Note:derivation of 3-31

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A. Substitue pressure modes for transverse velocity modes

1. mass equation

$$\begin{aligned}
&\frac{dU^a_\alpha}{ds} - \Psi_{\{\alpha\}\beta}[r]U^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa r cos(\phi))]\underline{\underline{V^a_\beta}} - \Psi_{(\alpha)\beta}[(1-\kappa r cos(\phi))]\underline{\underline{W^a_\beta}} \\
&= \Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa\underline{\underline{V^{a-b}_\beta V^b_\gamma}} - ib\kappa\underline{\underline{W^{a-b}_\beta W^b_\gamma}} - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma)
\end{aligned}\tag{1}$$

Transform:

$$\begin{aligned}
\frac{dU^a_\alpha}{ds} &:= \underline{\underline{I}}u'^a_\alpha \\
-\Psi_{\{\alpha\}\beta}[r]U^a_\beta &:= \underline{\underline{-\Psi_{\{\alpha\}\beta}[r]}}u^a_\beta \\
-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta &:= \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}p^a_\beta \\
-\Psi_{[\alpha]\beta}[r(1-\kappa r cos(\phi))]\underline{\underline{V^a_\beta}} &:= \underline{\underline{-\Psi_{[\alpha]\delta}[r(1-\kappa r cos(\phi))]}}V^a_\beta \\
-\Psi_{(\alpha)\beta}[(1-\kappa r cos(\phi))]\underline{\underline{W^a_\beta}} &:= \underline{\underline{-\Psi_{(\alpha)\delta}[(1-\kappa r cos(\phi))]}}W^a_\beta \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa r cos(\phi))]}}p^{a-b}_\beta p^b_\gamma = (B_1[p,p])^a_\alpha \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa U^{a-b}_\beta U^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa r cos(\phi))]}}u^{a-b}_\beta u^b_\gamma = (A_1[u,u])^a_\alpha \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa V^{a-b}_\beta V^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{\alpha\delta\epsilon}[r(1-\kappa r cos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa W^{a-b}_\beta W^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{\alpha\delta\epsilon}[r(1-\kappa r cos\phi)]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos(\phi))]\sum_{b=-\infty}^{+\infty}(-ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ia\kappa\frac{B}{2A}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa r cos(\phi))]}}p^{a-b}_\beta p^b_\gamma
\end{aligned}\tag{2}$$

2. momentum equation I

$$
\begin{aligned}
\frac{d}{ds}P^a_\alpha - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\
= \sum_{b=-\infty}^{+\infty}\underline{term\mathcal{D}1:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\underline{\underline{V^{a-b}_\beta}}U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\underline{\underline{W^{a-b}_\beta}}U^a_\gamma \\
+\underline{term(D1+\mathcal{P}1):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta U^b_\gamma \\
+\underline{term\mathcal{D}1:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\
+\underline{term\mathcal{X}1:}\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta\underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta\underline{\underline{W^b_\gamma}}
\end{aligned}
\tag{3}
$$

Transform:

$$
\begin{aligned}
\frac{d}{ds}P^a_\alpha := \underline{\underline{I}}{p'}^a_\alpha \\
-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta := \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}u^a_\beta \\
-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta := \underline{\underline{-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta}}p^a_\beta - \underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta = -\underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta \\
\Psi_{\{\alpha\}\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}U^{a-b}_\beta U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\beta\gamma}[r]}}u^{a-b}_\beta u^b_\gamma \\
\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{V^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}p^{a-b}_\beta}}u^b_\gamma \\
\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{W^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{(\alpha)\delta\epsilon}[(1-\kappa rcos\phi)]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}p^{a-b}_\beta}}u^b_\gamma \\
ia\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}P^{a-b}_\beta U^b_\gamma := ia\kappa\sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]}}p^{a-b}_\beta u^b_\gamma \\
-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])\sum_{b=-\infty}^{+\infty}U^{a-b}_\beta U^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])}}u^{a-b}_\beta u^b_\gamma \\
-\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) := \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\mathbf{G},\mathbf{I}] + \Psi_{\alpha\beta\gamma}[r][\mathbf{I},\mathbf{G}])u^{a-b}_\beta u^b_\gamma\blacksquare \\
\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta\underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta\underline{\underline{W^b_\gamma}} := \sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[rcos\phi][\mathbf{I},\mathbf{V}] - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi][\mathbf{I},\mathbf{W}]u^b_\gamma
\end{aligned}
\tag{4}
$$

■Proof:

$$G^a_{\alpha\beta} = -\Psi_{\{\alpha\}\beta}[r]$$

$$-\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\frac{dU^{a-b}_{\beta}}{ds}U^b_{\gamma}+\frac{dU^b_{\gamma}}{ds}U^{a-b}_{\beta}) := \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\mathbf{G},\mathbf{I}]+\Psi_{\alpha\beta\gamma}[r][\mathbf{I},\mathbf{G}])u^{a-b}_{\beta}u^b_{\gamma} \tag{5}$$

3. momentum equation II

$$\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_{\beta}\\
&+\int_0^{2\pi}[\psi_{\alpha}\psi_{\beta}r(1-\kappa rcos\phi)]^h_0 d\theta P^a_{\beta} - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P^a_{\beta} - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]P^a_{\beta}\\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_{\beta}V^a_{\gamma} + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_{\beta}V^a_{\gamma} + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_{\beta}V^a_{\gamma}\\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_{\beta}V^b_{\gamma}\\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_{\beta}V^b_{\gamma} - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_{\beta}}{ds}V^b_{\gamma}+\frac{dV^b_{\gamma}}{ds}U^{a-b}_{\beta})\\
&+\underline{term\mathcal{X}2:}\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_{\beta}W^b_{\gamma} - \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_{\beta}U^b_{\gamma}
\end{aligned} \tag{6}$$

Transform:

$$\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_{\beta} := -N^a_{\alpha\beta}V^a_{\beta}\\
&\{\int_0^{2\pi}[\psi_{\alpha}\psi_{\beta}r(1-\kappa rcos\phi)]^h_0 d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)] - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]\}P^a_{\beta}\\
&\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_{\beta}V^a_{\gamma} := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\delta\epsilon}[r]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}u^{a-b}_{\beta}p^b_{\gamma} = \sum_{b=-\infty}^{+\infty}\Psi_{\{\alpha\}\beta\gamma}[r][\mathbf{I},\mathbf{V}]u^{a-b}_{\beta}p^b_{\gamma}\\
&\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_{\beta}V^a_{\gamma} := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_{\beta}p^b_{\gamma}\\
&\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_{\beta}V^a_{\gamma} := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{(\alpha)\delta\epsilon}[1-\kappa rcos\phi]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_{\beta}p^b_{\gamma}\\
&i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_{\beta}V^b_{\gamma} := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{ia\kappa\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_{\beta}p^b_{\gamma}\\
&-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_{\beta}V^b_{\gamma} := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\delta\epsilon}[r])\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}u^{a-b}_{\beta}p^b_{\gamma}\\
&-\Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_{\beta}}{ds}V^b_{\gamma}+\frac{dV^b_{\gamma}}{ds}U^{a-b}_{\beta}) :== \Psi_{\alpha\beta\gamma}[r][G,I]+\Psi_{\alpha\beta\gamma}[r][I,G]
\end{aligned} \tag{7}$$

4. *momentum equation III*

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]W^a_\beta \\
&-\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta \\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta W^a_\gamma \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta W^b_\gamma \\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta V^b_\gamma
\end{aligned}
\tag{8}
$$