_1^2$

$KE_{cm} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v_1^2$

$KE_{cm} = \frac{m_2}{m_1 + m_2} KE_{lab}$

m_1 O → $v_1 - v_{cm}$ ← m_2 O $-v_{cm}$

$m_1 (v_1 - v_{cm}) = m_2 v_{cm}$

$v_{cm} = \frac{m_1}{m_1 + m_2} v_1$

$KE_{cm} = \frac{1}{2} m_1 (v_1 - v_{cm})^2 + \frac{1}{2} m_2 v_{cm}^2$

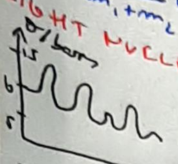
$= \frac{1}{2} m_1 (v_1 - v_{cm})^2 + \frac{1}{2} m_2 v_{cm}^2$

→ OFF RESONANCE { E do not match
— n doesn't enter nucleus (low %)
— low σ

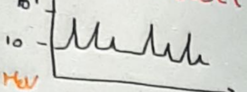
→ ON R. { energies match (within ΔE)
— n enters target → "long lived" compound nucleus
decay by break up or γ-decay

Resonance : initial KE matches E of excited state of the compound nucleus $^{A+1}_{Z}X$

LIGHT NUCLEI



HEAVY NUCLEI



KE (CM) MeV

widely spaced ← Peaks → closely spaced
broad ← → narrow
low (few barns) ← → high (10^3 – 10^4 barns)

Resonant peaks : caused by left over info from a compound nucleus

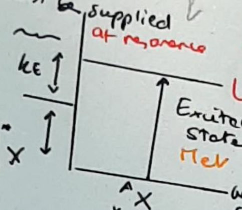
$n + {}^{A}_{Z}X \to {}^{A+1}_{Z}X^* \to$ products

neutron absorbed into nucleus : energy shared btw electrons

Note : Binding E of neutron in compound nucleus is available for excited of target nucleus ~ 4-8 MeV

Decay mode & format$^\circ$ process

KE must be supplied at resonance



React$^\circ$ product

└ Resonant Elastic scattering $n + {}^{16}_{8}O$

└ R Inelastic sct $n + {}^{16}_{8}O^*$

└ React$^\circ$ ${}^{13}_{6}C + \alpha$

└ Neutron capture ${}^{13}_{8}O + \gamma$

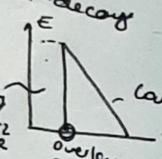
n + t space neutrons

Charged Particle react$^\circ$s ~ neutron react$^\circ$ + Coulomb barrier ≅ α-decay

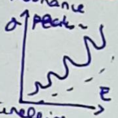
Experience Strong Force ⇒ nuclei touch

E supplied $= \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 (R_1 + R_2)}$

$\simeq \frac{Z_1 Z_2}{(A_1^{1/3} + A_2^{1/3})} \times 1.3$ MeV

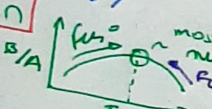


Fiss$^\circ$: Heavy Nucleus splits → 2 light nuclei + neutrons + E



Fiss$^\circ$
→ Spontaneous : unstable heavy nuclei, radioactive decay
→ induced : follows nuclear react$^\circ$ (neutron + heavy nucleus)

Fusion



Basic reactions :

$\boxed{4p \to {}^{4}_{2}He}$

$Q = 24.7$ MeV (+2 MeV)

HYDROGEN burning sun

↓

PPI chain

• E 1st step, weak int = slow
• $p + p \to {}^{2}_{1}H + e^+ + \nu_e$ $Q = 0.4$ MeV (+1 MeV from e^--e^+)
(${}^2_1H$) ~ weak int, slow
• $p + {}^{2}_{1}H \to {}^{3}_{2}He + \gamma$
$Q = 5.5$ MeV
• ${}^{3}_{2}He + {}^{3}_{2}He \to {}^{4}_{2}He + 2p$
$Q = 12.9$ MeV

Barrier height : $\frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 (R_1 + R_2)}$

$\simeq 1.44 \frac{Z_1 Z_2}{R_1 + R_2}$ MeV ≥ 1 MeV

High T ⇒ nuclei ionised

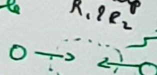
Gas ⇒ plasma (mix nuclei & e^-)

Distribut$^\circ$ → Maxwell Boltzmann

$\langle KE \rangle = \frac{3}{2} kT$

Typical KE << barrier height

↓



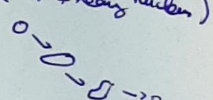
KE

<< Barrier height
• must tunnel to interact
• very small σ
• only light particle interact

>> Barrier height
• same as neutron react$^\circ$

Symmetry case

${}^{A}_{Z}X \to 2\,{}^{A/2}_{Z/2}Y$

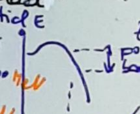


fiss$^\circ$ products
• ↑ surface E
• ↓ Coulomb E

$Q \geq 0$ i.e. $Z \geq 45$

↑Z ⇒ Q ~ 100-200 MeV

F$^\circ$ barrier ~ 5-6 MeV for A ~ 240



Energy released : $Q = 2B(\frac{A}{2}, \frac{Z}{2}) - B(A, Z)$

Liquid drop model ⇒ $B(Z, N) = aA - bA^{2/3} - s\frac{(N-Z)^2}{A} + d\frac{Z^2}{A^{1/3}} + \frac{\delta}{A^{1/2}}$

bulk term aA
Symmetry term $\frac{s(N-Z)^2}{A}$
} unaffected : only surface & coulomb term change

i.e. bulk term : aA & symmetry term : $\frac{s(N-Z)^2}{A}$ both unaffected

Fission tunnelling process ⇒ long lifetime

$Z \geq 144 \to$ barrier 0

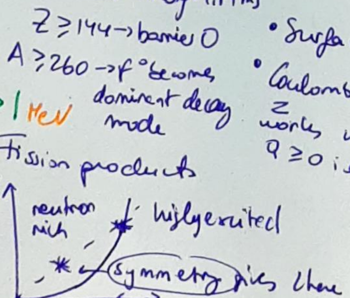
$A \geq 260 \to$ f$^\circ$ become dominant decay mode

• Surface E is ↑ : $bA^{2/3} = 2b(\frac{A}{2})^{2/3} = 1.26\, bA^{2/3}$

• Coulomb E ↓ : $\frac{dZ^2}{A^{1/3}} \Rightarrow \frac{2d(Z^2/4)}{(A/2)^{1/3}} \Rightarrow 0.62 \frac{dZ^2}{A^{1/3}}$

works when $Q \geq 0$ i.e. $Z \geq 45$

Fission products

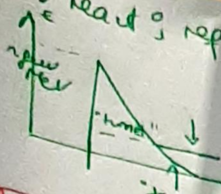


fragments usually asymmetric

Symmetric fiss = max E

Fission followed by
/ "prompt" neutrons "boiling off" ($\simeq 10^{-20}$ s)
— γ-decay to ground states ($\simeq 10^{-16}$ s)
— n-rich products : β-decay to β stability line (minutes, years)
\ 'delayed neutrons' sometimes emitted after β decay

React° rep QM tunnelling



Fus° can continue (nucleosynthesis) at higher T (higher barrier require ↑T)

heavy elements created supernova

Fusion reactors

↑PI to slow → D-D · high Q
D-T · lower barrier

Requirements

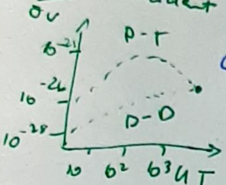
- T high ($>10^8$ K)
- Plasma density high (ions/m³)
- Hold plasma for long enough: τ: confinement time

$T > 10^8 K\,kT \to kT \sim 10$ keV

$n\tau > 10^{20}$ s m⁻³ : Lawson criter° higher ignit°

React° rate (for nucleus) $F\sigma$
per unit vol ($F = \sigma \rho_1$) n° density target nuclei $= \rho_1 \rho_2 \overline{\sigma v}$ → same nuclei $\frac{1}{2}\rho_1^2 \overline{\sigma v}$ → diff nuclei $\rho_1 \rho_2 \overline{\sigma v}$

For colliding beam particles $F = \rho_2 v$ ← velocity incident nuclei; incident



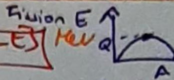
$$\overline{\sigma v} = \int_0^{+\infty} p(v)\,\sigma(v)\,v\,dv$$

tunnelling exp decreases at low v

$$\sim e^{\frac{-mv^2}{2kT}}$$

exp decreases at large v

Fis° PARTICLES



Q = BE(products) − BE(reactants)

Fis° fragments carry ~ 90% E

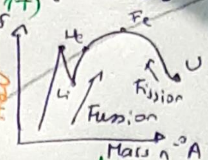
Neutron carry ~ 2% E

Spontaneous f° produces more neutrons: chain react°

Induced Fission by neutrons:

$n + {}^5U \to {}^6U + \text{MeV}$

needs to be higher than the barrier or QM tunnelling



Binding E per nucleon reaches peak around A.

$A > 56 \to$ Fiss° ${}^A_Z X \to {}^B_{Z_1} X + {}^{A-B}_{Z-Z_1} X$

Confinement methods → Magnetic Confinement Fus°
max output: 16 MW
power lost: 65%

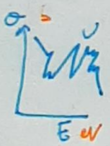
→ Inertial Confinement Fus°
$T \sim 10^8$ K
$\tau \sim 10^{-9}$ s

Nuclei III

NEUTRON DYNAMICS

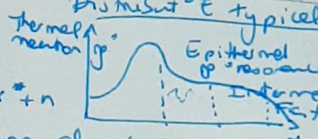
Can interact — elastic scattering (KE conserved) (Target recoil: neutron loses E)
— inelastic scattering: neutron loses KE: $n + {}^A_Z X \to {}^A_Z X^* + n$
— radioactive capture: incident neutron absorbed, compound nucleus forms & rapidly decays to it's ground state by gamma emmis°
$n + {}^A_Z X \to {}^{A+1}_Z X + \gamma$
— Fission

neutron leaves target at E state above spont fis° barrier



resonances occur: cross sect° peaks corresponding to the discrete state of E levels of the heavy nucleus

Distribut° E typical reactor:

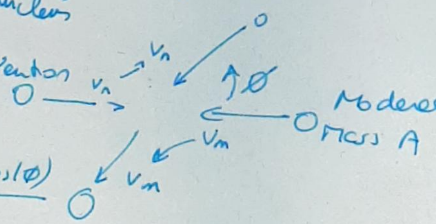


Moderator materials
↳ slow down fast neutrons
↳ lower E thermal regime at
cross sect° for inducing subsequent nuclear react° is higher

Read more before exam

- moderator materials
- neutron diffuss°
& leakage

CM frame



Final KE:

$$E_1 = E_0 \frac{1 + A^2 + 2A\cos(\phi)}{(1+A)^2}$$

E_{loss}: • minimum for grazing angle collis°
• max for head on collis°

(min) at grazing angle: $\phi = 0°$: $E_1 = E_0 \frac{1+A^2+2A}{(1+A)^2} = E_0$

(max) $\phi = 180°$: $E_1 = E_0 \frac{1+A^2-2A}{(1+A)^2} = E_0 \frac{(1-A)^2}{(1+A)^2} = E_0 \alpha$

Average logarithmic loss per collis°

$$\delta = \frac{1 + (1-A)^2}{2A} \log\left(\frac{A-1}{A+1}\right)$$: large A $\delta \to 0$

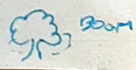
= mean E reduct° per event ≠ likelihood of scattering

Fiss° → neutron

Criticall mass

$$k = \frac{n° \text{ neutron at stage } (n+1)^{th}}{n° \text{ neutron at stage } n^{th}}$$

k=1: critical ✓
k<1: subcritical ✓
k>1: supercritical



4

Heisenberg uncertainty principle $\Delta x \times \Delta p \geq \hbar$ Prob density H_{α} - Born

de Broglie $p = \frac{h}{\lambda} = \hbar k$, $k = \frac{2\pi}{\lambda}$

Momentum $p = mv$

$P(x,t) = \Psi^*(x,t)\,\Psi(x,t) = |\Psi(x,t)|^2$

$P(x,t)\,dx = |\Psi(x,t)|^2 dx$

Find $\boxed{\text{posit}^{\circ}}$

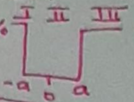
$\int_A^B P(x,t)\,dx = \int_A^B |\Psi(x,t)|^2 dx = 1$ normalisat' condit°

average posit°: $\langle x \rangle = \int_{-\infty}^{\infty} x\, P(x,t)\,dx = \int_{-\infty}^{\infty} \Psi^*(x,t)\, x\, \Psi(x,t)\,dx$

$\boxed{\text{Finite well}}$

$\beta = \hbar$, $\alpha = -\frac{\hbar^2}{2m}$ ← $\alpha \frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\Psi = \beta \frac{\partial \Psi}{\partial t}$

$V(x) = \begin{cases} 0, & -a \leq x \leq a \\ V_0, & x < -a \text{ or } x > a \end{cases}$



$\Psi_0 = \cos(kx - \omega t) + \gamma \sin(kx - \omega t)$

(II) $-a \leq x \leq a$, $V = 0$

TISE → $\frac{d^2\psi}{dx^2} = \frac{-2mE}{\hbar}\psi = \alpha^2\psi$ $\alpha = \sqrt{2mE}/\hbar$

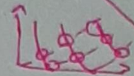
General sol: $\psi_{II} = A\sin(\alpha x) + B\cos(\alpha x)$ — even parity = 0, odd parity = 0

(I, III) $V = V_0$ → TISE: $\frac{d^2\psi}{dx^2} = \frac{2m(V_0 - E)}{\hbar^2} = \beta^2\psi$

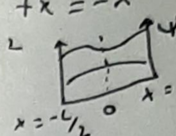
$\psi_{III} = Ce^{-\beta x}$

$\psi_I = Fe^{\beta x}$

$\Psi(x,t) \neq \psi(x)$



Symmetry { $|\psi(-x)|^2 = |\psi(x)|^2$ } Prob finding particle $+x = -x$



↑ γ ∝ γ bound states ↑

$\propto \sqrt{V_0}$ ↑ depth well; $\propto a$ ↑ width well

Parity $\psi_{II} = A$ or B — sin or cos ∝ Parity

[minimum i.e bound state] ← when Even --- solut°, Odd when → $\gamma > \pi/2$ => $V_0 a^2 \geq \frac{\hbar^2\pi^2}{8m}$ ↓

Even: $f(-x) = -f(x)$; $-\frac{1}{\alpha}\cot(\alpha a) = -1/\beta$; $\therefore \tan(\alpha a) = \beta/\alpha$

Odd: $f(-x) = f(x)$; $1/\alpha \tan(\alpha a) = -1/\beta$; $\therefore \cot(\alpha a) = \beta/\alpha$

n° bound sts = finite param ⇓

Calc γ → express as a multiple of $\pi/2$ → round nearest int

$\gamma = \sqrt{2mV_0 a^2}/\hbar$, $x = \gamma\sqrt{\frac{E}{V_0}}$ ⇒ $\tan(x) = \sqrt{(\gamma/x)^2 - 1}$ EVEN; $-\cot(x) = \sqrt{(\gamma/x)^2 - 1}$ ODD

TDSE solut° $\Psi = \psi\phi = \psi e^{-iEt/\hbar}$

Prob density $|\Psi|^2 = |\psi|^2$

solut° TISE eigenf° (stationary state)

$\boxed{\text{Schrö Eq}^{t°}}$

TISE $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$

$\hat{E} = i\hbar\frac{\partial}{\partial t}$

$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$

TDSE $-\frac{\hbar^2}{2m}\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} + V(x) = i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt} = C$

=> $\phi(t) = e^{-iCt/\hbar}$, $\omega = \frac{C}{\hbar}$ => $C = \hbar\omega$, $\phi(t) = e^{-iEt/\hbar}$

Condit°:

- → de Broglie Einstein: $\frac{\hbar^2}{2m\lambda^2} + V = hf$ } $k = 2\pi/\lambda$, $\omega = 2\pi f$
- → superposit° solut°;
- → EOM: $\frac{p^2}{2m} + V = E$ — $\frac{\hbar^2 k^2}{2m} + V = \hbar\omega$ ↓
- → Sinusoidal travelling waves: $\frac{d^2}{dx^2}\sin(kx - \omega t) \propto k^2$; $\frac{\partial}{\partial x}\sin(kx-\omega t) \propto -\omega$ — TE const deri
  - ↳ $F = -dU/dx = 0$
  - ↳ $\Psi_0(x,t) = Ae^{i(kx - \omega t)}$

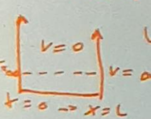
TISE: Substitute specific $V(x)$ ↓ General solut° ↓

BC

- I. Finite: ψ normalised, $\psi \neq 0$ everywhere
- II. Single valued prob. density has 1 val per posit°
- III. Continuous: $\frac{d\psi}{dx}$ finite

$\boxed{\infty \text{ well}}$

Infinite square well potential: $V = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & x < 0 \text{ or } x > L \end{cases}$



$\psi(x<0) = \psi(x>L) = 0$

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$ ⇓ $V = 0$ $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$\langle x \rangle = \int_{-\infty}^{\infty}\psi^* e^{iEt/\hbar}\, x\, \psi e^{-iEt/\hbar}dx = \int_{-\infty}^{\infty}\psi^* x\,\psi\,dx$ t indp

General solut° $\frac{d^2\psi}{dx^2}$ finite; $k_n = \frac{n\pi}{L} = \sqrt{2mE}/\hbar$

$\psi = A\sin(kx) + B\cos(kx)$

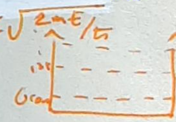
$\psi' = Ak\cos(kx) - Bk\sin(kx)$

$\psi'' = -k^2\psi$

$x = 0$: $A\sin(0) + B\cos(0) => B = 0$

$x = L$: $A\sin(kL) = 0 => \sin(kL) = 0$ $\therefore kL = n\pi => k_n = \frac{n\pi}{L}$ } $\psi = A_n\sin(k_n x) = A_n\sin\left(\frac{n\pi}{L}x\right)$

$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}$

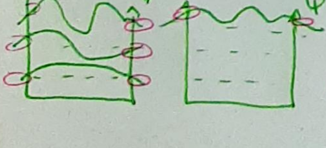


$E_2 = \frac{4\hbar^2\pi^2}{2mL^2}$, $E_1 = \frac{\hbar^2\pi^2}{2mL^2}$

UV catastrophe, Atomic model

$\boxed{\text{Quantisat}^{\circ}\text{ E levels}}$ $\Delta E = hf = \hbar\omega$

Allowed E Finite V: $E_n^{\infty} = \frac{\hbar^2\pi^2 n^2}{8ma^2}$



$a_{eff} > a$

$E^{V_0} < E^{\infty}$ as $V_0 \to \infty \to E^{V_0} \to E^{\infty}$

Normalisat° const

$\int_{-\infty}^{\infty}|\psi|^2 dx = 1$ + General solut° => $A_n = \sqrt{2/L}$

- odd n → even eigf° $\psi(-x) = \psi(x)$
- even n → odd eigf° $\psi(-x) = -\psi(x)$

$\psi_n = \sqrt{2/L}\sin(n\pi x/L)$

$E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$