Note:derivation of 3-31

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A. Substitue pressure modes for transverse velocity modes

1. *mass equation*

$$
\begin{aligned}
&\frac{dU_\alpha^a}{ds} - \Psi_{\{\alpha\}\beta}[r]U_\beta^a - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{\underline{V_\beta^a}}} - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{\underline{W_\beta^a}}} \\
&= \Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa \underline{\underline{\underline{V_\beta^{a-b}V_\gamma^b}}} - ib\kappa \underline{\underline{\underline{W_\beta^{a-b}W_\gamma^b}}} - ia\kappa\frac{B}{2A}P_\beta^{a-b}P_\gamma^b)
\end{aligned} \tag{1}
$$

Transform:

$$
\begin{aligned}
\frac{dU_\alpha^a}{ds} &:= \underline{\underline{I}}u_\alpha'^a \\
-\Psi_{\{\alpha\}\beta}[r]U_\beta^a &:= \underline{\underline{\underline{-\Psi_{\{\alpha\}\beta}[r]}}}u_\beta^a \\
-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P_\beta^a &:= \underline{\underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}}p_\beta^a \\
-\Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{\underline{V_\beta^a}}} &:= \underline{\underline{\underline{-\Psi_{[\alpha]\delta}[r(1-\kappa rcos(\phi))]}}}V_\beta^a \\
-\Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{\underline{W_\beta^a}}} &:= \underline{\underline{\underline{-\Psi_{(\alpha)\delta}[(1-\kappa rcos(\phi))]}}}W_\beta^a \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b}P_\gamma^b) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty} \underline{\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}}p_\beta^{a-b}p_\gamma^b = (B_1[p,p])_\alpha^a \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa U_\beta^{a-b}U_\gamma^b) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty} \underline{\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}}u_\beta^{a-b}u_\gamma^b = (A_1[u,u])_\alpha^a \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa V_\beta^{a-b}V_\gamma^b) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty} \underline{\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty} \Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b}}}p_\beta^{a-b}p_\gamma^b \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa W_\beta^{a-b}W_\gamma^b) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty} \underline{\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty} \Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{W}_{\delta\beta}^{a-b}\mathbf{W}_{\epsilon\gamma}^b}}}p_\beta^{a-b}p_\gamma^b \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))] \sum_{b=-\infty}^{+\infty} (-ia\kappa\frac{B}{2A}P_\beta^{a-b}P_\gamma^b) &:= \sum_{b=-\infty}^{+\infty} -ia\kappa\frac{B}{2A} \sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty} \underline{\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}}p_\beta^{a-b}p_\gamma^b
\end{aligned} \tag{2}
$$

2. momentum equation I

$$\begin{aligned}
&\frac{d}{ds}P^a_\alpha - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\
&= \sum_{b=-\infty}^{+\infty} \underline{term\mathcal{D}1:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\underline{\underline{V^{a-b}_\beta}}U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\underline{\underline{W^{a-b}_\beta}}U^a_\gamma \\
&+\underline{term(D1+\mathcal{P}1):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta U^b_\gamma \\
&+\underline{term\mathcal{D}1:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}1:}\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta\underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta\underline{\underline{W^b_\gamma}}
\end{aligned} \tag{3}$$

Transform:

$$\begin{aligned}
&\frac{d}{ds}P^a_\alpha := \underline{\underline{I}}p'^a_\alpha \\
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta := \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}u^a_\beta \\
&-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta := \underline{\underline{-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta}}p^a_\beta - \underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta = -\underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta \\
&\Psi_{\{\alpha\}\beta\gamma}[r]\sum_{b=-\infty}^{+\infty} U^{a-b}_\beta U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\beta\gamma}[r]}}u^{a-b}_\beta u^b_\gamma \\
&\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{V^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}}}p^{a-b}_\beta u^b_\gamma \\
&\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{W^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{(\alpha)\delta\epsilon}[(1-\kappa rcos\phi)]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}}}p^{a-b}_\beta u^b_\gamma \\
&ia\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}P^{a-b}_\beta U^b_\gamma := ia\kappa\sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]}}p^{a-b}_\beta u^b_\gamma \\
&-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])\sum_{b=-\infty}^{+\infty}U^{a-b}_\beta U^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\frac{\partial}{\partial s}}}(\Psi_{\alpha\beta\gamma}[r])u^{a-b}_\beta u^b_\gamma \\
&-\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) := \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][\mathbf{G},\mathbf{I}] + \Psi_{\alpha\beta\gamma}[r][\mathbf{I},\mathbf{G}])u^{a-b}_\beta u^b_\gamma \\
&\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta\underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta\underline{\underline{W^b_\gamma}} := \sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[rcos\phi][\mathbf{I},\mathbf{V}] - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi][\mathbf{I},\mathbf{W}]u^b_\gamma
\end{aligned} \tag{4}$$

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_\beta \\
&+\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)]P^a_\beta - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]P^a_\beta \\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta V^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta V^a_\gamma \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta V^b_\gamma \\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}V^b_\gamma + \frac{dV^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}2:}\Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta W^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta U^b_\gamma
\end{aligned} \tag{5}
$$

Transform:

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]V^a_\beta := -N^a_{\alpha\beta}V^a_\beta \\
&\{\int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa rcos\phi)]_0^h d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos\phi)] - \Psi_{\alpha\beta}[1-2\kappa rcos\phi]\}P^a_\beta \\
&\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\delta\epsilon}[r]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}u^{a-b}_\beta p^b_\gamma = \sum_{b=-\infty}^{+\infty}\Psi_{\{\alpha\}\beta\gamma}[r][\mathbf{I},\mathbf{V}]u^{a-b}_\beta p^b_\gamma \\
&\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta V^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \\
&\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta V^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{(\alpha)\delta\epsilon}[1-\kappa rcos\phi]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \\
&i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta V^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{ia\kappa\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{I}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \\
&-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta U^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])u^{a-b}_\beta u^b_\gamma \\
&-\Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}V^b_\gamma + \frac{dV^b_\gamma}{ds}U^{a-b}_\beta) :== \Psi_{\alpha\beta\gamma}[r][G,I] + \Psi_{\alpha\beta\gamma}[r][I,G]
\end{aligned} \tag{6}
$$

4. momentum equation III

$$
\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]W^a_\beta \\
&-\Psi_{(\alpha)\beta}[(1-\kappa rcos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[rsin\phi]P^a_\beta \\
&= \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]W^{a-b}_\beta W^a_\gamma \\
&+\underline{term(D2+\mathcal{P}2):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta W^b_\gamma \\
&+\underline{term\mathcal{D}2:}-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta) \\
&+\underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa rcos\phi]W^{a-b}_\beta V^b_\gamma
\end{aligned}
\tag{7}
$$