SUPPLEMENTARY INFORMATION

Supplementary Note 1. Entropy of "entanglement"

We consider a superposition of states represented by the following wave function:

$$\Psi_{6x1} = Ae_2 \otimes \begin{pmatrix} \sqrt{\omega + \beta k_2} \\ \pm\sqrt{\omega - \beta k_2} \end{pmatrix} e^{ik_2 x} e^{i\omega t} + Be_3 \otimes \begin{pmatrix} \sqrt{\omega + \beta k_3} \\ \pm\sqrt{\omega - \beta k_3} \end{pmatrix} e^{ik_3 x} e^{i\omega t}, \quad \text{(S1)}$$

where A and B are complex amplitudes, and e_2 and e_3 are normalized OAM eigen vectors. We also denote by $f_2 = \frac{1}{\sqrt{2\omega}} \begin{pmatrix} \sqrt{\omega + \beta k_2} \\ \pm\sqrt{\omega - \beta k_2} \end{pmatrix} e^{ik_2 x}$ and $f_3 = \frac{1}{\sqrt{2\omega}} \begin{pmatrix} \sqrt{\omega + \beta k_3} \\ \pm\sqrt{\omega - \beta k_3} \end{pmatrix} e^{ik_3 x}$, the normalized spinorial part of the wave functions associated with the directional degrees of freedom. We can form a basis for the states of the coupled elastic waveguides in the form of the four tensor products:

$\phi_1 = e_2 \otimes f_2$; $\phi_2 = e_2 \otimes f_3$; $\phi_3 = e_3 \otimes f_2$; $\phi_4 = e_3 \otimes f_3$. In that basis the state given by Eq. (S1) reads:

$$\Psi_{2N\times 1} = \sqrt{2\omega_n}(Ae_2 \otimes f_2 + Be_3 \otimes f_3)e^{i\omega t} = \sqrt{2\omega_n}(A\phi_1 + B\phi_4)e^{i\omega t}. \quad \text{(S2)}$$

It is clear that this state cannot be written as a tensor product in the basis $\{\phi_1, \phi_2, \phi_3, \phi_4\}$. Again, we can say that this state is nonseparable. To quantify the degree of nonseparability of this state, we can calculate the "entanglement entropy" [1]. First, we normalize the wave function of Eq. (S2):

$$\widetilde{\Psi} = \frac{1}{\sqrt{(|A|^2 + |B|^2)}} (A\phi_1 + B\phi_4)e^{i\omega t} . \quad \text{(S3)}$$

In obtaining the normalizing factor in Eq. (S3) we have used the fact that $\phi_i \phi_j^* = (e_i \otimes f_I)\left(e_j \otimes f_I\right)^* = (e_i \otimes f_I)(e_i^* \otimes f_I^*) = (e_i e_i^*) \otimes (f_I f_I^*)$ and that the OAM eigen vectors e_i form an orthonormal basis. We also note that the amplitude of f_I is real.

We can also define the basis $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ in terms of the 4×1 vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. We construct the density matrix associated with that state as the outer product of $\widetilde{\Psi}$ and its complex conjugate $\widetilde{\Psi}^*$:

$$\rho_{OAM-s} = \widetilde{\Psi} \otimes \widetilde{\Psi}^* = \frac{1}{|A|^2+|B|^2} \begin{pmatrix} AA^* & 0 & 0 & AB^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ BA^* & 0 & 0 & BB^* \end{pmatrix}. \tag{S4}$$

In Eq. (S4), we have used the notation ρ_{OAM-s} to highlight that the density of states is expressed in the tensor product Hilbert space of the directional and OAM subspaces. The reduced density matrix in the Hilbert space of OAM is obtained by taking the partial trace of the density matrix over the directional states:

$$\rho_{OAM} = \frac{1}{|A|^2+|B|^2} \begin{pmatrix} AA^* & 0 \\ 0 & BB^* \end{pmatrix}. \tag{S5}$$

The entropy of "entanglement" is now obtained from the relation:

$$S(\rho_{OAM}) = -Tr(\rho_{OAM} \ln \rho_{OAM}). \tag{S6}$$

Using Eq. (S5), we get:

$$S(\rho_{OAM}) = -\frac{1}{|A|^2+|B|^2}\left(AA^* ln\frac{AA^*}{|A|^2+|B|^2} + BB^* ln\frac{BB^*}{|A|^2+|B|^2}\right). \quad \text{(S7)}$$

We can also calculate the entropy of “entanglement” by calculating the reduced density matrix in the Hilbert space of directions, ρ_s. The two entropies are equal. We note that if one chooses $A = B$, $S(\rho_{OAM}) = ln2$. The state $\widetilde{\Psi}$ is maximally “entangled.” The state $\widetilde{\Psi} = \frac{1}{\sqrt{2}}(\phi_1 + \phi_4)e^{i\omega t}$ is equivalent to a Bell state. By controlling the amplitude and phase of A and B, one can control the degree of nonseparability of the $\widetilde{\Psi}$ as well as generate other Bell states.

Supplementary Note 2. Coupled three-chain mass-spring waveguides

For three coupled mass-spring chains with a total of N_m identical masses in each chain, the discrete elastic equations of motion are:

$$m\ddot{u}_n - k_{nn}(u_{n+1} - 2u_n + u_{n-1}) - k_c(v_n - u_n) + \eta\dot{u}_n = 0, \quad (S8a)$$

$$m\ddot{v}_n - k_{nn}(v_{n+1} - 2v_n + v_{n-1}) - k_c(u_n - v_n) - k_c(w_n - v_n) + \eta\dot{v}_n = 0, \quad (S8b)$$

$$m\ddot{w}_n - k_{nn}(w_{n+1} - 2w_n + w_{n-1}) - k_c(v_n - w_n) + \eta\dot{w}_n = 0. \quad (S8c)$$

In equations (S8), u_n, v_n and w_n are the displacements of n^{th} mass of chain 1, 2, and 3, respectively, and $n = 2, \dots, N_m - 1$. The term k_{nn} describes the coupling constant of the nearest-neighbor interaction, and k_c describes the stiffness of the springs that couples the chains. m is the mass, and the viscous damping coefficient η models the dissipation. To model the excitation applied by the transducer, the prescribed base periodic excitation $F^u(t) = F_0^u \sin(\omega t)$, $F^v(t) = F_0^v \sin(\omega t)$ and $F^w(t) = F_0^w \sin(\omega t)$ are applied to the first mass of each chain, where F_0^u, F_0^v and F_0^w are the amplitudes applied force to chain 1, 2 and 3, respectively. To mimic the physical experiment, we should use $m = m_R/N_m$ and $k_{nn} = EA/a$, where $a(= L/N_m)$ is the inter-mass spacing, and $m_R(= \rho AL)$, $E(= 60\ \text{GPa})$ and $A(= \pi d^2/4)$ are the mass, Young's modulus and cross-sectional area of the aluminum rod, respectively.

In the limit of long wavelength compared to the inter-mass spacing a, the equations of motion (S8) of the three coupled linear harmonic chains of masses and springs become:

$$\frac{\partial^2 u}{\partial t^2} - \beta^2 \frac{\partial^2 u}{\partial x^2} - \alpha^2 (v - u) + \bar{\eta} \frac{\partial u}{\partial t} = F_0^u e^{i\omega t} \delta_{x=0}, \tag{S9a}$$

$$\frac{\partial^2 v}{\partial t^2} - \beta^2 \frac{\partial^2 v}{\partial x^2} - \alpha^2 (u - v) - \alpha^2 (w - v) + \bar{\eta} \frac{\partial v}{\partial t} = F_0^v e^{i\omega t} \delta_{x=0}, \tag{S9b}$$

$$\frac{\partial^2 w}{\partial t^2} - \beta^2 \frac{\partial^2 w}{\partial x^2} - \alpha^2 (v - w) + \bar{\eta} \frac{\partial w}{\partial t} = F_0^w e^{i\omega t} \delta_{x=0}, \tag{S9c}$$

where $\beta^2 = k_{nn} a^2/m$, $\alpha^2 = k_c/m$ and $\bar{\eta} = \eta/m$. In a simple form:

$$\left(\frac{\partial^2}{\partial t^2} - \beta^2 \frac{\partial^2}{\partial x^2}\right) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \alpha^2 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \bar{\eta} \frac{\partial}{\partial t} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0.$$

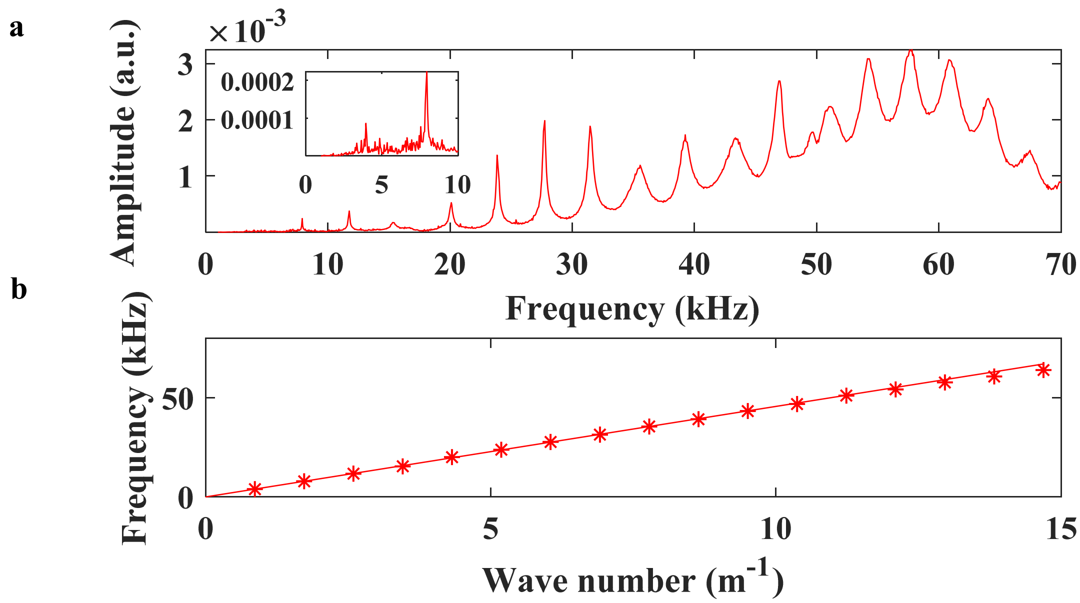
By comparing equation (1) of the manuscript, we obtain the coupling matrix as

$$M_{3\times3} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

By choosing $k_c = 33.5 \times 10^6\ \text{N/m}, N_m = 48$, we can numerically calculate the dispersion relation that also fits the experimental band structure for three coupled rods with cutoff frequencies 14.08 kHz and 24.43 kHz.

Supplementary Note 3. Single rod acoustic waveguide, $N = 1$

In Supplementary Figure 1, we establish the characteristics of a single ($N = 1$) free standing aluminum rod. Supplementary Figure 1a reports the measured experimental transmission spectrum and shows well defined resonances corresponding to the standing wave modes supported by the waveguide. The peak positions define the resonance frequencies (i.e., the modes) of the system. Supplementary Figure 1b shows the resultant dispersion relation.



Supplementary Figure 1 Identification of elastic wave states of a single rod. **a,** Transmission spectrum of the single rod waveguide ($N = 1$). The inset shows the transmission amplitudes at low frequency <10 kHz, identifying the first two resonances. The transmission amplitude is in arbitrary units. **b,** Band structure determined and calculated from Supplementary Figure 1a. The asterisks are obtained from the resonances of Supplementary Figure 1a and the solid line associated with the asterisks is a fit to resonances with frequency less than 60 kHz.

Supplementary References:

1. Janzing, D. Entropy of Entanglement. in *Compendium of Quantum Physics* (eds. Greenberger, D., Hentschel, K. & Weinert, F.) 205–209 (Springer Berlin Heidelberg, 2009). doi:10.1007/978-3-540-70626-7_66