PH20014: Electromagnetism 1

FULL NOTES

§1: Electromagnetism in Vacuum.

Learning Outcomes:

After taking this unit the student should be able to:

- derive and interpret Maxwell's equations and their solution in vacuum,
- list the distinguishing features of electromagnetic plane waves and write down a mathematical expression for a linearly or circularly polarised light wave,
- analyse in detail the propagation of vectorial plane waves in vacuum

Content:

Introduction to Maxwell's equations (6 hours):

> Derivation of integral and differential forms of Maxwell's equations and continuity equation. The wave equation in source-free vacuum. Plane wave solutions.

Electromagnetic plane waves (4 hours):

> 3D plane waves, vector nature of electromagnetic waves, relationships between $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{k}$. Impedance. Electromagnetic energy and the Poynting vector. Radiation pressure. Polarisation; Law of Malus, circular and elliptical polarisation. Birefringence, wave plates.

Contents

Recommended Reading / References.

[1] Electricity and Magnetism, Edward M. Purcell and David J. Morin. Cambridge: Cambridge University Press, Third edition. 2013

[2] Electromagnetism, I.S. Grant and W.R. Philips. Chichester: Wiley Second edition. 1990

[3] Elements of Electromagnetics. Matthew N. O. Sadiku. New York: Oxford University Press. Seventh edition. 2018.

[4] Optics, Hecht, Boston, Mass.: Pearson Education Limited, 5th ed. 2017

Lecture 01: Overview.

Electromagnetism is primarily concerned with the vector quantities:

$\mathbf{E}(\mathbf{r},t)$ **The electric field**

and

$\mathbf{B}(\mathbf{r},t)$ **The magnetic field**

They are continuous functions of position $\mathbf{r}$ and time t, defined over a given region, or **field**.

Consider:

Definition: Electric Field

An electric field is a region of space where a stationary charged particle experiences a force.

Region of space: We will have $f(\mathbf{r})$ or $f(x,y,z)$.

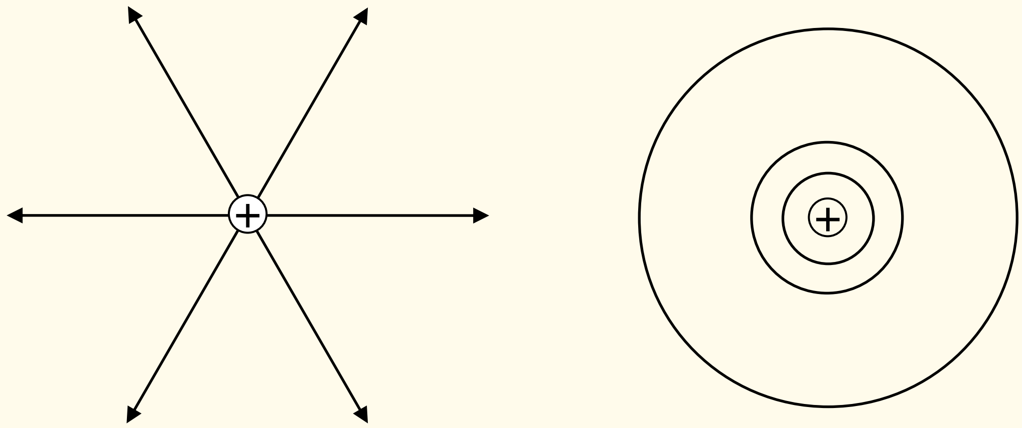
A force is a rate of change of momentum so generally we expect $f(x,y,z,t)$.

A force is a vector, so our function needs a vector magnitude and direction at every position in time and space (x,y,z,t).

We end with $\mathbf{E}(\mathbf{r},t)$ or $\mathbf{E}(x,y,z,t)$.

Other **vector fields**, such as the current density $\mathbf{J}(\mathbf{r},t)$ will be used, as will some **scalar fields** such as the electric charge density $\rho(\mathbf{r})$ and the electrostatic potential $\phi(\mathbf{r})$.

Recall: Vector fields can be visualised using **field lines** or **flux lines**, and scalar fields by contour lines or surfaces.



1.1 Maxwell's Equations.

Electromagnetism is governed by four differential equations in $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, called the Maxwell equations.

In a vacuum, they are,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad\qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad\qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

where t is time, ε_0 and μ_0 are constants associated with the vacuum.

$\rho(\mathbf{r})$ is electric charge density,

$\mathbf{J}(\mathbf{r},t)$ is electric current density.

Note that these are the only **source terms.**

The nature of the source terms tells us that all electromagnetic fields are caused by **charges**.

We will assume that the charge-carrying particle is much smaller than any length scale of interest. That is that they are **point charges**.

Hence, we have **Classical Electrodynamics**, and not **Quantum Electrodynamics** (QED).

Maxwell's equations can almost entirely be derived from the elementary laws of electricity and magnetism, i.e.

- Coulomb's Law
- Gauss's Law
- The Biot-Savart Law
- Ampère's Law
- Faraday's Law

But we first need these in their most general (integral) form. So, to get started, we will need some …

1.2 Vector Calculus.

See the appendix for a full summary. Here are the exciting highlights:

Volume integral of a scalar field: $\int_V \psi \, dV \equiv \lim_{\substack{N\to\infty \\ \Delta V_i \to 0}} \sum_{i=1}^{N} \psi(\mathbf{r}_i)\Delta V_i$.

Tangential line integral of a vector field: $\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} \equiv \lim_{\substack{N\to\infty \\ \Delta s_i \to 0}} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i)\cdot \Delta\mathbf{r}_i$.

If C forms a closed loop, we usually write $\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}$.

Flux integral of a vector field: $\int_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} \equiv \lim_{\substack{N\to\infty \\ \Delta S_i \to 0}} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i)\cdot \Delta\mathbf{S}_i$.

This integral is often called the **flux** of $\mathbf{F}$ across S . If S forms a closed surface, write $\oint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}$; $\mathrm{d}\mathbf{S}$ then points **outwards**.

The gradient of a scalar field: $\mathrm{grad}\,\psi \equiv \dfrac{\partial \psi}{\partial n}\hat{\mathbf{n}}.$

Note: $\mathrm{grad}\,\psi$ is a **vector field** derived from a **scalar field**.

The divergence of a vector field: $\mathrm{div}\,\mathbf{F} = \lim_{V\to 0}\left\{\dfrac{1}{V}\oint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}\right\}.$

$\mathrm{div}\,\mathbf{F}$ is a scalar field derived from a vector field, and measures the net flux originating or disappearing at a point.

The curl of a vector field: $\mathrm{curl}\,\mathbf{F} = \hat{\mathbf{n}}\lim_{S\to 0}\left\{\dfrac{1}{S}\oint_C \mathbf{F}\cdot \mathrm{d}\mathbf{r}\right\}.$

A non-zero curl at a point indicates a rotational element to the field at that point. The direction of $\mathrm{curl}\mathbf{F}$ is along the axis of rotation, and the modulus is a measure of local rotation rate.

The divergence theorem: $\oint_S \mathbf{F}\cdot d\mathbf{S} = \int_V \operatorname{div}\mathbf{F}\, dV$

Stokes' theorem: $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$

In Cartesian coordinates as follows:

$$\nabla\psi = \hat{\mathbf{i}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\psi}{\partial z}$$

gradient of a scalar field - ψ

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

divergence of a vector field $\mathbf{F}$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

curl of a vector field $\mathbf{F}$

Linking the macroscopic to microscopic …

The key to why the vector calculus above is important for us for electromagnetism is embedded above in the definitions like:

$$\operatorname{div}\mathbf{F} = \lim_{V\to 0}\left\{\frac{1}{V}\oint_S \mathbf{F}\cdot d\mathbf{S}\right\}.$$

Macroscopic net values over a region of field, like flux, are useful to us.

It is useful to have Gauss's Law as

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}.$$

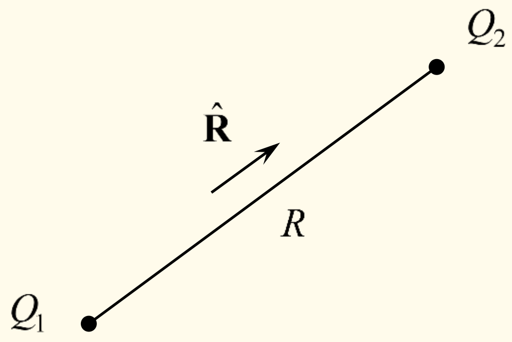
It is also simultaneously and equivalently true on a microscopic step-by-step scale that $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$.

1.3 Electrostatic fields.

Coulomb's Law (experimental).

The force between two **stationary** point charges is given by

$$\mathbf{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}\hat{\mathbf{R}}.$$



ε_0 is the **permittivity of free space** $\approx 8.854\times10^{-12}$ $C^2N^{-1}m^{-2}$ or Fm^{-1}.

Officially, standards organisations refer to ε_0 as the **electric constant.**

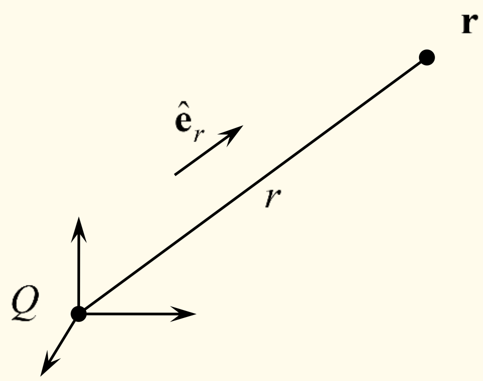
We also generally have $\mathbf{F} = q\mathbf{E}$ to define the relationship between the force on a charge q and the electric field.

Above, the force on a charge Q_2 at $\mathbf{r}$ is $\mathbf{F}_e = Q_2\mathbf{E}$. Another way to state this is:

> The **Electric Field** $\mathbf{E}(\mathbf{r})$ is the force experienced by a unit positive charge placed at $\mathbf{r}$.

Therefore, from Coulomb's Law, the $\mathbf{E}$-field due to a point charge $Q_1 = Q$ at the origin of coordinates is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{e}}_r,$$

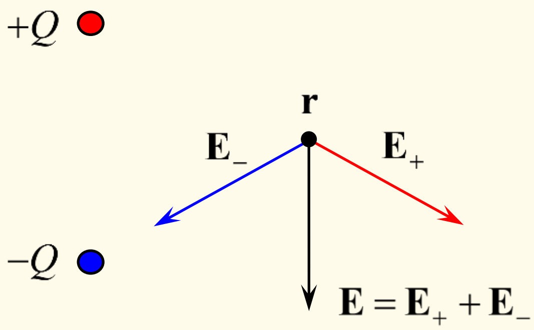


serves as a definition of $\mathbf{E}$.

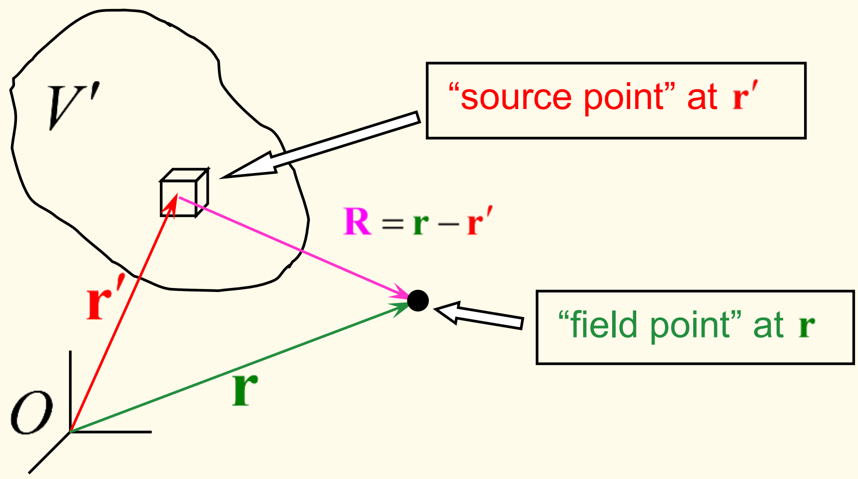
The units of electric field are Vm^{-1}

The **Principle of Superposition** states that the $\mathbf{E}$-field due to a distribution of charges is the vector sum of the fields due to individual charges:

For two charges:



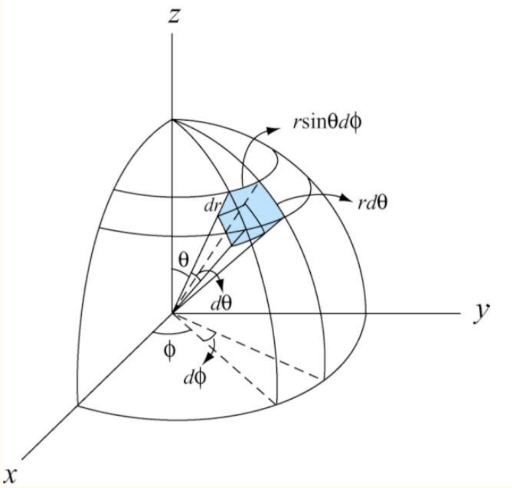
We use $\rho(\mathbf{r}')$ to describe a general charge distribution inside volume V':



Charge in $\mathrm{d}V'$ is $\rho(\mathbf{r}')\,\mathrm{d}V'$ so, by superposition,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\,\mathrm{d}V'.$$

Exercise 01.

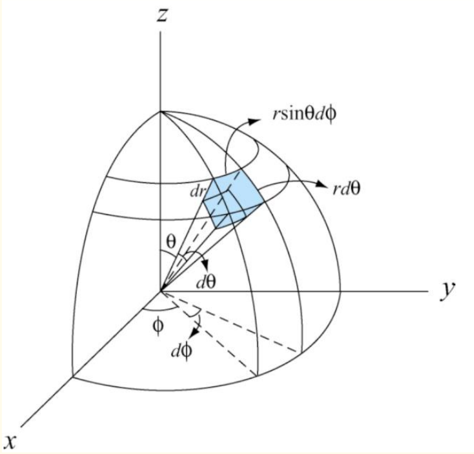


A solid hemisphere has radius R and uniform charge density ρ. Find the electric field at the centre of the hemisphere.

Hint: Consider $\mathrm{d}V$ in spherical polar coordinates.

Exercise 01 [Solution].

A solid hemisphere has radius R and uniform charge density ρ. Find the electric field at the centre of the hemisphere.



Answer.

Given the diagram, let the hemisphere be the half sphere above the x-y plane.

We will consider the standard spherical coordinates volume element (see diagram or Formula Book)

$$\mathrm{d}V = r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = (r\sin\theta\,\mathrm{d}\phi)(r\,\mathrm{d}\theta)(\mathrm{d}r)$$

For a volume $\mathrm{d}V$, we will have $\mathrm{d}Q = \rho\,\mathrm{d}V = \rho r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

This gives us an element of electric field at the origin,

$$\mathrm{d}\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}Q}{r^2}\hat{\mathbf{r}}.$$

This will point diagonally downwards at an angle θ to the vertical. The component in the x-y plane will be cancelled by a contribution from an equivalent element on the opposite side of the hemisphere. Thus, we only need to consider the component in the -z direction.

$$\mathrm{dE}_z = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}Q}{r^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}Q}{r^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\rho r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi}{r^2}\cos\theta$$

$$\mathrm{dE}_z = \frac{\rho}{4\pi\varepsilon_0}\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$\mathrm{dE}_z = \frac{\rho}{4\pi\varepsilon_0}\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Note – no *r* dependence. We'll discuss this more later.

To find the net field, integrate over the hemisphere. Thus,

$$\mathrm{E}_z = \int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\int_0^R \mathrm{d}r\int_0^{2\pi}\mathrm{d}\phi$$

$$= \frac{\rho}{4\pi\varepsilon_0}\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}[R][2\pi] = \frac{\rho R}{4\varepsilon_0}.$$

Note on symmetry.

By considering elements opposite each other we can decide that the field at the centre of the hemisphere is directed along the z-axis.

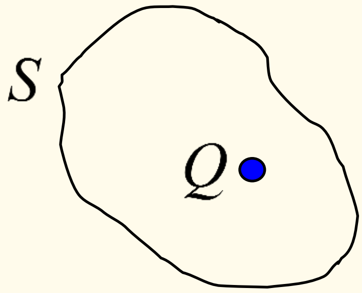
We can argue this from considerations of symmetry.

Assume – looking for a contradiction – that the electric field at the centre is not vertical. It then points at some angle to the surface – let's say it points to the left. Then if we rotate the hemisphere by a half turn then it points to the right. But, we have exactly the same hemisphere after the rotation so the field must point to the left. The only solution that is no affected by rotation is the field points along the axis.

Q. What direction will the field point if we consider point a distance $R/2$ away from the centre on the flat surface of the hemisphere.

Lecture 02: Gauss's Law.

2.1 Gauss's Law.



Gauss's Law, derived from Coulomb's Law, states that if charge Q is **completely** surrounded by a surface S **of any shape**, then

$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0}.$$

To show Gauss's law is true…

With Q at the origin, $\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{e}}_r$, so

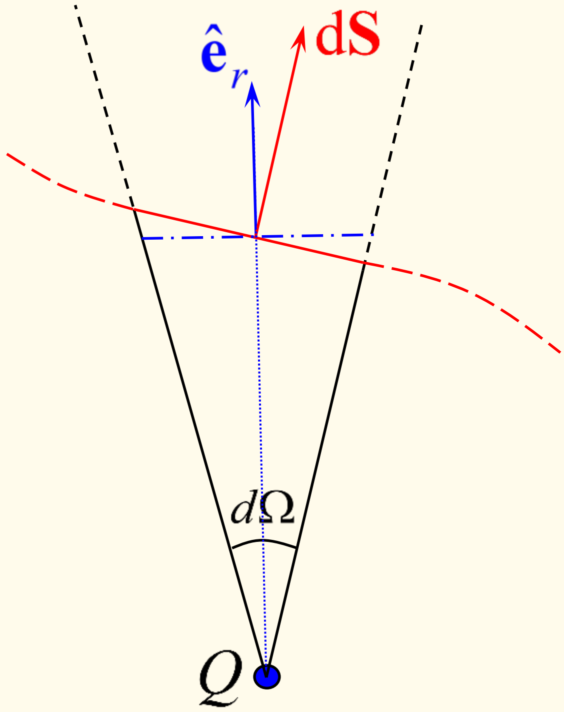
$$\oint_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{4\pi\varepsilon_0}\oint_S \frac{\hat{\mathbf{e}}_r \cdot \mathrm{d}\mathbf{S}}{r^2}.$$

For some shapes (e.g., a sphere, see Problem Sheet 1) this is an easy integral.

For general shapes, we use **solid angles…**

Aside on planar and solid angles.

Consider an arbitrary part of a general surface S:



$$\hat{\mathbf{e}}_r \cdot \mathrm{d}\mathbf{S} = \mathrm{d}S\cos\theta$$
$$= \mathrm{d}S',$$

= perpendicular area.

Therefore

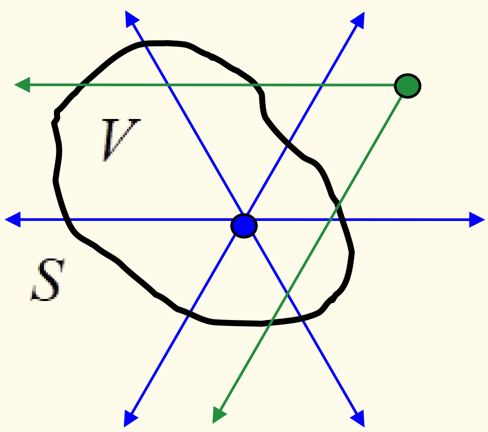
$$\frac{\hat{\mathbf{e}}_r \cdot \mathrm{d}\mathbf{S}}{r^2} = \frac{\mathrm{d}S'}{r^2} = \mathrm{d}\Omega,$$

so

$$\oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = \frac{Q}{4\pi\varepsilon_0}\oint_S \mathrm{d}\Omega = \frac{Q}{\varepsilon_0}.$$

Conclusion: the shape of S does not matter!

This result is not surprising in terms of flux lines.



Any charge **outside** S contributes nothing to the flux integral – all its flux lines enter then leave the enclosed volume V.

Finally, extend to a general charge distribution $\rho(\mathbf{r})$ inside V.

Total charge within S, $Q = \int_V \rho(\mathbf{r})\,\mathrm{d}V$, so

$$\oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \rho(\mathbf{r})\,\mathrm{d}V.$$ This is **Gauss's Law in integral form**.

Now apply the **divergence theorem** to the left-hand side.

$$\oint_S \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \int_V [\nabla\cdot\mathbf{E}]\mathrm{d}V$$

This means that we have,

$$\int_V [\nabla\cdot\mathbf{E}]\mathrm{d}V = \int_V \left[\frac{\rho(\mathbf{r})}{\varepsilon_0}\right]\mathrm{d}V.$$

We have made no assumption about the size and shape of S and V, or the function $\rho(\mathbf{r})$. These volume integrals can only be equal for all cases if the integrands are equal. i.e.:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}.$$ This is **Gauss's Law in differential form**.

Notes:

1. Though derived for static charges, the result is always true. This is our first Maxwell Equation.
2. Divergence measures direct sources and sinks of vector fields. So $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$ states

   The direct source of electric field is electric charge.

3. The result is true at all positions $\mathbf{r}$, not just in "V". Where there is charge, $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$, away from charge, $\nabla\cdot\mathbf{E} = 0$, though this **does not** mean $\mathbf{E} = \mathbf{0}$.

Here is an interesting fact that we can now prove

2.2 Earnshaw's Theorem.

It is impossible to construct an electrostatic field that will hold a charged particle in stable equilibrium in empty space.

Proof:

Assume that, contrary to the theorem, there is a point P at which a positively charged particle would be in stable equilibrium. This means that for any small displacement of the particle from P there must be a force to push it back to P. Thus, there would have to be an electric field pointing towards P for all small displacements.

If we construct a Gaussian surface around P then there would have to be a net flux into the surface. This contradicts Gauss's law if the point P is in empty space.

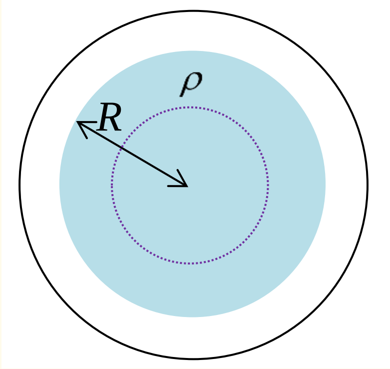
Thus, there cannot be an electrostatic field that will hold a charged particle in stable equilibrium in empty space.

Exercise 02: Field of a uniform sphere of charge.

A spherical charge distribution has a density ρ that is constant from $r = 0$ to $r = R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R?

Exercise 02 [Answer]: Field of a uniform sphere of charge.

A spherical charge distribution has a density ρ that is constant from $r=0$ to $r=R$ and is zero beyond. What is the electric field for all values of r, both less than and greater than R?



Use Gauss's Law: $\oint_S \mathbf{E}\cdot d\mathbf{S} = \dfrac{Q_{enc}}{\varepsilon_0}$.

By symmetry, the field will be of the form $\mathbf{E}=E(r)\hat{\mathbf{r}}$. There is no θ or φ dependence.

For the inner sphere, $r<R$.

Gauss: $E(r)\left(4\pi r^2\right)=\dfrac{1}{\varepsilon_0}\left(\dfrac{4}{3}\pi r^3\rho\right)$ gives $\mathbf{E}=\dfrac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}}$ for $r<R$.

For sphere outside the charge where $r>R$.

Gauss: $E(r)\left(4\pi r^2\right)=\dfrac{1}{\varepsilon_0}\left(\dfrac{4}{3}\pi R^3\rho\right)$ gives $\mathbf{E}=\dfrac{\rho R^3}{3\varepsilon_0 r^2}\hat{\mathbf{r}}$ for $r>R$

We would normally write this second result using the total charge in the sphere of charge density ρ.

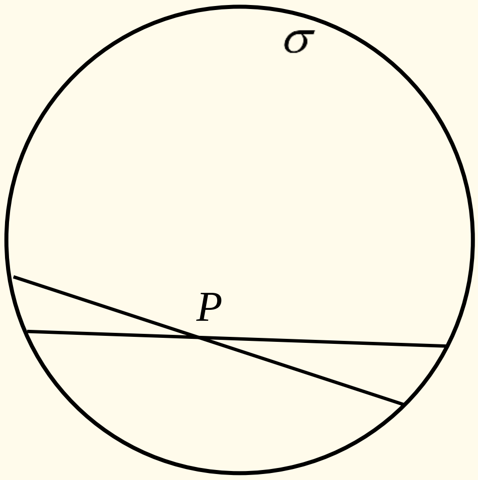
$$Q=\frac{4}{3}\pi R^3\rho \quad \text{so for } r>R,\ \mathbf{E}=\frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

If we use $Q=\dfrac{4}{3}\pi R^3\rho$ and return to the first part then the charge enclosed for a radius $r<R$ is $(r/R)^3 Q$ so we can also write, $\mathbf{E}=\dfrac{Qr}{4\pi\varepsilon_0 R^3}\hat{\mathbf{r}}$ for $r<R$.

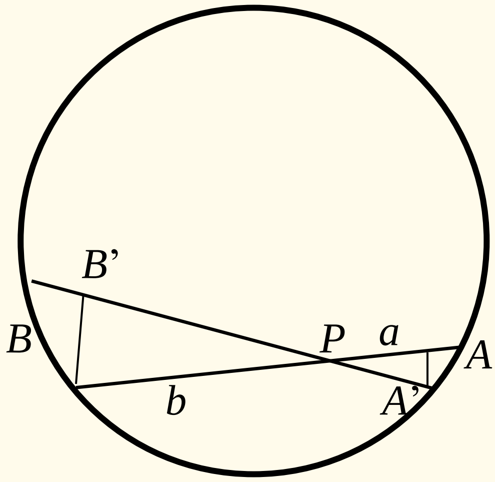
The two expressions agree at $r=R$.

Exercise 03: Zero field inside a spherical shell.

Consider a hollow spherical shell with uniform surface charge density σ. By considering the two small patches at the ends of the thin cones shown in the figure (or otherwise), show that the electric field at any point in the interior of the shell is zero.



Exercise 03 [Answer]: Zero field inside a spherical shell.



Consider a hollow spherical shell with uniform surface charge density σ. By considering the two small patches at the ends of the thin cones shown in the figure (or otherwise), show that the electric field at any point in the interior of the shell is zero.

Answer

[I] Discursive. Let distance a be the distance from point P to patch A, and let b be the distance from P to patch B. The cones are thin (small solid angle) so we don't need to exactly define the middle of the patch compared to the edge.

We construct perpendicular bases to the cones and call them A' and B'. The ratio of the area of A' to B' is a^2/b^2 because we have similar shapes and the areas are proportional to length squared.

A chord of the circle that passes perpendicularly through A' and B' makes the same angle to A and B [Sketch a nice symmetric chord on a circle to see this if this sounds wrong]. Thus, from projection of areas, the ratio of A to B is a^2/b^2. Thus, the charge on patch A is a^2/b^2 times the charge on patch B.

The $\mathbf{E}$-field at P scales as q/r^2. We can see that at P,

$\dfrac{E_A}{E_B} = \dfrac{Q_A}{Q_B}\dfrac{b^2}{a^2} = \dfrac{a^2}{b^2}\dfrac{b^2}{a^2} = 1$. Thus, the fields cancel for any position P.

[II] More formal.

You can be more formal with this using the maths we used for solid angle above. The cones have an equal solid angle.

We have above, $\frac{\hat{\mathbf{e}}_r \cdot \mathrm{d}\mathbf{S}}{r^2} = \frac{\mathrm{d}S'}{r^2} = \mathrm{d}\Omega.$

Ignoring the constant factor between $\mathrm{d}S'$ and $\mathrm{d}S$.

Thus, $\mathrm{d}S' = r^2 \mathrm{d}\Omega$ and so here for the surface $\mathrm{d}Q = \sigma \mathrm{d}S' = \sigma r^2 \mathrm{d}\Omega$

and, $\mathrm{d}E = \frac{1}{4\pi\varepsilon_0}\frac{\mathrm{d}Q}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma r^2 \mathrm{d}\Omega}{r^2} = \frac{\sigma}{4\pi\varepsilon_0}\mathrm{d}\Omega$

We see that the contribution will be equal and opposite from the two cones.

[III] Gauss's Law

For any Gaussian surface inside the sphere, the enclosed charge is zero and so the net flux through the surface is zero.

Can we think of a field, fitting the spherical symmetry, that we can have this property? All field lines – in electrostatics – much begin or end on charges.

If no, then the field is zero throughout the inside of the sphere.

Note: this also implies that the electric potential is constant throughout the interior.

Now – more on electrostatics …

Lecture 03: The curl of $\mathbf{E}$. (Electrostatic case).

3.1 Electrostatics starting from $\nabla\times\mathbf{E}$.

Derivation of the curl of E.

Stokes's theorem states that the curl of a vector field $\mathbf{F}$ is related to its closed line integral by,

$$\int_S \operatorname{curl}\mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_S \nabla\times\mathbf{F}\cdot\mathrm{d}\mathbf{S} = \oint_C \mathbf{F}\cdot\mathrm{d}\mathbf{l}$$

where S is the surface enclosed by C.

The electric field is given – for a single charge – by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}.$$

The line element in spherical polar coordinates is

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\hat{\mathbf{r}} + r\mathrm{d}\theta\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\phi\hat{\phi}$$

Thus, if we look at an element $\mathbf{E}\cdot\mathrm{d}\mathbf{l}$, we see that $\hat{\mathbf{r}}$ is orthogonal to $\hat{\boldsymbol{\theta}}$ and $\hat{\phi}$, and so only the $\hat{\mathbf{r}}$ term remains in the integral. Thus,

$$\int_S \nabla\times\mathbf{E}\cdot\mathrm{d}\mathbf{S} = \oint_C \mathbf{E}\cdot\mathrm{d}\mathbf{l} = \oint_C \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}\cdot\mathrm{d}r\hat{\mathbf{r}}$$

$$= \oint_C \frac{q}{4\pi\varepsilon_0 r^2}\mathrm{d}r = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r=a}^{r=a}$$

assuming a for the starting, and so also finishing, radius.

$$\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{a}+\frac{1}{a}\right] = 0$$

and so, $\nabla \times \mathbf{E} = 0$.

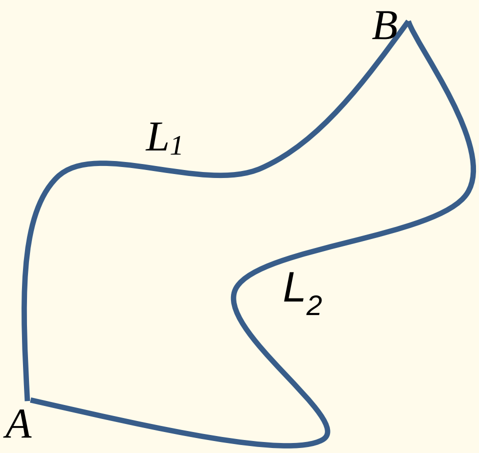
We get the same result if we use a combination of charges producing the electrostatic electric field where, as before,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho(\mathbf{r}')}{R^2}\hat{\mathbf{R}}\, dV'.$$

Curl measures “rotational” sources; this result implies that there are not any rotational sources in electrostatics. The source of an electrostatic field must be irrotational.

The path integral between two points is path independent.

If $\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0$ for a closed loop then if we consider two points A and B and two different paths between them, L_1 and L_2.



Then,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_A^B \mathbf{E} \cdot d\mathbf{l}_1 + \int_B^A \mathbf{E} \cdot d\mathbf{l}_2 = 0$$

$$\therefore \int_A^B \mathbf{E} \cdot d\mathbf{l}_1 = -\int_B^A \mathbf{E} \cdot d\mathbf{l}_2$$

$$= \int_A^B \mathbf{E} \cdot d\mathbf{l}_2$$

We have shown that the line integral of the electric field from A to B is the same whether the path is line 1 or line 2. The line integral is independent of the integration path taken. This means that we can define a **scalar field** whose value at a point is equal to the line integral of electric field from a fixed reference point O to the point. We can write,

$$\int_O^r \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \phi(r) - \phi(0)$$

which defines the electric potential ϕ.

We normally take the reference point O to be the point at which $\phi(O) = 0$ and a minus sign is added so that,

$$\int_O^r \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\phi(r)$$

Example 01: Potential due to a uniform sphere of charge.

A spherical charge distribution has a density ρ that is constant from $r=0$ to $r=R$ and is zero beyond. Use the results from Exercise 02 to find the potential for all values or r. Take the reference point O to be infinitely far away.

The fields are,

For, $r<R$, $\mathbf{E}=\frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}}$ for $r<R$.

For $r>R$, $\mathbf{E}=\frac{\rho R^3}{3\varepsilon_0 r^2}\hat{\mathbf{r}}$ for $r>R$.

For the potential outside the sphere:

$$\phi_{out}=-\int_{\infty}^{r}E(r')\mathrm{d}r'=-\int_{\infty}^{r}\frac{\rho R^3}{3\varepsilon_0 r'^2}\mathrm{d}r'=\frac{\rho R^3}{3\varepsilon_0}\left[\frac{1}{r'}\right]_{\infty}^{r}=\frac{\rho R^3}{3\varepsilon_0 r}\quad \text{or}\quad \frac{Q}{4\pi\varepsilon_o r}$$

For the potential inside the sphere, we split up the integral,

$$\phi_{in}=-\int_{\infty}^{R}E(r')\mathrm{d}r'-\int_{R}^{r}E(r')\mathrm{d}r'=-\int_{\infty}^{R}\frac{\rho R^3}{3\varepsilon_0 r'^2}\mathrm{d}r'-\int_{R}^{r}\frac{\rho r'}{3\varepsilon_0}\mathrm{d}r'$$

$$=\frac{\rho R^2}{3\varepsilon_0}-\frac{\rho}{6\varepsilon_0}\left(r^2-R^2\right)$$

$$=\frac{\rho R^2}{2\varepsilon_0}-\frac{\rho r^2}{6\varepsilon_0}$$

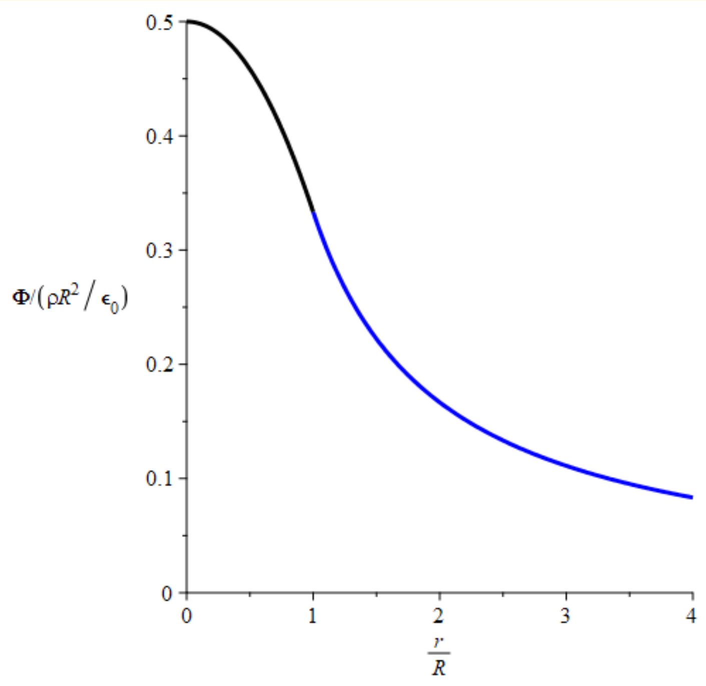
Note: the two expressions agree: $\phi(r=R)=\frac{\rho R^2}{3\varepsilon_0}$. Also, $\phi(r=0)=\frac{\rho R^2}{2\varepsilon_0}$.

Note we can write

$$\phi_{in} = \frac{\rho R^2}{\varepsilon_0}\left(\frac{1}{2} - \frac{1}{6}\left(\frac{r}{R}\right)^2\right)$$

$$\phi_{out} = \frac{\rho R^2}{\varepsilon_0}\left(\frac{1}{3\frac{r}{R}}\right)$$

These are easy to plot as a function of $\frac{r}{R}$.



Note that the potential is smooth and continuous. Also, the gradient is continuous at $r = R$ as the electric fields are continuous at $r = R$ as shown in Exercise 02.

How much work does it take to move a charge form the surface of the sphere to the centre of the sphere?

3.2 The gradient theorem.

The exact differential of ϕ is

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial x}\mathrm{d}x + \frac{\partial \phi}{\partial y}\mathrm{d}y + \frac{\partial \phi}{\partial z}\mathrm{dz} \quad \text{or}$$

$$\mathrm{d}\phi = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial y \end{bmatrix} \phi \cdot \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{dz} \end{bmatrix} = \nabla\phi \cdot \mathrm{d}\mathbf{l}$$

From $\int_O^r \mathbf{E}\cdot \mathrm{d}\mathbf{l} = -\phi(r)$ and fundamental theory of calculus, $\mathbf{E}\cdot\mathrm{d}\mathbf{l} = -\mathrm{d}\phi$.

Thus,

$$\mathbf{E}\cdot\mathrm{d}\mathbf{l} = -\mathrm{d}\phi = -\nabla\phi\cdot\mathrm{d}\mathbf{l}$$

This means that:

$$\mathbf{E} = -\nabla\phi \quad \text{or } \mathbf{E}(\mathbf{r}) = -\frac{\mathrm{d}\phi(\mathbf{r})}{\mathrm{d}r} \text{ etc.}$$

Summary.

This way of arguing started with finding that for an electrostatic electric field, $\mathbf{E}_s$, that $\nabla\times\mathbf{E}_s = 0$. The electrostatic field is irrotational. This means that the field has path independent path integrals which means that a scalar potential can be used to describe the field. We then see that $\mathbf{E} = -\nabla\phi$.

Alternative way of arguing about electrostatic fields.

Electrostatic fields are **conservative** fields**.** We start with this concept from mechanics and demonstrate it for electric fields.

Work done $=\oint_C \mathbf{F}_e \cdot \mathrm{d}\mathbf{r}$ and from earlier $\mathbf{F}_e = Q\mathbf{E}_S$, so

$Q\oint_C \mathbf{E}_S \cdot \mathrm{d}\mathbf{r} = 0$ for a conservative field - giving:

$$\oint_C \mathbf{E}_S \cdot \mathrm{d}\mathbf{r} = 0.$$

Now apply Stokes' Theorem to the LHS:

$$\oint_C \mathbf{E}_S \cdot \mathrm{d}\mathbf{r} = \int_S \left[\nabla \times \mathbf{E}_S\right] \cdot \mathrm{d}\mathbf{S} = 0,$$

where S is any surface enclosed by path C. Since C and S are "arbitrary", we have

$$\nabla \times \mathbf{E}_S = \mathbf{0}.$$

This is not quite a Maxwell Equation – it's only valid for electrostatic fields – but it useful to know.

Since $\nabla \times \nabla \psi = \mathbf{0}$ for any scalar field ψ (general maths result – see formula book), this also shows us that we can choose to write $\mathbf{E}_S$ in terms of a potential function:

$$\mathbf{E}_S = -\nabla \phi.$$

The minus sign is a convention and $\phi(\mathbf{r})$ is the **electrostatic potential**.

What we do here is use two mathematical results to derive what must be true about electrostatic fields, starting with the fact that $\oint_C \mathbf{E}_S \cdot \mathrm{d}\mathbf{r} = 0.$

3.3 Electrostatics via the electrostatic potential.

All **electrostatic** fields $\mathbf{E}_S(\mathbf{r})$ are solutions to our two differential equations,

$$\nabla \cdot \mathbf{E}_S = \frac{\rho}{\varepsilon_0} \text{ and } \nabla \times \mathbf{E}_S = \mathbf{0}.$$

Then $\nabla \cdot \mathbf{E}_S = \nabla \cdot (-\nabla \phi) = \dfrac{\rho}{\varepsilon_0}$.

But $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$, the Laplacian of ϕ, so

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}.$$ **The Poisson equation.**

This scalar, second order PDE governs electrostatics.

If we want a solution in a region away from charges, where $\rho(\mathbf{r}) = 0$, then the equation to solve is

$$\nabla^2 \phi = 0.$$ **Laplace's equation.**

Laplace's equation is encountered in many branches of physics. Purcell [1] comments that mathematically the theory of classical fields is mostly a study of the solutions of this equation. The class of functions that satisfy Laplace's equation are called harmonic functions.

Exercise 04: The Poisson equation.

Verify that the Poisson's equation holds for the potential due to a sphere with radius R and uniform charge density ρ. [See Example 01]. Use the Laplacian in spherical polar coordinates.

$$\phi_{out} = \frac{\rho R^3}{3\varepsilon_0 r} \quad \text{and} \quad \phi_{in} = \frac{\rho R^2}{2\varepsilon_0} - \frac{\rho r^2}{6\varepsilon_0}.$$

Exercise 04 [Answer]: The Poisson equation.

Verify that the Poisson's equation holds for the potential due to a sphere with radius R and uniform charge density ρ. [See Example 01]. Use the Laplacian in spherical polar coordinates.

$$\phi_{out} = \frac{\rho R^3}{3\varepsilon_0 r} \quad \text{and} \quad \phi_{in} = \frac{\rho R^2}{2\varepsilon_0} - \frac{\rho r^2}{6\varepsilon_0}.$$

From the Formula Book: Spherical polars:

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}$$

As we have only a dependence on r.

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right)$$

For $\phi_{out} = \dfrac{\rho R^3}{3\varepsilon_0 r}$;

$$r^2 \frac{\partial \phi_{out}}{\partial r} = r^2\left(-\frac{\rho R^3}{3\varepsilon_0 r^2}\right) = -\frac{\rho R^3}{3\varepsilon_0}$$ and thus, $\nabla^2 \phi_{out} = 0$ as expected as $\rho = 0$ for $r > R$.

For $\phi_{in} = \dfrac{\rho R^2}{2\varepsilon_0} - \dfrac{\rho r^2}{6\varepsilon_0}$, we see that $\dfrac{\rho R^2}{2\varepsilon_0}$ is a constant and so vanishes. For the second term.

$$r^2 \frac{\partial \phi_{in}}{\partial r} = r^2\left(-\frac{2r\rho}{6\varepsilon_0}\right) = -r^3 \frac{\rho}{3\varepsilon_0}$$

Thus,

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_{in}}{\partial r}\right) = -r^2 \frac{\rho}{\varepsilon_0},$$ and so,

$$\nabla^2 \phi_{in} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi_{in}}{\partial r}\right) = -\frac{\rho}{\varepsilon_0}$$ as expected.

3.4 Theorem: Laplace's equation and average potentials.

Theorem: If ϕ satisfies Laplace's equation, then the average value of ϕ over the surface of any sphere (not necessarily a small sphere) is equal to the value of ϕ at the centre of the sphere.

Proof.

Consider a function f and its average value over the surface of a sphere of radius r. We choose the origin of the coordinate system to be the centre of the sphere. The area of the sphere is $A = 4\pi r^2$. We can then write the average of the surface of the sphere as

$$f_{avg,r} = \frac{1}{A}\int f \, \mathrm{dA} = \frac{1}{4\pi r^2}\int f r^2 \mathrm{d}\Omega = \frac{1}{4\pi}\int f \, \mathrm{d}\Omega,$$

where $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the solid-angle element. We can take the r^2 outside the integral and cancel as it is constant over the surface that we defined. Now, this is trickier, we will differentiate both sides with $\mathrm{d}/\mathrm{d}r$ and then we will be able to use the divergence theorem. We also use $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = 1$.

$$\frac{\mathrm{d}f_{avg,r}}{\mathrm{d}r} = \frac{1}{4\pi}\int \frac{\mathrm{d}f}{\mathrm{d}r}\mathrm{d}\Omega = \frac{1}{4\pi}\int \hat{\mathbf{r}}\frac{\mathrm{d}f}{\mathrm{d}r}\cdot\hat{\mathbf{r}}\mathrm{d}\Omega = \frac{1}{4\pi r^2}\int \hat{\mathbf{r}}\frac{\mathrm{d}f}{\mathrm{d}r}\cdot\hat{\mathbf{r}}r^2\mathrm{d}\Omega$$

We have moved around an r^2 again and we have constructed $\hat{\mathbf{r}}r^2\mathrm{d}\Omega = \mathrm{d}\mathbf{S}$, the vector area element of the surface.

We also have $\hat{\mathbf{r}}\frac{\mathrm{d}f}{\mathrm{d}r}$ which is the $\hat{\mathbf{r}}$ component of ∇f in spherical coordinates and the other components would give *0* when dotted with $\mathrm{d}\mathbf{S}$. We can thus write, using the divergence theorem,

$$\frac{\mathrm{d}f_{avg,r}}{\mathrm{d}r} = \frac{1}{4\pi r^2}\int \nabla f\cdot d\mathbf{S} = \frac{1}{4\pi r^2}\int \nabla\cdot\nabla f \, \mathrm{d}V$$

So, $$\underline{\frac{\mathrm{d}f_{avg,r}}{\mathrm{d}r} = \frac{1}{4\pi r^2}\int \nabla^2 f \, \mathrm{d}V}$$

This is totally general for our function f and our sphere with radius r.

Now, For Laplace's equation as $\nabla^2 f = 0$ everywhere then $\mathrm{d}f_{avg,r}/\mathrm{d}r = 0$ everywhere. In other words, the average value of the function does not change with the radius of the sphere. All spheres centred at the same point have the same average value of f. But – the average over the surface of an infinitesimal sphere is the value at the centre of the sphere. Therefore, if $\nabla^2 f = 0$ at the centre of the sphere, the average value of f over the surface of a sphere (of any size) equals the value at the centre,

$$\nabla^2 f = 0 \quad \Rightarrow \quad f_{avg,r} = f_{centre.}$$ This is the theorem stated above.

Exercise 05: Earnshaw again.

Explain why the theorem above, proves Earnshaw's theorem that it is impossible to construct an electrostatic field that will hold a charged particle in stable equilibrium in empty space

Exercise 05 [Solution]: Earnshaw again.

Explain why the theorem above, proves Earnshaw's theorem that it is impossible to construct an electrostatic field that will hold a charged particle in stable equilibrium in empty space.

Proof.

An electrostatic field in empty space obeys the Laplace equation that $\nabla^2\phi = 0$. Thus, it follows that as ϕ satisfies Laplace's equation, then the average value of ϕ over the surface of any sphere (not necessarily a small sphere) is equal to the value of ϕ at the centre of the sphere.

A stable position for a charged particle must be one where the potential ϕ is either lower than the surrounding points (if the particle is positively charged) or higher than all neighbouring points (negatively charged particle).

This is not possible for a function whose average value over a sphere is always equal to it value at the centre. Thus, it is impossible to construct an electrostatic potential to give the electric field to maintain a charge in equilibrium.

Average value theorem: Why is this important?

Following on from the discussion in the exercise, the consequence is that for a potential satisfying Laplace's equation there can be no local maximum or minima: the extreme values of ϕ must occur at the boundaries.

Additionally, the fact that the average value of ϕ over the surface of any sphere (not necessarily a small sphere) is equal to the value of ϕ at the centre of the sphere suggests the **method of relaxation** which can be used for computational solutions to Laplace's equation.

You start with specified values for the potential at the boundary and reasonable guesses for the potential on an **appropriate** grid of points. You then iterate assigning to each point the average of its nearest neighbours, and so on. After a few iterations the numbers settle down so that subsequent iterations produce negligible changes. A numerical solution to Laplace's equation has been achieved.

3.5 Solutions to the Poisson equation.

For a single charge Q at the origin,

$$\mathbf{E}_S(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{\mathbf{e}}_r,$$

and we now have $\mathbf{E}_S = -\nabla\phi$. In this case we can find $\phi(\mathbf{r})$ by inspection:

Since,

$$-\nabla\left(\frac{1}{r}\right) = -\hat{\mathbf{e}}_r\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = +\frac{\hat{\mathbf{e}}_r}{r^2},$$ [Problem sheet]

it follows that the electrostatic potential due to a single charge at the origin is

$$\phi(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r}.$$

For a general charge distribution $\rho(\mathbf{r}')$ within volume V', superposition of electrostatic potentials yields,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V'.$$

Example 02: Laplace: Conducting sphere in a uniform field.

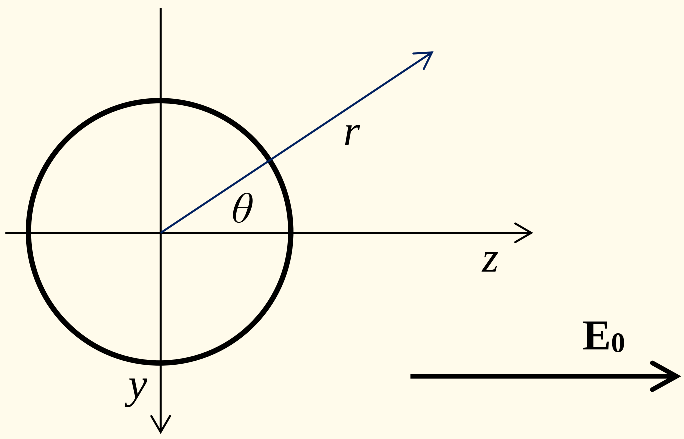
Suppose an earthed conducting sphere of radius R is placed in a uniform field of strength $\mathbf{E_0}$. The field immediately around the sphere will become distorted owing to the induced charges on the surface of the sphere, but the field at large distances will approach $\mathbf{E_0}$. Find an expression for the potential around the sphere and hence the field.

Proper Answer!

Let the potential be Φ. There are no charges, for $r > R$ so we look for a solution to Laplace's equation, $\nabla^2\Phi = 0$, in spherical coordinates assuming a solution $\Phi(r,\theta) = S(r)T(\theta)$. We place the centre of the sphere at the origin of the coordinate. We orient the coordinates so that the z-axis is in the direction of the electric field $\mathbf{E_0}$. By azimuthal symmetry, there will be no function for the coordinate ϕ. The maths of the solution will be in spherical harmonic functions – and that's a maths problem you can do when it's been covered in your maths unit.

Educated Guess Answer.

Coordinates as above. Look at a slice through the middle of the sphere (z-y plane).



At long distances from the sphere ($r \to \infty$), the electric field will tend to the uniform field $\mathbf{E_0}$.

If for $r \to \infty$, $\mathbf{E} = E_0\hat{\mathbf{k}}$ then this gives a contribution $\Phi_1 = -E_0 z = -E_0 r\cos\theta$ as we have $z = r\cos\theta$.

We will have induced charges on the sphere. We can guess that the effect of the polarised charges will fall off with distance, but it is likely that it will have the same dependence on θ so the boundary conditions will work out. We can guess for this contribution: $\Phi_2 = Ar^{-2}\cos\theta$.

We have a boundary condition to match that $V = 0$ at $r = R$ for the total potential. We have

$$\Phi = \Phi_1 + \Phi_2 = -E_0 r\cos\theta + Ar^{-2}\cos\theta$$

and the boundary condition, for $r = R$

$$\left(-E_0 R + AR^{-2}\right)\cos\theta = 0 \text{ giving } A = E_0 R^3$$

Thus, we decide

$$\Phi = -E_0 r\cos\theta + E_0 R^3 r^{-2}\cos\theta$$

$$= -E_0 r\cos\theta\left(1 - \frac{R^3}{r^3}\right)$$

This expression for the potential shows that the potential outside the sphere is due to the uniform field and the potential from a dipole of moment $\mathbf{p} = 4\pi\varepsilon_0 \mathbf{E_0} R^3$ situated at the centre of the sphere. [See Y1 E&M – dipoles]

Inside the earthed sphere, it is metallic so all at the same potential, $\Phi = 0$ for $r \le R$.

The electric field.

$$E_r = -\frac{\partial\Phi}{\partial r} = E_0\cos\theta\left(1 + 2\frac{R^3}{r^3}\right)$$

and

$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -E_0\sin\theta\left(1 - \frac{R^3}{r^3}\right)$$

Now, we can look at the surface of the sphere where $r = R$ using the expressions above.

We see that for $r = R$, $E_\theta = 0$. This is because the charge moves around the sphere until it is in equilibrium (electrostatics here) and the field produced is not exerting a force to cause charge to move.

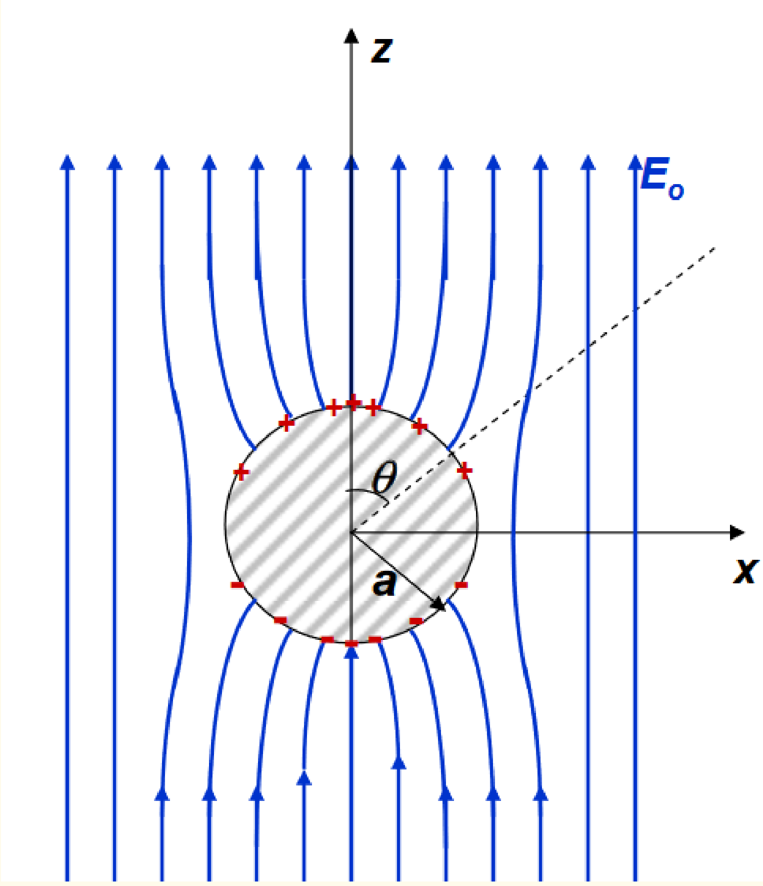
At the surface we can see that,

$$E_r = 3E_0 \cos\theta .$$

This field is due to the charges on the surface and so using a Gaussian surface (small pillbox) we can work out the induced charge density on the surface. This gives us,

$$\sigma(\theta) = 3\varepsilon_0 E_0 \cos\theta .$$

The solution looks like this:



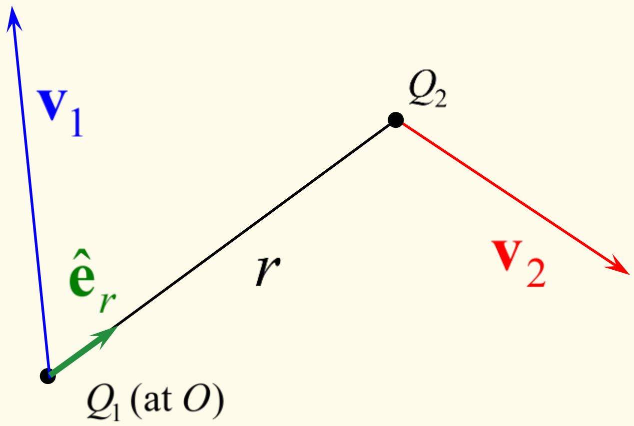
[Image from Farhan Rana – Cornell University, Boundary Value Problems 2007

https://courses.cit.cornell.edu/ece303/Lectures/lecture8.pdf]

Lecture 04: The magnetic field $\mathbf{B}$.

To deduce the origin of the magnetic field, consider a "thought experiment", in which we revisit the two charges Q_1 and Q_2 used to write down Coulomb's Law.

This time, let each charge move with constant velocity ($\mathbf{v}_1$ and $\mathbf{v}_2$) and re-measure the force between them:



If we could do this experiment, we would find an **additional** force $\mathbf{F}_m$ between the charges.

Total force:

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m.$$

Experimental Force	$\mathbf{F}_e = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_1 Q_2}{r^2}\hat{\mathbf{e}}_r$	$\mathbf{F}_m = \dfrac{\mu_0}{4\pi}\dfrac{Q_1 Q_2}{r^2}\mathbf{v}_2 \times (\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$ $\mu_0 =$ permeability of free space $= 4\pi \times 10^{-7}$ $\mathrm{Ns^2C^{-2}}$ or $\mathrm{Hm^{-1}}$
Use Q_2 as a test charge	$\mathbf{F}_e = Q_2\mathbf{E}$ defines $\mathbf{E}(\mathbf{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{\mathbf{e}}_r.$	$\mathbf{F}_m = Q_2(\mathbf{v}_2 \times \mathbf{B})$ defines $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi}\dfrac{Q_1}{r^2}(\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$

Pb 1

Note. Examining our new term:

$$\mathbf{F}_m = Q_2(\mathbf{v}_2 \times \mathbf{B}) \text{ defines where } \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Q_1}{r^2}(\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$$

1. $\mathbf{B}(\mathbf{r})$ is the **magnetic field** due to a single charge Q_1 at the origin, moving at constant velocity $\mathbf{v}_1$. Units are Ns/(Cm) or Tesla T.

2. $\mathbf{B} = \mathbf{0}$ if $\mathbf{v}_1 = \mathbf{0}$ or if $\mathbf{v}_1$ is parallel to $\hat{\mathbf{e}}_r$.

3. $\mathbf{B}$ is perpendicular to $\hat{\mathbf{e}}_r$ (and to $\mathbf{v}_1$)

   What does this imply about $\nabla \cdot \mathbf{B}$?

4. The total force on charge Q_2 is

   $$\mathbf{F} = Q_2\left(\mathbf{E} + \mathbf{v}_2 \times \mathbf{B}\right) \qquad \textbf{The Lorentz force}$$

$\mathbf{E}$ and $\mathbf{B}$ and special relativity.

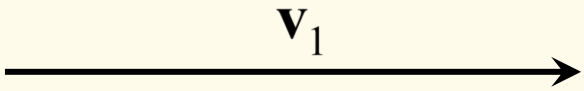
So far, we have

Two static charges: $\rightarrow \mathbf{F}_e \quad \rightarrow \mathbf{E}$

Giving each a constant velocity $\rightarrow \mathbf{F}_m \rightarrow \mathbf{B}$

where

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{Q_1}{r^2} (\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$$



What happens if we move to a new **inertial reference frame** (IRF), moving along with the same velocity as Q_1?

Now $\mathbf{v}_1' = \mathbf{0}$ which $\Rightarrow \mathbf{B}' = \mathbf{0}$!!

We just transformed away our new field!

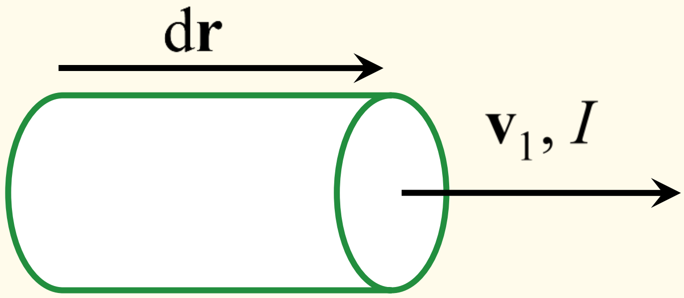
Special Relativity tells us:

The Laws of Physics are the **same** in all IRFs.

Therefore, electric fields and magnetic fields must essentially be the same thing, viewed from different frames.

4.1 $\mathbf{B}$-field due to a current element.

A current element is a small region within which all charges are moving with the same velocity:



For a single charge at the origin

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Q_1}{r^2}(\mathbf{v}_1 \times \hat{\mathbf{e}}_r)$$

$Q_1\mathbf{v}_1 =$ charge x distance / time

$=$ current x distance

So, replace $Q_1\mathbf{v}_1$ by $I\,d\mathbf{r}$ to find $\mathbf{B}$ from a current element at the origin:

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \frac{d\mathbf{r} \times \hat{\mathbf{e}}_r}{r^2}.$$

factoring out the element

This is the Biot-Savart Law: (Experimental result of magnetostatics (1820))

- Confirms that $\mathbf{B}$ from our thought experiment is the magnetic field
- $\mathbf{v}_1$ = constant, so only Direct Current (DC) so far.

≠ which velocities aren't

Divergence of $\mathbf{B}$ - an informal approach.

Relativity argument $\Rightarrow$ $\mathbf{B}$-fields have no direct "point source" of their own; they arise from moving sources of $\mathbf{E}$-fields.

Also, the Biot-Savart Law $\Rightarrow$ $\mathbf{B}$ - field lines are perpendicular to $I\,\mathrm{d}\mathbf{r}$ and to $\hat{\mathbf{e}}_r$, so they cannot come directly to or from the current element.

Intuition would therefore lead us to expect

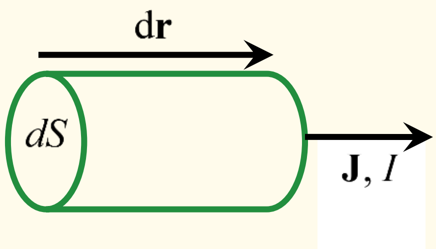
$\nabla \cdot \mathbf{B} = 0$ (i.e. "There are no magnetic charges")

There are no sources or sinks of magnetic field

This is, in fact, a correct field equation, and is our 2nd "complete" Maxwell Equation.

4.2 Generalising the Biot - Savart Law.

Define the **vector current density** $\mathbf{J}(\mathbf{r}')$:



- Direction of $\mathbf{J}$ $\rightarrow$ direction of current flow at $\mathbf{r}'$
- Magnitude of $\mathbf{J}$ $\rightarrow$ current crossing unit area perpendicular to $\mathbf{J}$

Thus $I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$ - current is the flux of the vector current density

(noted later in my handwriting)

A current element at the origin ($\mathbf{r}'=\mathbf{0}$) gives rise to

$$\mathrm{d}\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\,\mathrm{d}V\,\frac{\mathbf{J}\times\hat{\mathbf{e}}_r}{r^2}.$$

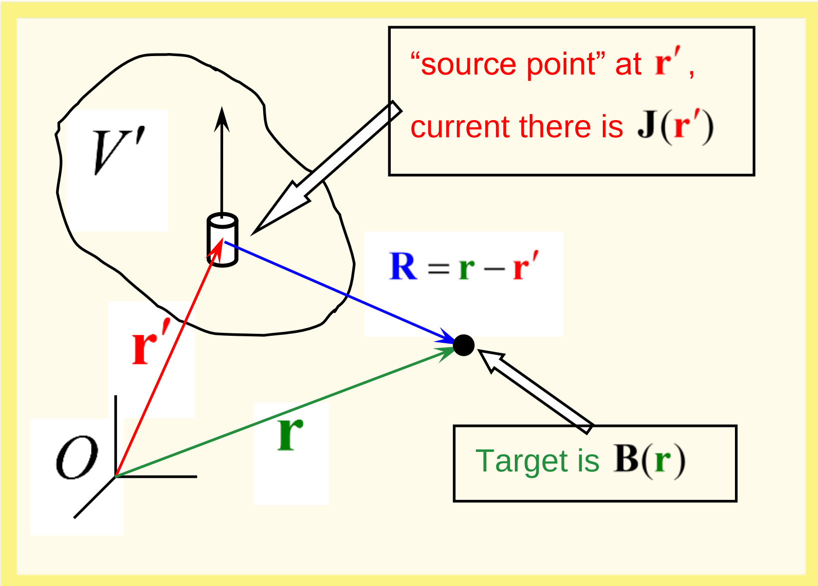
To verify this, we know that $I=|\mathbf{J}|\mathrm{d}S$ and $\mathbf{J}$ is parallel to $\mathrm{d}\mathbf{r}$. Therefore $I\,\mathrm{d}\mathbf{r}=\mathbf{J}\,\mathrm{d}S\mathrm{d}r=\mathbf{J}\,\mathrm{d}V$.

Aside: You can also potentially have a surface current density $\mathbf{K}$ where the surface current density is in amperes per meter. Then the source element is $\mathbf{K}\mathrm{d}S$. Overall, for the source elements of the magnetic field, we have $I\mathrm{d}\mathbf{l}\equiv\mathbf{K}\mathrm{d}S\equiv\mathbf{J}\mathrm{d}V$. All with units of amp-meter.

Now for the general current distribution…

For a single current element

$$\mathrm{d}\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\,\mathrm{d}V'\,\frac{\mathbf{J}(\mathbf{r}')\times\hat{\mathbf{R}}}{R^2}.$$



Then, by the superposition principle,

$$\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\int_{V'}\mathbf{J}(\mathbf{r}')\times\frac{\hat{\mathbf{R}}}{R^2}\,\mathrm{d}V'.$$

Lecture 05: Divergence of $\mathbf{B}$.

We have $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2} \mathrm{d}V'$.

5.1 Divergence of $\mathbf{B}$ - a more formal approach.

Find $\nabla \cdot \mathbf{B}$ for a general current distribution as found above.

We now find the divergence (w.r.t. $\mathbf{r}$, not $\mathbf{r}'$). Assume it is OK to swap the order of $\nabla \cdot$ and $\int_{V'}$:

vector current density

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \nabla_{\mathbf{r}} \cdot \left\{ \mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2} \right\} \mathrm{d}V'.$$

Concentrate on the integrand. Use the vector identity (see Formula Book)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}):$$

$$\nabla_{\mathbf{r}} \cdot \left\{ \mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2} \right\} = \frac{\hat{\mathbf{R}}}{R^2} \cdot \{\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}')\} - \mathbf{J}(\mathbf{r}') \cdot \left\{ \nabla_{\mathbf{r}} \times \frac{\hat{\mathbf{R}}}{R^2} \right\}.$$

This is always zero. Why?

$\nabla_{\mathbf{r}} \times \frac{\hat{\mathbf{R}}}{R^2} = \mathbf{0}$ as $\frac{\hat{\mathbf{R}}}{R^2} = -\nabla_{\mathbf{r}} \frac{1}{R}$ and $\nabla \times \nabla \psi = \mathbf{0} \quad \forall \psi$

vector current density

$\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{r}') = \mathbf{0}$ as $\mathbf{J}$ is not a function of $\mathbf{r}$.

Hence the integrand is 0 and $\nabla \cdot \mathbf{B} = 0$ as expected.

(N.B. true at all $\mathbf{r}$ - even inside V'.)

- Direct' of J → direct current flow at r'
- Magnitude of J → current crossing unit area ⊥ J

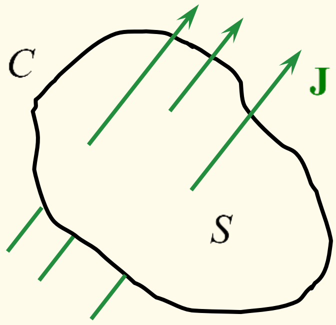
• $I = \int_S J \, dS$ current flux of the

5.2 The curl of $\mathbf{B}$ (DC only).

We start from Ampère's law, another law derived from experiments:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 \,(\text{Total DC current passing through } C)$$

where C is any closed curve bounding a surface S.



i.e. $\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$

Use Stokes' Theorem on the LHS:

$$\rightarrow \quad \int_S \nabla \times \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

Again, since C, S, $\mathbf{J}$ are arbitrary, we must have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Summary of this result

"Current is a rotational source of magnetic field"

Not quite a Maxwell Equation – restricted to DC.

Electrostatic → $\underline{\nabla} \cdot \underline{E} = \rho$
→ $\underline{\nabla} \times \underline{E} = 0$

Magnetostatics → $\underline{\nabla} \cdot \underline{B} = 0$
→ $\underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$

5.3 Time-varying currents and fields.

So far, we have for steady currents $\mathbf{J}$ and static charge distributions ρ placed in vacuum. We found:

1	$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$	Coulomb's Law / Gauss's Law
2	$\nabla \times \mathbf{E} = \mathbf{0}$	Electrostatic field is conservative
3	$\nabla \cdot \mathbf{B} = 0$	No magnetic monopoles
4	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	Ampère's Law

Note that $\mathbf{B}$ and $\mathbf{E}$ are still uncoupled.

That is, we still have "electricity" and "magnetism".

1 and 3 are completely correct (we think!)

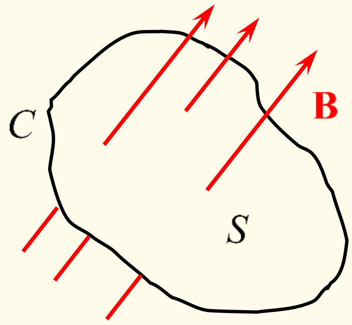
2 and 4 require modification for time-varying fields.

Faraday's Law.

A static $\mathbf{B}$ field induces current in a moving circuit (or a changing $\mathbf{B}$ field in a fixed circuit). The induced "emf" ε (actually an electric potential difference) is

$$\varepsilon = -\frac{\partial \Phi_B}{\partial t}, \qquad \textbf{Faraday's Law of Induction}$$

where



$$\Phi_B = \text{magnetic flux through circuit } C$$

$$\Phi_B = \int_S \mathbf{B}\cdot d\mathbf{S}.$$

ε is an electric potential difference **not** derived from Coulomb's Law – it must be due to an extra force $\mathbf{F}'$.

Assume that the induced current consists of a single charge Q moving around C. Then,

$$Q\varepsilon = \text{work} = \oint_C \mathbf{F}'\cdot d\mathbf{r}\text{, so}$$

$$\varepsilon = \frac{1}{Q}\oint_C \mathbf{F}'\cdot d\mathbf{r} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S}$$

We now have two choices:

1. $\mathbf{B}$ is static; circuit moves

2. Circuit is static; $\mathbf{B}$ changes with time.

In 1., $\mathbf{F}' = Q(\mathbf{u}\times\mathbf{B})$ where $\mathbf{u}$ is extra velocity of Q due to motion of circuit

In 2., $\mathbf{u} = \mathbf{0}$, so force must be due to an extra electric field $\mathbf{F}' = Q\mathbf{E}'$.

Make choice 2.

Expression for ε now reads, with $\mathbf{E}' = \mathbf{F}'/Q$.

$$\varepsilon = \oint_C \mathbf{E}' \cdot d\mathbf{r} = -\frac{\partial}{\partial t}\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

Use Stokes' Theorem on LHS.

Change order of $\partial / \partial t$ and $\int_S$ on RHS – these coordinates are not linked.

$$\int_S \nabla \times \mathbf{E}' \cdot \mathrm{d}\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S}$$

Since S is arbitrary, we have

$$\nabla \times \mathbf{E}' = -\frac{\partial \mathbf{B}}{\partial t}.$$

If an electrostatic field $\mathbf{E}_S$ is also present, superposition gives the total electric field as

$$\mathbf{E} = \mathbf{E}_S + \mathbf{E}'$$

So

$$\begin{aligned} \nabla \times \mathbf{E} &= \nabla \times \mathbf{E}_S + \nabla \times \mathbf{E}' \\ &= \quad 0 \quad + \quad -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned}$$

Thus

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$ **Third of Maxwell's equations,**

Notes: Maxwell's Third Law (Faraday's Law of Induction)

- A general electric field is not conservative. We cannot therefore use electrostatic potentials in the general, time-varying case.
- A time-varying $\mathbf{B}$ is a rotational source of $\mathbf{E}$.
- This is a first link between $\mathbf{B}$ and $\mathbf{E}$.

Paul's Extra Comments.

Inductors: $V = L\dfrac{\mathrm{d}I}{\mathrm{d}t}$.

Exercise 06: PH10006: Q6, 2017.

(a) Write down expressions for Ampère's Law and Gauss's Law for the magnetic field, illustrating them with diagrams and clearly describing the symbols you use. (5)

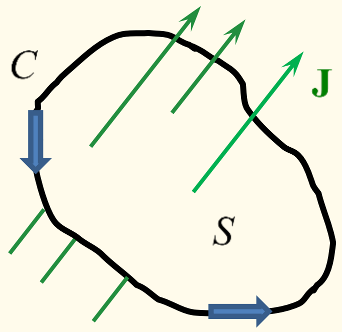
(b) Show that the magnetic field inside a long cylindrical solenoid with n turns per unit length, carrying a current I, can be written $B = \mu_0 nI$, where μ_0 is the permeability of free space, clearly explaining how the appropriate physical laws are used and stating what approximations are made. (5)

(c) An oscillating current $I = I_0 \sin \omega t$ is driven through the solenoid which has radius R, where t is the time elapsed and ω is the angular frequency of the current. Determine the magnitude of the electric field $E(r)$ for $r < R$ and $r > R$ at time $t = 2\pi/\omega$ and make a clearly labelled sketch of $E(r)$. (5)

Exercise 06 [Solutions]: PH10006/51: Q6, 2017.

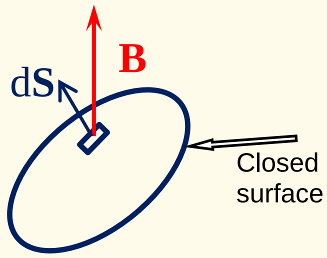
(a) Write down expressions for Ampère's Law and Gauss's Law for the magnetic field, illustrating them with diagrams and clearly describing the symbols you use. (5)

Ampère's law: $\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \sum_i I_i = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{S}$



Note sign convention: Right-hand-screw relation between the surface normal and the direction in which the circulation is taken.

Gauss's law: $\int_S \mathbf{B}\cdot d\mathbf{S} = 0$



Note sign convention: Flux is measured out of the closed surface.

(b) Show that the magnetic field inside a long cylindrical solenoid with n turns per unit length, carrying a current I, can be written $B = \mu_0 nI$, where μ_0 is the permeability of free space, clearly explaining how the appropriate physical laws are used and stating what approximations are made. (5)

There are three different ways that I know to do this. I think it's useful to look at all of them to compare them.

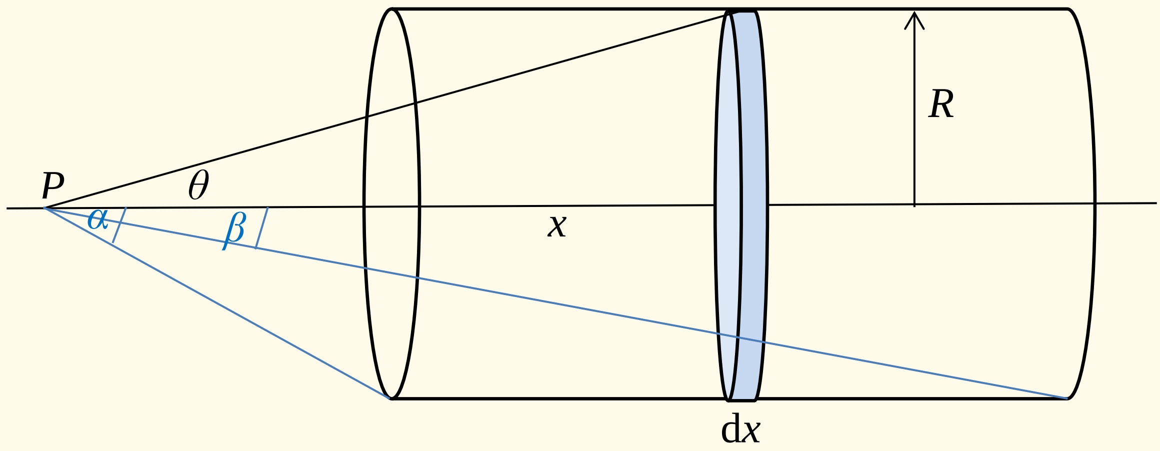
[A] [Back to basics] Use the Biot-Savart law and integrate all the current carrying elements. This is long winded – but makes few assumptions.

We have to start: $d\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} I \dfrac{d\mathbf{r}\times\hat{\mathbf{e}}_r}{r^2}$ or $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} I \int \dfrac{d\mathbf{r}\times\hat{\mathbf{e}}_r}{r^2}$.

If you use this to find the magnetic field on the axis of a circular coil of radius R, where the coil is centred on the origin and is located in the y-z plane. Then you get,

$$\mathbf{B}(x) = \frac{\mu_0 I}{4\pi} \frac{R^2}{\left(R^2 + x^2\right)^{\frac{3}{2}}} \hat{\mathbf{i}}$$

You can then integrate this over the length of a solenoid where you consider the solenoid as a collection of coils with n turns per unit length. You can do this from a point outside the solenoid:



The angle from the point of calculation, P, on the axis to the nearest coil is α and to the farthest coil the angle is β.

Integration gives, $B(P) = \frac{1}{2}\mu_0 nI\left(\cos\beta - \cos\alpha\right)$.

For a very long solenoid, for the field inside the solenoid, $\alpha \to \pi$ and $\beta \to 0$. This gives,

$$B_{sol} = \mu_0 nI$$

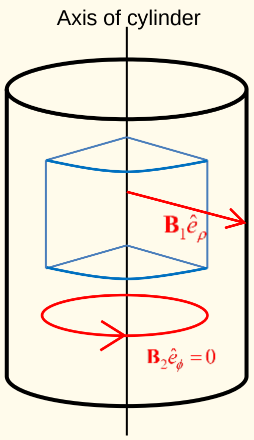
[B] Ampere's law and some assumptions – or starting points.

We neglect any end effects and assume that the current flowing parallel to the axis of the solenoid makes negligible contribution. [Comment: We also did this above – we never considered any angle to the current turns that threads them down the cylinder].

Y1 argument: The symmetry of the geometry and Gauss's law imply that there is no radial component of magnetic field. The field is then parallel to the axis.

[Year 2 - Comment: The cylindrical symmetry suggests a field that can be represented as $\mathbf{B}(\rho,\phi,z)=\mathbf{B}_1\hat{e}_\rho+\mathbf{B}_2\hat{e}_\phi+\mathbf{B}_3\hat{e}_z$.

We know from Bio-Savart that $\mathrm{d}\mathbf{B}=\frac{\mu_0}{4\pi}I\frac{\mathrm{d}\mathbf{l}\times\hat{\mathbf{e}}_r}{r^2}$. Here the current elements are of the form $I\mathrm{d}l\hat{\mathbf{e}}_\phi$ and so the cross product means that there can be no $\mathbf{B}_2\hat{e}_\phi$.

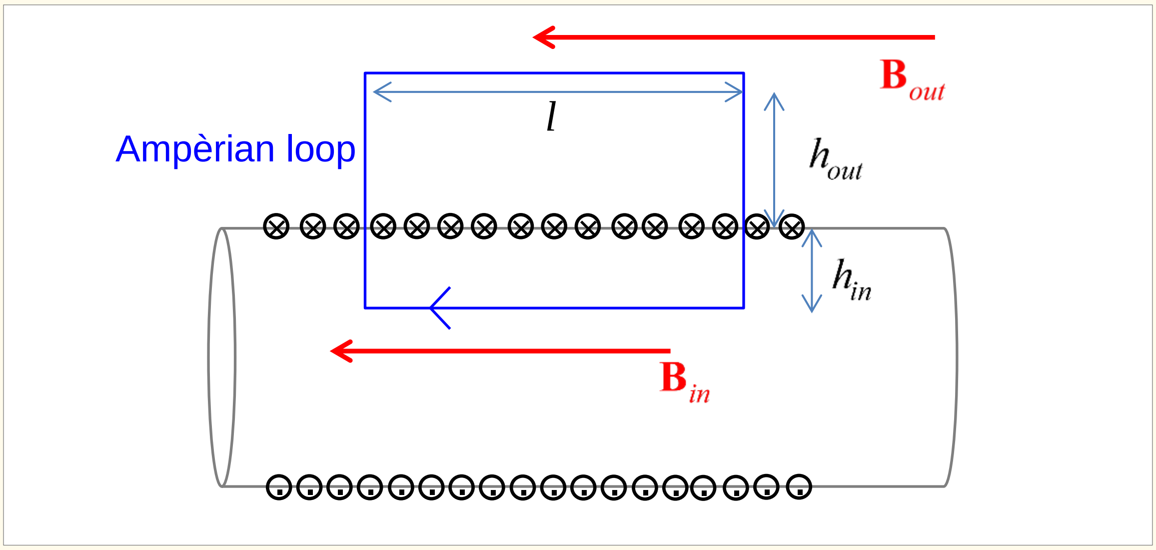


Gauss's law that $\int_S\mathbf{B}\cdot\mathrm{d}\mathbf{S}=0$ must be true for **any** Gaussian surface in the solenoid and this means that $\mathbf{B}_1\hat{e}_\rho=0$. Or, we can argue that the $\mathbf{B}$-field must form loops – what form of loop can we imagine that has a radial component that is constant for all z? None.

Thus, the field is parallel to the axis.

Alternatively, cite Bio-Savart for a single loop to show that the field is axial and extend the concept to say that we assume an axial field for the solenoid.

Now – a diagram to consider what the field could be.



The solenoid has n turns per unit length, carrying a current I.

By Ampère's law, $$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \sum_i I_i$$

$$B_{in} l - B_{out} l = \mu_0 n l I$$

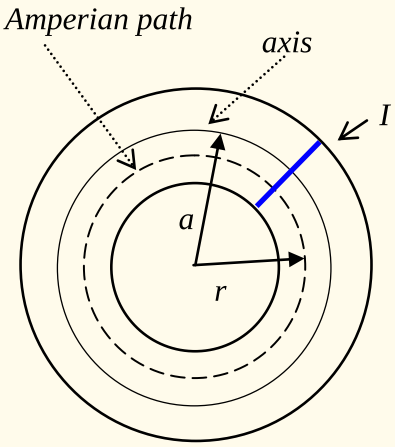
This expression is independent of h_{in} and h_{out} and the physics of the field can't depend on where we place h_{in} and h_{out}.

In the limit of large h_{out}, $(h_{out} \to \infty)$ the field outside must be zero, and so $\mathbf{B}_{out} = 0$ for all values of h_{out}. Thus,

$$B_{in} = \mu_0 n I \ .$$

This is constant within the cylinder as it does not depend on h_{in}.

[C] [A clever trick] Consider a toroidal solenoid.



The toroidal solenoid has a radius to its axis of a, with n turns per unit length around the solenoid, each carrying current I. Thus, the total number of turns on the solenoid is $2\pi an$.

Consider an Ampèrian loop in a circle at radius r.

We assume that – from Biot-Savart – that the field is axial along the toroid. We don't need to consider the field outside the solenoid. Applying Ampere's law:

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \sum_i I_i \quad \rightarrow B(2\pi r) = \mu_0 (2\pi anI)$$

$$B = \mu_0 nI \frac{a}{r}$$

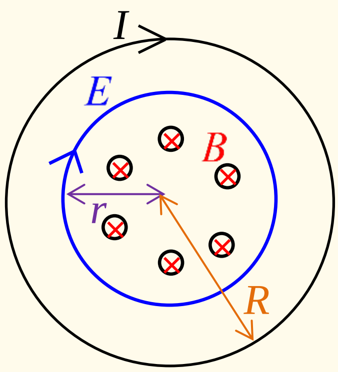
Thus, there is some variation of the field away from the axis of the toroid. On the axis, $B = \mu_0 nI$.

To consider a very long straight solenoid, we have $a \rightarrow \infty$. The toroid has finite width so $a/r \rightarrow 1$. We find $B_{solenoid} = \mu_0 nI$

(c) An oscillating current $I = I_0 \sin \omega t$ is driven through the solenoid which has radius R, where t is the time elapsed and ω is the angular frequency of the current. Determine the magnitude of the electric field $E(r)$ for $r < R$ and $r > R$ at time $t = 2\pi/\omega$ and make a clearly labelled sketch of $E(r)$. (5)

The oscillating current gives a changing flux. We use Faraday's law.

$$\varepsilon = \oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{r} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} .$$



The $\mathbf{B}$-field only exists inside the solenoid and is uniform down the solenoid. Thus, the flux down the solenoid is changing and so our closed loop will be circles around the axis of the solenoid through which the flux is flowing.

Note: $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$. The changing magnetic field is a rotational source of electric field. We now have loops of electric field.

We have for Faraday's law in integral form:

$$r < R, \quad 2\pi r E = -\frac{\partial}{\partial t}\left(\pi r^2 B\right) = --\frac{\partial}{\partial t}\left(\pi r^2 \mu_0 n I\right) = -\pi r^2 \mu_0 n \frac{\partial}{\partial t}\left(I_0 \sin \omega t\right)$$

$$E = -\frac{1}{2}\mu_0 n \omega r I_0 \cos \omega t$$

and

$$r > R, \quad 2\pi r E = -\frac{\partial}{\partial t}\left(\pi R^2 B\right) = --\frac{\partial}{\partial t}\left(\pi R^2 \mu_0 n I\right) = -\pi R^2 \mu_0 n \frac{\partial}{\partial t}\left(I_0 \sin \omega t\right)$$

$$E = -\frac{R^2}{2r}\mu_0 n \omega I_0 \cos \omega t$$

At time $t = 2\pi/\omega$, $\cos\omega t = 1$, so we get.

$$E_{r<R} = -\frac{\mu_0 n\omega I_0}{2} r \text{ and } E_{r>R} = -\frac{1}{r}\frac{R^2 \mu_0 n\omega I_0}{2}$$

At $r = R$, $E_{r=R} = -\frac{\mu_0 n\omega I_0 R}{2}$ in both expressions so the field is continuous at the boundary. [I'll leave the sketch for you to do]

Lecture 06: The Maxwell-Ampère Law.

To see that $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$ is incomplete, take the divergence of both sides…

$$\underbrace{\nabla\cdot(\nabla\times\mathbf{B})}=\mu_0\nabla\cdot\mathbf{J}$$

always 0 - a vector identity ∇ =

(But $\nabla\cdot(\nabla\times\mathbf{A})=0$ always 0 for any $\mathbf{A}$ - a vector identity.)

$\therefore\quad \nabla\times\mathbf{B}=\mu_0\mathbf{J}$ can only be correct when $\nabla\cdot\mathbf{J}=0$; i.e. when there are no direct sources of current.

To find a general expression for $\nabla\cdot\mathbf{J}$, consider…

Conservation of Charge.

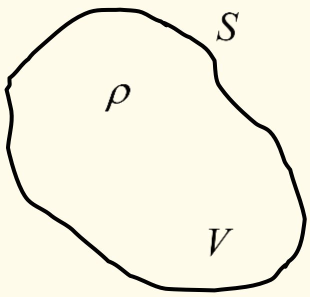
In English:

Electric charge is neither created nor destroyed.

This has been verified experimentally to at least 1 in 10^{20} 21

In Vector Calculus?

Consider an arbitrary volume V enclosed by surface S, containing charge with density $\rho(\mathbf{r},t)$.



Conservation of charge implies…

[Rate of flow of charge across surface S]

=

[Rate of change of charge contained within V]

or

$$\oint_S J\, ds = \frac{-\partial}{\partial t}\int_V \rho\, dV$$

$$\oint_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} = -\frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V\,.$$

Use the divergence theorem on the LHS

$$\rightarrow \quad \int_V \nabla\cdot\mathbf{J}\,\mathrm{d}V = \int_V\left(-\frac{\partial\rho}{\partial t}\right)\mathrm{d}V.$$

Since V is arbitrary, we obtain,

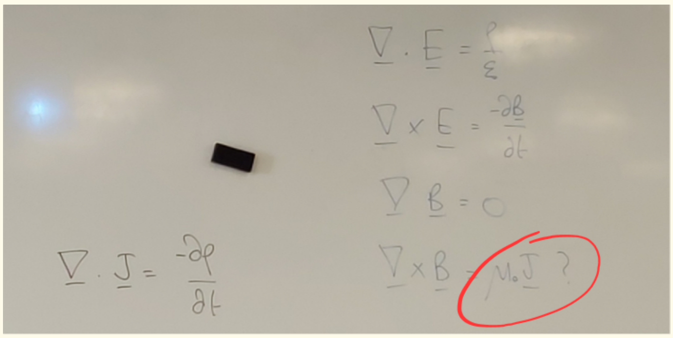
$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$$

The **Continuity Equation** – a mathematical expression of charge conservation.

From this, we see that $\nabla\cdot\mathbf{J}$ is **not** always 0 (as required by Ampère's Law), but $\nabla\cdot\mathbf{J}+\dfrac{\partial\rho}{\partial t}$ **is.**

To correct the Law, substitute for ρ from Gauss's Law:

$$\begin{aligned}0 &= \nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}\\ &= \nabla\cdot\mathbf{J}+\varepsilon_0\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E})\\ &= \nabla\cdot\left(\mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right).\end{aligned}$$



So to make Ampère's Law consistent with charge conservation, we can simply replace $\mathbf{J}$ with $\mathbf{J}+\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$; then the divergence of each side will be 0.

This yields the "Maxwell-Ampère Law".

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$ (Our 4th Maxwell Equation)

The extra term $\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ has same dimensions as $\mathbf{J}$. It is called the "vacuum displacement current density" - it acts just like a current density.

This was Maxwell's sole contribution to the Laws of Electricity and Magnetism, but it is a crucial one; it leads to the notion of **electromagnetism** and to **electromagnetic waves**.

Aside: Why did Faraday miss the displacement current?

Integrating the Maxwell-Ampere law over an arbitrary surface S enclosed by path C yields (see problem sheet question):

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{r} = \mu_0 I + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$

which shows that a time-varying $\mathbf{E}$-flux must induce a magnetic field.

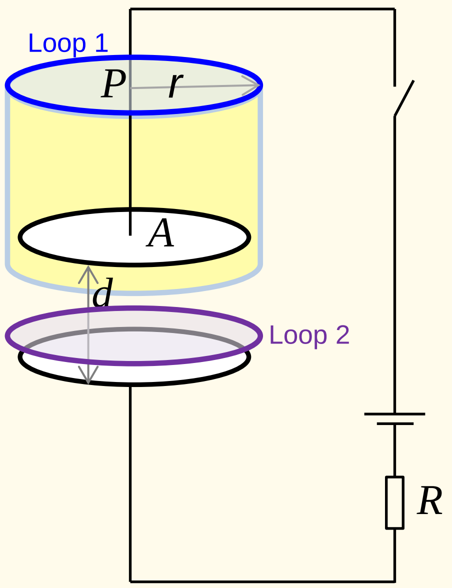
BUT $\varepsilon_0 \mu_0$ is a tiny number, so $\mathbf{E}$-flux must vary extremely rapidly to induce a detectable $\mathbf{B}$.

This happens in for example electromagnetic waves, which Faraday didn't know about. If we have $E = E_0 \sin \omega t$ then $\partial E / \partial t = \omega E_0 \cos \omega t$ and so high frequency oscillations make this more observable.

The effect was verified by Hertz in 1887, long after Maxwell predicted it.

Maxwell-Ampère Law for a Charging Capacitor.

This may have been briefly mentioned in Year 1. It's discussed in the Feynman lectures as an example to consider. Let's go through it more carefully now. Below is a diagram of a capacitor in a circuit with a cell of potential V. The switch is closed and the capacitor charges.



The capacitor is a parallel plate capacitor with circular plates with area A. We ignore edge effects etc. We assume that all lengths in the circuit are actually much greater than d, the separation of the capacitor plates.

All the resistance in the circuit is represented by a lumped resistance R.

For the capacitor: $C = \varepsilon_o \frac{A}{d}$

When the switch closes at $t = 0$ the capacitor charges according to,

$$Q = CV\left(1 - e^{-\frac{t}{CR}}\right).$$

From this we can see that the current flowing during the capacitor charging is given by

$$I = \frac{dQ}{dt} = +\frac{CV}{CR}\left(e^{-\frac{t}{CR}}\right) = \frac{V}{R}\left(e^{-\frac{t}{CR}}\right)$$

Thus, we can find the magnetic field around the wire as a function of time. We use loop 1 at point P in the figure. We assume that this is a long way from the capacitor that the wire long and straight at that point. Using the simple flat surface across the circle, Maxwell- Ampere's law for current density $\mathbf{J}$,

$$B(2\pi r) = \mu_0 I = \mu_0\left(\frac{V}{R}\left(e^{-\frac{t}{CR}}\right)\right) \quad \rightarrow B = \frac{\mu_0}{2\pi r}\frac{V}{R}\left(e^{-\frac{t}{CR}}\right)$$

At the capacitor, at any time the total charge is $Q = CV\left(1-e^{-\frac{t}{CR}}\right)$. This means that the charge density on each plate of the capacitor is

$$\sigma = \frac{Q}{A} = \frac{CV}{A}\left(1-e^{-\frac{t}{CR}}\right).$$

The electric field above a sheet of constant charge density of σ is by Gauss's law,

$$E = \frac{\sigma}{2\varepsilon_0}.$$

In the capacitor one plate is positively charge, the other negatively charged – the total electric field is.

$$E = \frac{\sigma}{\varepsilon_0} = \frac{CV}{A\varepsilon_0}\left(1-e^{-\frac{t}{CR}}\right) \text{ but } C = \varepsilon_o\frac{A}{d} \text{ so } E = \frac{V}{d}\left(1-e^{-\frac{t}{CR}}\right)$$

Now look at loop 2 on the diagram, it slices across the inside of the capacitor. We can now use the Maxell-Ampère law for the field inside the capacitor.

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 I + \varepsilon_0\mu_0\frac{\partial}{\partial t}\int_S \mathbf{E}\cdot d\mathbf{S}$$

There are no "real" current densities $\mathbf{J}$ here to give a current I. We do have a vacuum displacement current.

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{V}{d}\left(1-e^{-\frac{t}{CR}}\right)A \quad \rightarrow \frac{\partial}{\partial t}\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{VA}{dCR}e^{-\frac{t}{CR}}$$

Thus, using Maxwell-Ampere for loop 2 with a radius r.

$$B(2\pi r) = \mu_0\varepsilon_0\frac{VA}{dCR}e^{-\frac{t}{CR}}$$ but $C = \varepsilon_o\frac{A}{d}$ gives,

$$B = \frac{\mu_0}{2\pi r}\frac{V}{R}e^{-\frac{t}{CR}}$$, which is the same result for loop 2 as for loop 1.

You measure the same magnetic field at any point along the circuit.

There is one final and important point to be made here. When we write the integral form of the fourth Maxwell equation.

$$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad \Rightarrow \oint_C \mathbf{B}\cdot\mathrm{d}\mathbf{r} = \mu_0\int_S \mathbf{J}\cdot\mathrm{d}\mathbf{S} + \varepsilon_0\mu_0\frac{\partial}{\partial t}\int_S \mathbf{E}\cdot\mathrm{d}\mathbf{S}$$

The surface through which we calculate the flux can be **any** surface that spans the loop C. Thus, for Loop 1 to calculate the magnetic field at P, we can equally use the yellow open cylindrical surface that does not stretch across the loop but is like a mug shape with the bottom surface across the capacitor. This surface has no $\mathbf{J}$ flux through it but does have the vacuum displacement current through it.

Thus, we can see that $\mathbf{J}$ and $\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$ give the same result when used to find the magnetic field.

We now have:

Maxwell's Equations in a vacuum.

1	$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$	Gauss's Law
2	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	Faraday's Law
3	$\nabla \cdot \mathbf{B} = 0$	Absence of magnetic monopoles
4	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$	Maxwell-Ampère Law

KNOW THEM!

All electromagnetic fields in a vacuum, on all length scales down to those governed by quantum mechanics, are solutions to these equations.

Note we now have equations for **electromagnetism**. $\mathbf{E}$ is linked to $\mathbf{B}$ in two of the Maxwell equations. A changing magnetic field is a rotational source of electric field and, a changing electric field is a rotational source of magnetic field?

Could we have a self-supporting linked electromagnetic field? Well, we will now concentrate on wave solutions…

§ 2. Electromagnetic waves in a vacuum.

Lecture 07: The Wave Equation.

Maxwell's equations 1-4 are coupled, but may be de-coupled to give separate equations for $\mathbf{E}$ and $\mathbf{B}$:

start from an equation of the curl of ? $(\nabla \times ?)$ & curl both sides

For $\mathbf{E}$, take curl of equation 2:

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla \times \frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}).$$

$\nabla \times (\nabla \times ?) = \nabla \times \dots$

Use vector identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and substitute from the Maxwell equations: $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ and $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$:

$$\nabla\left(\frac{\rho}{\varepsilon_0}\right) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

- no $\mathbf{B}$

source terms

From now on we look only for solutions in **source-free** vacuum – we set $\rho = 0$; $\mathbf{J} = \mathbf{0}$, so

$$\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

compare the wave equation: $\nabla^2 \psi = \dfrac{1}{v^2} \dfrac{\partial^2 \psi}{\partial t^2}$

This is the wave equation for $\mathbf{E}$ in source-free vacuum.

Similarly, for $\mathbf{B}$, take curl of equation 4…

Substitute from Maxell equations for $\nabla \cdot \mathbf{B}$ and $\nabla \times \mathbf{E}$…

Set $\mathbf{J} = \mathbf{0}$…

to obtain the wave equation for $\mathbf{B}$ in source-free vacuum:

$$\nabla^2 \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Identical in form to the wave equation for $\mathbf{E}$. This implies both will have the same form of solution.

Among these solutions will be waves…

Remember that the general solution to the wave equation is (in one dimension).

$$\psi(x,t) = F(x - ct) + G(x + ct)$$

7.1 Revision of Waves.

In one dimension,

$$f(x,t) = A\cos(kx - \omega t),$$

represents a harmonic wave travelling in the $+x$ direction. It has **wavelength** $\lambda = 2\pi/k$ where k is the **wavenumber**; **frequency,** $\nu = \omega/2\pi$ where ω is the **angular frequency**; **amplitude** A and **phase speed** $v = \lambda\nu = \omega/k$.

Note that the sign of the phase $(kx - \omega t)$ is a matter of choice; $(\omega t - kx)$ is just as valid, as $\cos(\theta) = \cos(-\theta)$. However, having chosen the former, we must stick with it throughout our calculations.

As $\cos\theta$ is the real part of $\exp(i\theta)$, $f(x,t)$ can be written in exponential notation as

$$f(x,t) = \Re e\left(Ae^{i(kx-\omega t)}\right).$$

$\cos\theta + i\sin\theta = e^{i\theta}$

It is usual to omit the $\Re e(\ )$, and write

$$f(x,t) = Ae^{i(kx-\omega t)}$$

: we mean the real part ($\cos$)

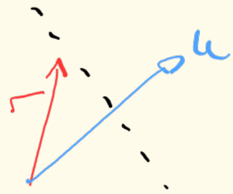
Exponential notation is preferred because many operations, such as differentiation and integration, are made trivial. However, it should be remembered that $\Re e(\ )$ is really still there but we're not bothering to write it! This becomes important when, for example, two waves are multiplied, as will happen later in the course.

put $Re(\ )$ back!

In three dimensions, the equivalent of $f(x,t)$ is the scalar **plane wave**

$$f(\mathbf{r},t) = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.$$

The **wavevector** $\mathbf{k}$ points in the direction of propagation. From vector algebra, $\mathbf{k}\cdot\mathbf{r}$ is constant across any infinite plane perpendicular to $\mathbf{k}$. (Hence the name "plane wave".)



At a given t, $f(\mathbf{r},t)$ is constant across a **plane wavefront**. The wavelength of $f(\mathbf{r},t)$ is $\lambda = 2\pi/|\mathbf{k}| = 2\pi/k$ etc.

In electromagnetism, we need plane waves with vector amplitude:

$$f(\mathbf{r},t) = \mathbf{A}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.$$

$\mathbf{A}$ is a constant vector, representing the amplitude of the wave. The oscillating exponential part is exactly the same as for scalar waves.

Aside: Vector Waves.

Vector waves come in two varieties: longitudinal and transverse. In isotropic (same in all directions) media these have a simple mathematical definition. If the vector amplitude is $\boldsymbol{\psi}$, then

Transverse wave: $\mathrm{div}\,\boldsymbol{\psi} = 0$.

Longitudinal wave: $\mathrm{curl}\,\boldsymbol{\psi} = 0$.

7.2 Wave solutions for $\mathbf{E}$.

Look for plane wave solutions $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$:

(E_x, E_y, E_z) (E_{0x}, E_{0y}, E_{0z})

Substitute into $\nabla^2\mathbf{E}=\varepsilon_0\mu_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ and see if it works…

Remember for a vector,

$$\nabla^2\mathbf{F}\equiv\hat{\mathbf{i}}\nabla^2F_x+\hat{\mathbf{j}}\nabla^2F_y+\hat{\mathbf{k}}\nabla^2F_z=\hat{\mathbf{i}}\left(\frac{\partial^2F_x}{\partial x^2}+\frac{\partial^2F_x}{\partial y^2}+\frac{\partial^2F_x}{\partial z^2}\right)+\cdots$$

Thus LHS: $\nabla^2\mathbf{E}=\hat{\mathbf{i}}\nabla^2E_x+\hat{\mathbf{j}}\nabla^2E_y+\hat{\mathbf{k}}\nabla^2E_z$

1

Look at this first term…

$$E_x=E_{0x}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}=E_{0x}e^{i(k_xx+k_yy+k_zz-\omega t)}$$

so

$$\begin{aligned}\nabla^2E_x&=\frac{\partial^2E_x}{\partial x^2}+\frac{\partial^2E_x}{\partial y^2}+\frac{\partial^2E_x}{\partial z^2}\\&=-(k_x^2+k_y^2+k_z^2)E_{0x}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\&=-k^2E_x.\end{aligned}$$

Similarly, $\nabla^2E_y=-k^2E_y$ and $\nabla^2E_z=-k^2E_z$ so overall

$$\begin{aligned}\nabla^2\mathbf{E}&=-k^2(\hat{\mathbf{i}}E_x+\hat{\mathbf{j}}E_y+\hat{\mathbf{k}}E_z)\\&=-k^2\mathbf{E}.\end{aligned}$$

RHS is easier: $\dfrac{\partial^2\mathbf{E}}{\partial t^2}=(-i\omega)^2\,\mathbf{E}=-\omega^2\mathbf{E}.$

Thus, the wave equation becomes

(w.E)

$$\boxed{-k^2\mathbf{E} = -\varepsilon_0\mu_0\omega^2\mathbf{E}.}$$

This means that plane waves are solutions to the wave equation, provided their wavenumber k and angular frequency ω are related by

$$k^2 = \varepsilon_0\mu_0\omega^2.$$

wave eqt° condit°

BUT

For any wave $\dfrac{\omega}{k} = f\lambda = v$, the phase speed.

Therefore, only waves in source-free vacuum with phase speed

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$ are allowed.

$\varepsilon_0 = 8.854\times10^{-12}$ Fm^{-1}, and $\mu_0 = 4\pi\times10^{-7}$ Hm^{-1} so the allowed v is

$v = 3\times10^8$ ms^{-1} $= c$, the speed of **light** in a vacuum.

From theory, Maxwell asserted that light is a form of electromagnetic wave, and that other forms (different ω, k; same c) should exist.

Hertz found another in 1887.

7.3 Monochromatic electromagnetic plane waves.

Consider an **electric plane wave** travelling in the $+z$ direction:

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0 \mathrm{e}^{i(k_z z-\omega t)}.$$

What direction is $\mathbf{E}_0$ in?

What $\mathbf{B}$ is associated with $\mathbf{E}$?

$$\mathbf{E}(\mathbf{r},t)=\hat{\mathbf{i}}\,E_{0x}\mathrm{e}^{i(k_z z-\omega t)}+\hat{\mathbf{j}}\,E_{0y}\mathrm{e}^{i(k_z z-\omega t)}+\hat{\mathbf{k}}\,E_{0z}\mathrm{e}^{i(k_z z-\omega t)}.$$

Therefore,

$$\begin{aligned}\nabla\cdot\mathbf{E}&=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\\ &=\ 0\ +\ 0\ +ik_z E_{0z}\mathrm{e}^{i(k_z z-\omega t)}\\ &=ik_z E_z.\end{aligned}$$

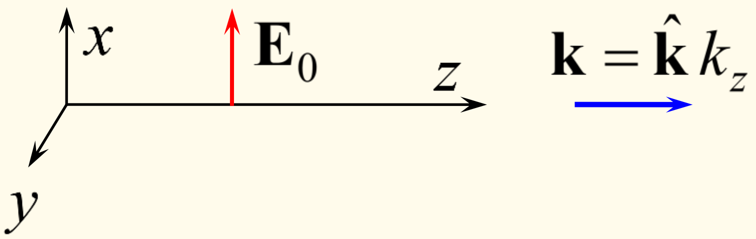
In source-free vacuum, $\rho(\mathbf{r})=0$ so $\nabla\cdot\mathbf{E}=0$ for this field as for any other.

Clearly $k_z\neq 0$, so we must have $E_z=0$ for this wave.

i.e., The electric field in a plane wave in source-free vacuum is **transverse** and we guarantee this by $\mathbf{E}\cdot\mathbf{k}=0$

What is $\mathbf{B}$ linked to the transverse plane wave in E?

Let $\mathbf{E}_0 = \hat{\mathbf{i}}\, E_0$.



Then our wave solution to Maxwell's equations is $\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{i}}\, E_0 \mathrm{e}^{i(k_z z - \omega t)}$.

Use the Maxwell equation, $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ to find $\mathbf{B}$ …

In component form, this is:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_0 \mathrm{e}^{i(k_z z - \omega t)} & 0 & 0 \end{vmatrix} = \hat{\mathbf{j}}\, i k_z E_0 \mathrm{e}^{i(k_z z - \omega t)} = -\frac{\partial \mathbf{B}}{\partial t}$$

Integrate this with respect to t (choosing integration constant $= 0$) to give

$$\mathbf{B} = -\hat{\mathbf{j}} \left[\frac{i k_z}{-i\omega} E_0 \mathrm{e}^{i(k_z z - \omega t)} \right] = \hat{\mathbf{j}} \left[\frac{k_z}{\omega} E_0 \mathrm{e}^{i(k_z z - \omega t)} \right]$$

Hence $\quad \underline{\mathbf{B} = \hat{\mathbf{j}}\, B_0 \mathrm{e}^{i(k_z z - \omega t)}}$

with $B_0 = \dfrac{E_0}{c}$ as $\omega / k = c$.

Important points from this: $\mathbf{B} = \hat{\mathbf{j}} B_0 e^{i(k_z z - \omega t)}$.

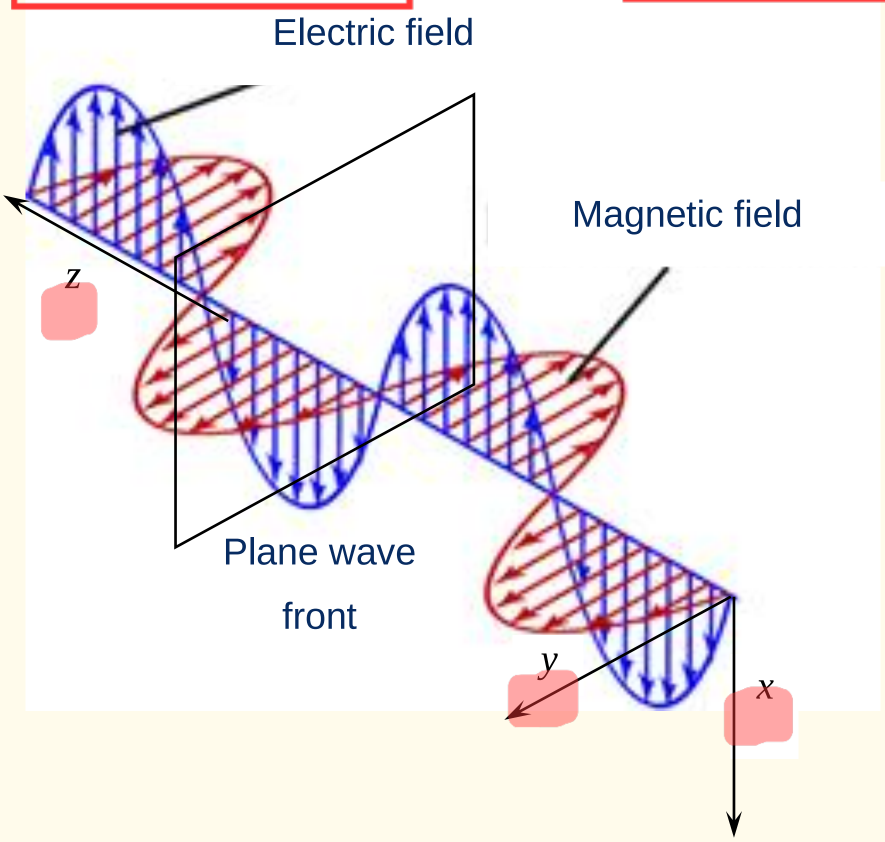
$\mathbf{B}$-wave is also transverse and is perpendicular to $\mathbf{E}$. In source-free vacuum, $\mathbf{E}$ & $\mathbf{B}$ are in phase. There is no phase difference between them.

$B_0 = \dfrac{E_0}{c}$. Does **not** imply $\mathbf{B}$ is less important than $\mathbf{E}$.

Summary.

In source-free vacuum, we have transverse electromagnetic waves. If we fix $\mathbf{k}$ in the z-direction, $\mathbf{E}_0$ in the x-direction, then

$$\boxed{\mathbf{E}(\mathbf{r},t) = \hat{\mathbf{i}} E_0 e^{i(k_z z - \omega t)}} \qquad \boxed{\mathbf{B}(\mathbf{r},t) = \hat{\mathbf{j}} B_0 e^{i(k_z z - \omega t)}}$$



Note that $\mathbf{E} \times \mathbf{B}$ points in the direction of energy flow. This is true for all electromagnetic fields.

In this case, $\mathbf{E} \times \mathbf{B} \propto \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$.

A note on polarisation.

Maxwell's equations require $\mathbf{E}\cdot\mathbf{k}=\mathbf{B}\cdot\mathbf{k}=\mathbf{E}\cdot\mathbf{B}=0$.

BUT

nothing in them prevents $\mathbf{E}$ and $\mathbf{B}$ rotating in the plane perpendicular to the wavevector $\mathbf{k}$.

So far, we've assumed that $\mathbf{E}_0$ is unchanging with time. We have therefore assumed we have **linearly polarised** or **plane polarised** waves.

By convention, the plane of polarisation is that of $\mathbf{E}_0$ and $\mathbf{k}$.

Other polarisations are possible, even useful, as we shall see later…

Lecture 08: Impedance and Energy.

8.1 $\mathbf{H}$-fields and wave impedance.

$\mathbf{H}$-fields ($\mathbf{H}$: magnetic field strength) are often used to describe magnetic fields in materials. In a vacuum, their relation to $\mathbf{B}$-fields is very simple:

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} \qquad \text{or} \qquad \mathbf{B} = \mu_0 \mathbf{H}.$$

Tweaking Ampere's Law.

If we use the $\mathbf{H}$-field in Ampere's Law.

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{S} \quad \to \oint_C \mu_0 \mathbf{H}\cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{S}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = \int_S \mathbf{J}\cdot d\mathbf{S} = \sum_{enclosed} I$$

From this we can see that the units of $\mathbf{H}$ must be Am^{-1}.

Impedance.

In terms of $\mathbf{H}$, our plane-polarised wave moving in the $+z$ direction is

$$\mathbf{E} = \hat{\mathbf{i}}\, E_0 e^{i(k_z z - \omega t)} \qquad\qquad \mathbf{H} = \hat{\mathbf{j}}\left(\frac{k_z E_0}{\mu_0 \omega}\right) e^{i(k_z z - \omega t)}$$

Therefore $\dfrac{|\mathbf{E}|}{|\mathbf{H}|} = \dfrac{\mu_0 \omega}{k_z} = \mu_0 c = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}.$ ratio magnitude E & H fields = const = impedance

$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

This is the same for all waves in source-free vacuum. The value of $\dfrac{|\mathbf{E}|}{|\mathbf{H}|}$ gives the "wave **impedance**". (Think: impedance to the wave)

Impedance?

The electric field has units of Vm^{-1} and we have just seen that the $\mathbf{H}$-field has units of Am^{-1}.

Thus, remembering Ohm's law:

$$\frac{|\mathbf{E}|}{|\mathbf{H}|} = \frac{\left[\mathrm{Vm}^{-1}\right]}{\left[\mathrm{Am}^{-1}\right]} = [\Omega]$$

For all electromagnetic waves **in source-free vacuum**, the wave impedance is

$$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377\Omega.$$

8.2 Electromagnetic energy (in a vacuum).

Look at examples:

1. Energy stored in a capacitor $W = \frac{1}{2}CV^2$.

e.g., a large parallel-plate capacitor.

$C = \frac{\varepsilon_0 A}{d}$; $V = Ed$. [A = plate area; d = spacing]

Therefore $W = \frac{1}{2}\varepsilon_0 E^2 \cdot (Ad)$ so the energy density in this purely electric field is $U = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 |\mathbf{E}|^2$. The units of U are joules per cubic meter.

[J/m³]

mag E field

2. Energy stored in an inductor $W = \frac{1}{2}LI^2$.

e.g., a long circular solenoid; radius a, length b.

$L = \mu_0 \pi a^2 b n^2$; $B = \mu_0 n I$. [n = turns per m]

Therefore $W = \frac{1}{2}\frac{1}{\mu_0}B^2 \cdot (\pi a^2 b)$ so the energy density in this purely magnetic field is $U = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2}\mu_0 H^2 = \frac{1}{2}\mu_0 |\mathbf{H}|^2$.

These specific examples reveal a general result. The energy density of an em field in vacuum is

VERY IMP EQT° (A4) !!!

$$U = \frac{1}{2}\varepsilon_0 |\mathbf{E}|^2 + \frac{1}{2}\mu_0 |\mathbf{H}|^2, \quad \text{or} \quad U = \frac{1}{2}\varepsilon_0 \mathbf{E}\cdot\mathbf{E} + \frac{1}{2}\mu_0 \mathbf{H}\cdot\mathbf{H}$$

U = energy density in $\mathbf{E}$-field + energy density in the $\mathbf{H}$-field

A More Formal Examination: Electrostatic Energy

The following is the usual argument used to find the result that we inferred above from a more general picture of an electromagnetic system.

If we have a system of n point charges in vacuum the energy W_E stored in the set of charges from assembling them is:

$$W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k \phi_k$$

where Q_k is the k_{th} charge brought into the system and ϕ_k is the potential at the point where Q_k is placed.

We convert this into a system with a continuous charge distribution by converting to an integral:

$$W_E = \frac{1}{2}\int \rho\phi \, \mathrm{d}V$$ where we have $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$ or $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$. So,

$$W_E = \frac{1}{2}\varepsilon_0 \int (\nabla \cdot \mathbf{E})\phi \, \mathrm{d}V$$

We use a standard vector identity (Formula Book)

$$\nabla \cdot (\phi\mathbf{A}) = \phi(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla\phi \quad \Rightarrow \quad (\nabla \cdot \mathbf{A})\phi = \nabla \cdot (\phi\mathbf{A}) - \mathbf{A} \cdot \nabla\phi$$

$$W_E = \frac{1}{2}\varepsilon_0 \int \nabla \cdot (\phi\mathbf{E}) \, \mathrm{d}V - \frac{1}{2}\varepsilon_0 \int \mathbf{E} \cdot \nabla\phi \, \mathrm{d}V$$

We use the divergence theorem on the first integral and use $E = -\nabla\phi$.

$$W_E = \frac{1}{2}\varepsilon_0 \int (\phi\mathbf{E}) \, \mathrm{dS} + \frac{1}{2}\varepsilon_0 \int \mathbf{E} \cdot \mathbf{E} \, \mathrm{d}V$$

$$W_E = \frac{1}{2}\varepsilon_0 \int (\phi\mathbf{E})\,\mathrm{dS} + \frac{1}{2}\varepsilon_0 \int \mathbf{E}\cdot\mathbf{E}\,\mathrm{d}V$$

We are doing this for an arbitrary volume and surface and so we move the surface to being at infinity and we decide $\int (\phi\mathbf{E})\,\mathrm{dS} \to 0$.

This may be a reasonable assumption. The general pattern we see for sets of points charges is:

Monopole: $\phi \sim 1/r$ and $\mathbf{E} \sim 1/r^2$.

Dipoles: $\phi \sim 1/r^2$ and $\mathbf{E} \sim 1/r^3$.

Quadrupoles: $\phi \sim 1/r^3$ and $\mathbf{E} \sim 1/r^4$.

We can express a general charge distribution as an expansion of monopoles, dipoles etc. It seems reasonable that $(\phi\mathbf{E})\,\mathrm{dS} \to 0$ and $\int (\phi\mathbf{E})\,\mathrm{dS} \to 0$.

This leaves us with,

$$W_E = \frac{1}{2}\varepsilon_0 \int \mathbf{E}\cdot\mathbf{E}\,\mathrm{d}V = \frac{1}{2}\int \varepsilon_0 \,|\mathbf{E}|^2\,\mathrm{d}V$$

U_E = energy density in $\mathbf{E}$-field $= \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 |\mathbf{E}|^2$

8.3 The Poynting vector

The conclusion.

The Poynting vector is $\mathbf{N} = \mathbf{E} \times \mathbf{H}$.

At any point,

- The direction of the Poynting vector $\mathbf{N}$ is the direction of energy flow.
- The magnitude of the Poynting vector $\mathbf{N}$ is the rate of energy flow per unit area perpendicular to $\mathbf{N}$.

This is a general result that we infer for all fields rather than rigorously prove but it is empirically correct as far as we can tell. For example, radio engineering for aerials and antennas is consistent with the Poynting vector.

N does for energy what J does for charge

Consider energy – stored or flowing.

Above, we have noted that we have a field U for stored energy. We can relate this to an energy flux $\mathbf{N}$ of the field. This is the flow of energy per unit time across a unit area perpendicular to the flow. We can write a law that it an analogy with the law of conservation of charge. We have:

$$-\frac{\partial U}{\partial t} = \nabla \cdot \mathbf{N} \qquad \text{compare to } -\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J}$$

Justification. If we have an electromagnetic field then the stored energy in a given volume is, $Q = \int_V U \, \mathrm{d}V$.

The rate of increase or decrease of this stored energy is the time derivative of the integral and is the flow of field energy out of the volume through its surface area.

$$\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial t}\int_V U\,\mathrm{d}V = \oint_S \mathbf{N}\cdot\mathrm{d}\mathbf{S}$$

Using the divergence theorem,

$$-\int_V \frac{\partial U}{\partial t}\mathrm{d}V = \oint_S \mathbf{N}\cdot\mathrm{d}\mathbf{S} = \int_V \nabla\cdot\mathbf{N}\,\mathrm{d}V \quad \text{giving } -\frac{\partial U}{\partial t} = \nabla\cdot\mathbf{N}.$$

A Poynting vector derivation.

Consider a charge q in an electromagnetic field. The force on the charge, with velocity $\mathbf{v}$, is $\mathbf{F} = q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$.

The rate at which this force does work is,

$$\mathbf{v}\cdot\mathbf{F} = q\left(\mathbf{v}\cdot\mathbf{E}+\mathbf{v}\cdot\left(\mathbf{v}\times\mathbf{B}\right)\right) = q\left(\mathbf{v}\cdot\mathbf{E}+\left(\mathbf{v}\times\mathbf{v}\right)\cdot\mathbf{B}\right) = q\mathbf{v}\cdot\mathbf{E}$$

Using rules for scalar triple product and cross product.

When we have many charges, we have;

$$\text{rate at which work is done by fields on charges} = \sum_i q_i\mathbf{v}_i\cdot\mathbf{E}_i\,.$$

If we change to a macroscopic view of all the charges then,

$$\sum_i q_i\mathbf{v}_i\cdot\mathbf{E}_i \rightarrow \int \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V\,.$$

We now use a Maxwell equation, with $\mathbf{B} = \mu_0\mathbf{H}$

$$\nabla\times\mathbf{H} = \mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \quad \text{or} \quad \mathbf{J} = \nabla\times\mathbf{H}-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

The rate at which work is done **by** the charges **on** the field is,

$$-\int \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V = -\int\left(\nabla\times\mathbf{H}-\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cdot\mathbf{E}\,\mathrm{d}V$$
$$=\int\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}\,\mathrm{d}V-\int\mathbf{E}\cdot(\nabla\times\mathbf{H})\,\mathrm{d}V$$

Using a standard result, see Formula Book

$$\nabla\cdot(\mathbf{E}\times\mathbf{H})=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{H})$$

giving

$$-\int \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V=\int\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}\,\mathrm{d}V-\int\mathbf{H}\cdot(\nabla\times\mathbf{E})\,\mathrm{d}V-\int\nabla\cdot(\mathbf{E}\times\mathbf{H})\,\mathrm{d}V$$

We know $\nabla\times\mathbf{E}=-\dfrac{\partial\mathbf{B}}{\partial t}$ and using $\mathbf{B}=\mu_0\mathbf{H}$

$$-\int \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V=\int\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}\,\mathrm{d}V+\int\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t}\mathrm{d}V-\int\nabla\cdot(\mathbf{E}\times\mathbf{H})\,\mathrm{d}V$$

This gives,

$$-\int \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}V=\int\left(\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}+\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t}\right)\mathrm{d}V-\int(\mathbf{E}\times\mathbf{H})\cdot\mathrm{d}\mathbf{S}$$

The left-hand side is an expression for work is done by the charges on the field. Can we relate the right-hand side to expressions for energy? Consider the following manipulation,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}|\mathbf{E}|^2\right)=\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E})=\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}.$$

We can do the same for the field $\mathbf{H}$. This means that we can write,

$$\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}+\mu_0\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t}=\frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon_0|\mathbf{E}|^2\right)+\frac{\partial}{\partial t}\left(\tfrac{1}{2}\mu_0|\mathbf{H}|^2\right)=\frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon_0|\mathbf{E}|^2+\tfrac{1}{2}\mu_0|\mathbf{H}|^2\right)$$

The right-hand side is the rate of change of the energy stored in the em field.

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} + \mu_0 \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon_0 \left|\mathbf{E}\right|^2 + \tfrac{1}{2}\mu_0 \left|\mathbf{H}\right|^2\right) = \frac{\partial U}{\partial t}$$

Putting this together,

$$-\int \mathbf{J} \cdot \mathbf{E}\,\mathrm{d}V = \int \frac{\partial U}{\partial t}\mathrm{d}V - \int \left(\mathbf{E} \times \mathbf{H}\right) \cdot \mathrm{d}\mathbf{S}$$

We have two terms about energy in the system – and we know that, when no work is being added to a system that $-\int_V \frac{\partial U}{\partial t}\mathrm{d}V = \oint_S \mathbf{N} \cdot \mathrm{d}\mathbf{S}$.

Here, we see that the LHS is the work is done by the charges on the field and this is balanced by the change of stored energy **in** the system and the flow of energy **out** of the system through its surface. This requires that Poynting vector is $\mathbf{N} = \mathbf{E} \times \mathbf{H}$ and, as stated above, the magnitude of the Poynting vector $\mathbf{N}$ is the rate of energy flow per unit area perpendicular to $\mathbf{N}$.

This is true for all electromagnetic fields.

Poynting vector: An alternative approach.

(i) Maxwell's Equations are true for all electromagnetic fields.

(ii) The rules of vector calculus are true for vector fields.

(iii) Apply vector calculus to Maxwell's Equations.

One result that you can get for a source free vacuum is [Sadiku /3e 10.7]:

$$\nabla\times\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} \quad \text{and} \quad \nabla\times\mathbf{H} = \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Thus, using $\operatorname{curl}\mathbf{H}$,

$$\mathbf{E}\cdot(\nabla\times\mathbf{H}) = \mathbf{E}\cdot\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \qquad [1]$$

For any vector field,

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}),\ \text{and with } \mathbf{A}=\mathbf{H} \text{ and } \mathbf{B}=\mathbf{E}$$

[1] becomes,

$$\mathbf{H}\cdot(\nabla\times\mathbf{E}) + \nabla\cdot(\mathbf{H}\times\mathbf{E}) = \mathbf{E}\cdot\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{2}\varepsilon_0\frac{\partial E^2}{\partial t}$$

Using Maxwell equation for $\operatorname{curl}\mathbf{E}$,

$$\nabla\cdot(\mathbf{H}\times\mathbf{E}) = \frac{1}{2}\varepsilon_0\frac{\partial E^2}{\partial t} - \mathbf{H}\cdot\left(-\mu_0\frac{\partial\mathbf{H}}{\partial t}\right) = \frac{1}{2}\varepsilon_0\frac{\partial E^2}{\partial t} + \frac{1}{2}\mu_0\frac{\partial H^2}{\partial t}$$

Thus,

$$-\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0E^2 + \frac{1}{2}\mu_0H^2\right) \text{or} \oint_S(\mathbf{E}\times\mathbf{H})\mathrm{d}\mathbf{S} = -\frac{\partial}{\partial t}\int_v\left(\frac{1}{2}\varepsilon_0E^2 + \frac{1}{2}\mu_0H^2\right)\mathrm{d}v$$

Now – start interpreting what these terms must mean. To me this is a "physics after" rather than "physics first" approach.

$$\frac{\partial}{\partial t}(|a|^2) = \frac{\partial}{\partial t}(a\cdot a) = a\frac{\partial a}{\partial t} + \frac{\partial a}{\partial t}a$$

$$= 2a\frac{\partial a}{\partial t}$$

$$P = \frac{1}{\mu_0}(\vec{E}\times\vec{B})$$

$$\vec{N} = \vec{E}\times\vec{H}$$

$$\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B})$$

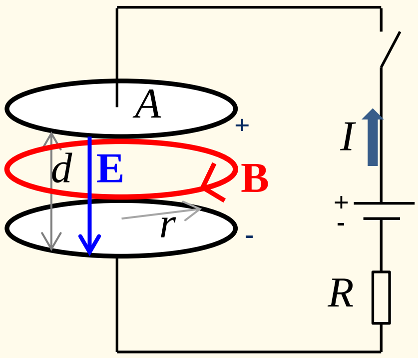
$$= \vec{B}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) \ldots$$

$$\left(\frac{\partial B}{\partial t}\right) - \vec{E}\cdot\left(\varepsilon_0\mu_0\frac{\partial E}{\partial t}\right)$$

Rearrange: $\nabla\cdot(\vec{E}\times\vec{B}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \varepsilon_0\mu_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{\partial U}{\partial t}$

Example of Poynting vector: A capacitor charging.

We will use results from our previous example in Lecture 5.



The capacitor is a capacitor with circular plates radius r. The separation of the capacitor plates is d. All the resistance in the circuit is represented by a lumped resistance R.

We have found $B = \frac{\mu_0}{2\pi r}\frac{V}{R}\left(e^{-\frac{t}{CR}}\right)$ at the edge of the capacitor and thus, $\mathbf{H} = \frac{1}{2\pi r}\frac{V}{R}\left(e^{-\frac{t}{CR}}\right)\left(-\hat{\boldsymbol{\theta}}\right)$, and we have an electric field between the plates, $\mathbf{E} = \frac{V}{d}\left(1-e^{-\frac{t}{CR}}\right)\left(-\hat{\mathbf{k}}\right)$. Thus,

$$\mathbf{N} = \mathbf{E}\times\mathbf{H} = \frac{V}{d}\left(1-e^{-\frac{t}{CR}}\right)\frac{1}{2\pi r}\frac{V}{R}\left(e^{-\frac{t}{CR}}\right)\left(-\hat{\mathbf{r}}\right) = -\frac{1}{2\pi rd}\frac{V^2}{R}\left(e^{-\frac{t}{CR}} - e^{-\frac{2t}{CR}}\right)\hat{\mathbf{r}}$$

Remember, the Poynting vector has a magnitude of the rate of energy flow per unit area perpendicular to $\mathbf{N}$. We have a surface for the outside of the capacitor of $2\pi rd$. Thus, the Poynting vector gives us the rate of energy flowing into the capacitor from outside as the direction is $-\hat{\mathbf{r}}$.

$$\frac{dE}{dt} = \frac{V^2}{R}\left(e^{-\frac{t}{CR}} - e^{-\frac{2t}{CR}}\right)$$

If we integrate over time to find the stored energy from $t=0$ to $t=\infty$ then

$$W = \frac{V^2}{R}\frac{CR}{2} = \frac{1}{2}CV^2 = \frac{1}{2}QV.$$ Just as expected.

8.4 Energy flow in an electromagnetic plane wave.

Take our previous example (monochromatic, source free vacuum):

$$\mathbf{E} = \hat{\mathbf{i}} E_0 e^{i(k_z z - \omega t)} \qquad \mathbf{H} = \hat{\mathbf{j}} H_0 e^{i(k_z z - \omega t)}$$

The wave impedance is $Z_0 = \dfrac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$.

Therefore

$\sqrt{\varepsilon_0}\,|\mathbf{E}| = \sqrt{\mu_0}\,|\mathbf{H}|$ and, squaring, $\frac{1}{2}\varepsilon_0 |\mathbf{E}|^2 = \frac{1}{2}\mu_0 |\mathbf{H}|^2$.

i.e., the electric and magnetic energy densities are **equal** in a vacuum plane wave (VPW).

So, the total energy density in a VPW is twice that in (say) the electric field:

$$U_{VPW} = \varepsilon_0 |\mathbf{E}|^2 .$$

As $|\mathbf{E}| = E_0 \cos(k_z z - \omega t)$, $U_{VPW} = \varepsilon_0 E_0^2 \cos^2(k_z z - \omega t)$

[N.B. Don't confuse $|\mathbf{E}|$ = magnitude of $\mathbf{E}$ at a given $(\mathbf{r}, t)$ and E_0 = constant wave amplitude]

Instantaneous values of energy density vary too quickly to measure; it is more useful to consider the average energy density. As the average of $\cos^2\theta$ is $\frac{1}{2}$, we have $\langle U_{VPW} \rangle = \frac{1}{2}\varepsilon_0 E_0^2$.

2 2

For our example plane wave,

$$\mathbf{N} = \mathbf{E}\times\mathbf{H} = \hat{\mathbf{i}}\, E_0 \mathrm{e}^{i(k_z z-\omega t)} \times \hat{\mathbf{j}}\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0 \mathrm{e}^{i(k_z z-\omega t)},$$ or

$$\mathbf{N} = \hat{\mathbf{k}}\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2 \cos^2(k_z z-\omega t).$$

As expected, $\mathbf{N}$ points in the direction of wave (& energy) propagation. But the speed of light $c = \dfrac{1}{\sqrt{\varepsilon_0\mu_0}}$, so

$$\mathbf{N} = \hat{\mathbf{k}}\, c\varepsilon_0 E_0^2 \cos^2(k_z z-\omega t) = \hat{\mathbf{k}}\, cU_{VPW}.$$

(With a bit of thought) this confirms that $|\mathbf{N}|$ =energy crossing unit area per unit time.

The average value of $|\mathbf{N}| = N$ is

$$\langle N\rangle = c\langle U_{VPW}\rangle = \tfrac{1}{2} c\varepsilon_0 E_0^2.$$

We can use $\langle N\rangle$ to find E_0 and H_0 for a given light source.

[Though note that we should be careful not to confuse “beams” and “plane waves”.]

Example.

Consider a 25W continuous-wave laser with a spot size of diameter 1mm. What are the amplitudes of the electric and magnetic fields in the laser light?

25W is the average power. Assume that the laser profile is a $d = 1\text{mm}$ circle cut out of a vacuum plane wave. The spot area is $A = \dfrac{\pi d^2}{4}$.

$\langle N \rangle$ gives us the average power per unit area:

$$\langle N \rangle \approx \frac{25 \cdot 4}{\pi \cdot 10^{-6}} \approx 3.2 \times 10^7 \, \mathrm{Wm}^{-2}.$$

Since $\langle N \rangle = \frac{1}{2} c \varepsilon_0 E_0^2$ we get the amplitude of the electric field as $E_0 \approx 1.5 \times 10^5 \, \mathrm{Vm}^{-1}$.

In a VPW $\mathbf{E}$ and $\mathbf{H}$ are in phase, and $\dfrac{E_0}{H_0} = Z_0 = 377\Omega$. This gives for the amplitude of the magnetic field $H_0 \approx 410 \, \mathrm{Am}^{-1}$.

8.5 Photons and Radiation Pressure.

This unit: e.m. waves are classical & continuous in energy. Any $\omega = ck$ is allowed.

Quantum theory: light "wave" consists of "particles"; photons of energy $\mathcal{E} = \hbar\omega$. [$\omega$ same as ours]

Special relativity: $\mathcal{E}^2 = p^2c^2 + m^2c^4$.

Here m is the "rest mass" of a particle and is zero for photons, so the momentum of a photon is

$$p = \frac{\mathcal{E}}{c} = \hbar\left(\frac{\omega}{c}\right) = \hbar k\,.$$

[Same as de Broglie result for $m \neq 0$; in 3D $\mathbf{p} = \hbar\mathbf{k}$.]

Through its momentum a photon can exert a **radiation pressure**, P_r

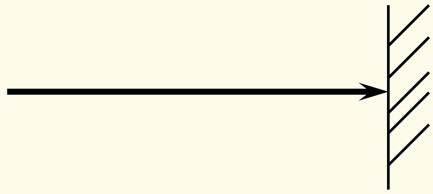
Example. Consider a light beam, cross-sectional area A cut out of a VPW, consisting of identical photons, each with $\mathcal{E} = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$, $p = \dfrac{\mathcal{E}}{c}$.

Radiation pressure $P_r = \dfrac{\text{Force}}{\text{Area}} = \dfrac{1}{A}\dfrac{\partial p}{\partial t} = \dfrac{1}{Ac}\dfrac{\partial \mathcal{E}}{\partial t}$.

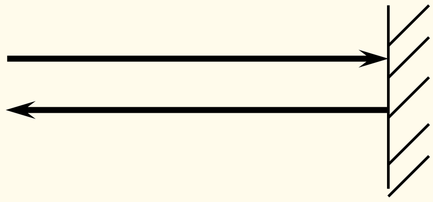
But $\dfrac{1}{A}\dfrac{\partial \mathcal{E}}{\partial t} =$ power per unit area in beam. Treated classically, this is $\langle N \rangle$.

So $P_r = \dfrac{\langle N \rangle}{c} = \langle U_{VPW} \rangle = \tfrac{1}{2}\varepsilon_0 E_0^2$.

If the beam strikes a surface and is perfectly absorbed, it exerts a radiation pressure $\langle U_{VPW} \rangle$ on the surface.



If it strikes a perfectly reflecting surface, it exerts a pressure
$\langle U_{VPW} \rangle - \left(-\langle U_{VPW} \rangle\right) = 2\langle U_{VPW} \rangle$.



Electromagnetic momentum density.

We can identify $\mathbf{g} = \frac{1}{c^2}\left(\mathbf{E} \times \mathbf{H}\right)$ as the electromagnetic momentum density of the field. For a VPW moving at c, this then gives.

$$\mathbf{P}_r = c\mathbf{g} = \frac{1}{c}\left(\mathbf{E} \times \mathbf{H}\right) = \frac{\mathbf{N}}{c}.$$

The full Maxwell equations and vector calculus proof for $\mathbf{g} = \left(\mathbf{E} \times \mathbf{H}\right) / c^2$ is given in Jackson [Ref 1]. Feynman notes [Ref 2] that is one version of a more general rule.

> "There is an important theorem in mechanics which is this: whenever there is a flow of energy in any circumstance at all (field energy or any other kind of energy) the energy flowing through a unit area per unit time, when multiplied by $1/c^2$, is equal to the momentum per unit volume in the space."

One of Feynman's examples: particles in a box.

Imagine that we have a population of particles in a box. Let's say C particles per cubic meter and they are all moving at a velocity $\mathbf{v}$. Now consider an imaginary plane surface perpendicular to $\mathbf{v}$.

The energy flow through a unit area of this surface per second is equal to $C\mathbf{v}$, the number which flow through the surface per second, times the energy carried by each one. The energy in each particle is

$$mc^2 = m_0 c^2 \Big/ \sqrt{1 - v^2/c^2}$$

Thus, the energy flow per unit area per second is

$$\mathbf{N} = C\mathbf{v}\, m_0 c^2 \Big/ \sqrt{1 - v^2/c^2}\ .$$

The momentum of each particle is $\mathbf{p} = m_0 \mathbf{v} \Big/ \sqrt{1 - v^2/c^2}$ which gives a density of momentum,

$$\mathbf{g} = C m_0 \mathbf{v} \Big/ \sqrt{1 - v^2/c^2}\ .$$

Thus, we see that $\mathbf{g} = \dfrac{\mathbf{N}}{c^2}$.

The particles – as a flow of total energy – have the same relationship as the photons.

Lecture 09: Polarisation of Light.

9.1 Polarisation of electromagnetic waves.

For convenience, consider an e.m. wave travelling in the $+z$ direction:

$\mathbf{E}(z,t) = \mathbf{E}_0(t)\cos(kz - \omega t)$ (and associated $\mathbf{H}$)

The **polarisation state** is determined by the behaviour of $\mathbf{E}$ in the transverse (xy) plane as a function of time.

(This is **not** fixed by M.Es. It is our way to categorise waves. The x and y axes are our construction and not part of the inherent physics.)

We write $\mathbf{E}$ (in this section) as the superposition

$$\mathbf{E}(z,t) = \mathbf{E}_x(z,t) + \mathbf{E}_y(z,t),$$

with

$$\mathbf{E}_x(z,t) = \hat{\mathbf{i}}E_{0x}\cos(kz - \omega t) \qquad (1)$$
$$\mathbf{E}_y(z,t) = \hat{\mathbf{j}}E_{0y}\cos(kz - \omega t + \beta) \qquad (2)$$

Note that E_{0x}, E_{0y}, β, k, ω, are all constants. The first three of these determine the polarisation state.

If $\beta > 0$, wave (2) reaches a given value of $\cos$ a time $\tau = \dfrac{\beta}{\omega}$ **later** than (1), so $\mathbf{E}_y$ **lags** $\mathbf{E}_x$ by β.

1. Plane Polarisation.

$\mathbf{E}_0$ is a constant vector.

To achieve this, set $\beta = 0$ (or 2π etc); then

$$\mathbf{E}(z,t) = \underbrace{(\hat{\mathbf{i}}E_{0x} + \hat{\mathbf{j}}E_{0y})}_{\mathbf{E}_0}\cos(kz - \omega t) = \mathbf{E}_0 \cos(kz - \omega t)$$

The **plane of polarisation** is the $(\mathbf{E}_0, \hat{\mathbf{k}})$ plane.

Note that $\beta = \pi$ (or 3π etc) also gives plane polarisation,

$$\mathbf{E}_0 = (\hat{\mathbf{i}}E_{0x} - \hat{\mathbf{j}}E_{0y}).$$

Alternatively, a wave-first approach:

If we start with the idea that there is a plane-polarised (aka linearly polarised) wave moving through space, then we can write that as:

$$\boldsymbol{\psi} = \mathbf{E}\exp i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)$$

The wave is moving in the direction of $\mathbf{k}$.

The plane of polarisation is $(\mathbf{E}, \mathbf{k})$.

We know that this is a transverse wave so $\mathbf{E}\cdot\mathbf{k} = 0$. Thus, $\mathbf{E}$ lies in the plane that is perpendicular to $\mathbf{k}$.

In that plane, $\mathbf{E}$ will have components in two orthogonal directions but that depends on the directions selected.

2. Circular Polarisation.

In our two orthogonal directions x and y we find,

$$E_{0x} = E_{0y} = E_0, \text{ and } \beta = \pm\frac{\pi}{2}.$$

The effect of the phase is

$$\cos(x \pm \tfrac{\pi}{2}) = \cos x \cos\tfrac{\pi}{2} \mp \sin x \sin\tfrac{\pi}{2} = \mp \sin x,$$

So …

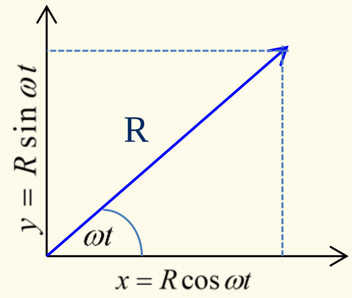
For $\beta = -\dfrac{\pi}{2}$ we have

$$\mathbf{E}_x = \hat{\mathbf{i}} E_0 \cos(kz - \omega t); \qquad \mathbf{E}_y = \hat{\mathbf{j}} E_0 \sin(kz - \omega t) \text{ and}$$

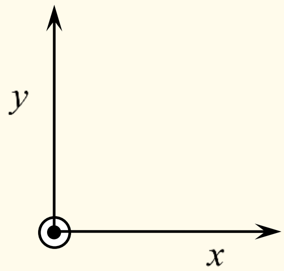
$$\mathbf{E} = E_0 \left[\hat{\mathbf{i}} \cos(kz - \omega t) + \hat{\mathbf{j}} \sin(kz - \omega t) \right].$$

Then $|\mathbf{E}| = E_0$, but the direction of $\mathbf{E}_0$ **rotates** in the transverse plane. The endpoint of $\mathbf{E}_0$ traces out a circle.

Revision: motion in a circle



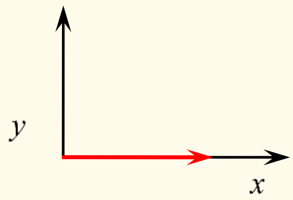
If we observe the wave at $z = 0$.



The wave is travelling <u>towards</u> us in the $+z$ direction.

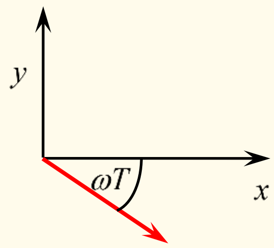
$$\mathbf{E} = E_0 \left[\hat{\mathbf{i}} \cos(kz - \omega t) + \hat{\mathbf{j}} \sin(kz - \omega t) \right]$$

At $z=0$ when $t=0$, $\mathbf{E}=E_0\hat{\mathbf{i}}$



At a later time $t=T$,

$$\mathbf{E}=E_0\left[\hat{\mathbf{i}}\cos(\omega T)-\hat{\mathbf{j}}\sin(\omega T)\right]$$



So, the $\mathbf{E}$-vector rotates **clockwise** at angular frequency ω according to our observer looking towards the oncoming wave.

This is a **right-circularly** polarised wave. [Though beware a confusing array of conventions.]

Circular polarisation: From the point of view of the receiver

In this convention, polarization is defined from the point of view of the receiver. To determine if the wave is clockwise or anti-clockwise circularly polarized, you take the point of view of the receiver and, while looking **toward** the source, against the direction of propagation, you observe the direction of the field's rotation in time. Using this convention, left- or right-handedness is determined by pointing one's left or right thumb **toward** the source, against the direction of propagation, and then matching the curling of one's fingers to the temporal rotation of the field.

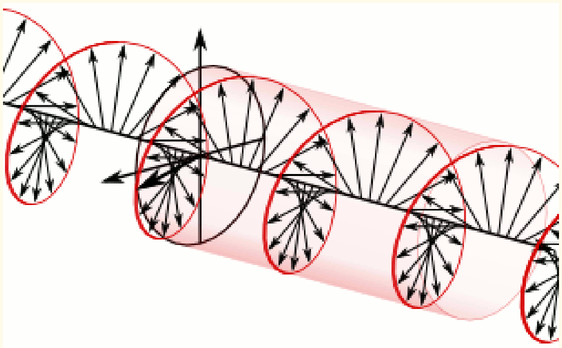
Many optics textbooks use this convention. It is also used by SPIE (The International Society for Optical Engineering) as well as the International Union of Pure and Applied Chemistry (IUPAC).

The other option …

$\beta = +\dfrac{\pi}{2}$ gives $\mathbf{E} = E_0\left[\hat{\mathbf{i}}\cos(kz-\omega t) - \hat{\mathbf{j}}\sin(kz-\omega t)\right]$, a **left-circularly** polarised wave, in which $\mathbf{E}$ rotates anti-clockwise at angular frequency ω according to our stationary observer.

Taking a snapshot.

If we take a snapshot of light at a time $t=0$ and look at the direction of $\mathbf{E}$ for values of z. Then the direction makes a full rotation over the wavelength of the light. [Image: https://en.wikipedia.org/wiki/Circular_polarization]

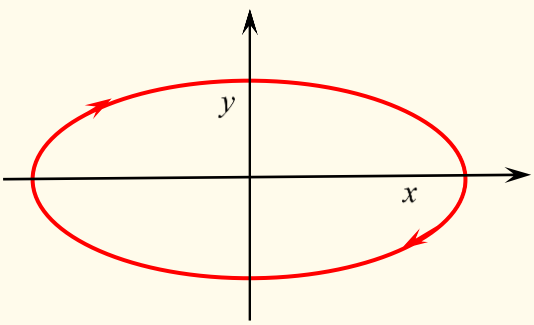


3. Elliptical Polarisation.

The tip of the $\mathbf{E}$-vector traces out an ellipse in the transverse plane.

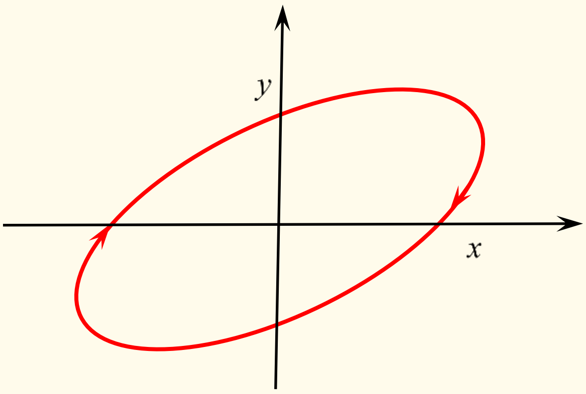
This is the general case and includes the special conditions of plane- and circular polarisation.

e.g. $\beta = -\dfrac{\pi}{2}$, but $E_{0x} \neq E_{0y}$:



$(E_{0x} > E_{0y})$

For most values of β, ellipse will be tilted:



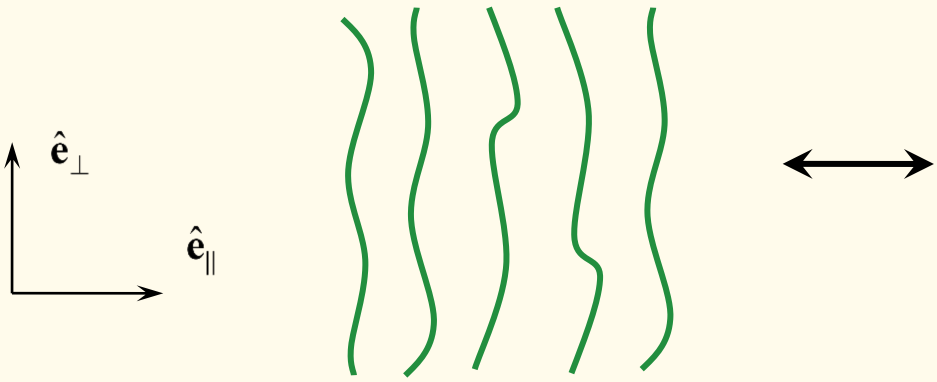
4. Natural light.

Often called “unpolarised”. “Randomly polarised” would be better; rapidly (and randomly) changing polarisation state. For a blackbody source, e.g. the sun, each wave emitted will be linearly polarised but of finite duration and have no correlation with the next wave emitted. There is no coherent superposition of the many waves to a consistent polarisation state.

9.2 Polarisers.

These are optical devices used to manipulate and analyse polarisation states.

Simple example (optical frequencies) – one or more sheets of polaroid. Essentially consists of aligned conducting polymer chains…



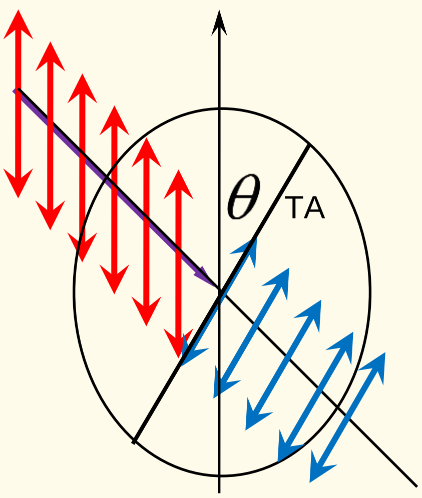
Transmission axis

(TA)

Ideal Polariser.

For an ideal polariser, only the component of $\mathbf{E}_0$ parallel to the transmission gets through.

Consider, for example, a **plane polarised wave**, at normal incidence on an "ideal" polaroid sheet. Let θ=angle (in plane of polariser) between $\mathbf{E}_0$ and the transmission axis.



$$\mathbf{E}_0 = E_0 \cos\theta\, \hat{\mathbf{e}}_{||} + E_0 \sin\theta\, \hat{\mathbf{e}}_\perp$$

Field component on $\hat{\mathbf{e}}_{||}$ transmitted with $T=1$

Field component on $\hat{\mathbf{e}}_\perp$ transmitted with $T=0$

Transmitted Intensity.

What is the transmitted intensity?

Intensity = av. power per unit area $=\langle N\rangle=\frac{1}{2}c\varepsilon_0 E_0^2$,

so $I(\theta)=\frac{1}{2}c\varepsilon_0\left|E_{||}\right|^2=\frac{1}{2}c\varepsilon_0 E_0^2\cos^2\theta.$

Maximum $(\theta=0)$ intensity is $I(0)=\frac{1}{2}c\varepsilon_0 E_0^2$ so

$$I(\theta)=I(0)\cos^2\theta. \qquad \text{Malus' Law}$$

[Often written, $I(\theta)=I_0\cos^2\theta.$]

Malus' Law can be used to analyse the effect of our ideal polariser on other forms of light.

9.3 Polarisers interacting with polarised Light.

Let I_{inc} = intensity of incident light

I_{tr} = intensity of transmitted light

Circularly polarised light: $\mathbf{E}_0$ has constant magnitude, but is rotating rapidly (at angular frequency ω)

$$\therefore \quad I_{tr}=I_{inc}\left\langle\cos^2\theta\right\rangle=\frac{1}{2}I_{inc}$$

Conclusion, half the light gets through.

Natural light: a similar story. The direction of $\mathbf{E}_0$ varies rapidly and randomly, so again

$$I_{tr} = I_{inc}\left\langle \cos^2\theta \right\rangle = \frac{1}{2} I_{inc}.$$

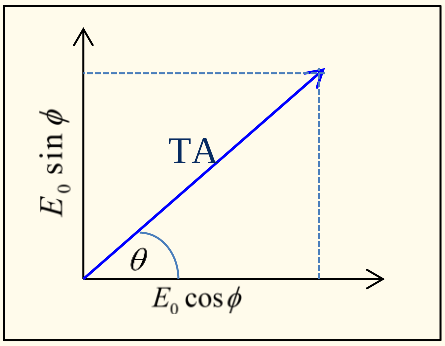
What if we rotate the polariser?

Simple answer: the values given immediately above didn't depend on the angle of the polariser so the answer will be $I_{tr} = \frac{1}{2} I_{inc}$ for any angle of the polariser. Let's see if that's consistent with our representation of polarised light above.

Circular polarisation.

$$\mathbf{E} = E_0\left[\hat{\mathbf{i}}\cos(kz-\omega t) + \hat{\mathbf{j}}\sin(kz-\omega t)\right]$$

If we place the transmission axis of the polariser at an angle θ to the x-axis and abbreviate with $\phi = kz - \omega t$.



Then the transmitted electric field is the superposition of two components.

$$E_{TA} = E_0 \cos\phi \cos\theta + E_0 \sin\phi \sin\theta$$

We will have a transmitted intensity

$$I = \frac{1}{2} c\varepsilon_0 \left\langle E^2 \right\rangle \quad \text{where}$$

$$\begin{aligned} E^2 &= \left(E_0 \cos\phi\cos\theta + E_0 \sin\phi \sin\theta\right)^2 \\ &= E_0^2 \cos^2\phi \cos^2\theta + 2E_0^2 \cos\phi\cos\theta\sin\phi\sin\theta + E_0^2 \sin^2\phi \sin^2\theta \end{aligned}$$

$$E^2 = E_0^2\cos^2\phi\cos^2\theta + E_0^2\sin^2\phi\sin^2\theta + \frac{E_0^2}{2}\sin 2\phi\sin 2\theta$$

Remembering that $\phi = kz - \omega t$ when we average over a period of the wave.

$$\langle\cos^2\phi\rangle = \langle\sin^2\phi\rangle = \frac{1}{2}; \quad \langle\sin 2\phi\rangle = 0$$

Thus,

$$\langle E^2\rangle = E_0^2\frac{1}{2}\cos^2\theta + E_0^2\frac{1}{2}\sin^2\theta = \frac{E_0^2}{2}\left(\cos^2\theta + \sin^2\theta\right) = \frac{E_0^2}{2}$$

giving,

$I_{tr} = \frac{1}{2}I_{inc}$ at all values of θ for the position of the polariser.

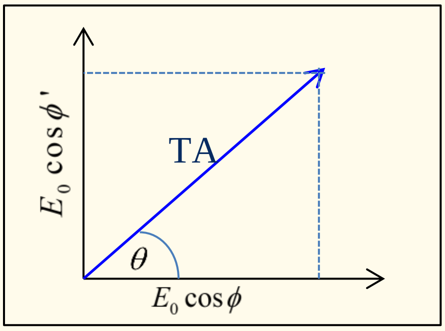
Natural light.

How do we represent natural light with our above components on axes? We depict natural light as two perpendicular linearly polarised waves of equal amplitude with **no phase coherence** between them.

$$\mathbf{E} = E_0\left[\hat{\mathbf{i}}\cos(kz-\omega t) + \hat{\mathbf{j}}\cos(kz-\omega t)'\right]$$

Or

$$\mathbf{E} = E_0\left[\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\cos\phi'\right]$$



As there is no phase coherence – or if you wish because $\langle\cos\phi\cos\phi'\rangle = 0$ - we can just add the intensities

$$E^2 = E_0^2\left(\cos^2\phi\cos^2\theta + \cos^2\phi'\sin^2\theta\right) = \frac{E_0^2}{2} \text{ and } I_{tr} = \frac{1}{2}I_{inc}$$

at all values of θ for the position of the polariser.

9.4 Angular momentum and the photon picture.

We have seen that em waves carry energy and linear momentum. We have also seen that for a right-handed circularly polarised wave,

$$\mathbf{E} = E_0\left[\hat{\mathbf{i}}\cos(kz-\omega t)+\hat{\mathbf{j}}\sin(kz-\omega t)\right].$$

If this acts on an electron the force, $\mathbf{F} = q\mathbf{E}$, will drive the electron in a circular motion. The $\mathbf{E}$-field drives the electron with an angular velocity ω equal to the frequency of the em wave. Angular momentum will be given to the electrons driven by the wave.

The following manipulation shows how we can think about the problem but really what we are doing is giving some intuitive feel to a known result from quantum mechanics.

The power obtained from a force in linear Newtonian mechanics is:

$\dfrac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{F}\cdot\mathbf{v}$ where $\mathbf{F}$ is a force acting on a particle with velocity $\mathbf{v}$.

The rotational mechanics version of this is $\dfrac{\mathrm{d}E}{\mathrm{d}t} = \boldsymbol{\Gamma}\cdot\boldsymbol{\omega}$ and the torque $\boldsymbol{\Gamma}$ is also the rate of change of angular momentum $\mathbf{L}$. Thus, dropping vector notation:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \omega\frac{\mathrm{d}L}{\mathrm{d}t} \qquad \text{or} \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{1}{\omega}\frac{\mathrm{d}E}{\mathrm{d}t}$$

An electron that absorbs energy E from the incident field will also absorb an amount of angular momentum L, where $L = E/\omega$.

The quantum mechanical view of light is that photons have an energy $E = \hbar\omega$ and the intrinsic spin of the photon, which is a Boson, is $-\hbar$ or $+\hbar$, where the signs indicate right- or left- handedness of the photon. The angular momentum of a photon is independent of its energy and the vector direction is along the direction of propagation (c.f. $\mathbf{L} = I\boldsymbol{\omega}$)

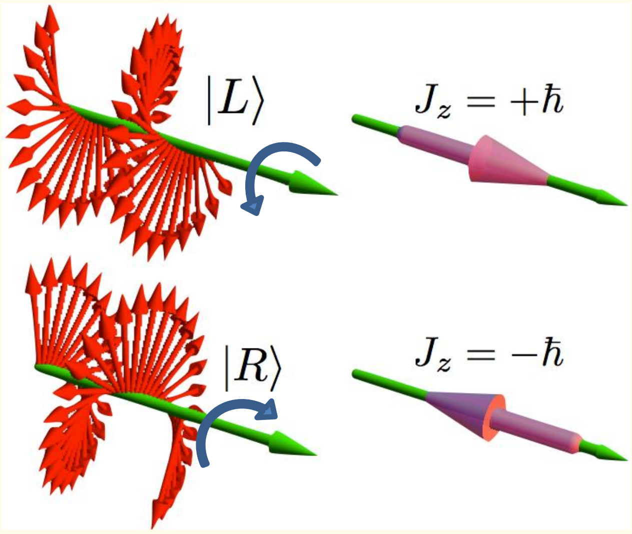


IMAGE FROM: SPIN-ANGULAR-MOMENTUM.PNG (1186×618) (WORDPRESS.COM) [1]

Using left- and right-handed photons as a building block, linearly polarised light is a coherent superposition of equal amplitudes of left and right-handed light.

Elliptically polarised light is a superposition of unequal amounts of left and right-handed polarised light with a set phase relationship in the superposition.

[1] spin-angular-momentum.png (1186×618) (wordpress.com)

Exercise 07.

Show that a superposition of an equal amplitude of left and right-handed circularly polarised em waves gives linearly polarised light.

Exercise 07: Solution

Show that a superposition of an equal amplitude of left and right-handed circularly polarised em waves gives linearly polarised light.

Using the notation used in the lecture notes.

$\mathbf{E}_R = E_0\left[\hat{\mathbf{i}}\cos(kz-\omega t)+\hat{\mathbf{j}}\sin(kz-\omega t)\right]$ is right circularly polarised.

$\mathbf{E}_L = E_0\left[\hat{\mathbf{i}}\cos(kz-\omega t)-\hat{\mathbf{j}}\sin(kz-\omega t)\right]$ is left circularly polarised

The vector superposition for the total field is,

$$\mathbf{E}_{Tot} = \mathbf{E}_L + \mathbf{E}_R$$

From above,

$$\mathbf{E}_{Tot} = \mathbf{E}_L + \mathbf{E}_R = 2E_0\cos(kz-\omega t)\hat{\mathbf{i}}$$

This is linearly polarised light with the electric field oriented along the x-axis (in this case).

Lecture 10: Birefringence.

10.1 Birefringence (anisotropy of refractive index).

This is a brief review of what you have learnt in Year 1 and a taster of details you will understand after the second half of this unit.

Summary: Refractive index of birefringent crystals.

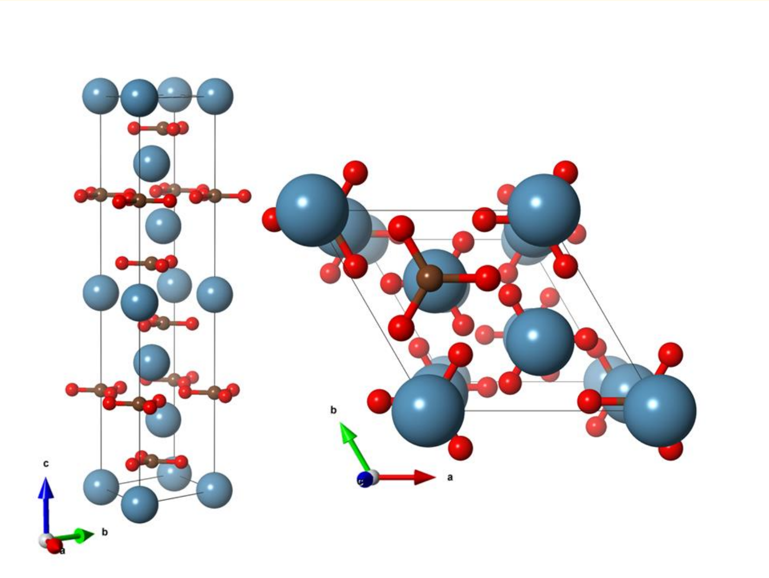
- An electric field can induce a dipole moment on an atom.
- The net result of these dipoles leads to the bulk property, the relative permittivity ε_r of different materials. [Seen in $C = \varepsilon_r \varepsilon_0 A/d$]
- [New] For a non-magnetic material, $n = \sqrt{\varepsilon_r}$, where n is the refractive index of the material.
- [New] In a crystal, we can observe different planes of atoms in different directions. Thus, the refractive index can be different for different direction of the electric field – i.e. different polarisations of light.
- This is optical anisotropy. Different optical effects for different directions in the crystal.

The optical axis.

If the crystal has one direction for which all linear polarisations of light experience the same refractive index when light is propagating along that direction, then that is the optic axis of the crystal. (i.e., the wavevector $\mathbf{k}$ of light is in the same direction as the optic axis, $\mathbf{E}$ always perpendicular to optic axis.) [It's also a **uniaxial** birefringent crystal as it has this **one** axis]

Example of uniaxial birefringence: calcite ($CaCO_3$).

The calcite structure is made up of just three atoms, calcium (**blue**), carbon (**brown**) and oxygen (**red**). The structure consists of calcium (Ca^{2+}) cations sandwiched between flat layers of carbonate (CO_3^{2-}) anions.



//CRYSTALLOGRAPHY365.WORDPRESS.COM/2014/02/17/SEEING-DOUBLE-CALCITE/]

The optic axis is the direction of the c-axis (blue).

For light propagating along the c-axis ($\mathbf{k}$ along c-axis), the electric field is always perpendicular to the optic axis. The wave experiences the same refractive index n_o for all directions of the $\mathbf{E}$-field of plane polarised light in the $\mathbf{a}$-$\mathbf{b}$ plane. This results from the three-fold symmetry in the $\mathbf{a}$-$\mathbf{b}$ plane. These "rays" are known as ordinary rays (or o-rays) as they behave as they would if passing through (isotropic) glass.

For light propagating with its wavevector in the $\mathbf{a}$-$\mathbf{b}$ plane, the refractive index depends on the orientation of the $\mathbf{E}$-field of the plane polarised light. In general, the light will have a component of its $\mathbf{E}$-field perpendicular to the optic axis and a component parallel to the optic axis.

These rays are extraordinary rays (or e-waves) as they do not behave as light does in glass. Light with its $\mathbf{E}$-field parallel to the optic axis has the biggest change in index to the extraordinary refractive index n_e.

[See Hecht. Chapter 8. Section 8.4 for more details]

Birefringence: details, examples, and numbers.

The $\mathbf{E}$-field of the o-wave is normal to the optic axis everywhere and moves at speed, $v_\perp = c/n_o$.

The e-wave has a component for which $\mathbf{E}$ is parallel to the optic axis and that portion of the wave expands at a speed of $v_{||} = c/n_e$.

This is illustrated in the wavelets diagram from Hecht, on the right.

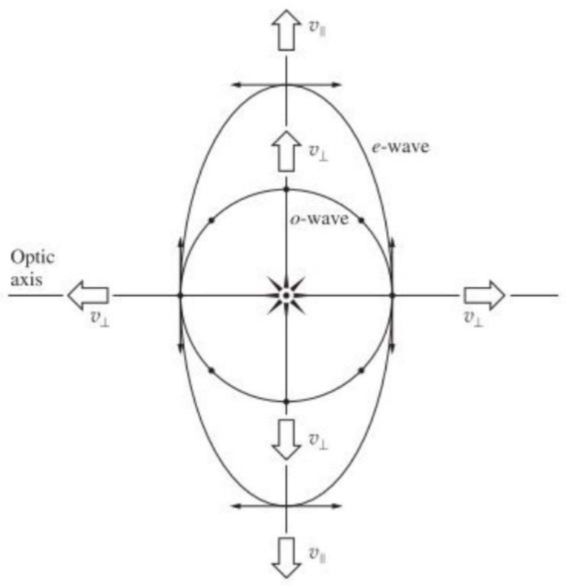


Figure 8.28 Wavelets in a negative uniaxial crystal (their differences much exaggerated). The arrows and dots represent the $\vec{\mathbf{E}}$-fields of the extraordinary and ordinary waves, respectively. The $\vec{\mathbf{E}}$-field of the o-wave is everywhere perpendicular to the optic axis. At these particular locations on the wavelets the $\vec{\mathbf{E}}$- and $\vec{\mathbf{D}}$-fields are parallel. A line from the center point to the ellipse corresponds to a ray in that direction whose length indicates the wave's speed in that direction. A tangent to the ellipse at the point where that ray intersects the e-wave is the direction of $\vec{\mathbf{D}}$. And the same is true for the o-wave where $\vec{\mathbf{E}}$ and $\vec{\mathbf{D}}$ are parallel and perpendicular to the plane of the drawing.

The difference $\Delta n = n_e - n_0$ is a measure of the birefringence – and often called the **birefringence** of the crystal.

In calcite, $\Delta n = n_e - n_0 = -0.172$ and so $v_{||} > v_\perp$. It is a **negative uniaxial** crystal.

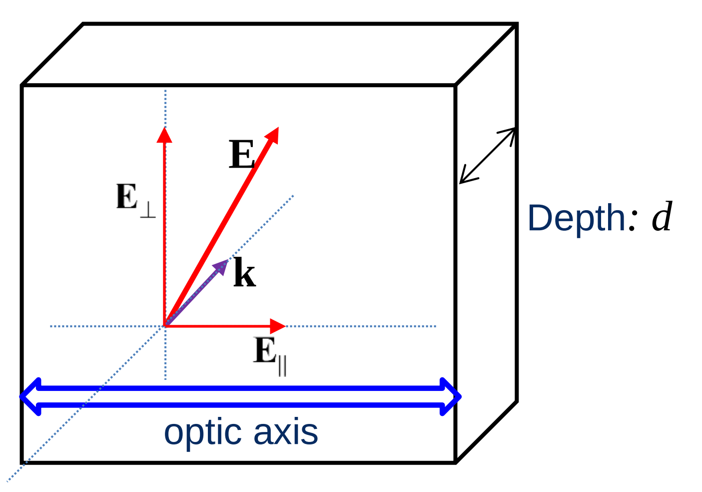
Refractive index of some uniaxial birefringent crystals (Measured at $\lambda = 589.3\,\mathrm{nm}$).

Crystal	n_o	n_e
Calcite	1.6584	1.4864
Quartz	1.5443	1.5534
Rutile (TiO_2)	2.616	2.903

10.2 Retarders and Waveplates.

Retarders are optical devices used to change the polarisation of an incoming wave. They are key components in optics research laboratories and are an application of birefringence.

Consider a slab of uniaxial birefringent crystal, as in the diagram below. The crystal has been cut so that optic axis of the material is parallel to the front face of the slab which has a thickness d. (I have shown a square slab to help with visualising the orientation, but the shape of the front face is unimportant.) We consider a plane polarised wave passing through the retarder with its wavevector perpendicular to the surface.



We take components of the $\mathbf{E}$-field of the incident wave.

$\mathbf{E}_\perp$ is perpendicular to the optical axis and so this component of the wave will experience a refractive index n_o.

$\mathbf{E}_{||}$ is parallel to the optical axis and so this component of the wave will experience n_e.

The optical thickness is nd and so the two components experience a different optical thickness for the slab. This causes a phase difference $\Delta\phi$ between them, $\Delta\phi = \frac{2\pi}{\lambda}(n_0 - n_e)d = \frac{2\pi}{\lambda}\Delta n\,d$. The axis with the smallest refractive index and thus the highest phase velocity for light is the **fast axis** of the waveplate.

For a set wavelength, we can set the phase difference by choosing the thickness of the material.

Full-wave plate.

If $\Delta\phi = \frac{2\pi}{\lambda}\Delta n\,d = 2m\pi$ where m is an integer, which requires $d = m\frac{\lambda}{\Delta n}$.

The relative retardation of the two components is equivalent to 2π, a whole wavelength of path difference. Hence, why this is called a full-wave plate. The e-wave and o-wave are back in phase as they exit the plate. There is no observable effect on the polarisation for any angle of the wave plate to the plane of polarisation.

However, if Δn does not depend on wavelength at these wavelength – or only changes slowly, then $\Delta\phi \sim \frac{1}{\lambda}$. The phase shift depends on wavelength and the retarder is called **chromatic** for its wavelength dependence.

The half-wave plate.

A retarder that introduces a relative phase difference of π radians between the two components is known as a half-wave plate.

Note. If linearly polarised light strikes the retarder with its $\mathbf{E}$-field aligned with fast axis or slow axis of the retarder then there is only one component not two components. No relative phase difference can be given to one component, and there will be no effect from the retarder.

For the half-wave plate, if the plane of polarisation of linear polarised light makes an angle θ to the fast axis of the polariser then the plane of polarisation of the outgoing wave is rotated by 2θ compared to the incident wave. The half-wave plate is a polarisation rotator for linearly polarised waves.   [See question on problem sheet]

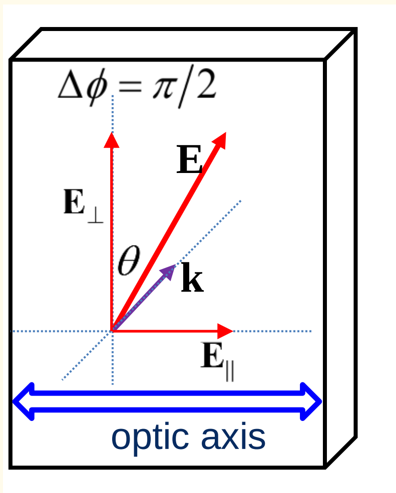
For a half-wave plate $\Delta\phi = \dfrac{2\pi}{\lambda}\Delta n d = \pi$. This requires

$\Delta n d = (2m+1)\lambda/2$ where m is an integer for a net effect of a retardation by π radians.

Quarter-wave plate.

The quarter-wave plate introduces a relative phase shift of $\pi/2$ radians between the two orthogonal components of the wave.

From the details above in Lecture 8, compare this to the way we described polarisation with the equivalence: $\mathbf{E}_x \equiv \mathbf{E}_{||}$ and $\mathbf{E}_y \equiv \mathbf{E}_{\perp}$.



For an incident plane polarised wave with its plane of polarisation at an angle θ to the fast axis of the quarter-wave plate.

If $\theta = 0^\circ$ or $\theta = 90^\circ$ then linearly polarised light exits the waveplate.

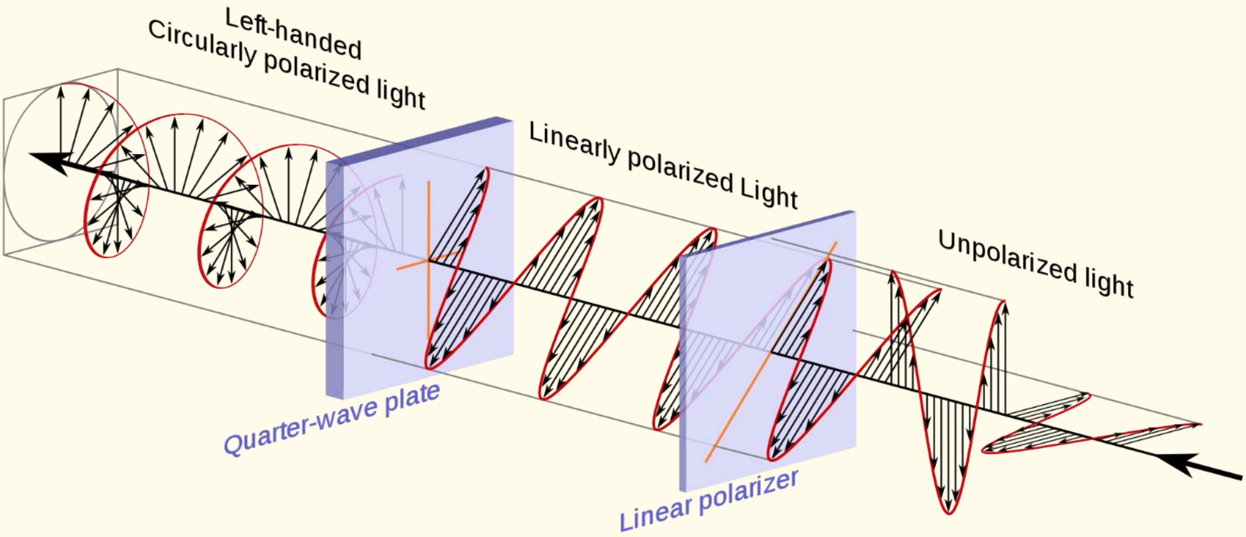
If $\theta = 45^\circ$, $\mathbf{E}_x = \mathbf{E}_y$ with $\Delta\phi = \pi/2$, then circularly polarised light is produced.

Other angles, $\mathbf{E}_x \neq \mathbf{E}_y$ with $\Delta\phi = \pi/2$, then elliptically polarised light is produced.

Question: What would we see if we reversed the flow of time in the situation above? Is this a reasonable thing to consider?

If we consider that that we can represent incoming elliptically polarised light and circularly polarised light as two orthogonal components of plane waves with $\Delta\phi = \pi/2$ between them. We can see that we can use a quarter-wave plate to convert elliptical light and circularly polarised to linearly polarised light.

Wikipedia for nice picture. [Public Domain. File:Circular.Polarization]

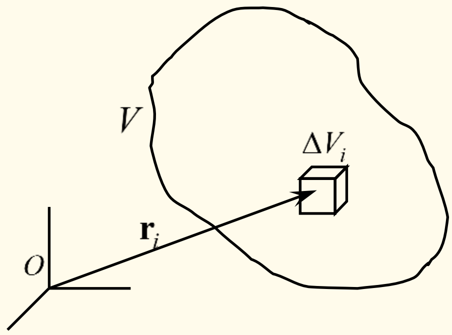


End of Lectures.

Appendix: Mathematical operations on field quantities.

A. Integration.

A1. Volume integral of a scalar field.



Consider a volume V in a region in which a scalar field $\psi(\mathbf{r})$ is defined.

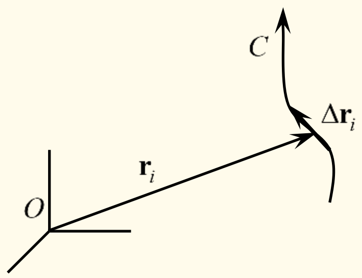
Divide V into N small volume elements ΔV_i, centred around points with position vectors $\mathbf{r}_i$, $.i = 1, \ldots, N$.

Then form the summation $\sum_{i=1}^{N} \psi(\mathbf{r}_i)\, \Delta V_i$.

Take the limit as $N \to \infty$, $\Delta V_i \to 0$ to obtain the **volume integral,**

$$\int_V \psi \, dV \equiv \lim_{\substack{N \to \infty \\ \Delta V_i \to 0}} \sum_{i=1}^{N} \psi(\mathbf{r}_i) \Delta V_i .$$

A2. Tangential line integral of a vector field.



We consider a path C in a vector field $\mathbf{F}(\mathbf{r})$.

Divide C into N small tangential elements $\Delta\mathbf{r}_i$, also written $\Delta s_i\hat{\mathbf{T}}_i$, at position vectors $\mathbf{r}_i$, $i=1,\ldots,N$.

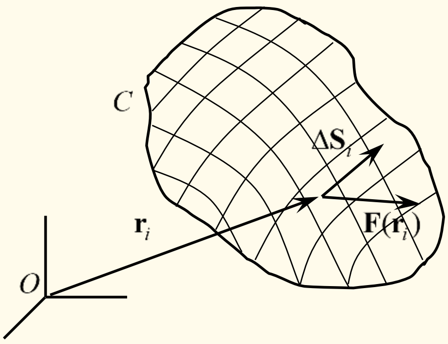
Then form the summation $\sum_{i=1}^{N}\mathbf{F}(\mathbf{r}_i)\cdot\Delta\mathbf{r}_i$.

Take the limit as $N\to\infty$, $\Delta s_i\to 0$ to obtain the **tangential line integral**

$$\int_C \mathbf{F}\cdot d\mathbf{r} \equiv \lim_{\substack{N\to\infty \\ \Delta s_i\to 0}} \sum_{i=1}^{N}\mathbf{F}(\mathbf{r}_i)\cdot\Delta\mathbf{r}_i .$$

If C forms a closed loop, we usually write $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

A3. Flux integral of a vector field.



Consider a surface S bounded by the closed curve C in a vector field $\mathbf{F}(\mathbf{r})$.

Divide the surface into N small patches. Let $\Delta\mathbf{S}_i$ be the vector area of patch i, equal to the area of the patch $\times$ the unit vector normal to the patch: $\Delta\mathbf{S}_i = \Delta S_i\hat{\mathbf{n}}_i$, at position vectors $\mathbf{r}_i$, $i=1,\ldots,N$.

Then form the summation $\sum_{i=1}^{N}\mathbf{F}(\mathbf{r}_i)\cdot\Delta\mathbf{S}_i$.

The required integral is then $\int_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} \equiv \lim_{\substack{N\to\infty \\ \Delta S_i\to 0}} \sum_{i=1}^{N}\mathbf{F}(\mathbf{r}_i)\cdot\Delta\mathbf{S}_i$. This integral is often called the **flux** of $\mathbf{F}$ across S.

If S forms a closed surface, write $\oint_S \mathbf{F}\cdot d\mathbf{S}$; $d\mathbf{S}$ then points **outwards**.

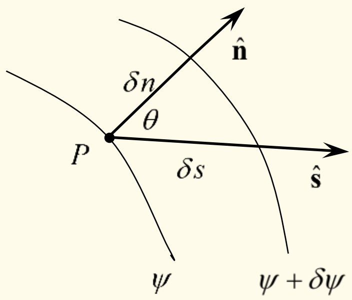
B. Differentiation.

B1. Differentiation of a vector by a scalar.

$\mathbf{F}(\mathbf{r},t)$ is a vector quantity. Its time derivative, for example, may be defined by $\mathbf{F}(\mathbf{r},t+\delta t)=\mathbf{F}(\mathbf{r},t)+\left(\frac{\partial \mathbf{F}}{\partial t}\right)\delta t$ in the limit $\delta t \to 0$.

B2. The gradient of a scalar field.

Here we consider the rate of change of a **scalar** field $\psi(\mathbf{r})$ with distance.



Take two neighbouring contours, with values ψ and $\psi+\delta\psi$.

Along a general direction $\hat{\mathbf{s}}$, the **directional derivative** is defined by

$$\frac{\partial \psi}{\partial s}=\lim_{\delta s\to 0}\frac{\delta\psi}{\delta s}.$$

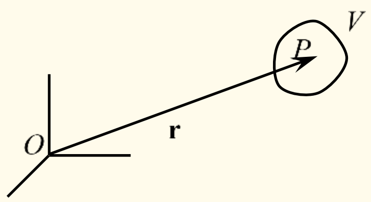
The **maximum** value this derivative can take occurs when δs is taken along the normal direction $\hat{\mathbf{n}}$ between the contours. We define the **gradient** of ψ at point P using the vector

$$\operatorname{grad}\psi \equiv \frac{\partial \psi}{\partial n}\hat{\mathbf{n}}.$$

The modulus of this vector gives the maximum spatial derivative of ψ at point P; the direction in which the maximum derivative occurs is the direction of the vector $\mathrm{grad}\,\psi$.

Note that $\mathrm{grad}\,\psi$ is a **vector field** derived from a **scalar field**.

B3. The divergence of a vector field.



Let P be a point in a **vector** field $\mathbf{F}(\mathbf{r})$. Around P construct the small volume V having (closed) surface S, and calculate the net **flux** of $\mathbf{F}(\mathbf{r})$ across S.

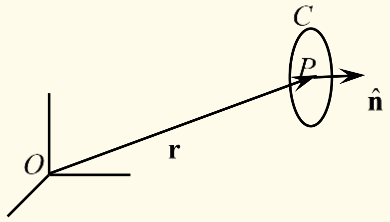
The **divergence** of $\mathbf{F}(\mathbf{r})$ at P is then defined as

$$\mathrm{div}\mathbf{F} = \lim_{V\to 0}\left\{\frac{1}{V}\oint_S \mathbf{F}\cdot d\mathbf{S}\right\}.$$

$\mathrm{div}\mathbf{F}$ is a scalar field derived from a vector field, and measures the net flux originating or disappearing at a point.

For the vector fields of interest in this unit, a non-zero divergence implies a **source** (if positive) or **sink** (if negative) of field at that point.

B4. The curl of a vector field.



Again, let P be a point in a **vector** field $\mathbf{F}(\mathbf{r})$. Around P construct a small loop C enclosing (open) surface S. Calculate the **tangential line integral** of $\mathbf{F}(\mathbf{r})$ around C

and **rotate** the loop until the integral has a maximum value.

In this orientation, let $\hat{\mathbf{n}}$ be the direction normal to the loop. The **curl** of $\mathbf{F}(\mathbf{r})$ at P is then defined as

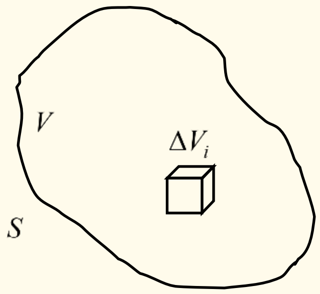
$$\mathrm{curl}\mathbf{F} = \hat{\mathbf{n}} \lim_{S\to 0}\left\{\frac{1}{S}\oint_C \mathbf{F}\cdot d\mathbf{r}\right\}.$$

$\mathrm{curl}\mathbf{F}$ is a vector field derived from a vector field. A non-zero curl at a point indicates a rotational element to the field at that point. The direction of $\mathrm{curl}\mathbf{F}$ is along the axis of rotation, and the modulus is a measure of local rotation rate.

C. Two important integral theorems.

We will use each of these integral theorems a lot…

C1. The divergence theorem.



Consider a volume V enclosed by surface S in a vector field $\mathbf{F}(\mathbf{r})$.

Let γ denote the flux of $\mathbf{F}$ out of S:

$$\gamma = \oint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}.$$

Divide V into N small volume elements ΔV_i, each enclosed by surface S_i.

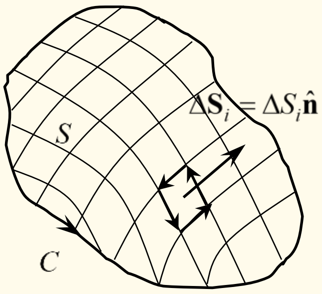
The flux of $\mathbf{F}$ from a single element is $\oint_{S_i} \mathbf{F}\cdot \mathrm{d}\mathbf{S}_i$. The sum of all N terms in V is equal to γ, since contributions from internal surfaces cancel. Thus

$$\gamma = \sum_{i=1}^{N}\oint_{S_i} \mathbf{F}\cdot \mathrm{d}\mathbf{S}_i = \sum_{i=1}^{N}\left\{\frac{1}{\Delta V_i}\oint_{S_i} \mathbf{F}\cdot \mathrm{d}\mathbf{S}_i\right\}\Delta V_i.$$

Now take the limit as $N \to \infty$, $\Delta V_i \to 0$. As $\Delta V_i \to 0$, the term in brackets becomes $\mathrm{div}\mathbf{F}$. As $N \to \infty$, the sum on the right becomes a volume integral. Thus

$$\oint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_V \mathrm{div}\mathbf{F}\,\mathrm{d}V\,. \quad \textbf{The divergence theorem}$$

C2. Stokes' theorem.



Let C be a closed curve around the edge of an arbitrary (but not closed) surface S in a vector field $\mathbf{F}(\mathbf{r})$.

Divide the surface into N small patches and calculate the tangential line integral of $\mathbf{F}$ around the perimeter of each patch.

The sum of all N of these integrals is equal to the tangential line integral of $\mathbf{F}$ around C since contributions on internal edges cancel. Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \sum_{i=1}^{N} \oint_{C_i} \mathbf{F}\cdot d\mathbf{r}_i = \sum_{i=1}^{N} \left\{ \frac{1}{\Delta S_i} \oint_{C_i} \mathbf{F}\cdot d\mathbf{r}_i \right\} \Delta S_i$$

$$= \sum_{i=1}^{N} \{\text{component of curl}\mathbf{F}\text{ along normal to }\Delta S_i\}\,\Delta S_i$$

$$= \sum_{i=1}^{N} \mathrm{curl}\mathbf{F}\cdot d\mathbf{S}_i\,.$$

In the limit as $N \to \infty$, $\Delta S_i \to 0$,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_S \mathrm{curl}\mathbf{F}\cdot d\mathbf{S}\,. \qquad \textbf{Stokes' theorem}$$

Each of these theorems helps us to relate physics inside a region to physics on the boundary of that region. This is extremely useful in any field theory. In this sense the theorems are just extensions of the basic result of integral calculus

$$\int_a^b d\phi = \int_a^b \frac{d\phi}{dx}dx = \phi(b) - \phi(a).$$

D. Evaluation of derivatives in Cartesian coordinates.

All the maths in this handout has so far been independent of any coordinate system. This is one of the major advantages of using vector calculus; general results and proofs, valid for all shapes and (well-behaved) fields, in all (orthonormal) coordinate systems, may be obtained in one go.

However, to **evaluate** fields their derivatives and integrals we need to adopt a coordinate system. In this unit we shall nearly always use Cartesian coordinates (x, y, z) with basis vectors $\hat{\mathbf{e}}_x = \hat{\mathbf{i}}$, $\hat{\mathbf{e}}_y = \hat{\mathbf{j}}$ and $\hat{\mathbf{e}}_z = \hat{\mathbf{k}}$. Here we look at the way to evaluate various derivatives in Cartesian coordinates.

D1. Differentiation of a vector by a scalar.

In this unit, the scalar is usually time, t. If in component form, $\mathbf{F} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_x\hat{\mathbf{k}}$, then

$$\frac{\partial \mathbf{F}}{\partial t} = \frac{\partial F_x}{\partial t}\hat{\mathbf{i}} + \frac{\partial F_y}{\partial t}\hat{\mathbf{j}} + \frac{\partial F_z}{\partial t}\hat{\mathbf{k}}.$$

N.B. It is important to remember that F_x is not necessarily a **function** of x alone; it is the x-**component** of $\mathbf{F}$. In general, F_x is a function of (x, y, z, t).

D2. Spatial derivatives of fields.

The three key spatial derivatives can all be written in terms of the **vector operator** ∇**:**

$$\mathrm{grad}\,\psi = \nabla\psi \qquad \mathrm{div}\mathbf{F} = \nabla\cdot\mathbf{F} \qquad \mathrm{curl}\mathbf{F} = \nabla\times\mathbf{F}\,.$$

This is the notation that will be used in this unit. In Cartesian coordinates we have

$$\nabla \equiv \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\,,$$

so that the derivatives can be evaluated in Cartesian coordinates as follows:

$$\nabla\psi = \hat{\mathbf{i}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\psi}{\partial z}$$ gradient of a scalar field - ψ

$$\nabla\cdot\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$ divergence of a vector field $\mathbf{F}$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$ curl of a vector field $\mathbf{F}$

D3. Triple products.

We will often apply ∇ twice to form triple products. An important example is the **Laplacian** operator $\nabla \cdot \nabla \psi = \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$.

In addition (and a bit confusingly) we **define** ∇^2 acting on a vector field component by component:

$$\nabla^2 \mathbf{F} \equiv \hat{\mathbf{i}} \nabla^2 F_x + \hat{\mathbf{j}} \nabla^2 F_y + \hat{\mathbf{k}} \nabla^2 F_z = \hat{\mathbf{i}} \left(\frac{\partial^2 F_x}{\partial x^2} + \frac{\partial^2 F_x}{\partial y^2} + \frac{\partial^2 F_x}{\partial z^2} \right) + \cdots$$

Other useful triple products are

$\nabla \times \nabla \psi$ which is zero for any scalar field ψ,

$\nabla \cdot (\nabla \times \mathbf{F})$ which is zero for any vector field $\mathbf{F}$, and

$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$, which is useful when deriving wave equations.

An excellent source of information for the vector calculus needed on this unit is to check §10.4 Vector Calculus on p23 of the University of Bath Formulae and Statistical Tables.