Note:Governing Euqations of General 3D duct flow

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I. Mass equation

Mass consevation:

$$-ia\kappa P^a + \nabla \cdot \mathbf{U}^a = \sum_{b=-\infty}^{+\infty} (-P^{a-b}\nabla \cdot \mathbf{U}^b - \mathbf{U}^{a-b} \cdot \nabla P^b - \frac{B}{2A} iakP^b P^{a-b}) \tag{1}$$

First, derivation of eq1:

We know that:

$$h_s = 1 - \kappa rcos(\phi), h_r = 1, h_\theta = r \tag{2}$$

Then,

$$\begin{aligned} \nabla \cdot \mathbf{U}^a = \frac{1}{h_1 h_2 h_3}[\frac{\partial(v_1 h_2 h_3)}{\partial w_1} + \frac{\partial(v_2 h_3 h_1)}{\partial w_2} + \frac{\partial(v_3 h_1 h_2)}{\partial w_3}] \\ = \frac{1}{r(1-\kappa rcos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa rcos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa rcos(\phi)))}{\partial \theta}] \end{aligned} \tag{3}$$

Thus, we have the mass equation, approximate RHS by:

$$\begin{aligned} \nabla \cdot \mathbf{U}^b = ib\kappa P^b + o(M^2) \\ \nabla P^b = ib\kappa \mathbf{U}^b + o(M^2) \end{aligned} \tag{4}$$

Then we have

$$\begin{aligned} -ia\kappa P^a + \frac{1}{r(1-\kappa rcos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa rcos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa rcos(\phi)))}{\partial \theta}] \\ = \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A} iakP^b P^{a-b}) \end{aligned} \tag{5}$$

The fourier harmonics are expanded as follows:

$$\begin{aligned} P^a = \sum_{\beta=0}^{\infty} P^a_\beta(s)\psi_\beta(s,r,\theta) \\ U^a = \sum_{\beta=0}^{\infty} U^a_\beta(s)\psi_\beta(s,r,\theta) \\ V^a = \sum_{\beta=0}^{\infty} V^a_\beta(s)\psi_\beta(s,r,\theta) \\ W^a = \sum_{\beta=0}^{\infty} W^a_\beta(s)\psi_\beta(s,r,\theta) \end{aligned} \tag{6}$$

with normalized relation:

$$\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r dr d\theta = \delta_{\alpha\beta} \tag{7}$$

Reorganize the eq5:

$$\begin{aligned}
&-ia\kappa P^a(1-\kappa r cos(\phi)) + \frac{1}{r}\frac{\partial(U^a r)}{\partial s} + \frac{1}{r}\frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{1}{r}\frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial\theta} \\
&= (1-\kappa r cos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A}iakP^bP^{a-b})
\end{aligned} \tag{8}$$

Intergal and insert eq 6, 7 into eq 8:

1. the first term:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[-ia\kappa(1-\kappa r cos(\phi))P^a] dr d\theta \\
&= -ia\kappa\sum_\beta^\infty\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r[(1-\kappa r cos(\phi))] dr d\theta P_\beta^a \\
&= -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P_\beta^a \\
&(summationconvention)
\end{aligned} \tag{9}$$

2. the second term:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(U^a r)}{\partial s}] dr d\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta U_\beta^a)}{\partial s} dr d\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r\frac{\partial(U_\beta^a)}{\partial s} dr d\theta + \int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta)}{\partial s} dr d\theta U_\beta^a \\
&= \frac{dU_\beta^a}{ds}\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r dr d\theta + \int_0^{2\pi}\int_0^h r\frac{\partial(\psi_\alpha\psi_\beta)}{\partial s} dr d\theta U_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta r\frac{\partial(\psi_\alpha)}{\partial s} dr d\theta U_\beta^a \\
&= \frac{dU_\beta^a}{ds}\delta_{\alpha\beta} + \frac{\partial(\int_0^{2\pi}\int_0^h r\psi_\alpha\psi_\beta dr d\theta)}{\partial s}U_\beta^a - \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial s}\psi_\beta r dr d\theta U_\beta^a \\
&= \frac{dU_\alpha^a}{ds} + 0 - \Psi_{\{\alpha\}\beta}[r]U_\beta^a
\end{aligned} \tag{10}$$

3. the third term:

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r}] dr d\theta \\
&= \int_0^{2\pi}\int_0^h \psi_\alpha\frac{\partial(\psi_\beta r(1-\kappa r cos(\phi)))}{\partial r} dr d\theta V_\beta^a \\
&= \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha\psi_\beta r(1-\kappa r cos(\phi)))}{\partial r} dr d\theta V_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta r(1-\kappa r cos(\phi))\frac{\partial(\psi_\alpha)}{\partial r} dr d\theta V_\beta^a \\
&= 0(periodic) - \Psi_{[\alpha]\beta}[r(1-\kappa r cos(\phi))]V_\beta^a
\end{aligned} \tag{11}$$

4. the fourth term:

$$\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(W^a(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta$$
$$=\int_0^{2\pi}\int_0^h \psi_\alpha[\frac{\partial(\psi_\beta(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta W^a_\beta$$
$$=\int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha\psi_\beta(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta W^a_\beta-\int_0^{2\pi}\int_0^h \psi_\beta(1-\kappa rcos(\phi))[\frac{\partial(\psi_\alpha)}{\partial\theta}]drd\theta W^a_\beta \tag{12}$$
$$=\int_0^h [\psi_\alpha\psi_\beta(1-\kappa rcos(\phi))]_0^{2\pi}drW^a_\beta-\int_0^{2\pi}\int_0^h \psi_\beta(1-\kappa rcos(\phi))[\frac{\partial(\psi_\alpha)}{\partial\theta}]drd\theta W^a_\beta$$
$$=0-\Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]W^a_\beta$$

5. the RHS term:

$$\int_0^{2\pi}\int_0^h \psi_\alpha r[(1-\kappa rcos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b-ib\kappa U^{a-b}U^b-ib\kappa V^{a-b}V^b-ib\kappa W^{a-b}W^b-\frac{B}{2A}iakP^bP^{a-b})]drd\theta$$
$$=\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta\psi_\gamma r(1-\kappa rcos(\phi))drd\theta$$
$$\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma-ib\kappa U^{a-b}_\beta U^b_\gamma-ib\kappa V^{a-b}_\beta V^b_\gamma-ib\kappa W^{a-b}_\beta W^b_\gamma-ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma)$$
$$=\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma-ib\kappa U^{a-b}_\beta U^b_\gamma-ib\kappa V^{a-b}_\beta V^b_\gamma-ib\kappa W^{a-b}_\beta W^b_\gamma-ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma) \tag{13}$$

Finally, we obtain the mass equation in the form of eigenfunction, the idea is same as Galerkin method:

$$\frac{dU^a_\alpha}{ds}-\Psi_{\{\alpha\}\beta}[r]U^a_\beta-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta-\Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]V^a_\beta-\Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]W^a_\beta$$
$$=\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma-ib\kappa U^{a-b}_\beta U^b_\gamma-ib\kappa V^{a-b}_\beta V^b_\gamma-ib\kappa W^{a-b}_\beta W^b_\gamma-ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma) \tag{14}$$

II. Momentum equation

Momentum consevation:

$$-ia\kappa\mathbf{U}^a+\nabla P^a=\sum_{b=-\infty}^{\infty}(-\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^b+P^{a-b}\nabla P^b) \tag{15}$$

First, we know that

$$\nabla P^a=\sum_i\frac{1}{h_i}\frac{\partial f}{\partial w_i}\hat{h}_i=\frac{1}{1-\kappa rcos\phi}\frac{\partial P^a}{\partial s}\hat{e}_s+\frac{\partial P^a}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial P^a}{\partial\theta}\hat{e}_\theta \tag{16}$$

The RHS term is a bit complex, with the divergence of a vector U with its gradient, with

First, we know that

$$(\mathbf{v}\cdot\nabla)\mathbf{v}^b = \begin{cases} term1: \mathcal{D}v_1^b + \frac{v_2^b}{h_2h_1}(v_1\frac{\partial h_1}{\partial \xi_2} - v_2\frac{\partial h_2}{\partial \xi_1}) + \frac{v_3^b}{h_3h_1}(v_1\frac{\partial h_1}{\partial \xi_3} - v_2\frac{\partial h_3}{\partial \xi_1}) \\ term2: \mathcal{D}v_2^b + \frac{v_3^b}{h_3h_2}(v_2\frac{\partial h_2}{\partial \xi_3} - v_3\frac{\partial h_3}{\partial \xi_2}) + \frac{v_1^b}{h_1h_2}(v_2\frac{\partial h_2}{\partial \xi_1} - v_1\frac{\partial h_1}{\partial \xi_2}) \\ term3: \mathcal{D}v_3^b + \frac{v_1^b}{h_1h_3}(v_3\frac{\partial h_3}{\partial \xi_1} - v_1\frac{\partial h_1}{\partial \xi_3}) + \frac{v_2^b}{h_2h_3}(v_3\frac{\partial h_3}{\partial \xi_2} - v_3\frac{\partial h_2}{\partial \xi_3}) \end{cases} \tag{17}$$

Besides,

$$\mathcal{D} = \frac{v_1}{h_1}\frac{\partial}{\partial \xi_1} + \frac{v_2}{h_2}\frac{\partial}{\partial \xi_2} + \frac{v_3}{h_3}\frac{\partial}{\partial \xi_3} \tag{18}$$

Thus, we have:

$$\begin{aligned} -\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^b = \\ -\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{D}1: \frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s} + V^{a-b}\frac{\partial U^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2: \frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s} + V^{a-b}\frac{\partial V^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3: \frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s} + V^{a-b}\frac{\partial W^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases} \\ -\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{X}1: \frac{V^b}{1-\kappa rcos\phi}(U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial r} - V^{a-b}\frac{\partial 1}{\partial s}) + \frac{W^b}{r(1-\kappa rcos\phi)}(U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial \theta} - V^{a-b}\frac{\partial r}{\partial s}) \\ term\mathcal{X}2: \frac{W^b}{r}(V^{a-b}\frac{\partial 1}{\partial \theta} - W^{a-b}\frac{\partial r}{\partial r}) + \frac{U^b}{(1-\kappa rcos\phi)1}(V^{a-b}\frac{\partial 1}{\partial s} - U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial r}) \\ term\mathcal{X}3: \frac{U^b}{(1-\kappa rcos\phi)r}(W^{a-b}\frac{\partial h_3}{\partial s} - U^{a-b}\frac{\partial(1-\kappa rcos\phi)}{\partial \theta}) + \frac{V^b}{1h_3}(W^{a-b}\frac{\partial h_3}{\partial r} - V^{a-b}\frac{\partial 1}{\partial \theta}) \end{cases} = \\ \sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{D}1: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3: -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases} \\ +\sum_{b=-\infty}^{\infty}\begin{cases} term\mathcal{X}1: \frac{\kappa cos\phi}{1-\kappa rcos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}W^b \\ term\mathcal{X}2: \frac{W^{a-b}W^b}{r} - \frac{\kappa cos\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b \\ term\mathcal{X}3: \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \end{cases} \end{aligned} \tag{19}$$

Finally, we could derive the momentum conservation equation, with final term approximate by eq 4:

$$\begin{cases} -ia\kappa U^a + \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s} \\ -ia\kappa V^a + \frac{\partial P^a}{\partial r} \\ -ia\kappa W^a + \frac{1}{r}\frac{\partial P^a}{\partial \theta} \end{cases}$$

$$= \sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases} \tag{20}$$

$$+ \sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa rcos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}W^b \\ term\mathcal{X}2 : \frac{W^{a-b}W^b}{r} - \frac{\kappa cos\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b \\ term\mathcal{X}3 : \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \end{cases} + \sum_{b=-\infty}^{\infty} \begin{cases} ib\kappa P^{a-b}U^b \\ ib\kappa P^{a-b}V^b \\ ib\kappa P^{a-b}W^b \end{cases}$$

Now, we are going to project on ψ, it may be a little complex, we will doing step by step.

A. Momentum e^s term

First, deal with the e^s term:

$$\begin{aligned} & -ia\kappa U^a + \frac{1}{1-\kappa rcos\phi}\frac{\partial P^a}{\partial s} \\ = & \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ & + \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa rcos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}W^b \\ & + ib\kappa P^{a-b}U^b \end{aligned} \tag{21}$$

Multiply $(1-\kappa rcos\phi)$, we have:

$$\begin{aligned} & -ia\kappa(1-\kappa rcos\phi)U^a + \frac{\partial P^a}{\partial s} \\ = & \, term\mathcal{D}1 : \sum_{b=-\infty}^{\infty} -U^{a-b}\frac{\partial U^b}{\partial s} - (1-\kappa rcos\phi)V^{a-b}\frac{\partial U^b}{\partial r} - (1-\kappa rcos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ & + term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b + \\ & term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b \end{aligned} \tag{22}$$

$\int\int XXr\psi_\alpha drd\theta$, we have:

$$\underline{RHS = \int_0^{2\pi}\int_0^h [term\mathcal{D}1 + term\mathcal{X}1 + term\mathcal{P}1]r\psi_\alpha drd\theta} \tag{23}$$

1. the first $\mathcal{D}1$ tems:

We ref the wiki $https://en.wikipedia.org/wiki/Leibniz\_integral\_rule$

General form: Differentiation under the integral sign:

$$\frac{d}{dx}(\int_{a(x)}^{b(x)} f(x,t)dt) - f(x,b(x)) \cdot \frac{d}{dx}b(x) + f(x,a(x)) \cdot \frac{d}{dx}a(x) = \int_{a(x)}^{b(x)} \frac{d}{dx}f(x,t)dt \tag{24}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{25}$$

For partial difference, for a given β, the derivation of the fucntion $g(\alpha) = \int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx$ is

$$\frac{d}{d\alpha}(\int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx) - 0 + \frac{da(\alpha)}{d\alpha}f(a(\alpha),\alpha) = \int_{a(\alpha)}^{b(\beta)} \frac{\partial}{\partial\alpha}f(x,\alpha)dx \tag{26}$$

1.1

$$\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [-rU^{a-b}\frac{\partial U^b}{\partial s}]\psi_\alpha dr d\theta \\
&= \sum_{b=-\infty}^{\infty} -\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]dr d\theta \\
&\quad + \int_0^{2\pi}\int_0^h U^b\frac{\partial}{\partial s}[rU^{a-b}\psi_\alpha]dr d\theta \\
= \sum_{b=-\infty}^{\infty} -\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]dr d\theta &+ \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta \\
&\quad + \int_0^{2\pi}\int_0^h \frac{r\partial U^{a-b}}{\partial s}U^b\psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s}rU^{a-b}U^b dr d\theta
\end{aligned} \tag{27}$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$h'U^{a-b} = (1 - \kappa h cos\phi)V^{a-b} \tag{28}$$

1.2

$$\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h[-(1-\kappa rcos\phi)V^{a-b}\frac{\partial U^b}{\partial r}]r\psi_\alpha drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h\frac{\partial(r(1-\kappa rcos\phi)V^{a-b}U^b\psi_\alpha)}{\partial r}drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b}\psi_\alpha)}{\partial r}drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}[(r(1-\kappa rcos\phi)V^{a-b}U^b)\psi_\alpha]_0^h d\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa rcos\phi)V^{a-b}drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}[(hh'U^{a-b}U^b)\psi_\alpha]_0^h d\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa rcos\phi)V^{a-b}drd\theta
\end{aligned}\tag{29}$$

1.3

$$\begin{aligned}
&\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h[-(1-\kappa rcos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial\theta}]r\psi_\alpha drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h\frac{\partial((1-\kappa rcos\phi)W^{a-b}U^b\psi_\alpha)}{\partial\theta}drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta \\
&=-\sum_{b=-\infty}^{\infty}\int_0^h[(1-\kappa rcos\phi)W^{a-b}U^b\psi_\alpha]_0^{2\pi}dr \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta \\
&=0(periodic)+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b})drd\theta
\end{aligned}\tag{30}$$

Combine together:

$$
\begin{aligned}
&\underline{\int_0^{2\pi}\int_0^h [term\mathcal{D}1] r\psi_\alpha drd\theta} = \sum_{b=-\infty}^{\infty} \\
&\{(\int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s} rU^{a-b}U^b drd\theta \\
&+\int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r(1-\kappa rcos\phi)V^{a-b}U^b drd\theta \\
&+\int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial\theta}((1-\kappa rcos\phi)W^{a-b}U^b)drd\theta) \\
&+(\int_0^{2\pi}\int_0^h U^b\frac{\partial U^{a-b}}{\partial s} r\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial(r(1-\kappa rcos\phi)V^{a-b})}{\partial r}\psi_\alpha drd\theta \\
&+\int_0^{2\pi}\int_0^h U^b\frac{\partial((1-\kappa rcos\phi)W^{a-b})}{\partial\theta}\psi_\alpha drd\theta) \\
&-\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta\}
\end{aligned} \tag{31}
$$

We apply eq 5, find that:

$$
\begin{aligned}
-i(a-b)\kappa r(1-\kappa rcos(\phi))P^{a-b} + [\frac{\partial(U^{a-b}r)}{\partial s} + \frac{\partial(V^{a-b}r(1-\kappa rcos(\phi)))}{\partial r} + \frac{\partial(W^{a-b}(1-\kappa rcos(\phi)))}{\partial\theta}] \\
= o(M^2)
\end{aligned} \tag{32}
$$

We have the second terms in eq(31) are equal to:

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h U^b[-i(a-b)\kappa r(1-\kappa rcos(\phi))P^{a-b}]\psi_\alpha drd\theta \\
&= i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P_\beta^{a-b}U_\gamma^b
\end{aligned} \tag{33}
$$

And, the longitudinal derivation s can also be expand about the duct modes, with note $[r],(\theta),\{s\}$:

$$
\begin{aligned}
&\frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta \\
&= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]U_\beta^{a-b}U_\gamma^b drd\theta \\
&= \frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta)U_\beta^{a-b}U_\gamma^b \\
&+\frac{\partial U_\beta^{a-b}U_\gamma^b}{\partial s}\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta \\
&= \frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta)U_\beta^{a-b}U_\gamma^b \\
&+(\frac{dU_\beta^{a-b}}{ds}U_\gamma^b + \frac{dU_\gamma^b}{ds}U_\beta^{a-b})\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]drd\theta
\end{aligned} \tag{34}
$$

2. the second $\mathcal{X}1$ tems:

$$term\mathcal{X}1: \sum_{b=-\infty}^{\infty} \kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b$$

$$\begin{aligned}\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [\kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b] r\psi_\alpha drd\theta \\ = \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U_\beta^{a-b}V_\gamma^b - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U_\beta^{a-b}W_\gamma^b\end{aligned} \tag{35}$$

3. the second $\mathcal{P}1$ tems:

$$term\mathcal{P}1: \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b$$

$$\begin{aligned}\sum_{b=-\infty}^{\infty}\int_0^{2\pi}\int_0^h [ib\kappa(1-\kappa rcos\phi)P^{a-b}U^b] r\psi_\alpha drd\theta \\ = ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P_\beta^{a-b}U_\gamma^b\end{aligned} \tag{36}$$

4. The LHS terms:

$$\frac{\partial P^a}{\partial s} - ia\kappa(1-\kappa rcos\phi)U^a$$

From 1.1 as example, we know that

$$\begin{aligned}\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\ = \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta\end{aligned} \tag{37}$$

4.1

$$\begin{aligned}\int_0^{2\pi}\int_0^h [\frac{\partial P^a}{\partial s}] r\psi_\alpha drd\theta \\ = \int_0^{2\pi}\int_0^h [\frac{\partial(P_\beta^a\psi_\beta)}{\partial s}] r\psi_\alpha drd\theta \\ = \int_0^{2\pi}\int_0^h [\frac{\partial(P_\beta^a\psi_\beta)}{\partial s} r\psi_\alpha] drd\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a \\ = \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h (P_\beta^a\psi_\beta) r\psi_\alpha drd\theta - \frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h} - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a \\ = \frac{\partial}{\partial s}(P_\beta^a\delta_{\alpha\beta}) - \int_0^{2\pi}\frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h}d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a \\ = \frac{d}{ds}P_\alpha^a - \int_0^{2\pi}\frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h}d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta drd\theta P_\beta^a\end{aligned} \tag{38}$$

$$= \frac{d}{ds}P^a_\alpha - \int_0^{2\pi}\int_0^h \frac{\partial \psi_\beta \psi_\alpha}{\partial r} hh' dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P^a_\beta$$

4.2

$$\begin{aligned} &\int_0^{2\pi}\int_0^h [-ia\kappa(1-\kappa r cos\phi)U^a] r\psi_\alpha dr d\theta \\ &= -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta \end{aligned} \tag{39}$$

Finally, putting all together becomes:

$$\begin{aligned} \frac{d}{ds}P^a_\alpha - \int_0^{2\pi}\frac{dh(s)}{ds}[P^a_\beta\psi_\beta r\psi_\alpha]_{r=h}d\theta - \int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta \\ = \frac{d}{ds}P^a_\alpha - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta - \int_0^{2\pi} hh'[P^a_\beta\psi_\beta\psi_\alpha]_{r=h}d\theta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\ = \sum_{b=-\infty}^{\infty} \\ (eq31): \{(\int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s} rU^{a-b}U^b dr d\theta \\ + \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r(1-\kappa r cos\phi)V^{a-b}U^b dr d\theta \\ +(eq33): i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\ eq(34): -\frac{\partial}{\partial s}(\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]dr d\theta)U^{a-b}_\beta U^b_\gamma \\ -(\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta)\int_0^{2\pi}\int_0^h [r\psi_\beta\psi_\gamma\psi_\alpha]dr d\theta \\ +eq(35): \kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma \\ +eq(36): ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\ = (abbreviation): \\ \Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta U^a_\gamma \\ +i(a-b)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\ -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\ +\kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma \\ +ib\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \end{aligned} \tag{40}$$

To be conclude, with the e^s term:

$$\begin{aligned}
&-ia\kappa U^a + \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{P}1 : ib\kappa P^{a-b}U^b
\end{aligned} \tag{41}$$

We have:

$$\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U^a_\beta + \frac{d}{ds}P^a_\alpha - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\
&= \underline{term\mathcal{D}1 :}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta U^a_\gamma \\
&+ \underline{term(D1+\mathcal{P}1) :}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta U^b_\gamma \\
&- \underline{term\mathcal{D}1 :}\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\
&+ \underline{term\mathcal{X}1 :}\kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta V^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta W^b_\gamma
\end{aligned} \tag{42}$$

B. Momentum e^r term

Second, deal with the e^r term:

$$\begin{aligned}
&-ia\kappa V^a + \frac{\partial P^a}{\partial r} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}2 : \frac{W^{a-b}W^b}{r} - \frac{\kappa cos\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b \\
&+ term\mathcal{P}2 : \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b}V^b
\end{aligned} \tag{43}$$

LHS-2: $\frac{\partial P^a}{\partial r}(1-\kappa rcos\phi)$

$$\begin{aligned}
&\int_0^{2\pi}\int_0^h[\frac{\partial P^a}{\partial r}(1-\kappa rcos\phi)]r\psi_\alpha drd\theta \\
&=\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\beta(1-\kappa rcos\phi)r\psi_\alpha}{\partial r}]drd\theta P^a_\beta-\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\alpha(1-\kappa rcos\phi)r}{\partial r}]\psi_\beta drd\theta P^a_\beta \\
&=\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\beta(1-\kappa rcos\phi)r\psi_\alpha}{\partial r}]drd\theta P^a_\beta \\
&-\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\alpha}{\partial r}](1-\kappa rcos\phi)r\psi_\beta drd\theta P^a_\beta-\int_0^{2\pi}\int_0^h[\frac{\partial(1-\kappa rcos\phi)r}{\partial r}]\psi_\alpha\psi_\beta drd\theta P^a_\beta \\
&=\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta \\
&-\int_0^{2\pi}\int_0^h[\frac{\partial\psi_\alpha}{\partial r}](1-\kappa rcos\phi)r\psi_\beta drd\theta P^a_\beta-\int_0^{2\pi}\int_0^h[1-2\kappa rcos\phi]\psi_\alpha\psi_\beta drd\theta P^a_\beta \\
&=\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta-\Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P^a_\beta-\Psi_{\alpha\beta}[1-2\kappa rcos\phi]P^a_\beta
\end{aligned} \tag{44}$$

The derivation of $term\mathcal{D}2$ is identical to A, we are not prove it again. $Term\mathcal{P}2$ also could be combine with the part separated term of $term\mathcal{D}2$ with V^b. $Term\mathcal{X}2$ is also easy to derive.

Thus, we have the final equation:

$$\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_\beta \\
&\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta-\Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P^a_\beta-\Psi_{\alpha\beta}[1-2\kappa rcos\phi]P^a_\beta \\
&=\underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma+\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta V^a_\gamma+\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta V^a_\gamma \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta V^b_\gamma \\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma-\Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}V^b_\gamma+\frac{dV^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta W^b_\gamma-\kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta U^b_\gamma
\end{aligned} \tag{45}$$

C. Momentum e^θ term

Third, deal with the e^θ term:

$$
\begin{aligned}
& -ia\kappa W^a + \frac{1}{r}\frac{\partial P^a}{\partial \theta} \\
= \sum_{b=-\infty}^{\infty} term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa rcos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \\
+ \sum_{b=-\infty}^{\infty} term\mathcal{X}3 : \frac{\kappa sin\phi}{(1-\kappa rcos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \\
+term\mathcal{P}3 : \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b}W^b
\end{aligned}
\tag{46}
$$

LHS-2: $\frac{\partial P^a}{\partial \theta}\frac{(1-\kappa rcos\phi)}{r}$

$$
\begin{aligned}
& \int_0^{2\pi}\int_0^h [\frac{\partial P^a}{\partial \theta}\frac{(1-\kappa rcos\phi)}{r}] r\psi_\alpha dr d\theta \\
= & \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\beta (1-\kappa rcos\phi)\psi_\alpha}{\partial \theta}] dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha (1-\kappa rcos\phi)}{\partial \theta}]\psi_\beta dr d\theta P^a_\beta \\
= 0 - & \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha}{\partial \theta}](1-\kappa rcos\phi)\psi_\beta dr d\theta P^a_\beta - \int_0^{2\pi}\int_0^h [\frac{\partial (1-\kappa rcos\phi)}{\partial \theta}]\psi_\alpha\psi_\beta dr d\theta P^a_\beta \\
= - & \int_0^{2\pi}\int_0^h [\frac{\partial \psi_\alpha}{\partial \theta}](1-\kappa rcos\phi)\psi_\beta dr d\theta P^a_\beta + \kappa\int_0^{2\pi}\int_0^h [rsin\phi]\psi_\alpha\psi_\beta dr d\theta P^a_\beta \\
= & -\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta
\end{aligned}
\tag{47}
$$

The derivation of $term\mathcal{D}3$ is identical to A, we are not prove it again. $Term\mathcal{P}3$ also could be combine with the part separated term of $term\mathcal{D}3$ with W^b. $Term\mathcal{X}3$ is also easy to derive.

Thus, we have the final equation:

$$
\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]W^a_\beta \\
& -\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta \\
= & \underline{term\mathcal{D}2 :}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta W^a_\gamma \\
& +\underline{term(D2+\mathcal{P}2) :}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta W^b_\gamma \\
& +\underline{term\mathcal{D}2 :} -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta) \\
& +\underline{term\mathcal{X}2 :}\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta V^b_\gamma
\end{aligned}
\tag{48}
$$

III. Merge the four equations and eliminate the V_γ^b and W_γ^b

A. V_α^a & W_α^a

Using the linear relationships:

$$
\begin{aligned}
ia\kappa V^a &= \frac{\partial P^a}{\partial r} \\
:= \int\int iakV_\beta^a \psi_\beta r\psi_\alpha drd\theta &= \int\int \frac{\partial P_\beta^a \psi_\beta}{\partial r} r\psi_\alpha drd\theta \\
= ia\kappa V_\beta^a \delta_{\alpha\beta} = \Psi_{\alpha[\beta]}[r]P_\beta^a = \int_0^{2\pi} [r\psi_\alpha\psi_\beta]_0^h d\theta P_\beta^a - \int\int \psi_\alpha\psi_\beta drd\theta P_\beta^a &- \int\int \frac{\partial\psi_\alpha}{\partial r}\psi_\beta rdrd\theta P_\beta^a
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
iakW^a &= \frac{1}{r}\frac{\partial P^a}{\partial\theta} \\
:= \int\int iakW_\beta^a \psi_\beta r\psi_\alpha drd\theta &= \int\int \frac{\partial P_\beta^a \psi_\beta}{\partial\theta}\frac{1}{r} r\psi_\alpha drd\theta \\
= ia\kappa W_\beta^a \delta_{\alpha\beta} = \Psi_{\alpha(\beta)}[r]P_\beta^a &= 0 - \Psi_{(\alpha)\beta}[r]P_\beta^a
\end{aligned}
\tag{50}
$$

Thus, we can establish relationships between the tranverse modes and pressure modes (no summation over a)

$$
V_\alpha^a = \frac{1}{ia\kappa}[\int_0^{2\pi} [r\psi_\alpha\psi_\beta]_0^h d\theta P_\beta^a - \Psi_{\alpha\beta} - \Psi_{[\alpha]\beta}[r]]P_\beta^a = \mathbf{V}_{\alpha\beta}^a P_\beta^a \tag{51}
$$

$$
W_\alpha^a = -\frac{1}{ia\kappa}\Psi_{(\alpha)\beta}P_\beta^a = \mathbf{W}_{\alpha\beta}^a P_\beta^a \tag{52}
$$

B. $\frac{d}{ds}V_\alpha^a$ & $\frac{d}{ds}W_\alpha^a$

We also require modal expressions for $\frac{d}{ds}V_\alpha^a$ and $\frac{d}{ds}W_\alpha^a$.

We differentiate eq49 with respect to s:

$$
\begin{aligned}
\frac{\partial V^a}{\partial s} &= \frac{1}{ia\kappa}\frac{\partial^2 P^a}{\partial s\partial r} \\
&= \frac{\partial}{\partial r}((1 - \kappa r cos\phi)U^a)
\end{aligned}
\tag{53}
$$

where we have used symmetry of mixed partials and the linear expression for $\frac{\partial P^a}{\partial s}$ from eq 21.

From 1.1 as example, we know that

$$
\begin{aligned}
&\int_0^{2\pi}\int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]drd\theta \\
&= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [rU^{a-b}U^b\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta
\end{aligned}
\tag{54}
$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$h'U^a_\beta = (1-\kappa hcos\phi)V^a_\beta \tag{55}$$

Multiplying this expression by $r\phi_\alpha$ and integrating across section of the duct, we obtain:

$$\begin{aligned}
\int_0^{2\pi}\int_0^h \frac{\partial V^a}{\partial s} r\psi_\alpha drd\theta &= \int_0^{2\pi}\int_0^h \frac{\partial}{\partial r}((1-\kappa rcos\phi)U^a)r\psi_\alpha drd\theta \\
LHS := \int_0^{2\pi}\int_0^h \frac{\partial[V^a_\beta\psi_\beta r\psi_\alpha]}{\partial s} drd\theta &- \int_0^{2\pi}\int_0^h \frac{\partial r\psi_\alpha}{\partial s}\psi_\beta drd\theta V^a_\beta \\
= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [V^a_\beta\psi_\beta r\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[V^a_\beta\psi_\beta r\psi_\alpha]_0^h d\theta &- \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s}\psi_\beta rdrd\theta V^a_\beta \\
= \frac{\partial}{\partial s}V^a_\beta\delta_{\alpha\beta} - \int_0^{2\pi}\frac{h'^2}{1-\kappa hcos\phi}[r\psi_\beta\psi_\alpha]_0^h d\theta &- \Psi_{\{\alpha\}\beta}[r]V^a_\beta \\
RHS := \int_0^{2\pi}\int_0^h \frac{\partial}{\partial r}((1-\kappa rcos\phi)U^a_\beta\psi_\beta r\psi_\alpha)drd\theta &- \int_0^{2\pi}\int_0^h \frac{\partial(r\psi_\alpha)}{\partial r}(1-\kappa rcos\phi)U^a_\beta\psi_\beta drd\theta \\
= \int_0^{2\pi}[r(1-\kappa rcos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U^a_\beta - \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial r} r(1-\kappa rcos\phi)\psi_\beta drd\theta U^a_\beta &- \int_0^{2\pi}\int_0^h (1-\kappa rcos\phi)\psi_\alpha\psi_\beta drd\theta U^a_\beta \\
= \int_0^{2\pi}[r(1-\kappa rcos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]U^a_\beta &- \Psi_{\alpha\beta}[(1-\kappa rcos\phi)]U^a_\beta
\end{aligned} \tag{56}$$

Thus, LHS=RHS, we have:

$$\begin{aligned}
\frac{d}{ds}V^a_\alpha = \int_0^{2\pi}\frac{h'^2}{1-\kappa hcos\phi}[r\psi_\beta\psi_\alpha]_0^h d\theta + \Psi_{\{\alpha\}\beta}[r]V^a_\beta \\
+\int_0^{2\pi}[r(1-\kappa rcos\phi)\psi_\beta\psi_\alpha]_0^h d\theta U^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]U^a_\beta - \Psi_{\alpha\beta}[(1-\kappa rcos\phi)]U^a_\beta
\end{aligned} \tag{57}$$

Similarly for W^a, differentiating eq50 with respect to s and substituting the linear expression for $\frac{\partial P^a}{\partial s}$ by eq21:

$$\begin{aligned}
\frac{\partial W^a}{\partial s} &= \frac{1}{ia\kappa}\frac{\partial^2 P^a}{\partial s\partial\theta} \\
&= \frac{1}{r}\frac{\partial}{\partial\theta}((1-\kappa rcos\phi)U^a)
\end{aligned} \tag{58}$$

Multiplying this expression by $r\phi_\alpha$ and integrating across section of the duct, we obtain:

$$\int_0^{2\pi}\int_0^h \frac{\partial W^a}{\partial s} r\psi_\alpha drd\theta = \int_0^{2\pi}\int_0^h \frac{1}{r}\frac{\partial}{\partial\theta}((1-\kappa rcos\phi)U^a)r\psi_\alpha drd\theta$$

$$\begin{aligned}
LHS &:= \int_0^{2\pi}\int_0^h \frac{\partial[W^a_\beta\psi_\beta r\psi_\alpha]}{\partial s}drd\theta - \int_0^{2\pi}\int_0^h \frac{\partial r\psi_\alpha}{\partial s}\psi_\beta drd\theta W^a_\beta \\
&= \frac{\partial}{\partial s}\int_0^{2\pi}\int_0^h [W^a_\beta\psi_\beta r\psi_\alpha]drd\theta - \int_0^{2\pi}\frac{dh(s)}{ds}[W^a_\beta\psi_\beta r\psi_\alpha]_0^h d\theta - \int_0^{2\pi}\int_0^h \frac{\partial\psi_\alpha}{\partial s}\psi_\beta rdrd\theta W^a_\beta \\
&= \frac{d}{ds}W^a_\alpha - \int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]_0^h d\theta W^a_\beta - \Psi_{\{\alpha\}\beta}[r]W^a_\beta \\
RHS &:= \int_0^{2\pi}\int_0^h \frac{\partial}{\partial\theta}((1-\kappa rcos\phi)U^a_\beta\psi_\beta\psi_\alpha)drd\theta - \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial\theta}(1-\kappa rcos\phi)U^a_\beta\psi_\beta drd\theta \\
&= 0(periodic) - \Psi_{(\alpha)\beta}[1-\kappa rcos\phi]U^a_\beta
\end{aligned} \tag{59}$$

Thus, LHS=RHS, we have:

$$\frac{d}{ds}V^a_\alpha = \int_0^{2\pi}\frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]_0^h d\theta W^a_\beta + \Psi_{\{\alpha\}\beta}[r]W^a_\beta - \Psi_{(\alpha)\beta}[1-\kappa rcos\phi]U^a_\beta \tag{60}$$

C. Substitue pressure modes for transverse velocity modes

1. mass equation

$$\begin{aligned}
&\frac{dU^a_\alpha}{ds} - \Psi_{\{\alpha\}\beta}[r]U^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{\underline{V^a_\beta}}} - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{\underline{W^a_\beta}}} \\
&= \Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa\underline{\underline{\underline{V^{a-b}_\beta V^b_\gamma}}} - ib\kappa\underline{\underline{\underline{W^{a-b}_\beta W^b_\gamma}}} - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma)
\end{aligned} \tag{61}$$

Transform:

$$\begin{aligned}
\Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{\underline{V^a_\beta}}} &:= \Psi_{[\alpha]\delta}[r(1-\kappa rcos(\phi))]\mathbf{V}^a_{\delta\beta}P^a_\beta \\
\Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{\underline{W^a_\beta}}} &:= \Psi_{(\alpha)\delta}[(1-\kappa rcos(\phi))]\mathbf{W}^a_{\delta\beta}P^a_\beta
\end{aligned} \tag{62}$$

$$
\begin{aligned}
-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U_\beta^a + \frac{d}{ds}P_\alpha^a - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h} d\theta P_\beta^a - \Psi_{\{\alpha\}\beta}[r]P_\beta^a \\
= \underline{term\mathcal{D}1:}\Psi_{\{\alpha\}\beta\gamma}[r]U_\beta^{a-b}U_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]\underline{\underline{V_\beta^{a-b}}}U_\gamma^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]\underline{\underline{W_\beta^{a-b}}}U_\gamma^a \\
+\underline{term(D1+\mathcal{P}1):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}U_\gamma^b \\
+\underline{term\mathcal{D}1:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_\beta^{a-b}U_\gamma^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_\beta^{a-b}}{ds}U_\gamma^b + \frac{dU_\gamma^b}{ds}U_\beta^{a-b}) \\
+\underline{term\mathcal{X}1:}\kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U_\beta^{a-b}\underline{\underline{V_\gamma^b}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U_\beta^{a-b}\underline{\underline{W_\gamma^b}}
\end{aligned} \tag{63}
$$

$$
\begin{aligned}
-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]V_\beta^a \\
\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa r cos\phi)]_0^h d\theta P_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]P_\beta^a - \Psi_{\alpha\beta}[1-2\kappa r cos\phi]P_\beta^a \\
= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U_\beta^{a-b}V_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V_\beta^{a-b}V_\gamma^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W_\beta^{a-b}V_\gamma^a \\
+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}V_\gamma^b \\
+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U_\beta^{a-b}U_\gamma^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_\beta^{a-b}}{ds}V_\gamma^b + \frac{dV_\gamma^b}{ds}U_\beta^{a-b}) \\
+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[1-\kappa r cos\phi]W_\beta^{a-b}W_\gamma^b - \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U_\beta^{a-b}U_\gamma^b
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]W_\beta^a \\
-\Psi_{(\alpha)\beta}[(1-\kappa r cos\phi)]P_\beta^a - \kappa\Psi_{\alpha\beta}[rsin\phi]P_\beta^a \\
= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U_\beta^{a-b}W_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V_\beta^{a-b}W_\gamma^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W_\beta^{a-b}W_\gamma^a \\
+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}W_\gamma^b \\
+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U_\beta^{a-b}V_\gamma^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_\beta^{a-b}}{ds}W_\gamma^b + \frac{dW_\gamma^b}{ds}U_\beta^{a-b}) \\
+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U_\beta^{a-b}U_\gamma^b - \Psi_{\alpha\beta\gamma}[1-\kappa r cos\phi]W_\beta^{a-b}V_\gamma^b
\end{aligned} \tag{65}
$$

IV. Basis function for the duct-Orthogonal Series

$$
\begin{aligned}
P^a &= \sum_{\alpha=0}^{\infty} P^a_\alpha(s)\psi_\alpha(s,r,\theta) \\
U^a &= \sum_{\alpha=0}^{\infty} U^a_\alpha(s)\psi_\alpha(s,r,\theta) \\
V^a &= \sum_{\alpha=0}^{\infty} V^a_\alpha(s)\psi_\alpha(s,r,\theta) \\
W^a &= \sum_{\alpha=0}^{\infty} W^a_\alpha(s)\psi_\alpha(s,r,\theta)
\end{aligned}
\tag{66}
$$

and can be normalized according to

$$\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r dr d\theta = \delta_{\alpha\beta} \tag{67}$$

and satify the no penetration condtion on the duct wall

$$\frac{\partial\psi_\alpha}{\partial r}|_{r=h} = 0, \forall\theta \tag{68}$$

They are gievn by

$$\psi_\alpha = C_\alpha J_p(\frac{j_{pq}r}{h})cos(p\phi - \frac{\xi\pi}{2}) \tag{69}$$

with normalization factor given by

$$C_\alpha = \begin{cases} \frac{1}{\sqrt{\pi h^2 J_0^2(j_{0q})}} & p = 0 \\ \frac{1}{\sqrt{\frac{\pi h^2}{2}(1-\frac{p^2}{j_{pq}^2})J_0^2(j_{0q})}} & p > 0 \end{cases} \tag{70}$$

and eigenvalues

$$\lambda_\alpha = \frac{j_{pq}}{h} \tag{71}$$

Here, J_p denotes the ordinary Bessel function of the first kind and j_{pq} denotes the q^{th} zero of the derivative of $J_p(r)$, noting that $j_{00} = 0$ (corresponding to plane wave modes) and $j_{0q} > 0$ for $q > 0$. The index α maps to the triplet of integers (p, q, ξ) with $p \in [0, \infty)$, $q \in [0, \infty)$ and $\xi = 0, 1$. The basis functions can be well ordered by pairing modes differing only by the ξ index(aside forom the $p = 0$ modes which do not come in a pair) and ordering them by the size of their eigenvalues $\lambda_\alpha \leq \lambda_{\alpha+2}$.

A. Detail introduction of Basis functions

B. Example of Helmoholtz's equation in a hollow cylinder

Consider Laplace's equation in a hollow cylinder of radius h with endcaps at $z = 0$and $z = a$,

$$\nabla^2 V = 0 \tag{72}$$

with boundary conditions

$$V(a, \psi, z) = 0, V(r, \psi, 0) = 0, V(r, \psi, h) = f(r, \psi) \tag{73}$$

for a given potential function $f(r, \psi)$. In cylindrical coordinates, we know that:

$$\begin{aligned} \nabla^2 f = \nabla \cdot \nabla f \\ = \frac{1}{h_1 h_2 h_3}[\frac{\partial(\nabla f)_1 h_2 h_3]}{\partial w_1} + \frac{\partial(\nabla f)_2 h_1 h_3]}{\partial w_2} + \frac{\partial(\nabla f)_3 h_1 h_2]}{\partial w_3}] \\ = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial w_1}(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial w_1}) + \frac{\partial}{\partial w_2}(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial w_2}) + \frac{\partial}{\partial w_3}(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial w_3})] \end{aligned} \tag{74}$$

In straight cylindical coordinates, http://mathworld.wolfram.com/CylindricalCoordinates.html

$$h_z = 1, h_r = 1, h_\psi = r \tag{75}$$

we get:

$$\begin{aligned} \nabla^2 f \\ = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial w_1}(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial w_1}) + \frac{\partial}{\partial w_2}(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial w_2}) + \frac{\partial}{\partial w_3}(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial w_3})] \\ = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \psi^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned} \tag{76}$$

Laplace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \psi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{77}$$

Using separation of variables, we assume a solution of the form $V(r, \psi, z) = P(r)\Phi(\psi)Z(z)$,

Substituting this from into the PDE and dividing by V yields,

$$\frac{1}{rP}\frac{\partial}{\partial r}(r\frac{\partial P}{\partial r}) + \frac{1}{r^2\Phi}\frac{\partial^2 \Phi}{\partial \psi^2} + \frac{1}{Z}\frac{\partial^2 V}{\partial z^2} = 0 \tag{78}$$

Due to periodicity, we require that Φ satisfy,

$$\frac{d^2\Phi}{d\psi^2} = -m^2\Phi \tag{79}$$

where m is an integer, $\Phi(\psi) = e^{im\psi}$.

We also have

$$\frac{d^2 Z}{dz^2} = -\lambda_\alpha^2 \Phi \tag{80}$$

which has solution $e^{\lambda_\alpha z}$. Because of the boundary condition at z=0, we take linear combinations of solutions so that $Z(z) = 1/2e^{\lambda_\alpha z} - 1/2e^{-\lambda_\alpha z} = sinh(\lambda_\alpha z)$

Then, we have

$$\frac{d^2 P}{dr^2} + \frac{1}{r}\frac{dP}{dr} + (\lambda_\alpha^2 - \frac{m^2}{r^2})P = 0 \tag{81}$$

This is also proved to be the **Bessel equation** of the oder m, which has solutions $J_m(\lambda_\alpha r)$.

In order to satisfy the boundary condtion at $r = h$, We then have

$$V(r, \psi, z) = J_m(\frac{j_{mn}}{h} r) e^{im\psi} sinh(\frac{j_{mn}}{h} z) \tag{82}$$

and eigenvalues

$$\lambda_\alpha = \frac{j_{mn}}{h} \tag{83}$$

Here, J_m denotes the ordinary Bessel function of the first kind and j_{mn} denotes the n^{th} zero of the derivative of $J_m(r)$, noting that $j_{00} = 0$ (corresponding to plane wave modes) and $j_{0n} > 0$ for $n > 0$.

To satisfy the boundary condition at $z = h$, we take a linear combination of solutions of this and seek the coefficients of the expansion:

$$f(r, \psi) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} c_{mn} J_m(\frac{j_{mn}}{h} r) e^{im\psi} sinh(\frac{j_{mn}}{h} a) \tag{84}$$

From the orthogonality relation for Bessel functions:

$$\int_0^h r J_m(\frac{j_{mi}}{h} r) J_m(\frac{j_{mj}}{h} r) dr = \frac{h^2}{2} [J_{v+1}(j_{mj})]^2 \delta_{ij} \tag{85}$$

and also,

$$\int_0^{2\pi} e^{im\psi} e^{im'\psi} d\psi = 2\pi \delta_{mm'} \tag{86}$$

We know that