Note:Governing Euqations of General 3D duct flow

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I. Hard-walled cylinderical ducts as basis function

A. Infinite straight duct mode

We began from the Helmholtz equation:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = -\alpha^2\psi \tag{1}$$

Using separation of variables, Circular symmetry: modes have the from : $\psi = F(r)G(\theta)$,

Then we have:

$$\begin{gathered}(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r})G + \frac{F}{r^2}\frac{\partial^2 G}{\partial\theta^2} = -\alpha^2 FG \\ Then, \\ \frac{(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r})}{F} + \frac{1}{r^2}\frac{\frac{\partial^2 G}{\partial\theta^2}}{G} = -\alpha^2\end{gathered} \tag{2}$$

We assume that:

Due to periodicity, we require that Φ satisfy,

$$\frac{d^2G}{d\theta^2} = -m^2G \to \Phi(\theta) = e^{\pm im\theta} \tag{3}$$

Thus, we have

$$F'' + \frac{1}{r}F' + (\alpha^2 - \frac{m^2}{r^2}) = 0 \to F(r) = J_m(\alpha r) \tag{4}$$

Circular symmetry $\psi = F(r)G(\theta)$: modes explicitly given by:

$$\psi = J_m(\alpha_{m\mu}r)e^{\pm im\theta} \tag{5}$$

Hard walls:

$$J_m'(\alpha R) = 0 \to \alpha_{m\mu} = \frac{j'_{m\mu}}{R} \tag{6}$$

Soft walls without flow:

$$Z\alpha_{m\mu}J'_m(\alpha_{m\mu}R) = -iw\rho_0 J_m(\alpha_{m\mu}R) \rightarrow \alpha_{m\mu}(Z) \tag{7}$$

Soft walls with flow:

$$Z\alpha_{m\mu}J'_m(\alpha_{m\mu}R) = (w - U_0\kappa_{m\mu})J_m(\alpha_{m\mu}R) \rightarrow \alpha_{m\mu}(Z) \tag{8}$$

A complete solution may be writtern as:

$$p(x, r, \theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}x} + B_{m\mu}e^{i\kappa_{m\mu}x})U_{m\mu}(r)e^{im\phi} \tag{9}$$

In a hard-walled duct $U_{m\mu}e^{-im\theta}$ are orthogonal. Normalise such that:

$$\int_0^{2\pi}\int_0^R U_{m\mu}(r)e^{-im\theta}U_{nv}(r)e^{-in\theta}rdr = 2\pi\delta_{\mu v}\delta_{mn} \tag{10}$$

Source expansion If $p(0, t, \theta) = p_0(r, \theta)$ is source in hard-walled duct, then for x>0

$$\begin{aligned}
p_0(r,\theta) &= \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta} \\
p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}r} &= \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}r} \\
\underline{\int_0^{2\pi}\int_0^R} p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}r}\underline{drd\theta} &= \underline{\int_0^{2\pi}\int_0^R}\sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}r}\underline{drd\theta} \\
\underline{\int_0^{2\pi}\int_0^R} p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}r}\underline{drd\theta} &= \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}A_{m\mu}\underline{\int_0^{2\pi}\int_0^R}U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}r}\underline{drd\theta} \\
A_{nv} &= \frac{1}{2\pi}\underline{\int_0^{2\pi}\int_0^R} p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}r}\underline{drd\theta}
\end{aligned} \tag{11}$$

and $B_{nv} = 0$. The same for $x < 0$ with A_{nv} and B_{nv} interchanged.

A finite number of modes (cut-on modes) survive at large distrances. Just 1 mode if kR«1: only A_{01} important.

B. General duct mode

The pressure and velocity can now be expressed as Fourier series. Upper indices shall be used to denote temporal decompositions:

$$\begin{aligned}
\widehat{p} &= \sum_{a=-\infty}^{\infty} P^a(\mathbf{x})e^{-iawt} \\
\widehat{u} &= \sum_{a=-\infty}^{\infty} U^a(\mathbf{x})e^{-iawt} \\
\widehat{v} &= \sum_{a=-\infty}^{\infty} V^a(\mathbf{x})e^{-iawt} \\
\widehat{w} &= \sum_{a=-\infty}^{\infty} W^a(\mathbf{x})e^{-iawt}
\end{aligned} \tag{12}$$

$$\begin{aligned}
P^a &= \sum_{\alpha=0}^{\infty} P^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} P^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) \\
U^a &= \sum_{\alpha=0}^{\infty} U^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} U^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) \\
V^a &= \sum_{\alpha=0}^{\infty} V^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} V^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) \\
W^a &= \sum_{\alpha=0}^{\infty} W^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} W^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta)
\end{aligned} \tag{13}$$

1. Discussion of property of ψ

In Jams's thesis, $\psi_\alpha = C_{\alpha_{m\mu}} J_m(\frac{j'_{m\mu}r}{h}) cos(m\phi - \xi\pi/2)$.As he mentioned, it was merely a matter of preference - I find it easier to visualise the modes as being "symmetric" and "anti-symmetric" along the plane of torsion free ducts, but the other method is equally as valid.

However, there still a problem, which may cause error, but is not mentioned in the thesis:

$$\int_0^{2\pi} cos(m\phi - \xi\pi/2)^2 d\theta = \frac{1 + cos(2m\phi - \xi\pi)}{2}|_0^{2\pi} = \begin{cases} \pi, m \neq 0 \\ 2\pi, m = 0, \xi = 0 \\ 0, m = 0, \xi = 1 \end{cases} \tag{14}$$

In the note, we try to introduce the common solution of ψ may have the form the same as the hard walls modes:

$$\psi_{m\mu}(r) = C_{\alpha_{m\mu}} J_m(\frac{j'_{m\mu}r}{h})e^{im\phi} \tag{15}$$

where may be normalized according to:

$$\int_0^{2\pi}\int_0^h \psi_{\alpha_{m\mu}}\psi_{\beta_{nv}} r dr d\theta = \delta_{\mu v}\delta_{mn} \tag{16}$$

In fact, we know that:

$$\int_0^{2\pi} e^{im\phi}e^{im\phi}d\theta = 0$$
$$\int_0^{2\pi} e^{im\phi}e^{-im\theta}d\theta = 2\pi \tag{17}$$

The orthogonality relation of Bessel function, with $J_{-m}(z) = (-1)^m J_m(z)$

$$\int_0^h rJ_m(\frac{j'_{m\mu}r}{h})J_m(\frac{j'_{mv}r}{h})dr = 0, \mu \neq v$$
$$\int_0^h rJ_m(\frac{j'_{m\mu}r}{h})J_{-m}(\frac{j'_{-mv}r}{h})dr \tag{18}$$
$$= (-1)^m \int_0^h rJ_m(\frac{j'_{m\mu}r}{h})J_m(\frac{j'_{mv}r}{h})dr = 0, \mu \neq v$$

That changes our idea of normalization to:

$$\underline{\underline{\int_0^{2\pi}\int_0^h \psi_{\alpha_{m\mu}}\psi_{\beta_{nv}}rdrd\theta = (-1)^m\delta_{\mu v}\delta_{m,-n}}} \tag{19}$$

C. Normalised Modes$\rightarrow C_{\alpha_{m\mu}}$

Relation involving intergrals:

$$\underline{2\int \alpha^2 xJ_m(\alpha x)^2dx = (\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2}$$
$$\rightarrow 2\int_0^h \alpha^2 xJ_m(\alpha x)^2dx = [(\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2]|_0^h$$
$$= [(\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2]|_h - [(\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2]|_0 \tag{20}$$
$$= [(\alpha^2h^2 - m^2)J_m(\alpha h)^2 + \alpha^2h^2J'_m(\alpha h)^2] - [(\alpha^2 0^2 - m^2)J_m(\alpha 0)^2 + \alpha^2 0^2J'_m(\alpha 0)^2]$$
$$= (\alpha^2h^2 - m^2)J_m(\alpha h)^2 + \alpha^2h^2J'_m(\alpha h)^2$$

With hard-walled boudary condition:

$$J'_m(\alpha h) = 0 \rightarrow \alpha_{m\mu} = \frac{j'_{m\mu}}{h}(eigenvalues) \tag{21}$$

Then, we have(ref: Rienstra-Fundamentals of Duct Acoustics-(55)):

$$\int_0^h rJ_m(\alpha r)J_{-m}(\alpha r)dr = (-1)^m\int_0^h rJ_m(\alpha r)^2dr$$
$$= (-1)^m\frac{1}{2\alpha_{m\mu}^2}(\alpha_{m\mu}^2h^2 - m^2)J_m(\alpha_{m\mu}h)^2 \tag{22}$$
$$= (-1)^m(\frac{J_m(\alpha_{m\mu}h)\sqrt{(h^2 - \frac{m^2}{\alpha_{m\mu}^2})}}{\sqrt{2}})^2$$
$$= (-1)^m(\frac{h^2}{2}(1 - \frac{m^2}{j'^2_{m\mu}})J_m^2(j'_{m\mu}))$$

Thus, with $\int_0^{2\pi} e^{im\phi}e^{-im\theta}d\theta = 2\pi$, $\psi_{\alpha_{m\mu}}(r) = C_{\alpha_{m\mu}}J_m(\frac{j'_{m\mu}r}{h})e^{im\phi}$,we have:

$$\begin{aligned} C_{\alpha_{m\mu}} &= \frac{i^m}{\sqrt{(\pi h^2(1-\frac{m^2}{j'^2_{m\mu}})J_m^2(j'_{m\mu}))}} \\ except for: C_{\alpha_{01}} &= \frac{1}{\sqrt{\pi}h} \end{aligned} \tag{23}$$

for $\int_0^{2\pi}\int_0^h \psi_{\alpha_{m\mu}}\psi_{\beta_{nv}}rdrd\theta = \delta_{\mu v}\delta_{m,-n} = \widehat{\delta}_{\alpha\beta} = I.$

D. Slowly varying ducts

waiting for updating......

E. Orthogonal-eigenvector

ref:https:

www.mathworks.com/help/matlab/ref/eigs.html

Eigenvectors, returned as a matrix. The columns in V correspond to the eigenvalues along the diagonal of D. The form and normalization of V depends on the combination of input arguments:

[V,D] = eigs(A) returns matrix V, whose columns are the eigenvectors of A such that A*V = V*D. The eigenvectors in V are normalized so that the 2-norm of each is 1.

If A is symmetric, then the eigenvectors, V, are orthonormal.

[V,D] = eigs(A,B) returns V as a matrix whose columns are the generalized eigenvectors that satisfy A*V = B*V*D. The 2-norm of each eigenvector is not necessarily 1.

If B is symmetric positive definite, then the eigenvectors in V are normalized so that the B-norm of each is 1. If A is also symmetric, then the eigenvectors are B-orthonormal.

We could further study this question!!

if we can use the GramSchmidt mode as basis??

II. Mass equation

Mass consevation:

$$-ia\kappa P^a + \nabla\cdot\mathbf{U}^a = \sum_{b=-\infty}^{+\infty}(-P^{a-b}\nabla\cdot\mathbf{U}^b - \mathbf{U}^{a-b}\cdot\nabla P^b - \frac{B}{2A}iakP^bP^{a-b}) \tag{24}$$

First, derivation of eq1:

We know that:

$$h_s = 1 - \kappa r cos(\phi), h_r = 1, h_\theta = r \tag{25}$$

Then,

$$\begin{aligned} \nabla \cdot \mathbf{U}^a = \frac{1}{h_1 h_2 h_3}[\frac{\partial(v_1 h_2 h_3)}{\partial w_1} + \frac{\partial(v_2 h_3 h_1)}{\partial w_2} + \frac{\partial(v_3 h_1 h_2)}{\partial w_3}] \\ = \frac{1}{r(1-\kappa r cos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}] \end{aligned} \tag{26}$$

Thus, we have the mass equation, approximate RHS by:

$$\begin{aligned} \nabla \cdot \mathbf{U}^b = ib\kappa P^b + o(M^2) \\ \nabla P^b = ib\kappa \mathbf{U}^b + o(M^2) \end{aligned} \tag{27}$$

Then we have

$$\begin{aligned} -ia\kappa P^a + \frac{1}{r(1-\kappa r cos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}] \\ = \sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A} iakP^b P^{a-b}) \end{aligned} \tag{28}$$

The fourier harmonics are expanded as follows:

$$\begin{aligned} P^a = \sum_{\beta=0}^{\infty} P^a_\beta(s)\psi_\beta(s,r,\theta) \\ U^a = \sum_{\beta=0}^{\infty} U^a_\beta(s)\psi_\beta(s,r,\theta) \\ V^a = \sum_{\beta=0}^{\infty} V^a_\beta(s)\psi_\beta(s,r,\theta) \\ W^a = \sum_{\beta=0}^{\infty} W^a_\beta(s)\psi_\beta(s,r,\theta) \end{aligned} \tag{29}$$

with normalized relation:

$$\int_0^{2\pi}\int_0^h \psi_\alpha \psi_\beta r dr d\theta = \widehat{\delta}_{\alpha\beta} \tag{30}$$

Reorganize the eq5:

$$\begin{aligned} -ia\kappa P^a(1-\kappa r cos(\phi)) + \frac{1}{r}\frac{\partial(U^a r)}{\partial s} + \frac{1}{r}\frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{1}{r}\frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta} \\ = (1-\kappa r cos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A} iakP^b P^{a-b}) \end{aligned} \tag{31}$$

Intergal and insert eq 6, 7 into eq 8:

1. the first term:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[-ia\kappa(1-\kappa rcos(\phi))P^a]drd\theta \\
&= -ia\kappa\sum_{\beta=0}^{\infty}\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r[(1-\kappa rcos(\phi))]drd\theta P^a_\beta \\
&\Rightarrow -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta \\
&(summationconvention)
\end{aligned} \tag{32}$$

2. the second term:

From 1.1 as example, we know that

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\
&= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta
\end{aligned} \tag{33}$$

$$\frac{d}{d\alpha}(\int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx) - 0 + \frac{da(\alpha)}{d\alpha}f(a(\alpha),\alpha) = \int_{a(\alpha)}^{b(\beta)}\frac{\partial}{\partial\alpha}f(x,\alpha)dx \tag{34}$$

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(U^a r)}{\partial s}]drd\theta \\
&= \sum_{\beta=0}^{\infty}\int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta U^a_\beta)}{\partial s}drd\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r\frac{\partial(U^a_\beta)}{\partial s}drd\theta + \int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta)}{\partial s}drd\theta U^a_\beta \\
&= \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha\psi_\beta rU^a_\beta}{\partial s}drd\theta - \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha\psi_\beta r}{\partial s}U^a_\beta drd\theta \\
&+ \int_0^{2\pi}\int_0^h r\frac{\partial(\psi_\alpha\psi_\beta)}{\partial s}drd\theta U^a_\beta - \int_0^{2\pi}\int_0^h \psi_\beta r\frac{\partial(\psi_\alpha)}{\partial s}drd\theta U^a_\beta \\
&= \sum_{\beta=0}^{\infty}\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [\psi_\alpha\psi_\beta rU^a_\beta]drd\theta - \sum_{\beta=0}^{\infty}\int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\alpha\psi_\beta rU^a_\beta]_{r=h}d\theta \\
&- \sum_{\beta=0}^{\infty}\int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial s}\psi_\beta rdrd\theta U^a_\beta \\
&\Rightarrow \frac{d}{ds}(U^a_\beta\widehat{\delta}_{\alpha\beta}) + 0(periodic) - \Psi_{\{\alpha\}\beta}[r]U^a_\beta
\end{aligned} \tag{35}$$

mark:0(periodic),which is similar in the follow momentum equation, but not eliminate in time.

3. the third term:

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r}]drd\theta \\
&= \sum_{\beta=0}^{\infty}\int_0^{2\pi}\int_0^h \psi_\alpha \frac{\partial(\psi_\beta r(1-\kappa r cos(\phi)))}{\partial r} drd\theta V_\beta^a \\
&= \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha\psi_\beta r(1-\kappa r cos(\phi)))}{\partial r} drd\theta V_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta r(1-\kappa r cos(\phi))\frac{\partial(\psi_\alpha)}{\partial r} drd\theta V_\beta^a \\
&= 0(periodic) - \Psi_{[\alpha]\beta}[r(1-\kappa r cos(\phi))]V_\beta^a
\end{aligned}
\tag{36}
$$

4. the fourth term:

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}]drd\theta \\
&= \sum_{\beta=0}^{\infty}\int_0^{2\pi}\int_0^h \psi_\alpha [\frac{\partial(\psi_\beta (1-\kappa r cos(\phi)))}{\partial \theta}] drd\theta W_\beta^a \\
&= \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha\psi_\beta (1-\kappa r cos(\phi)))}{\partial \theta}] drd\theta W_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta (1-\kappa r cos(\phi))[\frac{\partial(\psi_\alpha)}{\partial \theta}] drd\theta W_\beta^a \\
&= \int_0^h [\psi_\alpha\psi_\beta(1-\kappa r cos(\phi))]_0^{2\pi} dr W_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta (1-\kappa r cos(\phi))[\frac{\partial(\psi_\alpha)}{\partial \theta}] drd\theta W_\beta^a \\
&= 0 - \Psi_{(\alpha)\beta}[(1-\kappa r cos(\phi))]W_\beta^a
\end{aligned}
\tag{37}
$$

5. the RHS term:

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[(1-\kappa r cos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A} iak P^b P^{a-b})]drd\theta \\
&= \sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty}\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta\psi_\gamma r(1-\kappa r cos(\phi))drd\theta \\
&\sum_{b=-\infty}^{+\infty}(-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa V_\beta^{a-b}V_\gamma^b - ib\kappa W_\beta^{a-b}W_\gamma^b - ia\kappa\frac{B}{2A}P_\beta^{a-b}P_\gamma^b) \\
&= \sum_{\beta=0}^{\infty}\sum_{\gamma=0}^{\infty}\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa V_\beta^{a-b}V_\gamma^b - ib\kappa W_\beta^{a-b}W_\gamma^b - ia\kappa\frac{B}{2A}P_\beta^{a-b}P_\gamma^b)
\end{aligned}
\tag{38}
$$

Finally, we obtain the mass equation in the form of eigenfunction, the idea is same as Galerkin method:

$$
\begin{aligned}
&\frac{dU_\beta^a}{ds}\widehat{\delta}_{\alpha\beta} - \Psi_{\{\alpha\}\beta}[r]U_\beta^a - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa r cos(\phi))]V_\beta^a - \Psi_{(\alpha)\beta}[(1-\kappa r cos(\phi))]W_\beta^a \\
&= \Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa V_\beta^{a-b}V_\gamma^b - ib\kappa W_\beta^{a-b}W_\gamma^b - ia\kappa\frac{B}{2A}P_\beta^{a-b}P_\gamma^b)
\end{aligned}
\tag{39}
$$

III. Momentum equation

Momentum consevation:

$$-ia\kappa\mathbf{U}^a + \nabla P^a = \sum_{b=-\infty}^{\infty}(-\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^b + P^{a-b}\nabla P^b) \tag{40}$$

First, we know that

$$\nabla P^a = \sum_i \frac{1}{h_i}\frac{\partial f}{\partial w_i}\hat{h}_i = \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s}\hat{e}_s + \frac{\partial P^a}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial P^a}{\partial\theta}\hat{e}_\theta \tag{41}$$

The RHS term is a bit complex, with the divergence of a vector U with its gradient, with

First, we know that

$$(\mathbf{v}\cdot\nabla)\mathbf{v}^b = \begin{cases} term1 : \mathcal{D}v_1^b + \frac{v_2^b}{h_2h_1}(v_1\frac{\partial h_1}{\partial\xi_2} - v_2\frac{\partial h_2}{\partial\xi_1}) + \frac{v_3^b}{h_3h_1}(v_1\frac{\partial h_1}{\partial\xi_3} - v_2\frac{\partial h_3}{\partial\xi_1}) \\ term2 : \mathcal{D}v_2^b + \frac{v_3^b}{h_3h_2}(v_2\frac{\partial h_2}{\partial\xi_3} - v_3\frac{\partial h_3}{\partial\xi_2}) + \frac{v_1^b}{h_1h_2}(v_2\frac{\partial h_2}{\partial\xi_1} - v_1\frac{\partial h_1}{\partial\xi_2}) \\ term3 : \mathcal{D}v_3^b + \frac{v_1^b}{h_1h_3}(v_3\frac{\partial h_3}{\partial\xi_1} - v_1\frac{\partial h_1}{\partial\xi_3}) + \frac{v_2^b}{h_2h_3}(v_3\frac{\partial h_3}{\partial\xi_2} - v_3\frac{\partial h_2}{\partial\xi_3}) \end{cases} \tag{42}$$

Besides,

$$\mathcal{D} = \frac{v_1}{h_1}\frac{\partial}{\partial\xi_1} + \frac{v_2}{h_2}\frac{\partial}{\partial\xi_2} + \frac{v_3}{h_3}\frac{\partial}{\partial\xi_3} \tag{43}$$

Thus, we have:

$$-\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^{b} =$$

$$-\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{D}1: \frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s}+V^{a-b}\frac{\partial U^b}{\partial r}+\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta}\\ term\mathcal{D}2: \frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s}+V^{a-b}\frac{\partial V^b}{\partial r}+\frac{W^{a-b}}{r}\frac{\partial V^b}{\partial\theta}\\ term\mathcal{D}3: \frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s}+V^{a-b}\frac{\partial W^b}{\partial r}+\frac{W^{a-b}}{r}\frac{\partial W^b}{\partial\theta}\end{cases}$$

$$-\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{X}1: \frac{V^b}{1-\kappa rcos\phi}(U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial r}-V^{a-b}\frac{\partial 1}{\partial s})+\frac{W^b}{r(1-\kappa rcos\phi)}(U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial\theta}-V^{a-b}\frac{\partial r}{\partial s})\\ term\mathcal{X}2: \frac{W^b}{r}(V^{a-b}\frac{\partial 1}{\partial\theta}-W^{a-b}\frac{\partial r}{\partial r})+\frac{U^b}{(1-\kappa rcos\phi)1}(V^{a-b}\frac{\partial 1}{\partial s}-U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial r})\\ term\mathcal{X}3: \frac{U^b}{(1-\kappa rcos\phi)r}(W^{a-b}\frac{\partial h_3}{\partial s}-U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial\theta})+\frac{V^b}{1h_3}(W^{a-b}\frac{\partial h_3}{\partial r}-V^{a-b}\frac{\partial 1}{\partial\theta})\end{cases} =$$

$$\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{D}1: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s}-V^{a-b}\frac{\partial U^b}{\partial r}-\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta}\\ term\mathcal{D}2: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s}-V^{a-b}\frac{\partial V^b}{\partial r}-\frac{W^{a-b}}{r}\frac{\partial V^b}{\partial\theta}\\ term\mathcal{D}3: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s}-V^{a-b}\frac{\partial W^b}{\partial r}-\frac{W^{a-b}}{r}\frac{\partial W^b}{\partial\theta}\end{cases}$$

$$+\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{X}1: \frac{\kappa cos\phi}{1-\kappa rcos\phi}U^{a-b}V^b-\frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}W^b\\ term\mathcal{X}2: \frac{W^{a-b}W^b}{r}-\frac{\kappa cos\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b\\ term\mathcal{X}3: \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b-\frac{W^{a-b}V^b}{r}\end{cases} \tag{44}$$

Finally, we could derive the momentum conservation equation, with final term approximate by eq 4:

$$\begin{cases} -ia\kappa U^a+\frac{1}{1-\kappa rcos\phi}\frac{\partial P^a}{\partial s}\\ -ia\kappa V^a+\frac{\partial P^a}{\partial r}\\ -ia\kappa W^a+\frac{1}{r}\frac{\partial P^a}{\partial\theta}\end{cases}$$

$$=\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{D}1: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s}-V^{a-b}\frac{\partial U^b}{\partial r}-\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta}\\ term\mathcal{D}2: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s}-V^{a-b}\frac{\partial V^b}{\partial r}-\frac{W^{a-b}}{r}\frac{\partial V^b}{\partial\theta}\\ term\mathcal{D}3: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s}-V^{a-b}\frac{\partial W^b}{\partial r}-\frac{W^{a-b}}{r}\frac{\partial W^b}{\partial\theta}\end{cases} \tag{45}$$

$$+\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{X}1: \frac{\kappa cos\phi}{1-\kappa rcos\phi}U^{a-b}V^b-\frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}W^b\\ term\mathcal{X}2: \frac{W^{a-b}W^b}{r}-\frac{\kappa cos\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b\\ term\mathcal{X}3: \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b-\frac{W^{a-b}V^b}{r}\end{cases} +\sum_{b=-\infty}^{\infty}\begin{cases} ib\kappa P^{a-b}U^b\\ ib\kappa P^{a-b}V^b\\ ib\kappa P^{a-b}W^b\end{cases}$$

Now, we are going to project on ψ, it may be a little complex, we will doing step by step.

A. Momentum e^s term

First, deal with the e^s term:

$$\begin{aligned}
&-ia\kappa U^a + \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s} \\
=& \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b \\
&+ib\kappa P^{a-b}U^b
\end{aligned} \tag{46}$$

Multiply $(1-\kappa r cos\phi)$, we have:

$$\begin{aligned}
&-ia\kappa(1-\kappa r cos\phi)U^a + \frac{\partial P^a}{\partial s} \\
= term\mathcal{D}1 : &\sum_{b=-\infty}^{\infty} -U^{a-b}\frac{\partial U^b}{\partial s} - (1-\kappa r cos\phi)V^{a-b}\frac{\partial U^b}{\partial r} - (1-\kappa r cos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\
&+term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b + \\
&term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa r cos\phi)P^{a-b}U^b
\end{aligned} \tag{47}$$

$\int\int XXr\psi_\alpha drd\theta$, we have:

$$\underline{RHS = \int_0^{2\pi}\int_0^h [term\mathcal{D}1 + term\mathcal{X}1 + term\mathcal{P}1]r\psi_\alpha drd\theta} \tag{48}$$

1. the first $\mathcal{D}1$ tems:

We ref the wiki $https:$

$en.wikipedia.org/wiki/Leibniz\_integral\_rule$

General form: Differentiation under the integral sign:

$$\frac{d}{dx}(\int_{a(x)}^{b(x)} f(x,t)dt) - f(x,b(x))\cdot\frac{d}{dx}b(x) + f(x,a(x))\cdot\frac{d}{dx}a(x) = \int_{a(x)}^{b(x)}\frac{d}{dx}f(x,t)dt \tag{49}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{50}$$

For partial difference, for a given β, the derivation of the fucntion $g(\alpha) = \int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx$ is

$$\frac{d}{d\alpha}(\int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx) - 0 + \frac{da(\alpha)}{d\alpha}f(a(\alpha),\alpha) = \int_{a(\alpha)}^{b(\beta)}\frac{\partial}{\partial\alpha}f(x,\alpha)dx \tag{51}$$

1.1

$$
\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h[-rU^{a-b}\frac{\partial U^b}{\partial s}]\psi_\alpha drd\theta \\
&=\sum_{b=-\infty}^{\infty}-\int_0^{2\pi}\int_0^h\frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial}{\partial s}[rU^{a-b}\psi_\alpha]drd\theta \\
&=\sum_{b=-\infty}^{\infty}-\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h[rU^{a-b}U^b\psi_\alpha]drd\theta+\int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta \\
&+\int_0^{2\pi}\int_0^h\frac{r\partial U^{a-b}}{\partial s}U^b\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h\frac{\partial\psi_\alpha}{\partial s}rU^{a-b}U^b drd\theta
\end{aligned}
\tag{52}
$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$
h'U^{a-b}=(1-\kappa hcos\phi)V^{a-b} \tag{53}
$$

1.2

$$
\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h[-(1-\kappa rcos\phi)V^{a-b}\frac{\partial U^b}{\partial r}]r\psi_\alpha drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h\frac{\partial(r(1-\kappa rcos\phi)V^{a-b}U^b\psi_\alpha)}{\partial r}drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b}\psi_\alpha)}{\partial r}drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}[(r(1-\kappa rcos\phi)V^{a-b}U^b)\psi_\alpha]_0^h d\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa rcos\phi)V^{a-b}drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}[(hh'U^{a-b}U^b)\psi_\alpha]_0^h d\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa rcos\phi)V^{a-b}drd\theta
\end{aligned}
\tag{54}
$$

1.3

$$
\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h[-(1-\kappa rcos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta}]r\psi_\alpha drd\theta\\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h\frac{\partial((1-\kappa rcos\phi)W^{a-b}U^b\psi_\alpha)}{\partial\theta}drd\theta\\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta\\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta\\
&=-\sum_{b=-\infty}^{\infty}\int_0^h[(1-\kappa rcos\phi)W^{a-b}U^b\psi_\alpha]_0^{2\pi}dr\\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta\\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta\\
&=0(periodic)+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta\\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta
\end{aligned}
\tag{55}
$$

Combine together:

$$
\begin{aligned}
&\underline{\int_0^{2\pi}\int_0^h[term\mathcal{D}1]r\psi_\alpha drd\theta}=\sum_{b=-\infty}^{\infty}\\
&\{(\int_0^{2\pi}\int_0^h\frac{\partial\psi_\alpha}{\partial s}rU^{a-b}U^b drd\theta\\
&+\int_0^{2\pi}\int_0^h\frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa rcos\phi)V^{a-b}U^b drd\theta\\
&+\int_0^{2\pi}\int_0^h\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b}U^b)drd\theta)\\
&+(\int_0^{2\pi}\int_0^h U^b\frac{\partial U^{a-b}}{\partial s}r\psi_\alpha drd\theta\\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta\\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta)\\
&-\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h[rU^{a-b}U^b\psi_\alpha]drd\theta\}
\end{aligned}
\tag{56}
$$

We apply eq 5, find that:

$$
\begin{aligned}
-i(a-b)\kappa r(1-\kappa rcos(\phi))P^{a-b}+[\frac{\partial(U^{a-b}r)}{\partial s}+\frac{\partial(V^{a-b}r(1-\kappa rcos(\phi)))}{\partial r}+\frac{\partial(W^{a-b}(1-\kappa rcos(\phi)))}{\partial\theta}]\\
=o(M^2)
\end{aligned}
\tag{57}
$$

We have the second terms in eq(31) are equal to:

$$\begin{aligned}&\int_0^{2\pi}\int_0^h U^b[-i(a-b)\kappa r(1-\kappa rcos(\phi))P^{a-b}]\psi_\alpha drd\theta\\ &= i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P_\beta^{a-b}U_\gamma^b\end{aligned} \tag{58}$$

And, the longitudinal derivation s can also be expand about the duct modes, with note $[r], (\theta), \{s\}$:

$$\begin{aligned}&\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta\\ &= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]U_\beta^{a-b}U_\gamma^b drd\theta\\ &= \frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta)U_\beta^{a-b}U_\gamma^b\\ &+\frac{\partial U_\beta^{a-b}U_\gamma^b}{\partial s}\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta\\ &= \frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta)U_\beta^{a-b}U_\gamma^b\\ &+(\frac{dU_\beta^{a-b}}{ds}U_\gamma^b+\frac{dU_\gamma^b}{ds}U_\beta^{a-b})\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta\end{aligned} \tag{59}$$

2. the second $\mathcal{X}1$ tems:

$$term\mathcal{X}1: \sum_{b=-\infty}^{\infty}\kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b$$

$$\begin{aligned}&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [\kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b]r\psi_\alpha drd\theta\\ &= \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U_\beta^{a-b}V_\gamma^b - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U_\beta^{a-b}W_\gamma^b\end{aligned} \tag{60}$$

3. the second $\mathcal{P}1$ tems:

$$term\mathcal{P}1: \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b$$

$$\begin{aligned}&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b]r\psi_\alpha drd\theta\\ &= ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P_\beta^{a-b}U_\gamma^b\end{aligned} \tag{61}$$

4. The LHS terms:

$$\frac{\partial P^a}{\partial s} - ia\kappa(1-\kappa rcos\phi)U^a$$

From 1.1 as example, we know that

$$\begin{aligned}\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\ = \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta\end{aligned} \tag{62}$$

4.1

$$\begin{aligned}\int_0^{2\pi}\int_0^h [\frac{\partial P^a}{\partial s}]r\psi_\alpha drd\theta \\ = \int_0^{2\pi}\int_0^h [\frac{\partial(P^a_\beta\psi_\beta)}{\partial s}]r\psi_\alpha drd\theta \\ = \int_0^{2\pi}\int_0^h [\frac{\partial(P^a_\beta\psi_\beta)}{\partial s}r\psi_\alpha]drd\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}]r\psi_\beta drd\theta P^a_\beta \\ = \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h (P^a_\beta\psi_\beta)r\psi_\alpha drd\theta - \frac{dh(s)}{ds}[P^a_\beta\psi_\beta r\psi_\alpha]_{r=h} - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}]r\psi_\beta drd\theta P^a_\beta \\ = \frac{d}{ds}(P^a_\beta\widehat{\delta}_{\alpha\beta}) - \int_0^{2\pi}\frac{dh(s)}{ds}[P^a_\beta\psi_\beta r\psi_\alpha]_{r=h}d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}]r\psi_\beta drd\theta P^a_\beta\end{aligned} \tag{63}$$

4.2

$$\begin{aligned}\int_0^{2\pi}\int_0^h [-ia\kappa(1-\kappa rcos\phi)U^a]r\psi_\alpha drd\theta \\ = -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta\end{aligned} \tag{64}$$

Finally, putting all together becomes:

$$
\begin{aligned}
&\frac{d}{ds}(P^a_\beta)\widehat{\delta}_{\alpha\beta} - \int_0^{2\pi} \frac{dh(s)}{ds}[P^a_\beta \psi_\beta r \psi_\alpha]_{r=h} d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta \\
&= \frac{d}{ds}(P^a_\beta)\widehat{\delta}_{\alpha\beta} - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta - \int_0^{2\pi} hh'[P^a_\beta\psi_\beta\psi_\alpha]_{r=h} d\theta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\
&= \sum_{b=-\infty}^{\infty} \\
&(eq31) : \{(\int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s} r U^{a-b}U^b dr d\theta \\
&+ \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r(1-\kappa r cos\phi)V^{a-b}U^b dr d\theta \\
&+(eq33) : i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\
&eq(34) : -\frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta)U^{a-b}_\beta U^b_\gamma \\
&-(\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta)\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta \\
&+eq(35) : \kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma \\
&+eq(36) : ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\
&= (abbreviation) : \\
&\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta U^a_\gamma \\
&+i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\
&-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma \\
&+ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma
\end{aligned}
\tag{65}
$$

with the e^s term:

$$
\begin{aligned}
&-ia\kappa U^a + \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{P}1 : ib\kappa P^{a-b}U^b
\end{aligned}
\tag{66}
$$

We have:

$$-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta+\frac{d}{ds}(P^a_\beta)\widehat{\delta}_{\alpha\beta}-\int_0^{2\pi}hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta-\Psi_{\{\alpha\}\beta}[r]P^a_\beta$$

$$=\underline{term\mathcal{D}1:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma+\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta U^a_\gamma+\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta U^a_\gamma$$
$$+\underline{term(D1+\mathcal{P}1):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta U^b_\gamma$$
$$-\underline{term\mathcal{D}1:}\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma-\Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma+\frac{dU^b_\gamma}{ds}U^{a-b}_\beta)$$
$$+\underline{term\mathcal{X}1:}\kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta V^b_\gamma-\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta W^b_\gamma \tag{67}$$

B. Momentum e^r term

Second, deal with the e^r term:

$$\begin{aligned}&-ia\kappa V^a+\frac{\partial P^a}{\partial r}\\ =&\sum_{b=-\infty}^{\infty}term\mathcal{D}2:-\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s}-V^{a-b}\frac{\partial V^b}{\partial r}-\frac{W^{a-b}}{r}\frac{\partial V^b}{\partial\theta}\\ &+\sum_{b=-\infty}^{\infty}term\mathcal{X}2:\frac{W^{a-b}W^b}{r}-\frac{\kappa cos\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b\\ &+term\mathcal{P}2:\sum_{b=-\infty}^{\infty}ib\kappa P^{a-b}V^b\end{aligned} \tag{68}$$

LHS-2: $\frac{\partial P^a}{\partial r}(1-\kappa rcos\phi)$

$$\begin{aligned}&\int_0^{2\pi}\int_0^h[\frac{\partial P^a}{\partial r}(1-\kappa rcos\phi)]r\psi_\alpha drd\theta\\ =&\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\beta(1-\kappa rcos\phi)r\psi_\alpha}{\partial r}]drd\theta P^a_\beta-\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\alpha(1-\kappa rcos\phi)r}{\partial r}]\psi_\beta drd\theta P^a_\beta\\ =&\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\beta(1-\kappa rcos\phi)r\psi_\alpha}{\partial r}]drd\theta P^a_\beta\\ &-\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\alpha}{\partial r}](1-\kappa rcos\phi)r\psi_\beta drd\theta P^a_\beta-\int_0^{2\pi}\int_0^h[\frac{\partial(1-\kappa rcos\phi)r}{\partial r}]\psi_\alpha\psi_\beta drd\theta P^a_\beta\\ =&\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta\\ &-\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\alpha}{\partial r}](1-\kappa rcos\phi)r\psi_\beta drd\theta P^a_\beta-\int_0^{2\pi}\int_0^h[1-2\kappa rcos\phi]\psi_\alpha\psi_\beta drd\theta P^a_\beta\\ =&\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta-\Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P^a_\beta-\Psi_{\alpha\beta}[1-2\kappa rcos\phi]P^a_\beta\end{aligned} \tag{69}$$

The derivation of $term\mathcal{D}2$ is identical to A, we are not prove it again. $Term\mathcal{P}2$ also could be combine with the part separated term of $term\mathcal{D}2$ with V^b. $Term\mathcal{X}2$ is also easy to derive.

Thus, we have the final equation:

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_\beta \\
&\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P^a_\beta - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]P^a_\beta \\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta V^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta V^a_\gamma \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta V^b_\gamma \\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}V^b_\gamma + \frac{dV^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}2:}\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta W^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta U^b_\gamma
\end{aligned}
\tag{70}
$$

C. Momentum e^θ term

Third, deal with the e^θ term:

$$
\begin{aligned}
&-ia\kappa W^a + \frac{1}{r}\frac{\partial P^a}{\partial\theta} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}3: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial\theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}3: \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \\
&+term\mathcal{P}3: \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b}W^b
\end{aligned}
\tag{71}
$$

LHS-2: $\frac{\partial P^a}{\partial\theta}\frac{(1-\kappa rcos\phi)}{r}$

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h [\frac{\partial P^a}{\partial\theta}\frac{(1-\kappa rcos\phi)}{r}]r\psi_\alpha drd\theta \\
&= \int_0^{2\pi}\int_0^h [\frac{\partial\psi_\beta(1-\kappa rcos\phi)\psi_\alpha}{\partial\theta}]drd\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial\psi_\alpha(1-\kappa rcos\phi)}{\partial\theta}]\psi_\beta drd\theta P^a_\beta \\
&= 0 - \int_0^{2\pi}\int_0^h [\frac{\partial\psi_\alpha}{\partial\theta}](1-\kappa rcos\phi)\psi_\beta drd\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial(1-\kappa rcos\phi)}{\partial\theta}]\psi_\alpha\psi_\beta drd\theta P^a_\beta \\
&= -\int_0^{2\pi}\int_0^h [\frac{\partial\psi_\alpha}{\partial\theta}](1-\kappa rcos\phi)\psi_\beta drd\theta P^a_\beta + \kappa\int_0^{2\pi}\int_0^h [rsin\phi]\psi_\alpha\psi_\beta drd\theta P^a_\beta \\
&= -\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta
\end{aligned}
\tag{72}
$$

The derivation of $term\mathcal{D}3$ is identical to A, we are not prove it again. $Term\mathcal{P}3$ also could be combine with the part separated term of $term\mathcal{D}3$ with W^b. $Term\mathcal{X}3$ is also easy to derive.

Thus, we have the final equation:

$$-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]W^a_\beta$$

$$-\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta$$

$$= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta W^a_\gamma$$

$$+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta W^b_\gamma$$

$$+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta)$$

$$+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta V^b_\gamma \tag{73}$$

IV. Merge the four equations and eliminate the V^b_γ and W^b_γ

A. V^a_α & $W^a_\alpha forRHS$

Using the linear relationships:

$$ia\kappa V^a = \frac{\partial P^a}{\partial r}$$

$$:= \int\int iakV^a_\beta\psi_\beta r\psi_\alpha drd\theta = \int\int \frac{\partial P^a_\beta\psi_\beta}{\partial r}r\psi_\alpha drd\theta$$

$$= ia\kappa V^a_\beta\widehat{\delta}_{\alpha\beta} = \Psi_{\alpha[\beta]}[r]P^a_\beta = \int_0^{2\pi}[r\psi_\alpha\psi_\beta]^h_0 d\theta P^a_\beta - \int\int\psi_\alpha\psi_\beta drd\theta P^a_\beta - \int\int\frac{\partial\psi_\alpha}{\partial r}\psi_\beta rdrd\theta P^a_\beta \tag{74}$$

$$iakW^a = \frac{1}{r}\frac{\partial P^a}{\partial\theta}$$

$$:= \int\int iakW^a_\beta\psi_\beta r\psi_\alpha drd\theta = \int\int\frac{\partial P^a_\beta\psi_\beta}{\partial\theta}\frac{1}{r}r\psi_\alpha drd\theta \tag{75}$$

$$= ia\kappa W^a_\beta\widehat{\delta}_{\alpha\beta} = \Psi_{\alpha(\beta)}[r]P^a_\beta = 0 - \Psi_{(\alpha)\beta}[r]P^a_\beta$$

Thus, we can establish relationships between the tranverse modes and pressure modes (no summation over a)

$$\underline{\underline{V^a_\beta\widehat{\delta}_{\alpha\beta} = \frac{1}{ia\kappa}[\int_0^{2\pi}[r\psi_\alpha\psi_\beta]^h_0 d\theta - \Psi_{\alpha\beta} - \Psi_{[\alpha]\beta}[r]]P^a_\beta = \mathbf{V}^a_{\alpha\beta}P^a_\beta}} \tag{76}$$

$$\underline{\underline{W^a_\beta\widehat{\delta}_{\alpha\beta} = -\frac{1}{ia\kappa}\Psi_{(\alpha)\beta}P^a_\beta = \mathbf{W}^a_{\alpha\beta}P^a_\beta}} \tag{77}$$

B. $\frac{d}{ds}V_{\alpha}^{a}$ **&** $\frac{d}{ds}W_{\alpha}^{a} for RHS$

We also require modal expressions for $\frac{d}{ds}V_{\alpha}^{a}$ and $\frac{d}{ds}W_{\alpha}^{a}$.

We differentiate eq71 with respect to s:

$$\begin{aligned}\frac{\partial V^a}{\partial s} &= \frac{1}{ia\kappa}\frac{\partial^2 P^a}{\partial s \partial r} \\ &= \frac{\partial}{\partial r}((1-\kappa r cos\phi)U^a)\end{aligned} \tag{78}$$

where we have used symmetry of mixed partials and the linear expression for $\frac{\partial P^a}{\partial s}$ from eq 21.

From 1.1 as example, we know that

$$\begin{aligned}&\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\ &= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta\end{aligned} \tag{79}$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$h'U_\beta^a = (1-\kappa h cos\phi)V_\beta^a \tag{80}$$

Multiplying this expression by $r\phi_\alpha$ and integrating across section of the duct, we obtain:

$$\begin{aligned}\int_0^{2\pi}\int_0^h \frac{\partial V^a}{\partial s}r\psi_\alpha drd\theta &= \int_0^{2\pi}\int_0^h \frac{\partial}{\partial r}((1-\kappa r cos\phi)U^a)r\psi_\alpha drd\theta \\ LHS := \int_0^{2\pi}\int_0^h \frac{\partial[V_\beta^a\psi_\beta r\psi_\alpha]}{\partial s}drd\theta &- \int_0^{2\pi}\int_0^h \frac{\partial r\psi_\alpha}{\partial s}\psi_\beta drd\theta V_\beta^a \\ = \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [V_\beta^a\psi_\beta r\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[V_\beta^a\psi_\beta r\psi_\alpha]_0^h d\theta &- \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s}\psi_\beta r drd\theta V_\beta^a \\ = \frac{d}{ds}V_\beta^a\widehat{\delta}_{\alpha\beta} - \int_0^{2\pi}\frac{h'^2}{1-\kappa h cos\phi}[r\psi_\beta\psi_\alpha]_0^h d\theta U_\beta^a &- \Psi_{\{\alpha\}\beta}[r]V_\beta^a\end{aligned}$$

$$\begin{aligned}RHS := \int_0^{2\pi}\int_0^h \frac{\partial}{\partial r}((1-\kappa r cos\phi)U_\beta^a\psi_\beta r\psi_\alpha)drd\theta &- \int_0^{2\pi}\int_0^h \frac{\partial(r\psi_\alpha)}{\partial r}(1-\kappa r cos\phi)U_\beta^a\psi_\beta drd\theta \\ = \int_0^{2\pi}[r(1-\kappa r cos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U_\beta^a - \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial r}r(1-\kappa r cos\phi)\psi_\beta drd\theta U_\beta^a &- \int_0^{2\pi}\int_0^h (1-\kappa r cos\phi)\psi_\alpha\psi_\beta drd\theta U_\beta^a \\ = \int_0^{2\pi}[r(1-\kappa r cos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]U_\beta^a &- \Psi_{\alpha\beta}[(1-\kappa r cos\phi)]U_\beta^a\end{aligned} \tag{81}$$

Thus, LHS=RHS, we have:

$$\begin{aligned}\frac{d}{ds}V_\beta^a\widehat{\delta}_{\alpha\beta} = \int_0^{2\pi}\frac{h'^2}{1-\kappa h cos\phi}[r\psi_\beta\psi_\alpha]_0^h d\theta U_\beta^a + \Psi_{\{\alpha\}\beta}[r]V_\beta^a \\ + \int_0^{2\pi}[r(1-\kappa r cos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]U_\beta^a - \Psi_{\alpha\beta}[(1-\kappa r cos\phi)]U_\beta^a\end{aligned} \tag{82}$$

Similarly for W^a, differentiating eq50 with respect to s and substituting the linear expression for $\frac{\partial P^a}{\partial s}$ by eq21:

$$\begin{aligned}\frac{\partial W^a}{\partial s} &= \frac{1}{ia\kappa}\frac{\partial^2 P^a}{\partial s \partial \theta} \\ &= \frac{1}{r}\frac{\partial}{\partial \theta}((1-\kappa r cos\phi)U^a)\end{aligned} \tag{83}$$

Multiplying this expression by $r\phi_\alpha$ and integrating across section of the duct, we obtain:

$$\begin{aligned}\int_0^{2\pi}\int_0^h \frac{\partial W^a}{\partial s} r\psi_\alpha dr d\theta &= \int_0^{2\pi}\int_0^h \frac{1}{r}\frac{\partial}{\partial \theta}((1-\kappa r cos\phi)U^a) r\psi_\alpha dr d\theta \\ LHS := \int_0^{2\pi}\int_0^h \frac{\partial [W^a_\beta \psi_\beta r\psi_\alpha]}{\partial s} dr d\theta &- \int_0^{2\pi}\int_0^h \frac{\partial r\psi_\alpha}{\partial s}\psi_\beta dr d\theta W^a_\beta \\ = \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [W^a_\beta \psi_\beta r\psi_\alpha] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds}[W^a_\beta \psi_\beta r \psi_\alpha]_0^h d\theta &- \int_0^{2\pi}\int_0^h \frac{\partial \psi_\alpha}{\partial s}\psi_\beta r dr d\theta W^a_\beta \\ = \frac{d}{ds}W^a_\beta \widehat{\delta}_{\alpha\beta} - \int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]_0^h d\theta W^a_\beta &- \Psi_{\{\alpha\}\beta}[r]W^a_\beta \\ RHS := \int_0^{2\pi}\int_0^h \frac{\partial}{\partial \theta}((1-\kappa r cos\phi)U^a_\beta \psi_\beta \psi_\alpha) dr d\theta - \int_0^{2\pi}\int_0^h \frac{\partial (\psi_\alpha)}{\partial \theta} &(1-\kappa r cos\phi)U^a_\beta \psi_\beta dr d\theta \\ = 0(periodic) &- \Psi_{(\alpha)\beta}[1-\kappa r cos\phi]U^a_\beta\end{aligned} \tag{84}$$

Thus, LHS=RHS, we have:

$$\underline{\underline{\frac{d}{ds}W^a_\beta \widehat{\delta}_{\alpha\beta} = \int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]_0^h d\theta W^a_\beta + \Psi_{\{\alpha\}\beta}[r]W^a_\beta - \Psi_{(\alpha)\beta}[1-\kappa r cos\phi]U^a_\beta = -\Psi_{\alpha\{\beta\}}[r]W^a_\beta - \Psi_{(\alpha)\beta}[1-\kappa r cos\phi]U^a_\beta}} \tag{85}$$

V. Substitue pressure modes for transverse velocity modes

A. mass equation

$$\begin{aligned}&\frac{dU^a_\beta}{ds}\widehat{\delta}_{\alpha\beta} - \Psi_{\{\alpha\}\beta}[r]U^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa r cos(\phi))]\underline{\underline{V^a_\beta}} - \Psi_{(\alpha)\beta}[(1-\kappa r cos(\phi))]\underline{\underline{W^a_\beta}} \\ &= \Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa \underline{\underline{V^{a-b}_\beta V^b_\gamma}} - ib\kappa \underline{\underline{W^{a-b}_\beta W^b_\gamma}} - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma)\end{aligned} \tag{86}$$

Transform:

$$\frac{dU^a_\beta}{ds}\widehat{\delta}_{\alpha\beta} := \widehat{\mathbf{I}^a_{\alpha\beta}}u'^a_\beta$$

$$-\Psi_{\{\alpha\}\beta}[r]U^a_\beta := \underline{\underline{\underline{-\Psi_{\{\alpha\}\beta}[r]}}}u^a_\beta \rightarrow \mathcal{G}$$

$$\begin{cases} -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta := \sum_{\beta=0}^{+\infty} \underline{\underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}}p^a_\beta \rightarrow \mathcal{M}_1 \\ -\Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{\underline{V^a_\beta}}} := \sum_{\beta=0}^{+\infty} \underline{\underline{\underline{-\Psi_{[\alpha]\delta}[r(1-\kappa rcos(\phi))]\mathbf{V}^a_{\delta\beta}}}}p^a_\beta \rightarrow \mathcal{M}_2 + \Psi_{[\alpha]\delta}[r(1-krcos\phi)](N^{-1})(o(M_2^2)) \\ -\Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{\underline{W^a_\beta}}} := \sum_{\beta=0}^{+\infty} \underline{\underline{\underline{-\Psi_{(\alpha)\delta}[(1-\kappa rcos(\phi))]\mathbf{W}^a_{\delta\beta}}}}p^a_\beta \rightarrow \mathcal{M}_3 + \Psi_{[\alpha]\delta}[r(1-krcos\phi)](N^{-1})(o(M_3^2)) \end{cases}$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma) := \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}}p^{a-b}_\beta p^b_\gamma \rightarrow \mathcal{B}_2$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa U^{a-b}_\beta U^b_\gamma) := \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}}u^{a-b}_\beta u^b_\gamma \rightarrow \mathcal{A}_1$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa V^{a-b}_\beta V^b_\gamma) := \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}p^{a-b}_\beta p^b_\gamma}}} \rightarrow \mathcal{B}_3$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa W^{a-b}_\beta W^b_\gamma) := \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}p^{a-b}_\beta p^b_\gamma}}} \rightarrow \mathcal{B}_4$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma) := \sum_{b=-\infty}^{+\infty} -ia\kappa\frac{B}{2A}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}}p^{a-b}_\beta p^b_\gamma \rightarrow \mathcal{B}_1 \tag{87}$$

Here, may be a little question of transform with P, think about N^{-1}, transform it as matrix we could solve it:

$$\because ia\kappa V^a = \frac{\partial P^a}{\partial r} + o(M^2)$$

$$\int\int ia\kappa V^a\phi_\alpha drd\theta = \int\int ia\kappa\phi_\alpha\phi_\beta drd\theta V^a_\beta = \mathbf{I}^a_{\alpha\beta}V^a_\beta = \mathbf{V}^a_{\alpha\beta}p^a_\beta + \int\int o(M^2)r\phi_\alpha drd\theta$$

$$\because ia\kappa(1-\kappa rcos\phi)V^a = (1-\kappa rcos\phi)\frac{\partial P^a}{\partial r} + (1-\kappa rcos\phi)o(M^2)$$

$$\int\int ia\kappa(1-\kappa rcos\phi)V^a\phi_\alpha drd\theta = \mathbf{N}^a_{\alpha\beta}V^a_\beta = \int\int(1-\kappa rcos\phi)\frac{\partial P^a}{\partial r}r\phi_\alpha drd\theta + \int\int(1-\kappa rcos\phi)o(M^2)drd\theta$$

$$\therefore (\mathbf{N}^{-1})^a_{\beta\alpha}\int\int(1-\kappa rcos\phi)\frac{\partial P^a}{\partial r}r\phi_\alpha drd\theta = (\mathbf{I}^{-1})^a_{\beta\alpha}\mathbf{I}^a_{\alpha\beta}V^a_\beta$$

$$(\mathbf{N}^{-1})^a_{\beta\alpha}\int\int(1-\kappa rcos\phi)o(M^2)drd\theta = (\mathbf{I}^{-1})^a_{\beta\alpha}\int\int o(M^2)r\phi_\alpha drd\theta \tag{88}$$

B. momentum equation I

$$\frac{d}{ds}P^a_\beta\widehat{\delta}_{\alpha\beta} - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta$$
$$= \sum_{b=-\infty}^{+\infty} \underline{term\mathcal{D}1:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]\underline{\underline{V^{a-b}_\beta}}U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]\underline{\underline{W^{a-b}_\beta}}U^a_\gamma$$
$$+\underline{term(D1+\mathcal{P}1):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma$$
$$+\underline{term\mathcal{D}1:} - \frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta)$$
$$+\underline{term\mathcal{X}1:}\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta\underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta\underline{\underline{W^b_\gamma}} \tag{89}$$

Transform:

$$\frac{d}{ds}P^a_\beta\widehat{\delta}_{\alpha\beta} := \widehat{\mathbf{I}^a_{\alpha\beta}}p'^a_\beta$$
$$-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta := \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}u^a_\beta \to -\mathcal{N}$$
$$-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta := \underline{\underline{-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta}}p^a_\beta - \underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta = \underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta \to -\mathcal{H}$$
$$\Psi_{\{\alpha\}\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}U^{a-b}_\beta U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\beta\gamma}[r]}}u^{a-b}_\beta u^b_\gamma \to \mathcal{D}_4$$
$$\Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{V^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa r cos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}}}p^{a-b}_\beta u^b_\gamma \to \mathcal{C}_4$$
$$\Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{W^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{(\alpha)\delta\epsilon}[(1-\kappa r cos\phi)]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}}}p^{a-b}_\beta u^b_\gamma \to \mathcal{C}_5$$
$$ia\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]\sum_{b=-\infty}^{+\infty}P^{a-b}_\beta U^b_\gamma := ia\kappa\sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]}}p^{a-b}_\beta u^b_\gamma \to \mathcal{C}_3$$
$$-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])\sum_{b=-\infty}^{+\infty}U^{a-b}_\beta U^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])}}u^{a-b}_\beta u^b_\gamma \to \mathcal{D}_1$$
$$-\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) := \Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}([(Mp)^{a-b}_\beta + (Gu)^{a-b}_\beta]U^b_\gamma + [(Mp)^b_\gamma + (Gu)^b_\gamma]U^{a-b}_\beta)$$
$$= \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\mathbf{G},\mathbf{I}] + \Psi_{\alpha\beta\gamma}[r][\mathbf{I},\mathbf{G}])u^{a-b}_\beta u^b_\gamma \to \mathcal{D}_{2,3}$$
$$+ \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\mathbf{M},\mathbf{I}] + \Psi_{\alpha\beta\gamma}[r][\mathbf{I},\mathbf{M}])u^{a-b}_\beta p^b_\gamma \to \mathcal{C}_{1,2}$$
$$\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta\underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta\underline{\underline{W^b_\gamma}} := \{\sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[rcos\phi][\mathbf{I},\mathbf{V}] - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi][\mathbf{I},\mathbf{W}]\}u^{a-b}_\gamma p^b_\beta \to \mathcal{C}_{6,7} \tag{90}$$

Here, little transform easy to be proved:

$$\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{V_{\beta}^{a-b}}}U_{\gamma}^{a} := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}_{\delta\beta}^{a-b}\mathbf{I}_{\epsilon\gamma}^{b}}}p_{\beta}^{a-b}u_{\gamma}^{b} \to \mathcal{C}_4$$

$$\therefore \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)][I,V][p_{\beta}^{a-b},u_{\gamma}^{b}] = \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)][V,I][u_{\beta}^{a-b},p_{\gamma}^{b}]$$

$$\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{W_{\beta}^{a-b}}}U_{\gamma}^{a} := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{(\alpha)\delta\epsilon}[(1-\kappa rcos\phi)]\mathbf{W}_{\delta\beta}^{a-b}\mathbf{I}_{\epsilon\gamma}^{b}}}p_{\beta}^{a-b}u_{\gamma}^{b} \to \mathcal{C}_5$$

$$\therefore \Psi_{(\alpha)\beta\gamma}[1-\kappa rcos\phi][I,W][p_{\beta}^{a-b},u_{\gamma}^{b}] = \Psi_{[\alpha]\beta\gamma}[1-\kappa rcos\phi][W,I][u_{\beta}^{a-b},p_{\gamma}^{b}]$$

$$ia\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}P_{\beta}^{a-b}U_{\gamma}^{b} := ia\kappa\sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]}}p_{\beta}^{a-b}u_{\gamma}^{b} \to \mathcal{C}_3$$

$$\therefore ia\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)][I,I][p_{\beta}^{a-b},u_{\gamma}^{b}] = ia\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)][I,I][u_{\beta}^{a-b},p_{\gamma}^{b}] \tag{91}$$

C. momentum equation II

$$\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V_{\beta}^{a} \\
&+\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P_{\beta}^{a} - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P_{\beta}^{a} - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]P_{\beta}^{a} \\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^{a} + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V_{\beta}^{a-b}V_{\gamma}^{a} + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W_{\beta}^{a-b}V_{\gamma}^{a} \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P_{\beta}^{a-b}V_{\gamma}^{b} \\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_{\beta}^{a-b}V_{\gamma}^{b} - \Psi_{\alpha\beta\gamma}[r](\frac{dU_{\beta}^{a-b}}{ds}V_{\gamma}^{b} + \frac{dV_{\gamma}^{b}}{ds}U_{\beta}^{a-b}) \\
&+\underline{term\mathcal{X}2:}\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W_{\beta}^{a-b}W_{\gamma}^{b} - \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U_{\beta}^{a-b}U_{\gamma}^{b}
\end{aligned} \tag{92}$$

With:

$$\begin{aligned}
&\frac{d}{ds}V_{\beta}^{a}\widehat{\delta}_{\alpha\beta} = \{\int_0^{2\pi}\frac{h'^2}{1-\kappa hcos\phi}[r\psi_\beta\psi_\alpha]_0^h d\theta \\
&+\int_0^{2\pi}[r(1-\kappa rcos\phi)\psi_\beta\psi_\alpha]_0^h d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)] - \Psi_{\alpha\beta}[(1-\kappa rcos\phi)]\}U_{\beta}^{a} \\
&-G_{\alpha\beta}^{a}V_{\beta}^{a}
\end{aligned} \tag{93}$$

Transform:

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_\beta := -N^a_{\alpha\beta}V^a_\beta\\
&\{\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]^h_0 d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)] - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]\}P^a_\beta\\
&\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\delta\epsilon}[r]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}u^{a-b}_\beta p^b_\gamma = \sum_{b=-\infty}^{+\infty}\Psi_{\{\alpha\}\beta\gamma}[r][\mathbf{I},\mathbf{V}]u^{a-b}_\beta p^b_\gamma \to \varepsilon_{1.4}\\
&\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta V^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{5.3}\\
&\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta V^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{(\alpha)\delta\epsilon}[1-\kappa rcos\phi]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{5.4}\\
&i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta V^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{ia\kappa\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{5.2}\\
&-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta V^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\delta\epsilon}[r])\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}u^{a-b}_\beta p^b_\gamma \to \varepsilon_{1.1}\\
&-\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\frac{dU^{a-b}_\beta}{ds}V^b_\gamma + \frac{dV^b_\gamma}{ds}U^{a-b}_\beta)\\
&:= -\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\widehat{\mathbf{I}^{a-b}}^{-1}[-(\mathbf{M}p)^{a-b}_\beta - (\mathbf{G}u)^{a-b}_\beta]V^b_\gamma\\
&+\widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}\{\int_0^{2\pi}\frac{h'^2}{1-\kappa hcos\phi}[r\psi_\beta\psi_\alpha]^h_0 d\theta\\
&+\int_0^{2\pi}[r(1-\kappa rcos\phi)\psi_\beta\psi_\alpha]^h_0 d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)] - \Psi_{\alpha\beta}[(1-\kappa rcos\phi)]\}U^b_\beta U^{a-b}_\beta\\
&-(\widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}G^b_{\alpha\beta}V^b_\beta)U^{a-b}_\beta)\\
&= \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\widehat{\mathbf{I}}^{-1}\mathbf{G},\mathbf{V}]])u^{a-b}_\beta p^b_\gamma \to \varepsilon_{1.2}\\
&+\sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\widehat{\mathbf{I}}^{-1}\mathbf{M},\mathbf{V}]])p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{5.1}\\
&-\sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[r]\{\widehat{\mathbf{I}}^{-1}(\int_0^{2\pi}\frac{h'^2}{1-\kappa hcos\phi}[r\psi_\beta\psi_\alpha)]^h_0 d\theta\\
&+\int_0^{2\pi}[r(1-\kappa rcos\phi)\psi_\beta\psi_\alpha]^h_0 d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)] - \Psi_{\alpha\beta}[(1-\kappa rcos\phi)]\}u^{a-b}_\beta u^b_\gamma \to \mathcal{A}_{2.1,2.2,2.4,2.3}\\
&+\sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[r][\mathbf{I},\widehat{\mathbf{I}}^{-1}\mathbf{G}\mathbf{V}])u^{a-b}_\beta p^b_\gamma \to \varepsilon_{1.3}\\
&\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta W^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\delta\epsilon}[1-\kappa rcos\phi]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{5.5}\\
&-\kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta U^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\kappa\Psi_{\alpha\beta\gamma}[rcos\phi]}}u^{a-b}_\beta u^b_\gamma \to \mathcal{A}_{2.5}
\end{aligned}
\tag{94}
$$

D. momentum equation III

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]W^a_\beta \\
&-\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta \\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta W^a_\gamma \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta W^b_\gamma \\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta V^b_\gamma
\end{aligned}
\tag{95}
$$

With:

$$
\underline{\underline{\frac{d}{ds}W^a_\beta\widehat{\delta}_{\alpha\beta} = \int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]^h_0 d\theta W^a_\beta + \Psi_{\{\alpha\}\beta}[r]W^a_\beta - \Psi_{(\alpha)\beta}[1-\kappa rcos\phi]U^a_\beta = -\Psi_{\alpha\{\beta\}}[r]W^a_\beta - \Psi_{(\alpha)\beta}[1-\kappa rcos\phi]U^a_\beta}}
\tag{96}
$$

Transform:

$$-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]W^a_\beta := -N^a_{\alpha\beta}W^a_\beta$$

$$-\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]p^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]p^a_\beta$$

$$\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\delta\epsilon}[r]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}u^{a-b}_\beta p^b_\gamma = \sum_{b=-\infty}^{+\infty}\Psi_{\{\alpha\}\beta\gamma}[r][\mathbf{I},\mathbf{W}]u^{a-b}_\beta p^b_\gamma \to \varepsilon_{2.4}$$

$$\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta W^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{6.3}$$

$$\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta W^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{(\alpha)\delta\epsilon}[1-\kappa rcos\phi]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{6.4}$$

$$i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta W^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{ia\kappa\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{6.2}$$

$$-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta W^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\delta\epsilon}[r])\mathbf{I}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}u^{a-b}_\beta p^b_\gamma \to \varepsilon_{2.1}$$

$$-\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta)$$

$$:= -\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\widehat{\mathbf{I}}^{-1}[-(\mathbf{M}p)^{a-b}_\beta - (\mathbf{G}u)^{a-b}_\beta]W^b_\gamma - \{\mathbf{H}W^a_\beta - \Psi_{(\alpha)\beta}[1-\kappa rcos\phi]U^a_\beta\}U^{a-b}_\beta$$

$$= \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\widehat{\mathbf{I}}^{-1}\mathbf{G},\mathbf{W}]])u^{a-b}_\beta p^b_\gamma \to \varepsilon_{2.2}$$

$$+\sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\widehat{\mathbf{I}}^{-1}\mathbf{M},\mathbf{W}]])p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{6.1}$$

$$-\sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[r][\mathbf{I},\widehat{\mathbf{I}}^{-1}\mathbf{H}\mathbf{W}]u^{a-b}_\beta p^b_\gamma \to \varepsilon_{2.3}$$

$$+\sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[r](\widehat{\mathbf{I}}^{-1}\Psi_{(\alpha)\beta}[1-\kappa rcos\phi])u^a_\beta u^{a-b}_\beta \to \mathcal{A}_{3.1}$$

$$\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta U^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]}}u^{a-b}_\beta u^b_\gamma \to \mathcal{A}_{3.2}$$

$$-\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta V^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\Psi_{\alpha\delta\epsilon}[1-\kappa rcos\phi]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \to \mathcal{B}_{6.5} \tag{97}$$

Finally, we obtain two equations involving just pressure and longitudinal velocity modes. Here, $\mathbf{M}$,$\mathbf{N}$,$\mathcal{A}$,$\mathcal{B}$,and $\mathcal{C}$ encoding the curvature of the duct. The terms $\mathbf{G}$,$\mathbf{H}$,$\mathbf{D}$ encode the variation in duct diameter as well as the torsion. The term ε encodes variation of diameter and the torsion together with curvature if either of the first two are present.

$$
\begin{aligned}
\widehat{\mathbf{I}^a_{\alpha\beta}} {u'}^a_\beta + \mathbf{M}^a_{\alpha\beta} p^a_\beta + \mathbf{G}^a_{\alpha\beta} u^a_\beta &= \mathcal{A}^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, u^b_\gamma] + \mathcal{B}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \varepsilon^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, p^b_\gamma] \\
\widehat{\mathbf{I}^a_{\alpha\beta}} {p'}^a_\beta - \mathbf{N}^a_{\alpha\beta} u^a_\beta - \mathbf{H}^a_{\alpha\beta} p^a_\beta &= \mathcal{C}^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, p^b_\gamma] + \mathcal{D}^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, u^b_\gamma]
\end{aligned}
\tag{98}
$$

VI. Introduce the admittance matrix

Due to the present of evanescent modes these equations are numerically unstable and cannot be integrated directly. Define a relation between the pressure and velicity in terms of the admittance. When solving for pressure, it is easier to work with the addmittance rather than the impedance Z ($Y = Z^{-1}$), to avoid inverting large matrices in the work that will follow.

The following relationship is defined:

$$
\widehat{\mathbf{I}^a_{\alpha\beta}} u^a_\beta = Y^a_{\alpha\beta} p^a_\beta + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] \tag{99}
$$

where $Y = Y(s)$ is the linear part of the admittance and $\mathcal{Y} = \mathcal{Y}(s)$ is the second order non-linear correction to the admittance, henceforth referred to as the nonlinear admittance term. We differentiate it:

$$
\widehat{\mathbf{I}^a_{\alpha\beta}} u' = {Y'}^a_{\alpha\beta} p^a_\beta + Y^a_{\alpha\beta} {p'}^a_\beta + {\mathcal{Y}'}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[{p'}^{a-b}_\beta, p^b_\gamma] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, {p'}^b_\gamma] \tag{100}
$$

Substitute in eq98 of u',

$$
\begin{aligned}
&-\mathbf{M}^a_{\alpha\beta} p^a_\beta - \mathbf{G}^a_{\alpha\beta} u^a_\beta + \mathcal{A}^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, u^b_\gamma] + \mathcal{B}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \varepsilon^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, p^b_\gamma] \\
&= {Y'}^a_{\alpha\beta} p^a_\beta + Y^a_{\alpha\beta} {p'}^a_\beta + {\mathcal{Y}'}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[{p'}^{a-b}_\beta, p^b_\gamma] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, {p'}^b_\gamma]
\end{aligned}
\tag{101}
$$

Then, p'

$$
\begin{aligned}
-\mathbf{M}^a_{\alpha\beta} p^a_\beta - \mathbf{G}^a_{\alpha\beta} u^a_\beta + \mathcal{A}^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, u^b_\gamma] + \mathcal{B}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \varepsilon^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, p^b_\gamma] \\
= {Y'}^a_{\alpha\beta} p^a_\beta \\
+ Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} (\mathbf{N}^a_{\alpha\beta} u^a_\beta + \mathbf{H}^a_{\alpha\beta} p^a_\beta + \mathcal{C}^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, p^b_\gamma] + \mathcal{D}^{ab}_{\alpha\beta\gamma}[u^{a-b}_\beta, u^b_\gamma]) \\
+ {\mathcal{Y}'}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] \\
+ \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{N}^{a-b}_{\beta\delta} u^{a-b}_\delta + \mathbf{H}^{a-b}_{\beta\delta} p^{a-b}_\delta, p^b_\gamma] \\
+ \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, \mathbf{N}^b_{\gamma\delta} u^b_\delta + \mathbf{H}^b_{\gamma\delta} p^b_\delta]
\end{aligned}
\tag{102}
$$

Use eq99 to express u in terms of p,$\widehat{\mathbf{I}^a_{\alpha\beta}} u^a_\beta = Y^a_{\alpha\beta} p^a_\beta + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma]$

$$
\begin{aligned}
-\mathbf{M}^a_{\alpha\beta} p^a_\beta - \mathbf{G}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} p^a_\beta - \mathbf{G}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] \\
+\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1} Y^{\pm b}_{\alpha\gamma} p^b_\gamma] + \mathcal{B}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \varepsilon^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, p^b_\gamma] \\
= Y'^a_{\alpha\beta} p^a_\beta + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} (\mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} p^a_\beta + \mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \mathbf{H}^a_{\alpha\beta} p^a_\beta \\
+\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, p^b_\gamma] + \mathcal{D}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1} Y^{\pm b}_{\alpha\gamma} p^b_\gamma]) \\
+\mathcal{Y}'^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{N}^{a-b}_{\beta\delta} \widehat{\mathbf{I}^{a-b}_{\delta\alpha}}^{-1} Y^{a-b}_{\alpha\delta} p^{a-b}_\delta + \mathbf{H}^{a-b}_{\beta\delta} p^{a-b}_\delta, p^b_\gamma] \\
+\mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, \mathbf{N}^b_{\gamma\delta} \widehat{\mathbf{I}^b_{\delta\alpha}}^{-1} Y^b_{\alpha\delta} p^b_\delta + \mathbf{H}^b_{\gamma\delta} p^b_\delta]
\end{aligned} \tag{103}
$$

This equation has two distinct orders of magnitude: terms linear in p, and terms quadratic in p. We can equate linear terms and the quadratic terms separately to grt two distinct equations:

$$
linear: Y'^a_{\alpha\beta} p^a_\beta + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} p^a_\beta + \mathbf{M}^a_{\alpha\beta} p^a_\beta + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{H}^a_{\alpha\beta} p^a_\beta + \mathbf{G}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} p^a_\beta = 0, \tag{104}
$$

$$
\begin{aligned}
quadratic: \mathcal{Y}'^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{N}^{a-b}_{\beta\delta} \widehat{\mathbf{I}^{a-b}_{\delta\alpha}}^{-1} Y^{a-b}_{\alpha\delta} p^{a-b}_\delta, p^b_\gamma] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, \mathbf{N}^b_{\gamma\delta} \widehat{\mathbf{I}^b_{\delta\alpha}}^{-1} Y^b_{\alpha\delta} p^b_\delta] \\
+Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, p^b_\gamma] \\
-\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1} Y^{\pm b}_{\alpha\gamma} p^b_\gamma] - \mathcal{B}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] - \varepsilon^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, p^b_\gamma] \\
+\mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, \mathbf{H}^b_{\gamma\delta} p^b_\delta] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{H}^{a-b}_{\beta\delta} p^{a-b}_\delta, p^b_\gamma] + \mathbf{G}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] \\
+Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathcal{D}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1} Y^{\pm b}_{\alpha\gamma} p^b_\gamma] = 0
\end{aligned} \tag{105}
$$

As both of these equation hold true for a general p we can eliminate it to obtain an equation for the linear part of the admittance and an equation for the nonlinear part of the admittance:

$$
linear-2D: Y'^a_{\alpha\beta} + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} + \mathbf{M}^a_{\alpha\beta} + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{H}^a_{\alpha\beta} + \mathbf{G}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} = 0, \tag{106}
$$

For quadratic, with:

$$
\begin{aligned}
(\mathcal{A}[x,y])^a_\alpha &= (\mathcal{A}^{ab}_{\alpha\beta\gamma}[x^{a-b}_\beta, y^b_\gamma])^a_\alpha = \sum_{b=-\infty}^{\infty}\sum_{\beta,\gamma=0}^{\infty} A^{ab}_{\alpha\beta\gamma}x^{a-b}_\beta y^b_\gamma \\
(\mathcal{A}[X,Y])^{ab}_{\alpha\beta\gamma} &= (\mathcal{A}^{ab}_{\alpha\delta\epsilon}[X^{a-b}_{\delta\beta}, Y^b_{\epsilon\gamma}])^{ab}_{\alpha\beta\gamma} = \sum_{\delta,\epsilon=0}^{\infty} A^{ab}_{\alpha\delta\epsilon}X^{a-b}_{\delta\beta}Y^b_{\epsilon\gamma} \\
Thus, \{(\mathcal{A}[X,Y])^{ab}_{\alpha\beta\gamma}[x,y]\}^a_\alpha &= \{(\mathcal{A}^{ab}_{\alpha\delta\epsilon}[X^{a-b}_{\delta\beta}, Y^b_{\epsilon\gamma}])^{ab}_{\alpha\beta\gamma}[x^{a-b}_\beta, y^b_\gamma]\}^a_\alpha \\
&= \sum_{b=-\infty}^{\infty}\sum_{\beta,\gamma=0}^{\infty}(\sum_{\delta,\epsilon=0}^{\infty} A^{ab}_{\alpha\delta\epsilon}X^{a-b}_{\delta\beta}Y^b_{\epsilon\gamma})x^{a-b}_\beta y^b_\gamma \\
= \sum_{b=-\infty}^{\infty}\{\sum_{\delta,\epsilon=0}^{\infty} A^{ab}_{\alpha\delta\epsilon}(\sum_{\beta} X^{a-b}_{\delta\beta}x^{a-b}_\beta)^{a-b}_\delta & (\sum_{\gamma=0}^{\infty} Y^b_{\epsilon\gamma})y^b_\gamma)^b_\epsilon\} = \mathcal{A}[Xx, Yy]
\end{aligned}
\tag{107}
$$

We now can eliminate p, remaining 3-D tensors:

$$
\begin{aligned}
quadratic: \mathcal{Y}'^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}p^{a-b}_\delta, I_{\gamma\epsilon}p^b_\epsilon] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{N}^{a-b}_{\beta\delta}\widehat{\mathbf{I}^{a-b}_{\delta\alpha}}^{-1} Y^{a-b}_{\alpha\delta}p^{a-b}_\delta, I_{\gamma\epsilon}p^b_\epsilon] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}p^{a-b}_\delta, \mathbf{N}^b_{\gamma\delta}\widehat{\mathbf{I}^b_{\delta\alpha}}^{-1} Y^b_{\alpha\delta}p^b_\delta] \\
+Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}p^{a-b}_\delta, I_{\gamma\epsilon}p^b_\epsilon] + Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}p^{a-b}_\beta, I_{\gamma\epsilon}p^b_\epsilon] \\
-\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}p^{a-b}_\beta, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}Y^{\pm b}_{\alpha\gamma}p^b_\gamma] - \mathcal{B}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}p^{a-b}_\delta, I_{\gamma\epsilon}p^b_\epsilon] - \varepsilon^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}p^{a-b}_\beta, I_{\gamma\epsilon}p^b_\epsilon] \\
+\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}p^{a-b}_\delta, \mathbf{H}^b_{\gamma\delta}p^b_\delta] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{H}^{a-b}_{\beta\delta}p^{a-b}_\delta, I_{\gamma\epsilon}p^b_\epsilon] + \mathbf{G}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}p^{a-b}_\delta, I_{\gamma\epsilon}p^b_\epsilon] \\
+Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{D}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}p^{a-b}_\beta, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}Y^{\pm b}_{\alpha\gamma}I_{\gamma\epsilon}p^b_\epsilon] = 0
\end{aligned}
\tag{108}
$$

$\Rightarrow$ quadratic-rank 5 tensor(2 upper, 3 lower):

$$
\begin{aligned}
(\mathcal{Y}'^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, I_{\gamma\epsilon}]^{ab}_{\alpha\beta\gamma})\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{N}^{a-b}_{\beta\delta}\widehat{\mathbf{I}^{a-b}_{\delta\alpha}}^{-1}Y^{a-b}_{\alpha\delta}, I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, \mathbf{N}^b_{\gamma\delta}\widehat{\mathbf{I}^b_{\delta\alpha}}^{-1}Y^b_{\alpha\delta}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} \\
+Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} + Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}, I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} \\
-\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}Y^{\pm b}_{\alpha\gamma}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} - \mathcal{B}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} - \varepsilon^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}, I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} \\
+\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, \mathbf{H}^b_{\gamma\delta}p^b_\delta] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{H}^{a-b}_{\beta\delta}p^{a-b}_\delta, I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} + \mathbf{G}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} \\
+Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{D}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}Y^{\pm b}_{\alpha\gamma}I_{\gamma\epsilon}]\underline{\underline{[p^{a-b}_\delta, p^b_\epsilon]}} = 0
\end{aligned}
\tag{109}
$$

These equation are solved from the outlet of the duct, applying the appropriate radiation boundray condition at the duct exit. Once Y(s) and $\mathcal{Y}(s)$ are found throught the duct, eq99 can then be used to replace the velocity modes with pressure modes in eq98, to obtain a numerically stable first order ODE for the pressure modes:

$$\begin{aligned}\widehat{\mathbf{I}^a_{\alpha\beta}} {p'}^a_\beta &= \mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} p^a_\beta + \mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathcal{Y}^{ab}_{\alpha\beta\gamma}[p^{a-b}_\beta, p^b_\gamma] \\ +\mathbf{H}^a_{\alpha\beta} p^a_\beta + \mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, p^b_\gamma] &+ \mathcal{D}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta} p^{a-b}_\beta, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1} Y^{\pm b}_{\alpha\gamma} p^b_\gamma]) \end{aligned} \tag{110}$$

This equation can be solved from the source to the outlet. As the equation involves the local admittance at each point, the solution includes both forward and backwards propagating waves together with their nonlinear interaction.

VII. Boundary Conditions for an infinite uniform duct oulet

The simplest boundary condition to consider for the adimittance is that of an outlet consisting of an infinitely long uniform duct of constant curvature for which we have only propagating waves and decaying evanescent waves. In such a duct no point can be distinguished from another longitudinally, therefore we must have the adimittance being a fixed point of the admittance equations. To find the fixed points, we begin by combing eq99, ignoring the quadratic terms for the moment, to form a second order ODE for the pressure modes, G,H, the derivatives of M and N vanish:

$$\begin{aligned} \widehat{\mathbf{I}^a_{\alpha\beta}} {u'}^a_\beta + \mathbf{M}^a_{\alpha\beta} p^a_\beta &= 0 \\ \widehat{\mathbf{I}^a_{\alpha\beta}} {p'}^a_\beta - \mathbf{N}^a_{\alpha\beta} u^a_\beta &= 0 \end{aligned} \tag{111}$$

$$\widehat{\mathbf{I}^a_{\alpha\beta}} {p''}^a_\beta(s) + \mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{M}^a_{\alpha\beta} p^a_\beta(s) = 0 \tag{112}$$

$$\{v_i, -\lambda_i^2\} = eig(\mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathbf{M}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}), suppose \quad \alpha=\beta \tag{113}$$

In matrix,

$$[\mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathbf{M}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}]_{\alpha\times\alpha} V = V\Lambda^2, V = [v1, v2, v3, ...], \Lambda = diag(i\lambda_1, i\lambda_2, ...) \tag{114}$$

The solution in terms of forward and backward modes is given by:

$$p = p^+ + p^- = \sum_{k=1}^{\infty}(c_k^+ v_k e^{i\lambda_k s} + c_k^- v_k e^{-i\lambda_k s}) \tag{115}$$

where the v_i are the eigenvalue of NM with eigenvalues λ_i^2, with arbitary c_k^+ and c_k^-. Here, we have split the solution into forward and backward waves. The roots of the λ_k are chosen as follows:

$$\lambda_k = \begin{cases} (\lambda_k^2)^{1/2}, \lambda_k^2 > 0 \\ i(-\lambda_k^2)^{1/2}, \lambda_k^2 < 0 \end{cases} \tag{116}$$

to ensure either propagating or decaying evanescent modes in the positive direction. Based on extensive numerical evaluations, we observe that all of the eigenvectors of NM are real. We now introduce the characteristic forword and backwards admittance, linearly relating the forward and backwards modes:

$$\widehat{\mathbf{I}^a_{\alpha\beta}} u^{\pm a}_{\beta} = Y^{\pm a}_{\alpha\beta} p^{\pm a}_{\beta} \tag{117}$$

Using this, together with the linear equation relating pressure and velocity $(p^{\pm a}_{\beta})' = N^a_{\alpha\beta} u^{\pm a}_{\beta}$, we obtain and expression for $Y^{\pm a}_{\alpha\beta}$

$$(p^{\pm a}_{\beta}(s))' = N^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^{\pm a}_{\alpha\beta} p^{\pm a}_{\beta}(s) \tag{118}$$

Which is similar to above eig property, we have:

$$[N^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^{\pm a}_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}]_{\alpha\times\alpha} V = \pm V\Lambda, V = [v1, v2, v3, ...], \Lambda = diag(i\lambda_1, i\lambda_2, ...)$$
$$\Rightarrow Y^{\pm a}_{\alpha\beta} = \pm \widehat{\mathbf{I}^a_{\alpha\beta}} {N^a_{\beta\alpha}}^{-1} V\Lambda V^{-1} \widehat{\mathbf{I}^a_{\alpha\beta}} = \pm \widehat{\mathbf{I}^a_{\alpha\beta}} {N^a_{\beta\alpha}}^{-1} \sqrt{V\Lambda^2 V^{-1}} \widehat{\mathbf{I}^a_{\alpha\beta}} = \pm \widehat{\mathbf{I}^a_{\alpha\beta}} {N^a_{\beta\alpha}}^{-1} \sqrt{\mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{M}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}} \widehat{\mathbf{I}^a_{\alpha\beta}} \tag{119}$$

Subsitue into eq, again ignoring G and H as the duct is assumed uniform:

$$linear - 2D : Y'^a_{\alpha\beta} + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} + \mathbf{M}^a_{\alpha\beta} + Y^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{H}^a_{\alpha\beta} + \mathbf{G}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^a_{\alpha\beta} = 0, \tag{120}$$

$$Y'^a_{\alpha\beta} = \widehat{\mathbf{I}^a_{\alpha\beta}} {N^a_{\beta\alpha}}^{-1} \sqrt{\mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{M}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}} \widehat{\mathbf{I}^a_{\alpha\beta}} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \widehat{\mathbf{I}^a_{\alpha\beta}} {N^a_{\beta\alpha}}^{-1} \sqrt{\mathbf{N}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} \mathbf{M}^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}} \widehat{\mathbf{I}^a_{\alpha\beta}} - M^a_{\alpha\beta} = 0 \tag{121}$$

Therefore, $Y = Y^+$ is the boundary condition applied at exit, implying only outgoing (not ingoing) propagating waves and decaying evanescent waves in the outlet.

Now, we introduce a matrix W, =V with columns given by the eigevectors of $Y^{\pm}N$ with corresponding eigenvalue matrix $\pm\Lambda$:

$$Y^{\pm a}_{\alpha\beta} = \pm \widehat{\mathbf{I}^a_{\alpha\beta}} {N^a_{\beta\alpha}}^{-1} V^a_{\alpha\times\alpha} \Lambda^a_{\alpha\times\alpha} {V^a_{\alpha\times\alpha}}^{-1} \widehat{\mathbf{I}^a_{\alpha\beta}} \Rightarrow N^a_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} Y^{\pm a}_{\alpha\beta} \widehat{\mathbf{I}^a_{\beta\alpha}}^{-1} V^a_{\alpha\times\alpha} = \pm V^a_{\alpha\times\alpha} \Lambda^a_{\alpha\times\alpha}$$
$$Y^{\pm a}_{\alpha\beta} = \pm \widehat{\mathbf{I}^a_{\alpha\beta}} W^a_{\beta\times\beta} \Lambda^a_{\beta\times\beta} {W^a_{\beta\times\beta}}^{-1} {N^a_{\beta\alpha}}^{-1} \widehat{\mathbf{I}^a_{\alpha\beta}} \Rightarrow \tag{122}$$

We know that:

$$(\mathcal{A}[x,y])^a_\alpha = (\mathcal{A}^{ab}_{\alpha\beta\gamma}[x^{a-b}_\beta, y^b_\gamma])^a_\alpha = \sum_{b=-\infty}^{\infty}\sum_{\beta,\gamma=0}^{\infty} A^{ab}_{\alpha\beta\gamma}x^{a-b}_\beta y^b_\gamma$$

$$\underset{\sim\sim\sim\sim\sim}{(\mathcal{A}[X,Y])^{ab}_{\alpha\beta\gamma}} = (\mathcal{A}^{ab}_{\alpha\delta\epsilon}[X^{a-b}_{\delta\beta}, Y^b_{\epsilon\gamma}])^{ab}_{\alpha\beta\gamma} = \sum_{\delta,\epsilon=0}^{\infty} A^{ab}_{\alpha\delta\epsilon}X^{a-b}_{\delta\beta}Y^b_{\epsilon\gamma}$$

$$\begin{aligned} Thus, \{(\mathcal{A}[X1,Y1])^{ab}_{\alpha\beta\gamma}[X2,Y2]\}^{ab}_{\alpha\delta\epsilon} &= ((\mathcal{A}^{ab}_{\alpha\delta\epsilon}[X1^{a-b}_{\delta\beta}, Y1^b_{\epsilon\gamma}])^{ab}_{\alpha\beta\gamma}[X2^{a-b}_{\beta\delta}, Y2^b_{\gamma\epsilon}])^{ab}_{\alpha\delta\epsilon} \\ &= \sum_{\beta,\gamma=0}^{\infty}(\sum_{\delta,\epsilon=0}^{\infty} A^{ab}_{\alpha\delta\epsilon}X1^{a-b}_{\delta\beta}Y1^b_{\epsilon\gamma})X2^{a-b}_{\beta\delta}Y2^b_{\gamma\epsilon} \\ &= \sum_{\delta,\epsilon=0}^{\infty}\{A^{ab}_{\alpha\delta\epsilon}\sum_{\beta=0}^{\infty}(X1^{a-b}_{\delta\beta}X2^{a-b}_{\beta\delta})\sum_{\gamma=0}^{\infty}(Y1^b_{\epsilon\gamma}Y2^b_{\gamma\epsilon})\} \\ &= (\mathcal{A}^{ab}_{\alpha\delta\epsilon}[X1^{a-b}_{\delta\beta}X2^{a-b}_{\beta\delta}, Y1^b_{\epsilon\gamma}Y2^b_{\gamma\epsilon}])^{ab}_{\alpha\delta\epsilon} \end{aligned} \tag{123}$$

With $\varepsilon = 0, G = 0, H = 0, \mathcal{Y}' = 0$, fix points of the nonlinear admittance equation satisfy:

$$\begin{aligned} &\mathcal{Y}^{ab}_{\alpha\beta\gamma}[\mathbf{N}^{a-b}_{\beta\delta}\widehat{\mathbf{I}^{a-b}_{\delta\alpha}}^{-1}Y^{a-b}_{\alpha\delta}, I_{\gamma\epsilon}] + \mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, \mathbf{N}^b_{\gamma\delta}\widehat{\mathbf{I}^b_{\delta\alpha}}^{-1}Y^b_{\alpha\delta}] \\ &+Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, I_{\gamma\epsilon}] + Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}, I_{\gamma\epsilon}] \\ &-\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}Y^{\pm b}_{\alpha\gamma}] - \mathcal{B}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}, I_{\gamma\epsilon}] = 0 \end{aligned} \tag{124}$$

We apply $\{W^a_{\alpha\xi}\}^{-1}$ on the left of this equation and $V^a_{\alpha\beta}$ to the right on both terms in the square brackets:

$$\begin{aligned} &W^{-1}{}^a_{\xi\alpha}\mathcal{Y}^{\pm ab}_{\alpha\beta\gamma}[\mathbf{N}^{a-b}_{\beta\delta}\widehat{\mathbf{I}^{a-b}_{\delta\alpha}}^{-1}Y^{a-b}_{\alpha\delta}V, I_{\gamma\epsilon}V] + W^{-1}{}^a_{\xi\alpha}\mathcal{Y}^{\pm ab}_{\alpha\beta\gamma}[I_{\beta\delta}V, \mathbf{N}^b_{\gamma\delta}\widehat{\mathbf{I}^b_{\delta\alpha}}^{-1}Y^b_{\alpha\delta}V] \\ &+W^{-1}{}^a_{\xi\alpha}Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{Y}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}V, I_{\gamma\epsilon}V] + W^{-1}{}^a_{\xi\alpha}Y^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}V, I_{\gamma\epsilon}V] \\ &-W^{-1}{}^a_{\xi\alpha}\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}V, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}Y^{\pm b}_{\alpha\gamma}V] - W^{-1}{}^a_{\xi\alpha}\mathcal{B}^{ab}_{\alpha\beta\gamma}[I_{\beta\delta}V, I_{\gamma\epsilon}V] = 0 \end{aligned} \tag{125}$$

$$\begin{aligned} &W^{-1}{}^a_{\xi\alpha}\mathcal{Y}^{\pm ab}_{\alpha\beta\gamma}[\pm(V\Lambda)^{a-b}_{\beta\delta}, V^b_{\gamma\epsilon}] + W^{-1}{}^a_{\xi\alpha}\mathcal{Y}^{\pm ab}_{\alpha\beta\gamma}[V^{a-b}_{\beta\delta}, \pm(V\Lambda)^b_{\gamma\epsilon}] \\ &+W^{-1}{}^a_{\xi\alpha}Y^{\pm a}_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathbf{N}^a_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{Y}^{\pm ab}_{\alpha\beta\gamma}[V^{a-b}_{\beta\delta}, V^b_{\gamma\epsilon}] + W^{-1}{}^a_{\xi\alpha}Y^{\pm a}_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}V^{a-b}_{\beta\delta}, V^b_{\gamma\epsilon}] \\ &-W^{-1}{}^a_{\xi\alpha}\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1}Y^{\pm a-b}_{\alpha\beta}V^{a-b}_{\beta\delta}, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1}Y^{\pm b}_{\alpha\gamma}V^b_{\gamma\epsilon}] - W^{-1}{}^a_{\xi\alpha}\mathcal{B}^{ab}_{\alpha\beta\gamma}[V^{a-b}_{\beta\delta}, I_{\gamma\epsilon}V^b_{\gamma\epsilon}] = 0 \end{aligned} \tag{126}$$

Next, we transform $\mathcal{Y}^\pm$ in the following manner:

$$\mathcal{Y}^\pm[x,y] = W\widetilde{\mathcal{Y}}^\pm[V^{-1}x, V^{-1}y] \tag{127}$$

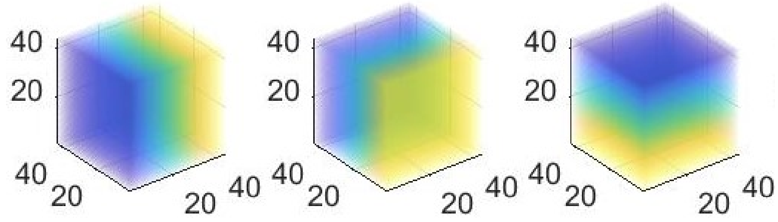


Fig. 1 $3D - model - lambda$

$$\begin{aligned}\widetilde{\mathcal{Y}}^{\pm ab}_{\alpha\beta\gamma}[\pm(\Lambda)^{a-b}_{\beta\delta}, I^b_{\gamma\epsilon}] + \widetilde{\mathcal{Y}}^{\pm ab}_{\alpha\beta\gamma}[I^{a-b}_{\beta\delta}, \pm(\Lambda)^b_{\gamma\epsilon}] \pm \Lambda_{\xi\alpha}\widetilde{\mathcal{Y}}^{\pm ab}_{\alpha\beta\gamma}[I^{a-b}_{\beta\delta}, I^b_{\gamma\epsilon}] \\ + {W^{-1}}^a_{\xi\alpha} Y^{\pm a}_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta}V^{a-b}_{\beta\delta}, V^b_{\gamma\epsilon}] \\ -{W^{-1}}^a_{\xi\alpha}\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta}V^{a-b}_{\beta\delta}, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1} Y^{\pm b}_{\alpha\gamma}V^b_{\gamma\epsilon}] - {W^{-1}}^a_{\xi\alpha}\mathcal{B}^{ab}_{\alpha\beta\gamma}[V^{a-b}_{\beta\delta}, I_{\gamma\epsilon}V^b_{\gamma\epsilon}] = 0\end{aligned} \tag{128}$$

$$\begin{aligned}\widetilde{\mathcal{Y}}^{\pm ab}_{\alpha\beta\gamma} = \frac{1}{\pm i\lambda^a_\alpha \pm i\lambda^{a-b}_\beta \pm i\lambda^b_\gamma}(-{W^{-1}}^a_{\xi\alpha} Y^{\pm a}_{\alpha\beta}\widehat{\mathbf{I}^a_{\beta\alpha}}^{-1}\mathcal{C}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta}V^{a-b}_{\beta\delta}, V^b_{\gamma\epsilon}] \\ +{W^{-1}}^a_{\xi\alpha}\mathcal{A}^{ab}_{\alpha\beta\gamma}[\widehat{\mathbf{I}^{a-b}_{\beta\alpha}}^{-1} Y^{\pm a-b}_{\alpha\beta}V^{a-b}_{\beta\delta}, \widehat{\mathbf{I}^b_{\gamma\alpha}}^{-1} Y^{\pm b}_{\alpha\gamma}V^b_{\gamma\epsilon}] + {W^{-1}}^a_{\xi\alpha}\mathcal{B}^{ab}_{\alpha\beta\gamma}[V^{a-b}_{\beta\delta}, I_{\gamma\epsilon}V^b_{\gamma\epsilon}])^{ab}_{\alpha\beta\gamma}\end{aligned} \tag{129}$$

VIII. Separating the Ψ intergrals into radial and angular parts

With $\int_0^{2\pi} e^{im\phi}e^{-im\theta}d\theta = 2\pi$, $\psi_{\alpha_{m\mu}}(r) = C_{\alpha_{m\mu}}J_m(\frac{{j'}_{m\mu}r}{h})e^{im\phi}$,we have:

$$\begin{aligned}C_{\alpha_{m\mu}} &= \frac{(i)^m}{\sqrt{(\pi h^2(1-\frac{m^2}{{j'}^2_{m\mu}})J^2_m({j'}_{m\mu}))}} \\ except for : C_{\alpha_{01}} &= \frac{1}{\sqrt{\pi}h}\end{aligned} \tag{130}$$

$$\psi_{\alpha_{m\mu}}(r) = C_{\alpha_{m\mu}}J_m(\frac{{j'}_{m\mu}r}{h})e^{im\phi} \tag{131}$$

$$\begin{aligned}\Psi_{[\alpha](\beta)\gamma}[r(1-\kappa r cos\phi)] &= \int_0^{2\pi}\int_0^h [\frac{\partial\psi_\alpha}{\partial r}][\frac{\partial\psi_\beta}{\partial\theta}]\psi_\gamma[r(1-\kappa r cos\phi)]drd\theta \\ &= \mathcal{X}_{[\alpha]\beta\gamma}[r]\Theta_{\alpha(\beta)\gamma} - \kappa\mathcal{X}_{[\alpha]\beta\gamma}[r^2]\Theta_{\alpha(\beta)\gamma}[cos\phi]\end{aligned} \tag{132}$$

with:

$$\begin{aligned}\mathcal{X}_{[\alpha]\beta\gamma} &= \int_0^h \frac{d}{dr}(C_{\alpha_{m\mu}}J_m(\frac{j'_{\alpha_{m\mu}}r}{h}))C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})C_{\gamma_{kw}}J_k(\frac{j'_{\gamma_{kw}}r}{h})dr \\ \Theta_{\alpha(\beta)\gamma} &= \int_0^{2\pi} e^{im\phi}\frac{d}{d\theta}(e^{in\phi})e^{ik\phi}d\theta\end{aligned} \tag{133}$$

Bessel function recurrence relations are:

$$J_{m-1}(x) + J_{m+1}(x) = 2m/xJ_m(x)$$
$$J_{m-1}(x) - J_{m+1}(x) = 2J'_m(x) \tag{134}$$

we can have:

$$2J_{m+1}(x) = 2m/xJ_m(x) - 2J'_m(x) \rightarrow$$
$$J'_m(x) = m/xJ_m(x) - J_{m+1}(x) \tag{135}$$

$$\begin{aligned} \mathcal{X}_{[\alpha]\beta}[r] &= \int_0^h \frac{d}{dr}(C_{\alpha_{m\mu}} J_m(\frac{j'_{\alpha_{m\mu}}r}{h}))C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}}r}{h})rdr \\ = \int_0^h \frac{j'_{\alpha_{m\mu}}}{h} C_{\alpha_{m\mu}}[\frac{m}{j'_{\alpha_{m\mu}}r/h} J_m(\frac{j'_{\alpha_{m\mu}}r}{h}) &- J_{m+1}(\frac{j'_{\alpha_{m\mu}}r}{h})]C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}}r}{h})rdr \\ = C_{\alpha_{m\mu}} C_{\beta_{nv}} \int_0^h [mJ_m(\frac{j'_{\alpha_{m\mu}}r}{h}) &- \frac{j'_{\alpha_{m\mu}}r}{h} J_{m+1}(\frac{j'_{\alpha_{m\mu}}r}{h})]J_n(\frac{j'_{\beta_{nv}}r}{h})dr \end{aligned} \tag{136}$$

The Θ intergrals can be calculated analytically:

$$\Theta_{\alpha\beta} = \int_0^{2\pi} e^{im\phi} e^{in\phi} d\theta = \begin{cases} 0 \\ 2\pi, m+n=0 \end{cases} = 2\pi\delta_{m,-n} \tag{137}$$

With Euler's equation:

$$e^{ix} = cosx + isinx$$
$$cosx = [e^{ix} + e^{-ix}]/2 \tag{138}$$
$$sinx = [e^{ix} - e^{-ix}]/2i$$

$$\begin{aligned} \Theta_{\alpha\beta}[cos\phi] &= \int_0^{2\pi} cos(\theta-\theta_0) e^{im\phi} e^{in\phi} d\theta = 1/2 \int_0^{2\pi} [e^{i(\theta-\theta_0)} + e^{-i(\theta-\theta_0)}] e^{im\phi} e^{in\phi} d\theta \\ &= 1/2 \int_0^{2\pi} e^{i(\theta-\theta_0)} e^{im\phi} e^{in\phi} d\theta + 1/2 \int_0^{2\pi} [e^{-i(\theta-\theta_0)}] e^{im\phi} e^{in\phi} d\theta \\ &= \pi e^{-i(1+m+n)\theta_0} \delta_{m,-n-1} + \pi e^{i(m+n-1)\theta_0} \delta_{m,-n+1} \\ &= \pi\delta_{m,-n-1} + \pi\delta_{m,-n+1} \end{aligned} \tag{139}$$

$$\Theta_{(\alpha)\beta} = \int_0^{2\pi} \frac{\partial}{\partial\theta}(e^{im\phi}) e^{in\phi} d\theta = 2\pi im\delta_{m,-n} \tag{140}$$

$$\Theta_{(\alpha)\beta}[cos\phi] = im[\pi\delta_{m,-n-1} + \pi\delta_{m,-n+1}] \tag{141}$$

$$\Theta_{\alpha\beta\gamma} = \int_0^{2\pi} e^{im\phi}e^{in\phi}e^{ik\theta}d\theta = 2\pi\delta_{m,-n,-k} \tag{142}$$

$$\Theta_{\alpha\beta\gamma}[cos\phi] = 1/2\int_0^{2\pi}[e^{i(\theta-\theta_0)} + e^{-i(\theta-\theta_0)}]e^{im\phi}e^{in\phi}e^{ik\theta}d\theta = \pi\delta_{m,-n,-k-1} + \pi\delta_{m,-n,-k+1} \tag{143}$$

$$\Theta_{\alpha\beta\gamma}[sin\phi] = 1/(2i)\int_0^{2\pi}[e^{i(\theta-\theta_0)} - e^{-i(\theta-\theta_0)}]e^{im\phi}e^{in\phi}e^{ik\theta}d\theta = -i[\pi\delta_{m,-n,-k-1} - \pi\delta_{m,-n,-k+1}] \tag{144}$$

$$\Theta_{(\alpha)\beta\gamma} = im\int_0^{2\pi} e^{im\phi}e^{in\phi}e^{ik\phi}d\theta = 2\pi im\delta_{m,-n,-k} \tag{145}$$

$$\Theta_{(\alpha)\beta\gamma}[cos\phi] = 1/2\int_0^{2\pi}[e^{i(\theta-\theta_0)} + e^{-i(\theta-\theta_0)}]e^{im\phi}e^{in\phi}e^{ik\theta}d\theta = im[\pi\delta_{m,-n,-k-1} + \pi\delta_{m,-n,-k+1}] \tag{146}$$

We also have:

$$\Psi_{\{\alpha\}\beta} = \mathcal{X}_{\{\alpha\}\beta}\Theta_{\alpha\beta} + \mathcal{X}_{\alpha\beta}\Theta_{\{\alpha\}\beta} \tag{147}$$

with:

$$\begin{aligned}
\mathcal{X}_{\{\alpha\}\beta} &= \int_0^h \frac{d}{ds}(C_{\alpha_{m\mu}}(s)J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}))C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr \\
&= \int_0^h[(\frac{dC_{\alpha_{m\mu}}(s)}{ds})J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}) + C_{\alpha_{m\mu}}J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)})\frac{-j'_{\alpha_{m\mu}}r}{h(s)^2}\frac{dh(s)}{ds}]C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr \\
&= \int_0^h[(\frac{d\frac{1}{h(s)}}{ds})C_{\alpha_{m\mu}}(s)J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}) + C_{\alpha_{m\mu}}J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)})\frac{-j'_{\alpha_{m\mu}}r}{h(s)^2}\frac{dh(s)}{ds}]C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr \\
&= \int_0^h[(-\frac{h'(s)}{h})C_{\alpha_{m\mu}}(s)J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}) - C_{\alpha_{m\mu}}\frac{\partial}{\partial r}(J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}))\frac{h'(s)}{h}r]C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr \\
&= -\frac{h'}{h}(\mathcal{X}_{[\alpha]\beta}[r] + \mathcal{X}_{\alpha\beta})
\end{aligned} \tag{148}$$

$$
\begin{aligned}
\mathcal{X}_{\{\alpha\}\beta\gamma} &= \int_0^h \frac{d}{ds}(C_{\alpha_{m\mu}}(s)J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}))C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})C_{\gamma_{kw}}J_n(\frac{j'_{\gamma_{kw}}r}{h})dr \\
&= \int_0^h [(\frac{dC_{\alpha_{m\mu}}(s)}{ds})J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}) + C_{\alpha_{m\mu}}J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)})\frac{-j'_{\alpha_{m\mu}}r}{h(s)^2}\frac{dh(s)}{ds}]C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})C_{\gamma_{kw}}J_n(\frac{j'_{\gamma_{kw}}r}{h})dr \\
&= \int_0^h [(\frac{d\frac{1}{h(s)}}{ds})C_{\alpha_{m\mu}}(s)J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}) + C_{\alpha_{m\mu}}J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)})\frac{-j'_{\alpha_{m\mu}}r}{h(s)^2}\frac{dh(s)}{ds}]C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})C_{\gamma_{kw}}J_n(\frac{j'_{\gamma_{kw}}r}{h})dr \\
&= \int_0^h [(-\frac{h'(s)}{h})C_{\alpha_{m\mu}}(s)J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}) - C_{\alpha_{m\mu}}\frac{\partial}{\partial r}(J_m(\frac{j'_{\alpha_{m\mu}}r}{h(s)}))\frac{h'(s)}{h}r]C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})C_{\gamma_{kw}}J_n(\frac{j'_{\gamma_{kw}}r}{h})dr \\
&= -\frac{h'}{h}(\mathcal{X}_{[\alpha]\beta\gamma}[r] + \mathcal{X}_{\alpha\beta\gamma})
\end{aligned} \tag{149}
$$

$$
\begin{aligned}
\Theta_{\{\alpha\}\beta} &= \int_0^{2\pi}\frac{\partial}{\partial s}(e^{im\phi})e^{in\phi}d\theta = -im\int_0^{2\pi}\frac{\partial\theta_0(s)}{\partial s}e^{im\phi}e^{in\phi}d\theta \\
&= -\tau\Theta_{(\alpha)\beta} = -\tau 2\pi im\delta_{m,-n}
\end{aligned} \tag{150}
$$

Similarly,

$$
\frac{d}{ds}(\Psi_{\alpha\beta\gamma}[r]) = \frac{d}{ds}(\mathcal{X}_{\alpha\beta\gamma}[r])\Theta_{\alpha\beta\gamma} \tag{151}
$$

with

$$
\begin{aligned}
\frac{d}{ds}(\mathcal{X}_{\alpha\beta\gamma}[r]) &= \mathcal{X}_{\{\alpha\}\beta\gamma}[r] + \mathcal{X}_{\alpha\{\beta\}\gamma}[r] + \mathcal{X}_{\alpha\beta\{\gamma\}}[r] \\
&= -\frac{h'}{h}(\mathcal{X}_{[\alpha]\beta\gamma}[r^2] - \frac{h'}{h}(\mathcal{X}_{\alpha[\beta]\gamma}[r^2] - \frac{h'}{h}(\mathcal{X}_{\alpha\beta[\gamma]}[r^2] - 3\frac{h'}{h}\mathcal{X}_{\alpha\beta\gamma}[r])
\end{aligned} \tag{152}
$$

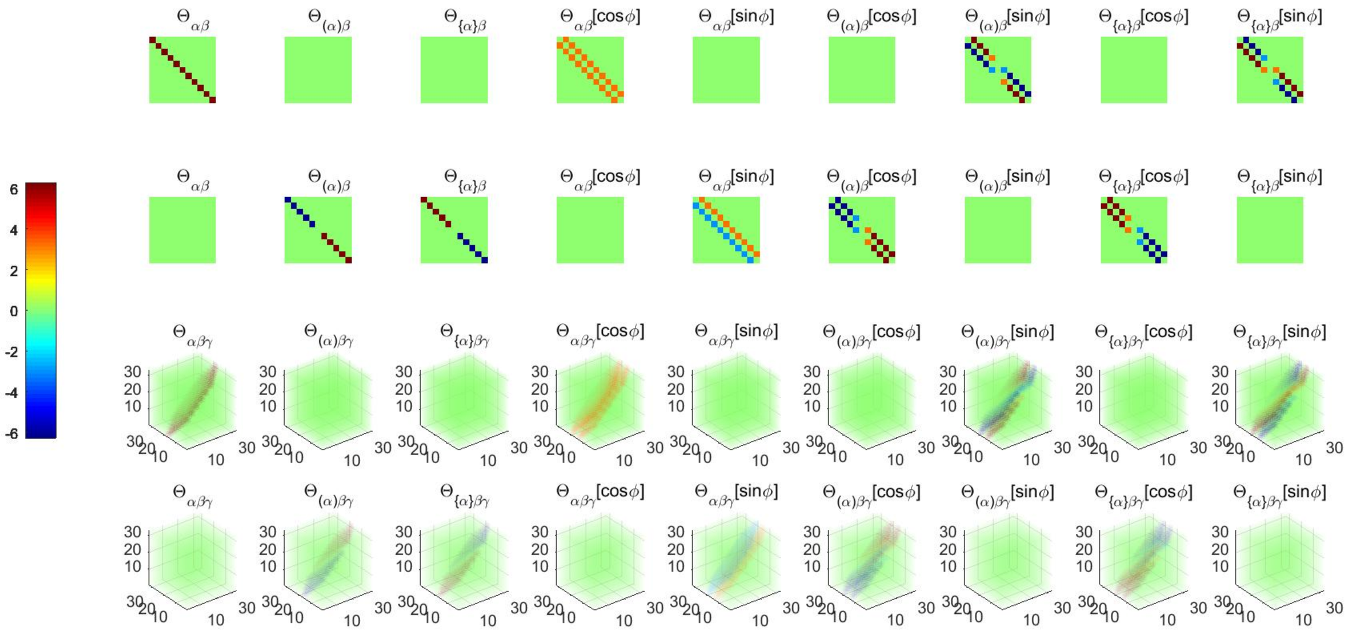


Fig. 2 Θ

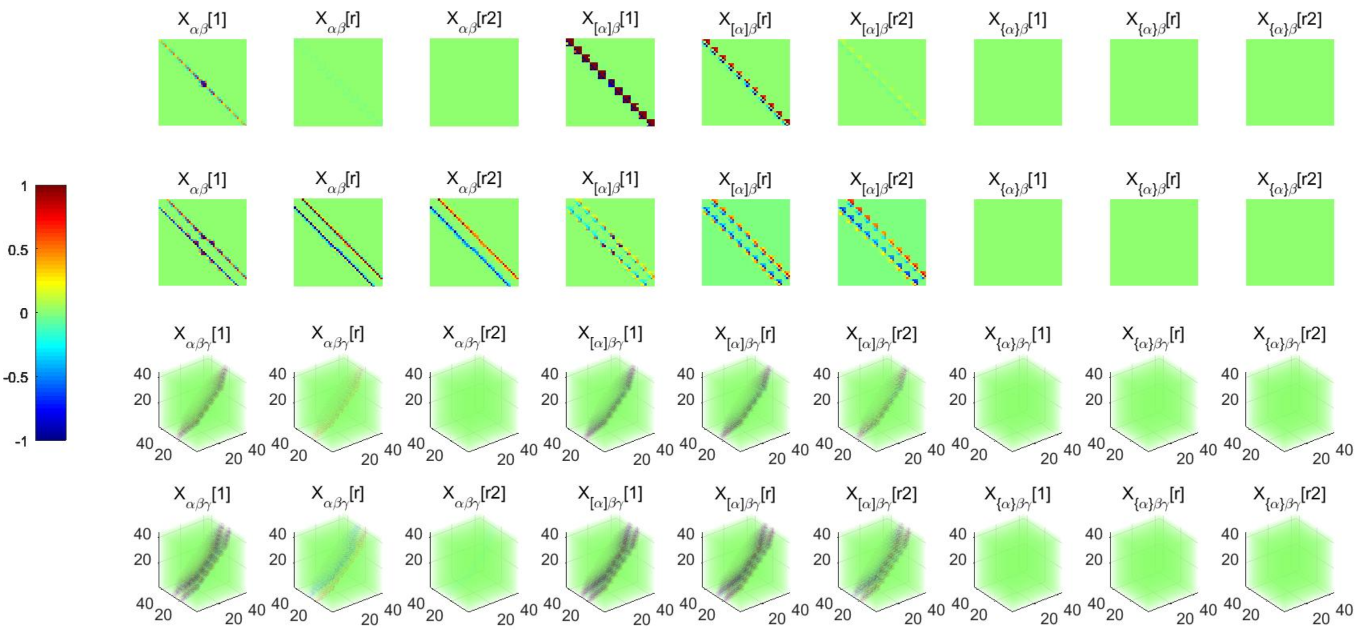
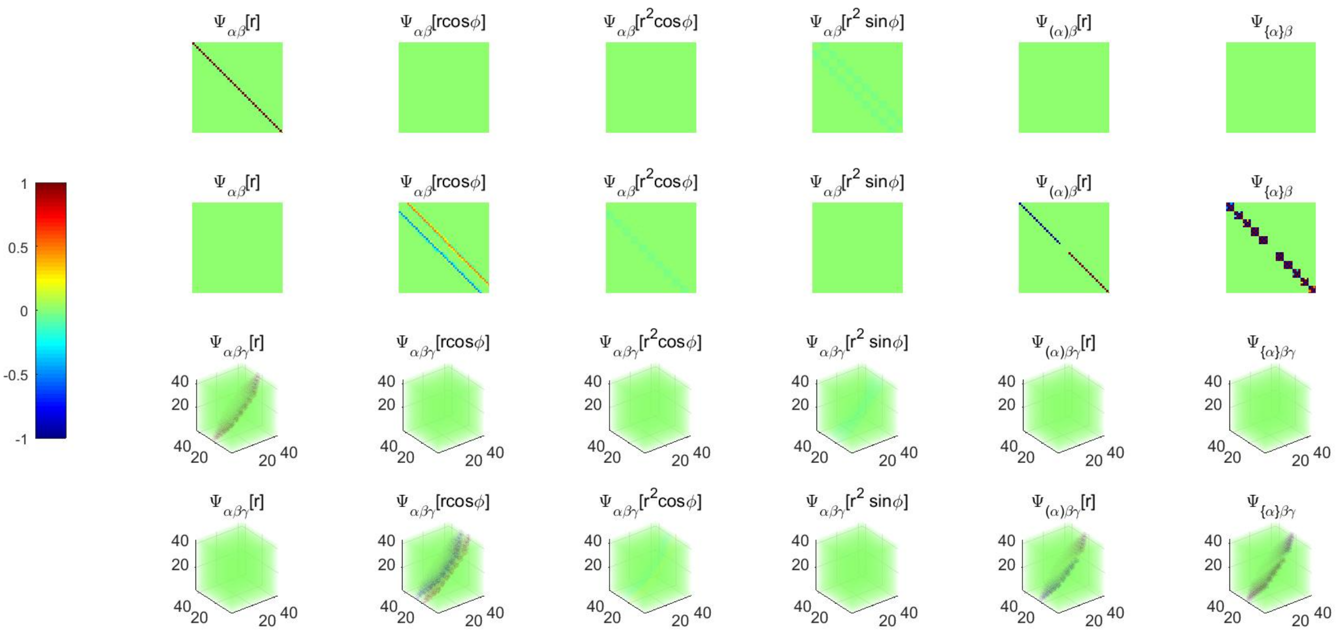


Fig. 3 X_Θ



$\Psi_{\alpha\beta}$[r]
$\Psi_{\alpha\beta}$[rcosϕ]
$\Psi_{\alpha\beta}$[r^{2}cosϕ]
$\Psi_{\alpha\beta}$[r^2 sinϕ]
$\Psi_{(\alpha)\beta}$[r]
$\Psi_{\{\alpha\}\beta}$
$\Psi_{\alpha\beta\gamma}$[r]
$\Psi_{\alpha\beta\gamma}$[rcosϕ]
$\Psi_{\alpha\beta\gamma}$[r^{2}cosϕ]
$\Psi_{\alpha\beta\gamma}$[r^2 sinϕ]
$\Psi_{(\alpha)\beta\gamma}$[r]
$\Psi_{\{\alpha\}\beta\gamma}$
1
0.5
0
-0.5
-1
40
20

Fig. 4 Ψ

A. Example for matlab simulation

B. Example for matlab simulation

C. Example for matlab simulation

$$
\begin{aligned}
Fig1: \Psi_{\alpha\beta}[r] = \mathcal{X}_{\alpha\beta}[r]\Theta_{\alpha\beta} := \int_0^h rC_{\alpha_{mu}}(s)J_m(\frac{j'_{\alpha_{mu}}r}{h})C_{\beta_{nv}}(scc)J_n(\frac{j'_{\beta_{nv}}r}{h})dr(2\pi\delta_{m,-n}) \\
Fig2: \Psi_{\alpha\beta}[rcos\phi] = \mathcal{X}_{\alpha\beta}[r]\Theta_{\alpha\beta}[cos\phi] := \int_0^h rC_{\alpha_{mu}}J_m(\frac{j'_{\alpha_{mu}}r}{h})C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr(\pi\delta_{m,-n-1} + \pi\delta_{m,-n+1}) \\
Fig3: \Psi_{\alpha\beta}[r^2cos\phi] = \mathcal{X}_{\alpha\beta}[r^2]\Theta_{\alpha\beta}[cos\phi] := \int_0^h r^2C_{\alpha_{mu}}J_m(\frac{j'_{\alpha_{mu}}r}{h})C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr(\pi\delta_{m,-n-1} + \pi\delta_{m,-n+1}) \\
Fig4: \Psi_{\alpha\beta}[r^2sins\phi] = \mathcal{X}_{\alpha\beta}[r^2]\Theta_{\alpha\beta}[sin\phi] := \int_0^h r^2C_{\alpha_{mu}}J_m(\frac{j'_{\alpha_{mu}}r}{h})C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr(-i(\pi\delta_{m,-n-1} - \pi\delta_{m,-n+1})) \\
Fig5: \Psi_{(\alpha)\beta}[r] = \mathcal{X}_{\alpha\beta}[r]\Theta_{(\alpha)\beta} := \int_0^h rC_{\alpha_{mu}}J_m(\frac{j'_{\alpha_{mu}}r}{h})C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr(2\pi im\delta_{m,-n}) \\
Fig6: \Psi_{\{\alpha\}\beta} = \mathcal{X}_{\{\alpha\}\beta}\Theta_{\alpha\beta} + \mathcal{X}_{\alpha\beta}\Theta_{\{\alpha\}\beta} \\
\mathcal{X}_{\{\alpha\}\beta} := -\frac{h'}{h}(\mathcal{X}_{[\alpha]\beta}[r] + \mathcal{X}_{\alpha\beta}) \\
\mathcal{X}_{[\alpha]\beta}[r] := \int_0^h r\frac{d}{dr}(C_{\alpha_{m\mu}}J_m(\frac{j'_{\alpha_{m\mu}}r}{h}))C_{\beta_{nv}}J_n(\frac{j'_{\beta_{nv}}r}{h})dr \\
\Theta_{\{\alpha\}\beta} = \int_0^{2\pi} \frac{\partial}{\partial s}(e^{im\phi})e^{in\phi}d\theta = -\tau 2\pi im\delta_{m,-n}
\end{aligned}
\tag{153}
$$

IX. Tensors in matlab for numerical simulation

A. Tensor times vectors: $\mathcal{A}\bar{\times}_n u$

Let $\mathcal{A}$ be a tensor of size $I_1 \times I_2 \times ... \times I_N$, u be a vector of size I_n.

We have:

$$
\begin{aligned}
ttv(\mathcal{A}, \{u\}, [n]) = (\mathcal{A}\bar{\times}_n u)(i_1, ..., i_{n-1}, i_{n+1}, ..., i_N) \\
\sum_{i_n=1}^{I_n} \mathcal{A}(i_1, i_2, ..., i_N)u(i_n)
\end{aligned}
\tag{154}
$$

$$
\begin{aligned}
ttv(A_{m\times n}, \{u_{m\times 1}\}, [1]) = A_{m\times n}\bar{\times}_1 u_{m\times 1} = A^T_{m\times n}u_{m\times 1} \\
ttv(A_{m\times n}, \{v_{n\times 1}\}, [2]) = A_{m\times n}\bar{\times}_2 v_{n\times 1} = A_{m\times n}v_{n\times 1}
\end{aligned}
\tag{155}
$$

Property:

$$
\begin{aligned}
ttv(\mathcal{A},\{u,v\},[m,n]) &= \mathcal{A}\bar{\times}_m u\bar{\times}_n v \\
= ttv(ttv(\mathcal{A},\{u\},[m]),\{v\},[n-1]) &= (\mathcal{A}\bar{\times}_m u)\bar{\times}_{n-1} v \\
= ttv(ttv(\mathcal{A},\{v\},[v]),\{u\},[m]) &= (\mathcal{A}\bar{\times}_n v)\bar{\times}_m u
\end{aligned} \tag{156}
$$

Multiplication with a sequence of vectors

$$
\begin{aligned}
\beta = \mathcal{A}\bar{\times}_1 u^{(1)}\bar{\times}_2 u^{(2)}...\bar{\times}_N u^{(N)} = \mathcal{A}\bar{\times}u \\
like: ttv(X,\{A,B,C,D\}) = ttv(X,\{A,B,C,D\},[1234]) = ttv(X,\{D,C,B,A\},[4321])
\end{aligned} \tag{157}
$$

Multiplication with **all but one** of a sequence of vectors

$$
\begin{aligned}
b = \mathcal{A}\bar{\times}_1 u^{(1)}\bar{\times}_2 u^{(2)}...\bar{\times}_{n-1} u^{(2)}\bar{\times}_{n+1} u^{(2)}...\bar{\times}_N u^{(N)} = \mathcal{A}\bar{\times}_{-n} u \\
like: X = tenrand([5,3,4,2]); \\
A = rand(5,1); B = rand(3,1); C = rand(4,1); D = rand(2,1); \\
Y = ttv(X,\{A,B,D\},-3) = ttv(X,\{A,B,C,D\},-3)
\end{aligned} \tag{158}
$$

B. Tensor times matrix (ttm): $\mathcal{A}\times_n u$

Let $\mathcal{A}$ be a tensor of size $I_1 \times I_2 \times ... \times I_N$, U be a matrix of size $J_n \times I_n$.

We have:

$$
\begin{aligned}
ttm(\mathcal{A},\{U\},[n]) = (\mathcal{A}\times_n U)(i_1,...,i_{n-1},j_n,i_{n+1},...,i_N) \\
\sum_{i_n=1}^{I_n} \mathcal{A}(i_1,i_2,...,i_N)U(j_n,i_n) \\
like: X = tensor(rand(5,3,4,2)); A = rand(4,5); \\
Y = ttm(X,A,1) = ttm(X,\{A,B,C,D\},1) = ttm(X,A',1,'t')
\end{aligned} \tag{159}
$$

Matrix Interpretation

$$
\begin{aligned}
ttm(A_{m\times n},\{U^T_{m\times k}\},[1]) = A\times_1 U^T = U^T A \\
ttm(A_{m\times n},\{V^T_{m\times k}\},[2]) = A\times_2 V^T = AV \\
ttm(A,\{U,V\},[1,2]) = UAV^T
\end{aligned} \tag{160}
$$

$$Y = ttm(X, A, B, C, D, [1234]); \% <-- 4 - waymutliply.$$

$$Y = ttm(X, D, C, B, A, [4321]); \% <-- Sameasabove.$$

$$Y = ttm(X, A, B, C, D); \% <-- Sameasabove.$$

$$Y = ttm(X, A', B', C', D',' t')\% <-- Sameasabove.$$

$$\begin{aligned} Y = ttm(X, C, D, [34]); \% <-- XtimesCinmode - 3Dinmode - 4 \\ Y = ttm(X, A, B, C, D, [34])\% <-- Sameasabove. \end{aligned} \tag{161}$$

$$Y = ttm(X, A, B, D, [124]); \% <-- 3 - waymultiply.$$

$$Y = ttm(X, A, B, C, D, [124]); \% <-- Sameasabove.$$

$$Y = ttm(X, A, B, D, -3); \% <-- Sameasabove.$$

$$Y = ttm(X, A, B, C, D, -3)\% <-- Sameasabove.$$

Property

$$\begin{aligned} ttm(\mathcal{A}, \{u, v\}, [m, n]) = \mathcal{A} \times_m u \bar{\times}_n v \\ = ttm(ttm(\mathcal{A}, \{u\}, [m]), \{v\}, [n]) = (\mathcal{A} \times_m u) \times_n v \\ = ttm(ttm(\mathcal{A}, \{v\}, [v]), \{u\}, [m]) = (\mathcal{A} \times_n v) \times_m u \end{aligned} \tag{162}$$

C. Tensor times tensor (ttt): $< \mathcal{A}, \mathcal{B} >$

Let $\mathcal{A}$ and $\mathcal{B}$ be a tensor of size $I_1 \times I_2 \times ... \times I_N$.

$$\begin{aligned} < \mathcal{A}, \mathcal{B} >= \\ \beta = \sum_{i_1=1}^{I_1} \sum_{i_1=1}^{I_2} ... \sum_{i_1=1}^{I_N} \mathcal{A}(i_1, i_2, ..., i_N) \mathcal{B}(i_1, i_2, ..., i_N) \end{aligned} \tag{163}$$

$$X = tensor(rand(4,2,3)); Y = tensor(rand(3,4,2));$$

$$Z = ttt(X,Y); \% <-- OuterproductofXandY.$$

$$size(Z)$$

$$Z = ttt(X,X,1:3)\% <-- InnerproductofXwithitself.$$

$$(164)$$

$$Z = ttt(X,Y,[123],[231])\% <-- InnerproductofXY.$$

$$Z = innerprod(X, permute(Y,[231]))\% <-- Sameasabove.$$

$$Z = ttt(X,Y,[13],[21])\% <-- ProductofXYalongspecifieddims.$$

X. model of helical duct

A. w

The duct is described by its centreline $\mathbf{q}(s)$ at arclength s from the inlet of the duct adn the radial distance from the centreline $h = h(s)$. The general position vector (x) in the duct is given in terms of (s, r, θ):

$$\mathbf{x} = \mathbf{q}(s) + rcos(\theta - \theta_0)\widehat{\mathbf{n}} + rsin(\theta - \theta_0)\widehat{\mathbf{b}} \tag{165}$$

where $\widehat{\mathbf{n}} = \widehat{\mathbf{n}}(s)$ is the normal to the centreline and $\widehat{\mathbf{b}} = \widehat{\mathbf{b}}(s)$ id the binormal given by $\widehat{\mathbf{b}} = \widehat{\mathbf{t}} \times \widehat{\mathbf{n}}$ for the tangent to the centreline $\widehat{\mathbf{t}} = \widehat{\mathbf{t}}(s)$. The vector $\widehat{\mathbf{n}}$,$\widehat{\mathbf{b}}$ and $\widehat{\mathbf{t}}$ are related by the Frenet-Serret formulas:

$$\frac{d\widehat{\mathbf{q}}}{ds} = \widehat{\mathbf{t}}, \quad \frac{d\widehat{\mathbf{t}}}{ds} = \kappa\widehat{\mathbf{n}}, \quad \frac{d\widehat{\mathbf{n}}}{ds} = -\kappa\widehat{\mathbf{t}} + \tau\widehat{\mathbf{b}}, \quad \frac{d\widehat{\mathbf{b}}}{ds} = -\tau\widehat{\mathbf{n}} \tag{166}$$

where $\kappa = \kappa(s)$ is the local curvature of the duct and $\tau = tau(s)$ is the torsion. Here, intorduce

$\theta_0' = \tau$, the cross-term differentials vanish and the metric reduces:

$$\begin{aligned} d\mathbf{x} &= d(\mathbf{q}(s)) + d(rcos(\theta-\theta_0)\widehat{\mathbf{n}}) + d(rsin(\theta-\theta_0)\widehat{\mathbf{b}}) \\ &= \widehat{\mathbf{t}}ds + drcos\phi\widehat{\mathbf{n}} - r\widehat{\mathbf{n}}sin\phi(d\theta-\tau ds) + rcos\phi(-\kappa\widehat{\mathbf{t}}+\tau\widehat{\mathbf{b}})ds \\ &\quad + drsin\phi\widehat{\mathbf{b}} + r\widehat{\mathbf{b}}cos\phi(d\theta-\tau ds) + rsin\phi(-\tau\widehat{\mathbf{n}})ds \\ &= \widehat{\mathbf{t}}(1-\kappa rcos\phi)ds + \widehat{\mathbf{n}}(drcos\phi - rsin\phi d\theta) + \widehat{\mathbf{b}}(drsin\phi + rcos\phi d\theta) \end{aligned} \tag{167}$$

Thus,

$$\begin{aligned} d\mathbf{x}\cdot d\mathbf{x} &= (1-\kappa rcos\phi)^2ds^2 + (drcos\phi - rsin\phi d\theta)^2 + (drsin\phi + rcos\phi d\theta)^2 \\ &= (1-\kappa rcos\phi)^2ds^2 + dr^2 + r^2d\theta \end{aligned} \tag{168}$$

As a result, we have an orthogonal coordinate system and as such do not need to distinguish between covariant and contravariant bases.

XI. funm-Evaluate general matrix function

https://ww2.mathworks.cn/help/matlab/ref/funm.html