Note:Lattice Bolzmann model for the simulation of the wave equation in curvilinear coordinates

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I. cylinderical ducts

A. metric tensor and Christoffel symbols

For the duct case, cylinderical coordinates are given by the transformation

$$
\begin{aligned}
x &= rcos\theta \\
y &= rsin\theta \\
z &= z
\end{aligned} \tag{1}
$$

The metric tensor are derived:

$$
\begin{aligned}
g_{rr} &= \frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial r}+\frac{\partial y}{\partial r}\cdot\frac{\partial y}{\partial r}+\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial r} \\
&= \frac{\partial rcos\theta}{\partial r}\cdot\frac{\partial rcos\theta}{\partial r}+\frac{\partial rsin\theta}{\partial r}\cdot\frac{\partial rsin\theta}{\partial r}+\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial r} \\
&= cos^2\theta+sin^2\theta=1
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
g_{r\theta} = g_{\theta r} &= \frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial y}{\partial r}\cdot\frac{\partial y}{\partial \theta}+\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial \theta} \\
&= \frac{\partial rcos\theta}{\partial r}\cdot\frac{\partial rcos\theta}{\partial \theta}+\frac{\partial rsin\theta}{\partial r}\cdot\frac{\partial rsin\theta}{\partial \theta}+\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial \theta} \\
&= -cos\theta\cdot rsin\theta+sin\theta\cdot rcos\theta+0=0
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
g_{rz} = g_{zr} &= \frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial z}+\frac{\partial y}{\partial r}\cdot\frac{\partial y}{\partial z}+\frac{\partial z}{\partial r}\cdot\frac{\partial z}{\partial z} \\
&= 0
\end{aligned} \tag{4}
$$

$$g_{\theta\theta} = \frac{\partial x}{\partial \theta} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \cdot \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial z}{\partial \theta}$$
$$= \frac{\partial rcos\theta}{\partial \theta} \cdot \frac{\partial rcos\theta}{\partial \theta} + \frac{\partial rsin\theta}{\partial \theta} \cdot \frac{\partial rsin\theta}{\partial \theta} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial z}{\partial \theta} \tag{5}$$
$$= (-rsin\theta)^2 + (rcos\theta)^2 + 0 = r^2$$

$$g_{\theta z} = g_{z\theta} = \frac{\partial x}{\partial \theta} \cdot \frac{\partial x}{\partial z} + \frac{\partial y}{\partial \theta} \cdot \frac{\partial y}{\partial z} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial z}{\partial z}$$
$$= 0 \tag{6}$$

$$g_{zz} = \frac{\partial x}{\partial z} \cdot \frac{\partial x}{\partial z} + \frac{\partial y}{\partial z} \cdot \frac{\partial y}{\partial z} + \frac{\partial z}{\partial z} \cdot \frac{\partial z}{\partial z}$$
$$= 1 \tag{7}$$

Thus, we have:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}$$

Next, we begin to derive the Christoffel symbols:

$$\Gamma^a_{bc} = 1/2g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}) \tag{10}$$

ref:https://www.youtube.com/watch?v=Axhz7NAk4BM

1. for a=r, b=c=θ, d can be r, θ , z, using Einstein summation convention:

$$\Gamma^r_{\theta\theta} = 1/2g^{rd}(g_{\theta d,\theta} + g_{\theta d,\theta} - g_{\theta\theta,d})$$
$$= 1/2g^{rr}(g_{\theta r,\theta} + g_{\theta r,\theta} - g_{\theta\theta,r}) + 1/2g^{r\theta}(g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}) + 1/2g^{rz}(g_{\theta z,\theta} + g_{\theta z,\theta} - g_{\theta\theta,z})$$
$$= 1/2g^{rr}(\frac{\partial g_{\theta r}}{\partial \theta} + \frac{\partial g_{\theta r}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r}) + 1/2g^{r\theta}(\frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{\partial g_{\theta\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial \theta}) + 1/2g^{rz}(\frac{\partial g_{\theta z}}{\partial \theta} + \frac{\partial g_{\theta z}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial z})$$
$$= 1/2 \cdot (-2r) = -r \tag{11}$$

2. for a=θ, b=r, c=θ, d can be r, θ , z, using Einstein summation convention:

$$\begin{aligned}
\Gamma^{\theta}_{r\theta} &= 1/2g^{\theta d}(g_{rd,\theta} + g_{\theta d,r} - g_{r\theta,d}) \\
&= 1/2g^{\theta r}(g_{rr,\theta} + g_{\theta r,r} - g_{r\theta,r}) + 1/2g^{\theta\theta}(g_{r\theta,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta}) + 1/2g^{\theta z}(g_{rz,\theta} + g_{\theta z,r} - g_{r\theta,z}) \\
&= 1/2g^{\theta r}(\frac{\partial g_{rr}}{\partial\theta} + \frac{\partial g_{\theta r}}{\partial r} - \frac{\partial g_{r\theta}}{\partial r}) + 1/2g^{\theta\theta}(\frac{\partial g_{r\theta}}{\partial\theta} + \frac{\partial g_{\theta\theta}}{\partial r} - \frac{\partial g_{r\theta}}{\partial\theta}) + 1/2g^{\theta z}(\frac{\partial g_{rz}}{\partial\theta} + \frac{\partial g_{\theta z}}{\partial r} - \frac{\partial g_{r\theta}}{\partial z}) \\
&= 0 + 1/2 \cdot (1/r^2) \cdot (2r) = 1/r
\end{aligned} \tag{12}$$

3.symmetry-> for a=θ, b=θ, c=r, d can be r, θ , z, using Einstein summation convention:

$$\begin{aligned}
\Gamma^{\theta}_{\theta r} &= 1/2g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}) = 1/2g^{ad}(g_{cd,b} + g_{bd,c} - g_{cb,d}) \\
&= \Gamma^{\theta}_{r\theta} = 1/r
\end{aligned} \tag{13}$$

4.for a=z,

$$\Gamma^{z}_{bc} = 0 \tag{14}$$

5.for b=z,

$$\Gamma^{a}_{zc} = 1/2g^{ad}(g_{zd,c} + g_{cd,z} - g_{zc,d}) = 0 \tag{15}$$

6. for a=r, b=r,:

$$\Gamma^{r}_{rc} = 1/2g^{rd}(g_{rd,c} + g_{cd,r} - g_{rc,d}) => (d = r) => 1/2g^{rr}(g_{rr,c} + g_{cr,r} - g_{rc,r}) = 0 \tag{16}$$

7. for b=r, c=r,:

$$\Gamma^{a}_{rr} = 1/2g^{ad}(g_{rd,r} + g_{rd,r} - g_{rr,d}) = 0 \tag{17}$$

Thus, we can conclude the 3D Christoffel symbols:

$$\Gamma^{r}_{bc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{18}$$

$$\Gamma^{\theta}_{bc} = \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

$$\Gamma^{z}_{bc} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{20}$$

B. numerical implement by matlab

For the duct case, cylinderical coordinates are given by the transformation