An Introductory Course in Duct Acoustics in the context of Aeroacoustics

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Technische Universiteit Eindhoven
University of Technology

Outline

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Modes in General

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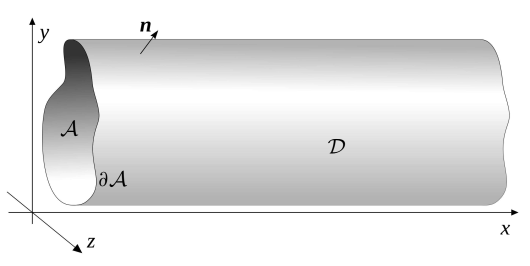

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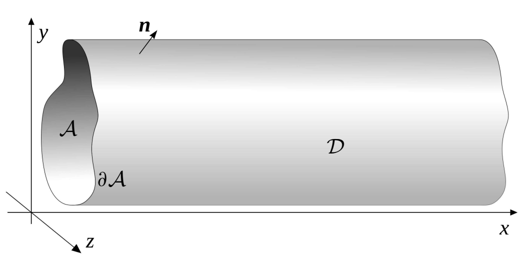

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Basic example

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0 \quad \& \quad \text{BC}$$

$$p(\boldsymbol{x}, t) = f(x - vt, y, z) = \mathrm{e}^{\mathrm{i}\omega(t - x/v)}\, P(y, z) = \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\kappa x}\, P(y, z)$$

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$$-\kappa^2 P + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} + \frac{\omega^2}{c_0^2} P = 0$$

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2D eigenvalue problem: $\quad \nabla^2 P = -\Big(\frac{\omega^2}{c_0^2} - \kappa^2\Big) P = -\alpha^2 P \quad \& \quad \text{BC}$

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- For given ω, all possible $\alpha = \alpha_n$ form a countable discrete set.

- By Fourier synthesis in ω we can construct general time dependent "modal" solutions.
  - $\kappa = \kappa(\omega)$
  - Without ω-dependent BC: α and P independent of ω and so

$$p(x, y, z, t) = \int_{-\infty}^{\infty} A(\omega)\, e^{i\omega t - i\kappa(\omega)x}\, d\omega\, P(y, z)$$

    For example: hard walls.
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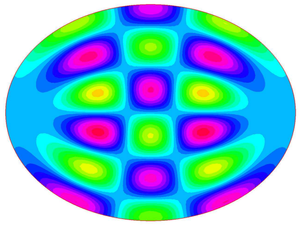
Modes: examples

Snapshot of cross section, modal shape function $P(y, z)$



Snapshot of side view, modal pressure distribution $\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\kappa x}\, P(y, z)\right)$

Modes: examples

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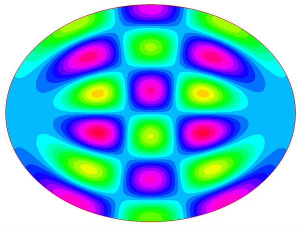

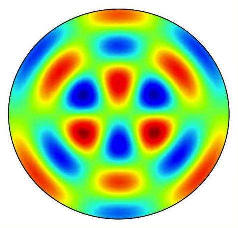

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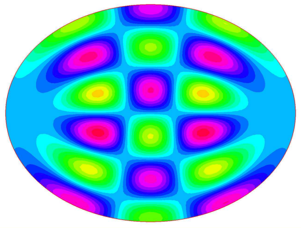
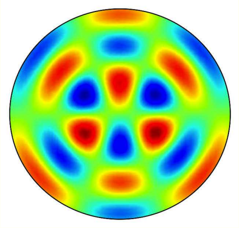

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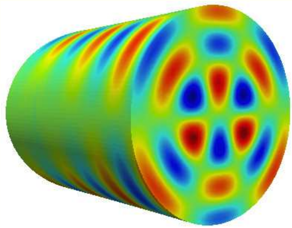
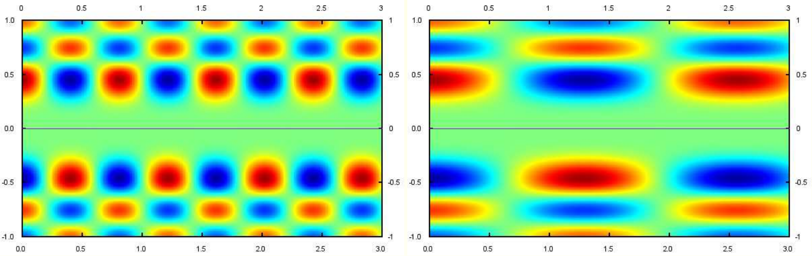
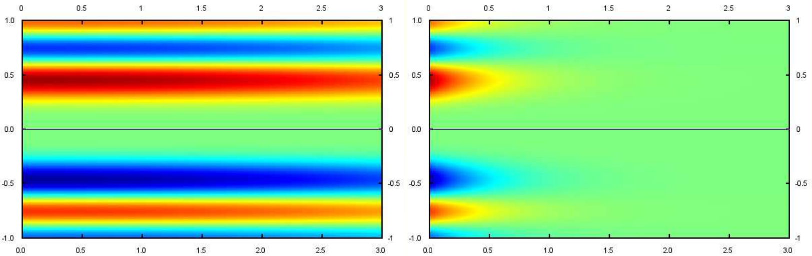


Mode patterns for decreasing frequency



Propagating modes: real κ large, real κ small



Resonance: $\kappa = 0$. Decaying mode: imaginary κ

Outline

Summary of equations for fluid motion

Acoustics is a branch of fluid mechanics!

$$
\begin{aligned}
\text{mass:} \quad & \left(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\rho = -\rho\nabla\cdot\boldsymbol{v} \\
\text{momentum:} \quad & \rho\left(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = -\nabla p + \nabla\cdot\boldsymbol{\tau} \\
\text{energy:} \quad & \rho T\left(\tfrac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)s = -\nabla\cdot\boldsymbol{q} + \boldsymbol{\tau}:\nabla\boldsymbol{v}.
\end{aligned}
$$

where

$$
T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}\rho^{-1} = \mathrm{d}h - \rho^{-1}\mathrm{d}p, \quad h = e + \frac{p}{\rho}, \quad p = \rho RT,
$$

$$
\mathrm{d}e = C_V\mathrm{d}T, \quad \mathrm{d}h = C_P\mathrm{d}T, \quad \mathrm{d}s = C_V\frac{\mathrm{d}p}{p} - C_P\frac{\mathrm{d}\rho}{\rho}, \quad c^2 = \gamma RT, \quad \gamma = \frac{C_P}{C_V}
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for a perfect gas (i.e. ideal + C_V, C_P constant).

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$$\boxed{\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot\boldsymbol{v}, \qquad \rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla p, \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = 0}$$

Mean flow and small perturbations:

$$\boldsymbol{v} = \boldsymbol{V}_0 + \boldsymbol{v}', \quad p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad s = s_0 + s'$$

linearise,

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and

$$\begin{aligned}
\big(\tfrac{\partial}{\partial t} + \boldsymbol{V}_0\cdot\nabla\big)\rho' + \rho'\nabla\cdot\boldsymbol{V}_0 + \nabla\cdot\big(\rho_0\boldsymbol{v}'\big) &= 0\\
\rho_0\big(\tfrac{\partial}{\partial t} + \boldsymbol{V}_0\cdot\nabla\big)\boldsymbol{v}' + \rho_0\big(\boldsymbol{v}'\cdot\nabla\big)\boldsymbol{V}_0 + \rho'\big(\boldsymbol{V}_0\cdot\nabla\big)\boldsymbol{V}_0 &= -\nabla p'\\
\frac{\partial p'}{\partial t} + \boldsymbol{V}_0\cdot\nabla p' - c_0^2\left(\frac{\partial\rho'}{\partial t} + \boldsymbol{V}_0\cdot\nabla\rho' + \boldsymbol{v}'\cdot\nabla\rho_0\right) &= 0
\end{aligned}$$

With source and force:

Conservation laws

mass: $\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\boldsymbol{v}) = q$

momentum: $\frac{\partial}{\partial t}(\rho\boldsymbol{v}) + \nabla\cdot(\rho\boldsymbol{v}\boldsymbol{v}) + \nabla p = \boldsymbol{f}$

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momentum: $$\rho\frac{\partial}{\partial t}\boldsymbol{v} + \rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} + \nabla p = \boldsymbol{f} - \boldsymbol{v}q$$

After linearisation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{V}_0\cdot\nabla\right)\rho' + \rho'\nabla\cdot\boldsymbol{V}_0 + \nabla\cdot\left(\rho_0\boldsymbol{v}'\right) = q'$$

$$\rho_0\left(\frac{\partial}{\partial t} + \boldsymbol{V}_0\cdot\nabla\right)\boldsymbol{v}' + \rho_0\left(\boldsymbol{v}'\cdot\nabla\right)\boldsymbol{V}_0 + \rho'\left(\boldsymbol{V}_0\cdot\nabla\right)\boldsymbol{V}_0 + \nabla p' = \boldsymbol{f}' - \boldsymbol{V}_0 q'$$

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with $\boldsymbol{V}_0 = 0$

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ACOUSTIC FIELD REDUCE TO:

$$D_0^3 p' + 2c_0^2 \frac{\partial}{\partial x}\left(\nabla U_0 \cdot \nabla p'\right) - D_0 \nabla\cdot\left(c_0^2 \nabla p'\right) = 0,$$

where $D_0 = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}$.

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Most general form of acoustic equation in parallel flow.
Includes various special cases.

Modes: special cases with flow

If we assume MODES $p'(x, y, z, t) = \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\kappa x} P(y, z)$ and $\Omega = \dfrac{\omega - \kappa U_0}{c_0}$, then

$$\boxed{\Omega^2 \nabla \cdot \left(\frac{\nabla P}{\Omega^2}\right) + \left(\Omega^2 - \kappa^2\right) P = 0.}$$

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In ducts, κ is determined by the boundary condition (eigenvalue problem).

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Variable sound speed c_0, no flow ($U_0 = 0$):

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Uniform & stationary medium:

$$\boxed{\nabla^2 P + \left(\frac{\omega^2}{c_0^2} - \kappa^2\right)P = 0,}$$

A Lorentz transformation:

Note that the convected wave equation (uniform mean flow)

$$\nabla^2 p' - \frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)^2 p' = 0.$$

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may be transformed by

$$p'(x, y, z, t) = \widetilde{p}(X, y, z, T), \quad x = \beta X, \;\; T = \beta t + \frac{M}{c_0 \beta} x, \;\; \beta = \sqrt{1 - M^2}$$

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to the ordinary wave equation

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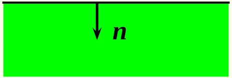
Similar in frequency domain.

Impedance boundary conditions (no mean flow)

Frequency domain:

$$\hat{p} = Z(\omega)(\hat{\boldsymbol{v}} \cdot \boldsymbol{n})$$



- Hard wall : $Z = \infty$ and $\hat{\boldsymbol{v}} \cdot \boldsymbol{n} = 0$.
- Pressure-release condition : $Z = 0$ and $p = 0$.

- In general: $Z(\omega) \in \mathbb{C}$ is *essentially* frequency dependent!

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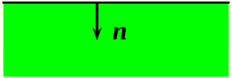

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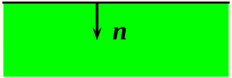

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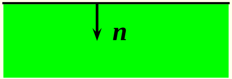

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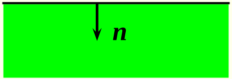

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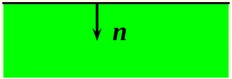

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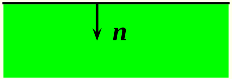

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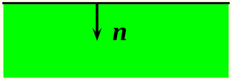

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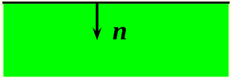

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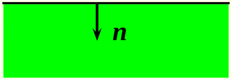

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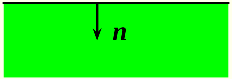

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so $\boldsymbol{Z(\omega) = Z^*(-\omega)}$.

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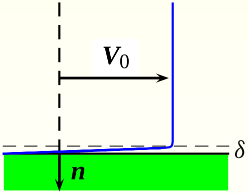

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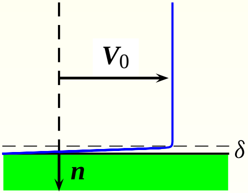

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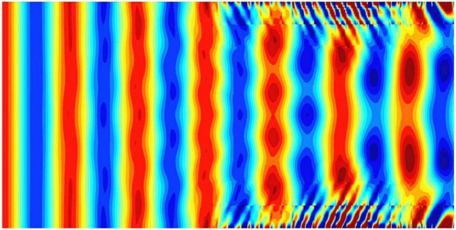
- For a point near the wall but still (just) inside the mean flow (Ingard 1959)

$$\mathrm{i}\omega Z(\hat{\boldsymbol{v}} \cdot \boldsymbol{n}) = \left(\mathrm{i}\omega + U_\infty \frac{\partial}{\partial x}\right)\hat{p}$$

Instability problems

- Ingard-Myers condition is OK in frequency domain,
  but ill-posed in time domain (absolute instabilities blow up; Brambley):



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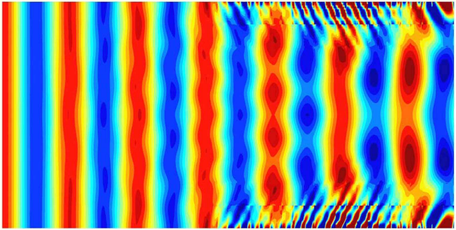

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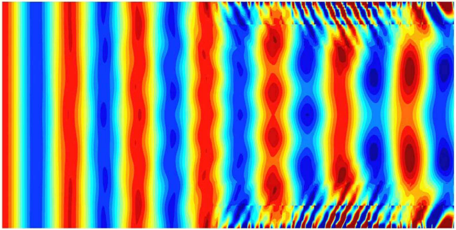

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Outline

Duct modes 2D, no flow

$$p(x, y, z, t) = \mathrm{e}^{\mathrm{i}\omega t - \mathrm{i}\kappa x}\, P(y), \;\rightarrow\; P_{yy} + \left(k^2 - \kappa^2\right) P = 0, \; k = \frac{\omega}{c_0}$$

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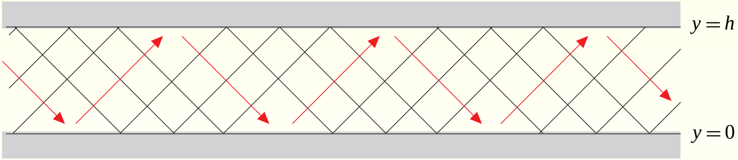

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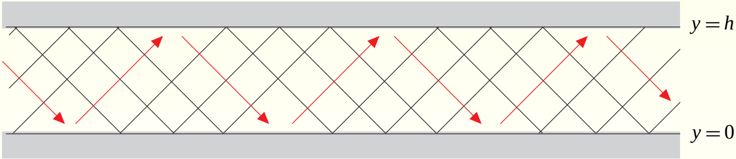

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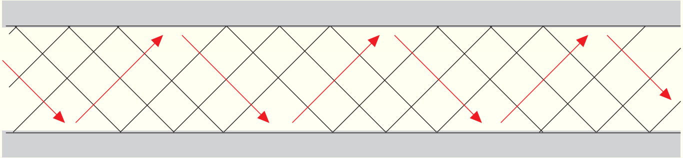

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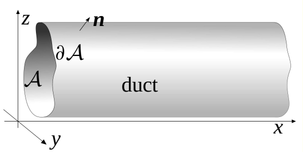
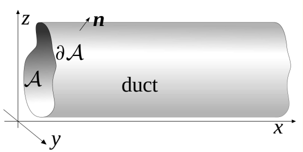


$$\nabla^2 p + k^2 p = 0, \quad \text{with} \quad Z(\nabla p \cdot \boldsymbol{n}) = -\mathrm{i}\omega\rho_0 p \quad \text{at wall}$$

with $k = \dfrac{\omega}{c_0}$ and solution

$$p(x, y, z) = \sum_{n=0}^{\infty} A_n \psi_n(y, z)\, \mathrm{e}^{-\mathrm{i}\kappa_n^+ x} + B_n \psi_n(y, z)\, \mathrm{e}^{-\mathrm{i}\kappa_n^- x}$$



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Mode ψ_n is n-th *eigensolution* of Laplace *eigenvalue problem* on $\mathcal{A}$:

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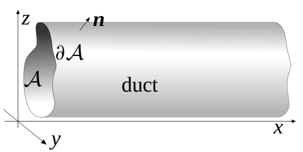

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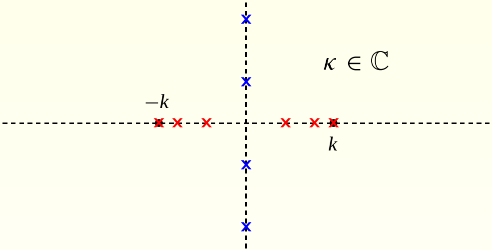
Note that $Z = Z(\omega)$ and therefore $\alpha = \alpha(\omega)$, $\psi = \psi(y, z; \omega)$.

Physical relevance of modal wave numbers

Hard walls: all α_n real $\rightarrow \kappa_n = \pm\sqrt{k^2 - \alpha_n^2}$: real (cut-on) or imaginary (cut-off)

Note: $\alpha_0 = 0$. If $k \to 0$, all modes cut-off except $\kappa_0 = \pm k$



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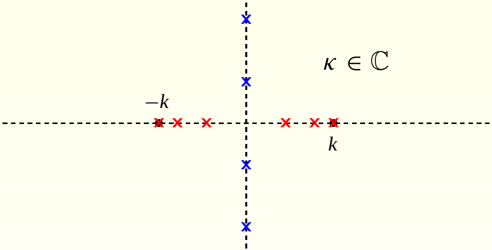
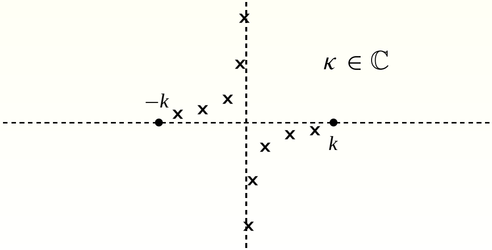


Soft walls: α_n and κ_n complex



Low frequency waves in ducts

Hard-walled ducts allow **plane wave** solutions (right-, left-running):

$$p(x, y, z, t) = f(x - c_0 t) + g(x + c_0 t)$$

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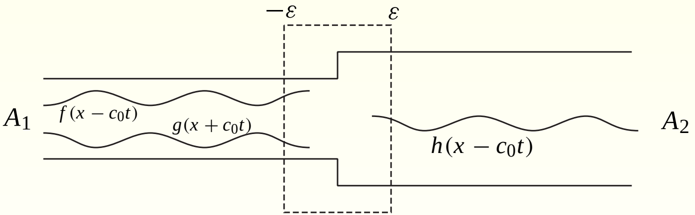
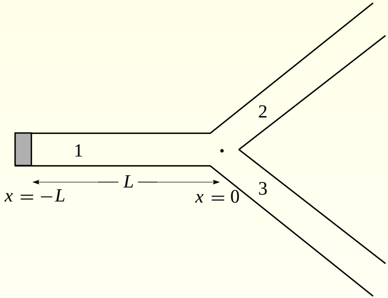


Coupling is approximately possible:

$$\int_{-\varepsilon}^{\varepsilon} \left[\mathrm{i}\omega\rho_0 \boldsymbol{v} + \nabla p \right] \mathrm{d}x = 0 \quad \Rightarrow \quad p(\varepsilon) - p(-\varepsilon) \to 0$$

$$\iiint_{\Omega} \left[\mathrm{i}\omega\rho + \rho_0 \nabla \cdot \boldsymbol{v} \right] \mathrm{d}\boldsymbol{x} = 0 \quad \Rightarrow \quad A_2 \rho_0 u(\varepsilon) - A_1 \rho_0 u(-\varepsilon) \to 0$$

Example: piston at $x = -L$ and branching at $x = 0$



$$1: \quad p_1 = f_1(x - c_0 t) + g_1(x + c_0 t)$$
$$2: \quad p_2 = f_2(x - c_0 t)$$
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at $\quad x = -L: \quad \frac{1}{\rho_0 c_0}\big(f_1(-L - c_0 t) - g_1(-L + c_0 t)\big) = U_0(t)$

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results into

$$f_1(-c_0 t) = -3 g_1(c_0 t) = \tfrac{3}{2} f_2(-c_0 t) = \rho_0 c_0 \sum_{n=0}^{\infty} \left(-\tfrac{1}{3}\right)^n U_0\left(t - (2n+1)\tfrac{L}{c_0}\right)$$

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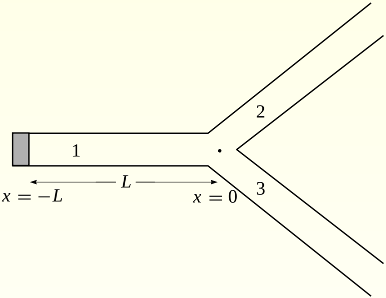

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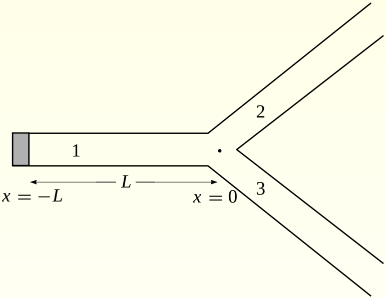

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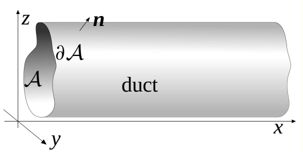
Straight duct with uniform mean flow



$$\nabla^2 p - \frac{1}{c_0^2}\left(\mathrm{i}\omega + U_0\frac{\partial}{\partial x}\right)^2 p = 0, \text{ with } \mathrm{i}\omega Z(\nabla p \cdot \boldsymbol{n}) = -\rho_0(\mathrm{i}\omega + U_0\tfrac{\partial}{\partial x})^2 p \text{ at wall.}$$

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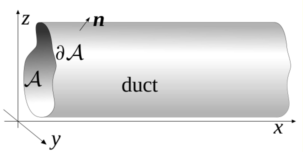

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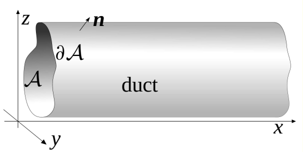

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Modal solution

$$p(x, y, z) = \sum_{n=0}^{\infty} A_n \psi_n^+(y, z)\, \mathrm{e}^{-\mathrm{i}\kappa_n^+ x} + B_n \psi_n^-(y, z)\, \mathrm{e}^{-\mathrm{i}\kappa_n^- x}$$

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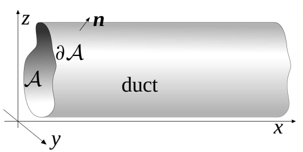

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Mode ψ_n is n-th eigensolution of Laplace problem on duct cross section $\mathcal{A}$

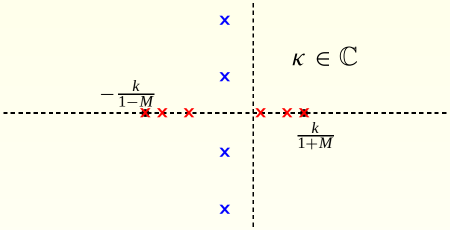
$$\nabla^2 \psi_n = -\alpha_n^2 \psi_n, \quad (+\text{ B.C. containing } \alpha_n)$$

where

$$\alpha_n^{\pm 2} = (k - M\kappa_n^{\pm})^2 - \kappa_n^{\pm 2}, \quad \kappa_n^{\pm} = -\frac{kM}{\beta^2} \pm \frac{1}{\beta^2}\sqrt{k^2 - \beta^2 \alpha_n^{\pm 2}}, \quad \beta = \sqrt{1 - M^2}$$

Modal wave numbers

Hard walls: all α_n real $\rightarrow \kappa_n$ real (cut-on) or (Doppler-shifted) complex (cut-off)



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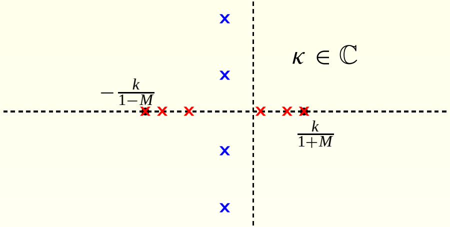
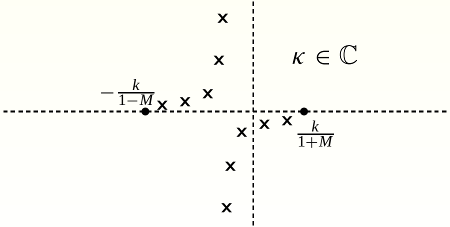


Soft walls: α_n and κ_n complex (but there is more . . .)



Modal Phase and Group Velocity (no flow)

(Wave crest and signal velocity.)

- Hard wall, cut-on.
- $v_f v_g = c_0^2$,
- Dispersion (a wave packet breaks up into distinct modes) $\longleftrightarrow$ $v_f \neq v_g$.
- $$\begin{array}{lll} v_f^- < -c_0 < v_g^- < 0 & \text{on} & \alpha_n c_0 < \omega \\ 0 < v_g^+ < c_0 < v_f^+ & \text{on} & \alpha_n c_0 < \omega \end{array}$$
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Outline

Important special case: circular duct $r = R$

$$\nabla^2 \psi = \frac{\partial^2}{\partial r^2}\psi + \frac{1}{r}\frac{\partial}{\partial r}\psi + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\psi = -\alpha^2 \psi$$

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$$F'' + \frac{1}{r}F' + \left(\alpha^2 - \frac{m^2}{r^2}\right)F = 0 \quad \rightarrow \quad F(r) = J_m(\alpha r)$$

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$$\psi_n(y, z) = J_m(\alpha_{m\mu} r)\, \mathrm{e}^{-\mathrm{i}m\theta}, \qquad m \in \mathbb{Z}.$$

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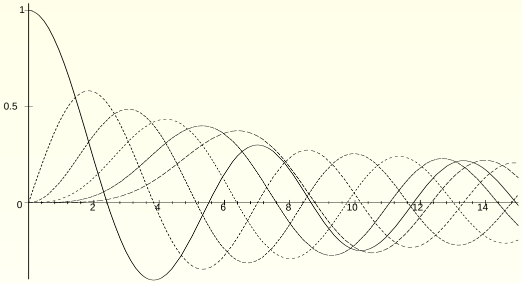
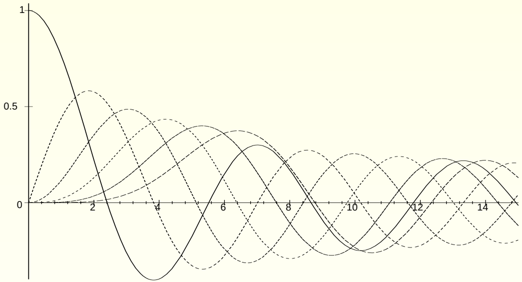
$$\mathrm{i}\omega Z\alpha_{m\mu} J_m'(\alpha_{m\mu} R) = (\omega - U_0\kappa_{m\mu})^2 J_m(\alpha_{m\mu} R) \;\to\; \alpha_{m\mu}(Z)$$

Bessel functions



Bessel function $J_m(x)$ as function of order and real argument.

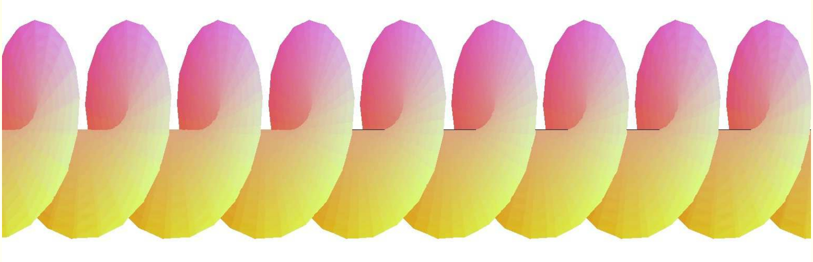


Bessel function $J_m(x)$ as function of order and real argument.

Note: $j'_{01} = 0$, $j'_{m1} \simeq m + 0.8m^{\frac{1}{3}} \quad \rightarrow \quad \kappa_{m1} = \sqrt{k^2 - \alpha_{m1}^2} = \dfrac{1}{R}\sqrt{\left(\dfrac{\omega R}{c_0}\right)^2 - {j'_{m1}}^2}$

$$\boxed{m\text{-modes cut-on if} \quad \frac{\omega R}{c_0} \gtrsim m}$$

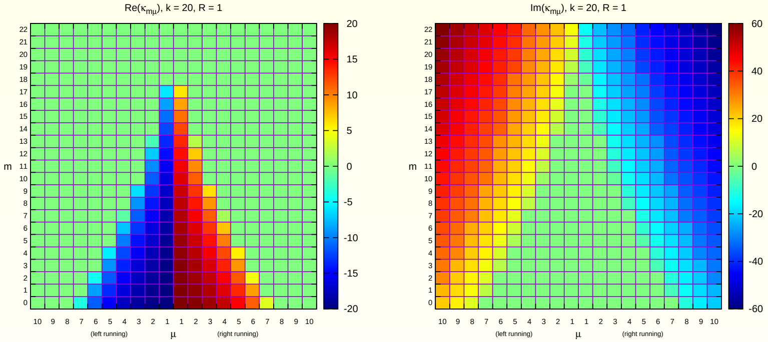
Surface of constant modal phase



Surface of constant phase $\omega t - m\vartheta - \mathrm{Re}(\kappa_{m\mu})x = \text{constant}$

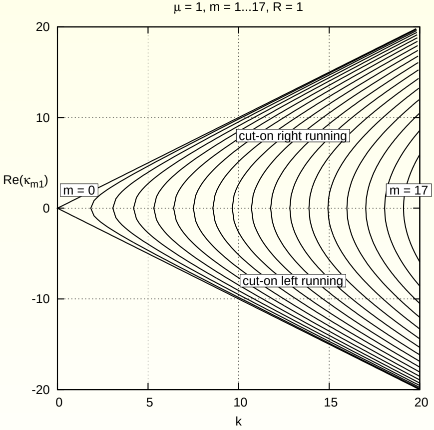
is a spiralling helicoid of pitch $2\pi m / \mathrm{Re}(\kappa_{m\mu})$

Modal wave numbers $\kappa_{m\mu}$, no flow



$$\kappa_{m\mu} = \pm\sqrt{k^2 - \alpha_{m\mu}^2}$$

Modal wave numbers $\kappa_{m1}(k)$, no flow



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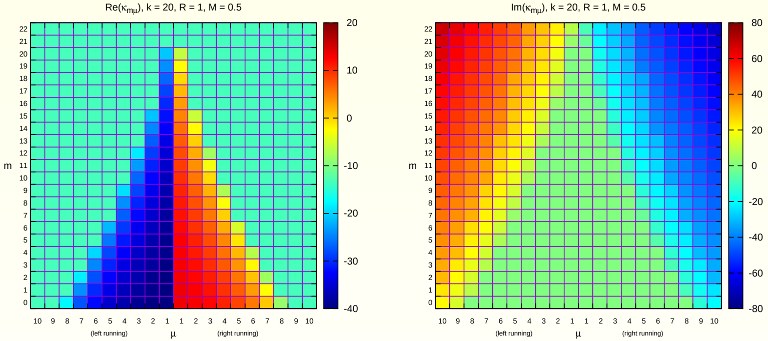
Modal wave numbers $\kappa_{m\mu}$, with flow



$$\kappa_{m\mu} = -\frac{kM}{\beta^2} \pm \frac{1}{\beta^2}\sqrt{k^2 - \beta^2\alpha_{m\mu}^2}$$

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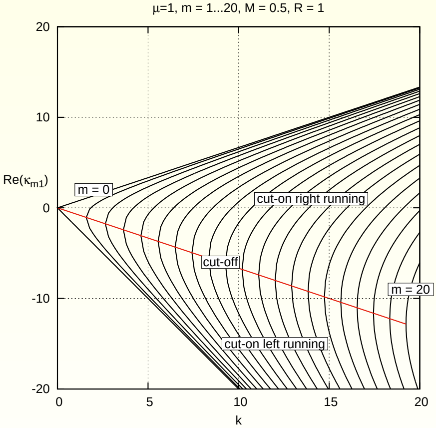

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Normalised modes

A complete solution may be written like

$$p(x,r,\vartheta) = \sum_{m=-\infty}^{\infty}\sum_{\mu=1}^{\infty}(A_{m\mu}\,\mathrm{e}^{-\mathrm{i}\kappa_{m\mu}x} + B_{m\mu}\,\mathrm{e}^{\mathrm{i}\kappa_{m\mu}x})U_{m\mu}(r)\,\mathrm{e}^{-\mathrm{i}m\vartheta}\,.$$

$$U_{m\mu}(r) = N_{m\mu}\,J_m(\alpha_{m\mu}r), \qquad N_{m\mu} = \left\{\tfrac{1}{2}(R^2 - m^2/\alpha_{m\mu}^2)\,J_m(\alpha_{m\mu}R)^2\right\}^{-\frac{1}{2}}$$

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In a hard-walled duct $U_{m\mu}\,\mathrm{e}^{-\mathrm{i}m\theta}$ are orthogonal. Normalise such that:

$$\int_0^{2\pi}\int_0^R U_{m\mu}(r)\,\mathrm{e}^{-\mathrm{i}m\theta}\,U_{n\nu}(r)\,\mathrm{e}^{\mathrm{i}n\theta}\,r\,\mathrm{d}r = 2\pi\delta_{\mu\nu}\delta_{mn}$$

Modal power in hard-walled duct

Time-averaged axial intensity

$$\langle \boldsymbol{I} \cdot \boldsymbol{e}_x \rangle = \tfrac{1}{2}\,\mathrm{Re}(pu^*)$$

Transmitted acoustic power

$$\mathcal{P} = \frac{\pi}{\rho_0 \omega} \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} \Big[\mathrm{Re}(\kappa_{m\mu})(|A_{m\mu}|^2 - |B_{m\mu}|^2) + 2\,\mathrm{Im}(\kappa_{m\mu})\,\mathrm{Im}(A^*_{m\mu} B_{m\mu}) \Big] \,.$$

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Tunneling: all modes cut-off & $\mathcal{P} \neq 0$.

Source expansion

If $p(0,t,\theta) = p_0(r,\theta)$ is source in hard-walled duct, then for $x > 0$

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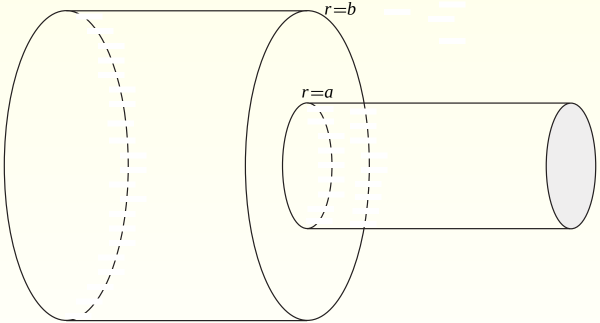
A **finite** *number of modes (cut-on modes) survive at large distances.*
Just 1 mode if $kR \ll 1$: only A_{01} important.

Mode matching

Using similar ideas we can connect duct sections by mode matching



Duct with discontinuous diameter

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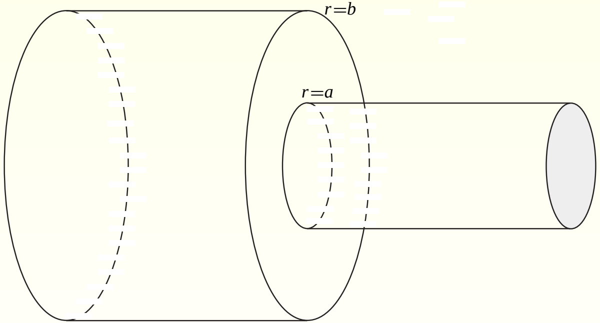

Duct with discontinuous diameter

Exercise in linear algebra. Know how to chop off the infinite series

Outline

Surface wave behaviour ("strange modes")

From the asymptotic behaviour

$$J_m(\alpha_{m\mu} r) \simeq \frac{\mathrm{e}^{\mathrm{i}\alpha_{m\mu} r - \frac{1}{2} m\pi \mathrm{i} - \frac{1}{4}\pi \mathrm{i}}}{\sqrt{2\pi \alpha_{m\mu} r}}, \qquad \alpha_{m\mu} r \to \infty,\ \operatorname{Im} \alpha_{m\mu} < 0$$

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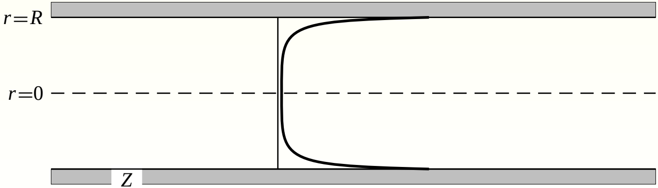
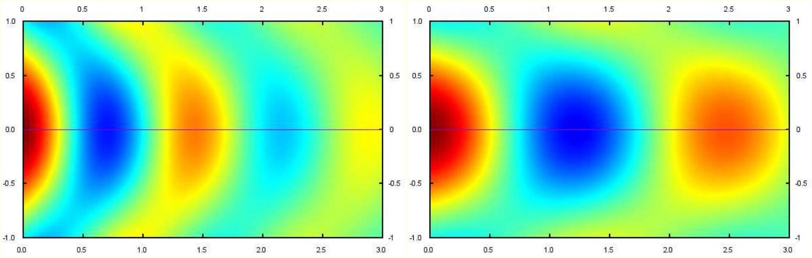
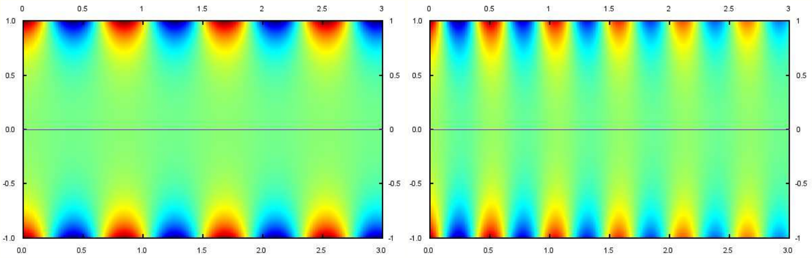
$\mathrm{Im}(\alpha_{m\mu}R) \gg 1$: $J_m(\alpha_{m\mu}r)$ exponentially close to $r = R$: surface wave.



Mode patterns for impedance walls



Acoustic modes: $\mathrm{Im}\,\alpha$ not large ($Z = 1.2 - 0.5\mathrm{i}$, $M = 0$ and $Z = 0.4 - 0.4\mathrm{i}$, $M = 0.5$)



Surface waves: $\mathrm{Im}\,\alpha$ large ($Z = 0.001 - \mathrm{i}$, $M = 0$ and $Z = 0.4 - 6\mathrm{i}$, $M = -0.5$)

In more detail: we recall the theory for lined wall modes

$$\text{equations} \begin{cases} \left(\mathrm{i}k + M\dfrac{\partial}{\partial x}\right)^2 p(x,r,\theta) - \nabla^2 p(x,r,\theta) = 0 \\ r = 1 : \qquad \rho_0 c_0 \left(\mathrm{i}k + M\dfrac{\partial}{\partial x}\right)^2 p = -\mathrm{i}kZ\dfrac{\partial}{\partial r} p \end{cases}$$

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LORENTZ TRANSFORMATION CLEANS UP THE FORMULAS:

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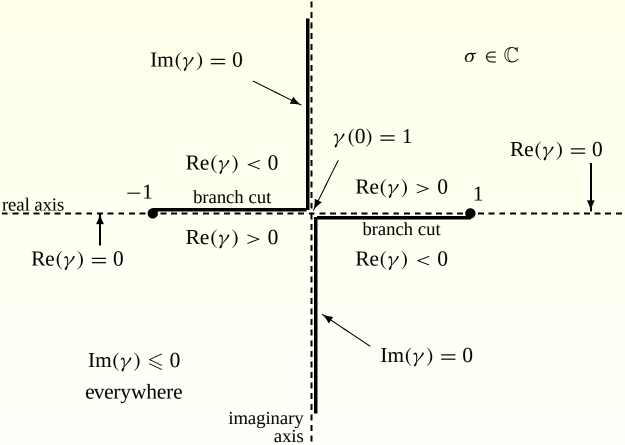
$$\boxed{(1 - M\sigma)^2 - \mathrm{i}\beta^3\tilde{Z}\gamma\,\frac{J_m'(\Omega\gamma)}{J_m(\Omega\gamma)} = 0}$$

Important square root γ

SQUARE ROOT $\gamma(\sigma)$ WITH BRANCH CUTS AND BRANCH:



Branch cuts and signs of $\gamma = \sqrt{1-\sigma^2}$ in complex σ-plane.
The definition adopted is the branch with $\mathrm{Im}(\gamma) \leqslant 0$ for all σ.
$\mathrm{Im}(\gamma) = 0$ along the branch cuts. $\gamma(\sigma) \simeq -\mathrm{i}\sigma\, \mathrm{sign}(\mathrm{Re}\,\sigma)$ if $|\sigma| \gg 1$.

Analysis of eigenvalue equation: with/without flow

$$(1 - M\sigma)^2 - \mathrm{i}\beta^3 \tilde{Z}\gamma \, \frac{J_m'(\Omega\gamma)}{J_m(\Omega\gamma)} = 0$$

$\mathrm{Im}(\Omega\gamma) = O(1)$: $J_m(\Omega\gamma)$, $J_m(\Omega\gamma r)$ oscillatory, $O(1)$: acoustic wave.

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Surface waves equation (2D wall limit):

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$\mathrm{Im}(\Omega\gamma) \gg O(1)$: $J_m(\Omega\gamma) \sim$ exponentially increasing: $\dfrac{J_m'(\Omega\gamma)}{J_m(\Omega\gamma)} \to \mathrm{i}$

$J_m(\Omega\gamma r)$ exp. restricted to $r \sim 1$: surface wave.

Surface waves equation (2D wall limit):

$$\boxed{(1 - M\sigma)^2 - \mathrm{i}\beta^3 \tilde{Z}\gamma \quad \mathrm{i} \qquad = 0}$$

Analysis of eigenvalue equation: with/without flow

$$(1-M\sigma)^2 - \mathrm{i}\beta^3\tilde{Z}\gamma\,\frac{J_m'(\Omega\gamma)}{J_m(\Omega\gamma)} = 0$$

From the asymptotic behaviour

$$J_m(z) \simeq \frac{\mathrm{e}^{\mathrm{i}z-\frac{1}{2}m\pi\mathrm{i}-\frac{1}{4}\pi\mathrm{i}}}{\sqrt{2\pi z}}, \qquad z\to\infty,\ \mathrm{Im}\,z<0$$

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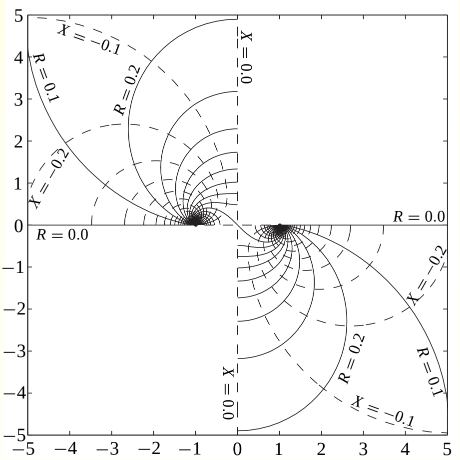
Occurrence only for $\mathrm{Im}\,\tilde{Z} \leqslant 0$, existence in 2nd and 4th quadrant of σ-plane.

Trajectories of 2D surface waves (no flow):



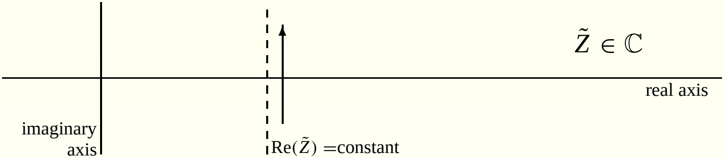
Trajectories of σ for varying $\tilde{Z} = R + \mathrm{i}\,X$; $M = 0$.

Fixed R & $X = 0 : -0.1 : -\infty$ ———

Fixed X & $R = 0 : \quad 0.1 : \quad \infty$ – – – – –

Behaviour of duct modes $\kappa_{m\mu}$ as a function of $\tilde{Z}$.

Consider $\tilde{Z}$ along paths parallel to imaginary axis, *i.e.*,$\mathrm{Re}(\tilde{Z}) = \text{constant}$:



Complex impedance plane.

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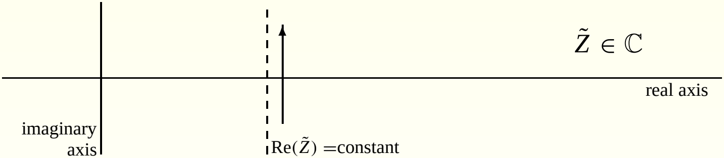

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Trajectories of all $\kappa_{m\mu}$ for $\mathrm{Re}(\tilde{Z})$ fixed, $-\infty < \mathrm{Im}(\tilde{Z}) < \infty$:

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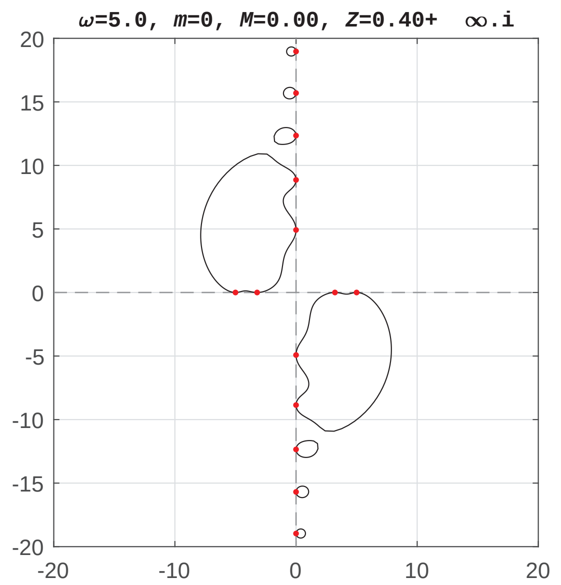
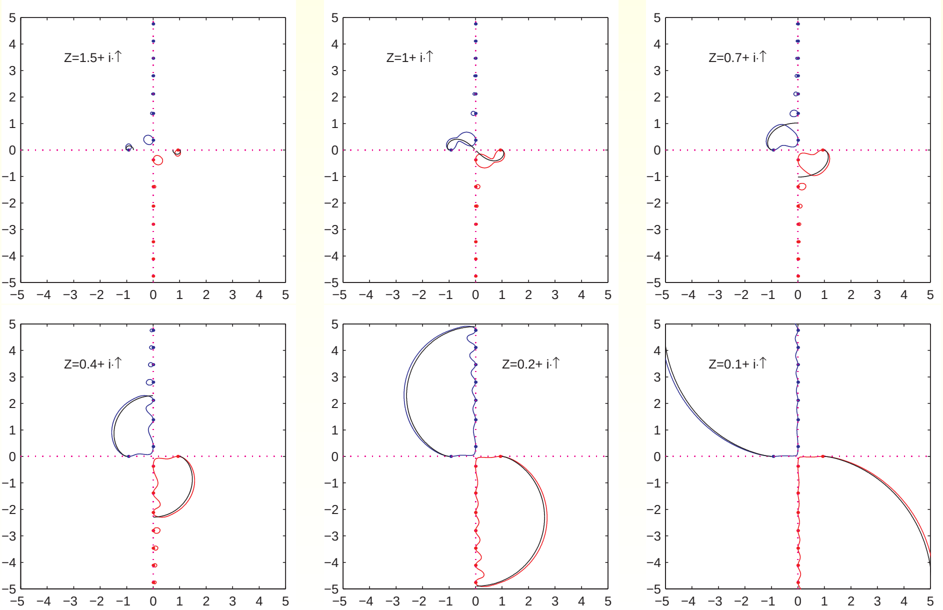
Tracing axial wave numbers $\kappa_{m\mu}$



9 left- and right-running modes with $\omega = 5$, $m = 1$, $M = 0$ tracing along Re $\tilde{Z} = 0.4$

Trajectories of $\kappa_{m\mu}$ for several Re $\tilde{Z}$



Analysis of 2D eigenvalue equation

ANALYSIS WITH FLOW IS MORE INVOLVED:

- $\mathrm{Re}(\tilde{Z}) = -\,\mathrm{Re}\left[\dfrac{(1-M\sigma)^2}{\beta^3\gamma}\right] = 0$, border of possible impedances.
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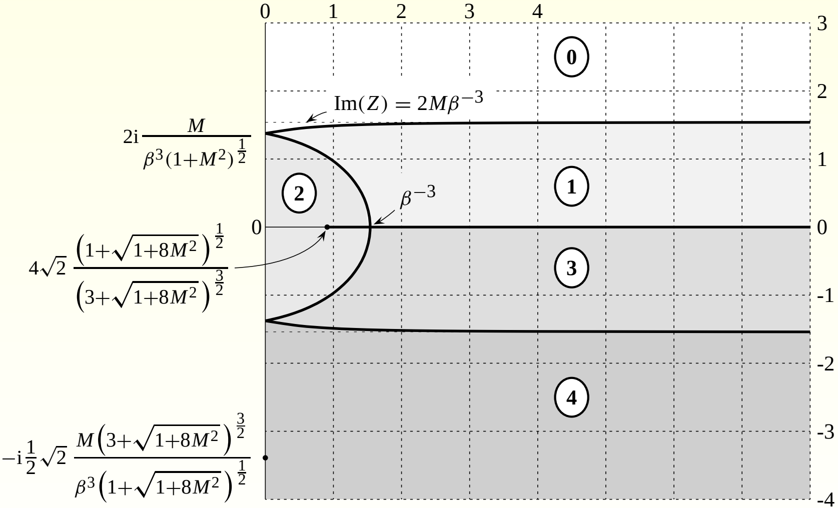
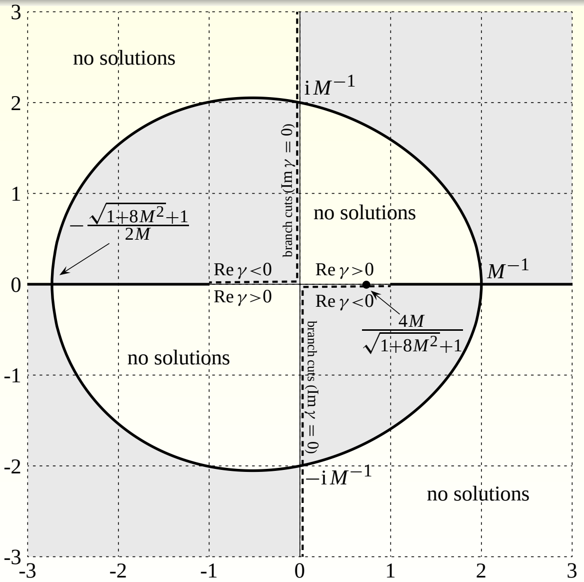
Regions in $\tilde{Z}$-plane of occurrence of 2D surface waves

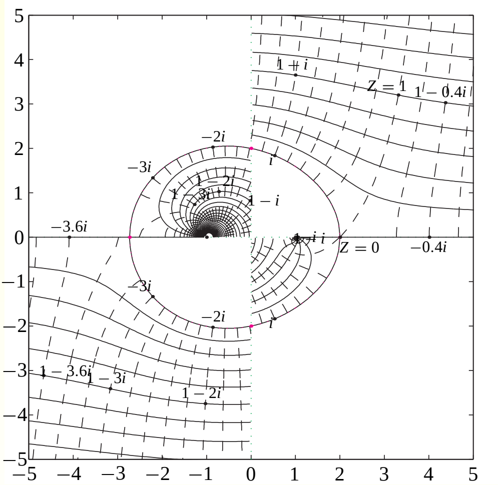


Regions of different numbers of surface waves in complex $\tilde{Z}$-plane ($M = 0.5$)
Thick lines map to the branch cuts in σ-plane.

Regions of existence of 2D surface waves in σ-plane



Regions of existence of surface waves σ $(M = 0.5)$
Thick lines map to the imaginary $\tilde{Z}$-axis.



Trajectories of σ for varying $\tilde{Z} = R + \mathrm{i}\, X$; $M = 0.5$

Fixed R & $X = -\infty : 0.2 : \infty$ ———

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Trajectories of modes $\kappa_{m\mu}$ with mean flow for several Re $\tilde{Z}$

THE FULL MODAL PICTURE WITH MEAN FLOW

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Remember the Lorentz transform to make the pictures clean:

$$\kappa^{\pm}_{m\mu} = \frac{k(\pm\sigma_{m\mu} - M)}{\beta^2}, \quad \sigma_{m\mu} = \sqrt{1 - \gamma^2_{m\mu}}, \quad \gamma_{m\mu} = \frac{\beta\alpha_{m\mu}}{k}$$

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Trajectories of all $\sigma_{m\mu}$ for $\mathrm{Re}(\tilde{Z})$ fixed, $-\infty < \mathrm{Im}(\tilde{Z}) < \infty$:

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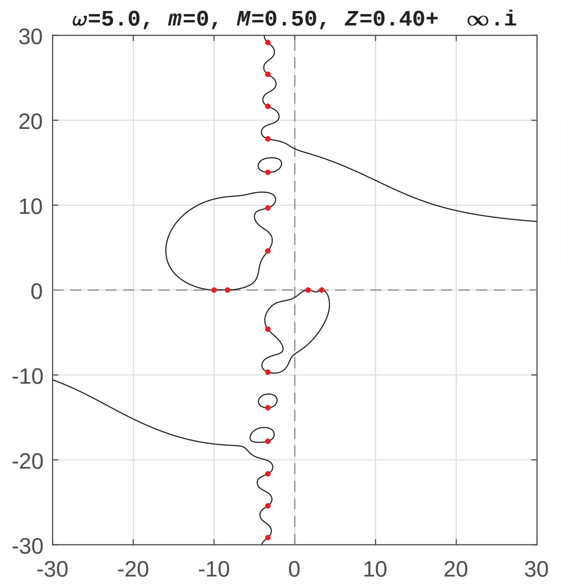
Note: 4 surface modes!

Tracing axial wave numbers $\kappa_{m\mu}$ with mean flow



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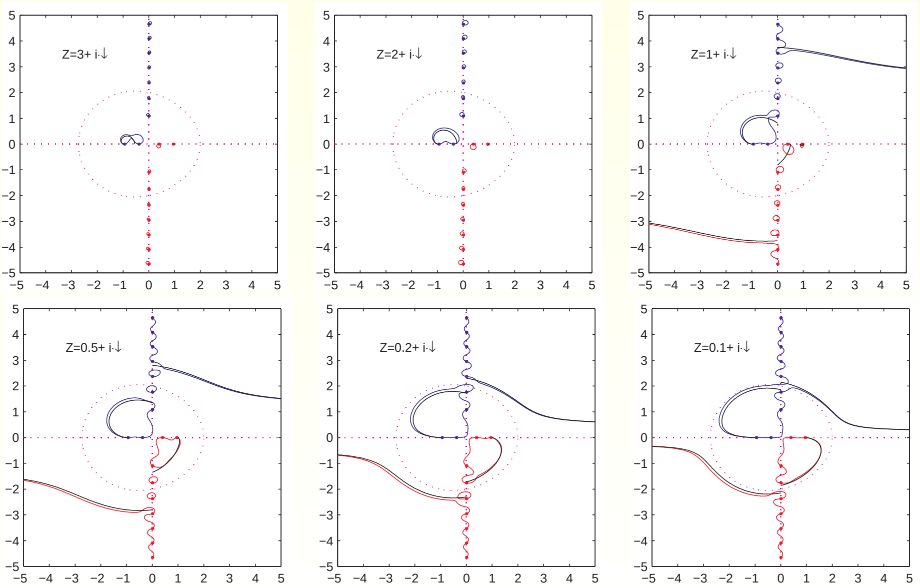
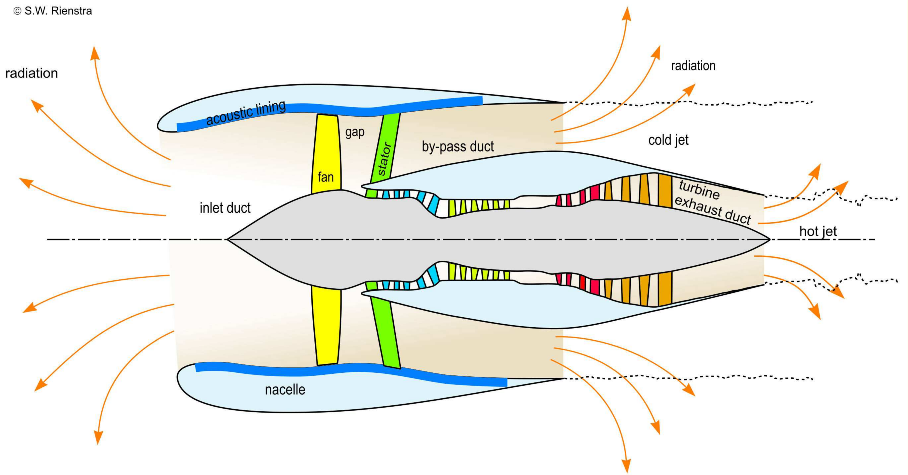


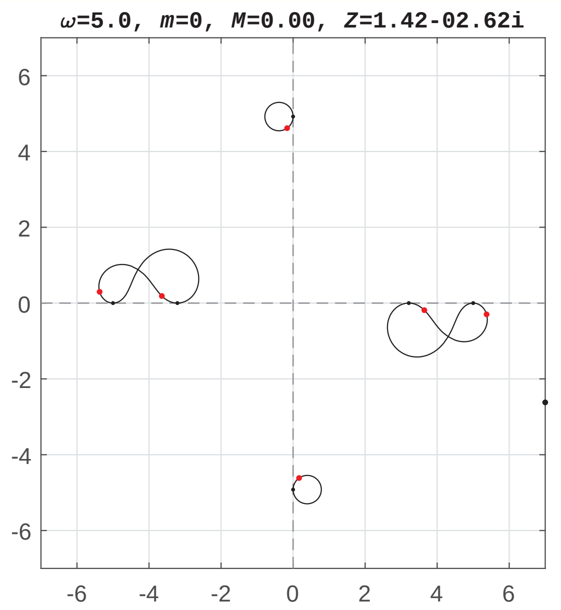
Outline

Important applications of basic* duct acoustics



*relatively…

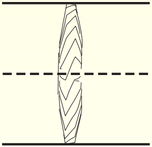
Cremer’s Optimum



Coalescing 1st and 2nd mode $\omega = 5$, $m = 0$, $M = 0$ at $\tilde{Z} = 1.4165 - 0.6082\mathrm{i}$ gives highest modal attenuation

Propeller in duct

B blades, B-periodic with period $\Delta\vartheta = 2\pi/B$:



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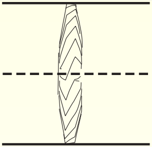

B blades, B-periodic with period $\Delta\vartheta = 2\pi/B$:

Any periodic function can be written as a Fourier series.

$$p(0, r, \vartheta, 0) = \sum_{m=-\infty}^{\infty} q_m(r)\, \mathrm{e}^{-\mathrm{i}m2\pi\frac{\vartheta}{\Delta\vartheta}} = \sum_{m=-\infty}^{\infty} q_m(r)\, \mathrm{e}^{-\mathrm{i}mB\vartheta}$$

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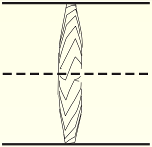

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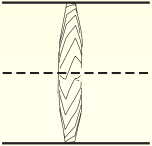

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$\Downarrow$ (in whatever way ...)

$$p(x, r, \vartheta, t) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} Q_{m\mu}\, J_{mB}(\alpha_{mB,\mu} r)\, \mathrm{e}^{-\mathrm{i}\kappa_{mB,\mu} x - \mathrm{i}mB\vartheta + \mathrm{i}mB\Omega t}$$

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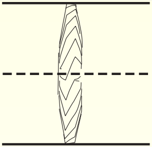

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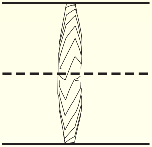

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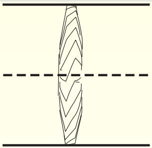

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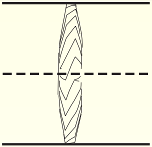

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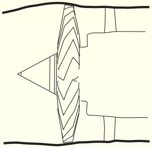
Supersonic tip speed: buzz-saw noise

Tyler & Sofrin's selection rule

Ingenious manipulation of periodicity in B rotor blades and V stator vanes:



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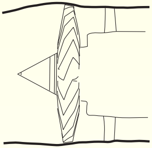

$$p(r, \vartheta, t) = \sum_{n=-\infty}^{\infty} P_n(r, \vartheta)\, \mathrm{e}^{\mathrm{i}nB\Omega t}$$

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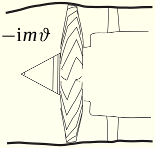

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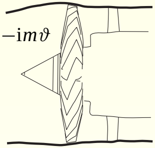

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with $\Delta\vartheta = 2\pi/V$ in a time step $\Delta t = \Delta\vartheta/\Omega$ because vanes are fixed.

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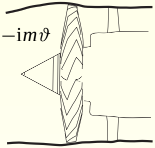

$$
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\end{aligned}
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$$
\mathrm{e}^{-\mathrm{i}nB\Omega\Delta t+\mathrm{i}m\Delta\vartheta} = 1 = \mathrm{e}^{2\pi\mathrm{i}k}, \qquad \text{or:} \quad \boxed{m = kV + nB,} \quad k\in\mathbb{Z}.
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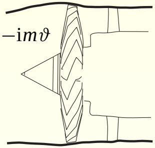

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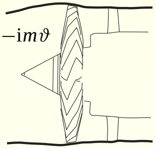

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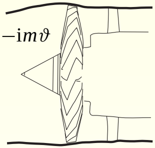

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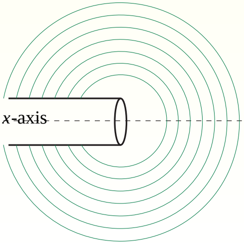
(and this is how real engines are designed!)

Reflection at/radiation from open end

SOLUTION IS CLASSIC EXAMPLE OF WIENER-HOPF TECHNIQUE

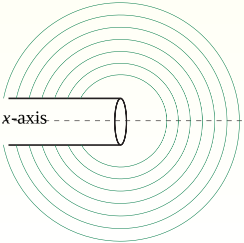


Reflection at/radiation from open end

SOLUTION IS CLASSIC EXAMPLE OF WIENER-HOPF TECHNIQUE

Inside duct: *incident* mode + *reflected* field $p(x, r, \vartheta) = p_{m\mu}(x, r)\, \mathrm{e}^{-\mathrm{i}m\vartheta}$

$$p_{m\mu}(x, r) = U_{m\mu}(r)\, \mathrm{e}^{-\mathrm{i}\kappa_{m\mu} x} + \sum_{\nu=1}^{\infty} R_{m\mu\nu} U_{m\nu}(r)\, \mathrm{e}^{\mathrm{i}\kappa_{m\nu} x}\,.$$



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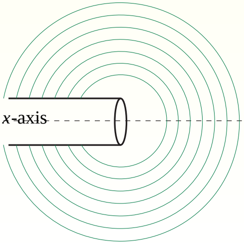
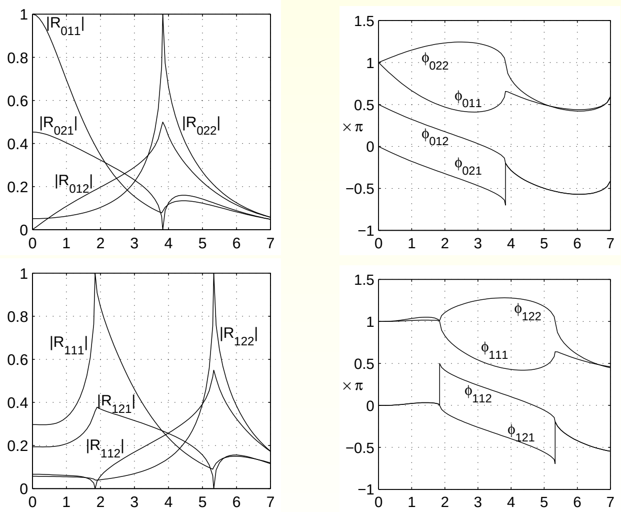
Outside far field: for $k\varrho \to \infty$

$$p_{m\mu}(x, r) \simeq D_{m\mu}(\xi)\frac{\mathrm{e}^{-\mathrm{i}k\varrho}}{k\varrho}, \qquad (x, r) = \varrho(\cos\xi, \sin\xi)$$

$D_{m\mu}(\xi)$ is: *directivity* function,
$|D_{m\mu}(\xi)|$ is: *radiation* pattern.



Reflection matrix



Modulus and phase of reflection coefficients $R_{m\mu\nu}$ for $m = 0 \ldots 2$, μ, $\nu = 1, 2$, as a function of $kR = 0 \ldots 7$.

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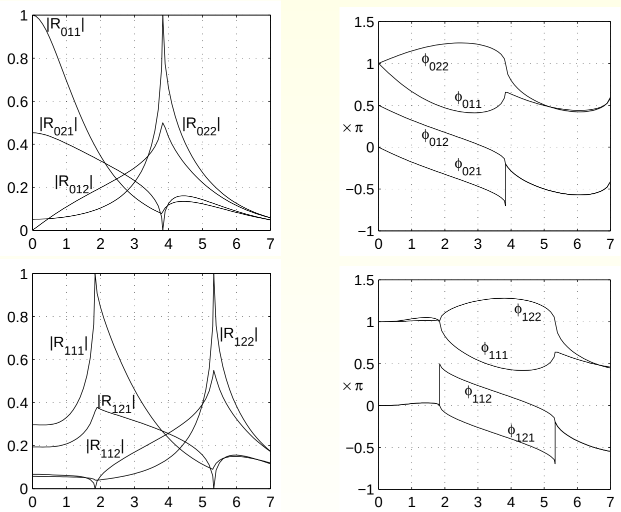

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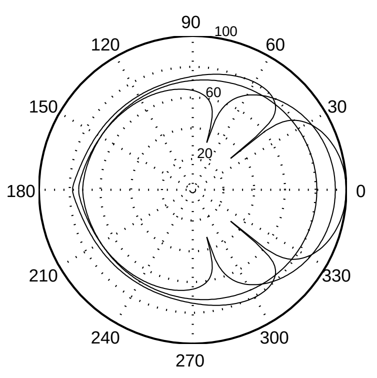
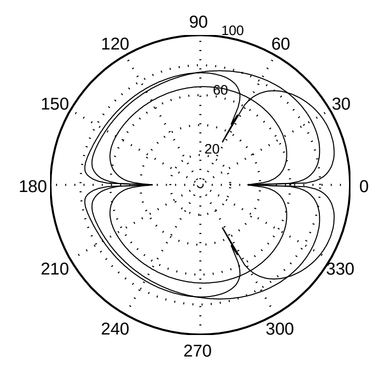
Note: $|R_{m\mu\nu}| = 1$: total reflection.

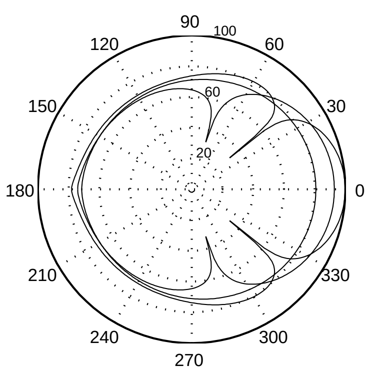
Far field radiation pattern

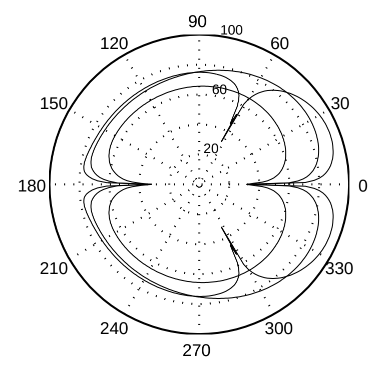




Radiation pattern $20 \log_{10} |D_{m\mu}|$
for $m\mu = 01, = 11$ and $kR = 2, 4, 6$.

Far field radiation pattern





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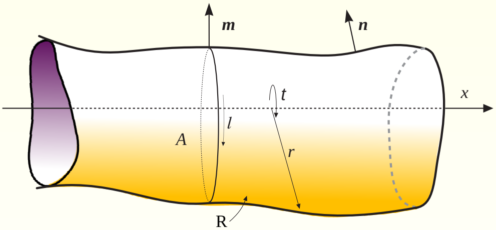
Note: lobes!

Outline

Slowly varying ducts



$$\nabla \cdot \left(c_0^2 \nabla p\right) + \omega^2 p = 0$$

A canonical model problem

Model equation $\quad \frac{\partial^2}{\partial t^2} y - \frac{\partial}{\partial x}\left(c(\varepsilon x)^2 \frac{\partial}{\partial x} y\right) = 0, \quad \varepsilon \ll 1$

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Expand: $A(X;\varepsilon) = A_0(X) + \varepsilon A_1(X) + \dots,$

$$O(1): \quad (\omega^2 - c^2\gamma^2)A_0 = 0$$
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yielding:

$$O(1): \quad \gamma(X)^2 = \omega^2/c^2(X)$$

$$O(\varepsilon): \quad \left(c^2\gamma A_0^2\right)' = 0 \quad \rightarrow \quad A_0(X) = \frac{\text{const}}{c(X)\sqrt{\gamma(X)}}$$

Classical straight-duct solution

Classical straight-duct solution as a modal sum:

$$p(x,r,\theta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_{mn} \Phi_{mn}(x,r,\theta)$$

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Boundary conditions:

$$J_m'(\alpha_{m\mu} R) = 0 \quad \rightarrow \quad \alpha_{mn} = j_{m\mu}'/R.$$

$$\kappa_{mn}^2 + \alpha_{mn}^2 = \frac{\omega^2}{c_0^2}, \qquad \kappa_{mn} \in \begin{cases} \mathrm{IV} & \text{right running} \\ \mathrm{I\,I} & \text{left running} \end{cases}$$

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$$\nabla \cdot \left(c_0^2 \nabla p\right) + \omega^2 p = 0$$

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Let's start with c_0 constant. Assume the mode-like WKB-Ansatz:

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Use $\displaystyle \frac{\partial}{\partial x} \sim -\mathrm{i}\gamma(X) + \varepsilon \frac{\partial}{\partial X}$, and substitute:

$$\boxed{\mathcal{L}(A) = \mathrm{i}\varepsilon(\gamma_X A + 2\gamma A_X) + O(\varepsilon^2) = \frac{\mathrm{i}\varepsilon}{A} \frac{\partial}{\partial X}\left(\gamma A^2\right) + O(\varepsilon^2)}$$

where $\displaystyle \mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\omega^2}{c_0^2} - \gamma^2 - \frac{m^2}{r^2}.$

(Similar for the boundary conditions.

Expand: $A(X, r; \varepsilon) = A_0(X, r) + \varepsilon A_1(X, r) + \ldots,$

$$O(1): \qquad \mathcal{L}(A_0) = 0$$

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$O(1): A_0(X,r) = N(X) J_m(\alpha(X) r)$

Dispersion relation: $\qquad \alpha^2 + \gamma^2 = \dfrac{\omega^2}{c_0^2}.$

Boundary condition: $\frac{\partial}{\partial r} A_0 = N(X)\alpha J_m'(\alpha R) = 0$

$$\rightarrow \text{determines } \alpha(X) = \frac{j'_{m\mu}}{R(X)}$$

$$\rightarrow \text{determines } \gamma(X) = \sqrt{\frac{\omega^2}{c_0^2} - \alpha^2}$$

Finally determine $N(X)$

Solvability condition for A_1: manipulate

$$\int_0^R \Big[A_0\mathcal{L}(A_1) - A_1\mathcal{L}(A_0)\Big] r\,\mathrm{d}r = \int_0^R \mathrm{i}\,\frac{\partial}{\partial X}(\gamma A_0^2) r\,\mathrm{d}r$$

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$$-\mathrm{i}\, R R_X\, \gamma A_0^2(X, R) = \int_0^R \mathrm{i}\, \frac{\partial}{\partial X}(\gamma A_0^2) r \,\mathrm{d}r$$

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$$\frac{\mathrm{d}}{\mathrm{d}X}\left[\int_0^R \gamma(X) A_0^2(X,r) r\,\mathrm{d}r\right] \overset{\Downarrow}{=} \frac{\mathrm{d}}{\mathrm{d}X}\left[\gamma N^2 \int_0^R J_m(\alpha r)^2 r\,\mathrm{d}r\right] = 0.$$

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$$\Downarrow$$

$$\boxed{N(X) = \frac{N_0}{R(X)\sqrt{\gamma(X)}}}$$

Finally determine $N(X)$

Solvability condition for A_1: manipulate

$$-\mathrm{i}\, R R_X\, \gamma A_0^2(X, R) = \int_0^R \mathrm{i}\, \frac{\partial}{\partial X}(\gamma A_0^2) r\, \mathrm{d}r$$

to obtain adiabatic invariant:

$\Downarrow$

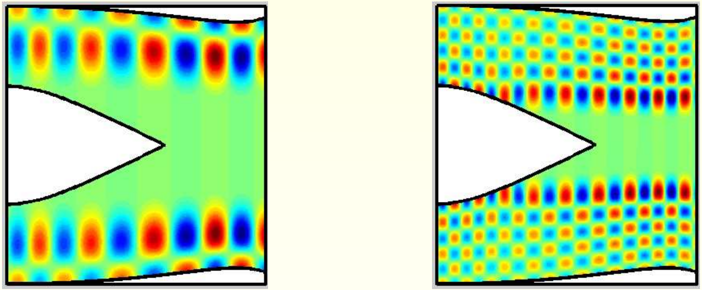
$$\frac{\mathrm{d}}{\mathrm{d}X}\left[\int_0^R \gamma(X) A_0^2(X, r) r\, \mathrm{d}r\right] = \frac{\mathrm{d}}{\mathrm{d}X}\left[\gamma N^2 \tfrac{1}{2} R^2 \left(1 - \frac{m^2}{{j'_{m\mu}}^2}\right) J_m(j'_{m\mu})^2\right] = 0.$$

$\Downarrow$

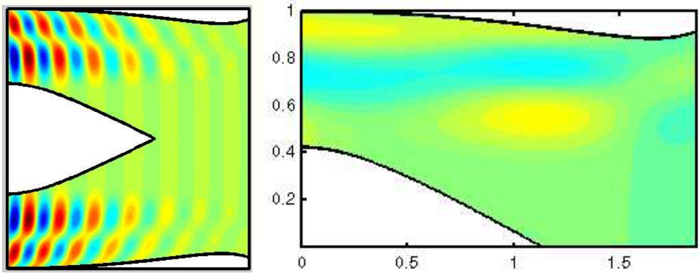
$$\boxed{N(X) = \frac{N_0}{R(X)\sqrt{\gamma(X)}}}$$

(Note possible cut-off or *turning* point $X = X_0$ where $\gamma(X) = 0$.)

Slowly varying modes in engine inlet: $m = 10$, $M = -0.5$

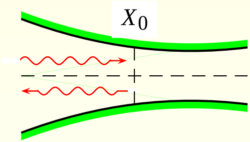


$\omega = 16$ and $\omega = 30$ (hard wall)



$\omega = 20$ (soft wall), well cut-on and near cut-off

Turning point reflection



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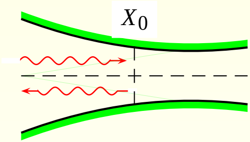

$X < X_0$:

$$p = \frac{n(X)}{\sqrt{\gamma(X)}} J_m(\alpha r) \Big[\mathrm{e}^{-\frac{\mathrm{i}}{\varepsilon} \int_{X_0}^{X} \gamma(\mathrm{z})\mathrm{dz}} + \mathcal{R}\, \mathrm{e}^{\frac{\mathrm{i}}{\varepsilon} \int_{X_0}^{X} \gamma(\mathrm{z})\mathrm{dz}} \Big]$$

$X > X_0$:

$$p = \mathcal{T} \frac{n(X)}{\sqrt{\gamma(X)}} J_m(\alpha r)\, \mathrm{e}^{-\frac{1}{\varepsilon} \int_{X_0}^{X} |\gamma(\mathrm{z})|\mathrm{dz}}$$

$$\gamma(X_0) = 0, \quad \alpha_0 = \alpha(X_0), \quad \alpha_0' = \alpha'(X_0), \quad n(X) = N_0/R(X)$$

$$\frac{\mathrm{d}}{\mathrm{d}X}\gamma(X_0)^2 < 0 \;\leftrightarrow\; \frac{\alpha_0'}{\alpha_0} > 0$$

Local analysis

Rescale

$$X = X_0 + \varepsilon^{\frac{2}{3}} \lambda^{-1} \xi, \quad \text{where} \quad \lambda = \frac{\omega}{c_0} \left(\frac{2\alpha_0'}{\alpha_0} \right)^{\frac{1}{3}}$$

$$\gamma(X) \simeq \varepsilon^{\frac{1}{3}} \lambda \sqrt{-\xi}$$

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$$p = \psi(\xi) J_m(\alpha(X_0 + \cdots)r)$$

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leading to:

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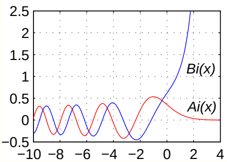

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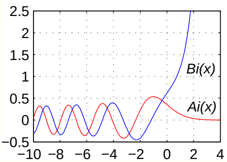

Matching:

$$\mathcal{R} = \mathrm{i}, \quad \mathcal{T} = 1, \quad b = 0, \quad a = \frac{n_0 2\sqrt{\pi}\,\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}}{\varepsilon^{\frac{1}{6}}\lambda^{\frac{1}{2}}}$$

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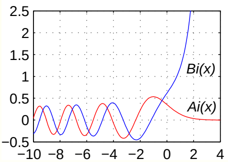

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(Note the $\frac{1}{2}\pi$ phase change by reflection.)

Outline

Summary

- Modes are solutions, self-similar in one direction
- Derivation of parallel flow acoustic equations:
  shear flow, uniform mean and no mean flow
- Boundary conditions: hard walls, soft walls without and with mean flow
- Duct modes: hard/soft walls, cut-on cut-off, group and phase velocity
- Circular duct: Bessel functions, source expansion, modal power
- Behaviour of soft-wall modes, surface waves
- Propeller in duct, Tyler-Sofrin rule
- Open end radiation, reflection coefficient
- Slowly varying modes & turning point at cut-off.
- **Most important**:

$$\text{Helmholtz number } kR = \frac{\omega R}{c_0}, \text{ and axial wave number } \kappa_n$$

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