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MAT 415/515
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Lecture 15 Notes

These notes correspond to Section 14.2 in the text.

Orthogonality of Bessel Functions

Since Bessel functions often appear in solutions of PDE, it is necessary to be able to compute coefficients of series whose terms include Bessel functions. Therefore, we need to understand their orthogonality properties.

Consider the Bessel equation

$$\rho^2 \frac{d^2 J_\nu(k\rho)}{d\rho^2} + \rho \frac{dJ_\nu(k\rho)}{d\rho} + (k^2\rho^2 - \nu^2) J_\nu(k\rho) = 0,$$

where $\nu \geq -1$. Rearranging yields

$$-\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}\right) J_\nu(k\rho) = k^2 J_\nu(k\rho).$$

Thus $J_\nu(k\rho)$ is an eigenfunction of the linear differential operator

$$\mathcal{L} = -\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{\nu^2}{\rho^2}\right)$$

with eigenvalue k^2.

The operator $\mathcal{L}$ is not self-adjoint with respect to the standard scalar product, as the coefficients $p_0(\rho) = -1$ and $p_1(\rho) = -1/\rho$ do not satisfy the condition $p_1'(\rho) = p_0(\rho)$, so we use the weight function

$$w(\rho) = \frac{1}{p_0(\rho)} e^{\int \frac{p_1(\rho)}{p_0(\rho)}\, d\rho} = -e^{\int 1/\rho\, d\rho} = -e^{\ln \rho} = -\rho.$$

It follows from the relation

$$(\lambda_u - \lambda_v) \int_a^b v^*(x) u(x) w(x)\, dx = \big(w(x) p_0(x)(v^*(x)u'(x) - (v^*)'(x)u(x))\big)\Big|_a^b$$

that

$$\int_0^a \rho J_\nu(k\rho) J_\nu(k'\rho)\, d\rho = \frac{a[k' J_\nu(ka) J_\nu'(k'a) - k J_\nu(k'a) J_\nu'(ka)]}{k^2 - k'^2}.$$

Therefore, in order to ensure orthogonality, we must have ka and $k'a$ be zeros of J_ν. Thus we have the orthogonality relation

$$\int_0^a \rho J_\nu\left(\alpha_{\nu i}\frac{\rho}{a}\right) J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right) d\rho = 0, \quad i \neq j,$$

where $\alpha_{\nu j}$ is the jth zero of J_ν.

It is worth noting that because of the weight function ρ being the Jacobian of the change of variable to polar coordinates, Bessel functions that are scaled as in the above orthogonality relation are also orthogonal with respect to the *unweighted* scalar product over a circle of radius a.

Normalization

Now that we have *orthogonal* Bessel functions, we seek *orthonormal* Bessel functions. From

$$\int_0^a \rho[J_\nu(k\rho)]^2\,d\rho = \lim_{k'\to k} \frac{a[k'J_\nu(ka)J'_\nu(k'a) - kJ_\nu(k'a)J'_\nu(ka)]}{k^2 - k'^2}.$$

Substituting $ka = \alpha_{\nu i}$ and applying l'Hospital's Rule yields

$$\int_0^a \rho[J_\nu(k\rho)]^2\,d\rho = \lim_{k'\to k} -\frac{kaJ_\nu(k'a)J'_\nu(\alpha_{\nu i})}{k^2 - k'^2} = \frac{a^2}{2}[J'_\nu(\alpha_{\nu i})]^2.$$

Using the recurrence relation

$$J_\nu(x) = \pm J'_{\nu\pm 1}(x) + \frac{\nu \pm 1}{x} J_{\nu\pm 1}(x),$$

we then obtain

$$\int_0^a \rho[J_\nu\left(\frac{\alpha_{\nu i}}{a}\rho\right)]^2\,d\rho = \frac{a^2}{2}[J_{\nu+1}(\alpha_{\nu i})]^2.$$

Bessel Series

Now we can easily describe functions as series of Bessel functions. If $f(\rho)$ has the expansion

$$f(\rho) = \sum_{j=1}^{\infty} c_{\nu j} J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right), \quad 0 \le \rho \le a, \quad \nu > -1,$$

Then, the coefficients $c_{\nu j}$ are given by

$$c_{\nu j} = \frac{\left\langle J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right)|f(\rho)\right\rangle}{\left\langle J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right)|J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right)\right\rangle} = \frac{2}{a^2[J_{\nu+1}(\alpha_{\nu j})]^2}\int_0^a \rho J_\nu\left(\alpha_{\nu j}\frac{\rho}{a}\right) f(\rho)\,d\rho.$$

It is worth noting that orthonormal sets of Bessel functions can also be obtained by imposing Neumann boundary conditions $J'_\nu(k\rho) = 0$ at $\rho = a$, in which case $ka = \beta_{\nu j}$, where $\beta_{\nu j}$ is the jth zero of J'_ν.

We now consider an example in which a Bessel series is used to describe a solution of a PDE.

Example Consider Laplace's equation in a hollow cylinder of radius a with endcaps at $z = 0$ and $z = h$,

$$\nabla^2 V = 0,$$

with boundary conditions

$$V(a,\varphi,z) = 0, \quad V(\rho,\varphi,0) = 0, \quad V(\rho,\varphi,h) = f(\rho,\varphi)$$

for a given potential function $f(\rho,\varphi)$. In cylindrical coordinates, Laplace's equation becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Using separation of variables, we assume a solution of the form

$$V(\rho,\varphi,z) = P(\rho)\Phi(\varphi)Z(z).$$

Substituting this form into the PDE and dividing by V yields

$$\frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0.$$

Due to periodicity, we require that Φ satisfy

$$\frac{d^2\Phi}{d\varphi^2}=-m^2\Phi,$$

where m is an integer, and therefore $\Phi(\varphi)=e^{im\varphi}$. We also have

$$\frac{d^2Z}{dz^2}=\ell^2Z,$$

which has solutions $e^{\ell z}$. Because of the boundary condition at $z=0$, we take linear combinations of solutions so that

$$Z(z)=\sinh lz.$$

Then, we have

$$\rho^2\frac{d^2P}{d\rho^2}+\rho\frac{dP}{d\rho}+(\ell^2\rho^2-m^2)P=0.$$

This is the Bessel equation of order m, which has solutions $J_m(\ell\rho)$. To satisfy the boundary condition at $\rho=a$, we set $\ell=\alpha_{mj}/a$, where α_{mj} is the jth zero of J_m. We then have

$$V(\rho,\varphi,z)=J_m\left(\alpha_{mj}\frac{\rho}{a}\right)e^{im\varphi}\sinh\left(\alpha_{mj}\frac{z}{a}\right).$$

To satisfy the boundary condition at $z=h$, we take a linear combination of solutions of this and seek the coefficients of the expansion

$$f(\rho,\varphi)=\sum_{m=-\infty}^{\infty}\sum_{j=1}^{\infty}c_{mj}J_m\left(\alpha_{mj}\frac{\rho}{a}\right)e^{im\varphi}\sinh\left(\alpha_{mj}\frac{h}{a}\right).$$

From the orthogonality relation for Bessel functions, as well as the orthogonality relation

$$\int_0^{2\pi}e^{-im\varphi}e^{im'\varphi}\,d\varphi=2\pi\delta_{mm'},$$

we obtain

$$c_{mj}=\frac{1}{\pi a^2\sinh\left(\alpha_{mj}\frac{h}{a}\right)J_{m+1}^2(\alpha_{mj})}\int_0^{2\pi}\int_0^a\rho e^{-im\varphi}J_m\left(\alpha_{mj}\frac{\rho}{a}\right)f(\rho,\varphi)\,d\rho\,d\varphi.$$

Thus the PDE and all boundary conditions are satisfied. $\square$