# Electric fields

1) Relative permitivity $\varepsilon_r = 1 + \chi_e$

electric susceptibility

$P = \chi_e \varepsilon_0 E$ — const, given

$P = np = nq_b r = 5\times10^{25} \times 10^{-18} \times 2e = 1.6\times10^{-11}$ $\text{cm}^{-1}$

n: atoms per cubic meter (given)

$p = q_b r$

r: e cloud shift (given)

$q_b$: charge dipole = $Ze$

Helium $Z = 2$

$\therefore \chi_e = \dfrac{P}{\varepsilon_0 E}$

$$\chi_e = \frac{1.602\times10^{-11}\,[\text{cm}^{-1}]}{8.854\times10^{-12}\,[\text{Fm}^{-1}]\;10\,[\text{kV}]}$$

$\hookrightarrow 10^4\,\text{V}$

$= 0.00018$

$\therefore \varepsilon_r = 1.00018$ ✓

2) Electric susceptibility: $\chi_e = 3.5$

$\vec{P} = \chi_e \varepsilon_0 \vec{E}$

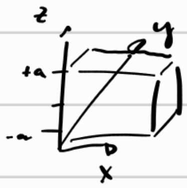
$= 3.5 \times \varepsilon_0 \times 15 = 4.65\times10^{-10}$ $\text{Cm}^{-2}$ ✓

$[\;\;]\;[\text{Fm}^{-1}]\;[\text{Vm}^{-1}]$

$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$

$\varepsilon_r = 1 + \chi_e = 4.5$

$\vec{D} = 5.97\times10^{-10}\ \text{Cm}^{-2}$ ✓

3) 


Gauss's Law: $\nabla \cdot \vec{D} = \rho_f$

$D_x = D_y = 0$

$D_z = \int \rho_f \, dz$

$= \rho_f z + c$ ← c=0 as D=0 when $\rho_f = 0$

$\vec{D}_z = \varepsilon \vec{E}$

$\therefore \vec{E} = \dfrac{\vec{D}_z}{\varepsilon} = \dfrac{\rho_f z}{\varepsilon}$ ✓

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

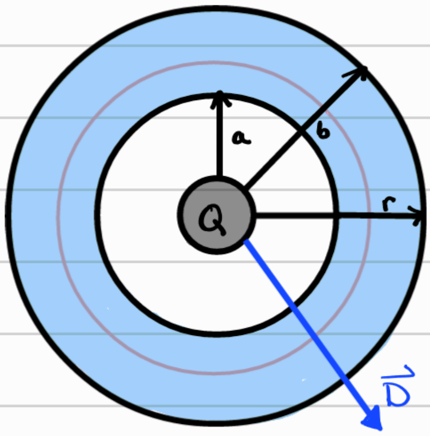
$\therefore \vec{P}$ ... $\rho_f z$ ... ✓

$\therefore P = \rho_f z - \frac{\rho_f z}{\varepsilon} = \rho_f z(1-\varepsilon)$

Polarisat° charge density: $\sigma_p$ top material $\sigma_p = |P| = \rho_f z(1-\varepsilon)$

bottom $\sigma_{p_{bottom}} = -\sigma_{p_{top}} = -\rho_f z(1-\varepsilon)$

4)




$Q_f = \int_V \rho_f \, dV$

$= \int_V (\nabla \cdot \vec{D}) \, dV$

$= \vec{D} \cdot \vec{A}$

a) $\therefore \vec{D} = \frac{Q}{4\pi r^2} \hat{r}$ ← unit vector r direct°

b)

i & iii) $r \begin{matrix} > a \\ < b \end{matrix}$ => vacuum $\varepsilon_r = 1 \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon_0 \varepsilon_r} = \frac{Q_f}{4\pi\varepsilon_0 r^2}$

ii) $a < r < b$ => dielectric with $\varepsilon_r$

$$\vec{E} = \frac{Q_f}{4\pi \varepsilon_r \varepsilon_0 r^2}$$

c) Electrostatic energy stored: $U = \int_{r=a}^{r=b} w \, dV$

Scalar energy density $w = \frac{1}{2} \vec{D} \cdot \vec{E}$

$$w = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1}{2} \frac{Q}{4\pi r^2} \frac{Q}{4\pi \varepsilon_0 \varepsilon_r r^2}$$

$$= \frac{Q^2}{32\pi^2 \varepsilon_0 \varepsilon_r r^4}$$

$$U = \int_a^b w \, dv = \int_a^b \frac{Q^2}{8\pi \varepsilon r^2} dr$$

word! $dv = 4\pi r^2 dr$ (no $\theta$ or $\phi$ dep)

$$= \frac{Q^2}{8\pi\varepsilon} \left[ \frac{-1}{r} \right]_a^b$$

$$= \frac{Q^2}{8\pi\varepsilon} \left( \frac{a-b}{ab} \right) [J]$$

## Magnetic fields

5) $\vec{B} = \mu_0 (\vec{H} + \vec{M})$

$\vec{M} = \chi_m \vec{H}$

$\vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H}$

$\chi_m = 0.01$

$\vec{M} = 0.01 \times 10^3 = 10 \ Am^{-1}$

$\vec{B} = \mu_0 (10^3 + 10) = 1.27 \times 10^{-3}$ T

6) Ampère's law : $\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

$J_f = \frac{I}{A} = \frac{I}{\pi a^2}$ ✓

$\oint_L \vec{H}\, d\vec{L} = \iint \vec{J}_f\, d\vec{S}$

$H(2\pi r) = J_f (\pi r^2) = \frac{I}{\cancel{\pi} a^2} \cancel{\pi} r$

$\therefore H = \frac{Ir}{2\pi a^2}$

$\vec{M} = \chi_m \vec{H} = \chi_m \frac{Ir}{2\pi a^2} \hat{\theta}$

7) $\vec{B} = 2\hat{\imath} - 5\hat{\jmath} + 4\hat{k}$

a) $\mu_r = 1 + \chi_m$

$\chi_m = 3$

$\vec{B} = \mu_0 (\vec{H} + \vec{M})$

b) $\vec{B} = \mu_0 (1 + \chi_m) \vec{H}$

$\vec{M} = \chi_m \vec{H}$

$\vec{H} = \frac{\vec{B}}{\mu_0 \mu_r}$

$\vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H}$

$= \frac{(2, -5, 4)}{4\mu_0}$ ma $Am^{-1}$

c) $\vec{M} = \chi_m \vec{H} = \frac{3(2, -5, 4)}{4\mu_0}$ m $Am^{-1}$

d) magnetic E per unit vol : $w$

$w = \frac{1}{2} \vec{B} \cdot \vec{H}$

$(2, -5, 4) \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} = 4 + 25 + 16$

$= 45$

$= \frac{1}{2} B \cdot \frac{3B}{4\mu_0} = \frac{3B^2}{8\mu_0}$

$= \frac{3 \cdot 45}{8\mu_0} \times 10^{-6} = 4.5$ $J.m^{-3}$

8) general use $\vec{B} = \mu_0 (\vec{H} + \vec{M})$

$\nabla \cdot \vec{B} = 0$

$\nabla \cdot [\mu_0 (\vec{H} + \vec{M})] = 0 \Rightarrow \nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$

A bar magnet is a magnetised bar of material. The vacuum surrounding it contains a magnetic field ($\vec{B}$) that originates from an $H$-field ($\vec{B} = \mu_0 \vec{H}$) as there can be no magnetisation of the vacuum. Where does this $H$-field come from? There are no real external currents flowing in a bar magnet and we start by associating the $H$-

field with currents.

The equation above shows that where there is a divergence of the magnetisation there is a (negative) divergence of the *H*-field. This means that a sink of magnetisation becomes a source of *H*-field. Thus, the boundaries of the magnet where there is spatial variation in magnetisation produce the *H*-field that gives the magnetic field around the bar magnet.

We noted in the lectures (section 'Macroscopic view: the magnetisation') that for a magnetised material there is a **magnetisation-induced volume current density** $\vec{J}_m = \nabla \times \vec{M}$ and a **magnetisation-induced surface current density** $\vec{k}_m = \vec{M} \times \hat{n}$, where $\hat{n}$ is the unit vector normal to the surface. The other way to understand the field around the magnet is that it is created by these currents related to the spatial configuration of the magnetisation.

$\nabla \cdot \vec{J}_f = -\frac{\partial \rho_f}{\partial t}$ (The continuity Equation),

$\vec{J}_f = \sigma \vec{E}$ (Ohm's law) and

$\nabla \cdot \vec{E} = \frac{\rho_f}{\varepsilon_0}$ (Gauss' Law),

erential equation which has the solution $\rho_f = \rho_{f0} e^{-t/\tau}$, where $\tau = \frac{\varepsilon_0}{\sigma}$,