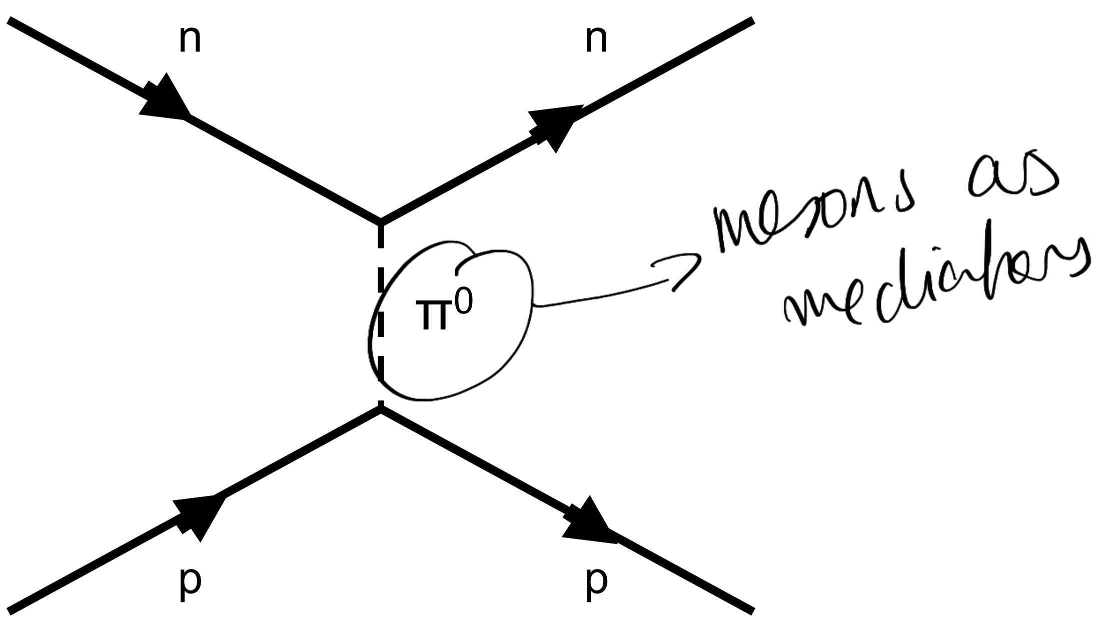
Binding and Decay

2.1. Nucleon–nucleon interactions

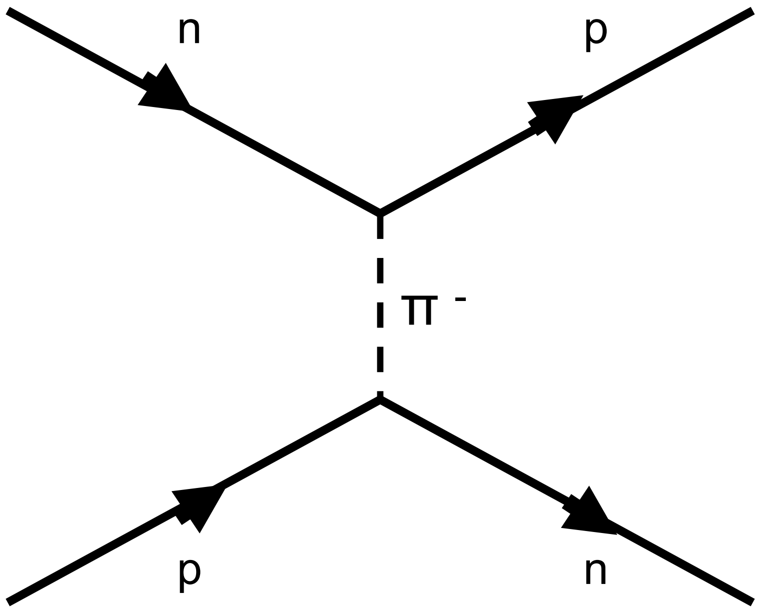
Protons and neutrons are held together by the strong force

but exchange particles are mesons, not gluons

For example:



It could also be:

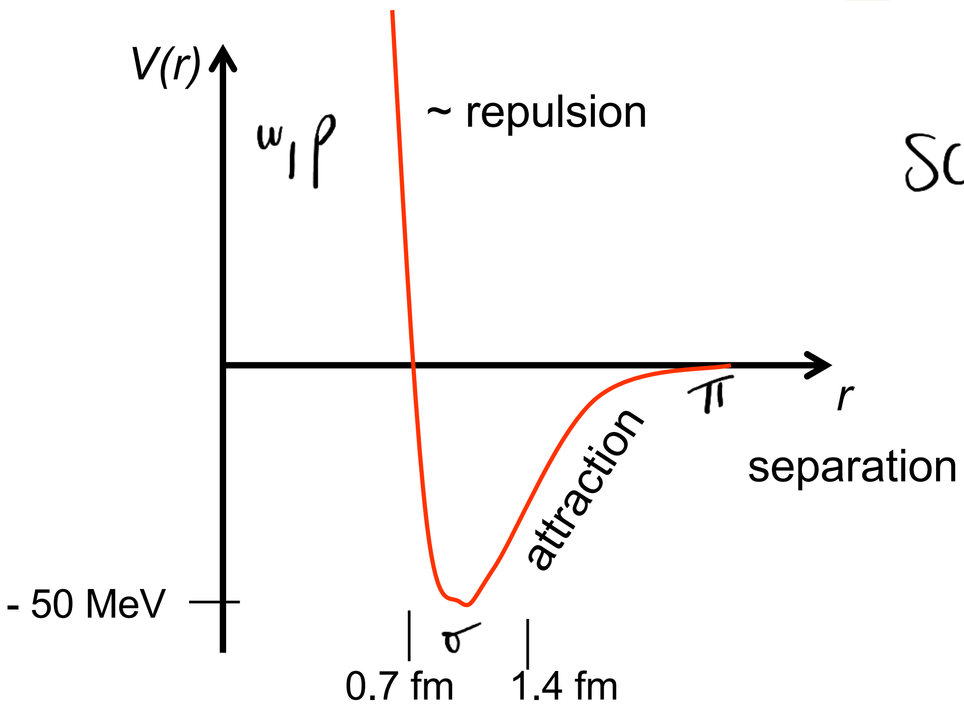


In both cases, the range is ~ $\dfrac{\hbar}{2\, m_0\, c} \sim c\,\Delta t \sim \dfrac{\hbar c}{2 E_0}$

Longest range comes from lightest meson (pion) ⇒ range ~ 1.4 fm

This gives a nucleon-nucleon potential like:

screened potential



p – p

n – p

n – n

STRONG INTERACTION
SIMILAR IN STRENGTH

(Coulomb forces are relatively weak on this scale)

The potential also depends on spin. $p \propto S$

<u>Observation:</u> a bound *n-p* state exists (deuteron).

But bound *p-p* and *n-n* states do not exist. **WHY ???**

This stems from the Pauli Exclusion Principle:
there cannot be 2 identical fermions in each quantum state

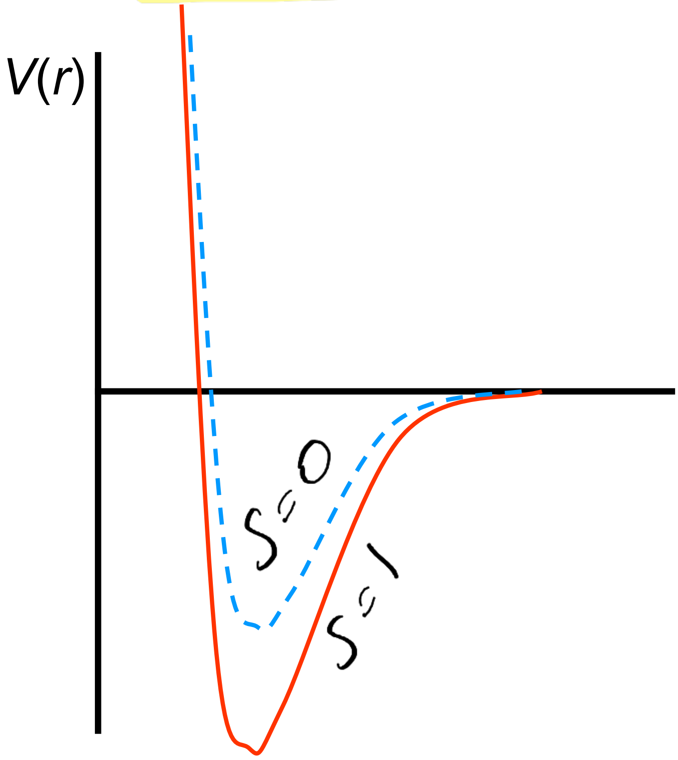
2 possibilites

	spin ↑		spin ↓	
	p-p	n-n	n-p	n-p
	↑ ↓	↓ ↑	↑ ↓	↑ ↑
net spin = S =	0	0	0	1

The {S = 0} potential well is not deep enough
to form a bound state.

But the {S = 1} potential well is deeper.

The {S = 0} potential well (– –) is not deep enough to form a bound state.



But the {S = 1} potential well is deeper.

The deuteron ${}_{1}^{2}H$ can exist, but *n*-*n* or *p*-*p* cannot.

The binding energy of a deuteron is ~ 2.2 MeV.

2.2. Nuclear size

Adding protons and neutrons produces heavier nuclei.

These nuclei are spherical (experimental evidence).

But they are not point objects.

How big are they ?

We can probe the mass (charge) distribution with scattering experiments firing a beam of high-energy electrons (for probing charge distribution) or high-energy nucleons (for probing mass distribution) at a tin foil composed of the nuclei whose structure you want to study → measure energies and deflection angles of outgoing particles → these observations can be inverted to yield the nuclear mass (charge) distribution

For that, we need:

$$\lambda_{\substack{\text{de Broglie} \\ (\text{for } e^-)}} \approx \text{nuclear size}$$

$$\lambda = \frac{2\pi\hbar}{p}$$

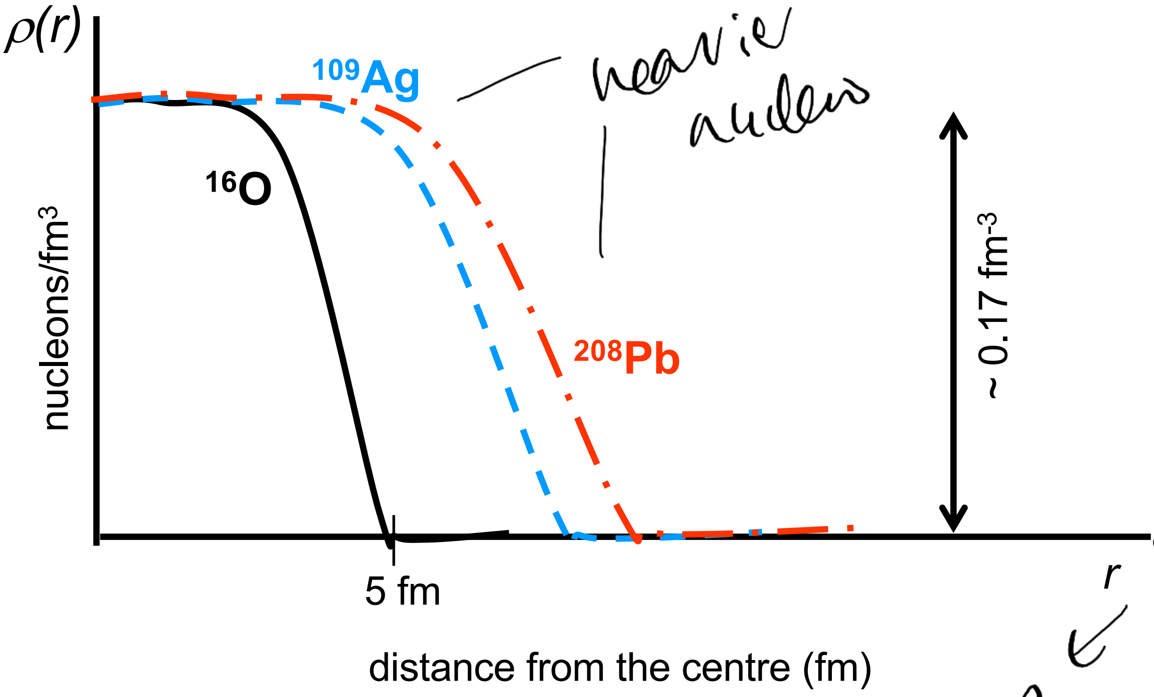
momentum

$$E^2 = p^2c^2 + m_0^2c^4$$

$$\Rightarrow \quad \underbrace{\lambda \leq 1\ \text{fm}}_{\text{resolution at which}} \rightarrow \underbrace{E \geq 100\ \text{MeV}}_{\text{need to accelerate } e^- \text{ to } E\ 100\,\text{MeV}}$$

map the density distribution

After inversion, the experimental results show:



heavier nuclei

total # nucleons

$^{A}_{Z}X$

protons

For all but light nuclei ($A \leq 12$):

- reasonably well defined radius
- narrow surface region (~ 2.5 fm)
- constant density of nucleons near the centre for all nuclei

$A = N + Z$

cst Z isotopes

cst A isobars

cst N isotones

Hence: R_{nuc} is related to A, since:

No. of nucleons = A = $(4\pi/3)\, R_{nuc}^{3} \times$ nucleon density

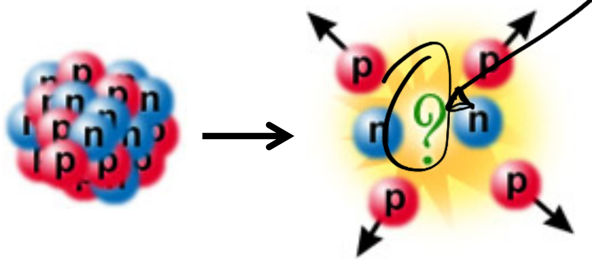
and: $\boxed{R_{nuc} = 1.1\, A^{1/3} \text{ fm}}$

(this shows that the force has a short range)

2.3. Binding energy of nuclei

The binding energy of a nucleus is the energy required to split it into its elementary constituents:

$$ {}^{A}_{Z}X \rightarrow Z\ p + N\ n $$



It is also the energy released
when free protons and neutrons form a nucleus

$B(Z,N) \geq 0$ by definition ← always +

For easier comparisons, we often use:

$$ \frac{B(Z,N)}{A} $$ = binding energy per nucleon

To calculate it, we can use the mass of the nucleus.

Using the nuclear mass:

$$m_{nuc}(Z,N) = Z\,m_p + N\,m_n - \frac{B(Z,N)}{c^2}$$

mass of the proton

mass of the neutron

But atomic masses are easier to measure in the laboratory.

Therefore, we use:

$$m_{atom}(Z,N) = Z\left(m_p + m_e\right) + N\,m_n - \frac{B(Z,N)}{c^2} - \frac{b(e^-)}{c^2}$$

binding energy of electrons
(usually small compared with *B(Z,N)*)

negligeble (very small)

The binding energies of light nuclei:

- increase to ~ 8MeV/nucleon
- see the binding energy of the last nucleon fluctuate
- are large for even-even nuclei

BINDING ENERGIES OF LIGHT NUCLEI

(low mass nuclei)

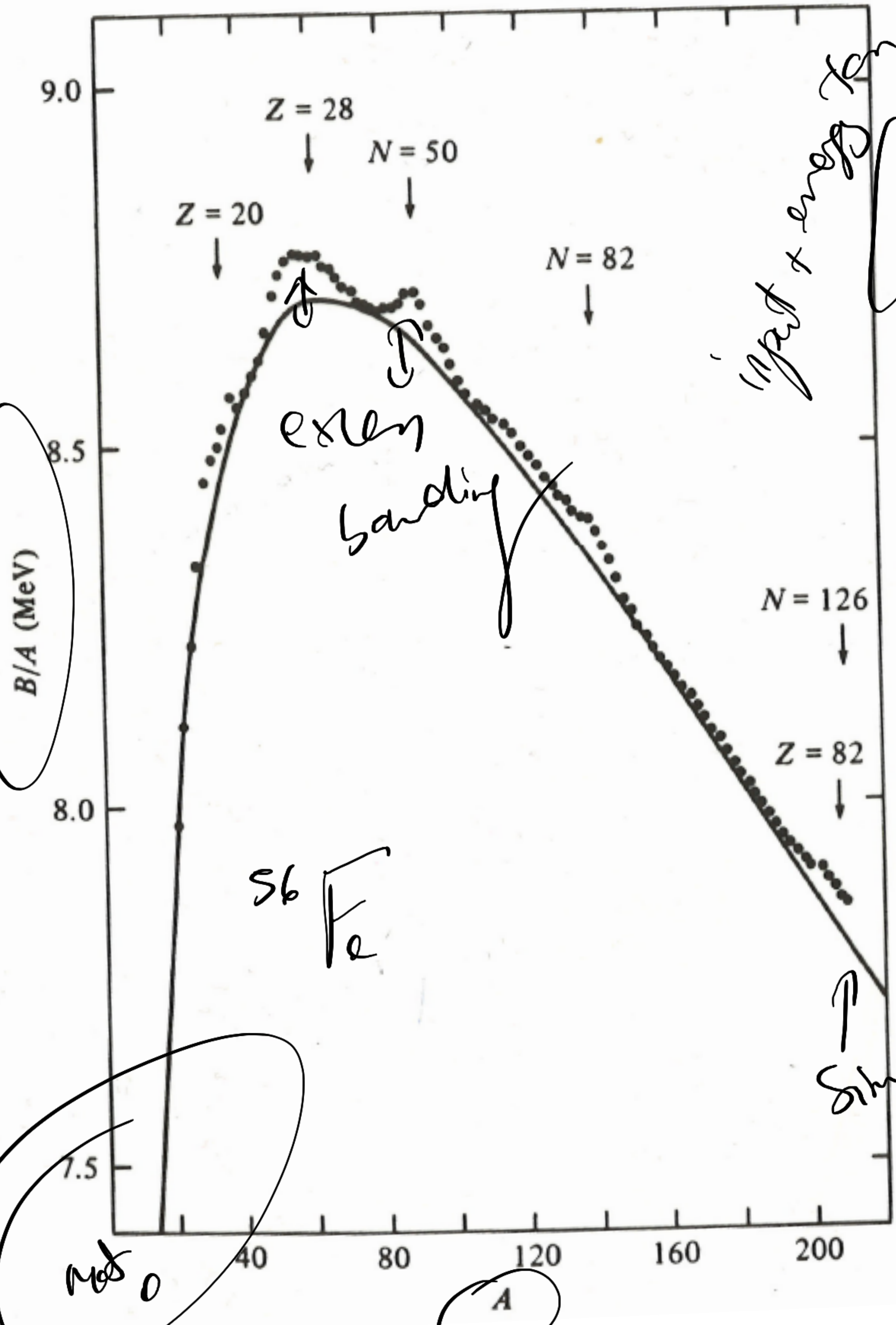
Nucleus	Binding energy (MeV)	Binding energy of last nucleon (MeV)	Binding energy per nucleon (MeV)	Spin
$^{2}_{1}H$	2.22	2.2	1.1	1
$^{3}_{1}H$	8.48	6.3	2.8	1/2
$^{4}_{2}He$	28.30	19.8	7.1	0
$^{5}_{2}He$	27.34	-1.0	5.5	3/2
$^{6}_{3}Li$	31.99	4.7	5.3	1
$^{7}_{3}Li$	39.25	7.3	5.6	3/2
$^{8}_{4}Be$	56.50	17.3	7.1	0
$^{9}_{4}Be$	58.16	1.7	6.5	3/2
$^{10}_{5}B$	64.75	6.6	6.5	3
$^{11}_{5}B$	76.21	11.5	6.9	3/2
$^{12}_{6}C$	92.16	16.0	7.7	0
$^{13}_{6}C$	97.11	5.0	7.5	1/2
$^{14}_{7}N$	104.66	7.6	7.5	1
$^{15}_{7}N$	115.49	10.8	7.7	1/2
$^{16}_{8}O$	127.62	12.1	8.0	0
$^{17}_{8}O$	131.76	4.1	7.8	5/2

unstable

half life

energetically unfavourable to break things up

4.7 The binding energy per nucleon of β-stable (odd-A) nuclei. Note the displaced origin. The smooth curve is from the semi-empirical mass formula with Z related to A by equation (4.14). Experimental values for odd-A nuclei are shown for comparison; the main deviations ($<1\%$) are due to 'shell' effects not included in our formula.



extra binding

^{56}Fe

not 0

input x energy for obtained

V lose breaking, but U want E by losing

simple model predicts

~see liquid drop model

2.4. Radioactive decay

Nuclei will decay if the rest mass of the nucleus is larger than the total rest mass of the products.

The energy released (Q) is calculated using: $Q > 0$

$$Q = \left(\begin{matrix} \text{mass of} \\ \text{nucleus} \end{matrix} - \begin{matrix} \text{mass of} \\ \text{products} \end{matrix} \right) \times c^2$$

always (+)

→ released as Kinetic Energy of the products

→ released as heat

The main decay modes are:

- β-decay
- α-decay
- fission
- γ-decay

Each has its own energetics and rates of decay.

2.4.1. β-decay

The basic reactions correspond to:

(a) $n \rightarrow p + e^- + \bar{\nu}_e$ **β^- decay: $Z \rightarrow Z+1$**

(b) $p \rightarrow n + e^+ + \nu_e$ **β^+ decay: $Z \rightarrow Z-1$**

(c) $p + e^- \rightarrow n + \nu_e$ **K-capture: $Z \rightarrow Z-1$**

atomic electron : does not come from free space was already bound

$L_e = 1$ $L_e = 1$

Note that:

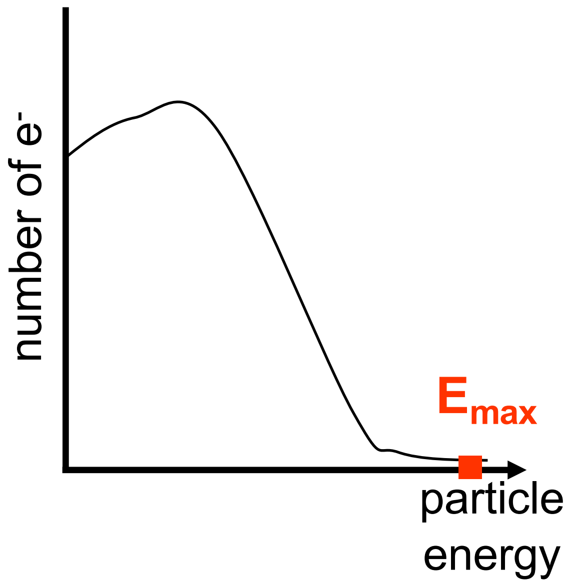
i. All are mediated by the weak interaction (ν's involved) neutrinos

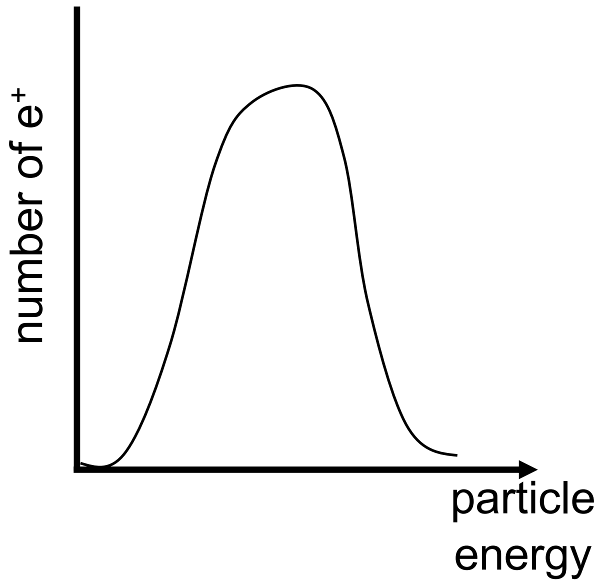
ii. The mass number A is conserved

iii. Only (a) can occur in free space;

(a), (b) and (c) are all observed in nuclei

iv. In (a) and (b), one can observe the energy of emitted e^- (e^+)



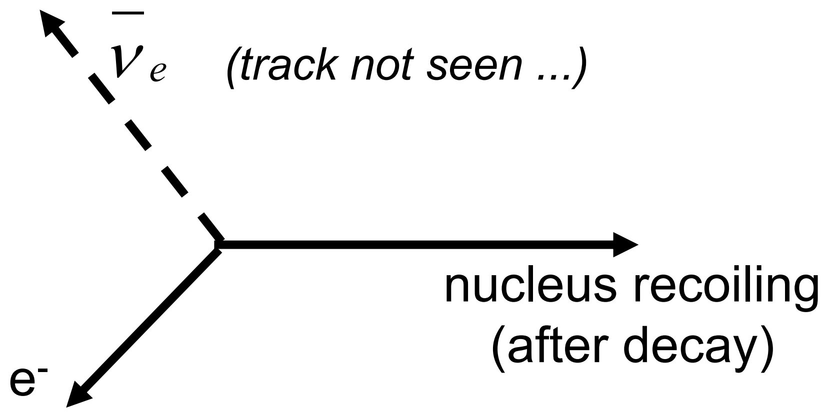


KE of e^-

The energy released in each reaction is fixed.

It is shared between e^- (e^+) and $\overline{\nu}_e$ (ν_e)

This was behind the discovery of the neutrino ...



Energies of β-decay

(a) β⁻–decay

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}Y + e^{-} + \overline{\nu}_{e} $$

For this reaction to occur, we need:

$$ m_{nuc}(A,Z) > m_{nuc}(A,Z+1) + m_e $$

Adding Z m_e to each side, we find:

$$ m_{atom}(A,Z) > m_{atom}(A,Z+1) $$

(ignoring the difference in electron binding energies)

The energy released is:

$$ \boxed{Q = \left(m_{atom}(A,Z) - m_{atom}(A,Z+1)\right) \times c^2} $$

Example: $ {}^{40}_{19}K \rightarrow {}^{40}_{20}Ca + e^{-} + \overline{\nu}_{e} $

$$ 39.9640 \rightarrow 39.9626 \text{ amu} $$

The difference corresponds to 0.0014 u ∴ Q = 1.32 MeV

$t_{1/2} \cong 10^9$ years

Energies of β-decay

(b) β⁺–decay

$$ {}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}Y + e^{+} + \nu_e $$

For this reaction to occur, we need:

$$ m_{nuc}(A,Z) > m_{nuc}(A,Z-1) + m_e $$

Adding Z m_e to each side, we find:

$$ m_{atom}(A,Z) > m_{atom}(A,Z-1) + 2\,m_e $$

(ignoring the difference in electron binding energies)

The term 2 m_e (=0.0011 amu) is a "penalty".

Example: $$ {}^{13}_{7}N \rightarrow {}^{13}_{6}C + e^{+} + \nu_e $$

$$ 13.0057 \rightarrow 13.0034 \text{ amu} $$

The difference is 0.0023 u > 2 m_e

This means the decay is possible *(it will have a $t_{1/2} \cong 10$ minutes)*

And the energy released is: Q = (0.0023 – 0.0011) × c^2

Energies of β-decay

(c) K–capture

$$ {}^{A}_{Z}X + e^{-} \rightarrow {}^{A}_{Z-1}Y + \nu_e $$

For this reaction to occur, we need:

$$ m_{nuc}(A,Z) + m_e > m_{nuc}(A,Z-1) $$

Adding (Z – 1) m_e to each side, we find:

$$ m_{atom}(A,Z) > m_{atom}(A,Z-1) $$

Example: $$ {}^{109}_{48}Cd + e^{-} \rightarrow {}^{109}_{47}Ag + \nu_e $$

$$ 108.9049 \rightarrow 108.9047 \text{ amu} $$

The difference is 0.0002 u

This decay can occur by K-capture.

But because Q < 2 m_e, it cannot occur by β^+ decay.

For two neighbouring isobars
(i.e. same atomic mass but different atomic numbers)

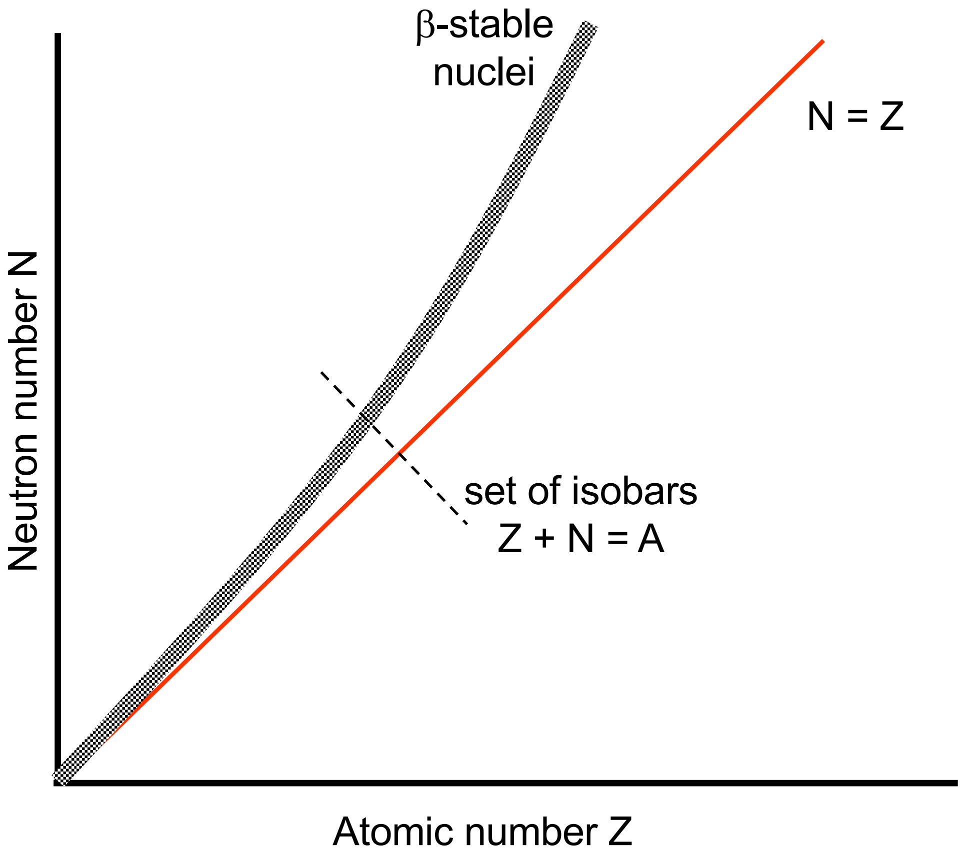
The heavier atom can always β-decay into the lighter.

⇒ For any A, β-decay tends toward the most stable neutron-proton ratio

For A small, $N \approx Z$

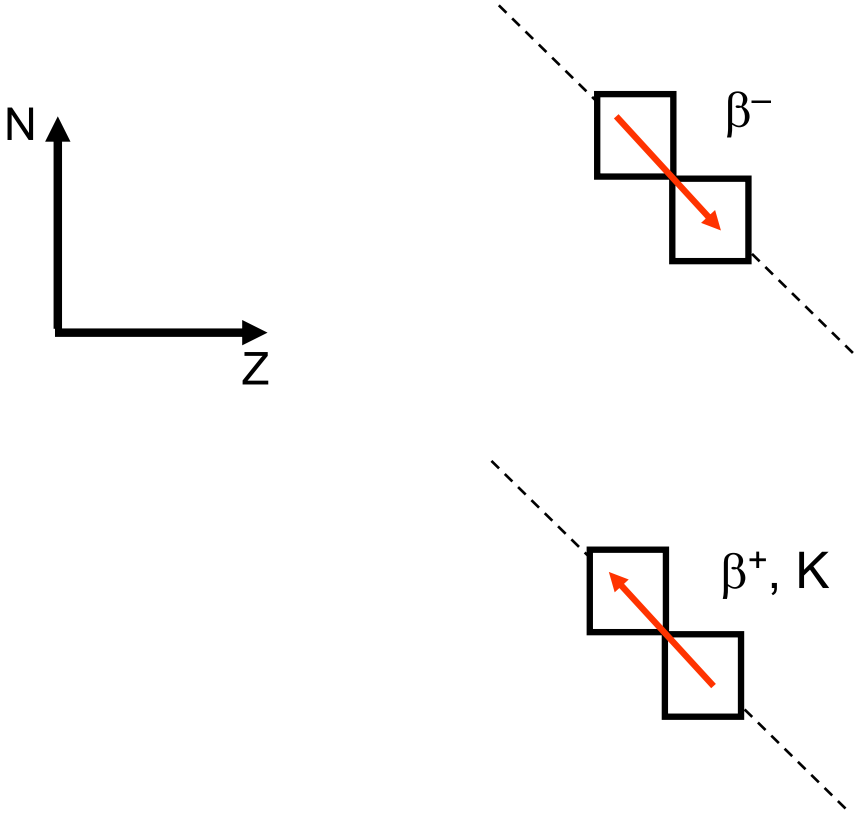
For A large, $N > Z$

This creates a “valley” of β-stable nuclei.



Obvious but worth remembering: these are nuclei.
Z and N take discrete values.

β-decay is a movement along the isobars



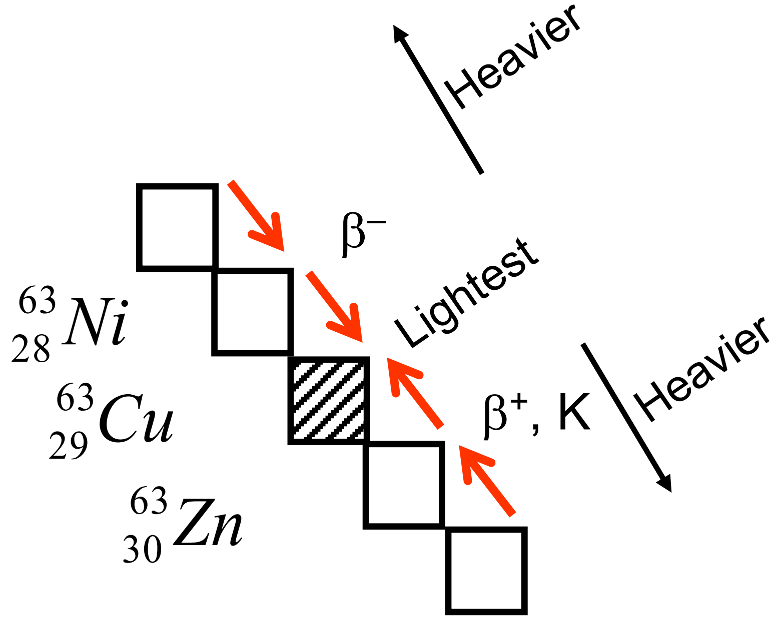
For two neighbours, one of the above is always allowed

(but it may take a long time to occur)

We could expect only 1 stable isobar for each value of A.

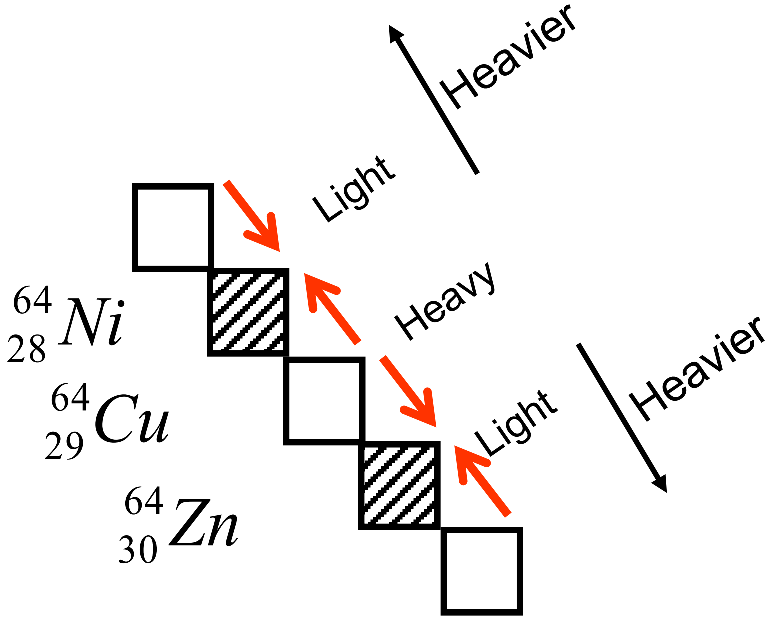
This is true for A odd

Example: A = 63



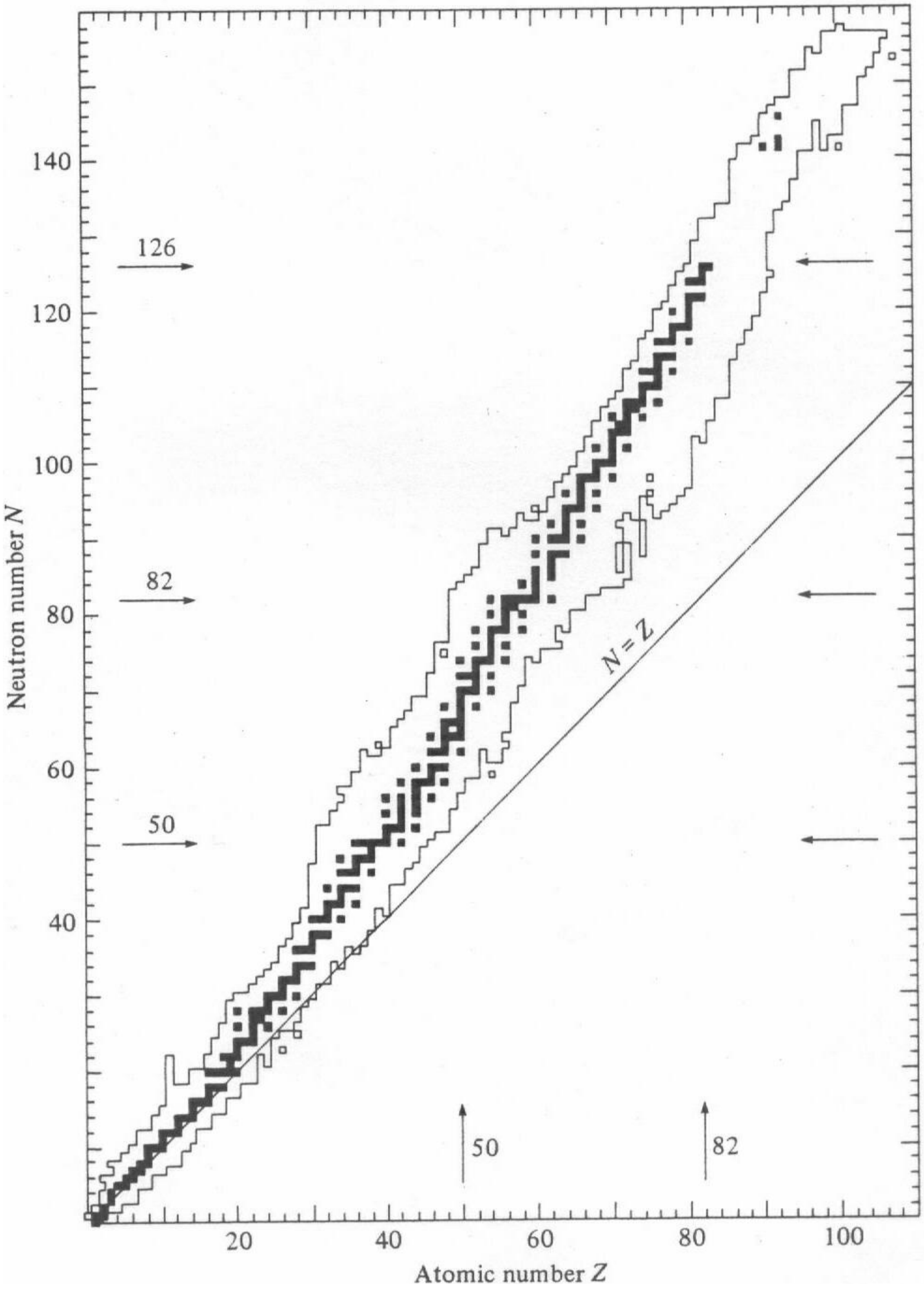
For A even, this may not be true (because of pairing: even-even combinations are more stable)

Example: A = 64



Two stable isobars
One isobar may decay both ways

4.6 The β-stability valley. Filled squares denote the stable nuclei and long-lived nuclei occurring in nature. Neighbouring nuclei are unstable. Those for which data on masses and mean lives are known fill the area bounded by the lines. For the most part these unstable nuclei have been made artificially. (Data taken from *Chart of the Nuclides* (1977), Schenectady: General Electric Company.)



2.4.2. α–decay

The basic reaction corresponds to:

$$^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}Y + {}^{4}_{2}He$$

Parent → Daughter + α–particle

Energy released Q equals:

$$Q = \left[m_{nuc}(A,Z) - m_{nuc}(A-4,Z-2) - m_{nuc}(He)\right] \times c^2$$

$$Q = B(A-4,Z-2) + B(He) - B(A,Z)$$

$$\Rightarrow$$

$$Q = B(A-4,Z-2) - B(A,Z) + 28.3\ \text{MeV}$$

For α-decay to be possible, we need $Q > 0$

This is only satisfied in the high *A* region.

<u>We should</u> find $Q > 0$ for $Z \geq 70$

But … low-Q α-decays are too slow to be observed …

So, <u>in practice</u>, we need Z > 83

(close to Bi in periodic table)

All heavier nuclei are unstable

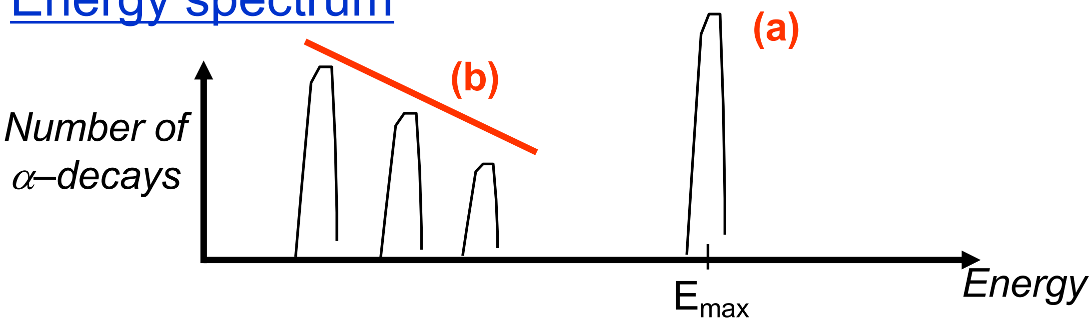
(some have very large $t_{½}$)

Because A → A–4, there are 4 decay series, based on:

$$\underbrace{{}^{235}_{92}U \qquad {}^{238}_{92}U \qquad {}^{232}_{90}Th}_{\textit{still observed}} \qquad \underset{\textit{extinct}}{{}^{237}_{93}Np}$$

(β–decay leaves *A* unchanged)

Energy spectrum



Small number of sharp peaks ⇒ single particle emission

(a) Daughter left in in ground state
i.e. all Q becomes K.E.

(b) Daughter nucleus left in excited state

We need to find
$$Q = E_{max} \times \left(\frac{m_\alpha}{m_{daughter}} + 1 \right)$$
(because of the recoil of the nucleus)
(neglected for β–decay: $m_e \ll m_d$)

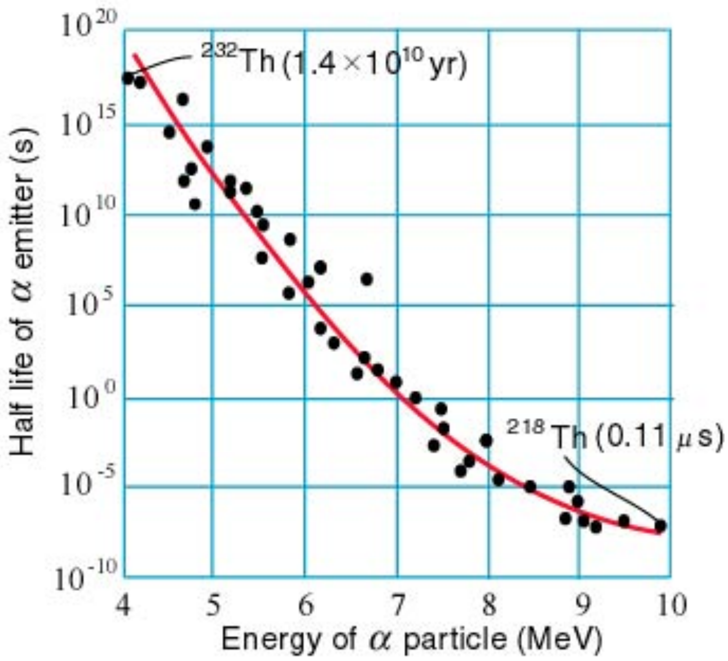
Half-lives

Q 4 MeV ⟶ 9 MeV
$t_{1/2}$ 10^{17} s ⟶ 10^{-7} s

Strong correlation observed between Q and $t_{1/2}$ …

not examinable material

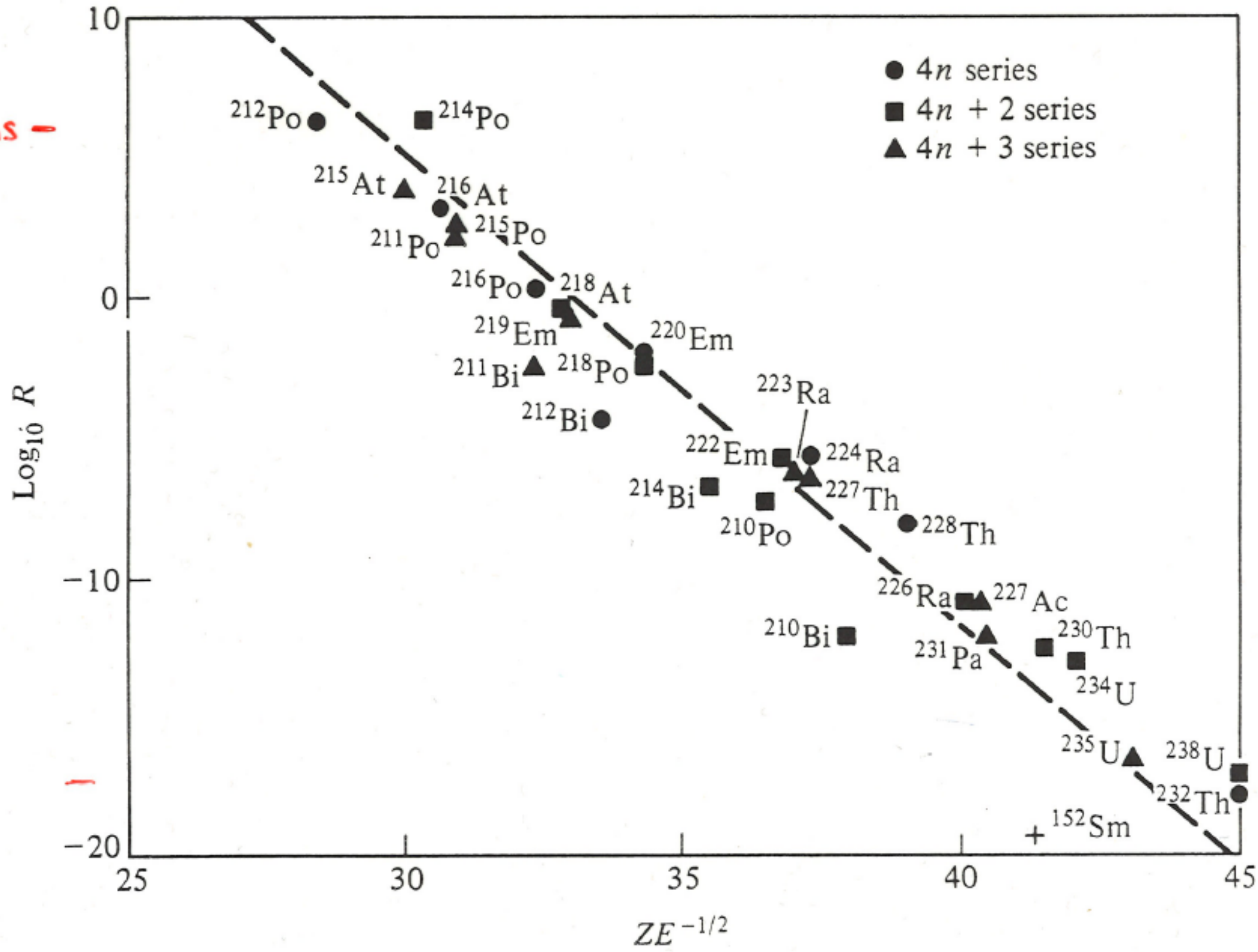
Example:
decay of the *Th* series



t 1/2

0.1ms –

10^17 s –



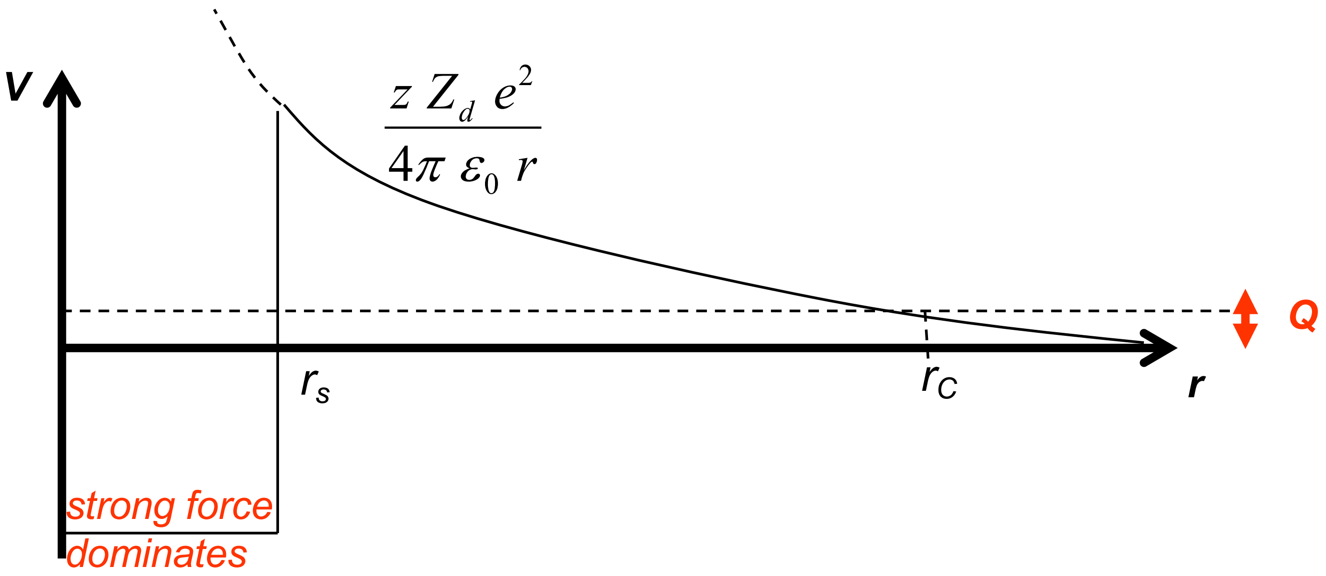
5.2. Plot of the logarithm of the decay rate R versus $ZE^{-1/2}$ for a number of α-decaying nuclei. The dashed line has the slope of -1.71 predicted by equation 5.3. (From Leighton, R. B. *Principles of Modern Physics*, McGraw-Hill (1959).)

2.4.2. Q.M. model of α–decay

Although we can often neglect electromagnetism in nuclear physics, it is Coulomb repulsion which explains α – decay rates.

Let us consider the Coulomb energy between:

the α–particle (Z = 2) and the daughter nucleus (Z_d, A_d)



At r_s: α and nucleus touch

$$r_s = r_d + r_\alpha = 1.1\left(A_d^{1/3} + \underbrace{4^{1/3}}_{A_\alpha}\right)\text{fm}$$

$$\sim 8\text{ fm}$$

At r_c: $$Q = \frac{z Z_d e^2}{4\pi \varepsilon_0 r_C}$$

for α–decay.
$z = 2$

This is the classical distance of closest approach

for Kinetic Energy Q $\qquad$ ($r_c \sim 50$ fm)

α–particles created at r_s cannot have a free existence until they get to r_C.

Classical interpretation: Coulomb barrier prevents α–decay

Q.M. interpretation: α–particle tunnels through the barrier

The relative motion of the α–particle and daughter are described by the wavefunction:

$$\psi(\underline{r}_d - \underline{r}_\alpha)$$

In the tunnelling region, ψ decays exponentially

So: $\boxed{\text{small } Q \rightarrow \text{long tunnelling path} \rightarrow \text{long half-life}}$

Let us look at the tunnelling probability in more detail.

The tunnelling probability can be expressed as:

$$\frac{\text{probability of finding } \alpha \text{ at } r_C}{\text{probability of finding } \alpha \text{ at } r_s}$$

$$= \frac{4\pi\, r_C^2\, |\psi(r_C)|^2}{4\pi\, r_s^2\, |\psi(r_s)|^2}$$

$$= e^{-G}$$

If $1/\tau_0$ α particles are produced per second at r_s, then

the probability of decay in a time δt becomes:

$$\frac{\delta t}{\tau_0} e^{-G} = \lambda\, \delta t$$

$$\Rightarrow t_{1/2} = \frac{\ln 2}{\lambda} = \ln 2\, \tau_0\, e^{+G}$$

By solving the time-independent Schrödinger equation,*
one can derive that the Gamov factor G is smaller (i.e., the probability of tunnelling larger)
when the energy released in the process Q is larger:

$$G \propto \sqrt{1/Q}$$

* The associated derivation is beyond the scope of this unit, but appended as non-examinable material for illustrative purposes.

not examinable material

Tunnelling theory of α–decay:

1. $r > r_s$ the strong force is not important but the Coulomb potential is

The Schrödinger equation for the α–particle reads:

$$-\frac{\hbar^2}{2m}\left(\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}\right)+\left(\frac{z\,Z_d\,e^2}{4\pi\,\varepsilon_0\,r}+\frac{\hbar^2}{2m}\frac{l\,(l+1)}{r^2}\right)\psi$$

$$=Q\,\psi \qquad \textbf{(1)}$$

Q energy

m reduced mass $\left(\dfrac{m_\alpha\,m_d}{m_\alpha+m_d}\right)\approx m_\alpha$

z = 2 for α–particle

Z_d atomic number of daughter

2. l angular momentum quantum number

For simplicity, let us set l = 0.

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3. We can now write:

$$\psi(r) = \frac{f(r)}{r}$$

$$\frac{z\, Z_d\, e^2}{4\pi\, \varepsilon_0} = Q\, r_C$$

$$\textbf{(1)} \Rightarrow \quad \frac{\hbar^2}{2m}\frac{d^2 f}{dr^2} = Q\left(\frac{r_C}{r} - 1\right) f \qquad \textbf{(2)}$$

4. We need to look for solutions of the form:

$$f(r) = e^{\phi(r)}$$

Substituting in **(2)** yields:

$$\frac{d^2\phi}{dr^2} + \left(\frac{d\phi}{dr}\right)^2 = \frac{2m\, Q}{\hbar^2}\left(\frac{r_C}{r} - 1\right) \qquad \textbf{(3)}$$

Let us assume that the Coulomb potential varies slowly, so that the term $\dfrac{d^2\phi}{dr^2}$ can be neglected.

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5.

$$\textbf{(3)} \Rightarrow \quad \frac{d\phi}{dr} = \pm\left(\frac{2m\,Q}{\hbar^2}\right)^{1/2}\left(\frac{r_C}{r}-1\right)^{1/2}$$

$$\text{i.e. } \phi(r) = \pm\int^{r}\left(\frac{2m\,Q}{\hbar^2}\right)^{1/2}\left(\frac{r_C}{r'}-1\right)^{1/2} dr' \quad \textbf{(4)}$$

6. The tunnelling probability equals:

$$e^{-G} = \frac{4\pi\, r_C^2}{4\pi\, r_s^2}\left|\frac{\psi(r_C)}{\psi(r_s)}\right|^2$$

$$= \left|\frac{f(r_C)}{f(r_s)}\right|^2 = \frac{e^{2\,\phi(r_C)}}{e^{2\,\phi(r_s)}} = e^{-2\,(\phi(r_s)-\phi(r_C))}$$

Hence:

$$G = 2\left(\frac{2m\,Q}{\hbar^2}\right)^{1/2}\int_{r_s}^{r_C}\left(\frac{r_C}{r'}-1\right)^{1/2} dr' \qquad \textbf{(5)}$$

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7. To evaluate, we substitute:

$$r' = r_C \cos^2 \theta$$

$$\Rightarrow G = 2\, r_C \left(\frac{2mQ}{\hbar^2}\right)^{1/2} \frac{\pi}{2}\, \mathcal{G}\left(\frac{r_s}{r_C}\right)$$

where

$$\boxed{\mathcal{G} = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{r_s}{r_C}\right)^{1/2} - \left\{\frac{r_s}{r_C}\left(1 - \frac{r_s}{r_C}\right)\right\}^{1/2}\right]}$$

$$0 < \frac{r_s}{r_C} < 1$$

$$0 < \mathcal{G} < 1$$

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8. For $\frac{r_s}{r_C} << 1$ $\mathcal{G} \sim 1$

Hence: $t_{1/2} = \ln 2 \, \tau_0 \, e^G$

$$\text{where } G \cong \pi \, r_C \left(\frac{2m \, Q}{\hbar^2} \right)^{1/2}$$

Remembering that:

$$r_C = \frac{z \, Z_d \, e^2}{4\pi \, \varepsilon_0 \, Q}$$

Then:

$$G \propto \left(\frac{1}{Q} \right)^{1/2}$$

Or:

$$G \propto r_C^{1/2}$$

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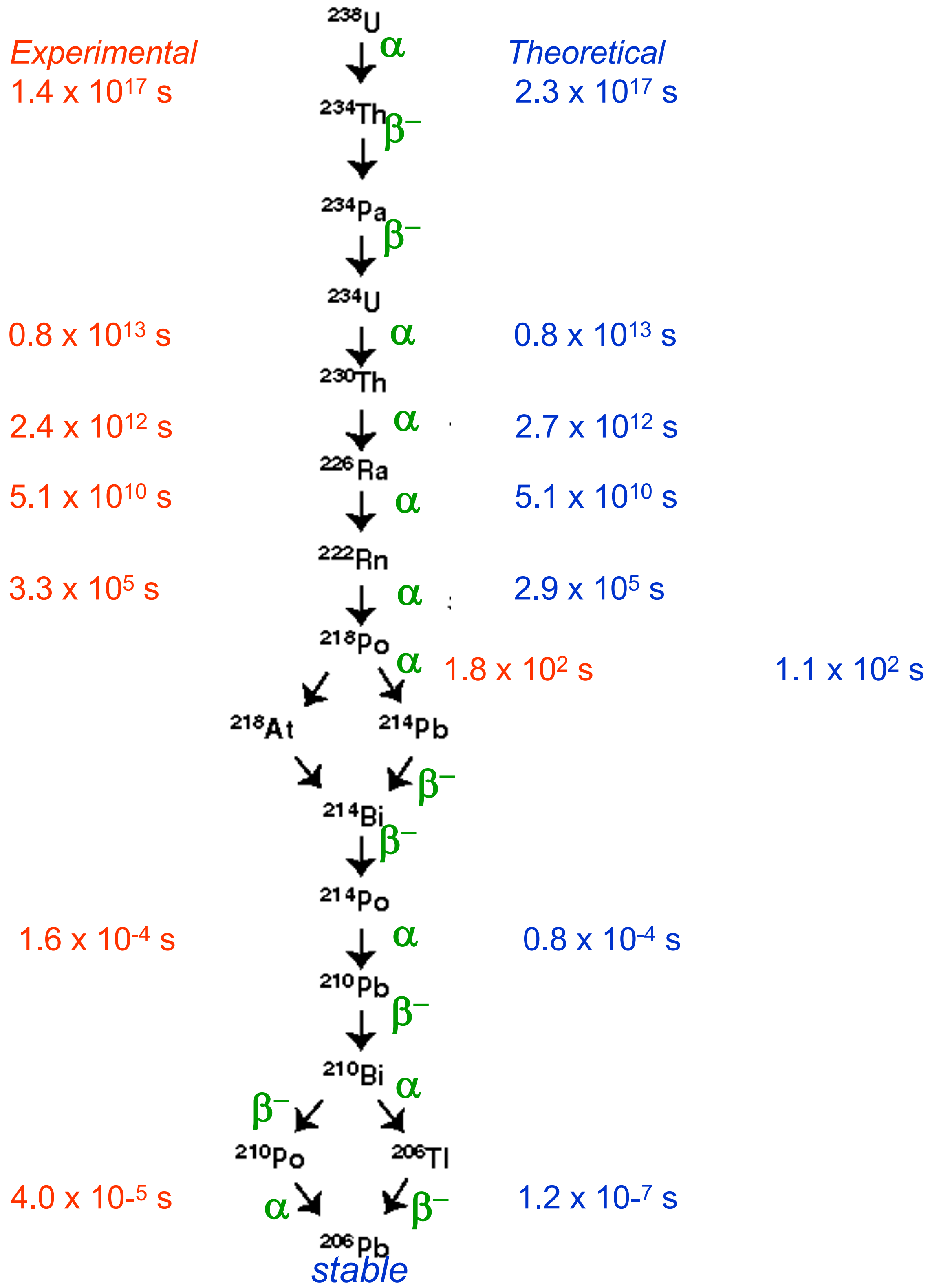
α–decay series from $^{238}_{92}U$



Comparison between theoretical and measured half-lives is reasonably accurate over 21 orders of magnitude