

# Note: Governing Equations of General 3D duct flow

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## I. Hard-walled cylindrical ducts as basis function

### A. Infinite straight duct mode

We began from the Helmholtz equation:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} = -\alpha^2 \psi \quad (1)$$

Using separation of variables, Circular symmetry: modes have the form :  $\psi = F(r)G(\theta)$ ,

Then we have:

$$\begin{aligned} \left( \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} \right) G + \frac{F}{r^2} \frac{\partial^2 G}{\partial \theta^2} &= -\alpha^2 FG \\ \text{Then,} \\ \frac{\left( \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} \right)}{F} + \frac{1}{r^2} \frac{\frac{\partial^2 G}{\partial \theta^2}}{G} &= -\alpha^2 \end{aligned} \quad (2)$$

We assume that:

Due to periodicity, we require that  $\Phi$  satisfy,

$$\frac{d^2 G}{d\theta^2} = -m^2 G \rightarrow \Phi(\theta) = e^{\pm im\theta} \quad (3)$$

Thus, we have

$$F'' + \frac{1}{r} F' + \left( \alpha^2 - \frac{m^2}{r^2} \right) F = 0 \rightarrow F(r) = J_m(\alpha r) \quad (4)$$

Circular symmetry  $\psi = F(r)G(\theta)$ : modes explicitly given by:

$$\psi = J_m(\alpha_{m\mu} r) e^{\pm im\theta} \quad (5)$$

Hard walls:

$$J'_m(\alpha R) = 0 \rightarrow \alpha_{m\mu} = \frac{j'_{m\mu}}{R} \quad (6)$$

Soft walls without flow:

$$Z\alpha_{m\mu}J'_m(\alpha_{m\mu}R) = -iw\rho_0J_m(\alpha_{m\mu}R) \rightarrow \alpha_{m\mu}(Z) \quad (7)$$

Soft walls with flow:

$$Z\alpha_{m\mu}J'_m(\alpha_{m\mu}R) = (w - U_0\kappa_{m\mu})J_m(\alpha_{m\mu}R) \rightarrow \alpha_{m\mu}(Z) \quad (8)$$

A complete solution may be writtern as:

$$p(x, r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} (A_{m\mu}e^{-i\kappa_{m\mu}x} + B_{m\mu}e^{i\kappa_{m\mu}x})U_{m\mu}(r)e^{im\theta} \quad (9)$$

In a hard-walled duct  $U_{m\mu}e^{-im\theta}$  are orthogonal. Normalise such that:

$$\int_0^{2\pi} \int_0^R U_{m\mu}(r)e^{-im\theta}U_{nv}(r)e^{-in\theta}rdr = 2\pi\delta_{\mu v}\delta_{mn} \quad (10)$$

Source expansion If  $p(0, t, \theta) = p_0(r, \theta)$  is source in hard-walled duct, then for  $x > 0$

$$\begin{aligned} p_0(r, \theta) &= \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} (A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta} \\ p_0(r, \theta)\underline{U_{nv}(r)e^{-in\theta}r} &= \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} (A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}r} \\ \underline{\int_0^{2\pi} \int_0^R p_0(r, \theta)\underline{U_{nv}(r)e^{-in\theta}r}drd\theta} &= \underline{\int_0^{2\pi} \int_0^R \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} (A_{m\mu}e^{-i\kappa_{m\mu}0})U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}r}drd\theta} \\ \underline{\int_0^{2\pi} \int_0^R p_0(r, \theta)\underline{U_{nv}(r)e^{-in\theta}r}drd\theta} &= \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} A_{m\mu} \underline{\int_0^{2\pi} \int_0^R U_{m\mu}(r)e^{-im\theta}\underline{U_{nv}(r)e^{-in\theta}r}drd\theta} \\ A_{nv} &= \frac{1}{2\pi} \underline{\int_0^{2\pi} \int_0^R p_0(r, \theta)\underline{U_{nv}(r)e^{-in\theta}r}drd\theta} \end{aligned} \quad (11)$$

and  $B_{nv} = 0$ . The same for  $x < 0$  with  $A_{nv}$  and  $B_{nv}$  interchanged.

A finite number of modes (cut-on modes) survive at large distances. Just 1 mode if  $kR \ll 1$ : only  $A_{01}$  important.

## B. General duct mode

The pressure and velocity can now be expressed as Fourier series. Upper indices shall be used to denote temporal decompositions:

$$\begin{aligned}
\hat{p} &= \sum_{a=-\infty}^{\infty} P^a(\mathbf{x})e^{-ia\omega t} \\
\hat{u} &= \sum_{a=-\infty}^{\infty} U^a(\mathbf{x})e^{-ia\omega t} \\
\hat{v} &= \sum_{a=-\infty}^{\infty} V^a(\mathbf{x})e^{-ia\omega t} \\
\hat{w} &= \sum_{a=-\infty}^{\infty} W^a(\mathbf{x})e^{-ia\omega t}
\end{aligned} \tag{12}$$

$$\begin{aligned}
P^a &= \sum_{\alpha=0}^{\infty} P_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} P_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta) \\
U^a &= \sum_{\alpha=0}^{\infty} U_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} U_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta) \\
V^a &= \sum_{\alpha=0}^{\infty} V_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} V_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta) \\
W^a &= \sum_{\alpha=0}^{\infty} W_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} W_{\alpha m \mu}^a(s) \psi_{\alpha m \mu}(s, r, \theta)
\end{aligned} \tag{13}$$

#### 1. Discussion of property of $\psi$

In Jams's thesis,  $\psi_{\alpha} = C_{\alpha m \mu} J_m(\frac{j'_{m \mu} r}{h}) \cos(m\phi - \xi\pi/2)$ . As he mentioned, it was merely a matter of preference - I find it easier to visualise the modes as being "symmetric" and "anti-symmetric" along the plane of torsion free ducts, but the other method is equally as valid.

However, there still a problem, which may cause error, but is not mentioned in the thesis:

$$\int_0^{2\pi} \cos(m\phi - \xi\pi/2)^2 d\theta = \frac{1 + \cos(2m\phi - \xi\pi)}{2} \Big|_0^{2\pi} = \begin{cases} \pi, m \neq 0 \\ 2\pi, m = 0, \xi = 0 \\ 0, m = 0, \xi = 1 \end{cases} \tag{14}$$

In the note, we try to introduce the common solution of  $\psi$  may have the form the same as the hard walls modes:

$$\psi_{m\mu}(r) = C_{\alpha m \mu} J_m(\frac{j'_{m \mu} r}{h}) e^{im\phi} \tag{15}$$

where may be normalized according to:

$$\int_0^{2\pi} \int_0^h \psi_{\alpha m \mu} \psi_{\beta n \nu} r dr d\theta = \delta_{\mu\nu} \delta_{mn} \tag{16}$$

In fact, we know that:

$$\begin{aligned} \int_0^{2\pi} e^{im\theta} e^{im\theta} d\theta &= 0 \\ \int_0^{2\pi} e^{im\theta} e^{-im\theta} d\theta &= 2\pi \end{aligned} \quad (17)$$

The orthogonality relation of Bessel function, with  $J_{-m}(z) = (-1)^m J_m(z)$

$$\begin{aligned} \int_0^h r J_m\left(\frac{j'_{m\mu} r}{h}\right) J_m\left(\frac{j'_{mv} r}{h}\right) dr &= 0, \mu \neq v \\ \int_0^h r J_m\left(\frac{j'_{m\mu} r}{h}\right) J_{-m}\left(\frac{j'_{-mv} r}{h}\right) dr & \\ = (-1)^m \int_0^h r J_m\left(\frac{j'_{m\mu} r}{h}\right) J_m\left(\frac{j'_{mv} r}{h}\right) dr &= 0, \mu \neq v \end{aligned} \quad (18)$$

That changes our idea of normalization to:

$$\underline{\underline{\int_0^{2\pi} \int_0^h \psi_{\alpha_{m\mu}} \psi_{\beta_{nv}} r dr d\theta = (-1)^m \delta_{\mu v} \delta_{m, -n}}} \quad (19)$$

### C. Normalised Modes $\rightarrow C_{\alpha_{m\mu}}$

Relation involving integrals:

$$\begin{aligned} & \frac{2 \int \alpha^2 x J_m(\alpha x)^2 dx}{\rightarrow 2 \int_0^h \alpha^2 x J_m(\alpha x)^2 dx} = \frac{(\alpha^2 x^2 - m^2) J_m(\alpha x)^2 + \alpha^2 x^2 J'_m(\alpha x)^2}{[(\alpha^2 x^2 - m^2) J_m(\alpha x)^2 + \alpha^2 x^2 J'_m(\alpha x)^2]_0^h} \\ & = [(\alpha^2 x^2 - m^2) J_m(\alpha x)^2 + \alpha^2 x^2 J'_m(\alpha x)^2]_h - [(\alpha^2 x^2 - m^2) J_m(\alpha x)^2 + \alpha^2 x^2 J'_m(\alpha x)^2]_0 \\ & = [(\alpha^2 h^2 - m^2) J_m(\alpha h)^2 + \alpha^2 h^2 J'_m(\alpha h)^2] - [(\alpha^2 0^2 - m^2) J_m(\alpha 0)^2 + \alpha^2 0^2 J'_m(\alpha 0)^2] \\ & = (\alpha^2 h^2 - m^2) J_m(\alpha h)^2 + \alpha^2 h^2 J'_m(\alpha h)^2 \end{aligned} \quad (20)$$

With hard-walled boudary condition:

$$J'_m(\alpha h) = 0 \rightarrow \alpha_{m\mu} = \frac{j'_{m\mu}}{h} (\text{eigenvalues}) \quad (21)$$

Then, we have(ref: Rienstra-Fundamentals of Duct Acoustics-(55)):

$$\begin{aligned} \int_0^h r J_m(\alpha r) J_{-m}(\alpha r) dr &= (-1)^m \int_0^h r J_m(\alpha r)^2 dr \\ &= (-1)^m \frac{1}{2\alpha_{m\mu}^2} (\alpha_{m\mu}^2 h^2 - m^2) J_m(\alpha_{m\mu} h)^2 \\ &= (-1)^m \left( \frac{J_m(\alpha_{m\mu} h) \sqrt{(h^2 - \frac{m^2}{\alpha_{m\mu}^2})}}{\sqrt{2}} \right)^2 \\ &= (-1)^m \left( \frac{h^2}{2} \left( 1 - \frac{m^2}{j'^2_{m\mu}} \right) J_m^2(j'_{m\mu}) \right) \end{aligned} \quad (22)$$

Thus, with  $\int_0^{2\pi} e^{im\theta} e^{-im\theta} d\theta = 2\pi$ ,  $\psi_{\alpha_{m\mu}}(r) = C_{\alpha_{m\mu}} J_m(\frac{j'_{m\mu} r}{h}) e^{im\phi}$ , we have:

$$C_{\alpha_{m\mu}} = \frac{1}{\sqrt{(\pi h^2 (1 - \frac{m^2}{j'^2_{m\mu}}) J_m^2(j'_{m\mu}))}} \quad (23)$$

*except for* :  $C_{\alpha_{01}} = \frac{1}{\sqrt{\pi h}}$

$$\text{for } \int_0^{2\pi} \int_0^h \psi_{\alpha_{m\mu}} \psi_{\beta_{nv}} r dr d\theta = (-1)^m \delta_{\mu\nu} \delta_{m,-n} = \hat{\delta}_{\alpha\beta}.$$

#### D. Slowly varying ducts

waiting for updating.....

#### E. Orthogonal-eigenvector

ref:https:

[www.mathworks.com/help/matlab/ref/eigs.html](http://www.mathworks.com/help/matlab/ref/eigs.html)

Eigenvectors, returned as a matrix. The columns in V correspond to the eigenvalues along the diagonal of D. The form and normalization of V depends on the combination of input arguments:

[V,D] = eigs(A) returns matrix V, whose columns are the eigenvectors of A such that  $A^*V = V^*D$ . The eigenvectors in V are normalized so that the 2-norm of each is 1.

If A is symmetric, then the eigenvectors, V, are orthonormal.

[V,D] = eigs(A,B) returns V as a matrix whose columns are the generalized eigenvectors that satisfy  $A^*V = B^*V^*D$ . The 2-norm of each eigenvector is not necessarily 1.

If B is symmetric positive definite, then the eigenvectors in V are normalized so that the B-norm of each is 1. If A is also symmetric, then the eigenvectors are B-orthonormal.

We could further study this question!!

if we can use the GramSchmidt mode as basis??

## II. Mass equation

Mass consevation:

$$-ia\kappa P^a + \nabla \cdot \mathbf{U}^a = \sum_{b=-\infty}^{+\infty} (-P^{a-b} \nabla \cdot \mathbf{U}^b - \mathbf{U}^{a-b} \cdot \nabla P^b - \frac{B}{2A} ia\kappa P^b P^{a-b}) \quad (24)$$

First, derivation of eq1:

We know that:

$$h_s = 1 - \kappa r \cos(\phi), h_r = 1, h_\theta = r \quad (25)$$

Then,

$$\begin{aligned} \nabla \cdot \mathbf{U}^a &= \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial(v_1 h_2 h_3)}{\partial w_1} + \frac{\partial(v_2 h_3 h_1)}{\partial w_2} + \frac{\partial(v_3 h_1 h_2)}{\partial w_3} \right] \\ &= \frac{1}{r(1 - \kappa r \cos(\phi))} \left[ \frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] \end{aligned} \quad (26)$$

Thus, we have the mass equation, approximate RHS by:

$$\begin{aligned} \nabla \cdot \mathbf{U}^b &= ib\kappa P^b + o(M^2) \\ \nabla P^b &= ib\kappa \mathbf{U}^b + o(M^2) \end{aligned} \quad (27)$$

Then we have

$$\begin{aligned} -iakP^a + \frac{1}{r(1 - \kappa r \cos(\phi))} \left[ \frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] \\ = \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b} P^b - ib\kappa U^{a-b} U^b - ib\kappa V^{a-b} V^b - ib\kappa W^{a-b} W^b - \frac{B}{2A} iakP^b P^{a-b}) \end{aligned} \quad (28)$$

The fourier harmonics are expanded as follows:

$$\begin{aligned} P^a &= \sum_{\beta=0}^{\infty} P_\beta^a(s) \psi_\beta(s, r, \theta) \\ U^a &= \sum_{\beta=0}^{\infty} U_\beta^a(s) \psi_\beta(s, r, \theta) \\ V^a &= \sum_{\beta=0}^{\infty} V_\beta^a(s) \psi_\beta(s, r, \theta) \\ W^a &= \sum_{\beta=0}^{\infty} W_\beta^a(s) \psi_\beta(s, r, \theta) \end{aligned} \quad (29)$$

with normalized relation:

$$\int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta r dr d\theta = \hat{\delta}_{\alpha\beta} \quad (30)$$

Reorganize the eq5:

$$\begin{aligned} -iakP^a(1 - \kappa r \cos(\phi)) + \frac{1}{r} \frac{\partial(U^a r)}{\partial s} + \frac{1}{r} \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{1}{r} \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \\ = (1 - \kappa r \cos(\phi)) \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b} P^b - ib\kappa U^{a-b} U^b - ib\kappa V^{a-b} V^b - ib\kappa W^{a-b} W^b - \frac{B}{2A} iakP^b P^{a-b}) \end{aligned} \quad (31)$$

Intergal and insert eq 6, 7 into eq 8:

1. the first term:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \psi_\alpha r [-ia\kappa(1 - \kappa r \cos(\phi)) P^a] dr d\theta \\
&= -ia\kappa \sum_\beta \int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta r [(1 - \kappa r \cos(\phi))] dr d\theta P_\beta^a \\
&= -ia\kappa \Psi_{\alpha\beta} [r(1 - \kappa \cos(\phi))] P_\beta^a \\
&\quad (summation convention)
\end{aligned} \tag{32}$$

2. the second term:

From 1.1 as example, we know that

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \frac{\partial}{\partial s} [r U^{a-b} U^b \psi_\alpha] dr d\theta \\
&= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds} [r U^{a-b} U^b \psi_\alpha]_{r=h} d\theta
\end{aligned} \tag{33}$$

$$\frac{d}{d\alpha} \left( \int_{a(\alpha)}^{b(\beta)} f(x, \alpha) dx \right) - 0 + \frac{da(\alpha)}{d\alpha} f(a(\alpha), \alpha) = \int_{a(\alpha)}^{b(\beta)} \frac{\partial}{\partial \alpha} f(x, \alpha) dx \tag{34}$$

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \psi_\alpha r \left[ \frac{1}{r} \frac{\partial(U^a r)}{\partial s} \right] dr d\theta \\
&= \int_0^{2\pi} \int_0^h \psi_\alpha r \frac{\partial(\psi_\beta U_\beta^a)}{\partial s} dr d\theta \\
&= \int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta r \frac{\partial(U_\beta^a)}{\partial s} dr d\theta + \int_0^{2\pi} \int_0^h \psi_\alpha r \frac{\partial(\psi_\beta)}{\partial s} dr d\theta U_\beta^a \\
&= \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha \psi_\beta r U_\beta^a}{\partial s} dr d\theta - \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha \psi_\beta r}{\partial s} U_\beta^a dr d\theta \\
&+ \int_0^{2\pi} \int_0^h r \frac{\partial(\psi_\alpha \psi_\beta)}{\partial s} dr d\theta U_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta r \frac{\partial(\psi_\alpha)}{\partial s} dr d\theta U_\beta^a \\
&= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [\psi_\alpha \psi_\beta r U_\beta^a] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds} [\psi_\alpha \psi_\beta r U_\beta^a]_{r=h} d\theta \\
&\quad - \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial s} \psi_\beta r dr d\theta U_\beta^a \\
&= \frac{d}{ds} (U_\beta^a \hat{\delta}_{\alpha\beta}) + 0(\text{periodic}) - \Psi_{\{\alpha\}\beta} [r] U_\beta^a
\end{aligned} \tag{35}$$

mark:0(periodic), which is similar in the follow momentum equation, but not eliminate in time.

3. the third term:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \psi_\alpha r \left[ \frac{1}{r} \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} \right] dr d\theta \\
&= \int_0^{2\pi} \int_0^h \psi_\alpha \frac{\partial(\psi_\beta r(1 - \kappa r \cos(\phi)))}{\partial r} dr d\theta V_\beta^a \\
&= \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha \psi_\beta r(1 - \kappa r \cos(\phi)))}{\partial r} dr d\theta V_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta r(1 - \kappa r \cos(\phi)) \frac{\partial(\psi_\alpha)}{\partial r} dr d\theta V_\beta^a \\
&= 0(\text{periodic}) - \Psi_{[\alpha]\beta}[r(1 - \kappa r \cos(\phi))] V_\beta^a
\end{aligned} \tag{36}$$

4. the fourth term:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \psi_\alpha r \left[ \frac{1}{r} \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] dr d\theta \\
&= \int_0^{2\pi} \int_0^h \psi_\alpha \left[ \frac{\partial(\psi_\beta(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] dr d\theta W_\beta^a \\
&= \int_0^{2\pi} \int_0^h \left[ \frac{\partial(\psi_\alpha \psi_\beta(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] dr d\theta W_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta(1 - \kappa r \cos(\phi)) \left[ \frac{\partial(\psi_\alpha)}{\partial \theta} \right] dr d\theta W_\beta^a \\
&= \int_0^h [\psi_\alpha \psi_\beta(1 - \kappa r \cos(\phi))]_0^{2\pi} dr W_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta(1 - \kappa r \cos(\phi)) \left[ \frac{\partial(\psi_\alpha)}{\partial \theta} \right] dr d\theta W_\beta^a \\
&= 0 - \Psi_{(\alpha)\beta}[(1 - \kappa r \cos(\phi))] W_\beta^a
\end{aligned} \tag{37}$$

5. the RHS term:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \psi_\alpha r [(1 - \kappa r \cos(\phi)) \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b} P^b - ib\kappa U^{a-b} U^b - ib\kappa V^{a-b} V^b - ib\kappa W^{a-b} W^b - \frac{B}{2A} iak P^b P^{a-b})] dr d\theta \\
&= \int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta \psi_\gamma r (1 - \kappa r \cos(\phi)) dr d\theta \\
&\quad \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b - ib\kappa U_\beta^{a-b} U_\gamma^b - ib\kappa V_\beta^{a-b} V_\gamma^b - ib\kappa W_\beta^{a-b} W_\gamma^b - iak \frac{B}{2A} P_\beta^{a-b} P_\gamma^b) \\
&= \Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b - ib\kappa U_\beta^{a-b} U_\gamma^b - ib\kappa V_\beta^{a-b} V_\gamma^b - ib\kappa W_\beta^{a-b} W_\gamma^b - iak \frac{B}{2A} P_\beta^{a-b} P_\gamma^b)
\end{aligned} \tag{38}$$

Finally, we obtain the mass equation in the form of eigenfunction, the idea is same as Galerkin method:

$$\begin{aligned}
& \frac{dU_\beta^a}{ds} \hat{\delta}_{\alpha\beta} - \Psi_{\{\alpha\}\beta}[r] U_\beta^a - iak \Psi_{\alpha\beta}[r(1 - \kappa r \cos(\phi))] P_\beta^a - \Psi_{[\alpha]\beta}[r(1 - \kappa r \cos(\phi))] V_\beta^a - \Psi_{(\alpha)\beta}[(1 - \kappa r \cos(\phi))] W_\beta^a \\
&= \Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b - ib\kappa U_\beta^{a-b} U_\gamma^b - ib\kappa V_\beta^{a-b} V_\gamma^b - ib\kappa W_\beta^{a-b} W_\gamma^b - iak \frac{B}{2A} P_\beta^{a-b} P_\gamma^b)
\end{aligned} \tag{39}$$



### III. Momentum equation

Momentum consevation:

$$-ia\kappa\mathbf{U}^a + \nabla P^a = \sum_{b=-\infty}^{\infty} (-\mathbf{U}^{a-b} \cdot \nabla \mathbf{U}^b + P^{a-b} \nabla P^b) \quad (40)$$

First, we know that

$$\nabla P^a = \sum_i \frac{1}{h_i} \frac{\partial f}{\partial w_i} \hat{h}_i = \frac{1}{1 - \kappa r \cos \phi} \frac{\partial P^a}{\partial s} \hat{e}_s + \frac{\partial P^a}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial P^a}{\partial \theta} \hat{e}_\theta \quad (41)$$

The RHS term is a bit complex, with the divergence of a vector  $\mathbf{U}$  with its gradient, with

First, we know that

$$(\mathbf{v} \cdot \nabla) \mathbf{v}^b = \begin{cases} term1 : \mathcal{D}v_1^b + \frac{v_2^b}{h_2 h_1} (v_1 \frac{\partial h_1}{\partial \xi_2} - v_2 \frac{\partial h_2}{\partial \xi_1}) + \frac{v_3^b}{h_3 h_1} (v_1 \frac{\partial h_1}{\partial \xi_3} - v_2 \frac{\partial h_3}{\partial \xi_1}) \\ term2 : \mathcal{D}v_2^b + \frac{v_3^b}{h_3 h_2} (v_2 \frac{\partial h_2}{\partial \xi_3} - v_3 \frac{\partial h_3}{\partial \xi_2}) + \frac{v_1^b}{h_1 h_2} (v_2 \frac{\partial h_2}{\partial \xi_1} - v_1 \frac{\partial h_1}{\partial \xi_2}) \\ term3 : \mathcal{D}v_3^b + \frac{v_1^b}{h_1 h_3} (v_3 \frac{\partial h_3}{\partial \xi_1} - v_1 \frac{\partial h_1}{\partial \xi_3}) + \frac{v_2^b}{h_2 h_3} (v_3 \frac{\partial h_3}{\partial \xi_2} - v_2 \frac{\partial h_2}{\partial \xi_3}) \end{cases} \quad (42)$$

Besides,

$$\mathcal{D} = \frac{v_1}{h_1} \frac{\partial}{\partial \xi_1} + \frac{v_2}{h_2} \frac{\partial}{\partial \xi_2} + \frac{v_3}{h_3} \frac{\partial}{\partial \xi_3} \quad (43)$$

Thus, we have:

$$\begin{aligned}
& -\mathbf{U}^{a-b} \cdot \nabla \mathbf{U}^b = \\
& - \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{D}1 : \frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial U^b}{\partial s} + V^{a-b} \frac{\partial U^b}{\partial r} + \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\ & term\mathcal{D}2 : \frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial V^b}{\partial s} + V^{a-b} \frac{\partial V^b}{\partial r} + \frac{W^{a-b}}{r} \frac{\partial V^b}{\partial \theta} \\ & term\mathcal{D}3 : \frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial W^b}{\partial s} + V^{a-b} \frac{\partial W^b}{\partial r} + \frac{W^{a-b}}{r} \frac{\partial W^b}{\partial \theta} \end{aligned} \right. \\
& - \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{X}1 : \frac{V^b}{1-\kappa r \cos \phi} (U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial r} - V^{a-b} \frac{\partial 1}{\partial s}) + \frac{W^b}{r(1-\kappa r \cos \phi)} (U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial \theta} - V^{a-b} \frac{\partial r}{\partial s}) \\ & term\mathcal{X}2 : \frac{W^b}{r} (V^{a-b} \frac{\partial 1}{\partial \theta} - W^{a-b} \frac{\partial r}{\partial r}) + \frac{U^b}{(1-\kappa r \cos \phi)1} (V^{a-b} \frac{\partial 1}{\partial s} - U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial r}) \\ & term\mathcal{X}3 : \frac{U^b}{(1-\kappa r \cos \phi)r} (W^{a-b} \frac{\partial h_3}{\partial s} - U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial \theta}) + \frac{V^b}{1h_3} (W^{a-b} \frac{\partial h_3}{\partial r} - V^{a-b} \frac{\partial 1}{\partial \theta}) \end{aligned} \right. = \\
& \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\ & term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial V^b}{\partial s} - V^{a-b} \frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial V^b}{\partial \theta} \\ & term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial W^b}{\partial s} - V^{a-b} \frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial W^b}{\partial \theta} \end{aligned} \right. \\
& + \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{X}1 : \frac{\kappa \cos \phi}{1-\kappa r \cos \phi} U^{a-b} V^b - \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} W^b \\ & term\mathcal{X}2 : \frac{W^{a-b} W^b}{r} - \frac{\kappa \cos \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b \\ & term\mathcal{X}3 : \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b - \frac{W^{a-b} V^b}{r} \end{aligned} \right. \quad (44)
\end{aligned}$$

Finally, we could derive the momentum conservation equation, with final term approximate by eq 4:

$$\begin{aligned}
& \left\{ \begin{aligned} & -ia\kappa U^a + \frac{1}{1-\kappa r \cos \phi} \frac{\partial P^a}{\partial s} \\ & -ia\kappa V^a + \frac{\partial P^a}{\partial r} \\ & -ia\kappa W^a + \frac{1}{r} \frac{\partial P^a}{\partial \theta} \end{aligned} \right. \\
& = \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\ & term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial V^b}{\partial s} - V^{a-b} \frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial V^b}{\partial \theta} \\ & term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial W^b}{\partial s} - V^{a-b} \frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial W^b}{\partial \theta} \end{aligned} \right. \quad (45) \\
& + \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{X}1 : \frac{\kappa \cos \phi}{1-\kappa r \cos \phi} U^{a-b} V^b - \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} W^b \\ & term\mathcal{X}2 : \frac{W^{a-b} W^b}{r} - \frac{\kappa \cos \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b \\ & term\mathcal{X}3 : \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b - \frac{W^{a-b} V^b}{r} \end{aligned} \right. + \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & ib\kappa P^{a-b} U^b \\ & ib\kappa P^{a-b} V^b \\ & ib\kappa P^{a-b} W^b \end{aligned} \right.
\end{aligned}$$

Now, we are going to project on  $\psi$ , it may be a little complex, we will doing step by step.

### A. Momentum $e^s$ term

First, deal with the  $e^s$  term:

$$\begin{aligned}
& -iakU^a + \frac{1}{1 - \kappa r \cos \phi} \frac{\partial P^a}{\partial s} \\
= & \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1 - \kappa r \cos \phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\
& + \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa \cos \phi}{1 - \kappa r \cos \phi} U^{a-b} V^b - \frac{\kappa \sin \phi}{(1 - \kappa r \cos \phi)} U^{a-b} W^b \\
& + ib\kappa P^{a-b} U^b
\end{aligned} \tag{46}$$

Multiply  $(1 - \kappa r \cos \phi)$ , we have:

$$\begin{aligned}
& -iak(1 - \kappa r \cos \phi)U^a + \frac{\partial P^a}{\partial s} \\
= & term\mathcal{D}1 : \sum_{b=-\infty}^{\infty} -U^{a-b} \frac{\partial U^b}{\partial s} - (1 - \kappa r \cos \phi) V^{a-b} \frac{\partial U^b}{\partial r} - (1 - \kappa r \cos \phi) \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\
& + term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa \cos \phi U^{a-b} V^b - \kappa \sin \phi U^{a-b} W^b + \\
& term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1 - \kappa r \cos \phi) P^{a-b} U^b
\end{aligned} \tag{47}$$

$\int \int XXr\psi_\alpha dr d\theta$ , we have:

$$RHS = \int_0^{2\pi} \int_0^h [term\mathcal{D}1 + term\mathcal{X}1 + term\mathcal{P}1] r\psi_\alpha dr d\theta \tag{48}$$

1. the first  $\mathcal{D}1$  tems:

We ref the wiki [https](https://en.wikipedia.org/wiki/Leibniz_integral_rule) :

[en.wikipedia.org/wiki/Leibniz\\_integral\\_rule](https://en.wikipedia.org/wiki/Leibniz_integral_rule)

General form: Differentiation under the integral sign:

$$\frac{d}{dx} \left( \int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x)) \cdot \frac{d}{dx} b(x) + f(x, a(x)) \cdot \frac{d}{dx} a(x) = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x, t) dt \tag{49}$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{50}$$

For partial difference, for a given  $\beta$ , the derivation of the fuction  $g(\alpha) = \int_{a(\alpha)}^{b(\beta)} f(x, \alpha) dx$  is

$$\frac{d}{d\alpha} \left( \int_{a(\alpha)}^{b(\beta)} f(x, \alpha) dx \right) = 0 + \frac{da(\alpha)}{d\alpha} f(a(\alpha), \alpha) = \int_{a(\alpha)}^{b(\beta)} \frac{\partial}{\partial \alpha} f(x, \alpha) dx \tag{51}$$

1.1

$$\begin{aligned}
& \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-rU^{a-b} \frac{\partial U^b}{\partial s}] \psi_\alpha dr d\theta \\
&= \sum_{b=-\infty}^{\infty} - \int_0^{2\pi} \int_0^h \frac{\partial}{\partial s} [rU^{a-b} U^b \psi_\alpha] dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial}{\partial s} [rU^{a-b} \psi_\alpha] dr d\theta \\
&= \sum_{b=-\infty}^{\infty} - \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [rU^{a-b} U^b \psi_\alpha] dr d\theta + \int_0^{2\pi} \frac{dh(s)}{ds} [rU^{a-b} U^b \psi_\alpha]_{r=h} d\theta \\
&\quad + \int_0^{2\pi} \int_0^h \frac{r \partial U^{a-b}}{\partial s} U^b \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} r U^{a-b} U^b dr d\theta
\end{aligned} \tag{52}$$

here, we gives a relationship between  $U^a$  and  $V^a$  at the boundary which to dliminate  $V^a$  tems:

$$h'U^{a-b} = (1 - \kappa h \cos \phi) V^{a-b} \tag{53}$$

1.2

$$\begin{aligned}
& \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-(1 - \kappa r \cos \phi) V^{a-b} \frac{\partial U^b}{\partial r}] r \psi_\alpha dr d\theta \\
&= - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h \frac{\partial (r(1 - \kappa r \cos \phi) V^{a-b} U^b \psi_\alpha)}{\partial r} dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial (r(1 - \kappa r \cos \phi) V^{a-b} \psi_\alpha)}{\partial r} dr d\theta \\
&= - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} [(r(1 - \kappa r \cos \phi) V^{a-b} U^b) \psi_\alpha]_0^h d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial (r(1 - \kappa r \cos \phi) V^{a-b})}{\partial r} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial (\psi_\alpha)}{\partial r} r(1 - \kappa r \cos \phi) V^{a-b} dr d\theta \\
&\quad = - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} [(h h' U^{a-b} U^b) \psi_\alpha]_0^h d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial (r(1 - \kappa r \cos \phi) V^{a-b})}{\partial r} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial (\psi_\alpha)}{\partial r} r(1 - \kappa r \cos \phi) V^{a-b} dr d\theta
\end{aligned} \tag{54}$$

$$\begin{aligned}
& \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-(1 - \kappa r \cos \phi) \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta}] r \psi_\alpha dr d\theta \\
&= - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h \frac{\partial((1 - \kappa r \cos \phi) W^{a-b} U^b \psi_\alpha)}{\partial \theta} dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b}) dr d\theta \\
&= - \sum_{b=-\infty}^{\infty} \int_0^h [(1 - \kappa r \cos \phi) W^{a-b} U^b \psi_\alpha]_0^{2\pi} dr \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b}) dr d\theta \\
&= 0(\text{periodic}) + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b}) dr d\theta
\end{aligned} \tag{55}$$

Combine together:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h [term \mathcal{D}1] r \psi_\alpha dr d\theta = \sum_{b=-\infty}^{\infty} \\
& \quad \{ (\int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} r U^{a-b} U^b dr d\theta \\
& \quad + \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r (1 - \kappa r \cos \phi) V^{a-b} U^b dr d\theta \\
& \quad + \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b} U^b) dr d\theta) \\
& \quad + (\int_0^{2\pi} \int_0^h U^b \frac{\partial U^{a-b}}{\partial s} r \psi_\alpha dr d\theta \\
& \quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1 - \kappa r \cos \phi) V^{a-b})}{\partial r} \psi_\alpha dr d\theta \\
& \quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta) \\
& \quad - \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta \}
\end{aligned} \tag{56}$$

We apply eq 5, find that:

$$\begin{aligned}
& -i(a-b)\kappa r(1 - \kappa r \cos(\phi)) P^{a-b} + [\frac{\partial(U^{a-b} r)}{\partial s} + \frac{\partial(V^{a-b} r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{\partial(W^{a-b}(1 - \kappa r \cos(\phi)))}{\partial \theta}] \\
& = o(M^2)
\end{aligned} \tag{57}$$

We have the second terms in eq(31) are equal to:

$$\begin{aligned} & \int_0^{2\pi} \int_0^h U^b [-i(a-b)\kappa r(1-\kappa r \cos(\phi)) P^{a-b}] \psi_\alpha dr d\theta \\ &= i(a-b)\kappa \Psi_{\alpha\beta\gamma} [r(1-\kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \end{aligned} \quad (58)$$

And, the longitudinal derivation s can also be expand about the duct modes, with note  $[r], (\theta), \{s\}$ :

$$\begin{aligned} & \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta \\ &= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] U_\beta^{a-b} U_\gamma^b dr d\theta \\ &= \frac{\partial}{\partial s} \left( \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right) U_\beta^{a-b} U_\gamma^b \\ & \quad + \frac{\partial U_\beta^{a-b} U_\gamma^b}{\partial s} \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \\ &= \frac{\partial}{\partial s} \left( \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right) U_\beta^{a-b} U_\gamma^b \\ & \quad + \left( \frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \end{aligned} \quad (59)$$

2. the second  $\mathcal{X}1$  tems:

$$\begin{aligned} & term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa \cos\phi U^{a-b} V^b - \kappa \sin\phi U^{a-b} W^b \\ & \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [\kappa \cos\phi U^{a-b} V^b - \kappa \sin\phi U^{a-b} W^b] r \psi_\alpha dr d\theta \\ &= \kappa \Psi_{\alpha\beta\gamma} [r \cos\phi] U_\beta^{a-b} V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma} [r \sin\phi] U_\beta^{a-b} W_\gamma^b \end{aligned} \quad (60)$$

3. the second  $\mathcal{P}1$  tems:

$$\begin{aligned} & term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa r \cos\phi) P^{a-b} U^b \\ & \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [ib\kappa(1-\kappa r \cos\phi) P^{a-b} U^b] r \psi_\alpha dr d\theta \\ &= ib\kappa \Psi_{\alpha\beta\gamma} [r(1-\kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \end{aligned} \quad (61)$$

4. The LHS terms:

$$\frac{\partial P^a}{\partial s} - ia\kappa(1-\kappa r \cos\phi) U^a$$

From 1.1 as example, we know that

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \frac{\partial}{\partial s} [rU^{a-b}U^b\psi_\alpha] dr d\theta \\
&= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [rU^{a-b}U^b\psi_\alpha] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds} [rU^{a-b}U^b\psi_\alpha]_{r=h} d\theta
\end{aligned} \tag{62}$$

4.1

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h [\frac{\partial P^a}{\partial s}] r\psi_\alpha dr d\theta \\
&= \int_0^{2\pi} \int_0^h [\frac{\partial(P_\beta^a\psi_\beta)}{\partial s}] r\psi_\alpha dr d\theta \\
&= \int_0^{2\pi} \int_0^h [\frac{\partial(P_\beta^a\psi_\beta)}{\partial s}] r\psi_\alpha dr d\theta - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P_\beta^a \\
&= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h (P_\beta^a\psi_\beta) r\psi_\alpha dr d\theta - \frac{dh(s)}{ds} [P_\beta^a\psi_\beta r\psi_\alpha]_{r=h} - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P_\beta^a \\
&= \frac{d}{ds} (P_\beta^a \widehat{\delta}_{\alpha\beta}) - \int_0^{2\pi} \frac{dh(s)}{ds} [P_\beta^a\psi_\beta r\psi_\alpha]_{r=h} d\theta - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P_\beta^a
\end{aligned} \tag{63}$$

4.2

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h [-ia\kappa(1 - \kappa r \cos\phi)U^a] r\psi_\alpha dr d\theta \\
&= -ia\kappa\Psi_{\alpha\beta}[r(1 - \kappa r \cos\phi)]U_\beta^a
\end{aligned} \tag{64}$$

Finally, putting all together becomes:

$$\begin{aligned}
& \frac{d}{ds}(P_\beta^a)\widehat{\delta}_{\alpha\beta} - \int_0^{2\pi} \frac{dh(s)}{ds} [P_\beta^a \psi_\beta r \psi_\alpha]_{r=h} d\theta - \int_0^{2\pi} \int_0^h \left[ \frac{\partial(\psi_\alpha)}{\partial s} \right] r \psi_\beta dr d\theta P_\beta^a - i a \kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos\phi)] U_\beta^a \\
&= \frac{d}{ds}(P_\beta^a)\widehat{\delta}_{\alpha\beta} - i a \kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos\phi)] U_\beta^a - \int_0^{2\pi} h h' [P_\beta^a \psi_\beta \psi_\alpha]_{r=h} d\theta - \Psi_{\{\alpha\}\beta} [r] P_\beta^a \\
&= \sum_{b=-\infty}^{\infty} \\
&\quad (eq31) : \left\{ \left( \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} r U^{a-b} U^b dr d\theta \right. \right. \\
&\quad \left. \left. + \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r (1 - \kappa r \cos\phi) V^{a-b} U^b dr d\theta \right. \right. \\
&\quad \left. \left. + (eq33) : i(a-b) \kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \right. \right. \\
&\quad \left. \left. eq(34) : -\frac{\partial}{\partial s} \left( \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right) U_\beta^{a-b} U_\gamma^b \right. \right. \\
&\quad \left. \left. - \left( \frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right. \right. \\
&\quad \left. \left. + eq(35) : \kappa \Psi_{\alpha\beta\gamma} [r \cos\phi] U_\beta^{a-b} V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma} [r \sin\phi] U_\beta^{a-b} W_\gamma^b \right. \right. \\
&\quad \left. \left. + eq(36) : i b \kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \right. \right. \\
&\quad \left. \left. = (abbreviation) : \right. \right. \\
&\quad \Psi_{\{\alpha\}\beta\gamma} [r] U_\beta^{a-b} U_\gamma^a + \Psi_{[\alpha]\beta\gamma} [r(1 - \kappa r \cos\phi)] V_\beta^{a-b} U_\gamma^a + \Psi_{(\alpha)\beta\gamma} [(1 - \kappa r \cos\phi)] W_\beta^{a-b} U_\gamma^a \\
&\quad + i(a-b) \kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \\
&\quad - \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma} [r] U_\beta^{a-b} U_\gamma^b - \Psi_{\alpha\beta\gamma} [r] \left( \frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right)) \\
&\quad + \kappa \Psi_{\alpha\beta\gamma} [r \cos\phi] U_\beta^{a-b} V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma} [r \sin\phi] U_\beta^{a-b} W_\gamma^b \\
&\quad + i b \kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \\
&\quad (65)
\end{aligned}$$

with the  $e^s$  term:

$$\begin{aligned}
& -i a \kappa U^a + \frac{1}{1 - \kappa r \cos\phi} \frac{\partial P^a}{\partial s} \\
&= \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1 - \kappa r \cos\phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa \cos\phi}{1 - \kappa r \cos\phi} U^{a-b} V^b - \frac{\kappa \sin\phi}{(1 - \kappa r \cos\phi)} U^{a-b} W^b \\
&+ \sum_{b=-\infty}^{\infty} term\mathcal{P}1 : i b \kappa P^{a-b} U^b \\
&\quad (66)
\end{aligned}$$



We have:

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]U_\beta^a + \frac{d}{ds}(P_\beta^a)\widehat{\delta}_{\alpha\beta} - \int_0^{2\pi} hh'[\psi_\beta\psi_\alpha]_{r=h}d\theta P_\beta^a - \Psi_{\{\alpha\}\beta}[r]P_\beta^a \\
& = \underline{term\mathcal{D}1}:\Psi_{\{\alpha\}\beta\gamma}[r]U_\beta^{a-b}U_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_\beta^{a-b}U_\gamma^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_\beta^{a-b}U_\gamma^a \\
& \quad + \underline{term(D1+\mathcal{P}1)}:i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_\beta^{a-b}U_\gamma^b \\
& \quad - \underline{term\mathcal{D}1}:\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U_\beta^{a-b}U_\gamma^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_\beta^{a-b}}{ds}U_\gamma^b + \frac{dU_\gamma^b}{ds}U_\beta^{a-b})) \\
& \quad + \underline{term\mathcal{X}1}:\kappa\Psi_{\alpha\beta\gamma}[r\cos\phi]U_\beta^{a-b}V_\gamma^b - \kappa\Psi_{\alpha\beta\gamma}[r\sin\phi]U_\beta^{a-b}W_\gamma^b
\end{aligned} \tag{67}$$

## B. Momentum $e^r$ term

Second, deal with the  $e^r$  term:

$$\begin{aligned}
& -ia\kappa V^a + \frac{\partial P^a}{\partial r} \\
& = \sum_{b=-\infty}^{\infty} term\mathcal{D}2: -\frac{U^{a-b}}{1-\kappa r\cos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\
& \quad + \sum_{b=-\infty}^{\infty} term\mathcal{X}2: \frac{W^{a-b}W^b}{r} - \frac{\kappa\cos\phi}{(1-\kappa r\cos\phi)}U^{a-b}U^b \\
& \quad + term\mathcal{P}2: \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b}V^b
\end{aligned} \tag{68}$$

$$LHS-2: \frac{\partial P^a}{\partial r}(1-\kappa r\cos\phi)$$

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h [\frac{\partial P^a}{\partial r}(1-\kappa r\cos\phi)]r\psi_\alpha dr d\theta \\
& = \int_0^{2\pi} \int_0^h [\frac{\partial\psi_\beta(1-\kappa r\cos\phi)r\psi_\alpha}{\partial r}]dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h [\frac{\partial\psi_\alpha(1-\kappa r\cos\phi)r}{\partial r}]\psi_\beta dr d\theta P_\beta^a \\
& \quad = \int_0^{2\pi} \int_0^h [\frac{\partial\psi_\beta(1-\kappa r\cos\phi)r\psi_\alpha}{\partial r}]dr d\theta P_\beta^a \\
& \quad - \int_0^{2\pi} \int_0^h [\frac{\partial\psi_\alpha}{\partial r}](1-\kappa r\cos\phi)r\psi_\beta dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h [\frac{\partial(1-\kappa r\cos\phi)r}{\partial r}]\psi_\alpha\psi_\beta dr d\theta P_\beta^a \\
& \quad = \int_0^{2\pi} [\psi_\alpha\psi_\beta r(1-\kappa r\cos\phi)]_0^h d\theta P_\beta^a \\
& \quad - \int_0^{2\pi} \int_0^h [\frac{\partial\psi_\alpha}{\partial r}](1-\kappa r\cos\phi)r\psi_\beta dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h [1-2\kappa r\cos\phi]\psi_\alpha\psi_\beta dr d\theta P_\beta^a \\
& = \int_0^{2\pi} [\psi_\alpha\psi_\beta r(1-\kappa r\cos\phi)]_0^h d\theta P_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)]P_\beta^a - \Psi_{\alpha\beta}[1-2\kappa r\cos\phi]P_\beta^a
\end{aligned} \tag{69}$$

The derivation of  $term\mathcal{D}2$  is identical to A, we are not prove it again.  $Term\mathcal{P}2$  also could be combine with the part separated term of  $term\mathcal{D}2$  with  $V^b$ .  $Term\mathcal{X}2$  is also easy to derive.

Thus, we have the final equation:

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]V_{\beta}^a \\
& \int_0^{2\pi} [\psi_{\alpha}\psi_{\beta}r(1-\kappa r\cos\phi)]_0^h d\theta P_{\beta}^a - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)]P_{\beta}^a - \Psi_{\alpha\beta}[1-2\kappa r\cos\phi]P_{\beta}^a \\
& = \underline{termD2} : \Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_{\beta}^{a-b}V_{\gamma}^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_{\beta}^{a-b}V_{\gamma}^a \\
& \quad + \underline{term(D2+\mathcal{P}2)} : i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_{\beta}^{a-b}V_{\gamma}^b \\
& \quad + \underline{termD2} : -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_{\beta}^{a-b}}{ds}V_{\gamma}^b + \frac{dV_{\gamma}^b}{ds}U_{\beta}^{a-b})) \\
& \quad + \underline{term\mathcal{X}2} : \Psi_{\alpha\beta\gamma}[1-\kappa r\cos\phi]W_{\beta}^{a-b}W_{\gamma}^b - \kappa\Psi_{\alpha\beta\gamma}[r\cos\phi]U_{\beta}^{a-b}U_{\gamma}^b
\end{aligned} \tag{70}$$

### C. Momentum $e^{\theta}$ term

Third, deal with the  $e^{\theta}$  term:

$$\begin{aligned}
& -ia\kappa W^a + \frac{1}{r}\frac{\partial P^a}{\partial\theta} \\
& = \sum_{b=-\infty}^{\infty} \underline{termD3} : -\frac{U^{a-b}}{1-\kappa r\cos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial\theta} \\
& \quad + \sum_{b=-\infty}^{\infty} \underline{term\mathcal{X}3} : \frac{\kappa\sin\phi}{(1-\kappa r\cos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \\
& \quad + \underline{term\mathcal{P}3} : \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b}W^b
\end{aligned} \tag{71}$$

$$\text{LHS-2: } \frac{\partial P^a}{\partial\theta} \frac{(1-\kappa r\cos\phi)}{r}$$

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h [\frac{\partial P^a}{\partial\theta} \frac{(1-\kappa r\cos\phi)}{r}] r\psi_{\alpha} dr d\theta \\
& = \int_0^{2\pi} \int_0^h [\frac{\partial\psi_{\beta}(1-\kappa r\cos\phi)\psi_{\alpha}}{\partial\theta}] dr d\theta P_{\beta}^a - \int_0^{2\pi} \int_0^h [\frac{\partial\psi_{\alpha}(1-\kappa r\cos\phi)}{\partial\theta}] \psi_{\beta} dr d\theta P_{\beta}^a \\
& = 0 - \int_0^{2\pi} \int_0^h [\frac{\partial\psi_{\alpha}}{\partial\theta}](1-\kappa r\cos\phi)\psi_{\beta} dr d\theta P_{\beta}^a - \int_0^{2\pi} \int_0^h [\frac{\partial(1-\kappa r\cos\phi)}{\partial\theta}] \psi_{\alpha}\psi_{\beta} dr d\theta P_{\beta}^a \\
& = - \int_0^{2\pi} \int_0^h [\frac{\partial\psi_{\alpha}}{\partial\theta}](1-\kappa r\cos\phi)\psi_{\beta} dr d\theta P_{\beta}^a + \kappa \int_0^{2\pi} \int_0^h [r\sin\phi]\psi_{\alpha}\psi_{\beta} dr d\theta P_{\beta}^a \\
& = -\Psi_{(\alpha)\beta}[(1-\kappa r\cos\phi)]P_{\beta}^a - \kappa\Psi_{\alpha\beta}[r\sin\phi]P_{\beta}^a
\end{aligned} \tag{72}$$

The derivation of  $\underline{termD3}$  is identical to A, we are not prove it again.  $\underline{Term\mathcal{P}3}$  also could be combine with the part separated term of  $\underline{termD3}$  with  $W^b$ .  $\underline{Term\mathcal{X}3}$  is also easy to derive.

Thus, we have the final equation:

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]W_\beta^a \\
& -\Psi_{(\alpha)\beta}[(1-\kappa r\cos\phi)]P_\beta^a - \kappa\Psi_{\alpha\beta}[r\sin\phi]P_\beta^a \\
= & \underline{\text{term}\mathcal{D}2}:\Psi_{\{\alpha\}\beta\gamma}[r]U_\beta^{a-b}W_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_\beta^{a-b}W_\gamma^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_\beta^{a-b}W_\gamma^a \\
& +\underline{\text{term}(D2+\mathcal{P}2)}:i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_\beta^{a-b}W_\gamma^b \\
& +\underline{\text{term}\mathcal{D}2}:-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U_\beta^{a-b}V_\gamma^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_\beta^{a-b}}{ds}W_\gamma^b + \frac{dW_\gamma^b}{ds}U_\beta^{a-b})) \\
& +\underline{\text{term}\mathcal{X}2}:\kappa\Psi_{\alpha\beta\gamma}[r\sin\phi]U_\beta^{a-b}U_\gamma^b - \Psi_{\alpha\beta\gamma}[1-\kappa r\cos\phi]W_\beta^{a-b}V_\gamma^b
\end{aligned} \tag{73}$$

#### IV. Merge the four equations and eliminate the $V_\gamma^b$ and $W_\gamma^b$

##### A. $V_\alpha^a$ & $W_\alpha^a$ for RHS

Using the linear relationships:

$$\begin{aligned}
& ia\kappa V^a = \frac{\partial P^a}{\partial r} \\
& := \int \int ia\kappa V_\beta^a \psi_\beta r \psi_\alpha dr d\theta = \int \int \frac{\partial P_\beta^a \psi_\beta}{\partial r} r \psi_\alpha dr d\theta \\
= & ia\kappa V_\beta^a \widehat{\delta}_{\alpha\beta} = \Psi_{\alpha[\beta]}[r]P_\beta^a = \int_0^{2\pi} [r\psi_\alpha \psi_\beta]_0^h d\theta P_\beta^a - \int \int \psi_\alpha \psi_\beta dr d\theta P_\beta^a - \int \int \frac{\partial \psi_\alpha}{\partial r} \psi_\beta r dr d\theta P_\beta^a
\end{aligned} \tag{74}$$

$$\begin{aligned}
& ia\kappa W^a = \frac{1}{r} \frac{\partial P^a}{\partial \theta} \\
& := \int \int ia\kappa W_\beta^a \psi_\beta r \psi_\alpha dr d\theta = \int \int \frac{\partial P_\beta^a \psi_\beta}{\partial \theta} \frac{1}{r} r \psi_\alpha dr d\theta \\
& = ia\kappa W_\beta^a \widehat{\delta}_{\alpha\beta} = \Psi_{\alpha(\beta)}[r]P_\beta^a = 0 - \Psi_{(\alpha)\beta}[r]P_\beta^a
\end{aligned} \tag{75}$$

Thus, we can establish relationships between the tranverse modes and pressure modes (no summation over  $\alpha$ )

$$\underline{\underline{V_\beta^a \widehat{\delta}_{\alpha\beta} = \frac{1}{ia\kappa} [\int_0^{2\pi} [r\psi_\alpha \psi_\beta]_0^h d\theta - \Psi_{\alpha\beta} - \Psi_{[\alpha]\beta}[r]] P_\beta^a = \mathbf{V}_{\alpha\beta}^a P_\beta^a}} \tag{76}$$

$$\underline{\underline{W_\beta^a \widehat{\delta}_{\alpha\beta} = -\frac{1}{ia\kappa} \Psi_{(\alpha)\beta} P_\beta^a = \mathbf{W}_{\alpha\beta}^a P_\beta^a}} \tag{77}$$

**B.**  $\frac{d}{ds}V_\alpha^a$  &  $\frac{d}{ds}W_\alpha^a$  for RHS

We also require modal expressions for  $\frac{d}{ds}V_\alpha^a$  and  $\frac{d}{ds}W_\alpha^a$ .

We differentiate eq71 with respect to s:

$$\begin{aligned}\frac{\partial V^a}{\partial s} &= \frac{1}{ia\kappa} \frac{\partial^2 P^a}{\partial s \partial r} \\ &= \frac{\partial}{\partial r}((1 - \kappa r \cos\phi)U^a)\end{aligned}\tag{78}$$

where we have used symmetry of mixed partials and the linear expression for  $\frac{\partial P^a}{\partial s}$  from eq 21.

From 1.1 as example, we know that

$$\begin{aligned}& \int_0^{2\pi} \int_0^h \frac{\partial}{\partial s} [rU^{a-b}U^b\psi_\alpha] dr d\theta \\ &= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [rU^{a-b}U^b\psi_\alpha] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds} [rU^{a-b}U^b\psi_\alpha]_{r=h} d\theta\end{aligned}\tag{79}$$

here, we gives a relationship between  $U^a$  and  $V^a$  at the boundary which to eliminate  $V^a$  tems:

$$h'U_\beta^a = (1 - \kappa h \cos\phi)V_\beta^a\tag{80}$$

Multiplying this expression by  $r\phi_\alpha$  and integrating across section of the duct, we obtain:

$$\begin{aligned}& \int_0^{2\pi} \int_0^h \frac{\partial V^a}{\partial s} r\psi_\alpha dr d\theta = \int_0^{2\pi} \int_0^h \frac{\partial}{\partial r}((1 - \kappa r \cos\phi)U^a) r\psi_\alpha dr d\theta \\ & LHS := \int_0^{2\pi} \int_0^h \frac{\partial [V_\beta^a \psi_\beta r\psi_\alpha]}{\partial s} dr d\theta - \int_0^{2\pi} \int_0^h \frac{\partial r\psi_\alpha}{\partial s} \psi_\beta dr d\theta V_\beta^a \\ &= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [V_\beta^a \psi_\beta r\psi_\alpha] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds} [V_\beta^a \psi_\beta r\psi_\alpha]_0^h d\theta - \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} \psi_\beta r dr d\theta V_\beta^a \\ &= \frac{d}{ds} V_\beta^a \widehat{\delta}_{\alpha\beta} - \int_0^{2\pi} \frac{h'^2}{1 - \kappa h \cos\phi} [r\psi_\beta \psi_\alpha]_0^h d\theta U_\beta^a - \Psi_{\{\alpha\}\beta}[r] V_\beta^a \\ & RHS := \int_0^{2\pi} \int_0^h \frac{\partial}{\partial r}((1 - \kappa r \cos\phi)U_\beta^a \psi_\beta r\psi_\alpha) dr d\theta - \int_0^{2\pi} \int_0^h \frac{\partial(r\psi_\alpha)}{\partial r} (1 - \kappa r \cos\phi) U_\beta^a \psi_\beta dr d\theta \\ &= \int_0^{2\pi} [r(1 - \kappa r \cos\phi) \psi_\beta \psi_\alpha]_0^h d\theta U_\beta^a - \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial r} r(1 - \kappa r \cos\phi) \psi_\beta dr d\theta U_\beta^a - \int_0^{2\pi} \int_0^h (1 - \kappa r \cos\phi) \psi_\alpha \psi_\beta dr d\theta U_\beta^a \\ &= \int_0^{2\pi} [r(1 - \kappa r \cos\phi) \psi_\beta \psi_\alpha]_0^h d\theta U_\beta^a - \Psi_{[\alpha]\beta}[r(1 - \kappa r \cos\phi)] U_\beta^a - \Psi_{\alpha\beta}[(1 - \kappa r \cos\phi)] U_\beta^a\end{aligned}\tag{81}$$

Thus, LHS=RHS, we have:

$$\begin{aligned}& \underline{\underline{\frac{d}{ds} V_\beta^a \widehat{\delta}_{\alpha\beta} = \int_0^{2\pi} \frac{h'^2}{1 - \kappa h \cos\phi} [r\psi_\beta \psi_\alpha]_0^h d\theta U_\beta^a + \Psi_{\{\alpha\}\beta}[r] V_\beta^a}} \\ & + \underline{\underline{\int_0^{2\pi} [r(1 - \kappa r \cos\phi) \psi_\beta \psi_\alpha]_0^h d\theta U_\beta^a - \Psi_{[\alpha]\beta}[r(1 - \kappa r \cos\phi)] U_\beta^a - \Psi_{\alpha\beta}[(1 - \kappa r \cos\phi)] U_\beta^a}}\end{aligned}\tag{82}$$

Similarly for  $W^a$ , differentiating eq50 with respect to  $s$  and substituting the linear expression for  $\frac{\partial P^a}{\partial s}$  by eq21:

$$\begin{aligned}\frac{\partial W^a}{\partial s} &= \frac{1}{ia\kappa} \frac{\partial^2 P^a}{\partial s \partial \theta} \\ &= \frac{1}{r} \frac{\partial}{\partial \theta} ((1 - \kappa r \cos \phi) U^a)\end{aligned}\tag{83}$$

Multiplying this expression by  $r\phi_\alpha$  and integrating across section of the duct, we obtain:

$$\begin{aligned}\int_0^{2\pi} \int_0^h \frac{\partial W^a}{\partial s} r \psi_\alpha dr d\theta &= \int_0^{2\pi} \int_0^h \frac{1}{r} \frac{\partial}{\partial \theta} ((1 - \kappa r \cos \phi) U^a) r \psi_\alpha dr d\theta \\ LHS &:= \int_0^{2\pi} \int_0^h \frac{\partial [W_\beta^a \psi_\beta r \psi_\alpha]}{\partial s} dr d\theta - \int_0^{2\pi} \int_0^h \frac{\partial r \psi_\alpha}{\partial s} \psi_\beta dr d\theta W_\beta^a \\ &= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [W_\beta^a \psi_\beta r \psi_\alpha] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds} [W_\beta^a \psi_\beta r \psi_\alpha]_0^h d\theta - \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} \psi_\beta r dr d\theta W_\beta^a \\ &= \frac{d}{ds} W_\beta^a \hat{\delta}_{\alpha\beta} - \int_0^{2\pi} \frac{dh(s)}{ds} [\psi_\beta r \psi_\alpha]_0^h d\theta W_\beta^a - \Psi_{\{\alpha\}\beta}[r] W_\beta^a \\ RHS &:= \int_0^{2\pi} \int_0^h \frac{\partial}{\partial \theta} ((1 - \kappa r \cos \phi) U_\beta^a \psi_\beta \psi_\alpha) dr d\theta - \int_0^{2\pi} \int_0^h \frac{\partial (\psi_\alpha)}{\partial \theta} (1 - \kappa r \cos \phi) U_\beta^a \psi_\beta dr d\theta \\ &= 0(\text{periodic}) - \Psi_{(\alpha)\beta}[1 - \kappa r \cos \phi] U_\beta^a\end{aligned}\tag{84}$$

Thus, LHS=RHS, we have:

$$\frac{d}{ds} W_\beta^a \hat{\delta}_{\alpha\beta} = \int_0^{2\pi} \frac{dh(s)}{ds} [\psi_\beta r \psi_\alpha]_0^h d\theta W_\beta^a + \Psi_{\{\alpha\}\beta}[r] W_\beta^a - \Psi_{(\alpha)\beta}[1 - \kappa r \cos \phi] U_\beta^a = -\Psi_{\alpha\{\beta\}}[r] W_\beta^a - \Psi_{(\alpha)\beta}[1 - \kappa r \cos \phi] U_\beta^a\tag{85}$$

## V. Substitutue pressure modes for transverse velocity modes

### A. mass equation

$$\begin{aligned}\frac{dU_\beta^a}{ds} \hat{\delta}_{\alpha\beta} - \Psi_{\{\alpha\}\beta}[r] U_\beta^a - ia\kappa \Psi_{\alpha\beta}[r(1 - \kappa \cos(\phi))] P_\beta^a - \Psi_{[\alpha]\beta}[r(1 - \kappa \cos(\phi))] \underline{\underline{V_\beta^a}} - \Psi_{(\alpha)\beta}[(1 - \kappa r \cos(\phi))] \underline{\underline{W_\beta^a}} \\ = \Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b - ib\kappa U_\beta^{a-b} U_\gamma^b - ib\kappa \underline{\underline{V_\beta^{a-b}}} \underline{\underline{V_\gamma^b}} - ib\kappa \underline{\underline{W_\beta^{a-b}}} \underline{\underline{W_\gamma^b}} - ia\kappa \frac{B}{2A} P_\beta^{a-b} P_\gamma^b)\end{aligned}\tag{86}$$

Transform:

$$\frac{dU_\beta^a}{ds} \widehat{\delta}_{\alpha\beta} := \widehat{\mathbf{I}_{\alpha\beta}^a} u_\beta'^a$$

$$-\Psi_{\{\alpha\}\beta}[r]U_\beta^a := \underline{\underline{-\Psi_{\{\alpha\}\beta}[r]u_\beta^a}} \rightarrow \mathcal{G}$$

$$\left\{ \begin{array}{l} -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa\cos(\phi))]P_\beta^a := \sum_{\beta=0}^{+\infty} \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1-\kappa\cos(\phi))]p_\beta^a}} \rightarrow \mathcal{M}_1 \\ -\Psi_{[\alpha]\beta}[r(1-\kappa\cos(\phi))]V_\beta^a := \sum_{\beta=0}^{+\infty} \underline{\underline{-\Psi_{[\alpha]\delta}[r(1-\kappa\cos(\phi))]\mathbf{V}_{\delta\beta}^a p_\beta^a}} \rightarrow \mathcal{M}_2 + \Psi_{[\alpha]\delta}[r(1-\kappa\cos(\phi))](N^{-1})(o(M_2^2)) \\ -\Psi_{(\alpha)\beta}[(1-\kappa\cos(\phi))]W_\beta^a := \sum_{\beta=0}^{+\infty} \underline{\underline{-\Psi_{(\alpha)\delta}[(1-\kappa\cos(\phi))]\mathbf{W}_{\delta\beta}^a p_\beta^a}} \rightarrow \mathcal{M}_3 + \Psi_{[\alpha]\delta}[r(1-\kappa\cos(\phi))](N^{-1})(o(M_3^2)) \end{array} \right.$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa\cos(\phi))]p_\beta^{a-b} p_\gamma^b}} \rightarrow \mathcal{B}_2$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa U_\beta^{a-b} U_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa\cos(\phi))]u_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{A}_1$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa V_\beta^{a-b} V_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta,\epsilon=0}^{\infty} \underline{\underline{\Psi_{\alpha\delta\epsilon}[r(1-\kappa\cos(\phi))]\mathbf{V}_{\delta\beta}^{a-b} \mathbf{V}_{\epsilon\gamma}^b p_\beta^{a-b} p_\gamma^b}} \rightarrow \mathcal{B}_3$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa W_\beta^{a-b} W_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta,\epsilon=0}^{\infty} \underline{\underline{\Psi_{\alpha\delta\epsilon}[r(1-\kappa\cos(\phi))]\mathbf{W}_{\delta\beta}^{a-b} \mathbf{W}_{\epsilon\gamma}^b p_\beta^{a-b} p_\gamma^b}} \rightarrow \mathcal{B}_4$$

$$\Psi_{\alpha\beta\gamma}[r(1-\kappa\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ia\kappa \frac{B}{2A} P_\beta^{a-b} P_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ia\kappa \frac{B}{2A} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[(1-\kappa\cos(\phi))]p_\beta^{a-b} p_\gamma^b}} \rightarrow \mathcal{B}_1$$

(87)

Here, may be a little question of transform with P, think about  $N^{-1}$ , transform it as matrix we could solve it:

$$\begin{aligned} \therefore ia\kappa V^a &= \frac{\partial P^a}{\partial r} + o(M^2) \\ \int \int ia\kappa V^a \phi_\alpha dr d\theta &= \int \int ia\kappa \phi_\alpha \phi_\beta dr d\theta V_\beta^a = \mathbf{I}_{\alpha\beta}^a V_\beta^a = \mathbf{V}_{\alpha\beta}^a p_\beta^a + \int \int o(M^2) r \phi_\alpha dr d\theta \\ \therefore ia\kappa(1-\kappa\cos\phi)V^a &= (1-\kappa\cos\phi) \frac{\partial P^a}{\partial r} + (1-\kappa\cos\phi)o(M^2) \\ \int \int ia\kappa(1-\kappa\cos\phi)V^a \phi_\alpha dr d\theta &= \mathbf{N}_{\alpha\beta}^a V_\beta^a = \int \int (1-\kappa\cos\phi) \frac{\partial P^a}{\partial r} r \phi_\alpha dr d\theta + \int \int (1-\kappa\cos\phi) o(M^2) dr d\theta \\ \therefore (\mathbf{N}^{-1})_{\beta\alpha}^a \int \int (1-\kappa\cos\phi) \frac{\partial P^a}{\partial r} r \phi_\alpha dr d\theta &= (\mathbf{I}^{-1})_{\beta\alpha}^a \mathbf{I}_{\alpha\beta}^a V_\beta^a \\ (\mathbf{N}^{-1})_{\beta\alpha}^a \int \int (1-\kappa\cos\phi) o(M^2) dr d\theta &= (\mathbf{I}^{-1})_{\beta\alpha}^a \int \int o(M^2) r \phi_\alpha dr d\theta \end{aligned}$$

(88)

## B. momentum equation I

$$\begin{aligned}
& \frac{d}{ds} P_\beta^a \widehat{\delta}_{\alpha\beta} - ia\kappa \Psi_{\alpha\beta}[r(1 - \kappa r \cos\phi)] U_\beta^a - \int_0^{2\pi} hh'[\psi_\beta \psi_\alpha]_{r=h} d\theta P_\beta^a - \Psi_{\{\alpha\}\beta}[r] P_\beta^a \\
= & \sum_{b=-\infty}^{+\infty} \underline{\text{term}\mathcal{D}1} : \Psi_{\{\alpha\}\beta\gamma}[r] U_\beta^{a-b} U_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1 - \kappa r \cos\phi)] \underline{\underline{V_\beta^{a-b} U_\gamma^a}} + \Psi_{(\alpha)\beta\gamma}[(1 - \kappa r \cos\phi)] \underline{\underline{W_\beta^{a-b} U_\gamma^a}} \\
& + \underline{\text{term}(D1 + \mathcal{P}1)} : i(a)\kappa \Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \\
& + \underline{\text{term}\mathcal{D}1} : - \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma}[r]) U_\beta^{a-b} U_\gamma^b - \Psi_{\alpha\beta\gamma}[r] \left( \frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) \\
& + \underline{\text{term}\mathcal{X}1} : \Psi_{\alpha\beta\gamma}[r \cos\phi] U_\beta^{a-b} \underline{\underline{V_\gamma^b}} - \kappa \Psi_{\alpha\beta\gamma}[r \sin\phi] U_\beta^{a-b} \underline{\underline{W_\gamma^b}} \\
& \quad \quad \quad (89)
\end{aligned}$$

Transform:

$$\begin{aligned}
& \frac{d}{ds} P_\beta^a \widehat{\delta}_{\alpha\beta} := \widehat{\mathbf{I}_{\alpha\beta}^a p_\beta'^a} \\
& -ia\kappa \Psi_{\alpha\beta}[r(1 - \kappa r \cos\phi)] U_\beta^a := \underline{\underline{-ia\kappa \Psi_{\alpha\beta}[r(1 - \kappa \cos(\phi))]} u_\beta^a} \rightarrow -\mathcal{N} \\
& - \int_0^{2\pi} hh'[\psi_\beta \psi_\alpha]_{r=h} d\theta P_\beta^a - \Psi_{\{\alpha\}\beta}[r] P_\beta^a := \underline{\underline{- \int_0^{2\pi} hh'[\psi_\beta \psi_\alpha]_{r=h} d\theta p_\beta^a - \Psi_{\{\alpha\}\beta}[r] p_\beta^a}} = \underline{\underline{\Psi_{\{\alpha\}\beta}[r] p_\beta^a}} \rightarrow -\mathcal{H} \\
& \Psi_{\{\alpha\}\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} U_\beta^{a-b} U_\gamma^a := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\{\alpha\}\beta\gamma}[r] u_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{D}_4 \\
& \Psi_{[\alpha]\beta\gamma}[r(1 - \kappa r \cos\phi)] \sum_{b=-\infty}^{+\infty} \underline{\underline{V_\beta^{a-b} U_\gamma^a}} := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta, \epsilon=0}^{+\infty} \underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1 - \kappa r \cos\phi)] \mathbf{V}_{\delta\beta}^{a-b} \mathbf{I}_{\epsilon\gamma} p_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{C}_4 \\
& \Psi_{(\alpha)\beta\gamma}[(1 - \kappa r \cos\phi)] \sum_{b=-\infty}^{+\infty} \underline{\underline{W_\beta^{a-b} U_\gamma^a}} := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta, \epsilon=0}^{+\infty} \underline{\underline{\Psi_{(\alpha)\delta\epsilon}[(1 - \kappa r \cos\phi)] \mathbf{W}_{\delta\beta}^{a-b} \mathbf{I}_{\epsilon\gamma} p_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{C}_5 \\
& ia\kappa \Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)] \sum_{b=-\infty}^{+\infty} P_\beta^{a-b} U_\gamma^b := ia\kappa \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)] p_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{C}_3 \\
& - \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma}[r]) \sum_{b=-\infty}^{+\infty} U_\beta^{a-b} U_\gamma^b := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{- \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma}[r]) u_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{D}_1 \\
& - \Psi_{\alpha\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} \left( \frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) := \Psi_{\alpha\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} [(Mp)_\beta^{a-b} + (Gu)_\beta^{a-b}] U_\gamma^b + [(Mp)_\gamma^b + (Gu)_\gamma^b] U_\beta^{a-b} \\
& \quad \quad \quad = \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma}[r][\mathbf{G}, \mathbf{I}] + \Psi_{\alpha\beta\gamma}[r][\mathbf{I}, \mathbf{G}]) u_\beta^{a-b} u_\gamma^b \rightarrow \mathcal{D}_{2,3} \\
& \quad \quad \quad + \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma}[r][\mathbf{M}, \mathbf{I}] + \Psi_{\alpha\beta\gamma}[r][\mathbf{I}, \mathbf{M}]) u_\beta^{a-b} p_\gamma^b \rightarrow \mathcal{C}_{1,2} \\
& \Psi_{\alpha\beta\gamma}[r \cos\phi] U_\beta^{a-b} \underline{\underline{V_\gamma^b}} - \kappa \Psi_{\alpha\beta\gamma}[r \sin\phi] U_\beta^{a-b} \underline{\underline{W_\gamma^b}} := \left\{ \sum_{b=-\infty}^{+\infty} \Psi_{\alpha\beta\gamma}[r \cos\phi][\mathbf{I}, \mathbf{V}] - \kappa \Psi_{\alpha\beta\gamma}[r \sin\phi][\mathbf{I}, \mathbf{W}] \right\} u_\gamma^{a-b} p_\beta^b \rightarrow \mathcal{C}_{6,7} \\
& \quad \quad \quad (90)
\end{aligned}$$

Here, little transform easy to be proved:

$$\begin{aligned}
\Psi_{[\alpha]\beta\gamma}[r(1 - \kappa r \cos\phi)] \sum_{b=-\infty}^{+\infty} \underline{\underline{V_{\beta}^{a-b} U_{\gamma}^a}} &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta, \epsilon=0}^{\infty} \Psi_{[\alpha]\delta\epsilon}[r(1 - \kappa r \cos\phi)] \underline{\underline{\mathbf{V}_{\delta\beta}^{a-b} \mathbf{I}_{\epsilon\gamma}^b p_{\beta}^{a-b} u_{\gamma}^b}} \rightarrow \mathcal{C}_4 \\
\therefore \Psi_{[\alpha]\beta\gamma}[r(1 - \kappa r \cos\phi)][I, V][p_{\beta}^{a-b}, u_{\gamma}^b] &= \Psi_{[\alpha]\beta\gamma}[r(1 - \kappa r \cos\phi)][V, I][u_{\beta}^{a-b}, p_{\gamma}^b] \\
\Psi_{(\alpha)\beta\gamma}[(1 - \kappa r \cos\phi)] \sum_{b=-\infty}^{+\infty} \underline{\underline{W_{\beta}^{a-b} U_{\gamma}^a}} &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta, \epsilon=0}^{\infty} \Psi_{(\alpha)\delta\epsilon}[(1 - \kappa r \cos\phi)] \underline{\underline{\mathbf{W}_{\delta\beta}^{a-b} \mathbf{I}_{\epsilon\gamma}^b p_{\beta}^{a-b} u_{\gamma}^b}} \rightarrow \mathcal{C}_5 \\
\therefore \Psi_{(\alpha)\beta\gamma}[1 - \kappa r \cos\phi][I, W][p_{\beta}^{a-b}, u_{\gamma}^b] &= \Psi_{[\alpha]\beta\gamma}[1 - \kappa r \cos\phi][W, I][u_{\beta}^{a-b}, p_{\gamma}^b] \\
ia\kappa\Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)] \sum_{b=-\infty}^{+\infty} P_{\beta}^{a-b} U_{\gamma}^b &:= ia\kappa \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)] p_{\beta}^{a-b} u_{\gamma}^b}} \rightarrow \mathcal{C}_3 \\
\therefore ia\kappa\Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)][I, I][p_{\beta}^{a-b}, u_{\gamma}^b] &= ia\kappa\Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)][I, I][u_{\beta}^{a-b}, p_{\gamma}^b] \\
&\quad (91)
\end{aligned}$$

### C. momentum equation II

$$\begin{aligned}
&-ia\kappa\Psi_{\alpha\beta}[r(1 - \kappa r \cos\phi)]V_{\beta}^a \\
&+ \int_0^{2\pi} [\psi_{\alpha}\psi_{\beta}r(1 - \kappa r \cos\phi)]_0^h d\theta P_{\beta}^a - \Psi_{[\alpha]\beta}[r(1 - \kappa r \cos\phi)]P_{\beta}^a - \Psi_{\alpha\beta}[1 - 2\kappa r \cos\phi]P_{\beta}^a \\
&= \underline{\underline{term\mathcal{D}2}} : \Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^a + \Psi_{[\alpha]\beta\gamma}[r(1 - \kappa r \cos\phi)]V_{\beta}^{a-b}V_{\gamma}^a + \Psi_{(\alpha)\beta\gamma}[(1 - \kappa r \cos\phi)]W_{\beta}^{a-b}V_{\gamma}^a \\
&\quad + \underline{\underline{term(D2 + \mathcal{P}2)}} : i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1 - \kappa r \cos\phi)]P_{\beta}^{a-b}V_{\gamma}^b \\
&\quad + \underline{\underline{term\mathcal{D}2}} : -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_{\beta}^{a-b}V_{\gamma}^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_{\beta}^{a-b}}{ds}V_{\gamma}^b + \frac{dV_{\gamma}^b}{ds}U_{\beta}^{a-b}) \\
&\quad + \underline{\underline{term\mathcal{X}2}} : \Psi_{\alpha\beta\gamma}[1 - \kappa r \cos\phi]W_{\beta}^{a-b}W_{\gamma}^b - \kappa\Psi_{\alpha\beta\gamma}[r \cos\phi]U_{\beta}^{a-b}U_{\gamma}^b \\
&\quad (92)
\end{aligned}$$

With:

$$\begin{aligned}
\frac{d}{ds}V_{\beta}^a\widehat{\delta}_{\alpha\beta} &= \left\{ \int_0^{2\pi} \frac{h'^2}{1 - \kappa h \cos\phi} [r\psi_{\beta}\psi_{\alpha}]_0^h d\theta \right. \\
&+ \int_0^{2\pi} [r(1 - \kappa r \cos\phi)\psi_{\beta}\psi_{\alpha}]_0^h d\theta - \Psi_{[\alpha]\beta}[r(1 - \kappa r \cos\phi)] - \Psi_{\alpha\beta}[(1 - \kappa r \cos\phi)] \left. \right\} U_{\beta}^a \\
&\quad (93)
\end{aligned}$$

$$-G_{\alpha\beta}^a V_{\beta}^a$$



Transform:

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]V_{\beta}^a := -N_{\alpha\beta}^a V_{\beta}^a \\
& \{ \int_0^{2\pi} [\psi_{\alpha}\psi_{\beta}r(1-\kappa r\cos\phi)]_0^h d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)] - \Psi_{\alpha\beta}[1-2\kappa r\cos\phi] \} P_{\beta}^a \\
\Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^a &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\{\alpha\}\delta\epsilon}[r]\mathbf{I}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b u_{\beta}^{a-b}p_{\gamma}^b}} = \sum_{b=-\infty}^{+\infty} \Psi_{\{\alpha\}\beta\gamma}[r][\mathbf{I}, \mathbf{V}]u_{\beta}^{a-b}p_{\gamma}^b \rightarrow \varepsilon_{1.4} \\
\Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_{\beta}^{a-b}V_{\gamma}^a &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa r\cos\phi)]\mathbf{V}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{5.3} \\
\Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_{\beta}^{a-b}V_{\gamma}^a &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{(\alpha)\delta\epsilon}[1-\kappa r\cos\phi]\mathbf{W}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{5.4} \\
i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_{\beta}^{a-b}V_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{ia\kappa\Psi_{\alpha\delta\epsilon}[r(1-\kappa r\cos\phi)]\mathbf{I}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{5.2} \\
-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_{\beta}^{a-b}V_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\delta\epsilon}[r])\mathbf{I}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b u_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \varepsilon_{1.1} \\
& -\Psi_{\alpha\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} \left( \frac{dU_{\beta}^{a-b}}{ds} V_{\gamma}^b + \frac{dV_{\gamma}^b}{ds} U_{\beta}^{a-b} \right) \\
& := -\Psi_{\alpha\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} (\widehat{\mathbf{I}^{a-b}}^{-1} [-(\mathbf{M}p)_{\beta}^{a-b} - (\mathbf{G}u)_{\beta}^{a-b}] V_{\gamma}^b \\
& \quad + \widehat{\mathbf{I}_{\gamma\alpha}^b}^{-1} \{ \int_0^{2\pi} \frac{h'^2}{1-\kappa h\cos\phi} [r\psi_{\beta}\psi_{\alpha}]_0^h d\theta \\
& \quad + \int_0^{2\pi} [r(1-\kappa r\cos\phi)\psi_{\beta}\psi_{\alpha}]_0^h d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)] - \Psi_{\alpha\beta}[(1-\kappa r\cos\phi)] \} U_{\beta}^b U_{\beta}^{a-b} \\
& \quad - (\widehat{\mathbf{I}_{\gamma\alpha}^b}^{-1} G_{\alpha\beta}^b V_{\beta}^b) U_{\beta}^{a-b} \\
& = \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma}[r][\widehat{\mathbf{I}}^{-1} \mathbf{G}, \mathbf{V}]) u_{\beta}^{a-b} p_{\gamma}^b \rightarrow \varepsilon_{1.2} \\
& + \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma}[r][\widehat{\mathbf{I}}^{-1} \mathbf{M}, \mathbf{V}]) p_{\beta}^{a-b} p_{\gamma}^b \rightarrow \mathcal{B}_{5.1} \\
& - \sum_{b=-\infty}^{+\infty} \Psi_{\alpha\beta\gamma}[r] \{ \widehat{\mathbf{I}}^{-1} ( \int_0^{2\pi} \frac{h'^2}{1-\kappa h\cos\phi} [r\psi_{\beta}\psi_{\alpha}]_0^h d\theta \\
& + \int_0^{2\pi} [r(1-\kappa r\cos\phi)\psi_{\beta}\psi_{\alpha}]_0^h d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)] - \Psi_{\alpha\beta}[(1-\kappa r\cos\phi)] \} u_{\beta}^{a-b} u_{\gamma}^b \rightarrow \mathcal{A}_{2.1,2.2,2.4,2.3} \\
& + \sum_{b=-\infty}^{+\infty} \Psi_{\alpha\beta\gamma}[r][\mathbf{I}, \widehat{\mathbf{I}}^{-1} \mathbf{G}\mathbf{V}] u_{\beta}^{a-b} p_{\gamma}^b \rightarrow \varepsilon_{1.3} \\
\Psi_{\alpha\beta\gamma}[1-\kappa r\cos\phi]W_{\beta}^{a-b}W_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\delta\epsilon}[1-\kappa r\cos\phi]\mathbf{W}_{\delta\beta}^{a-b}\mathbf{W}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{5.5} \\
-\kappa\Psi_{\alpha\beta\gamma}[r\cos\phi]U_{\beta}^{a-b}U_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{-\kappa\Psi_{\alpha\beta\gamma}[r\cos\phi]u_{\beta}^{a-b}u_{\gamma}^b}} \rightarrow \mathcal{A}_{2.5}
\end{aligned} \tag{94}$$

#### D. momentum equation III

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]W_\beta^a \\
& -\Psi_{(\alpha)\beta}[(1-\kappa r\cos\phi)]P_\beta^a - \kappa\Psi_{\alpha\beta}[r\sin\phi]P_\beta^a \\
= & \underline{term\mathcal{D}2}:\Psi_{\{\alpha\}\beta\gamma}[r]U_\beta^{a-b}W_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_\beta^{a-b}W_\gamma^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_\beta^{a-b}W_\gamma^a \\
& +\underline{term(D2+\mathcal{P}2)}:i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_\beta^{a-b}W_\gamma^b \\
& +\underline{term\mathcal{D}2}:-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_\beta^{a-b}V_\gamma^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_\beta^{a-b}}{ds}W_\gamma^b + \frac{dW_\gamma^b}{ds}U_\beta^{a-b}) \\
& +\underline{term\mathcal{X}2}:\kappa\Psi_{\alpha\beta\gamma}[r\sin\phi]U_\beta^{a-b}U_\gamma^b - \Psi_{\alpha\beta\gamma}[1-\kappa r\cos\phi]W_\beta^{a-b}V_\gamma^b
\end{aligned} \tag{95}$$

With:

$$\frac{d}{ds}W_\beta^a\widehat{\delta}_{\alpha\beta} = \int_0^{2\pi} \frac{dh(s)}{ds}[\psi_\beta r\psi_\alpha]_0^h d\theta W_\beta^a + \Psi_{\{\alpha\}\beta}[r]W_\beta^a - \Psi_{(\alpha)\beta}[1-\kappa r\cos\phi]U_\beta^a = -\Psi_{\alpha\{\beta\}}[r]W_\beta^a - \Psi_{(\alpha)\beta}[1-\kappa r\cos\phi]U_\beta^a$$


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(96)

Transform:

$$\begin{aligned}
& -iak\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]W_{\beta}^a := -N_{\alpha\beta}^a W_{\beta}^a \\
& -\Psi_{(\alpha)\beta}[(1-\kappa r\cos\phi)]p_{\beta}^a - \kappa\Psi_{\alpha\beta}[r\sin\phi]p_{\beta}^a \\
\Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}W_{\gamma}^a &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\{\alpha\}\delta\epsilon}[r]\mathbf{I}_{\delta\beta}^{a-b}\mathbf{W}_{\epsilon\gamma}^b u_{\beta}^{a-b}p_{\gamma}^b}} = \sum_{b=-\infty}^{+\infty} \Psi_{\{\alpha\}\beta\gamma}[r][\mathbf{I}, \mathbf{W}]u_{\beta}^{a-b}p_{\gamma}^b \rightarrow \varepsilon_{2.4} \\
\Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_{\beta}^{a-b}W_{\gamma}^a &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa r\cos\phi)]\mathbf{V}_{\delta\beta}^{a-b}\mathbf{W}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{6.3} \\
\Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_{\beta}^{a-b}W_{\gamma}^a &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{(\alpha)\delta\epsilon}[1-\kappa r\cos\phi]\mathbf{W}_{\delta\beta}^{a-b}\mathbf{W}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{6.4} \\
i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_{\beta}^{a-b}W_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{iak\Psi_{\alpha\delta\epsilon}[r(1-\kappa r\cos\phi)]\mathbf{I}_{\delta\beta}^{a-b}\mathbf{W}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{6.2} \\
-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_{\beta}^{a-b}W_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\delta\epsilon}[r])\mathbf{I}_{\delta\beta}^{a-b}\mathbf{W}_{\epsilon\gamma}^b u_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \varepsilon_{2.1} \\
& -\Psi_{\alpha\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} \left( \frac{dU_{\beta}^{a-b}}{ds} W_{\gamma}^b + \frac{dW_{\gamma}^b}{ds} U_{\beta}^{a-b} \right) \\
& := -\Psi_{\alpha\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} (\hat{\mathbf{I}}^{-1}[-(\mathbf{M}p)_{\beta}^{a-b} - (\mathbf{G}u)_{\beta}^{a-b}]W_{\gamma}^b - \{\mathbf{H}W_{\beta}^a - \Psi_{(\alpha)\beta}[1-\kappa r\cos\phi]U_{\beta}^a\}U_{\beta}^{a-b}) \\
& = \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma}[r][\hat{\mathbf{I}}^{-1}\mathbf{G}, \mathbf{W}])u_{\beta}^{a-b}p_{\gamma}^b \rightarrow \varepsilon_{2.2} \\
& + \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma}[r][\hat{\mathbf{I}}^{-1}\mathbf{M}, \mathbf{W}])p_{\beta}^{a-b}p_{\gamma}^b \rightarrow \mathcal{B}_{6.1} \\
& - \sum_{b=-\infty}^{+\infty} \Psi_{\alpha\beta\gamma}[r][\mathbf{I}, \hat{\mathbf{I}}^{-1}\mathbf{H}\mathbf{W}]u_{\beta}^{a-b}p_{\gamma}^b \rightarrow \varepsilon_{2.3} \\
& + \sum_{b=-\infty}^{+\infty} \Psi_{\alpha\beta\gamma}[r](\hat{\mathbf{I}}^{-1}\Psi_{(\alpha)\beta}[1-\kappa r\cos\phi])u_{\beta}^a u_{\beta}^{a-b} \rightarrow \mathcal{A}_{3.1} \\
\kappa\Psi_{\alpha\beta\gamma}[r\sin\phi]U_{\beta}^{a-b}U_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\kappa\Psi_{\alpha\beta\gamma}[r\sin\phi]u_{\beta}^{a-b}u_{\gamma}^b}} \rightarrow \mathcal{A}_{3.2} \\
-\Psi_{\alpha\beta\gamma}[1-\kappa r\cos\phi]W_{\beta}^{a-b}V_{\gamma}^b &:= \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{-\Psi_{\alpha\delta\epsilon}[1-\kappa r\cos\phi]\mathbf{W}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b p_{\beta}^{a-b}p_{\gamma}^b}} \rightarrow \mathcal{B}_{6.5}
\end{aligned} \tag{97}$$

Finally, we obtain two equations involving just pressure and longitudinal velocity modes. Here,  $\mathbf{M}, \mathbf{N}, \mathcal{A}, \mathcal{B}$ , and  $\mathcal{C}$  encoding the curvature of the duct. The terms  $\mathbf{G}, \mathbf{H}, \mathbf{D}$  encode the variation in duct diameter as well as the torsion. The term  $\varepsilon$  encodes variation of diameter and the torsion together with curvature if either of the first two are present.

$$\begin{aligned}\widehat{\mathbf{I}}_{\alpha\beta}^a u_\beta'^a + \mathbf{M}_{\alpha\beta}^a p_\beta^a + \mathbf{G}_{\alpha\beta}^a u_\beta^a &= \mathcal{A}[u, u] + \mathcal{B}[p, p] + \varepsilon[u, p] \\ \widehat{\mathbf{I}}_{\alpha\beta}^a p_\beta'^a - \mathbf{N}_{\alpha\beta}^a u_\beta^a - \mathbf{H}_{\alpha\beta}^a p_\beta^a &= \mathcal{C}[u, p] + \mathcal{D}[u, u]\end{aligned}\tag{98}$$

## VI. Separating the $\Psi$ integrals into radial and angular parts

With  $\int_0^{2\pi} e^{im\theta} e^{-im\theta} d\theta = 2\pi$ ,  $\psi_{\alpha m_\mu}(r) = C_{\alpha m_\mu} J_m(\frac{j'_{m_\mu} r}{h}) e^{im\phi}$ , we have:

$$\begin{aligned}C_{\alpha m_\mu} &= \frac{1}{\sqrt{(\pi h^2 (1 - \frac{m^2}{j'^2_{m_\mu}}) J_m^2(j'_{m_\mu}))}} \\ \text{except for : } C_{\alpha 01} &= \frac{1}{\sqrt{\pi h}}\end{aligned}\tag{99}$$

$$\psi_{\alpha m_\mu}(r) = C_{\alpha m_\mu} J_m(\frac{j'_{m_\mu} r}{h}) e^{im\theta}\tag{100}$$

$$C_{\alpha m_\mu} = \begin{cases} \frac{1}{\sqrt{\frac{\pi}{2} h^2 J_0^2(j'_{m_\mu})}}, m = 0 \\ \frac{1}{\sqrt{(\frac{\pi h^2}{2} (1 - \frac{m^2}{j'^2_{m_\mu}}) J_m^2(j'_{m_\mu}))}}, m \neq 0 \end{cases}\tag{101}$$

$$\begin{aligned}\Psi_{[\alpha](\beta)\gamma}[r(1 - \kappa r \cos\phi)] &= \int_0^{2\pi} \int_0^h [\frac{\partial \psi_\alpha}{\partial r}][\frac{\partial \psi_\beta}{\partial \theta}] \psi_\gamma[r(1 - \kappa r \cos\phi)] dr d\theta \\ &= \mathcal{X}_{[\alpha]\beta\gamma}[r] \Theta_{\alpha(\beta)\gamma} - \kappa \mathcal{X}_{[\alpha]\beta\gamma}[r^2] \Theta_{\alpha(\beta)\gamma}[\cos\phi]\end{aligned}\tag{102}$$

with:

$$\begin{aligned}\mathcal{X}_{[\alpha]\beta\gamma} &= \int_0^h \frac{d}{dr} (C_{\alpha m_\mu} J_m(\frac{j'_{\alpha m_\mu} r}{h})) C_{\beta n_\nu} J_n(\frac{j'_{\beta n_\nu} r}{h}) C_{\gamma k_w} J_k(\frac{j'_{\gamma k_w} r}{h}) dr \\ \Theta_{\alpha(\beta)\gamma} &= \int_0^{2\pi} e^{im\theta} \frac{d}{d\theta} (e^{in\theta}) e^{ik\theta} d\theta\end{aligned}\tag{103}$$

Bessel function recurrence relations are:

$$\begin{aligned}J_{m-1}(x) + J_{m+1}(x) &= 2m/x J_m(x) \\ J_{m-1}(x) - J_{m+1}(x) &= 2J'_m(x)\end{aligned}\tag{104}$$

we can have:

$$\begin{aligned}2J_{m+1}(x) &= 2m/x J_m(x) - 2J'_m(x) \rightarrow \\ J'_m(x) &= m/x J_m(x) - J_{m+1}(x)\end{aligned}\tag{105}$$

$$\begin{aligned}
\mathcal{X}_{[\alpha]\beta}[r] &= \int_0^h \frac{d}{dr} (C_{\alpha_{m\mu}} J_m(\frac{j'_{\alpha_{m\mu}} r}{h})) C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}} r}{h}) r dr \\
&= \int_0^h \frac{j'_{\alpha_{m\mu}}}{h} C_{\alpha_{m\mu}} [\frac{m}{j'_{\alpha_{m\mu}} r/h} J_m(\frac{j'_{\alpha_{m\mu}} r}{h}) - J_{m+1}(\frac{j'_{\alpha_{m\mu}} r}{h})] C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}} r}{h}) r dr \\
&= C_{\alpha_{m\mu}} C_{\beta_{nv}} \int_0^h [m J_m(\frac{j'_{\alpha_{m\mu}} r}{h}) - \frac{j'_{\alpha_{m\mu}} r}{h} J_{m+1}(\frac{j'_{\alpha_{m\mu}} r}{h})] J_n(\frac{j'_{\beta_{nv}} r}{h}) dr
\end{aligned} \tag{106}$$

The  $\Theta$  integrals can be calculated analytically:

$$\Theta_{\alpha\beta} = \int_0^{2\pi} e^{im\theta} e^{in\theta} d\theta = \begin{cases} 0 \\ 2\pi, m+n=0 \end{cases} = 2\pi\delta_{m,-n} \tag{107}$$

With Euler's equation:

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
\cos x &= [e^{ix} + e^{-ix}]/2 \\
\sin x &= [e^{ix} - e^{-ix}]/2i
\end{aligned} \tag{108}$$

$$\begin{aligned}
\Theta_{\alpha\beta}[\cos\phi] &= \int_0^{2\pi} \cos(\theta - \theta_0) e^{im\theta} e^{in\theta} d\theta = 1/2 \int_0^{2\pi} [e^{i(\theta-\theta_0)} + e^{-i(\theta-\theta_0)}] e^{im\theta} e^{in\theta} d\theta \\
&= 1/2 \int_0^{2\pi} e^{i(\theta-\theta_0)} e^{im\theta} e^{in\theta} d\theta + 1/2 \int_0^{2\pi} [e^{-i(\theta-\theta_0)}] e^{im\theta} e^{in\theta} d\theta \\
&= \pi e^{-i\theta_0} \delta_{m,-n-1} + \pi e^{i\theta_0} \delta_{m,-n+1}
\end{aligned} \tag{109}$$

$$\Theta_{(\alpha)\beta} = \int_0^{2\pi} \frac{\partial}{\partial\theta} (e^{im\theta}) e^{in\theta} d\theta = 2\pi im \delta_{m,-n} \tag{110}$$

$$\Theta_{(\alpha)\beta}[\cos\phi] = im[\pi e^{-i\theta_0} \delta_{m,-n-1} + \pi e^{i\theta_0} \delta_{m,-n+1}] \tag{111}$$

$$\Theta_{\alpha\beta\gamma} = \int_0^{2\pi} e^{im\theta} e^{in\theta} e^{ik\theta} d\theta = 2\pi \delta_{m,-n-k} \tag{112}$$

$$\Theta_{\alpha\beta\gamma}[\cos\phi] = 1/2 \int_0^{2\pi} [e^{i(\theta-\theta_0)} + e^{-i(\theta-\theta_0)}] e^{im\theta} e^{in\theta} e^{ik\theta} d\theta = \pi e^{-i\theta_0} \delta_{m,-n-k-1} + \pi e^{i\theta_0} \delta_{m,-n-k+1} \tag{113}$$

$$\Theta_{\alpha\beta\gamma}[\sin\phi] = 1/2i \int_0^{2\pi} [e^{i(\theta-\theta_0)} - e^{-i(\theta-\theta_0)}] e^{im\theta} e^{in\theta} e^{ik\theta} d\theta = -i[\pi e^{-i\theta_0} \delta_{m,-n-k-1} - \pi e^{i\theta_0} \delta_{m,-n-k+1}] \quad (114)$$

$$\Theta_{(\alpha)\beta\gamma} = im \int_0^{2\pi} e^{im\theta} e^{in\theta} e^{ik\theta} d\theta = 2\pi im \delta_{m,-n-k} \quad (115)$$

$$\Theta_{(\alpha)\beta\gamma}[\cos\phi] = 1/2 \int_0^{2\pi} [e^{i(\theta-\theta_0)} + e^{-i(\theta-\theta_0)}] e^{im\theta} e^{in\theta} e^{ik\theta} d\theta = im[\pi e^{-i\theta_0} \delta_{m,-n-k-1} + \pi e^{i\theta_0} \delta_{m,-n-k+1}] \quad (116)$$

We also have:

$$\Psi_{\{\alpha\}\beta} = \mathcal{X}_{\{\alpha\}\beta} \Theta_{\alpha\beta} + \mathcal{X}_{\alpha\beta} \Theta_{\{\alpha\}\beta} \quad (117)$$

with:

$$\begin{aligned} \mathcal{X}_{\{\alpha\}\beta} &= \int_0^h \frac{d}{ds} (C_{\alpha_{m\mu}}(s) J_m(\frac{j'_{\alpha_{m\mu}} r}{h(s)})) C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}} r}{h}) dr \\ &= \int_0^h [(\frac{dC_{\alpha_{m\mu}}(s)}{ds}) J_m(\frac{j'_{\alpha_{m\mu}} r}{h(s)}) + C_{\alpha_{m\mu}} J_m(\frac{j'_{\alpha_{m\mu}} r}{h(s)}) \frac{-j'_{\alpha_{m\mu}} r}{h(s)^2} \frac{dh(s)}{ds}] C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}} r}{h}) dr \\ &= \int_0^h [(\frac{d}{ds} \frac{1}{h(s)}) C_{\alpha_{m\mu}}(s) J_m(\frac{j'_{\alpha_{m\mu}} r}{h(s)}) + C_{\alpha_{m\mu}} J_m(\frac{j'_{\alpha_{m\mu}} r}{h(s)}) \frac{-j'_{\alpha_{m\mu}} r}{h(s)^2} \frac{dh(s)}{ds}] C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}} r}{h}) dr \\ &= \int_0^h [(-\frac{h'(s)}{h}) C_{\alpha_{m\mu}}(s) J_m(\frac{j'_{\alpha_{m\mu}} r}{h(s)}) - C_{\alpha_{m\mu}} \frac{\partial}{\partial r} (J_m(\frac{j'_{\alpha_{m\mu}} r}{h(s)})) \frac{h'(s)}{h} r] C_{\beta_{nv}} J_n(\frac{j'_{\beta_{nv}} r}{h}) dr \\ &= -\frac{h'}{h} (\mathcal{X}_{[\alpha]\beta}[r] + \mathcal{X}_{\alpha\beta}) \end{aligned} \quad (118)$$

$$\Theta_{\{\alpha\}\beta} = \int_0^{2\pi} \frac{\partial}{\partial s} (e^{im\theta}) e^{in\theta} d\theta = 2\pi im \delta_{m,-n} \quad (119)$$

## VII. Tensors in matlab for numerical simulation

### A. Tensor times vectors: $\mathcal{A} \bar{\times}_n u$

Let  $\mathcal{A}$  be a tensor of size  $I_1 \times I_2 \times \dots \times I_N$ ,  $u$  be a vector of size  $I_n$ .

We have:

$$\begin{aligned} ttv(\mathcal{A}, \{u\}, [n]) &= (\mathcal{A} \bar{\times}_n u)(i_1, \dots, i_{n-1}, i_{n+1}, \dots, i_N) \\ &= \sum_{i_n=1}^{I_n} \mathcal{A}(i_1, i_2, \dots, i_N) u(i_n) \end{aligned} \quad (120)$$

$$\begin{aligned}
ttv(A_{m \times n}, \{u_{m \times 1}\}, [1]) &= A_{m \times n} \bar{\times}_1 u_{m \times 1} = A_{m \times n}^T u_{m \times 1} \\
ttv(A_{m \times n}, \{v_{n \times 1}\}, [2]) &= A_{m \times n} \bar{\times}_2 v_{n \times 1} = A_{m \times n} v_{n \times 1}
\end{aligned} \tag{121}$$

Property:

$$\begin{aligned}
ttv(\mathcal{A}, \{u, v\}, [m, n]) &= \mathcal{A} \bar{\times}_m u \bar{\times}_n v \\
= ttv(ttv(\mathcal{A}, \{u\}, [m]), \{v\}, [n-1]) &= (\mathcal{A} \bar{\times}_m u) \bar{\times}_{n-1} v \\
= ttv(ttv(\mathcal{A}, \{v\}, [v]), \{u\}, [m]) &= (\mathcal{A} \bar{\times}_n v) \bar{\times}_m u
\end{aligned} \tag{122}$$

Multiplication with a sequence of vectors

$$\beta = \mathcal{A} \bar{\times}_1 u^{(1)} \bar{\times}_2 u^{(2)} \dots \bar{\times}_N u^{(N)} = \mathcal{A} \bar{\times} u \tag{123}$$

$$like : ttv(X, \{A, B, C, D\}) = ttv(X, \{A, B, C, D\}, [1234]) = ttv(X, \{D, C, B, A\}, [4321])$$

Multiplication with **all but one** of a sequence of vectors

$$\begin{aligned}
b &= \mathcal{A} \bar{\times}_1 u^{(1)} \bar{\times}_2 u^{(2)} \dots \bar{\times}_{n-1} u^{(2)} \bar{\times}_{n+1} u^{(2)} \dots \bar{\times}_N u^{(N)} = \mathcal{A} \bar{\times}_{-n} u \\
like : X &= tenrand([5, 3, 4, 2]); \\
A &= rand(5, 1); B = rand(3, 1); C = rand(4, 1); D = rand(2, 1); \\
Y &= ttv(X, \{A, B, D\}, -3) = ttv(X, \{A, B, C, D\}, -3)
\end{aligned} \tag{124}$$

## B. Tensor times matrix (ttm): $\mathcal{A} \times_n u$

Let  $\mathcal{A}$  be a tensor of size  $I_1 \times I_2 \times \dots \times I_N$ ,  $U$  be a matrix of size  $J_n \times I_n$ .

We have:

$$\begin{aligned}
ttm(\mathcal{A}, \{U\}, [n]) &= (\mathcal{A} \times_n U)(i_1, \dots, i_{n-1}, j_n, i_{n+1}, \dots, i_N) \\
&\sum_{i_n=1}^{I_n} \mathcal{A}(i_1, i_2, \dots, i_N) U(j_n, i_n)
\end{aligned} \tag{125}$$

$$like : X = tensor(rand(5, 3, 4, 2)); A = rand(4, 5);$$

$$Y = ttm(X, A, 1) = ttm(X, \{A, B, C, D\}, 1) = ttm(X, A', 1, 't')$$

## Matrix Interpretation

$$\begin{aligned}
ttm(A_{m \times n}, \{U_{m \times k}^T\}, [1]) &= A \times_1 U^T = U^T A \\
ttm(A_{m \times n}, \{V_{m \times k}^T\}, [2]) &= A \times_2 V^T = AV \\
ttm(A, \{U, V\}, [1, 2]) &= UAV^T
\end{aligned} \tag{126}$$

$$Y = ttm(X, A, B, C, D, [1234]); \% < - - 4 - waymutlply.$$

$$Y = ttm(X, D, C, B, A, [4321]); \% < - - Sameasabove.$$

$$Y = ttm(X, A, B, C, D); \% < - - Sameasabove.$$

$$Y = ttm(X, A', B', C', D', 't'); \% < - - Sameasabove.$$

$$Y = ttm(X, C, D, [34]); \% < - - XtimesCinmode - 3Dinmode - 4 \tag{127}$$

$$Y = ttm(X, A, B, C, D, [34]); \% < - - Sameasabove.$$

$$Y = ttm(X, A, B, D, [124]); \% < - - 3 - waymultiply.$$

$$Y = ttm(X, A, B, C, D, [124]); \% < - - Sameasabove.$$

$$Y = ttm(X, A, B, D, -3); \% < - - Sameasabove.$$

$$Y = ttm(X, A, B, C, D, -3); \% < - - Sameasabove.$$

## Property

$$\begin{aligned}
ttm(\mathcal{A}, \{u, v\}, [m, n]) &= \mathcal{A} \times_m u \bar{\times}_n v \\
&= ttm(ttm(\mathcal{A}, \{u\}, [m]), \{v\}, [n]) = (\mathcal{A} \times_m u) \times_n v \\
&= ttm(ttm(\mathcal{A}, \{v\}, [n]), \{u\}, [m]) = (\mathcal{A} \times_n v) \times_m u
\end{aligned} \tag{128}$$

## C. Tensor times tensor (ttt): $\langle \mathcal{A}, \mathcal{B} \rangle$

Let  $\mathcal{A}$  and  $\mathcal{B}$  be a tensor of size  $I_1 \times I_2 \times \dots \times I_N$ .

$$\begin{aligned}
\langle \mathcal{A}, \mathcal{B} \rangle &= \\
\beta &= \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \dots \sum_{i_N=1}^{I_N} \mathcal{A}(i_1, i_2, \dots, i_N) \mathcal{B}(i_1, i_2, \dots, i_N)
\end{aligned} \tag{129}$$



$$X = \text{tensor}(\text{rand}(4, 2, 3)); Y = \text{tensor}(\text{rand}(3, 4, 2));$$

$$Z = \text{ttt}(X, Y); \% < - - \text{Outerproduct of } X \text{ and } Y.$$

$$\text{size}(Z)$$

$$Z = \text{ttt}(X, X, 1 : 3) \% < - - \text{Innerproduct of } X \text{ with itself.}$$

(130)

$$Z = \text{ttt}(X, Y, [123], [231]) \% < - - \text{Innerproduct of } XY.$$

$$Z = \text{innerprod}(X, \text{permute}(Y, [231])) \% < - - \text{Same as above.}$$

$$Z = \text{ttt}(X, Y, [13], [21]) \% < - - \text{Product of } XY \text{ along specified dims.}$$