① $\psi(x) = Axe^{-\lambda x}$

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{V_0 x}{a} \quad , \quad x>0$$

$$\hat{H}\psi = -\frac{\hbar^2}{2m} A(-2\lambda + \lambda^2 x)e^{-\lambda x} + A\frac{V_0}{a}x^2 e^{-\lambda x}$$

ignore

Expectat° val: $\langle \hat{H} \rangle = \langle \hat{T} \rangle + \langle \hat{V} \rangle$

$\langle \hat{H} \rangle = \langle \psi | \hat{H} | \psi \rangle \qquad \langle \hat{T} \rangle = \langle \psi | \hat{T} | \psi \rangle \qquad \langle \hat{V} \rangle = \langle \psi | \hat{V} | \psi \rangle$

$$\langle \hat{T} \rangle = \int (Axe^{-\lambda x})\left[-\frac{\hbar^2}{2m}\left(2\frac{\lambda}{x} + \lambda^2\right)\right](Axe^{-\lambda x})$$

where is this coming from?
T = k.E

$$= A^2 \frac{\hbar^2}{2m}\left(\frac{1}{4\lambda}\right)$$

$$\langle \hat{V} \rangle = \int (Axe^{-\lambda x})\left[\frac{V_0}{a}x\right](Axe^{-\lambda x})$$

$$= A^2 \frac{V_0}{a}\frac{3}{8\lambda}$$

pre-set

$$\langle \hat{H} \rangle = \frac{A^2}{8\lambda}\left(\frac{\hbar^2}{m} + \frac{3V_0}{a}\right)$$

Normalisat°: $\langle \psi | \psi \rangle = 1$

$$= \int_0^\infty Ax e^{-\lambda x} Ax e^{-\lambda x} dx$$

$$= A^2 \int_0^\infty x^2 e^{-2\lambda x} dx$$

$$= \frac{A^2}{4\lambda^3}$$

$$\Rightarrow A = 4\lambda^3$$

So $$\langle \hat{H} \rangle = \frac{A^2}{8\lambda}\left(\frac{\hbar^2}{m} + \frac{3V_0}{a}\right)$$

$$= \frac{\hbar^2 \lambda^2}{2m} + \frac{3V_0}{2a}\frac{1}{\lambda}$$

Minimising: $$\frac{d\langle \hat{H} \rangle}{d\lambda} = \left(\frac{\hbar^2 \lambda}{m} - \frac{3V_0}{2a}\frac{1}{\lambda^2}\right) = 0$$

$$\lambda^3 = \frac{3mV_0}{2\hbar a}$$

$$\lambda_{min} = \sqrt[3]{\frac{3mV_0}{2\hbar a}}$$

So $$\langle \hat{H} \rangle_{min} = \frac{\hbar^2 \lambda_{min}^2}{2m} + \frac{3V_0}{2a}\frac{1}{\lambda}$$

$$= \frac{\hbar}{2\cancel{m}} \frac{3\cancel{m} V_0}{2\cancel{\hbar} a} \sqrt{\frac{3 m V_0}{2 \hbar a}} + \frac{3 V_0}{2a} \sqrt{\frac{2 \hbar a}{3 m V_0}}$$

$$= \frac{3 V_0}{4a} \sqrt{\frac{3 m V_0 \hbar}{2a}} + \sqrt{\frac{6 V_0}{4 a m}}$$

Grand state E: $E_{grand} \leq \langle \hat{H} \rangle_{min}$

$$E_{grand} \leq \sqrt{\frac{6 V_0}{4 a m}}$$

2 $\Psi(x) = Ax(a-x)$

$$\hat{H} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{V_0 x}{a}, \quad x > 0$$

$u = Ax \quad u' = A$

$(uv)' = uv' + u'v \qquad v = a - x \quad v' = -1$

$$\frac{d}{dx}\Psi = -Ax + A(a-x)$$

$$\frac{d^2}{dx^2}\Psi = -A\frac{x^2}{2} - A\frac{x^2}{2} = -Ax^2$$

$$H\Psi = \frac{\hbar^2}{2m} A x^2 + \frac{V_0 x}{a}$$

$$\hat{H} = \frac{-\hbar}{2m} \frac{\partial^2}{\partial x^2}$$

$$\langle \hat{H} \rangle = \langle \hat{T} \rangle + \langle \hat{V} \rangle$$

$$\langle \hat{T} \rangle = \int (Axe^{-\lambda x}) \left[\frac{\hbar^2}{2m} A x^2 \right] (Axe^{-\lambda x})$$

$$= A^3 \int \quad x^4 \hbar^2 e^{-2\lambda x} \, dx$$

$$\int x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}}$$

$$= A^3\left[\frac{\hbar^2}{2m}\,\frac{24}{32\,\lambda^5}\right]$$

$$= \frac{A^3\hbar^2}{3\lambda^5}$$

$$\langle \hat{V} \rangle = \int \left(A x e^{-\lambda x}\right)\left[\frac{V_0 x}{a}\right]\left(A x e^{-\lambda x}\right)$$

$$= \frac{A^2 V_0}{a}\int x^3 e^{-2\lambda x}\,dx$$

$$= \frac{A^2 V_0}{a}\left[\frac{6}{16\lambda^4}\right]$$

$$= \frac{A^2 V_0\, 3}{a\, 8\, \lambda^4}$$

$$\Rightarrow \langle \hat{H} \rangle = \frac{A^3\hbar^2}{3\lambda^5} + \frac{A^2 V_0\, 3}{a\, 8\, \lambda^4}$$

Minimise : $\frac{d\langle \hat{H} \rangle}{d\lambda} = \frac{-A^3\hbar^2}{12\lambda^4} - \frac{A^2 V_0 3}{24 a \lambda^3}$

$= 0$

$\therefore \quad \frac{-A^3\hbar^2}{12\lambda^4} = \frac{A^2 V_0 3}{24 a \lambda^3}$

$\frac{A\hbar}{3\lambda} = \frac{3 V_0}{2a}$

$\lambda_{min} = \frac{A\hbar 2a}{3V_0}$

2	3
4	9
8	27
16	81
32	243

so $\langle H_{min} \rangle = \frac{A^3\hbar^2}{3\lambda_{min}^5} + \frac{A^2 V_0 3}{a 8 \lambda_{min}^4}$

$= \frac{81 V_0^5}{A^2\hbar^3 32 a^5} + \frac{3}{128}\frac{81 V_0^5}{A^2\hbar^4 a^5}$

$= \frac{81 V_0^5}{A^2\hbar^3 a^5}\left(\frac{1}{32} + \frac{3}{128\hbar}\right)$

Grand state E : $E_{grand} \leq \langle \hat{H} \rangle_{min}$

$E \ldots 81 V_0^5 \left(\ldots + 3 \ldots \right)$

$$t_{ground} \leq \frac{\dots}{A^2 \hbar^3 a^5} \left(\frac{\dots}{32} + \frac{\dots}{128 \hbar} \right)$$

② $\Psi(x) = Ax(a-x)$

$$\hat{H} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{V_0 x}{a}$$

$$= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$$

BC

$V(x) = 0$ for $0 \leq x \leq a$ i.e inside box

$$\langle \hat{H} \rangle = \int_0^a \left[Ax(a-x) \right] \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right] \left[Ax(a-x) \right] dx$$

$$= -\frac{A^2 \hbar^2}{2m} \int_0^a (ax - x^2) \left(\frac{d^2 (ax - x^2)}{dx^2} \right) dx$$

$(ax - x^2)' = a - 2x$

$(a - 2x)' = -2$

$$= \frac{A^2 \hbar^2}{m} \int_0^a (ax - x^2)\, dx$$

$$= \frac{A^2 \hbar^2}{m} \left[\frac{ax^2}{2} - \frac{x^3}{3} \right]_0^a$$

$$\langle \hat{H} \rangle = \frac{A^2 \hbar^2 a^3}{6m}$$

Normalisat°: $\langle \psi | \psi \rangle = 1$

$$= \int_0^a Ax(a-x) \; Ax(a-x) \, dx$$

$$= A^2 \int_0^a x^4 + a^2 x^2 - 2x^3 a) \, dx$$

$$= A^2 \left[\frac{x^5}{5} + a^2 \frac{x^3}{3} - \frac{x^4 a}{2} \right]_0^a$$

$$= A^2 \left[\frac{a^5}{5} + \frac{a^5}{3} - \frac{a^5}{2} \right]$$

$$\therefore A^2 = \frac{30}{a^5}$$

Minimizat° not required: no dep λ ≈ other cnst
$d\langle \hat{H} \rangle / d\lambda$

$$\therefore \langle \hat{H} \rangle = \frac{5\hbar^2}{ma^2}$$

So the ground state energy $E_{\text{ground}} \le \frac{5\hbar^2}{ma^2}$. From the full solution of the TISE for the 1D infinite square well we know that $E_{ground} = \frac{\ }{2ma^2}$, i.e., the discrepancy in the estimated ground state

energy is $\frac{5-(\pi^2/2)}{5} = 0.013$ or 1.3%. The normalised trial wavefunction

$\psi(x) = \left(\frac{30}{a}\right)^{\frac{1}{2}} x(a-x)$ closely resembles the actual ground state wavefunction: -

if both T + 0 ≠ V then u do ⟨T⟩ & ⟨V⟩ separately & then add!

(3) $\Psi(x) = Ae^{-bx^2}$

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

$$\langle \hat{H} \rangle = \int (Ae^{-bx^2})\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)(Ae^{-bx^2})\,dx$$

$$(Ae^{-bx^2})' = -2xAe^{-bx^2}$$

$$(uv)' = uv' + u'v$$

$u = -2x \qquad v = Ae^{-bx^2}$

$u' = -2 \qquad v' = -2xAe^{-bx^2}$

$$(Ae^{-bx^2})'' = 4x^2Ae^{-bx^2} - 2Ae^{-bx^2}$$

$$\langle H \rangle = \int (Ae^{-bx^2})\left(\frac{-\hbar^2}{2m}\left[4x^2Ae^{-bx^2} - 2Ae^{-bx^2}\right] + \frac{1}{2}m\omega^2x^2Ae^{-bx^2}\right)dx$$

$$= \frac{A^2}{2}\int \frac{-\hbar^2}{m}4x^2e^{-2bx^2} + \frac{\hbar^2}{m}2e^{-2bx^2}$$

$$+ m\omega^2 x^2 e^{-bx^2} dx$$

$$= \frac{A^2}{2}\left[\frac{-\hbar^2}{mb}\sqrt{\frac{2b}{b}} + \frac{\hbar^2}{m}\sqrt{\frac{2\pi}{b}} + m\omega^2 \frac{1}{2b}\sqrt{\frac{\pi}{b}}\right]$$

Normalised: $\langle \psi | \psi \rangle = 1$

$$= \int e^{-2bx^2} dx$$

$$= A\left(\frac{\pi}{2b}\right)^{1/2}$$

$$\Rightarrow A^2 = \frac{2b}{\pi}$$

Thus $\langle H \rangle =$

$$\frac{b}{\pi}\left[\frac{-\hbar^2}{mb}\sqrt{\frac{2b}{b}} + \frac{\hbar^2}{m}\sqrt{\frac{2\pi}{b}} + m\omega^2 \frac{1}{2b}\sqrt{\frac{\pi}{b}}\right]$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x$$

$$\langle \hat{T} \rangle = A^2 \int e^{-bx^2} \left(\frac{-\hbar^2}{2m} \right. \rightarrow$$

$$\left(e^{-bx^2}\right)' = -2bxe^{-2bx^2}$$

$u = 2bx \quad u' = 2b \quad v = e^{-2bx^2} \quad v' = -4bxe^{-2bx^2}$

$$\left(e^{-bx^2}\right)'' = 8b^2x^2e^{-2bx^2} + 2be^{-2bx^2}$$

$$\rightarrow \left(8b^2x^2e^{-2bx^2} + 2be^{-2bx^2}\right) dx$$

$$= -\frac{A^2\hbar^2 b}{m} \int e^{-3bx^2} \left(4bx^2 + 1\right) dx$$

$$= \frac{-A^2\hbar^2 b}{m} \left[2b \left(\frac{\pi}{27b^3}\right)^{1/2} + \left(\frac{\pi}{3b}\right)^{1/2} \right]$$

$$= -\frac{A^2\hbar^2\sqrt{b}}{m} \left(2\sqrt{\frac{\pi}{27}} + \sqrt{\frac{\pi}{3}} \right)$$

Normalise 1st - might make things easier

$$\langle \Psi | \Psi \rangle = \int A^2 e^{-2bx^2} dx \ldots$$

$$= A^2\left(\frac{\pi}{2b}\right)^{1/2}$$

$$A^2 = \sqrt{\frac{2b}{\pi}}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx} + \frac{1}{2}m\omega^2 x$$

$$\langle \hat{T} \rangle = \int \left(Ae^{-2bx^2}\right)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\left(Ae^{-2bx^2}\right)$$

$$= \frac{-\hbar^2}{m}\sqrt{\frac{b}{2\pi}}\int e^{-2bx^2}\underbrace{\left(\frac{d^2}{dx^2}e^{-2bx^2}\right)}\,dx$$

$$\left(e^{-2bx^2}\right)' = -4bxe^{-2bx^2}$$

$u = -4bx \qquad v = e^{-2bx^2}$

$u' = -4b \qquad v' = -4bxe^{-2bx^2}$

$$= \frac{-\hbar^2}{m}\sqrt{\frac{b}{2\pi}}\int e^{-2bx^2}\left(16b^2x^2e^{-2bx^2} - 4be^{-2bx^2}\right)dx$$

$$= \frac{-\hbar^2}{m}\sqrt{\frac{b}{2\pi}}\int e^{-4bx^2}\cdot 4b\left(bx^2 - 1\right)dx$$

$$= -\frac{4b\hbar^2}{m}\sqrt{\frac{b}{2\pi}} \int (bx^2-1)e^{-4bx^2}dx$$

$$= -\frac{4b\hbar^2}{m}\sqrt{\frac{b}{2\pi}} \left[\frac{b}{2}\left(\frac{\pi}{(4b)^3}\right)^{1/2} - \left(\frac{\pi}{4b}\right)^{1/2} \right]$$

$$= \frac{7b\hbar}{2m}\sqrt{\frac{\pi}{4b}}$$

$$= \frac{7\hbar\sqrt{\pi b}}{4m}$$

I give up on this one, will try the next one

(4) $\psi = Ae^{-bx^2}$

Normalisat°: $\langle\psi|\psi\rangle = 1$

$$A^2\int e^{-2bx^2}dx = A^2\left(\frac{\pi}{2b}\right)^{1/2}$$

$$\therefore A^2 = \sqrt{\frac{2b}{\pi}}$$

BC: $V(0) = \infty$

$V(x) = 0$

$\langle \hat{H} \rangle = \langle \hat{T} \rangle + \langle \hat{V} \rangle$

$\langle \hat{T} \rangle = \langle \psi | -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} | \psi \rangle$

$= \sqrt{\frac{2b}{\pi}} \int (e^{-bx^2}) \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) (e^{-bx^2}) \, dx$

$= -2b e^{-2bx^2} + 4b^2 x^2 e^{-2bx^2}$

$= \sqrt{\frac{2b}{\pi}} \int -2b e^{-2bx^2} + 4b^2 x^2 e^{-2bx^2} \, dx$

$= \sqrt{\frac{2b}{\pi}} \left[-2b \left(\frac{\pi}{2b} \right)^{1/2} + \frac{4b^2}{2} \left(\frac{\pi}{4b^2} \right)^{1/2} \right]$

$= \sqrt{\frac{2b}{\pi}} \, 2b \left(\frac{\pi}{2b} \right)^{1/2} \left(-1 + \frac{b}{\sqrt{2b}} \right)$

$= \cancel{\sqrt{\frac{2b}{\pi}}} \, 2b \cancel{\sqrt{\frac{\pi}{2b}}} \left(\sqrt{\frac{b}{2}} - 1 \right)$

$= 2b \left(\sqrt{\frac{b}{2}} - 1 \right)$

If given $V(x) = -\alpha \, \delta(x)$

use it for $\langle \hat{V} \rangle = \langle \psi | V(x) | \psi \rangle$

(5) $V(x) = \alpha x^4$

$$\psi(x) = A e^{-bx^2}$$

Normalisat°: $\langle \psi | \psi \rangle = 1$

$$= \int A^2 e^{-2bx^2} dx$$

$$= A^2 \left(\frac{\pi}{2b}\right)^{1/2}$$

$$\Rightarrow A^2 = \left(\frac{2b}{\pi}\right)$$

$$\langle \hat{H} \rangle = \langle \hat{T} \rangle + \langle \hat{V} \rangle$$

given qst°

aternatives → = 0 inside area (BC)

→ given in hamitonian

def afte $\hat{T}$ = K.E $\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$

$$\langle \hat{V} \rangle = \langle \psi | \alpha x^4 | \psi \rangle$$

$$= \int A e^{-bx^2} \alpha x^4 A e^{-bx^2} dx$$

$$= \left(\frac{2b}{\pi}\right)^{1/2} \int \alpha x^4 e^{-2bx^2} dx$$

$$= \left(\frac{2b}{\pi}\right)^{1/2} \alpha \int x^4 e^{-2bx^2} dx$$

$$= \left(\frac{2b}{\pi}\right)^{1/2} \alpha \, \frac{3}{4} \left(\frac{\pi}{(2b)^5}\right)^{1/2}$$

$$= \frac{3\alpha}{\ldots}$$

$16b^2$

$$\langle \hat{T} \rangle = \int A e^{-bx^2} \left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) A e^{-bx^2} \, dx$$

$$= h^2 b / 2m$$

$$\therefore \langle \hat{H} \rangle = \frac{\hbar^2 b}{2m} + \frac{3\alpha}{16 b^2}$$

Minimise w.r.t b :

$$\frac{d \langle \hat{H} \rangle}{db} = 0$$

$$= \frac{\hbar^2}{2m} + \frac{3\alpha}{8} \frac{1}{b^3}$$

$$\Rightarrow b = \left(\frac{3\alpha m}{4\hbar^2} \right)^{1/3}$$

↓ input in $\langle \hat{H} \rangle$ for MIN

$$\therefore \langle \hat{H} \rangle_{min} = \frac{\hbar^2}{2m} \left(\frac{3\alpha m}{4\hbar^2} \right)^{1/3} + \frac{3\alpha}{16} \left(\frac{4\hbar^2}{3\alpha m} \right)^{2/3}$$

$$= \frac{3}{4} \left(\frac{3\hbar^4 \alpha}{4m^2} \right)$$

$$E_{ground} \leq \langle \hat{H} \rangle_{min} = \frac{3}{4}\left(\frac{3\hbar^4\alpha}{4m^2}\right)$$