Note:Governing Euqations of General 3D duct flow

Email: jiaqi_wang@sjtu.edu.cn

I. Hard-walled cylinderical ducts as basis function

A. Infinite straight duct mode

We began from the Helmholtz equation:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = -\alpha^2\psi \tag{1}$$

Using separation of variables, Circular symmetry: modes have the from : $\psi = F(r)G(\theta)$,

Then we have:

$$\begin{gathered}(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r})G + \frac{F}{r^2}\frac{\partial^2 G}{\partial\theta^2} = -\alpha^2 FG \\ Then, \\ \frac{(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r})}{F} + \frac{1}{r^2}\frac{\frac{\partial^2 G}{\partial\theta^2}}{G} = -\alpha^2\end{gathered} \tag{2}$$

We assume that:

Due to periodicity, we require that Φ satisfy,

$$\frac{d^2G}{d\theta^2} = -m^2G \to \Phi(\theta) = e^{\pm im\theta} \tag{3}$$

Thus, we have

$$F'' + \frac{1}{r}F' + (\alpha^2 - \frac{m^2}{r^2}) = 0 \to F(r) = J_m(\alpha r) \tag{4}$$

Circular symmetry $\psi = F(r)G(\theta)$: modes explicitly given by:

$$\psi = J_m(\alpha_{m\mu}r)e^{\pm im\theta} \tag{5}$$

Hard walls:

$$J'_m(\alpha R) = 0 \to \alpha_{m\mu} = \frac{j'_{m\mu}}{R} \tag{6}$$

Soft walls without flow:

$$Z\alpha_{m\mu}J'_m(\alpha_{m\mu}R) = -iw\rho_0 J_m(\alpha_{m\mu}R) \to \alpha_{m\mu}(Z) \tag{7}$$

Soft walls with flow:

$$Z\alpha_{m\mu}J'_m(\alpha_{m\mu}R) = (w - U_0\kappa_{m\mu})J_m(\alpha_{m\mu}R) \to \alpha_{m\mu}(Z) \tag{8}$$

A complete solution may be writtern as:

$$p(x, r, \theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}x} + B_{m\mu}e^{i\kappa_{m\mu}x})U_{m\mu}(r)e^{im\theta} \tag{9}$$

In a hard-walled duct $U_{m\mu}e^{-im\theta}$ are orthogonal. Normalise such that:

$$\int_0^{2\pi}\int_0^R U_{m\mu}(r)e^{-im\theta}U_{nv}(r)e^{-in\theta}rdr = 2\pi\delta_{\mu v}\delta_{mn} \tag{10}$$

Source expansion If $p(0, t, \theta) = p_0(r, \theta)$ is source in hard-walled duct, then for x>0

$$\begin{aligned}
p_0(r,\theta) &= \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta}\\
p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}r} &= \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}r}\\
\underline{\int_0^{2\pi}\int_0^R} p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}}r\underline{drd\theta} &= \underline{\int_0^{2\pi}\int_0^R}\sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}}r\underline{drd\theta}\\
\underline{\int_0^{2\pi}\int_0^R} p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}}r\underline{drd\theta} &= \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}A_{m\mu}\underline{\int_0^{2\pi}\int_0^R}U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}}r\underline{drd\theta}\\
A_{nv} &= \frac{1}{2\pi}\underline{\int_0^{2\pi}\int_0^R} p_0(r,\theta)\underline{U_{nv}(r)e^{-in\theta}}r\underline{drd\theta}
\end{aligned} \tag{11}$$

and $B_{nv} = 0$. The same for $x < 0$ with A_{nv} and B_{nv} interchanged.

A finite number of modes (cut-on modes) survive at large distrances. Just 1 mode if kR«1: only A_{01} important.

B. General duct mode

The pressure and velocity can now be expressed as Fourier series. Upper indices shall be used to denote temporal decompositions:

$$
\begin{aligned}
\widehat{p} &= \sum_{a=-\infty}^{\infty} P^a(\mathbf{x})e^{-iawt} \\
\widehat{u} &= \sum_{a=-\infty}^{\infty} U^a(\mathbf{x})e^{-iawt} \\
\widehat{v} &= \sum_{a=-\infty}^{\infty} V^a(\mathbf{x})e^{-iawt} \\
\widehat{w} &= \sum_{a=-\infty}^{\infty} W^a(\mathbf{x})e^{-iawt}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
P^a &= \sum_{\alpha=0}^{\infty} P^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} P^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) \\
U^a &= \sum_{\alpha=0}^{\infty} U^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} U^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) \\
V^a &= \sum_{\alpha=0}^{\infty} V^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} V^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) \\
W^a &= \sum_{\alpha=0}^{\infty} W^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty} W^a_{\alpha_{m\mu}}(s)\psi_{\alpha_{m\mu}}(s,r,\theta)
\end{aligned}
\tag{13}
$$

A solution of ψ may have the form the same as the hard walls modes:

$$
\psi_{m\mu}(r) = C_{\alpha_{m\mu}} J_m(\frac{j'_{m\mu} r}{h})e^{im\theta} \tag{14}
$$

where should be normalized according to:

$$
\int_0^{2\pi}\int_0^h \psi_{\alpha_{m\mu}}\psi_{\beta_{nv}} r dr d\theta = \delta_{\mu v}\delta_{mn} \tag{15}
$$

C. Normalised Modes$\rightarrow C_{\alpha_{m\mu}}$

Relation involving intergrals:

$$
\begin{aligned}
&\underline{2\int \alpha^2 x J_m(\alpha x)^2 dx = (\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2} \\
&\rightarrow 2\int_0^h \alpha^2 x J_m(\alpha x)^2 dx = [(\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2]|_0^h \\
&= [(\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2]|_h - [(\alpha^2x^2 - m^2)J_m(\alpha x)^2 + \alpha^2x^2J'_m(\alpha x)^2]|_0 \\
&= [(\alpha^2h^2 - m^2)J_m(\alpha h)^2 + \alpha^2h^2J'_m(\alpha h)^2] - [(\alpha^2 0^2 - m^2)J_m(\alpha 0)^2 + \alpha^2 0^2 J'_m(\alpha 0)^2] \\
&= (\alpha^2h^2 - m^2)J_m(\alpha h)^2 + \alpha^2h^2J'_m(\alpha h)^2
\end{aligned}
\tag{16}
$$

With hard-walled boudary condition:

$$
J'_m(\alpha h) = 0 \rightarrow \alpha_{m\mu} = \frac{j'_{m\mu}}{h}(eigenvalues) \tag{17}
$$

Then, we have:

$$\int_0^h rJ_m(\alpha r)^2 dr$$
$$= \begin{cases} \frac{1}{2\alpha_{m\mu}^2}(\alpha_{m\mu}^2 h^2 - m^2)J_m(\alpha_{m\mu}h)^2, m \neq 0 \\ \frac{1}{2}(h^2)J_0(\alpha_{m\mu}h)^2, m = 0 \end{cases}$$
$$= \begin{cases} (\frac{J_m(\alpha_{m\mu}h)\sqrt{(h^2-\frac{m^2}{\alpha_{m\mu}^2})}}{\sqrt{2}})^2, m \neq 0 \\ \frac{1}{2}(h^2)J_0(\alpha_{m\mu}h)^2, m = 0 \end{cases} \tag{18}$$
$$= \begin{cases} (\frac{h^2}{2}(1-\frac{m^2}{{j'}_{m\mu}^2})J_m^2({j'}_{m\mu})), m \neq 0 \\ \frac{1}{2}h^2 J_0^2({j'}_{m\mu}), m = 0 \end{cases}$$

Thus,

$$C_{\alpha_{m\mu}} = \begin{cases} \frac{1}{\sqrt{\frac{\pi}{2}h^2 J_0^2({j'}_{m\mu})}}, m = 0 \\ \frac{1}{\sqrt{(\frac{\pi h^2}{2}(1-\frac{m^2}{{j'}_{m\mu}^2})J_m^2({j'}_{m\mu}))}}, m \neq 0 \end{cases} \tag{19}$$

D. Slowly varying ducts

waiting for updating......

E. Orthogonal-eigenvector

ref:https:

www.mathworks.com/help/matlab/ref/eigs.html

Eigenvectors, returned as a matrix. The columns in V correspond to the eigenvalues along the diagonal of D. The form and normalization of V depends on the combination of input arguments:

[V,D] = eigs(A) returns matrix V, whose columns are the eigenvectors of A such that A*V = V*D. The eigenvectors in V are normalized so that the 2-norm of each is 1.

If A is symmetric, then the eigenvectors, V, are orthonormal.

[V,D] = eigs(A,B) returns V as a matrix whose columns are the generalized eigenvectors that satisfy A*V = B*V*D. The 2-norm of each eigenvector is not necessarily 1.

If B is symmetric positive definite, then the eigenvectors in V are normalized so that the B-norm of each is 1. If A is also symmetric, then the eigenvectors are B-orthonormal.

We could further study this question!!

if we can use the GramSchmidt mode as basis??

II. Mass equation

Mass consevation:

$$-ia\kappa P^a + \nabla \cdot \mathbf{U}^a = \sum_{b=-\infty}^{+\infty} (-P^{a-b}\nabla \cdot \mathbf{U}^b - \mathbf{U}^{a-b} \cdot \nabla P^b - \frac{B}{2A} iakP^b P^{a-b}) \tag{20}$$

First, derivation of eq1:

We know that:

$$h_s = 1 - \kappa r cos(\phi), h_r = 1, h_\theta = r \tag{21}$$

Then,

$$\begin{aligned} \nabla \cdot \mathbf{U}^a &= \frac{1}{h_1 h_2 h_3}[\frac{\partial(v_1 h_2 h_3)}{\partial w_1} + \frac{\partial(v_2 h_3 h_1)}{\partial w_2} + \frac{\partial(v_3 h_1 h_2)}{\partial w_3}] \\ &= \frac{1}{r(1-\kappa r cos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}] \end{aligned} \tag{22}$$

Thus, we have the mass equation, approximate RHS by:

$$\begin{aligned} \nabla \cdot \mathbf{U}^b &= ib\kappa P^b + o(M^2) \\ \nabla P^b &= ib\kappa \mathbf{U}^b + o(M^2) \end{aligned} \tag{23}$$

Then we have

$$\begin{aligned} &-ia\kappa P^a + \frac{1}{r(1-\kappa r cos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}] \\ &= \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A} iakP^b P^{a-b}) \end{aligned} \tag{24}$$

The fourier harmonics are expanded as follows:

$$\begin{aligned} P^a &= \sum_{\beta=0}^{\infty} P^a_\beta(s)\psi_\beta(s,r,\theta) \\ U^a &= \sum_{\beta=0}^{\infty} U^a_\beta(s)\psi_\beta(s,r,\theta) \\ V^a &= \sum_{\beta=0}^{\infty} V^a_\beta(s)\psi_\beta(s,r,\theta) \\ W^a &= \sum_{\beta=0}^{\infty} W^a_\beta(s)\psi_\beta(s,r,\theta) \end{aligned} \tag{25}$$

with normalized relation:

$$\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r dr d\theta = \delta_{\alpha\beta} \tag{26}$$

Reorganize the eq5:

$$\begin{aligned}
&-ia\kappa P^a(1-\kappa rcos(\phi)) + \frac{1}{r}\frac{\partial(U^a r)}{\partial s} + \frac{1}{r}\frac{\partial(V^a r(1-\kappa rcos(\phi)))}{\partial r} + \frac{1}{r}\frac{\partial(W^a(1-\kappa rcos(\phi)))}{\partial\theta} \\
&= (1-\kappa rcos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A}iakP^bP^{a-b})
\end{aligned} \tag{27}$$

Intergal and insert eq 6, 7 into eq 8:

1. the first term:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[-ia\kappa(1-\kappa rcos(\phi))P^a]drd\theta \\
&= -ia\kappa\sum_\beta^\infty\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r[(1-\kappa rcos(\phi))]drd\theta P^a_\beta \\
&= -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta \\
&(summationconvention)
\end{aligned} \tag{28}$$

2. the second term:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(U^a r)}{\partial s}]drd\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta U^a_\beta)}{\partial s}drd\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r\frac{\partial(U^a_\beta)}{\partial s}drd\theta + \int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta)}{\partial s}drd\theta U^a_\beta \\
&= \frac{dU^a_\beta}{ds}\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta rdrd\theta + \int_0^{2\pi}\int_0^h r\frac{\partial(\psi_\alpha\psi_\beta)}{\partial s}drd\theta U^a_\beta - \int_0^{2\pi}\int_0^h \psi_\beta r\frac{\partial(\psi_\alpha)}{\partial s}drd\theta U^a_\beta \\
&= \frac{dU^a_\beta}{ds}\delta_{\alpha\beta} + \frac{\partial(\int_0^{2\pi}\int_0^h r\psi_\alpha\psi_\beta drd\theta)}{\partial s}U^a_\beta - \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial s}\psi_\beta rdrd\theta U^a_\beta \\
&= \frac{dU^a_\alpha}{ds} + 0 - \Psi_{\{\alpha\}\beta}[r]U^a_\beta
\end{aligned} \tag{29}$$

3. the third term:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(V^a r(1-\kappa rcos(\phi)))}{\partial r}]drd\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha\frac{\partial(\psi_\beta r(1-\kappa rcos(\phi)))}{\partial r}drd\theta V^a_\beta \\
&= \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha\psi_\beta r(1-\kappa rcos(\phi)))}{\partial r}drd\theta V^a_\beta - \int_0^{2\pi}\int_0^h \psi_\beta r(1-\kappa rcos(\phi))\frac{\partial(\psi_\alpha)}{\partial r}drd\theta V^a_\beta \\
&= 0(periodic) - \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]V^a_\beta
\end{aligned} \tag{30}$$

4. the fourth term:

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(W^a(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha[\frac{\partial(\psi_\beta(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta W^a_\beta \\
&= \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha\psi_\beta(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta W^a_\beta - \int_0^{2\pi}\int_0^h \psi_\beta(1-\kappa rcos(\phi))[\frac{\partial(\psi_\alpha)}{\partial\theta}]drd\theta W^a_\beta \\
&= \int_0^h [\psi_\alpha\psi_\beta(1-\kappa rcos(\phi))]_0^{2\pi} dr W^a_\beta - \int_0^{2\pi}\int_0^h \psi_\beta(1-\kappa rcos(\phi))[\frac{\partial(\psi_\alpha)}{\partial\theta}]drd\theta W^a_\beta \\
&= 0 - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]W^a_\beta
\end{aligned}
\tag{31}
$$

5. the RHS term:

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[(1-\kappa rcos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A}iakP^bP^{a-b})]drd\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta\psi_\gamma r(1-\kappa rcos(\phi))drd\theta \\
&\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa V^{a-b}_\beta V^b_\gamma - ib\kappa W^{a-b}_\beta W^b_\gamma - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma) \\
&= \Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa V^{a-b}_\beta V^b_\gamma - ib\kappa W^{a-b}_\beta W^b_\gamma - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma)
\end{aligned}
\tag{32}
$$

Finally, we obtain the mass equation in the form of eigenfunction, the idea is same as Galerkin method:

$$
\begin{aligned}
&\frac{dU^a_\alpha}{ds} - \Psi_{\{\alpha\}\beta}[r]U^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]V^a_\beta - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]W^a_\beta \\
&= \Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa V^{a-b}_\beta V^b_\gamma - ib\kappa W^{a-b}_\beta W^b_\gamma - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma)
\end{aligned}
\tag{33}
$$

III. Momentum equation

Momentum consevation:

$$-ia\kappa\mathbf{U}^a + \nabla P^a = \sum_{b=-\infty}^{\infty}(-\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^b + P^{a-b}\nabla P^b) \tag{34}$$

First, we know that

$$\nabla P^a = \sum_i \frac{1}{h_i}\frac{\partial f}{\partial w_i}\hat{h}_i = \frac{1}{1-\kappa rcos\phi}\frac{\partial P^a}{\partial s}\hat{e}_s + \frac{\partial P^a}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial P^a}{\partial\theta}\hat{e}_\theta \tag{35}$$

The RHS term is a bit complex, with the divergence of a vector U with its gradient, with

First, we know that

$$(\mathbf{v}\cdot\nabla)\mathbf{v}^b = \begin{cases} term1: \mathcal{D}v_1^b + \frac{v_2^b}{h_2h_1}(v_1\frac{\partial h_1}{\partial \xi_2} - v_2\frac{\partial h_2}{\partial \xi_1}) + \frac{v_3^b}{h_3h_1}(v_1\frac{\partial h_1}{\partial \xi_3} - v_2\frac{\partial h_3}{\partial \xi_1}) \\ term2: \mathcal{D}v_2^b + \frac{v_3^b}{h_3h_2}(v_2\frac{\partial h_2}{\partial \xi_3} - v_3\frac{\partial h_3}{\partial \xi_2}) + \frac{v_1^b}{h_1h_2}(v_2\frac{\partial h_2}{\partial \xi_1} - v_1\frac{\partial h_1}{\partial \xi_2}) \\ term3: \mathcal{D}v_3^b + \frac{v_1^b}{h_1h_3}(v_3\frac{\partial h_3}{\partial \xi_1} - v_1\frac{\partial h_1}{\partial \xi_3}) + \frac{v_2^b}{h_2h_3}(v_3\frac{\partial h_3}{\partial \xi_2} - v_3\frac{\partial h_2}{\partial \xi_3}) \end{cases} \tag{36}$$

Besides,

$$\mathcal{D} = \frac{v_1}{h_1}\frac{\partial}{\partial \xi_1} + \frac{v_2}{h_2}\frac{\partial}{\partial \xi_2} + \frac{v_3}{h_3}\frac{\partial}{\partial \xi_3} \tag{37}$$

Thus, we have:

$$\begin{aligned} -\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^b = \\ -\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{D}1: \frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} + V^{a-b}\frac{\partial U^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2: \frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial V^b}{\partial s} + V^{a-b}\frac{\partial V^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3: \frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial W^b}{\partial s} + V^{a-b}\frac{\partial W^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases} \\ -\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{X}1: \frac{V^b}{1-\kappa r cos\phi}(U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial r} - V^{a-b}\frac{\partial 1}{\partial s}) + \frac{W^b}{r(1-\kappa r cos\phi)}(U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial \theta} - V^{a-b}\frac{\partial r}{\partial s}) \\ term\mathcal{X}2: \frac{W^b}{r}(V^{a-b}\frac{\partial 1}{\partial \theta} - W^{a-b}\frac{\partial r}{\partial r}) + \frac{U^b}{(1-\kappa r cos\phi)1}(V^{a-b}\frac{\partial 1}{\partial s} - U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial r}) \\ term\mathcal{X}3: \frac{U^b}{(1-\kappa r cos\phi)r}(W^{a-b}\frac{\partial h_3}{\partial s} - U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial \theta}) + \frac{V^b}{1h_3}(W^{a-b}\frac{\partial h_3}{\partial r} - V^{a-b}\frac{\partial 1}{\partial \theta}) \end{cases} = \\ \sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{D}1: -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2: -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3: -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases} \\ +\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{X}1: \frac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b \\ term\mathcal{X}2: \frac{W^{a-b}W^b}{r} - \frac{\kappa cos\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b \\ term\mathcal{X}3: \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \end{cases} \end{aligned} \tag{38}$$

Finally, we could derive the momentum conservation equation, with final term approximate by eq 4:

$$
\begin{aligned}
&\begin{cases} -ia\kappa U^a + \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s} \\ -ia\kappa V^a + \frac{\partial P^a}{\partial r} \\ -ia\kappa W^a + \frac{1}{r}\frac{\partial P^a}{\partial \theta} \end{cases} \\
&= \sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases} \\
&+ \sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa rcos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}W^b \\ term\mathcal{X}2 : \frac{W^{a-b}W^b}{r} - \frac{\kappa cos\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b \\ term\mathcal{X}3 : \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \end{cases} + \sum_{b=-\infty}^{\infty} \begin{cases} ib\kappa P^{a-b}U^b \\ ib\kappa P^{a-b}V^b \\ ib\kappa P^{a-b}W^b \end{cases}
\end{aligned} \tag{39}
$$

Now, we are going to project on ψ, it may be a little complex, we will doing step by step.

A. Momentum e^s term

First, deal with the e^s term:

$$
\begin{aligned}
&-ia\kappa U^a + \frac{1}{1-\kappa rcos\phi}\frac{\partial P^a}{\partial s} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa rcos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}W^b \\
&+ib\kappa P^{a-b}U^b
\end{aligned} \tag{40}
$$

Multiply $(1-\kappa rcos\phi)$, we have:

$$
\begin{aligned}
&-ia\kappa(1-\kappa rcos\phi)U^a + \frac{\partial P^a}{\partial s} \\
&= term\mathcal{D}1 : \sum_{b=-\infty}^{\infty} -U^{a-b}\frac{\partial U^b}{\partial s} - (1-\kappa rcos\phi)V^{a-b}\frac{\partial U^b}{\partial r} - (1-\kappa rcos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\
&+term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b + \\
&term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b
\end{aligned} \tag{41}
$$

$\int\int XXr\psi_\alpha drd\theta$, we have:

$$
\underline{RHS = \int_0^{2\pi}\int_0^h [term\mathcal{D}1 + term\mathcal{X}1 + term\mathcal{P}1]r\psi_\alpha drd\theta} \tag{42}
$$

1. the first $\mathcal{D}1$ tems:

We ref the wiki $https:$

$en.wikipedia.org/wiki/Leibniz\_integral\_rule$

General form: Differentiation under the integral sign:

$$\frac{d}{dx}(\int_{a(x)}^{b(x)} f(x,t)dt) - f(x,b(x)) \cdot \frac{d}{dx}b(x) + f(x,a(x)) \cdot \frac{d}{dx}a(x) = \int_{a(x)}^{b(x)} \frac{d}{dx}f(x,t)dt \tag{43}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{44}$$

For partial difference, for a given β, the derivation of the fucntion $g(\alpha) = \int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx$ is

$$\frac{d}{d\alpha}(\int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx) - 0 + \frac{da(\alpha)}{d\alpha}f(a(\alpha),\alpha) = \int_{a(\alpha)}^{b(\beta)} \frac{\partial}{\partial\alpha}f(x,\alpha)dx \tag{45}$$

1.1

$$\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [-rU^{a-b}\frac{\partial U^b}{\partial s}]\psi_\alpha dr d\theta \\
&= \sum_{b=-\infty}^{\infty} -\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\
&\quad + \int_0^{2\pi}\int_0^h U^b\frac{\partial}{\partial s}[rU^{a-b}\psi_\alpha]drd\theta \\
&= \sum_{b=-\infty}^{\infty} -\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta + \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta \\
&\quad + \int_0^{2\pi}\int_0^h \frac{r\partial U^{a-b}}{\partial s}U^b\psi_\alpha drd\theta \\
&\quad + \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s}rU^{a-b}U^b drd\theta
\end{aligned} \tag{46}$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$h'U^{a-b} = (1-\kappa h cos\phi)V^{a-b} \tag{47}$$

1.2

$$
\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h[-(1-\kappa rcos\phi)V^{a-b}\frac{\partial U^b}{\partial r}]r\psi_\alpha drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h\frac{\partial(r(1-\kappa rcos\phi)V^{a-b}U^b\psi_\alpha)}{\partial r}drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b}\psi_\alpha)}{\partial r}drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}[(r(1-\kappa rcos\phi)V^{a-b}U^b)\psi_\alpha]_0^h d\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa rcos\phi)V^{a-b}drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}[(hh'U^{a-b}U^b)\psi_\alpha]_0^h d\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa rcos\phi)V^{a-b}drd\theta
\end{aligned}
\tag{48}
$$

1.3

$$
\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h[-(1-\kappa rcos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta}]r\psi_\alpha drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h\frac{\partial((1-\kappa rcos\phi)W^{a-b}U^b\psi_\alpha)}{\partial\theta}drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^h[(1-\kappa rcos\phi)W^{a-b}U^b\psi_\alpha]_0^{2\pi}dr \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta \\
&=0(periodic)+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta
\end{aligned}
\tag{49}
$$

Combine together:

$$
\begin{aligned}
\underline{\int_0^{2\pi}\int_0^h [term\mathcal{D}1] r\psi_\alpha dr d\theta =} \sum_{b=-\infty}^{\infty} \\
\{(\int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s} rU^{a-b}U^b dr d\theta \\
+\int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r(1-\kappa r cos\phi)V^{a-b}U^b dr d\theta \\
+\int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa r cos\phi)W^{a-b}U^b) dr d\theta) \\
+(\int_0^{2\pi}\int_0^h U^b\frac{\partial U^{a-b}}{\partial s} r\psi_\alpha dr d\theta \\
+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa r cos\phi)V^{a-b})}{\partial r}\psi_\alpha dr d\theta \\
+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa r cos\phi)W^{a-b})}{\partial\theta}\psi_\alpha dr d\theta) \\
-\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha] dr d\theta\}
\end{aligned}
\tag{50}
$$

We apply eq 5, find that:

$$
\begin{aligned}
-i(a-b)\kappa r(1-\kappa r cos(\phi))P^{a-b} + [\frac{\partial(U^{a-b}r)}{\partial s} + \frac{\partial(V^{a-b}r(1-\kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^{a-b}(1-\kappa r cos(\phi)))}{\partial\theta}] \\
= o(M^2)
\end{aligned}
\tag{51}
$$

We have the second terms in eq(31) are equal to:

$$
\begin{aligned}
\int_0^{2\pi}\int_0^h U^b[-i(a-b)\kappa r(1-\kappa r cos(\phi))P^{a-b}]\psi_\alpha dr d\theta \\
= i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}U_\gamma^b
\end{aligned}
\tag{52}
$$

And, the longitudinal derivation s can also be expand about the duct modes, with note $[r], (\theta), \{s\}$:

$$
\begin{aligned}
\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha] dr d\theta \\
= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]U_\beta^{a-b}U_\gamma^b dr d\theta \\
= \frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta)U_\beta^{a-b}U_\gamma^b \\
+\frac{\partial U_\beta^{a-b}U_\gamma^b}{\partial s}\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta \\
= \frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta)U_\beta^{a-b}U_\gamma^b \\
+(\frac{dU_\beta^{a-b}}{ds}U_\gamma^b + \frac{dU_\gamma^b}{ds}U_\beta^{a-b})\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta
\end{aligned}
\tag{53}
$$

2. the second $\mathcal{X}1$ tems:

$$term\mathcal{X}1: \sum_{b=-\infty}^{\infty} \kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b$$

$$\begin{aligned} &\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [\kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b] r\psi_\alpha drd\theta \\ &= \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U_\beta^{a-b}V_\gamma^b - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U_\beta^{a-b}W_\gamma^b \end{aligned} \tag{54}$$

3. the second $\mathcal{P}1$ tems:

$$term\mathcal{P}1: \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b$$

$$\begin{aligned} &\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b] r\psi_\alpha drd\theta \\ &= ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P_\beta^{a-b}U_\gamma^b \end{aligned} \tag{55}$$

4. The LHS terms:

$$\frac{\partial P^a}{\partial s} - ia\kappa(1-\kappa rcos\phi)U^a$$

From 1.1 as example, we know that

$$\begin{aligned} &\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\ &= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta \end{aligned} \tag{56}$$

4.1

$$\begin{aligned} &\int_0^{2\pi}\int_0^h [\frac{\partial P^a}{\partial s}] r\psi_\alpha drd\theta \\ &= \int_0^{2\pi}\int_0^h [\frac{\partial (P_\beta^a\psi_\beta)}{\partial s}] r\psi_\alpha drd\theta \\ &= \int_0^{2\pi}\int_0^h [\frac{\partial (P_\beta^a\psi_\beta)}{\partial s} r\psi_\alpha] drd\theta - \int_0^{2\pi}\int_0^h [\frac{\partial (\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a \\ &= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h (P_\beta^a\psi_\beta) r\psi_\alpha drd\theta - \frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h} - \int_0^{2\pi}\int_0^h [\frac{\partial (\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a \\ &= \frac{\partial}{\partial s}(P_\beta^a\delta_{\alpha\beta}) - \int_0^{2\pi}\frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h}d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial (\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a \\ &= \frac{d}{ds}P_\alpha^a - \int_0^{2\pi}\frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h}d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial (\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a \end{aligned} \tag{57}$$

$$= \frac{d}{ds}P^a_\alpha - \int_0^{2\pi}\int_0^h \frac{\partial\psi_\beta\psi_\alpha}{\partial r} hh' dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P^a_\beta$$

4.2

$$\begin{aligned} &\int_0^{2\pi}\int_0^h [-ia\kappa(1-\kappa r cos\phi)U^a] r\psi_\alpha dr d\theta \\ &= -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta \end{aligned} \tag{58}$$

Finally, putting all together becomes:

$$\begin{aligned} &\frac{d}{ds}P^a_\alpha - \int_0^{2\pi}\frac{dh(s)}{ds}[P^a_\beta\psi_\beta r\psi_\alpha]_{r=h} d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta \\ &= \frac{d}{ds}P^a_\alpha - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta - \int_0^{2\pi} hh'[P^a_\beta\psi_\beta\psi_\alpha]_{r=h} d\theta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\ &= \sum_{b=-\infty}^{\infty} \\ &(eq31) : \{(\int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s} rU^{a-b}U^b dr d\theta \\ &+ \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r(1-\kappa r cos\phi)V^{a-b}U^b dr d\theta \\ &+(eq33) : i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\ &eq(34) : -\frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta)U^{a-b}_\beta U^b_\gamma \\ &-(\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta)\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha] dr d\theta \\ &+eq(35) : \kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma \\ &+eq(36) : ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\ &= (abbreviation) : \\ &\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta U^a_\gamma \\ &+i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\ &-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\ &+\kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma \\ &+ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \end{aligned} \tag{59}$$

To be conclude, with the e^s term:

$$
\begin{aligned}
&-ia\kappa U^a + \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{P}1 : ib\kappa P^{a-b}U^b
\end{aligned}
\tag{60}
$$

We have:

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta + \frac{d}{ds}P^a_\alpha - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\
&= \underline{term\mathcal{D}1 :}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta U^a_\gamma \\
&+\underline{term(D1+\mathcal{P}1) :}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\
&-\underline{term\mathcal{D}1 :}\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}1 :}\kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma
\end{aligned}
\tag{61}
$$

B. Momentum e^r term

Second, deal with the e^r term:

$$
\begin{aligned}
&-ia\kappa V^a + \frac{\partial P^a}{\partial r} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}2 : \frac{W^{a-b}W^b}{r} - \frac{\kappa cos\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b \\
&+term\mathcal{P}2 : \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b}V^b
\end{aligned}
\tag{62}
$$

LHS-2: $\frac{\partial P^a}{\partial r}(1-\kappa r cos\phi)$

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h [\frac{\partial P^a}{\partial r}(1-\kappa r cos\phi)] r\psi_\alpha dr d\theta \\
&= \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\beta (1-\kappa r cos\phi) r\psi_\alpha}{\partial r}] dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha (1-\kappa r cos\phi) r}{\partial r}]\psi_\beta dr d\theta P^a_\beta \\
&= \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\beta (1-\kappa r cos\phi) r\psi_\alpha}{\partial r}] dr d\theta P^a_\beta \\
&- \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha}{\partial r}](1-\kappa r cos\phi) r\psi_\beta dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial (1-\kappa r cos\phi) r}{\partial r}]\psi_\alpha\psi_\beta dr d\theta P^a_\beta \\
&= \int_0^{2\pi} [\psi_\alpha\psi_\beta r(1-\kappa r cos\phi)]_0^h d\theta P^a_\beta \\
&- \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha}{\partial r}](1-\kappa r cos\phi) r\psi_\beta dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [1-2\kappa r cos\phi]\psi_\alpha\psi_\beta dr d\theta P^a_\beta \\
&= \int_0^{2\pi} [\psi_\alpha\psi_\beta r(1-\kappa r cos\phi)]_0^h d\theta P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]P^a_\beta - \Psi_{\alpha\beta}[1-2\kappa r cos\phi]P^a_\beta
\end{aligned}
\tag{63}
$$

The derivation of $term\mathcal{D}2$ is identical to A, we are not prove it again. $Term\mathcal{P}2$ also could be combine with the part separated term of $term\mathcal{D}2$ with V^b. $Term\mathcal{X}2$ is also easy to derive.

Thus, we have the final equation:

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]V^a_\beta \\
&\int_0^{2\pi} [\psi_\alpha\psi_\beta r(1-\kappa r cos\phi)]_0^h d\theta P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]P^a_\beta - \Psi_{\alpha\beta}[1-2\kappa r cos\phi]P^a_\beta \\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta V^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta V^a_\gamma \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta V^b_\gamma \\
&+\underline{term\mathcal{D}2:} - \frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}V^b_\gamma + \frac{dV^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[1-\kappa r cos\phi]W^{a-b}_\beta W^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta U^b_\gamma
\end{aligned}
\tag{64}
$$

C. Momentum e^θ term

Third, deal with the e^θ term:

$$\begin{aligned}
& -ia\kappa W^a + \frac{1}{r}\frac{\partial P^a}{\partial \theta} \\
= \sum_{b=-\infty}^{\infty} term\mathcal{D}3 : & -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \\
+ \sum_{b=-\infty}^{\infty} term\mathcal{X}3 : & \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \\
& + term\mathcal{P}3 : \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b}W^b
\end{aligned} \tag{65}$$

LHS-2: $\frac{\partial P^a}{\partial \theta}\frac{(1-\kappa r cos\phi)}{r}$

$$\begin{aligned}
& \int_0^{2\pi}\int_0^h [\frac{\partial P^a}{\partial \theta}\frac{(1-\kappa r cos\phi)}{r}] r\psi_\alpha dr d\theta \\
&= \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\beta (1-\kappa r cos\phi)\psi_\alpha}{\partial \theta}] dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha (1-\kappa r cos\phi)}{\partial \theta}]\psi_\beta dr d\theta P^a_\beta \\
&= 0 - \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha}{\partial \theta}](1-\kappa r cos\phi)\psi_\beta dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial (1-\kappa r cos\phi)}{\partial \theta}]\psi_\alpha\psi_\beta dr d\theta P^a_\beta \\
&= -\int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha}{\partial \theta}](1-\kappa r cos\phi)\psi_\beta dr d\theta P^a_\beta + \kappa\int_0^{2\pi}\int_0^h [r sin\phi]\psi_\alpha\psi_\beta dr d\theta P^a_\beta \\
&= -\Psi_{(\alpha)\beta}[(1-\kappa r cos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[r sin\phi]P^a_\beta
\end{aligned} \tag{66}$$

The derivation of $term\mathcal{D}3$ is identical to A, we are not prove it again. $Term\mathcal{P}3$ also could be combine with the part separated term of $term\mathcal{D}3$ with W^b. $Term\mathcal{X}3$ is also easy to derive.

Thus, we have the final equation:

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]W^a_\beta \\
& -\Psi_{(\alpha)\beta}[(1-\kappa r cos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[r sin\phi]P^a_\beta \\
= \underline{term\mathcal{D}2 :} & \Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta W^a_\gamma \\
& + \underline{term(D2+\mathcal{P}2) :} i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta W^b_\gamma \\
& + \underline{term\mathcal{D}2 :} -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta) \\
& + \underline{term\mathcal{X}2 :} \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa r cos\phi]W^{a-b}_\beta V^b_\gamma
\end{aligned} \tag{67}$$

IV. Merge the four equations and eliminate the V_γ^b and W_γ^b

A. V_α^a & W_α^a

Using the linear relationships:

$$
\begin{aligned}
ia\kappa V^a &= \frac{\partial P^a}{\partial r} \\
:= \int\int iakV_\beta^a \psi_\beta r\psi_\alpha dr d\theta &= \int\int \frac{\partial P_\beta^a \psi_\beta}{\partial r} r\psi_\alpha dr d\theta \\
= ia\kappa V_\beta^a \delta_{\alpha\beta} = \Psi_{\alpha[\beta]}[r]P_\beta^a = \int_0^{2\pi} [r\psi_\alpha\psi_\beta]_0^h d\theta P_\beta^a - \int\int \psi_\alpha\psi_\beta dr d\theta P_\beta^a &- \int\int \frac{\partial \psi_\alpha}{\partial r}\psi_\beta r dr d\theta P_\beta^a
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
iakW^a &= \frac{1}{r}\frac{\partial P^a}{\partial \theta} \\
:= \int\int iakW_\beta^a \psi_\beta r\psi_\alpha dr d\theta &= \int\int \frac{\partial P_\beta^a \psi_\beta}{\partial \theta}\frac{1}{r} r\psi_\alpha dr d\theta \\
= ia\kappa W_\beta^a \delta_{\alpha\beta} = \Psi_{\alpha(\beta)}[r]P_\beta^a &= 0 - \Psi_{(\alpha)\beta}[r]P_\beta^a
\end{aligned} \tag{69}
$$

Thus, we can establish relationships between the tranverse modes and pressure modes (no summation over a)

$$
V_\alpha^a = \frac{1}{ia\kappa}[\int_0^{2\pi} [r\psi_\alpha\psi_\beta]_0^h d\theta P_\beta^a - \Psi_{\alpha\beta} - \Psi_{[\alpha]\beta}[r]]P_\beta^a = \mathbf{V}_{\alpha\beta}^a P_\beta^a \tag{70}
$$

$$
W_\alpha^a = -\frac{1}{ia\kappa}\Psi_{(\alpha)\beta}P_\beta^a = \mathbf{W}_{\alpha\beta}^a P_\beta^a \tag{71}
$$

B. $\frac{d}{ds}V_\alpha^a$ & $\frac{d}{ds}W_\alpha^a$

We also require modal expressions for $\frac{d}{ds}V_\alpha^a$ and $\frac{d}{ds}W_\alpha^a$.

We differentiate eq49 with respect to s:

$$
\begin{aligned}
\frac{\partial V^a}{\partial s} &= \frac{1}{ia\kappa}\frac{\partial^2 P^a}{\partial s \partial r} \\
&= \frac{\partial}{\partial r}((1 - \kappa r cos\phi)U^a)
\end{aligned} \tag{72}
$$

where we have used symmetry of mixed partials and the linear expression for $\frac{\partial P^a}{\partial s}$ from eq 21.

From 1.1 as example, we know that

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha] dr d\theta \\
= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha] dr d\theta &- \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h} d\theta
\end{aligned} \tag{73}
$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$h'U^a_\beta = (1 - \kappa h cos\phi) V^a_\beta \tag{74}$$

Multiplying this expression by $r\phi_\alpha$ and integrating across section of the duct, we obtain:

$$\begin{aligned}
\int_0^{2\pi}\int_0^h \frac{\partial V^a}{\partial s} r\psi_\alpha dr d\theta &= \int_0^{2\pi}\int_0^h \frac{\partial}{\partial r}((1-\kappa r cos\phi)U^a) r\psi_\alpha dr d\theta \\
LHS := \int_0^{2\pi}\int_0^h \frac{\partial [V^a_\beta \psi_\beta r\psi_\alpha]}{\partial s} dr d\theta &- \int_0^{2\pi}\int_0^h \frac{\partial r\psi_\alpha}{\partial s}\psi_\beta dr d\theta V^a_\beta \\
= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [V^a_\beta \psi_\beta r\psi_\alpha] dr d\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[V^a_\beta\psi_\beta r\psi_\alpha]_0^h d\theta &- \int_0^{2\pi}\int_0^h \frac{\partial \psi_\alpha}{\partial s}\psi_\beta r dr d\theta V^a_\beta \\
= \frac{\partial}{\partial s}V^a_\beta \delta_{\alpha\beta} - \int_0^{2\pi}\frac{h'^2}{1-\kappa h cos\phi}[r\psi_\beta\psi_\alpha]_0^h d\theta &- \Psi_{\{\alpha\}\beta}[r]V^a_\beta \\
RHS := \int_0^{2\pi}\int_0^h \frac{\partial}{\partial r}((1-\kappa r cos\phi)U^a_\beta\psi_\beta r\psi_\alpha) dr d\theta &- \int_0^{2\pi}\int_0^h \frac{\partial (r\psi_\alpha)}{\partial r}(1-\kappa r cos\phi)U^a_\beta\psi_\beta dr d\theta \\
= \int_0^{2\pi}[r(1-\kappa r cos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U^a_\beta - \int_0^{2\pi}\int_0^h \frac{\partial \psi_\alpha}{\partial r} r(1-\kappa r cos\phi)\psi_\beta dr d\theta U^a_\beta &- \int_0^{2\pi}\int_0^h (1-\kappa r cos\phi)\psi_\alpha\psi_\beta dr d\theta U^a_\beta \\
= \int_0^{2\pi}[r(1-\kappa r cos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]U^a_\beta &- \Psi_{\alpha\beta}[(1-\kappa r cos\phi)]U^a_\beta
\end{aligned} \tag{75}$$

Thus, LHS=RHS, we have:

$$\begin{aligned}
\frac{d}{ds}V^a_\alpha &= \int_0^{2\pi}\frac{h'^2}{1-\kappa h cos\phi}[r\psi_\beta\psi_\alpha]_0^h d\theta + \Psi_{\{\alpha\}\beta}[r]V^a_\beta \\
&+ \int_0^{2\pi}[r(1-\kappa r cos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]U^a_\beta - \Psi_{\alpha\beta}[(1-\kappa r cos\phi)]U^a_\beta
\end{aligned} \tag{76}$$

Similarly for W^a, differentiating eq50 with respect to s and substituting the linear expression for $\frac{\partial P^a}{\partial s}$ by eq21:

$$\begin{aligned}
\frac{\partial W^a}{\partial s} &= \frac{1}{ia\kappa}\frac{\partial^2 P^a}{\partial s \partial \theta} \\
&= \frac{1}{r}\frac{\partial}{\partial \theta}((1-\kappa r cos\phi)U^a)
\end{aligned} \tag{77}$$

Multiplying this expression by $r\phi_\alpha$ and integrating across section of the duct, we obtain:

$$\int_0^{2\pi}\int_0^h \frac{\partial W^a}{\partial s} r\psi_\alpha drd\theta = \int_0^{2\pi}\int_0^h \frac{1}{r}\frac{\partial}{\partial\theta}((1-\kappa rcos\phi)U^a)r\psi_\alpha drd\theta$$

$$\begin{aligned} LHS &:= \int_0^{2\pi}\int_0^h \frac{\partial[W^a_\beta\psi_\beta r\psi_\alpha]}{\partial s}drd\theta - \int_0^{2\pi}\int_0^h \frac{\partial r\psi_\alpha}{\partial s}\psi_\beta drd\theta W^a_\beta \\ = \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [W^a_\beta\psi_\beta r\psi_\alpha]drd\theta &- \int_0^{2\pi}\frac{dh(s)}{ds}[W^a_\beta\psi_\beta r\psi_\alpha]_0^h d\theta - \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s}\psi_\beta rdrd\theta W^a_\beta \\ &= \frac{d}{ds}W^a_\alpha - \int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]_0^h d\theta W^a_\beta - \Psi_{\{\alpha\}\beta}[r]W^a_\beta \\ RHS := \int_0^{2\pi}\int_0^h \frac{\partial}{\partial\theta}((1-\kappa rcos\phi)U^a_\beta\psi_\beta\psi_\alpha)drd\theta &- \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial\theta}(1-\kappa rcos\phi)U^a_\beta\psi_\beta drd\theta \\ &= 0(periodic) - \Psi_{(\alpha)\beta}[1-\kappa rcos\phi]U^a_\beta \end{aligned} \tag{78}$$

Thus, LHS=RHS, we have:

$$\frac{d}{ds}V^a_\alpha = \int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]_0^h d\theta W^a_\beta + \Psi_{\{\alpha\}\beta}[r]W^a_\beta - \Psi_{(\alpha)\beta}[1-\kappa rcos\phi]U^a_\beta \tag{79}$$

C. Substitue pressure modes for transverse velocity modes

1. *mass equation*

$$\begin{aligned} &\frac{dU^a_\alpha}{ds} - \Psi_{\{\alpha\}\beta}[r]U^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{V^a_\beta}} - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{W^a_\beta}} \\ &= \Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa\underline{\underline{V^{a-b}_\beta V^b_\gamma}} - ib\kappa\underline{\underline{W^{a-b}_\beta W^b_\gamma}} - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma) \end{aligned} \tag{80}$$

Transform:

$$\begin{aligned} \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{V^a_\beta}} &:= \Psi_{[\alpha]\delta}[r(1-\kappa rcos(\phi))]\mathbf{V}^a_{\delta\beta}P^a_\beta \\ \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{W^a_\beta}} &:= \Psi_{(\alpha)\delta}[(1-\kappa rcos(\phi))]\mathbf{W}^a_{\delta\beta}P^a_\beta \end{aligned} \tag{81}$$

$$-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta + \frac{d}{ds}P^a_\alpha - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta$$

$$= \underline{term\mathcal{D}1:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\underline{\underline{V^{a-b}_\beta}}U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\underline{\underline{W^{a-b}_\beta}}U^a_\gamma$$

$$+\underline{term(D1+\mathcal{P}1):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta U^b_\gamma$$

$$+\underline{term\mathcal{D}1:} - \frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta)$$

$$+\underline{term\mathcal{X}1:}\kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta \underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta \underline{\underline{W^b_\gamma}} \tag{82}$$

$$-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_\beta$$

$$\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P^a_\beta - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]P^a_\beta$$

$$= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta V^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta V^a_\gamma$$

$$+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta V^b_\gamma$$

$$+\underline{term\mathcal{D}2:} - \frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}V^b_\gamma + \frac{dV^b_\gamma}{ds}U^{a-b}_\beta)$$

$$+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta W^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta U^b_\gamma \tag{83}$$

$$-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]W^a_\beta$$

$$-\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta$$

$$= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta W^a_\gamma$$

$$+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta W^b_\gamma$$

$$+\underline{term\mathcal{D}2:} - \frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta)$$

$$+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta V^b_\gamma \tag{84}$$

V. Tensors in matlab for numerical simulation

A. Tensor times vectors: $\mathcal{A}\bar{\times}_n u$

Let $\mathcal{A}$ be a tensor of size $I_1 \times I_2 \times ... \times I_N$, u be a vector of size I_n.

We have:

$$ttv(\mathcal{A}, \{u\}, [n]) = (\mathcal{A}\bar{\times}_n u)(i_1, ..., i_{n-1}, i_{n+1}, ..., i_N)$$
$$\sum_{i_n=1}^{I_n} \mathcal{A}(i_1, i_2, ..., i_N)u(i_n) \tag{85}$$

$$ttv(A_{m\times n}, \{u_{m\times 1}\}, [1]) = A_{m\times n}\bar{\times}_1 u_{m\times 1} = A^T_{m\times n}u_{m\times 1}$$
$$ttv(A_{m\times n}, \{v_{n\times 1}\}, [2]) = A_{m\times n}\bar{\times}_2 v_{n\times 1} = A_{m\times n}v_{n\times 1} \tag{86}$$

Property:

$$ttv(\mathcal{A}, \{u, v\}, [m, n]) = \mathcal{A}\bar{\times}_m u\bar{\times}_n v$$
$$= ttv(ttv(\mathcal{A}, \{u\}, [m]), \{v\}, [n-1]) = (\mathcal{A}\bar{\times}_m u)\bar{\times}_{n-1} v \tag{87}$$
$$= ttv(ttv(\mathcal{A}, \{v\}, [v]), \{u\}, [m]) = (\mathcal{A}\bar{\times}_n v)\bar{\times}_m u$$

Multiplication with a sequence of vectors

$$\beta = \mathcal{A}\bar{\times}_1 u^{(1)}\bar{\times}_2 u^{(2)}...\bar{\times}_N u^{(N)} = \mathcal{A}\bar{\times}u$$
$$like : ttv(X, \{A, B, C, D\}) = ttv(X, \{A, B, C, D\}, [1234]) = ttv(X, \{D, C, B, A\}, [4321]) \tag{88}$$

Multiplication with **all but one** of a sequence of vectors

$$b = \mathcal{A}\bar{\times}_1 u^{(1)}\bar{\times}_2 u^{(2)}...\bar{\times}_{n-1} u^{(2)}\bar{\times}_{n+1} u^{(2)}...\bar{\times}_N u^{(N)} = \mathcal{A}\bar{\times}_{-n} u$$
$$like : X = tenrand([5, 3, 4, 2]);$$
$$A = rand(5, 1); B = rand(3, 1); C = rand(4, 1); D = rand(2, 1); \tag{89}$$
$$Y = ttv(X, \{A, B, D\}, -3) = ttv(X, \{A, B, C, D\}, -3)$$

B. Tensor times matrix (ttm): $\mathcal{A}\times_n u$

Let $\mathcal{A}$ be a tensor of size $I_1 \times I_2 \times ... \times I_N$, U be a matrix of size $J_n \times I_n$.

We have:

$$ttm(\mathcal{A}, \{U\}, [n]) = (\mathcal{A}\times_n U)(i_1, ..., i_{n-1}, j_n, i_{n+1}, ..., i_N)$$
$$\sum_{i_n=1}^{I_n} \mathcal{A}(i_1, i_2, ..., i_N)U(j_n, i_n)$$
$$like : X = tensor(rand(5, 3, 4, 2)); A = rand(4, 5); \tag{90}$$
$$Y = ttm(X, A, 1) = ttm(X, \{A, B, C, D\}, 1) = ttm(X, A', 1,' t')$$

Matrix Interpretation

$$
\begin{aligned}
ttm(A_{m\times n}, \{U^T_{m\times k}\}, [1]) = A \times_1 U^T = U^T A \\
ttm(A_{m\times n}, \{V^T_{m\times k}\}, [2]) = A \times_2 V^T = AV \\
ttm(A, \{U, V\}, [1, 2]) = UAV^T
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
Y = ttm(X, A, B, C, D, [1234]); \% <- - 4 - waymutliply. \\
Y = ttm(X, D, C, B, A, [4321]); \% <- - Sameasabove. \\
Y = ttm(X, A, B, C, D); \% <- - Sameasabove. \\
Y = ttm(X, A', B', C', D',' t')\% <- - Sameasabove.
\end{aligned}
$$

$$
\begin{aligned}
Y = ttm(X, C, D, [34]); \% <- - XtimesCinmode - 3Dinmode - 4 \\
Y = ttm(X, A, B, C, D, [34])\% <- - Sameasabove.
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
Y = ttm(X, A, B, D, [124]); \% <- - 3 - waymultiply. \\
Y = ttm(X, A, B, C, D, [124]); \% <- - Sameasabove. \\
Y = ttm(X, A, B, D, -3); \% <- - Sameasabove. \\
Y = ttm(X, A, B, C, D, -3)\% <- - Sameasabove.
\end{aligned}
$$

Property

$$
\begin{aligned}
ttm(\mathcal{A}, \{u, v\}, [m, n]) = \mathcal{A} \times_m u \bar{\times}_n v \\
= ttm(ttm(\mathcal{A}, \{u\}, [m]), \{v\}, [n]) = (\mathcal{A} \times_m u) \times_n v \\
= ttm(ttm(\mathcal{A}, \{v\}, [v]), \{u\}, [m]) = (\mathcal{A} \times_n v) \times_m u
\end{aligned}
\tag{93}
$$

C. Tensor times tensor (ttt): $< \mathcal{A}, \mathcal{B} >$

Let $\mathcal{A}$ and $\mathcal{B}$ be a tensor of size $I_1 \times I_2 \times ... \times I_N$.

$$
\begin{aligned}
< \mathcal{A}, \mathcal{B} >= \\
\beta = \sum_{i_1=1}^{I_1} \sum_{i_1=1}^{I_2} ... \sum_{i_1=1}^{I_N} \mathcal{A}(i_1, i_2, ..., i_N) \mathcal{B}(i_1, i_2, ..., i_N)
\end{aligned}
\tag{94}
$$

$$X = tensor(rand(4,2,3)); Y = tensor(rand(3,4,2));$$

$$Z = ttt(X,Y); \% <-- Outerproduct of X and Y.$$

$$size(Z)$$

$$Z = ttt(X,X,1:3)\% <-- Innerproduct of X with itself.$$

$$(95)$$

$$Z = ttt(X,Y,[123],[231])\% <-- Innerproduct of XY.$$

$$Z = innerprod(X, permute(Y,[231]))\% <-- Same as above.$$

$$Z = ttt(X,Y,[13],[21])\% <-- Product of XY along specified dims.$$