Note:derivation of 3-31

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A. Substitue pressure modes for transverse velocity modes

1. mass equation

$$\begin{aligned}&\frac{dU^a_\alpha}{ds} - \Psi_{\{\alpha\}\beta}[r]U^a_\beta - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{V^a_\beta}} - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{W^a_\beta}} \\ &= \Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma - ib\kappa U^{a-b}_\beta U^b_\gamma - ib\kappa\underline{\underline{V^{a-b}_\beta V^b_\gamma}} - ib\kappa\underline{\underline{W^{a-b}_\beta W^b_\gamma}} - ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma)\end{aligned} \tag{1}$$

Transform:

$$\begin{aligned}
\frac{dU^a_\alpha}{ds} &:= \underline{\underline{I}}u'^a_\alpha \\
-\Psi_{\{\alpha\}\beta}[r]U^a_\beta &:= \underline{\underline{-\Psi_{\{\alpha\}\beta}[r]}}u^a_\beta \\
-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P^a_\beta &:= \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}p^a_\beta \\
-\Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]\underline{\underline{V^a_\beta}} &:= \underline{\underline{-\Psi_{[\alpha]\delta}[r(1-\kappa rcos(\phi))]\mathbf{V}^a_{\delta\beta}}}p^a_\beta \\
-\Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]\underline{\underline{W^a_\beta}} &:= \underline{\underline{-\Psi_{(\alpha)\delta}[(1-\kappa rcos(\phi))]\mathbf{W}^a_{\delta\beta}}}p^a_\beta \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}_\beta P^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}p^{a-b}_\beta p^b_\gamma = (B_1[p,p])^a_\alpha \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa U^{a-b}_\beta U^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}u^{a-b}_\beta u^b_\gamma = (A_1[u,u])^a_\alpha \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa V^{a-b}_\beta V^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{V}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa W^{a-b}_\beta W^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ib\kappa\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{\alpha\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{W}^b_{\epsilon\gamma}}}p^{a-b}_\beta p^b_\gamma \\
\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ia\kappa\frac{B}{2A}P^{a-b}_\beta P^b_\gamma) &:= \sum_{b=-\infty}^{+\infty} -ia\kappa\frac{B}{2A}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa rcos(\phi))]}}p^{a-b}_\beta p^b_\gamma
\end{aligned} \tag{2}$$

$$
\begin{aligned}
&\frac{d}{ds}P^a_\alpha - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta \\
&= \sum_{b=-\infty}^{+\infty} \underline{term\mathcal{D}1:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta U^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\underline{\underline{V^{a-b}_\beta}}U^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\underline{\underline{W^{a-b}_\beta}}U^a_\gamma \\
&\qquad + \underline{term(D1+\mathcal{P}1):}i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]P^{a-b}_\beta U^b_\gamma \\
&\qquad + \underline{term\mathcal{D}1:} - \frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) \\
&\qquad + \underline{term\mathcal{X}1:}\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta \underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta \underline{\underline{W^b_\gamma}}
\end{aligned}
\tag{3}
$$

Transform:

$$
\begin{aligned}
&\frac{d}{ds}P^a_\alpha := \underline{\underline{I}}p'^a_\alpha \\
&-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa rcos\phi)]U^a_\beta := \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]}}u^a_\beta \\
&-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P^a_\beta - \Psi_{\{\alpha\}\beta}[r]P^a_\beta := \underline{\underline{-\int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta}} p^a_\beta - \underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta = -\underline{\underline{\Psi_{\{\alpha\}\beta}[r]}}p^a_\beta \\
&\Psi_{\{\alpha\}\beta\gamma}[r]\sum_{b=-\infty}^{+\infty} U^{a-b}_\beta U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\{\alpha\}\beta\gamma}[r]}}u^{a-b}_\beta u^b_\gamma \\
&\Psi_{[\alpha]\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{V^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa rcos\phi)]\mathbf{V}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}}}p^{a-b}_\beta u^b_\gamma \\
&\Psi_{(\alpha)\beta\gamma}[(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}\underline{\underline{W^{a-b}_\beta}}U^a_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\sum_{\delta,\epsilon=0}^{\infty}\Psi_{(\alpha)\delta\epsilon}[(1-\kappa rcos\phi)]\mathbf{W}^{a-b}_{\delta\beta}\mathbf{I}^b_{\epsilon\gamma}}}p^{a-b}_\beta u^b_\gamma \\
&ia\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]\sum_{b=-\infty}^{+\infty}P^{a-b}_\beta U^b_\gamma := ia\kappa\sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos\phi)]}}p^{a-b}_\beta u^b_\gamma \\
&-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])\sum_{b=-\infty}^{+\infty}U^{a-b}_\beta U^b_\gamma := \sum_{b=-\infty}^{+\infty}\sum_{\beta=0}^{+\infty}\sum_{\gamma=0}^{+\infty}\underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])}}u^{a-b}_\beta u^b_\gamma \\
&-\Psi_{\alpha\beta\gamma}[r]\sum_{b=-\infty}^{+\infty}(\frac{dU^{a-b}_\beta}{ds}U^b_\gamma + \frac{dU^b_\gamma}{ds}U^{a-b}_\beta) := \sum_{b=-\infty}^{+\infty}(\Psi_{\alpha\beta\gamma}[r][G,I] + \Psi_{\alpha\beta\gamma}[r][I,G])u^{a-b}_\beta u^b_\gamma \\
&\Psi_{\alpha\beta\gamma}[rcos\phi]U^{a-b}_\beta\underline{\underline{V^b_\gamma}} - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi]U^{a-b}_\beta\underline{\underline{W^b_\gamma}} := \sum_{b=-\infty}^{+\infty}\Psi_{\alpha\beta\gamma}[rcos\phi][I,V] - \kappa\Psi_{\alpha\beta\gamma}[rsin\phi][I,W]u^b_\gamma
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]V^a_\beta \\
& + \int_0^{2\pi}[\psi_\alpha\psi_\beta r(1-\kappa r cos\phi)]_0^h d\theta P^a_\beta - \Psi_{[\alpha]\beta}[r(1-\kappa r cos\phi)]P^a_\beta - \Psi_{\alpha\beta}[1-2\kappa r cos\phi]P^a_\beta \\
& = \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta V^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta V^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta V^a_\gamma \\
& + \underline{term(D2+\mathcal{P}2):} i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta V^b_\gamma \\
& + \underline{term\mathcal{D}2:} -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}V^b_\gamma + \frac{dV^b_\gamma}{ds}U^{a-b}_\beta) \\
& + \underline{term\mathcal{X}2:}\Psi_{\alpha\beta\gamma}[1-\kappa r cos\phi]W^{a-b}_\beta W^b_\gamma - \kappa\Psi_{\alpha\beta\gamma}[r cos\phi]U^{a-b}_\beta U^b_\gamma
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]W^a_\beta \\
& -\Psi_{(\alpha)\beta}[(1-\kappa r cos\phi)]P^a_\beta - \kappa\Psi_{\alpha\beta}[r sin\phi]P^a_\beta \\
& = \underline{term\mathcal{D}2:}\Psi_{\{\alpha\}\beta\gamma}[r]U^{a-b}_\beta W^a_\gamma + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V^{a-b}_\beta W^a_\gamma + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W^{a-b}_\beta W^a_\gamma \\
& + \underline{term(D2+\mathcal{P}2):} i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P^{a-b}_\beta W^b_\gamma \\
& + \underline{term\mathcal{D}2:} -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U^{a-b}_\beta V^b_\gamma - \Psi_{\alpha\beta\gamma}[r](\frac{dU^{a-b}_\beta}{ds}W^b_\gamma + \frac{dW^b_\gamma}{ds}U^{a-b}_\beta) \\
& + \underline{term\mathcal{X}2:}\kappa\Psi_{\alpha\beta\gamma}[r sin\phi]U^{a-b}_\beta U^b_\gamma - \Psi_{\alpha\beta\gamma}[1-\kappa r cos\phi]W^{a-b}_\beta V^b_\gamma
\end{aligned}
\tag{6}
$$

I. Tensors in matlab for numerical simulation

A. Tensor times vectors: $\mathcal{A}\bar{\times}_n u$

Let $\mathcal{A}$ be a tensor of size $I_1 \times I_2 \times ... \times I_N$, u be a vector of size I_n.

We have:

$$
\begin{aligned}
ttv(\mathcal{A}, \{u\}, [n]) &= (\mathcal{A}\bar{\times}_n u)(i_1, ..., i_{n-1}, i_{n+1}, ..., i_N) \\
& \sum_{i_n=1}^{I_n} \mathcal{A}(i_1, i_2, ..., i_N)u(i_n)
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
ttv(A_{m\times n}, \{u_{m\times 1}\}, [1]) &= A_{m\times n}\bar{\times}_1 u_{m\times 1} = A^T_{m\times n}u_{m\times 1} \\
ttv(A_{m\times n}, \{v_{n\times 1}\}, [2]) &= A_{m\times n}\bar{\times}_2 v_{n\times 1} = A_{m\times n}v_{n\times 1}
\end{aligned}
\tag{8}
$$

Property:

$$ttv(\mathcal{A}, \{u, v\}, [m, n]) = \mathcal{A} \bar{\times}_m u \bar{\times}_n v$$
$$= ttv(ttv(\mathcal{A}, \{u\}, [m]), \{v\}, [n-1]) = (\mathcal{A} \bar{\times}_m u) \bar{\times}_{n-1} v \tag{9}$$
$$= ttv(ttv(\mathcal{A}, \{v\}, [v]), \{u\}, [m]) = (\mathcal{A} \bar{\times}_n v) \bar{\times}_m u$$

Multiplication with a sequence of vectors

$$\beta = \mathcal{A} \bar{\times}_1 u^{(1)} \bar{\times}_2 u^{(2)} ... \bar{\times}_N u^{(N)} = \mathcal{A} \bar{\times} u$$
$$like : ttv(X, \{A, B, C, D\}) = ttv(X, \{A, B, C, D\}, [1234]) = ttv(X, \{D, C, B, A\}, [4321]) \tag{10}$$

Multiplication with **all but one** of a sequence of vectors

$$b = \mathcal{A} \bar{\times}_1 u^{(1)} \bar{\times}_2 u^{(2)} ... \bar{\times}_{n-1} u^{(2)} \bar{\times}_{n+1} u^{(2)} ... \bar{\times}_N u^{(N)} = \mathcal{A} \bar{\times}_{-n} u$$
$$like : X = tenrand([5, 3, 4, 2]);$$
$$A = rand(5, 1); B = rand(3, 1); C = rand(4, 1); D = rand(2, 1); \tag{11}$$
$$Y = ttv(X, \{A, B, D\}, -3) = ttv(X, \{A, B, C, D\}, -3)$$