$f(t+T) = f(t)$ $\forall t$

$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$

$= \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_0 t)$

- $\frac{a_0}{2} = c_0$ – average $f(t)$ over one period
- $n=1$: $\cos(\omega_0 t)$ – fundamental freq $\omega_0 = \frac{2\pi}{T}$
- $n>1$: $\cos(n\omega_0 t)$ – higher harmonics

$f(t) = \frac{1}{4} + \frac{1}{2}\sin(2t) + \frac{1}{6}\sin(6t) + \frac{1}{10}\sin(10t) + \ldots$

(average over one period; fundamental freq $\omega_0 = 2$ $(T=\pi)$; higher harmonics n.w. $6 = 3\cdot 2$, $10 = 5\cdot 2$, …)

Time ↔ Freq

$f(t) \rightarrow c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-in\omega_0 t}\, dt$

$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\omega_0 t}$ ← c_n

Coeffs: $a_n = 0$, $b_n = \frac{1}{n\pi a}$, ω_0

$c_n = \text{freq } \omega = n\omega_0$

$\sum_{n=1}^{\infty} \frac{1}{n\pi} \sin(2n\pi t)$

Spectra $= n\omega_0$

n	ω	c_n	$\lvert c_n\rvert$	$\arg(c_n)$

Phase: $\arg(c_n)$ vs ω

Amplitude: $\lvert c_n\rvert$ vs ω (unbounded: $\arg(0)$)

$c_n = \frac{-1}{i} = i = e^{i\frac{\pi}{2}}$, $\arg(i) = \frac{\pi}{2}$

$c_n = -1 = e^{i\pi}$, $\cos(-1) = \pi$

$c_n = -i \rightarrow \frac{3\pi}{2}$

Nyquist sampling criteria: $f_s \geq 2 f_{max}$ (sampling rate ≥ highest freq)

Aliasing: spurious low-freq components introduced within $-\omega_m > \omega > \omega_m$ when criterion not satisfied

Avoid:
- Find Nx freq
- Sampling rate = $2 \times f_{max}$
- $\Delta t = \frac{1}{\text{sampling rate}}$

Reminder – Freq: $\omega = \frac{2\pi}{\Delta t} = 2\pi f$ $[\text{rad s}^{-1}]$, $f = \frac{1}{\Delta t}$ $[\text{Hz}]$

Convolut° $f * g$ ← see eg.s° booklet

Special f°s

- Dirac comb: $\Delta_t(t) = \sum_{-\infty}^{\infty} \delta(t - \eta T)$ "same thing happens repeatedly"

$\omega_0 = 2\pi / T$

$\mathcal{F}[\Delta_T(t)] = \omega_0 \Delta_{\omega_0}(\omega) = \omega_0 \sum_{-\infty}^{\infty} \delta(\omega - n\omega_0)$

- Dirac Delta (Impulse) f° $\delta(t) = 0$ $\forall t \neq 0$ "smthing that happens once"

$\delta(t-a)$, $\delta(t) = \frac{dH(t)}{dt}$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

Sift property: $\int_{-\infty}^{\infty} f(t)\delta(t-a)\, dt = f(a)$

$\mathcal{F}[\delta(t-a)] = e^{-i\omega a}$

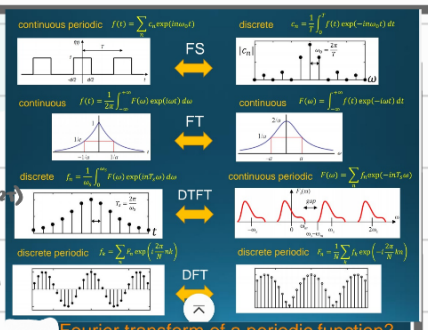
$\mathcal{F}^{-1}[e^{i\omega_0 t}] = 2\pi\delta(\omega - \omega_0)$

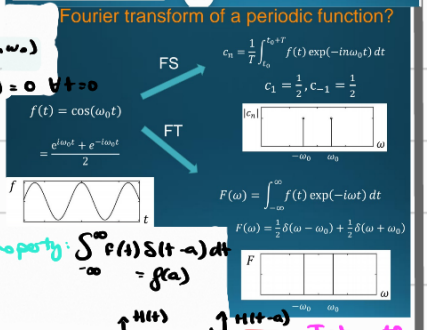
- Heavy side (Unit Step) f° $H(t)$

Integrat°: $\int ab\, dx = a\int b\, dx - \int a' (\int b\, dx)\, dx$

Int by parts: $\int uv\, dx = u\int v\, dx - \int u' (\int v\, dx)\, dx$

Div thm: $\int_S \vec{A}\, ds = \int_V \nabla \vec{A}\, dV$





MATRICES any matrix Σ $\begin{matrix}\text{Symm}\\ \text{anti-sym}\end{matrix}$

$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

$= \Sigma$ Hermitian 2 a-H

$= \frac{1}{2}(A+A^+) + \frac{1}{2}(A-A^+)$

Lo orthogonal matrix: $|A| = \pm 1$

Lo unitary $AA^+ = I$

Lo normal $AA^+ = A^+A$

$A^{-1}(A^{-1})^+ = A^{-1}(A^+)^{-1} = (A^+A)^{-1}$

Lo 2 matrices commute ∴ all their eigv

Lo 2 matrices = eig ∴ they comm

Bandwidth thm: width $F(\omega)$: $\Delta\omega \propto 1/\text{width } f(t)$, Δt

Hermitian conjugate: $A^+ = (A^*)^T = (A^T)^*$

$a^+ = (a_1^*, a_2^*, \ldots a_n^*)$

$a^+ b = \sum_{i=1}^{n} a_i^* b_i$

Properties
- Lo eigv real ⇒ observables are real
- Lo anti-H matrix eigv are imag
- Lo H ≥ a-H = normal matrix

$\langle ab \rangle = a^+ b = ab$

DETERMINANTS
$|A^T| = |A|$
$|A^+| = |A|^*$
$|A^{-1}| = |A|^{-1}$

INVERSE
$(A^{-1})^{-1} = A$
$(A^T)^{-1} = (A^{-1})^T$
$(A^+)^{-1} = (A^{-1})^+$

PRODUCT
$|AB| = |A||B| = |BA|$
$(AB)^T = A^T B^T$
$(AB)^+ = B^+ A^+$
$(AB)^{-1} = B^{-1} A^{-1}$
$(AB)^+ = B^+ A^+$

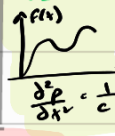
Temp distribut°

$D^2 T - \frac{\rho c}{k}\frac{\partial T}{\partial t} = -\frac{P(t)}{k}$

steady state: $\frac{\partial \rho}{\partial t} = 0$, $\frac{dT}{dt} = 0$

$\therefore D^2 \rho = \frac{-Q(\vec{r})}{D} \Rightarrow D^2 T = -P(\vec{r})/k$ = Poisson!

1D diffus°



$\frac{\partial^2 \rho}{\partial x^2} = \frac{1}{c}\frac{\partial \rho}{\partial t}$ $x=0$ $x=L$

Separation of variables method for solving PDEs:

You should be able to use the separation of variables method for solving linear homogeneous PDEs.

The procedure for this method is as follows:

1. Take the unknown function (say $u(x,t)$), and make the assumption that we can write this function as:
$$u(x,t) = X(x)T(t),$$
where $X(x)$ is a function of x only, and $T(t)$ is a function of t only.
2. Substitute this expression for $u(x,t)$ into the PDE, differentiating u as needed with respect to x and t. Note that this transforms the original PDE (containing partial derivatives of $u(x,t)$) into an ODE that contains ordinary derivatives of $X(x)$ and $T(t)$.
3. Separate the variables – i.e. move all functions that only depend on x onto one side of the equation, and all functions that only depend on t onto the other side.
4. One side of the equation now depends only on x and the other only on t. Since the equation has to be true for all values of x and t, the only way that this can be true is if both sides of the equation are equal to the same constant, known as the 'separation constant'.
5. Set both sides of the ODE to be equal to this separation constant. You need to think carefully about the sign for the separation constant (i.e. whether you set it to be $\pm k^2$). A physically meaningful solution will not shoot off to infinity within the region of interest, and will typically be oscillating/periodic and/or damped/decaying.
6. Solve each of the ODEs in x and t separately, to obtain expressions for $X(x)$ and $T(t)$.
7. Multiply these two expressions together to obtain a general solution for $u(x,t)$, using the principle of linear superposition to express it as a summation of all allowed values of the separation constant k.
8. Now apply the boundary conditions to find the required solutions.

Separat° of var ex

1D wave eq°: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

$u(x,t) = X(x)\,T(t)$

$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

$\frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$, $\frac{\partial^2 u}{\partial t^2} = X\frac{d^2T}{dt^2}$

$\Rightarrow \frac{\partial^2 X}{\partial x^2} - \frac{1}{c^2}X\frac{\partial^2 T}{\partial t^2} = 0$

$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2 T}\frac{d^2T}{dt^2} = -k^2$

Possible sols:

$\frac{d^2X}{dx^2} = -k^2 X$ $\quad$ $\frac{d^2T}{dt^2} = -k^2 c^2 T$

General sols:

$\begin{cases} X(x) = a_k e^{ikx} + b_k e^{-ikx} \\ T(t) = c_k e^{ikt} + d_k e^{-ikt} \end{cases}$

- L-HS: $T(x,y) = \sum_k B_k (C_k \sinh(ky) + D_k \cosh(ky))$
- Bottom: $T(x,y) = \sum_k A_k D_k \sin(kx) \cdot 0$
- RHS: $T(x,y) = \sum_k A_k (C_k \sin(k\alpha)\sinh(ky)$
- Top: $T(x,y) = \sum_n A_n C_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi b}{a}\right) f(x)$

Laplace's eq° $D^2 T = 0$ Laplace 2D

Separat° of: $T(x,y) = X(x)\,Y(y)$

$\Rightarrow \frac{1}{x}\frac{d^2x}{dx^2} = \frac{1}{y}\frac{d^2y}{dy^2} = k^2$

$\frac{d^2x}{dx^2} = -k^2 x \rightarrow x = A\sin(kx) + B\cos(kx)$

$\frac{d^2y}{dy^2} = k^2 y \Rightarrow y = C\sinh(ky) + D\cosh(kx)$

$T = \sum_k (A_k \sin(kx) + B\cos(kx))(C_k \sinh(ky) + D_k \cosh(ky))$

Solve by linearity:

Top A_n's $\rightarrow A_n$

① $T(x,y) = \sum_n A_n \sin(nx)\sinh(ny)$

we know $T(x,\pi) = \sin(2x)$

$= \sum A_n \sin(nx)\sinh(ny)$

$\therefore A_n = 0$ for $n \neq 2$ when $n=2$: $\begin{cases} 1 = A_2 \sinh(2\pi) \\ A_2 = 1/\sinh(2\pi) \end{cases}$

$T(x,y) = \sum_n A_n \sin(nx)\sinh(ny) = \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)}$

② by Analogy $x \leftrightarrow y$ noting $\sin(2x)$ as Top BC

$T(x,y) = \frac{\sin(y)\sinh(x)}{\sinh(\pi)}$

Full solut°: $T(x,y) = ① + ② = \frac{\sin(2x)\sinh(2y)}{\sinh(2\pi)} + \frac{\sin(y)\sinh(x)}{\sinh(\pi)}$

Diffus° chem species: Charge tot amount of species: $\Delta_m = \delta_c \int_V Q(\vec{r},t)\, dV$

Flux vector: $\vec{J}$

Charge due diffus°: $\Delta H_2 = \delta_t \int -\vec{J}\, ds = D\,\delta_t \int \nabla\rho\, ds$

tot charge: $\Delta H = H_1 + H_2$ we know $\Delta H = \delta_c \int \frac{\partial \rho}{\partial t} dV$ $\xrightarrow{\text{div thm}}$ $\int_V \frac{\partial \rho}{\partial t} dV = \int_V \vec{Q}(\vec{r},t)\, dV + D\int_V \nabla^2 \rho\, dV$

$\frac{\partial \rho}{\partial t} = Q(\vec{r},t) + D\nabla^2\rho \rightarrow \nabla^2\rho = \frac{1}{D}\frac{\partial \rho}{\partial t} = \frac{-Q(\vec{r},t)}{D}$

Homogeneous : all terms involve dependent var & / or it's derivs → Linear $L(a\psi + b\Phi) = aL(\psi) + bL(\Phi)$ → PDE linear if:

↑ arbitrary f° ↑ arb non-zero const

↳ dep var & derivs occur only 1st pwr

↳ no products dep var & / or it's deriv

↳ commutative $\sum L+M = M+L$

↳ associative $(L+M)+N = L+(M+N)$ & $(LM)N = L(MN)$

↳ distributive $L(c_1M + c_2N) = c_1LM + c_2LN$

↳ commutative × $LM = ML$

PDEs

$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y)$

elliptic $B^2 - 4AC < 0$ steady state

hyperbolic $B^2 - 4AC > 0$ waves

parabolic $B^2 - 4A = 0$ transf°

- Poisson $\nabla^2\Phi = -S(\vec{r})$
- Wave $\Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -S(\vec{r},t)$
- Helmholz $\nabla^2\Phi + \lambda\Phi = 0$
- Schrod $-\frac{\hbar^2}{2m}\nabla^2\Phi + V(\vec{r})\Phi = i\hbar\frac{\partial \Phi}{\partial t}$ $= \varepsilon\Phi$
- Laplace $\nabla^2\Phi = 0$
- Diffu° $\nabla^2\Phi - \frac{1}{k}\frac{\partial \Phi}{\partial t} = -\delta(\vec{r},t)$

Normalised f° $\hat{f} = f/\|f\|$

$L(u) = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$

$L(u) = G(x,y)$ = 0 homog, ≠0 inhom

$L\left\{\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0\right.$

$L\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right.$

$NL\left\{i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + k|\psi^2|\psi\right.$

$NL\left\{u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 = u^2\right.$

Divergent $S_n \to \infty$ as $n \to \infty$

Convergent $S = \lim_{n\to\infty} S_n$

Oscillatory partial sums oscillate but never converge

Test → $\lim_{n\to\infty} u_n \neq 0$: div

ratio test $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)$ $\begin{cases} <1 \to \text{conv} \\ <1 \to \text{div} \\ =1 \to \text{undet} \end{cases}$

bigger series converges ⇒ so does smaller

smaller series conv ⇒ … bigger

$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^a = 1$

Solving ODE's

Homogeneous $0 = \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y$

2 solut's comb → $y(x) = c_1y_1(x) + c_2y_2(x)$

- $P(x)=0$, $Q(x)=\omega^2$ → $\frac{d^2y}{dx^2} + \omega^2 y = 0$
- $P(x)=1/x$, $Q(x) = 1 - \frac{n^2}{x^2}$ → $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$
- Inhomogeneous $F(x) = \ldots$

y_p = sol ∴ General sol → $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$

$\frac{dy}{dx} = \pm ky \quad \Rightarrow \quad y = A\exp(\pm kx)$

$\frac{d^2y}{dx^2} = k^2y \quad \Rightarrow \quad y = A\exp(kx) + B\exp(-kx) \quad OR \quad y = C\cosh(kx) + D\sinh(kx)$

$\frac{d^2y}{dx^2} = -k^2y \quad \Rightarrow \quad y = A\exp(ikx) + B\exp(-ikx) \quad OR \quad y = C\cos(kx) + D\sin(kx)$

Points → ordinary P2Q finite at $x \to x_0$

↓ Singular P2Q div as $x \to x_0$

→ regular $(x-x_0)P$ & $(x-x_0)^2Q$ remain finite as $x \to x_0$

→ irregular … approaches ∞ ← expans° may fail

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

n° solut°s of indicial eqt°s $\begin{matrix} \nearrow q_1 \\ \searrow q_2 \end{matrix}$

- $q_1 = q_2$ → 1 sol
- $q_1 - q_2$ = int → larger q will give a smaller may not
- $q_1 - q_2 \neq$ int → 2 sols

Spherical harmonics

solut°: $\Psi = R \otimes \Phi$

$= R(r)\,Y(\theta,\phi)$

$Y_l^m = (-1)^m\left[\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)\,e$

Boundary Condit's - Wave eqt° 1D

$L(\phi) = 0$

$L = \frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c}\frac{\partial^2\phi}{\partial t} = 0$

GS: $\phi(x,t) = \sum_k (A_k e^{i(kx - kct)} + B_k e^{i(kx + kct)})$

BC: $\phi(x,0) = \sin(3x) \;\&\; 2\frac{\partial\phi}{\partial t}(x,0) = 0$

$\Rightarrow \phi(x,0) = \sum_k [-ikc\,A_k e^{ikx} + ikc\,B_k e^{-ikx}]$

$= \sum_k -ikc(A_k - B_k)e^{ikx}$

$\phi(x,t) = \sum_k A_k e^{ikx}(e^{-ikct} + e^{ikct})$

$= \sum_k 2A_k e^{ikx}\cos(kct)$

BC → $\phi(x,0) = \sum_k 2A_k e^{ikx} = \sin 3x$

$= \frac{e^{i3x}}{2i} - \frac{e^{-i3x}}{2i}$

$\begin{cases} 2A_3 = 1/2i \\ 2A_{-3} = -1/2i \\ \text{other } A_n = 0 \end{cases}$ $\Rightarrow \phi(x,t) = \sum_k 2A_n e^{ikx}\cos(kct)$

$= \sin(3x)\cos(3ct)$

Circle

Potential sphere surface $\Phi(a,\theta,\phi) = V_0\cos(2\theta)$ maintained at

GS Laplace spherical $\Phi(r,\theta,\phi) = (Ar^l + Br^{-(l+1)})P_l^m(\cos\theta)\cdot(C\cos(m\phi) + D\sin(m\phi))$

Potential only depends on θ (not ϕ)

$\Rightarrow$ GS: $\Phi(r,\theta) = (Ar^l + Br^{-(l+1)})P_l(\cos\theta)$ with $m = 0$

Reg° r.c.a include reg° as $r \to 0$ ∴ $B = 0$ (otherwise divergence)

Linear superposit° $\Rightarrow \Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

BC $r=a$ → $\Phi(a,\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = V_0\cos(2\theta) = V_0(2\cos^2\theta - 1)$

so $A_0a^0P_0(\cos\theta) + A_1a^1P_1(\cos\theta) + A_2a^2P_2(\cos\theta) + \ldots = V_0(2\cos^2\theta - 1)$

highest pwr $\cos\theta$ in the express° $P_l(\cos\theta)$ is l. Comparing the coeffs of pwrs of $\cos\theta$ on each side of this eqt° ∴ $A_l = 0$ for $l \geq 3$.

so $A_0P_0(\cos\theta) + A_1aP_1(\cos\theta) + A_2a^2P_2(\cos\theta) \equiv V_0(2\cos^2\theta - 1)$

$A_0 + A_1a\cos\theta + A_2a^2(\frac{3}{2}(\cos\theta)^2 - \frac{1}{2}) \equiv V_0(2\cos^2\theta - 1)$ ← Legendre Polynom Formula book p32

Comparing coeffs:

$\cos^2\theta$: $\frac{3}{2}A_2a^2 = 2V_0 \Rightarrow A_2 = \frac{4V_0}{3a^2}$

$\cos\theta$: $A_1a = 0 \Rightarrow A_1 = 0$

const: $A_0 - \frac{1}{2}A_2a^2 = -V_0 \Rightarrow A_0 = \frac{1}{2}A_2a^2 - V_0 = \frac{-V_0}{3}$

we can now substitute these coeffs into our express° for $\Phi(r,\theta)$

$\Phi(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

$= A_0r^0P_0(\cos\theta) + A_1r^1P_1(\cos\theta) + A_2r^2P_2(\cos\theta)$

$= -\frac{V_0}{3} + 4V_0(r^2/a^2)P_2(\cos\theta)$

$\Phi(r,\theta) = V_0\left[-\frac{1}{3} + \left(\frac{2r^2}{a^2}\right)(3\cos^2\theta - 1)\right]$

Displacement profile:

t=0

$u(x,0) = \sum_n A_n\sin(nx) = \frac{2}{\pi}\begin{cases} P_0 : 0 \leq x \leq \frac{\pi}{2} \\ P(\pi - x) : \frac{\pi}{2} \leq x \leq \pi \end{cases}$ (x=0, x=π/2, x=π)

$A_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)f(x)\,dx$

symmetry $f(x)$ → profile of spring t=0

integral will be non-zero for odd values

$A_n = \frac{2}{\pi}\cdot 2\int_0^{\pi/2}\sin(nx)\left[\frac{2P_0}{\pi}\right]dx$

integral restricted to $0 \leq x \leq \pi/2$; n odd only

$I = \int_0^{\pi/2} x\sin(nx)\,dx$ Integ by parts → $I = \frac{1}{n^2}[\sin(nx)]_0^{\pi/2}$

How $[\sin(nx)]_0^{\pi/2} = 1$ for $n = 1, 5, 9, \ldots$

$[\sin(nx)]_0^{\pi/2} = -1$ for $n = 3, 7, 11, 15, \ldots$

$[\sin(nx)]_0^{\pi/2} = 0$ for even values of n

general express°: $[\sin(nx)]_0^{\pi/2} = (-1)^m$, $m = \frac{n-1}{2}$ always odd!

$I = \frac{(-1)^m}{n^2}$ → $A_n = \frac{2}{\pi}\cdot 2\frac{2P}{\pi}\frac{(-1)^m}{n^2} = \frac{8P}{\pi^2}\frac{(-1)^m}{(2m+1)^2}$

Power series solution method for solving ODEs:

You should be able to use the power series solution method for solving 1st or 2nd order ODEs.

The procedure for this method is as follows (taking a 2nd order homogeneous ODE as an example):

1. Write the ODE in the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, where $P(x)$ and $Q(x)$ are arbitrary functions of x. $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + by = 0$
2. Make the assumption that we can express $y(x)$ in the form: $y(x) = \sum_{s=0}^{\infty} a_s x^{s+q}$, where q is chosen so that $a_0 \neq 0$. $\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n(n+q)x^{n+q-1}$ $\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n(n+q)(n+q-1)x^{n+q-2}$
3. Substitute this expression for $y(x)$ into the ODE, differentiating it as needed. $\sum_{n=0} a_n(n+q)(n+q-1)x^{n+q-2} - \sum_{n=0} a_n(n+q)x^{n+q} + b\sum_{n=0} a_n x^{n+q} = 0$
4. Since the resulting equation has to be true for all values of x, the coefficients of each power of x on each side of the equation have to be equal to each other. In other words, since we've set the RHS of the ODE to be equal to zero, the coefficient of each and every power of x on the LHS of the equation must also be equal to zero. $q-2$: $a_0(q)(q-1) = 0$ (lowest pwr of x: n=0 → x^{q-2})
5. Start with the lowest possible power of x on the LHS of the equation, and set the coefficient of this power of x to be equal to zero. The resulting equation is known as the *indicial equation*. Solving this equation gives the allowed values for q. So $q=0$ or $q=1$ $a_0 \neq 0$
6. Now carry out the same procedure for the next lowest power of x. This will give an expression for a_1. Repeat as need be for increasing powers of x.
7. Now consider the general power for x, i.e. x^{s+q}, and set coefficient of this power of x to be equal to zero. In this step (as in steps 5 and 6, you'll need to consider which values of s in which series will give you the power in question). $q=0 \Rightarrow \sum_n a_n n(n-1)x^{n-2} - \sum_n a_n n x^n + b\sum_n a_n x^n = 0$ (second lowest is x^{n-1}) $q+1+0 = 0 \to a_1$ is undef $\therefore a_1 = 0$
8. All being well, step 7 will lead you to the recurrence relation, which defines the sequence relating the coefficient a_i to one or more preceding or following coefficients.
9. The recurrence relation will generally be a function of q. The values of q already found by solving the indicial equation can therefore be substituted into the recurrence relation, to give two recurrence relations, one for each value of q. General pwr x^k occurs n=k+2 in the 1st series & n=k in second
10. You can then use these recurrence relations, along with the expression for $y(x)$: $y(x) = \sum_{s=0}^{\infty} a_s x^{s+q}$ to build up the power series solutions for each value of q. Finite polynomials $\frac{a_{n+2}}{a_n} = 0$
11. All being well, this will result in two distinct solutions to the original 2nd order ODE.
12. To check whether these solutions are physically valid, you need to check whether the power series solutions are convergent.
13. You should be able to define ordinary and singular (both regular and irregular) points, and to relate this to the number of solutions to the ODE found using the power series method.
14. You should also be able to recognise (from the recurrence relation) the instances when the power series solutions become finite polynomials, and you should be able to write down expressions for these polynomials.

3.2. Linear Superposition

Recall the general 2nd order linear PDE for two independent variables x and y, and dependent variable u:

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y) \quad (3.1)$$

This can be rewritten in the form

$$L(u) = G(x,y), \quad (3.2)$$

where $G=0$ ∴ homog, $G \neq 0$ ∴ inh

$$L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F. \quad (3.3)$$

Suppose that v is a particular solution of the inhomogeneous linear equation (3.2) (i.e. when $G \neq 0$), and that w is a particular solution of the associated homogeneous equation (i.e. when $G = 0$).

Then

$$L(v) = G(x,y) \quad (3.4)$$

and

$$L(w) = 0 \quad (3.5)$$

Therefore, if $u = v + w$, and adding equations (3.4) and (3.5):

$$L(u) = L(v+w) = L(v) + L(w) = G(x,y) + 0 = G(x,y) \quad (3.6)$$

In other words, if v is a particular solution of a linear equation, and w is a particular solution of the associated homogeneous linear equation, then $u = v + w$ is also a solution of the equation.

More generally, equation (3.5) can have many solutions (often infinitely many solutions), which we can write as w_k. So we can write the solution to equation (3.1) as

$$u = v + \sum_k c_k w_k \quad (3.7)$$

where $c_1, c_2, c_3, \ldots$ are constant coefficients.

To obtain a complete solution, we then have to apply the relevant boundary conditions to fix the values of the coefficients c_k.