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directed straight towards it. There is no doubt that these changes of tint serve to heighten the illusion of apparent motion when the eye is allowed to wander over the different parts of a complicated pattern. This phænomenon may perhaps be explained by the fact of the sight being most perfect in the axis of vision, or, as Sir David Brewster has expressed it, "the eye has the power of seeing objects with perfect distinctness only when it is directed straight upon them, so that all objects seen indirectly are seen indistinctly;" and it may be supposed that impressions received in those parts of the retina used in oblique vision are, as it were, diffused. Thus the red and blue spots, when viewed indirectly, appear tinged with the prevailing colour of the ground of the pattern,—the red spot becoming darker by the influence of the blue round it, and the blue spot lighter by the vicinity of the red; for it is remarkable that this illusion is not produced with single colours, only with spots of one colour surrounded by a field of the other.

In concluding these observations, I have only to add, that there cannot be much doubt of the correctness of the view which ascribes the illusory appearance of motion to the change of tint at the edges of the figures. These are matters of fact: but whether the theories offered in explanation of these facts are correct or not, I must leave to more competent observers to determine.

I have the honour to be, Gentlemen,

Yours faithfully,

Henry Taylor.

LIV. *On a difficulty in the Theory of Sound. By* G. G. Stokes, *M.A., Fellow of Pembroke College, Cambridge*.*

THE theoretical determination of the velocity of sound has recently been the occasion of a discussion between Professor Challis and the Astronomer Royal. It is not my intention to enter into the controversy, but merely to consider a very remarkable difficulty which Professor Challis has noticed in connexion with a known first integral of the accurate equations of motion for the case of plane waves.

I would first however observe, that I do not think that we are *obliged*, in treating the subject to a first approximation, to enter into the consideration of any difficulty which may arise when we come to employ exact equations. In neglecting the squares of small quantities, we adopt a consistent system of approximation, and we arrive at a precise result, namely, the

* Communicated by the Author.

laws which the actual motion tends indefinitely to obey when the magnitude of the velocity is indefinitely diminished. And if, moreover, we have every reason to believe that the first order of small quantities contains all that are sensible in practical cases, we have arrived, not merely at abstract mathematical results, but at important physical laws. Nevertheless it will be admitted on all hands, that any apparent contradiction arrived at by employing exact equations is deserving of serious consideration.

The difficulty already alluded to is to be found at page 496 of the preceding volume of this Magazine. In what follows I shall use Professor Challis's notation.

Without entering into the consideration of the mode in which Poisson obtained the particular integral

$$w = f\{z - (a + w)t\}, \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

it may be easily shown, by actual differentiation and substitution, that the integral does satisfy our equations. The function f being arbitrary, we may assign to it any form we please, as representing a particular possible motion, and may employ the result, *so long as no step tacitly assumed in the course of our reasoning fails.* The interpretation of the integral (1.) will be rendered more easy by the consideration of a curve. In fig. 1 let oz be the axis of z, and let the ordinates of the curve represent the values of w for $t = 0$. The equation (1.) merely

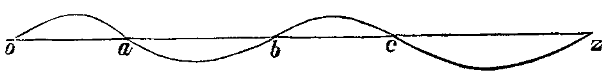
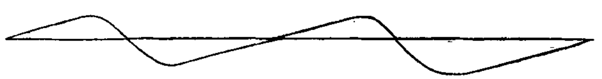
Fig. 1. 

Fig. 2. 

asserts that whatever value the velocity w may have at any particular point when $t = 0$, the same value will it have at the time t at a point in advance of the former by the space $(a + w)t$. Take any point P in the curve of fig. 1, and from it draw, in the positive direction, the right line PP′ parallel to the axis of z, and equal to $(a + w)t$. The locus of all the points P′ will be the velocity-curve for the time t. This curve is represented in fig. 2, except that the displacement at common to all points of the original curve is omitted, in order that the modification in the form of the curve may be more easily perceived. This comes to the same thing as drawing PP′ equal to wt instead of $(a + w)t$. Of course in this way P′ will lie on the positive or negative side of P, according as P lies above or below the axis of z. It is evident that in the neighbourhood of the points a, c the curve becomes more and more steep as t increases, while in the neighbourhood of the points o, b, z its inclination becomes more and more gentle.

The same result may easily be obtained mathematically. In fig. 1, take two points, infinitely close to each other, whose abscissæ are z and $z+dz$; the ordinates will be w and

$$w+\frac{dw}{dz}dz.$$

After the time t these same ordinates will belong to points whose abscissæ will have become (in fig. 2) $z+wt$ and

$$z+dz+\left(w+\frac{dw}{dz}dz\right)t.$$

Hence the horizontal distance between the points, which was dz, will have become

$$\left(1+\frac{dw}{dz}t\right)dz;$$

and therefore the tangent of the inclination, which was $\frac{dw}{dz}$, will have become

$$\frac{\frac{dw}{dz}}{1+\frac{dw}{dz}t}. \quad \ldots \ldots \ldots \quad \text{(A.)}$$

At those points of the original curve at which the tangent is horizontal, $\frac{dw}{dz}=0$, and therefore the tangent will constantly remain horizontal at the corresponding points of the altered curve. For the points for which $\frac{dw}{dz}$ is positive, the denominator of the expression (A.) increases with t, and therefore the inclination of the curve continually decreases. But when $\frac{dw}{dz}$ is negative, the denominator of (A.) decreases as t increases, so that the curve becomes steeper and steeper. At last, for a sufficiently large value of t, the denominator of (A.) becomes infinite for some value of z. Now the very formation of the differential equations of motion with which we start, tacitly supposes that we have to deal with finite and continuous functions; and therefore in the case under consideration we must not, without limitation, push our results beyond the least value of t which renders (A.) infinite. This value is evidently the reciprocal, taken positively, of the greatest negative value of $\frac{dw}{dz}$; w here, as in the whole of this paragraph, denoting the velocity when $t=0$.

By the term *continuous function*, I here understand a function whose value does not alter *per saltum*, and not (as the term is sometimes used) a function which preserves the same algebraical expression. Indeed, it seems to me to be of the utmost importance, in considering the application of partial differential equations to physical, and even to geometrical problems, to contemplate functions apart from all idea of algebraical expression.

In the example considered by Professor Challis,

$$w = m \sin \frac{2\pi}{\lambda} \{z-(a+w)t\},$$

where m may be supposed positive; and we get by differentiating and putting $t=0$,

$$\frac{dw}{dz} = \frac{2\pi m}{\lambda} \cos \frac{2\pi z}{\lambda},$$

the greatest negative value of which is $-\frac{2\pi m}{\lambda}$; so that the greatest value of t for which we are at liberty to use our results without limitation is $\frac{\lambda}{2\pi m}$, whereas the contradiction arrived at by Professor Challis is obtained by extending the result to a larger value of t, namely $\frac{\lambda}{4m}$.

Of course, after the instant at which the expression (A.) becomes infinite, some motion or other will go on, and we might wish to know what the nature of that motion was. Perhaps the most natural supposition to make for trial is, that a surface of discontinuity is formed, in passing across which there is an abrupt change of density and velocity. The existence of such a surface will presently be shown to be possible, on the two suppositions that the pressure is equal in all directions about the same point, and that it varies as the density. I have however convinced myself, by a train of reasoning which I do not think it worth while to give, inasmuch as the result is merely negative, that even on the supposition of the existence of a surface of discontinuity, it is not possible to satisfy all the conditions of the problem by means of a single function of the form $f\{z-(a+w)t\}$. Apparently, something like reflexion must take place. Be that as it may, it is evident that the change which now takes place in the nature of the motion, beginning with the particle (or rather plane of particles) for which (A.) first becomes infinite, cannot influence a particle at a finite distance from the former until after the expiration of a finite time. Consequently, even after the change in the nature of the motion, our original expressions are applicable, at least for a certain time, to a certain portion of the fluid. It was for this reason that I inserted the words "without limitation," in saying that we are not at liberty to use our original results without limitation beyond a certain value of t. The full discussion of the motion which would take place after the change above alluded to, if possible at all, would probably require more pains than the result would be worth.

I proceed now to consider the possibility of the existence of a surface of discontinuity, and the conditions which must be satisfied at such a surface. Although I was led to the subject by considering the interpretation of the integral (1.), the consideration of a discontinuous motion is not here introduced in connexion with that interpretation, but simply for its own sake; and I wish the two subjects to be considered as quite distinct.

Suppose that in passing across a point Q in the axis of z the velocity and density change suddenly from w, ρ to w', ρ', and let ϖ be the velocity of propagation of the surface of discontinuity. Let us first investigate the equation which expresses that there is no generation or destruction of mass at the surface of discontinuity, the equation in fact which takes the place of the equation of continuity, which has to be satisfied elsewhere.

Take two points A, B in the axis of z, the first on the negative and the second on the positive side of Q, and let $\mathrm{QA}=h$, $\mathrm{QB}=h'$. Take also $\mathrm{QQ}'=\varpi dt$, so that Q′ is the point where the surface of discontinuity cuts the axis of z at the time $t+dt$. The quantities h, h' are supposed to be very small, and will be made to vanish after QQ′. Consider the portion of space comprised within a cylinder whose ends consist of two planes, of area unity, drawn through the points A, B perpendicular to the axis. In the time dt, the mass of fluid which flows in at the plane A is ultimately $\rho w dt$, and that which flows out at the plane B is ultimately $\rho' w' dt$. Hence the gain of mass within the cylinder is ultimately $(\rho w-\rho' w')dt$. Now the mass at the time t is ultimately $\rho h+\rho' h'$, and that at the time $t+dt$ is

$$\left(\rho+\frac{d\rho}{dt}\,dt\right)(h+\varpi dt)+\left(\rho'+\frac{d\rho'}{dt}\,dt\right)(h'-\varpi dt);$$

and therefore the gain of mass is $\varpi(\rho-\rho')dt$, h and h' being omitted, since they are to be made to vanish in the end. Equating the two expressions for the gain of mass, we get

$$\rho w-\rho' w'=(\rho-\rho')\varpi. \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

It remains to form the equation of motion. Now we have

moving force on plane $A = a^2\rho$, ultimately;

moving force on plane $B = a^2\rho'$, ultimately;

$\therefore$ resultant moving force $= a^2(\rho - \rho')$,

$\therefore$ momentum generated in time $dt = a^2(\rho - \rho')dt$.

Now the momentum of the mass contained within the cylinder at the time t is ultimately $\rho w h + \rho' w' h'$, and the momentum of the same set of particles at the time $t + dt$ is

$$\left(\rho w + \frac{d\rho w}{dt}dt\right)(h + \overline{\varpi - w}dt) + \left(\rho' w' + \frac{d\rho' w'}{dt}dt\right)(h' + \overline{w' - \varpi}dt);$$

and therefore the gain of momentum is ultimately

$$\{(\rho w - \rho' w')\varpi - \rho w^2 + \rho' w'^2\}dt;$$

whence we have

$$(\rho w - \rho' w')\varpi - (\rho w^2 - \rho' w'^2) = a^2(\rho - \rho'). \quad . \quad . \quad . \quad (3.)$$

By eliminating ϖ between (2.) and (3.), we get

$$(w' - w)^2 \rho\rho' = a^2(\rho - \rho')^2, \quad . \quad . \quad . \quad . \quad (4.)$$

an equation which we may if we please employ instead of (3.).

The equations (2.), (3.) being satisfied, it appears that the discontinuous motion is dynamically possible. This result, however, is so strange, that it may be well to consider more in detail the simplest possible case of such a motion.

Conceive then an infinitely long straight tube filled with air, of which the portion to the left of a certain section s is of a uniform density ρ, and at rest, while the portion to the right is of a uniform smaller density ρ', and is moving in the positive direction with a uniform velocity w', the surface of separation s at the same time travelling backwards into the first portion with the uniform velocity ϖ. The conception of such a motion having been formed, consider next whether the motion is possible or impossible; that is to say, not whether it is possible or impossible in the actual state of elastic fluids, but whether it would or would not be consistent with dynamical principles in the case of an ideal elastic fluid, in which the pressure was equal in all directions about the same point, and varied as the density.

In the case under consideration, the fluid to the left of s is in equilibrium in the simplest way. The fluid to the right is of uniform pressure, and there is no generation or destruction of velocity. The only question, then, can be as to the possibility of the passage from the one state into the other taking place in the way supposed. In the first place, it is evident that, independently of any consideration of force, there must be a relation between ρ, ρ', w', and ϖ; for a length ϖt of condensed air comes to occupy, in the rarefied state, a length $(\varpi + w')t$, so that we must have $\rho\varpi = \rho'(\varpi + w')$. Next, if we take two sections s_1, s_2, the first to the left and the second to the right of s, and suppose the first to remain at rest, with the fluid in which it is situated, while the second moves, along with the fluid in which it is situated, with the velocity w', since the pressure on s_1 exceeds that on s_2 by $a^2(\rho - \rho')$, the surface of s_1 or s_2 being for simplicity's sake supposed equal to unity, there must in the time t be generated a momentum $a^2(\rho - \rho')t$ in the fluid lying between s_1 and s_2. But this will be the case in consequence of the velocity w' being communicated to a volume ϖt of air which was previously at rest and of density ρ, while the state of rest or motion of the remainder of the air between s_1 and s_2 has been unaltered, provided $a^2(\rho - \rho') = w'\varpi\rho$. These two relations being satisfied, it appears that the motion is dynamically possible. The two equations might have been obtained at once from (2.) and (3.) by writing $-\varpi$ for ϖ and putting $w = 0$, but I have preferred deducing them afresh from first principles in consequence of the novelty of the subject, and the reluctance with which the conclusions that I have arrived at are likely to be received by mathematicians.

These conclusions certainly seem sufficiently startling; yet a result still more extraordinary remains behind. By solving the two equations of the preceding paragraph with respect to w' and ϖ, we get

$$w' = \frac{\rho - \rho'}{\sqrt{\rho\rho'}}\,a, \qquad \varpi = \sqrt{\frac{\rho'}{\rho}}\,a^{*}.$$

Now let ρ' vanish; then w' becomes infinite and ϖ vanishes. Hence the rate at which the condensed air (which remains packed like the combustible matter in a rocket) is discharged decreases indefinitely as the space into which the discharge takes place approaches indefinitely to a vacuum. Of course the velocity of discharge becomes infinite, without which the requisite momentum could not be furnished. The quantity of air which passes in a unit of time across a plane, of area unity, taken at the positive side of the tube, is $w'\rho'$, which is easily seen to be a maximum, for a given value of ρ, when $\rho' = \frac{1}{3}\rho$.

A similar paradox is fully considered by MM. Barré de

* It is worthy of remark, that when ρ' is very nearly equal to ρ, and consequently w' very small, the velocity of propagation ϖ is very nearly equal to a, to which it approaches indefinitely when w' is indefinitely diminished. Thus even this discontinuous motion offers no exception to the theorem, at once proved by neglecting the squares of small quantities, that in very small motions any disturbance is propagated in the fluid with the velocity a.

Saint-Venant and Wantzel, in the 27th *Cahier* of the *Journal de l'Ecole Polytechnique.*

The strange results at which I have arrived appear to be fairly deducible from the two hypotheses already mentioned. It does not follow that the discontinuous motion considered can ever take place in nature, for we have all along been reasoning on an ideal elastic fluid which does not exist in nature. In the first place, it is not true that the pressure varies as the density, in consequence of the heat and cold produced by condensation and rarefaction respectively. But it will be easily seen that the discontinuous motion remains possible when we take account of the variation of temperature due to condensation and rarefaction, neglecting, however, the communication of heat from one part of the fluid to another. Indeed, so far as the possibility of discontinuity is concerned, it is immaterial according to what law the pressure may increase with the density.

Of course the communication of heat from one particle of the fluid to another would affect the result, though whether to the extent of preventing the possibility of discontinuity I am unable to say. But there is another supposition that we have made which is at variance with the actual state of elastic fluids. It is not true that one portion of an elastic fluid is incapable of exerting any tangential force on another portion on which it slides, even though the variation of velocity from the one portion to the other be not abrupt but continuous. In consequence of this tangential force, analogous in some respects to friction in the case of solids, the mutual pressure of two adjacent elements of a fluid is not accurately normal to the surface of separation, nor equal in all directions about the same point. In many cases the influence of this internal friction is insensible, while in other cases it is very important. Its general effect is to check the relative motion of the parts of a fluid. Suppose now that a surface of discontinuity is very nearly formed, that is to say, that in the neighbourhood of a certain surface there is a very rapid change of density and velocity. It may be easily shown, that in such a case the rapid condensation or rarefaction implies a rapid sliding motion of the fluid; and this rapid sliding motion would call into play a considerable tangential force, the effect of which would be to check the relative motion of the parts of the fluid. It appears, then, almost certain that the internal friction would effectually prevent the formation of a surface of discontinuity, and even render the motion continuous again if it were for an instant discontinuous.