

# Q1

Potassium (K) is monovalent and forms the bcc structure with lattice parameter  $a = 5.23 \text{ \AA}$ .

Calculate for K:

(i) The electron density  $n$ . [2]

(ii) Using Sommerfeld theory, the speed (in m/s) of the fastest moving electron at  $T = 0 \text{ K}$ . [2]

$$\text{Note: } m = 9.1094 \times 10^{-31} \text{ kg} \quad \hbar = 1.055 \times 10^{-34} \text{ J s}$$

$$e = 1.602 \times 10^{-19} \text{ C}$$

## Answer

(i) The conventional bcc unit cell contains 2 atoms. So

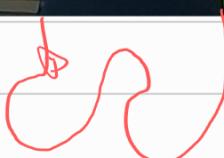
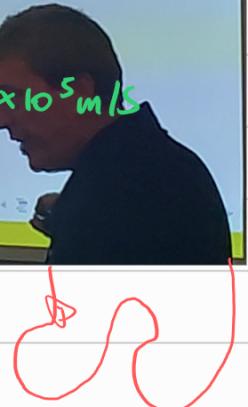
$$n = \frac{\text{no. of electrons}}{\text{volume}} = \frac{2 \times 1}{(5.23\text{\AA})^3} = 0.0140 \text{ \AA}^{-3} = 1.40 \times 10^{22} \text{ cm}^{-3}$$

(ii) The fastest moving electrons are at the surface of the Fermi sphere  
The Fermi wave vector is

$$k_F = (3\pi^2 n)^{1/3} = (3\pi^2 \times 1.40 \times 10^{22})^{1/3} = 7.46 \times 10^7 \text{ cm}^{-1} = 0.746 \text{ \AA}^{-1}$$

Equating momenta  $mv_F = \hbar k_F$  so [SI units throughout]

$$v_F = \frac{\hbar k_F}{m} = \frac{1.055 \times 10^{-34} \times 0.746 \times 10^{10}}{9.1094 \times 10^{-31}} = 8.64 \times 10^5 \text{ m/s}$$



i) bcc

$$a = 5.23 \text{ \AA}$$

$$\frac{2 \cdot 1}{a^3} = 1.4 \times 10^{-2} \text{ \AA}^{-3}$$

$$= 1.4 \times 10^{422} \text{ cm}^{-3}$$

ii)  $T = 0 \text{ K}$

$$V_F = (3\pi^2 n)^{1/3} = 0.746 \text{ \AA}^{-1}$$

$$V_F = \frac{\hbar k_F}{m} = \frac{\sqrt{m}}{m}$$

right technique

surface e = largest wavevector  
∴ move fastest

# Q2

A constant magnetic field  $\vec{B} = B\hat{z}$  and oscillating electric field  $\vec{E} = E e^{i\Omega t}\hat{x}$  are simultaneously applied to a material.

(i) Assume the electrons can be described by Drude theory. Write down an equation of motion for the drift velocity.

Let  $\vec{v} = (v_x, v_y)$ . Resolve the equation of motion into  $x$  and  $y$  components.

Let  $u = v_x + iv_y$ . Show  $\frac{du}{dt} + \frac{u}{\tau} - i\omega u = Ae^{i\Omega t}$ , where  $\omega = \frac{eB}{m}$ ,  $A = -\frac{eE}{m}$ . [4]

(ii) Assume  $u = \bar{u}e^{i\Omega t}$ . Find  $\bar{u}$ . If the kinetic energy  $T$  of the electrons is  $\propto |\bar{u}|^2$  ( $= v_x^2 + v_y^2$ ), show  $T(\omega) = \frac{T_{\max}/\tau^2}{(\Omega - \omega)^2 + (1/\tau)^2}$ . [3]

(iii) Where does this energy come from?  
Sketch  $T$  vs  $B$  labelling relevant features. [3]

i) Show  $\frac{du}{dt} + \frac{u}{\tau} - i\omega u = Ae^{i\Omega t}$

$$\omega = eB/m \quad 2\pi f = -e\varepsilon/m$$

$$\text{E.O.M.: } \frac{d\vec{p}}{dt} = \vec{F} - \frac{\vec{p}}{\tau} \Rightarrow m \frac{d\vec{v}}{dt} = -e(\vec{E} + \vec{v}_x \vec{B}) - \frac{m\vec{v}}{\tau}$$

$$\text{Cross product: } \vec{v} \times \vec{B} = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix}$$

$$= B(v_x v_y - v_x v_z)$$

$$\text{ii) } u = \bar{u} e^{i\Omega t} \therefore \bar{u} = \frac{u}{e^{i\Omega t}}$$

$$u = v_x + i v_y \quad \bar{u} = \frac{v_x + i v_y}{e^{i\Omega t}}$$

$$T \propto |\bar{u}|^2$$

**Answer**

(ii) Assume  $u = \bar{u} e^{i\Omega t}$ . Find  $\bar{u}$ . If the kinetic energy  $T$  of the electrons is  $\propto |\bar{u}|^2$ , show  $T(\omega) = \frac{T_{\max}/\tau^2}{(\Omega - \omega)^2 + (1/\tau)^2}$ . [3]

Substitute into  $\frac{du}{dt} + \frac{u}{\tau} - i\omega u = Ae^{i\Omega t}$ :  $i\Omega \bar{u} e^{i\Omega t} + \frac{\bar{u} e^{i\Omega t}}{\tau} - i\omega \bar{u} e^{i\Omega t} = Ae^{i\Omega t}$

Tidy:  $\left( i(\Omega - \omega) + \frac{1}{\tau} \right) \bar{u} = A \quad \text{or} \quad \bar{u} = \frac{A}{i(\Omega - \omega) + 1/\tau}$

Kinetic energy:

$$T \propto |\bar{u}|^2 = \frac{A}{i(\Omega - \omega) + 1/\tau} \times \frac{A}{-i(\Omega - \omega) + 1/\tau} = \frac{A^2}{(\Omega - \omega)^2 + (1/\tau)^2}$$

Max. value  $T_{\max} = A^2 \tau^2$  when  $\omega = \Omega$ , so  $T = \frac{T_{\max}/\tau^2}{(\Omega - \omega)^2 + (1/\tau)^2}$

**Answer**

(i) Write down an equation of motion for the drift velocity. Resolve the equation of motion into  $x$  and  $y$  components. Let  $u = v_x + iv_y$ . Show  $\frac{du}{dt} + \frac{u}{\tau} - i\omega u = Ae^{i\Omega t}$ , where  $\omega = eB/m$  and  $A = -eE/m$ . [4]

E.O.M.:  $\frac{d\vec{p}}{dt} = \vec{F} - \frac{\vec{p}}{\tau} \Rightarrow m \frac{d\vec{v}}{dt} = -e(\vec{E} + \vec{v}_x \vec{B}) - \frac{m\vec{v}}{\tau}$

Cross product:  $\vec{v}_x \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = \hat{x}(v_y B) + \hat{y}(-v_z B)$ .

$x$  and  $y$  components:  $\begin{aligned} \hat{x}: \quad m \frac{dv_x}{dt} &= -e(E_x e^{i\Omega t} + v_y B) - \frac{mv_x}{\tau} \quad (1) \\ \hat{y}: \quad m \frac{dv_y}{dt} &= -e(E_y e^{i\Omega t} - v_z B) - \frac{mv_y}{\tau} \quad (2) \end{aligned}$

(1)+(2), divide by  $m$ :  $\frac{d(v_x + iv_y)}{dt} = -\frac{e}{m} (\vec{E} e^{i\Omega t} + (v_y - iv_z) \vec{B}) - \frac{v_x + iv_y}{\tau}$

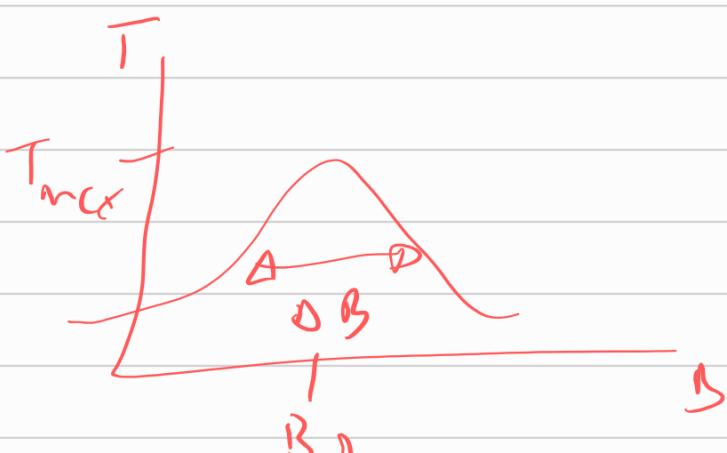
$v_y - iv_z = -i(v_z + iv_y)$  so  $\frac{du}{dt} = Ae^{i\Omega t} + i\omega u - \frac{u}{\tau}$ .

iii) Electric field does not work on a charged ions from applied  $E$  field

Sketch  $T = \frac{T_{\max}/\tau^2}{(\Omega - \omega)^2 + (1/\tau)^2}$  - resonance

Peak when  $\omega = \Omega \Rightarrow \frac{eB}{m} = \Omega \text{ or } B_\Omega = \frac{\Omega m}{e}$

Half width  $\Delta\omega = \frac{1}{f} \Rightarrow \frac{\Delta B}{B_\Omega} \approx \frac{1}{f}$



$$\Delta B = \frac{m}{e\Omega}$$

## Answer

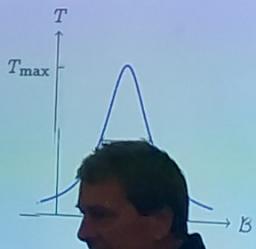
(iii) Where does this energy come from? Sketch  $T$  vs  $B$  labelling relevant features. [3]

The electric field does work on the electron  $\Rightarrow$  the energy comes from the applied electric field.

$$\text{Sketch } T = \frac{T_{\max}/\tau^2}{(\Omega - \omega)^2 + (1/\tau)^2} \quad \text{--- resonance.}$$

$$\text{Peak when } \omega = \Omega \Rightarrow \frac{eB}{m} = \Omega \quad \text{or} \quad B_\Omega = \frac{\Omega m}{e}.$$

$$\text{Half-width } \Delta\omega = \frac{1}{\tau} \Rightarrow \frac{e\Delta B}{m} = \frac{1}{\tau} \quad \text{or} \quad \Delta B = \frac{m}{e\tau}.$$



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