

Lecture 1:

Some basics of mathematical
modelling

The need for computer modelling & simulation

Analytical methods are very powerful, but they cannot begin to cover all phenomena we wish to understand.

There is a need to deal with

(ex)

- 1) equat^os with no analytical solut^o pendulum ex^o
- 2) manageable eqt^os with difficult BC heat distribut^o
- 3) situat^os where modelling via eqt^o is inappropriate Brownian mot^o

Analytic approximation techniques (perturbation, variation, separation of scales...) sometimes help. More often we will need to find numerical solutions.

This unit:

an introduction to some techniques of computational modelling

Setting up the model

- Models based on first principles (“*ab initio calculations*”)
e.g. Newton's laws, Maxwell equations
 - The governing equations are well-known **(I know stuff is right)**
 - Often resource-demanding (many degrees of freedom, large system sizes) **exponential growth**
- Reduced problem-specific models
 - derived from first-principles under certain approximations
 - Empirical or semi-empirical
 - Easy to handle, require minimum resources
 - Often linked to well-studied systems/equations
 - Require careful considerations of validity of approximations, empirical terms
- test before running **==**

Setting up the model

An example: Brownian motion

1^o principios distintos
'ab initio':

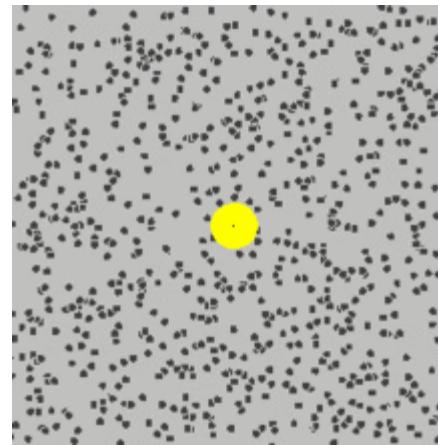
2nd law Newton for each

particle $m_1 \frac{d^2 \vec{r}_1}{dt^2} = \sum_j \vec{F}_{1j}$

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = \sum_j \vec{F}_{2j}$$

$$m_3 \frac{d^2 \vec{r}_3}{dt^2} = \sum_j \vec{F}_{3j}$$

...



Langevin equation:

$$m \frac{d^2 \vec{r}}{dt^2} = -\gamma \vec{v} + \vec{S}(t)$$

Empirical terms:

were not derived, but simply postulated

Another example: supercontinuum generation in an optical fibre

*newer reduction of the problem
done via deriving*

'ab initio':

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

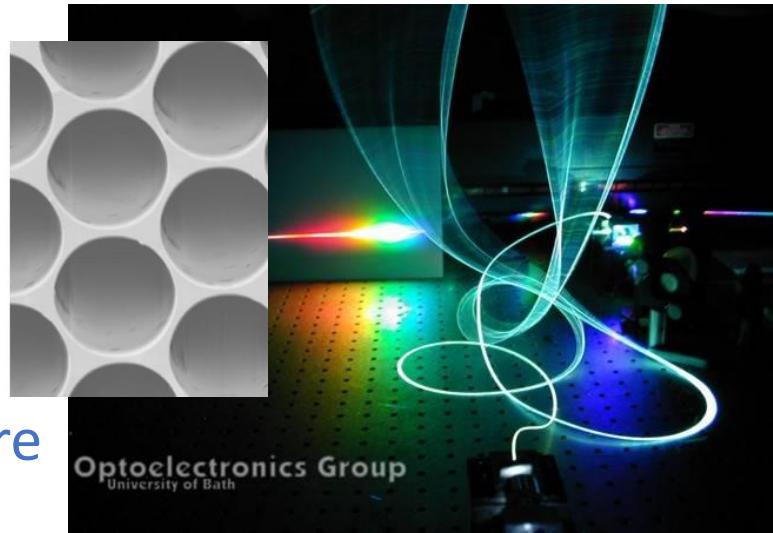
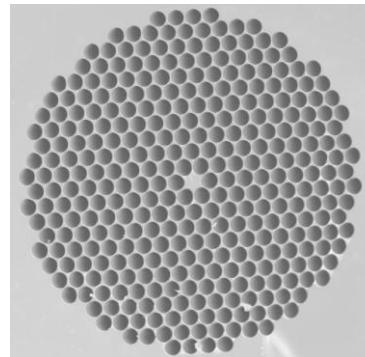
$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

$$\vec{D} = \epsilon_0 \epsilon(\vec{r}) \vec{E} + \vec{P}_{nl}(\vec{E})$$

$$\vec{B} = \mu_0 \vec{H}$$

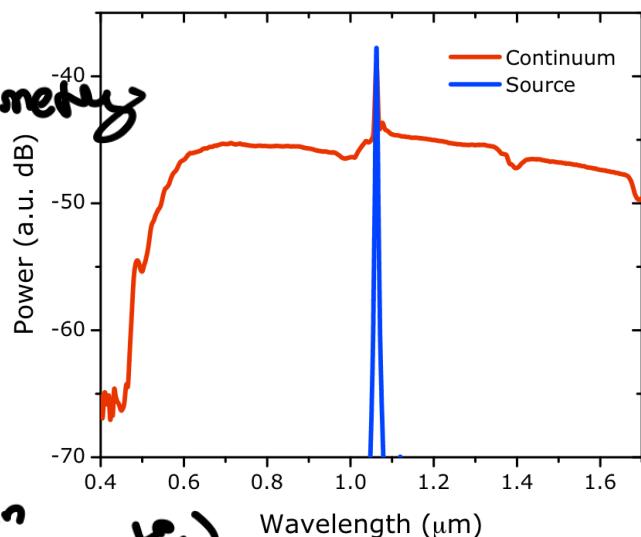
Reduced model (derived from Maxwell's eqs):

$$i \frac{\partial \Psi}{\partial z} = -\frac{\beta_2}{2} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\beta_3}{6} \frac{\partial^3 \Psi}{\partial t^3} + \gamma |\Psi|^2 \Psi$$



complex fibre geometry

reduced model ~~ageometry~~
as it was derived from initials
~~diff'ret~~
(with certain approx'mts)



Generalized Nonlinear Schrödinger equation

<https://en.wikipedia.org/wiki/Supercontinuum>

L1: math modelling

The Major Equations of Science

↑ not examinable

- often a useful starting point.

Broadly, can organise in terms of time dependence:

Time:	none	$\frac{\partial}{\partial t}$	$\frac{\partial^2}{\partial t^2}$
Single object or variable	Algebraic or transcendental equation	ODE(1)	ODE(2)
Coupled objects or variables	Simultaneous A/T Eqns	Simultaneous ODE(1)	Simultaneous ODE(2)
Continuum or field	Elliptic PDE(2)	Parabolic PDE(2)	Hyperbolic PDE(2)

+ exceptions !

Most major (linear) eqns of science; all contain $\nabla^2 \dots$

Some well-known equations

Elliptic:

$$\nabla^2 \Phi = 0$$

Laplace

$$\nabla^2 \Phi = f(\underline{r})$$

Poisson

$$(\nabla^2 + k) \Phi = 0$$

Helmholtz

Parabolic:

$$\nabla^2 \Phi + g(\underline{r}, t) = \frac{1}{D} \frac{\partial \Phi}{\partial t}$$

Diffusion/Heat

$$-\frac{\hbar^2}{2m} \nabla^2 \Phi + V(\underline{r}, t) \Phi = i\hbar \frac{\partial \Phi}{\partial t}$$

Schrödinger

Hyperbolic:

$$\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

Wave

A lot is known about these, so use them to guide you.

eg: if you see terms in $\partial^2 u / \partial t^2$ and u , expect oscillation (or exponential decay).

another consideration to keep in mind:

Linear vs Nonlinear equations

$$i \frac{\partial \Psi}{\partial z} = -\frac{\beta_2}{2} \frac{\partial^2 \Psi}{\partial t^2} - \frac{\beta_3}{6} \frac{\partial^3 \Psi}{\partial t^3} + \gamma |\Psi|^2 \Psi$$

$$-\frac{\hbar^2}{2m} \nabla^2 \Phi + V(r, t) \Phi = i\hbar \frac{\partial \Phi}{\partial t} \quad \text{Schrödinger}$$

linear

non-linear

+ a nonlinear term

+ higher order derivative(s)

if linear / non-linear
Check: substitute
 $\psi \rightarrow A\psi$, equation becomes different!

Generalized Nonlinear Schrödinger equation

Nonlinear equations often require special methods in computational physics!

- Establishing **connections** with the known in literature models is important for gaining insight into properties of the system and confidence in your modelling tools
- Use freedom in choosing values of model parameter/setting to zero some as the limiting case
- May be useful to introduce **artificial parameters** (equation terms) to establish a “**bridge**” between your model and a well-studied system

Another example: “bridging” two systems

- System to study:

$$\Phi^2 \cdot \nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

- Known in literature system (wave equation):

$$\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

- Modified system

$$(1 - \epsilon + \epsilon \Phi^2) \cdot \nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

$\epsilon = 0$ - standard wave equation

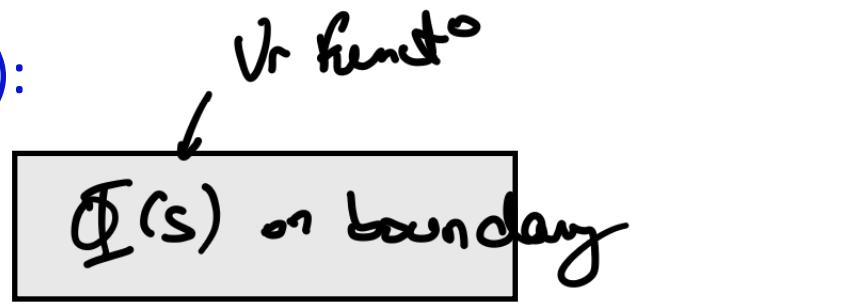
$\epsilon = 1$ - your system

Auxiliary conditions (ACs) - condit° \cup need to be aware exist

- No time dependence + at least one spatial coordinate

“Boundary Value Problems”(BVP):

(i) Dirichlet condition: specify



(ii) Neumann condition: specify

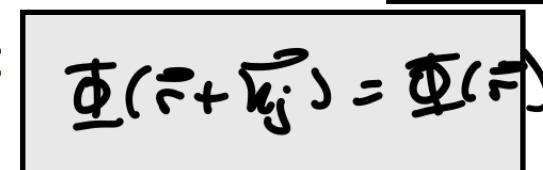


(iii) Mixed condition (Robin condition): specify



(iv) Periodic boundary condition:

↑
only case you can
model an ∞ system



Auxiliary conditions (ACs)

- First-order time derivative, constant coefficients, stationary problem

$$\Phi(t) \sim \exp(-i\omega t)$$

$$\frac{\partial}{\partial t} \rightarrow (-i\omega)$$

Eigen-value problem $\lambda\Phi = \hat{L}\Phi$

Example: Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{r}) \Psi$$
$$\Psi(t) \sim \exp\left(-\frac{iE}{\hbar}t\right) \Rightarrow E\Psi = \left(-\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\vec{r})\right)\Psi$$

Auxiliary conditions (ACs)

- Time-dependent, “Initial Value Problem” (IVP)

First-order, $\frac{\partial}{\partial t}$: specify

$$\Phi \text{ at } t = t_0$$

Second-order, $\frac{\partial^2}{\partial t^2}$: specify

$$\Phi \text{ & } \frac{\partial \Phi}{\partial t} \text{ at } t = t_0$$

Higher-orders, $\frac{\partial^m}{\partial t^m}$: specify

$$\Phi, \frac{\partial \Phi}{\partial t}, \dots, \frac{\partial^{m-1} \Phi}{\partial t^{m-1}} \text{ at } t = t_0$$

$m \text{ n}^{\circ}$ or m d.o.f.

- Correct ACs are vital in any mathematical or computational model
- When dealing with PDEs, often a combination of BVP (for spatial coordinates) and IVP (for time) is required
- Auxiliary conditions are set:
 - Explicitly (while setting up the model) *U choose*
 - Implicitly (via using specific numerical techniques such as FFT) *Ur tool choose*

Summary:

- **First-principles vs reduced** (approximate) models
 - Consider required resources, limitations of the model
- Establishing connections with known in literature models
 - Use freedom in setting parameter values
- Auxiliary conditions:
 - Boundary Value Problem
 - Eigenvalue Problem
 - Initial Value Problem

Lecture 2:

De-dimensionalization
(simplifying your model)

Dealing with physical units

- Each physical quantity has units (dimensions). Your starting model is likely to have variables and parameters measured in physical units.

Example (ODE): A train moving in x -direction on level ground.

$$m \frac{d^2x}{dt^2} = -A \frac{dx}{dt} - B \left(\frac{dx}{dt} \right)^2 + T$$

Annotations:

- A blue arrow points to the term $m \frac{d^2x}{dt^2}$ labeled "mass*acceleration".
- A green arrow points to the term $-A \frac{dx}{dt}$ labeled "rolling friction".
- A purple arrow points to the term $-B \left(\frac{dx}{dt} \right)^2$ labeled "air drag". A pink box labeled "non-linear" is placed above this term.
- A black arrow points to the term $+T$ labeled "tractive force". A pink box labeled "starting point" is placed below this term.

(semi-empirical model. Can you identify empirical terms?)

Dealing with physical units

$$m \frac{d^2x}{dt^2} = -A \frac{dx}{dt} - B \left(\frac{dx}{dt} \right)^2 + T$$

- Each term in this equation must have the same units of $\text{kg}\cdot\text{m}\cdot\text{s}^{-2} = \text{N}$
- Have a look at each variable and parameter separately:

m - Mass in kg

x - Position in m

t - Time in s

T - Force in N

$\frac{dx}{dt}$ - First derivative (velocity) in ...
 m/s $\text{& } \frac{d}{dt} = \text{s}^{-1}$

$\frac{d^2x}{dt^2}$ - Second derivative (acceleration) in ...
 m/s^2 $\text{& } \frac{d^2}{dt^2} = \text{s}^{-2}$

A - Rolling friction coefficient in ... kg/s

B - Air drag coefficient in ... kg/m

De-dimensionalisation (non-dimensionalisation)

- Systematically reduce number of **parameters** by making each term in equation dimensionless
- Also makes generic behaviour easier to spot.

$$m \frac{d^2x}{dt^2} = -A \frac{dx}{dt} - B \left(\frac{dx}{dt} \right)^2 + T$$

Note: only derivatives of x appear in this equation,
so simplify: $v = \dot{x}$, and

↓ *simplify*

$$m \frac{dv}{dt} + Av + Bv^2 = T \quad ① \quad \begin{matrix} 1^{st} \text{ order differential} \\ \text{eqn} \end{matrix}$$

$$m \frac{dv}{dt} + Av + Bv^2 = T \quad ①$$

Goal is $v(t)$:

t is the independent VARIABLE

v is the dependent VARIABLE

T, m, A, B are PARAMETERS

1. Define dimensionless versions of the variables (we have 2).

$$v = D_1 u$$

$$t = D_2 \tau$$

Here τ is dimensionless time, so D_2 is a time. Treat it as a timescale that we are free to choose when it suits us.

Similarly, u is dimensionless velocity, so D_1 is a velocity

NOTE: You only need to do it for variables which have dimensions! E.g. if the original equation has variable angle ϕ (measured in rads – i.e. dimensionless!) you should not touch it. Introducing another dimensionless angle would not help you in any way!

dimensionless - NO TOUCHING!

$$m \frac{dv}{dt} + Av + Bv^2 = T \quad (1)$$

2. Substitute into (1)

$$m \frac{D_1}{D_2} \frac{du}{d\tau} + ADu + B(D_1)^2 u^2 = T$$

3. Make the equation dimensionless. There are choices! eg

$$\div m \frac{D_1}{D_2}$$

↑
choice

$$\frac{du}{d\tau} + \frac{AD_2}{m} u + \frac{BD_1 D_2}{m} u^2 = \frac{T D_2}{D_1 m} \quad (2)$$

this term is dimensionless, so all are.

Moreover, since u is dimensionless (just a number), so are all the

"dimensionless products" $\frac{AD_2}{m}$, $\frac{BD_1 D_2}{m}$ & $\frac{T D_2}{D_1 m}$. Now choose

D_1 & D_2 to make (2) as simple as possible...

u can test the simulet by setting the $u^2=0$, and

Analytical predict that $\frac{du}{d\tau} + \frac{AD_2}{m}u + \frac{BD_1D_2}{m}u^2 = \frac{T D_2}{D_1 m}$. (2)

U can reverse ($\frac{du}{d\tau} + au = 0$)

If it works, U can turn u^2 "on"

4. Fix scales – usually some choice...

eg choose $\frac{AD_2}{m} = 1$; this sets timescale

$$D_2 = \frac{m}{A}$$

then choose $\frac{T D_2}{D_1 m} = 1$; this sets speed scale

$$D_1 = \frac{T}{A}$$

We've made all our choices, so any remaining products are fixed:

Here, $\frac{BD_1D_2}{m} = \frac{BT}{A^2} \equiv b$ is the single parameter left.

Our (1-parameter) governing equation is now

$$\frac{du}{d\tau} + u + bu^2 = 1$$

- a dimensionless equation with a dimensionless parameter b

amazing advancement

to one-to-one dependence: reduced problem

5. Consider dimensionless product(s)

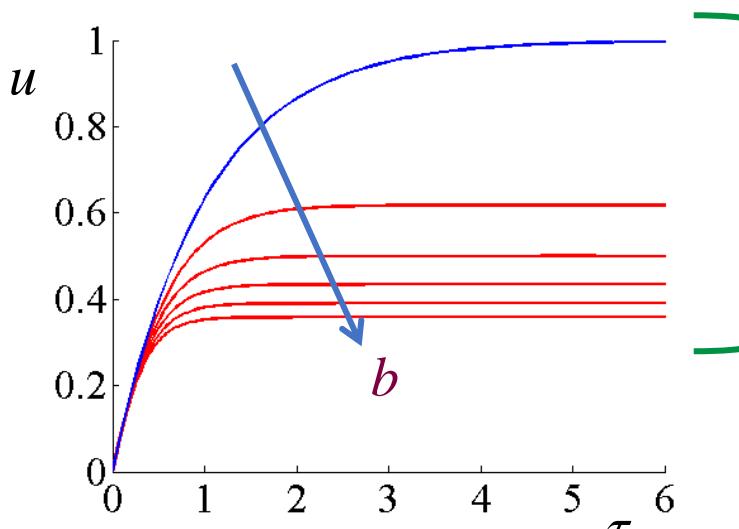
$$\frac{du}{d\tau} + u + bu^2 = 1$$

$$\frac{BT}{A^2} \equiv b$$

- They reveal generic behaviour
eg if $A \rightarrow A/2$ only need $T/4$ tractive force for same b
- If $b \ll 1$ may be able to neglect bu^2 term.
- Parameter reduction => easier to explore all solutions

6. Solve dimensionless equation

eg with
 $u(0) = 0$



terminal speeds

$$u_T(b)$$

Note: u_T can be obtained analytically!

Next slide: $\frac{du}{d\tau} + u + bu^2 = 1$ - a specific behaviour to consider

7. Take care to convert back to $v(t)$ for interpretation

re-translate back

$$v = D_1 u, \quad t = D_2 \tau$$

$$D_1 = \frac{T}{A} \quad \text{- units of speed}$$

$$D_2 = \frac{m}{A} \quad \text{- units of time}$$

may get
a surprise
when
re-converting.

$$v_T = D_1 u_T = \frac{T}{A} u_T$$

$$\frac{BT}{A^2} \equiv b$$

eg keep A & B the same; double T

$b \rightarrow 2b$ so u_T DEcreases.

(u_T is decreasing for larger b – check the plot on the previous slide!)

seems odd, but real terminal speed $v_T = \frac{T}{A} u_T$ INcreases!

more steps

De-dimensionisation of higher-order derivatives:

A man in a blue and white striped shirt stands in front of a projector screen, gesturing towards it. The screen displays a slide titled "De-dimensionisation of higher-order derivatives:" with handwritten-style text and equations. The text on the slide reads: "Take $v = D_1 u$, $t = D_2 \tau$ & compute $\frac{d^2 v}{dt^2}$:". Below this, the equation is derived step-by-step:
$$\begin{aligned} \frac{d^2 v}{dt^2} &= \frac{d^2(D_1 u)}{dt^2} \\ &= D_1 \frac{d}{dt} \left(\frac{du}{d\tau} \frac{d\tau}{dt} \right) \quad \frac{1}{D_2} \\ &= \frac{D_1}{D_2} \frac{d^2 u}{d\tau^2} \frac{d\tau}{dt} \\ &= \frac{D_1}{(D_2)^2} \frac{d^2 u}{d\tau^2}, \end{aligned}$$
 and in general
$$\frac{d^n v}{dt^n} = \frac{D_1}{(D_2)^n} \frac{d^n u}{d\tau^n}$$
. The projector screen is mounted on a white wall, and there is a small sign with a hearing aid icon and text below it.

& in general
$$\frac{d^n v}{dt^n} = \frac{D_1}{(D_2)^n} \frac{d^n u}{d\tau^n}$$

Alternative method of de-dimensionalisation:

Choose scales (suitable for the problem) at the outset.

So for our train model, perhaps $v = Vu$, $t = \theta\tau$

with $V = 125\text{mph}$ and $\theta = 1\text{h}$.

We cannot now minimise the number of dimensionless products

BUT we can ensure our dimensionless variables are $O(1)$

- useful when computing.

Summary:

- De-dimensionalising your model is useful for:
 - reducing the number of parameters;
 - identifying generic behaviour;
 - working with convenient variables of the order of $O(1)$.
- Remember to convert back to physical units for correct interpretation of your results!

Lecture 3:

Some further examples

An example: an optical pulse propagating in a fibre

$$i \frac{\partial A}{\partial z} = -\frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \gamma |A|^2 A$$

z - Length along the fibre (in m)

t - time (in s)



Independent
variables

$A(t, z)$ - Amplitude of the pulse (in ???) **Dependent variable (field)**

$\beta_2 = 2 \text{ ps}^2 \text{m}^{-1}$ - Dispersion parameter of the fibre

$\gamma = 0.01 \text{ W}^{-1} \text{m}^{-1}$ - Nonlinear parameter of the fibre



Parameters

An example: an optical pulse propagating in a fibre

$$i \frac{\partial A}{\partial z} = -\frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \gamma |A|^2 A$$

Geometry Parameters

$$\beta_2 = 2 \text{ ps}^2 \text{m}^{-1}$$

$$\gamma = 0.01 \text{ W}^{-1} \text{m}^{-1}$$

Input Parameters

- Fibre length
- Pulse duration
- Pulse peak amplitude

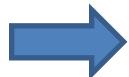
Potentially, a large parameter space to explore!

Reduce parameters by de-dimensionalization

$$i \frac{\partial A}{\partial z} = -\frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} + \gamma |A|^2 A$$

Introduce dimensionless variables

$$z = L\zeta \quad t = T\tau \quad A = U\psi$$



Reduce parameters by de-dimensionialization

$$i \frac{\partial \psi}{\partial \zeta} = -\frac{1}{2} \frac{\beta_2 L}{T^2} \frac{\partial^2 \psi}{\partial \tau^2} + \gamma L U^2 |\psi|^2 \psi$$

Can select L, T, and U to set all coefficients to 1



E.g.: 1) fix L to be the length of the fibre
(such that $0 < \zeta < 1$)

2) Set $\frac{|\beta_2|L}{T^2} = 1 \Rightarrow$



3) Set $|\gamma|LU^2 = 1 \Rightarrow$



OR: 1) fix T to be the input pulse width
(such that τ is O(1))

2) Set $\frac{|\beta_2|L}{T^2} = 1 \Rightarrow$



3) Set $|\gamma|LU^2 = 1 \Rightarrow$



$$i \frac{\partial \psi}{\partial \zeta} = -\frac{\text{sgn}(\beta_2)}{2} \frac{\partial^2 \psi}{\partial \tau^2} + \text{sgn}(\gamma) |\psi|^2 \psi$$

Note some important tendencies

$$\frac{|\beta_2|L}{T^2} = 1$$

$$|\gamma|LU^2 = 1$$

- Changing fibre dispersion $|\beta_2| \rightarrow 2|\beta_2|$ is equivalent to:
 - (whilst keeping the pulse duration fixed) reducing the fibre length L by half and increasing pulse peak amplitude by $\sqrt{2}$
- or
- (whilst keeping the fibre length fixed) stretching the pulse duration by $\sqrt{2}$
- Changing fibre nonlinearity $|\gamma| \rightarrow 2|\gamma|$ is equivalent to reducing the input pulse amplitude by $\sqrt{2}$

Theory – Experiment Phrasebook

“Set $T = 100 \text{ fs} = 0.1 \text{ ps}$;
Set input pulse duration = 1;
Set $L = \frac{T^2}{|\beta_2|} = \frac{0.1^2 \text{ ps}^2}{1 \text{ ps}^2/\text{m}} = 0.01 \text{ m}$;
Set propagation length $\xi_m = \frac{1 \text{ m}}{L} = 100$;
Set $U^2 = T^2 \left| \frac{\beta_2}{\gamma} \right| = 0.1^2 \left| \frac{1}{0.01} \right| = 1 \text{ W}$
Set pulse amplitude $\Psi_0 = \sqrt{\frac{1 \text{ kW}}{U^2}} = \sqrt{10^3}$ “
(note: amplitude² gives power)

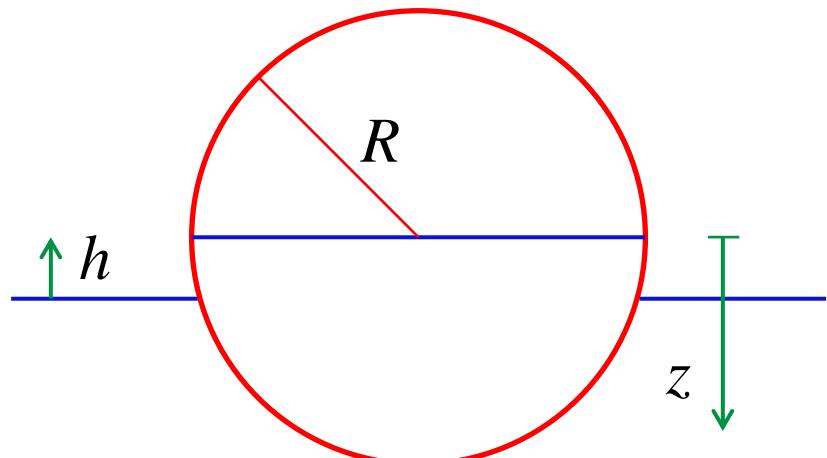
“My fibre has $\beta_2 = 1 \frac{\text{ps}^2}{\text{m}}$, $\gamma = 0.01 (\text{W m})^{-1}$, and it is 1 m long. My input pulse is 100 fs long with a peak power of 1 kW. What will I observe at the output?”

“I observe some interesting effects when I set pulse amplitude $\Psi_0 = 100$ and propagate over distance $\xi_m = 10$ ”

“You will observe some interesting effects if you set pulse peak power $P_0 = 100^2 \cdot T^2 \left| \frac{\beta_2}{\gamma} \right|$ and use fibre length $L = 10 \cdot T^2 / |\beta_2|$, where β_2 and γ are your fibre parameters, and T is the duration of your input pulse”

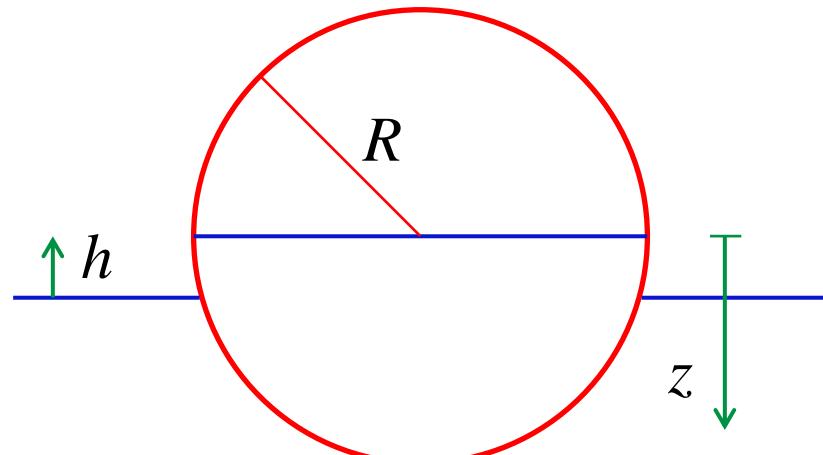
Case study: model the motion of a buoy half-filled with water

- “Ab initio” modelling must involve a combination of Newton’s laws (for the buoy) and hydrodynamics - **very resource demanding and complicated!**
- Focus on vertical oscillations of the buoy and **derive a simple model**
- Expect damped oscillations – hence an oscillator-type model ($d^2x/dt^2 = -\omega_0^2x + \text{damping}$)



Case study: model the motion of a buoy half-filled with water

- Forcing term: water waves? wind? Assume not
- Consider vertical motion only
- 2nd law of Newton:



$$m \frac{d^2h}{dt^2} = -mg + \rho g V_{disp} + \text{drag forces}$$

↑
gravity

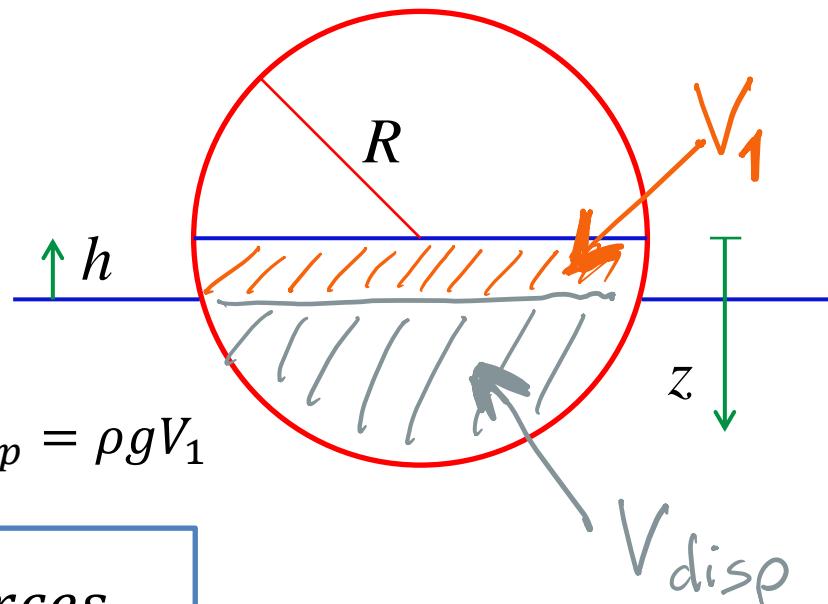
Restoring force
(=weight of displaced water)

$$m \frac{d^2h}{dt^2} = -mg + \rho g V_{disp} + \text{drag forces}$$

Neglect plastic shell,
assume water only

$$\rightarrow m = \rho \frac{V_{sphere}}{2}$$

$$mg - \rho g V_{disp} = \rho \frac{V_{sphere}}{2} g - \rho g V_{disp} = \rho g V_1$$



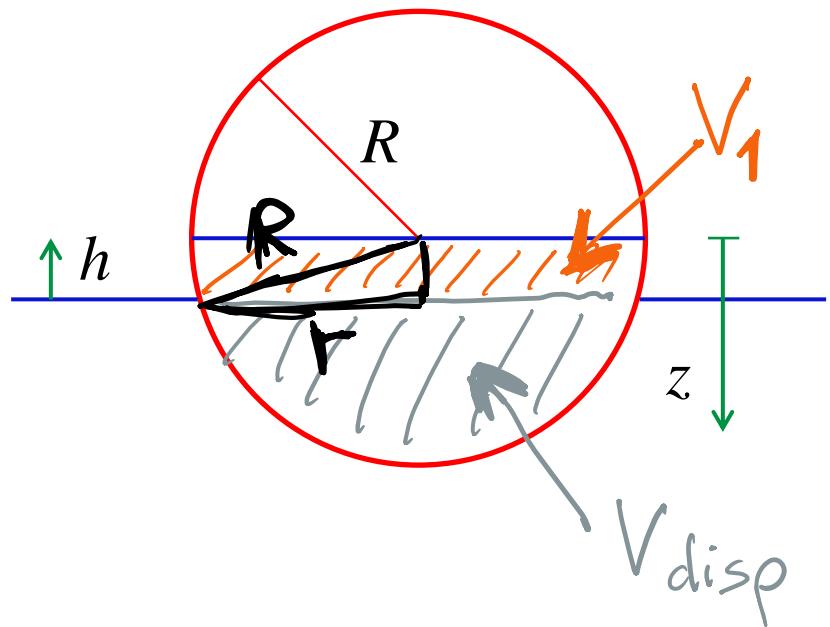
$$m \frac{d^2h}{dt^2} = -\rho g V_1 + \text{drag forces}$$

- Expect an oscillator-type model ($d^2x/dt^2 = -\omega_0^2 x + \text{damping}$)

=> Need to express V_1 (and drag forces) via h

$$m \frac{d^2 h}{dt^2} = -\rho g V_1 + \text{drag forces}$$

$$\begin{aligned} V_1 &= \int_0^h \pi r^2 dz \\ &= \pi \int_0^h (R^2 - z^2) dz = \pi \left(R^2 h - \frac{1}{3} h^3 \right) \end{aligned}$$



$$m \frac{d^2h}{dt^2} = -\rho g \pi \left(R^2 h - \frac{1}{3} h^3 \right) + \text{drag forces}$$

- Damping (energy loss) forces

Relevant science is fluid dynamics.

First-principles forces difficult. Semi-empirically, expect

“viscous drag” \propto speed $= \frac{dh}{dt}$ (eg Stokes: $F = 6\pi\eta vr$)

“inertial drag” $\propto \left(\frac{dh}{dt} \right)^2$ (eg cars: $F = \frac{1}{2} C v^2 A$)

- Damping via wave generation? Ignore.

Warning: careful with empirical terms!

- Drag forces should oppose motion

- "viscous drag" \sim speed $= \frac{dh}{dt}$

$$m \frac{d^2h}{dt^2} = -\alpha \frac{dh}{dt}$$

negative for positive velocity $\left(\frac{dh}{dt} > 0\right)$

and positive for negative velocity $\left(\frac{dh}{dt} < 0\right)$



- "inertial drag" \sim speed² $= \left(\frac{dh}{dt}\right)^2$

$$m \frac{d^2h}{dt^2} = -\beta \left(\frac{dh}{dt}\right)^2$$

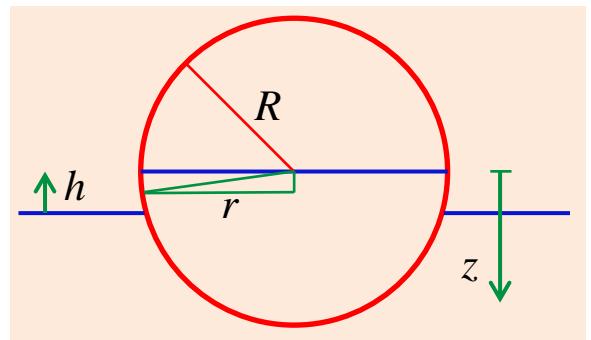
negative for positive velocity $\left(\frac{dh}{dt} > 0\right)$

negative for negative velocity $\left(\frac{dh}{dt} < 0\right)$



Fix: use $v|v|$ instead of v^2 !

$$m \frac{d^2h}{dt^2} = -\beta \left|\frac{dh}{dt}\right| \frac{dh}{dt}$$

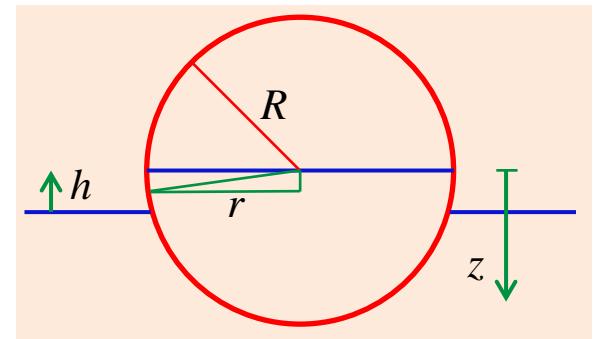


Our governing ODE is

$$\frac{2}{3}\pi R^3 \rho \frac{d^2h}{dt^2} + \alpha \frac{dh}{dt} + \beta \left| \frac{dh}{dt} \right| \frac{dh}{dt} + \rho g \pi R^2 h - \frac{1}{3} \rho g \pi h^3 = 0$$

- A damped oscillator equation
- ... with some nonlinear terms

Let's make it look neater now!



Our governing ODE is

$$\frac{2}{3}\pi R^3 \rho \frac{d^2h}{dt^2} + \alpha \frac{dh}{dt} + \beta \left| \frac{dh}{dt} \right| \frac{dh}{dt} + \rho g \pi R^2 h - \frac{1}{3} \rho g \pi h^3 = 0$$

Variables: h, t

Parameters: R, ρ, α, β (g)

De-dimensionalise: Let $h = Hy$, $t = Tx$, so

$$\frac{d^2y}{dx^2} + \frac{3\alpha T}{2\pi R^3 \rho} \frac{dy}{dx} + \frac{3\beta H}{2\pi R^3 \rho} \left| \frac{dy}{dx} \right| \frac{dy}{dx} + \frac{3g T^2}{2R} y - \frac{g H^2 T^2}{2R^3} y^3 = 0$$

Now simplify by choice of scales (not free choice this time).
eg: leave dimensionless products in damping terms by setting

$$\frac{3g T^2}{2R} = 1 \rightarrow \text{timescale } \boxed{}, \text{ then set}$$

$$\frac{g H^2 T^2}{2R^3} = 1 \rightarrow \text{length scale } \boxed{}$$

Dimensionless equation is then

$$\boxed{\frac{d^2y}{dx^2} + a \frac{dy}{dx} + b y = 0}$$

with dimensionless products $a = \frac{3\alpha\sqrt{2R}}{2\pi R^3 \rho \sqrt{3g}}, \quad b = \frac{3\sqrt{3}\beta}{2\pi R^2 \rho}.$

Solving the ODE

$$\frac{d^2y}{dx^2} + a \frac{dy}{dx} + b \left| \frac{dy}{dx} \right| \frac{dy}{dx} + y - y^3 = 0$$

- cannot be done analytically

However, as a useful first step, solve those parts that can:

If $a \ll 1$, $b \ll 1$ & $y \ll 1 \rightarrow y^3 \ll y$ so ignore y^3 ,

approx equation is $\frac{d^2y}{dx^2} + y = 0$ with solution $y = P \cos x + Q \sin x$

Should then expect oscillation at angular frequency

1 (dimensionless) or T^{-1} rad/s as $t = Tx$.

$$T = \sqrt{\frac{2R}{3g}}$$

For general values of a, b, y we need to solve the equation numerically

Summary:

- Considered two examples
- De-dimensionalizing helps to simplify the model and understand some important properties (before we set off with the numerical studies)