

Note: Governing Equations of General 3D duct flow

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I. Mass equation

Mass consevation:

$$-iakP^a + \nabla \cdot \mathbf{U}^a = \sum_{b=-\infty}^{+\infty} (-P^{a-b} \nabla \cdot \mathbf{U}^b - \mathbf{U}^{a-b} \cdot \nabla P^b - \frac{B}{2A} iakP^b P^{a-b}) \quad (1)$$

First, derivation of eq1:

We know that:

$$h_s = 1 - \kappa r \cos(\phi), h_r = 1, h_\theta = r \quad (2)$$

Then,

$$\begin{aligned} \nabla \cdot \mathbf{U}^a &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial(v_1 h_2 h_3)}{\partial w_1} + \frac{\partial(v_2 h_3 h_1)}{\partial w_2} + \frac{\partial(v_3 h_1 h_2)}{\partial w_3} \right] \\ &= \frac{1}{r(1 - \kappa r \cos(\phi))} \left[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] \end{aligned} \quad (3)$$

Thus, we have the mass equation, approximate RHS by:

$$\begin{aligned} \nabla \cdot \mathbf{U}^b &= ib\kappa P^b + o(M^2) \\ \nabla P^b &= ib\kappa \mathbf{U}^b + o(M^2) \end{aligned} \quad (4)$$

Then we have

$$\begin{aligned} -iakP^a + \frac{1}{r(1 - \kappa r \cos(\phi))} \left[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] \\ = \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b} P^b - ib\kappa U^{a-b} U^b - ib\kappa V^{a-b} V^b - ib\kappa W^{a-b} W^b - \frac{B}{2A} iakP^b P^{a-b}) \end{aligned} \quad (5)$$

The fourier harmonics are expanded as follows:

$$\begin{aligned} P^a &= \sum_{\beta=0}^{\infty} P_{\beta}^a(s) \psi_{\beta}(s, r, \theta) \\ U^a &= \sum_{\beta=0}^{\infty} U_{\beta}^a(s) \psi_{\beta}(s, r, \theta) \\ V^a &= \sum_{\beta=0}^{\infty} V_{\beta}^a(s) \psi_{\beta}(s, r, \theta) \\ W^a &= \sum_{\beta=0}^{\infty} W_{\beta}^a(s) \psi_{\beta}(s, r, \theta) \end{aligned} \quad (6)$$

with normalized relation:

$$\int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta r dr d\theta = \delta_{\alpha\beta} \quad (7)$$

Reorganize the eq5:

$$\begin{aligned} & -ia\kappa P^a(1 - \kappa r \cos(\phi)) + \frac{1}{r} \frac{\partial(U^a r)}{\partial s} + \frac{1}{r} \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{1}{r} \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \\ & = (1 - \kappa r \cos(\phi)) \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b} P^b - ib\kappa U^{a-b} U^b - ib\kappa V^{a-b} V^b - ib\kappa W^{a-b} W^b - \frac{B}{2A} iak P^b P^{a-b}) \end{aligned} \quad (8)$$

Intergal and insert eq 6, 7 into eq 8:

1. the first term:

$$\begin{aligned} & \int_0^{2\pi} \int_0^h \psi_\alpha r [-ia\kappa(1 - \kappa r \cos(\phi)) P^a] dr d\theta \\ & = -ia\kappa \sum_{\beta} \int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta r [(1 - \kappa r \cos(\phi))] dr d\theta P_\beta^a \\ & = -ia\kappa \Psi_{\alpha\beta} [r(1 - \kappa \cos(\phi))] P_\beta^a \\ & \quad (summationconvention) \end{aligned} \quad (9)$$

2. the second term:

$$\begin{aligned} & \int_0^{2\pi} \int_0^h \psi_\alpha r \left[\frac{1}{r} \frac{\partial(U^a r)}{\partial s} \right] dr d\theta \\ & = \int_0^{2\pi} \int_0^h \psi_\alpha r \frac{\partial(\psi_\beta U_\beta^a)}{\partial s} dr d\theta \\ & = \int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta r \frac{\partial(U_\beta^a)}{\partial s} dr d\theta + \int_0^{2\pi} \int_0^h \psi_\alpha r \frac{\partial(\psi_\beta)}{\partial s} dr d\theta U_\beta^a \\ & = \frac{dU_\beta^a}{ds} \int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta r dr d\theta + \int_0^{2\pi} \int_0^h r \frac{\partial(\psi_\alpha \psi_\beta)}{\partial s} dr d\theta U_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta r \frac{\partial(\psi_\alpha)}{\partial s} dr d\theta U_\beta^a \\ & = \frac{dU_\beta^a}{ds} \delta_{\alpha\beta} + \frac{\partial(\int_0^{2\pi} \int_0^h r \psi_\alpha \psi_\beta dr d\theta)}{\partial s} U_\beta^a - \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial s} \psi_\beta r dr d\theta U_\beta^a \\ & = \frac{dU_\alpha^a}{ds} + 0 - \Psi_{\{\alpha\}\beta} [r] U_\beta^a \end{aligned} \quad (10)$$

3. the third term:

$$\begin{aligned} & \int_0^{2\pi} \int_0^h \psi_\alpha r \left[\frac{1}{r} \frac{\partial(V^a r(1 - \kappa r \cos(\phi)))}{\partial r} \right] dr d\theta \\ & = \int_0^{2\pi} \int_0^h \psi_\alpha \frac{\partial(\psi_\beta r(1 - \kappa r \cos(\phi)))}{\partial r} dr d\theta V_\beta^a \\ & = \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha \psi_\beta r(1 - \kappa r \cos(\phi)))}{\partial r} dr d\theta V_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta r(1 - \kappa r \cos(\phi)) \frac{\partial(\psi_\alpha)}{\partial r} dr d\theta V_\beta^a \\ & = 0(periodic) - \Psi_{[\alpha]\beta} [r(1 - \kappa r \cos(\phi))] V_\beta^a \end{aligned} \quad (11)$$

4. the fourth term:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \psi_\alpha r \left[\frac{1}{r} \frac{\partial(W^a(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] dr d\theta \\
&= \int_0^{2\pi} \int_0^h \psi_\alpha \left[\frac{\partial(\psi_\beta(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] dr d\theta W_\beta^a \\
&= \int_0^{2\pi} \int_0^h \left[\frac{\partial(\psi_\alpha \psi_\beta(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] dr d\theta W_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta(1 - \kappa r \cos(\phi)) \left[\frac{\partial(\psi_\alpha)}{\partial \theta} \right] dr d\theta W_\beta^a \quad (12) \\
&= \int_0^h [\psi_\alpha \psi_\beta(1 - \kappa r \cos(\phi))]_0^{2\pi} dr W_\beta^a - \int_0^{2\pi} \int_0^h \psi_\beta(1 - \kappa r \cos(\phi)) \left[\frac{\partial(\psi_\alpha)}{\partial \theta} \right] dr d\theta W_\beta^a \\
&= 0 - \Psi_{(\alpha)\beta}[(1 - \kappa r \cos(\phi))] W_\beta^a
\end{aligned}$$

5. the RHS term:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \psi_\alpha r [(1 - \kappa r \cos(\phi)) \sum_{b=-\infty}^{+\infty} (-ib\kappa P^{a-b} P^b - ib\kappa U^{a-b} U^b - ib\kappa V^{a-b} V^b - ib\kappa W^{a-b} W^b - \frac{B}{2A} iak P^b P^{a-b})] dr d\theta \\
&= \int_0^{2\pi} \int_0^h \psi_\alpha \psi_\beta \psi_\gamma r (1 - \kappa r \cos(\phi)) dr d\theta \\
&\quad \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b - ib\kappa U_\beta^{a-b} U_\gamma^b - ib\kappa V_\beta^{a-b} V_\gamma^b - ib\kappa W_\beta^{a-b} W_\gamma^b - iak \frac{B}{2A} P_\beta^b P_\gamma^{a-b}) \\
&= \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b - ib\kappa U_\beta^{a-b} U_\gamma^b - ib\kappa V_\beta^{a-b} V_\gamma^b - ib\kappa W_\beta^{a-b} W_\gamma^b - iak \frac{B}{2A} P_\beta^b P_\gamma^{a-b}) \quad (13)
\end{aligned}$$

Finally, we obtain the mass equation in the form of eigenfunction, the idea is same as Galerkin

method:

$$\begin{aligned}
& \frac{dU_\alpha^a}{ds} - \Psi_{\{\alpha\}\beta} [r] U_\beta^a - iak \Psi_{\alpha\beta} [r(1 - \kappa r \cos(\phi))] P_\beta^a - \Psi_{[\alpha]\beta} [r(1 - \kappa r \cos(\phi))] V_\beta^a - \Psi_{(\alpha)\beta} [(1 - \kappa r \cos(\phi))] W_\beta^a \\
&= \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b} P_\gamma^b - ib\kappa U_\beta^{a-b} U_\gamma^b - ib\kappa V_\beta^{a-b} V_\gamma^b - ib\kappa W_\beta^{a-b} W_\gamma^b - iak \frac{B}{2A} P_\beta^b P_\gamma^{a-b}) \quad (14)
\end{aligned}$$

II. Momentum equation

Momentum consevation:

$$-ia\kappa \mathbf{U}^a + \nabla P^a = \sum_{b=-\infty}^{\infty} (-\mathbf{U}^{a-b} \cdot \nabla \mathbf{U}^b + P^{a-b} \nabla P^b) \quad (15)$$

First, we know that

$$\nabla P^a = \sum_i \frac{1}{h_i} \frac{\partial f}{\partial w_i} \hat{h}_i = \frac{1}{1 - \kappa r \cos \phi} \frac{\partial P^a}{\partial s} \hat{e}_s + \frac{\partial P^a}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial P^a}{\partial \theta} \hat{e}_\theta \quad (16)$$

The RHS term is a bit complex, with the divergence of a vector \mathbf{U} with its gradient, with

First, we know that

$$(\mathbf{v} \cdot \nabla) \mathbf{v}^b = \begin{cases} term1 : \mathcal{D}v_1^b + \frac{v_2^b}{h_2 h_1} (v_1 \frac{\partial h_1}{\partial \xi_2} - v_2 \frac{\partial h_2}{\partial \xi_1}) + \frac{v_3^b}{h_3 h_1} (v_1 \frac{\partial h_1}{\partial \xi_3} - v_2 \frac{\partial h_3}{\partial \xi_1}) \\ term2 : \mathcal{D}v_2^b + \frac{v_3^b}{h_3 h_2} (v_2 \frac{\partial h_2}{\partial \xi_3} - v_3 \frac{\partial h_3}{\partial \xi_2}) + \frac{v_1^b}{h_1 h_2} (v_2 \frac{\partial h_2}{\partial \xi_1} - v_1 \frac{\partial h_1}{\partial \xi_2}) \\ term3 : \mathcal{D}v_3^b + \frac{v_1^b}{h_1 h_3} (v_3 \frac{\partial h_3}{\partial \xi_1} - v_1 \frac{\partial h_1}{\partial \xi_3}) + \frac{v_2^b}{h_2 h_3} (v_3 \frac{\partial h_3}{\partial \xi_2} - v_2 \frac{\partial h_2}{\partial \xi_3}) \end{cases} \quad (17)$$

Besides,

$$\mathcal{D} = \frac{v_1}{h_1} \frac{\partial}{\partial \xi_1} + \frac{v_2}{h_2} \frac{\partial}{\partial \xi_2} + \frac{v_3}{h_3} \frac{\partial}{\partial \xi_3} \quad (18)$$

Thus, we have:

$$\begin{aligned} & -\mathbf{U}^{a-b} \cdot \nabla \mathbf{U}^b = \\ & - \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{D}1 : \frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial U^b}{\partial s} + V^{a-b} \frac{\partial U^b}{\partial r} + \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\ & term\mathcal{D}2 : \frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial V^b}{\partial s} + V^{a-b} \frac{\partial V^b}{\partial r} + \frac{W^{a-b}}{r} \frac{\partial V^b}{\partial \theta} \\ & term\mathcal{D}3 : \frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial W^b}{\partial s} + V^{a-b} \frac{\partial W^b}{\partial r} + \frac{W^{a-b}}{r} \frac{\partial W^b}{\partial \theta} \end{aligned} \right. \\ & - \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{X}1 : \frac{V^b}{1-\kappa r \cos \phi} (U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial r} - V^{a-b} \frac{\partial 1}{\partial s}) + \frac{W^b}{r(1-\kappa r \cos \phi)} (U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial \theta} - V^{a-b} \frac{\partial r}{\partial s}) \\ & term\mathcal{X}2 : \frac{W^b}{r} (V^{a-b} \frac{\partial 1}{\partial \theta} - W^{a-b} \frac{\partial r}{\partial r}) + \frac{U^b}{(1-\kappa r \cos \phi)1} (V^{a-b} \frac{\partial 1}{\partial s} - U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial r}) \\ & term\mathcal{X}3 : \frac{U^b}{(1-\kappa r \cos \phi)r} (W^{a-b} \frac{\partial h_3}{\partial s} - U^{a-b} \frac{\partial(1-\kappa r \cos \phi)}{\partial \theta}) + \frac{V^b}{1h_3} (W^{a-b} \frac{\partial h_3}{\partial r} - V^{a-b} \frac{\partial 1}{\partial \theta}) \end{aligned} \right. = \\ & \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\ & term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial V^b}{\partial s} - V^{a-b} \frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial V^b}{\partial \theta} \\ & term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial W^b}{\partial s} - V^{a-b} \frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial W^b}{\partial \theta} \end{aligned} \right. \\ & + \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{X}1 : \frac{\kappa \cos \phi}{1-\kappa r \cos \phi} U^{a-b} V^b - \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} W^b \\ & term\mathcal{X}2 : \frac{W^{a-b} W^b}{r} - \frac{\kappa \cos \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b \\ & term\mathcal{X}3 : \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b - \frac{W^{a-b} V^b}{r} \end{aligned} \right. \quad (19) \end{aligned}$$

Finally, we could derive the momentum conservation equation, with final term approximate by

eq 4:

$$\begin{aligned}
& \left\{ \begin{aligned} & -ia\kappa U^a + \frac{1}{1-\kappa r \cos \phi} \frac{\partial P^a}{\partial s} \\ & -ia\kappa V^a + \frac{\partial P^a}{\partial r} \\ & -ia\kappa W^a + \frac{1}{r} \frac{\partial P^a}{\partial \theta} \end{aligned} \right\} \\
& = \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\ & term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial V^b}{\partial s} - V^{a-b} \frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial V^b}{\partial \theta} \\ & term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial W^b}{\partial s} - V^{a-b} \frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial W^b}{\partial \theta} \end{aligned} \right\} \quad (20) \\
& + \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & term\mathcal{X}1 : \frac{\kappa \cos \phi}{1-\kappa r \cos \phi} U^{a-b} V^b - \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} W^b \\ & term\mathcal{X}2 : \frac{W^{a-b} W^b}{r} - \frac{\kappa \cos \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b \\ & term\mathcal{X}3 : \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} U^b - \frac{W^{a-b} V^b}{r} \end{aligned} \right\} + \sum_{b=-\infty}^{\infty} \left\{ \begin{aligned} & ib\kappa P^{a-b} U^b \\ & ib\kappa P^{a-b} V^b \\ & ib\kappa P^{a-b} W^b \end{aligned} \right\}
\end{aligned}$$

Now, we are going to project on ψ , it may be a little complex, we will doing step by step.

A. Momentum e^s term

First, deal with the e^s term:

$$\begin{aligned}
& -ia\kappa U^a + \frac{1}{1-\kappa r \cos \phi} \frac{\partial P^a}{\partial s} \\
& = \sum_{b=-\infty}^{\infty} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\
& + \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa \cos \phi}{1-\kappa r \cos \phi} U^{a-b} V^b - \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} W^b \\
& + ib\kappa P^{a-b} U^b \quad (21)
\end{aligned}$$

Multiply $(1-\kappa r \cos \phi)$, we have:

$$\begin{aligned}
& -ia\kappa(1-\kappa r \cos \phi) U^a + \frac{\partial P^a}{\partial s} \\
& = term\mathcal{D}1 : \sum_{b=-\infty}^{\infty} -U^{a-b} \frac{\partial U^b}{\partial s} - (1-\kappa r \cos \phi) V^{a-b} \frac{\partial U^b}{\partial r} - (1-\kappa r \cos \phi) \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\
& + term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa \cos \phi U^{a-b} V^b - \kappa \sin \phi U^{a-b} W^b + \\
& term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa r \cos \phi) P^{a-b} U^b \quad (22)
\end{aligned}$$

$\int \int XXr\psi_\alpha dr d\theta$, we have:

$$\underline{RHS = \int_0^{2\pi} \int_0^h [term\mathcal{D}1 + term\mathcal{X}1 + term\mathcal{P}1] r\psi_\alpha dr d\theta} \quad (23)$$

1. the first $\mathcal{D}1$ tems:

We ref the wiki https://en.wikipedia.org/wiki/Leibniz_integral_rule

General form: Differentiation under the integral sign:

$$\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x)) \cdot \frac{d}{dx} b(x) + f(x, a(x)) \cdot \frac{d}{dx} a(x) = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x, t) dt \quad (24)$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad (25)$$

For partial difference, for a given β , the derivation of the fuction $g(\alpha) = \int_{a(\alpha)}^{b(\beta)} f(x, \alpha) dx$ is

$$\frac{d}{d\alpha} \left(\int_{a(\alpha)}^{b(\beta)} f(x, \alpha) dx \right) = 0 + \frac{da(\alpha)}{d\alpha} f(a(\alpha), \alpha) = \int_{a(\alpha)}^{b(\beta)} \frac{\partial}{\partial \alpha} f(x, \alpha) dx \quad (26)$$

1.1

$$\begin{aligned} & \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-rU^{a-b} \frac{\partial U^b}{\partial s}] \psi_\alpha dr d\theta \\ &= \sum_{b=-\infty}^{\infty} - \int_0^{2\pi} \int_0^h \frac{\partial}{\partial s} [rU^{a-b} U^b \psi_\alpha] dr d\theta \\ & \quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial}{\partial s} [rU^{a-b} \psi_\alpha] dr d\theta \\ &= \sum_{b=-\infty}^{\infty} - \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [rU^{a-b} U^b \psi_\alpha] dr d\theta + \int_0^{2\pi} \frac{dh(s)}{ds} [rU^{a-b} U^b \psi_\alpha]_{r=h} d\theta \\ & \quad + \int_0^{2\pi} \int_0^h \frac{r \partial U^{a-b}}{\partial s} U^b \psi_\alpha dr d\theta \\ & \quad + \int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} rU^{a-b} U^b dr d\theta \end{aligned} \quad (27)$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$h'U^{a-b} = (1 - \kappa h \cos \phi) V^{a-b} \quad (28)$$

1.2

$$\begin{aligned}
& \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-(1 - \kappa r \cos \phi) V^{a-b} \frac{\partial U^b}{\partial r}] r \psi_\alpha dr d\theta \\
&= - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h \frac{\partial(r(1 - \kappa r \cos \phi) V^{a-b} U^b \psi_\alpha)}{\partial r} dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1 - \kappa r \cos \phi) V^{a-b} \psi_\alpha)}{\partial r} dr d\theta \\
&= - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} [(r(1 - \kappa r \cos \phi) V^{a-b} U^b) \psi_\alpha]_0^h d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1 - \kappa r \cos \phi) V^{a-b})}{\partial r} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial r} r(1 - \kappa r \cos \phi) V^{a-b} dr d\theta \\
&\quad = - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} [(h h' U^{a-b} U^b) \psi_\alpha]_0^h d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1 - \kappa r \cos \phi) V^{a-b})}{\partial r} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial r} r(1 - \kappa r \cos \phi) V^{a-b} dr d\theta
\end{aligned} \tag{29}$$

1.3

$$\begin{aligned}
& \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-(1 - \kappa r \cos \phi) \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta}] r \psi_\alpha dr d\theta \\
&= - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h \frac{\partial((1 - \kappa r \cos \phi) W^{a-b} U^b \psi_\alpha)}{\partial \theta} dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b}) dr d\theta \\
&\quad = - \sum_{b=-\infty}^{\infty} \int_0^{2\pi} [(1 - \kappa r \cos \phi) W^{a-b} U^b \psi_\alpha]_0^{2\pi} dr \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b}) dr d\theta \\
&= 0(\text{periodic}) + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta \\
&\quad + \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b}) dr d\theta
\end{aligned} \tag{30}$$

Combine together:

$$\begin{aligned}
& \frac{\int_0^{2\pi} \int_0^h [term \mathcal{D}1] r \psi_\alpha dr d\theta}{\sum_{b=-\infty}^{\infty}} = \sum_{b=-\infty}^{\infty} \\
& \left\{ \left(\int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} r U^{a-b} U^b dr d\theta \right. \right. \\
& + \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r (1 - \kappa r \cos \phi) V^{a-b} U^b dr d\theta \\
& + \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r \cos \phi) W^{a-b} U^b) dr d\theta) \\
& + \left(\int_0^{2\pi} \int_0^h U^b \frac{\partial U^{a-b}}{\partial s} r \psi_\alpha dr d\theta \right. \\
& + \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1 - \kappa r \cos \phi) V^{a-b})}{\partial r} \psi_\alpha dr d\theta \\
& + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r \cos \phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta) \\
& \left. - \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta \right\}
\end{aligned} \tag{31}$$

We apply eq 5, find that:

$$\begin{aligned}
& -i(a-b)\kappa r(1 - \kappa r \cos(\phi))P^{a-b} + \left[\frac{\partial(U^{a-b}r)}{\partial s} + \frac{\partial(V^{a-b}r(1 - \kappa r \cos(\phi)))}{\partial r} + \frac{\partial(W^{a-b}(1 - \kappa r \cos(\phi)))}{\partial \theta} \right] \\
& = o(M^2)
\end{aligned} \tag{32}$$

We have the second terms in eq(31) are equal to:

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h U^b [-i(a-b)\kappa r(1 - \kappa r \cos(\phi))P^{a-b}] \psi_\alpha dr d\theta \\
& = i(a-b)\kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos \phi)] P_\beta^{a-b} U_\gamma^b
\end{aligned} \tag{33}$$

And, the longitudinal derivation s can also be expand about the duct modes, with note

$[r], (\theta), \{s\}$:

$$\begin{aligned}
& \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta \\
& = \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] U_\beta^{a-b} U_\gamma^b dr d\theta \\
& = \frac{\partial}{\partial s} \left(\int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right) U_\beta^{a-b} U_\gamma^b \\
& \quad + \frac{\partial U_\beta^{a-b} U_\gamma^b}{\partial s} \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \\
& = \frac{\partial}{\partial s} \left(\int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right) U_\beta^{a-b} U_\gamma^b \\
& \quad + \left(\frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta
\end{aligned} \tag{34}$$

2. the second $\mathcal{X}1$ tems:

$$\begin{aligned}
term\mathcal{X}1 &: \sum_{b=-\infty}^{\infty} \kappa \cos\phi U^{a-b} V^b - \kappa \sin\phi U^{a-b} W^b \\
&\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [\kappa \cos\phi U^{a-b} V^b - \kappa \sin\phi U^{a-b} W^b] r \psi_\alpha dr d\theta \\
&= \kappa \Psi_{\alpha\beta\gamma} [r \cos\phi] U_\beta^{a-b} V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma} [r \sin\phi] U_\beta^{a-b} W_\gamma^b
\end{aligned} \tag{35}$$

3. the second $\mathcal{P}1$ tems:

$$\begin{aligned}
term\mathcal{P}1 &: \sum_{b=-\infty}^{\infty} ib\kappa(1 - \kappa r \cos\phi) P^{a-b} U^b \\
&\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [ib\kappa(1 - \kappa r \cos\phi) P^{a-b} U^b] r \psi_\alpha dr d\theta \\
&= ib\kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b
\end{aligned} \tag{36}$$

4. The LHS terms:

$$\frac{\partial P^a}{\partial s} - ia\kappa(1 - \kappa r \cos\phi) U^a$$

From 1.1 as example, we know that

$$\begin{aligned}
&\int_0^{2\pi} \int_0^h \frac{\partial}{\partial s} [r U^{a-b} U^b \psi_\alpha] dr d\theta \\
&= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta - \int_0^{2\pi} \frac{dh(s)}{ds} [r U^{a-b} U^b \psi_\alpha]_{r=h} d\theta
\end{aligned} \tag{37}$$

4.1

$$\begin{aligned}
&\int_0^{2\pi} \int_0^h \left[\frac{\partial P^a}{\partial s} \right] r \psi_\alpha dr d\theta \\
&= \int_0^{2\pi} \int_0^h \left[\frac{\partial(P_\beta^a \psi_\beta)}{\partial s} \right] r \psi_\alpha dr d\theta \\
&= \int_0^{2\pi} \int_0^h \left[\frac{\partial(P_\beta^a \psi_\beta)}{\partial s} r \psi_\alpha \right] dr d\theta - \int_0^{2\pi} \int_0^h \left[\frac{\partial(\psi_\alpha)}{\partial s} \right] r \psi_\beta dr d\theta P_\beta^a \\
&= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h (P_\beta^a \psi_\beta) r \psi_\alpha dr d\theta - \frac{dh(s)}{ds} [P_\beta^a \psi_\beta r \psi_\alpha]_{r=h} - \int_0^{2\pi} \int_0^h \left[\frac{\partial(\psi_\alpha)}{\partial s} \right] r \psi_\beta dr d\theta P_\beta^a \\
&= \frac{\partial}{\partial s} (P_\beta^a \delta_{\alpha\beta}) - \int_0^{2\pi} \frac{dh(s)}{ds} [P_\beta^a \psi_\beta r \psi_\alpha]_{r=h} d\theta - \int_0^{2\pi} \int_0^h \left[\frac{\partial(\psi_\alpha)}{\partial s} \right] r \psi_\beta dr d\theta P_\beta^a \\
&= \frac{d}{ds} P_\alpha^a - \int_0^{2\pi} \frac{dh(s)}{ds} [P_\beta^a \psi_\beta r \psi_\alpha]_{r=h} d\theta - \int_0^{2\pi} \int_0^h \left[\frac{\partial(\psi_\alpha)}{\partial s} \right] r \psi_\beta dr d\theta P_\beta^a
\end{aligned} \tag{38}$$

$$= \frac{d}{ds} P_\alpha^a - \int_0^{2\pi} \int_0^h \frac{\partial \psi_\beta \psi_\alpha}{\partial r} h h' dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h \left[\frac{\partial(\psi_\alpha)}{\partial s} \right] r \psi_\beta dr d\theta P_\beta^a$$

4.2

$$\begin{aligned} & \int_0^{2\pi} \int_0^h [-ia\kappa(1 - \kappa r \cos\phi) U^a] r \psi_\alpha dr d\theta \\ &= -ia\kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos\phi)] U_\beta^a \end{aligned} \quad (39)$$

Finally, putting all together becomes:

$$\begin{aligned} & \frac{d}{ds} P_\alpha^a - \int_0^{2\pi} \frac{dh(s)}{ds} [P_\beta^a \psi_\beta r \psi_\alpha]_{r=h} d\theta - \int_0^{2\pi} \int_0^h \left[\frac{\partial(\psi_\alpha)}{\partial s} \right] r \psi_\beta dr d\theta P_\beta^a - ia\kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos\phi)] U_\beta^a \\ &= \frac{d}{ds} P_\alpha^a - ia\kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos\phi)] U_\beta^a - \int_0^{2\pi} h h' [P_\beta^a \psi_\beta \psi_\alpha]_{r=h} d\theta - \Psi_{\{\alpha\}\beta} [r] P_\beta^a \\ &= \sum_{b=-\infty}^{\infty} \end{aligned}$$

$$\begin{aligned} & (eq31) : \left\{ \left(\int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} r U^{a-b} U^b dr d\theta \right. \right. \\ & \quad \left. \left. + \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r (1 - \kappa r \cos\phi) V^{a-b} U^b dr d\theta \right. \right. \\ & \quad \left. \left. + (eq33) : i(a-b)\kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \right. \right. \\ & \quad \left. \left. eq(34) : -\frac{\partial}{\partial s} \left(\int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right) U_\beta^{a-b} U_\gamma^b \right. \right. \\ & \quad \left. \left. - \left(\frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta \right. \right. \\ & \quad \left. \left. + eq(35) : \kappa \Psi_{\alpha\beta\gamma} [r \cos\phi] U_\beta^{a-b} V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma} [r \sin\phi] U_\beta^{a-b} W_\gamma^b \right. \right. \\ & \quad \left. \left. + eq(36) : ib\kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \right. \right. \\ & \quad \left. = (abbreviation) : \right. \end{aligned}$$

$$\begin{aligned} & \Psi_{\{\alpha\}\beta\gamma} [r] U_\beta^{a-b} U_\gamma^a + \Psi_{[\alpha]\beta\gamma} [r(1 - \kappa r \cos\phi)] V_\beta^{a-b} U_\gamma^a + \Psi_{(\alpha)\beta\gamma} [(1 - \kappa r \cos\phi)] W_\beta^{a-b} U_\gamma^a \\ & \quad + i(a-b)\kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b \\ & \quad - \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma} [r] U_\beta^{a-b} U_\gamma^b - \Psi_{\alpha\beta\gamma} [r] \left(\frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) \\ & \quad + \kappa \Psi_{\alpha\beta\gamma} [r \cos\phi] U_\beta^{a-b} V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma} [r \sin\phi] U_\beta^{a-b} W_\gamma^b \\ & \quad + ib\kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos\phi)] P_\beta^{a-b} U_\gamma^b) \end{aligned} \quad (40)$$

To be conclude, with the e^s term:

$$\begin{aligned}
& -ia\kappa U^a + \frac{1}{1-\kappa r \cos\phi} \frac{\partial P^a}{\partial s} \\
= & \sum_{b=-\infty}^{\infty} \text{term}\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r \cos\phi} \frac{\partial U^b}{\partial s} - V^{a-b} \frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial U^b}{\partial \theta} \\
& + \sum_{b=-\infty}^{\infty} \text{term}\mathcal{X}1 : \frac{\kappa \cos\phi}{1-\kappa r \cos\phi} U^{a-b} V^b - \frac{\kappa \sin\phi}{(1-\kappa r \cos\phi)} U^{a-b} W^b \\
& + \sum_{b=-\infty}^{\infty} \text{term}\mathcal{P}1 : ib\kappa P^{a-b} U^b
\end{aligned} \tag{41}$$

We have:

$$\begin{aligned}
& -ia\kappa \Psi_{\alpha\beta}[r(1-\kappa r \cos\phi)] U_{\beta}^a + \frac{d}{ds} P_{\alpha}^a - \int_0^{2\pi} hh'[\psi_{\beta}\psi_{\alpha}]_{r=h} d\theta P_{\beta}^a - \Psi_{\{\alpha\}\beta}[r] P_{\beta}^a \\
= & \underline{\text{term}\mathcal{D}1} : \Psi_{\{\alpha\}\beta\gamma}[r] U_{\beta}^{a-b} U_{\gamma}^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r \cos\phi)] V_{\beta}^{a-b} U_{\gamma}^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r \cos\phi)] W_{\beta}^{a-b} U_{\gamma}^a \\
& + \underline{\text{term}(\mathcal{D}1 + \mathcal{P}1)} : i(a)\kappa \Psi_{\alpha\beta\gamma}[r(1-\kappa r \cos\phi)] P_{\beta}^{a-b} U_{\gamma}^b \\
& - \underline{\text{term}\mathcal{D}1} : \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma}[r] U_{\beta}^{a-b} U_{\gamma}^b - \Psi_{\alpha\beta\gamma}[r] (\frac{dU_{\beta}^{a-b}}{ds} U_{\gamma}^b + \frac{dU_{\gamma}^b}{ds} U_{\beta}^{a-b})) \\
& + \underline{\text{term}\mathcal{X}1} : \kappa \Psi_{\alpha\beta\gamma}[r \cos\phi] U_{\beta}^{a-b} V_{\gamma}^b - \kappa \Psi_{\alpha\beta\gamma}[r \sin\phi] U_{\beta}^{a-b} W_{\gamma}^b
\end{aligned} \tag{42}$$

B. Momentum e^r term

Second, deal with the e^r term:

$$\begin{aligned}
& -ia\kappa V^a + \frac{\partial P^a}{\partial r} \\
= & \sum_{b=-\infty}^{\infty} \text{term}\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r \cos\phi} \frac{\partial V^b}{\partial s} - V^{a-b} \frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial V^b}{\partial \theta} \\
& + \sum_{b=-\infty}^{\infty} \text{term}\mathcal{X}2 : \frac{W^{a-b} W^b}{r} - \frac{\kappa \cos\phi}{(1-\kappa r \cos\phi)} U^{a-b} U^b \\
& + \text{term}\mathcal{P}2 : \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b} V^b
\end{aligned} \tag{43}$$

LHS-2: $\frac{\partial P^a}{\partial r}(1 - \kappa r \cos \phi)$

$$\begin{aligned}
& \int_0^{2\pi} \int_0^h \left[\frac{\partial P^a}{\partial r}(1 - \kappa r \cos \phi) \right] r \psi_\alpha dr d\theta \\
&= \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\beta (1 - \kappa r \cos \phi) r \psi_\alpha}{\partial r} \right] dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\alpha (1 - \kappa r \cos \phi) r}{\partial r} \right] \psi_\beta dr d\theta P_\beta^a \\
&= \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\beta (1 - \kappa r \cos \phi) r \psi_\alpha}{\partial r} \right] dr d\theta P_\beta^a \\
&- \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\alpha}{\partial r} \right] (1 - \kappa r \cos \phi) r \psi_\beta dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h \left[\frac{\partial (1 - \kappa r \cos \phi) r}{\partial r} \right] \psi_\alpha \psi_\beta dr d\theta P_\beta^a \quad (44) \\
&= \int_0^{2\pi} [\psi_\alpha \psi_\beta r (1 - \kappa r \cos \phi)]_0^h d\theta P_\beta^a \\
&- \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\alpha}{\partial r} \right] (1 - \kappa r \cos \phi) r \psi_\beta dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h [1 - 2\kappa r \cos \phi] \psi_\alpha \psi_\beta dr d\theta P_\beta^a \\
&= \int_0^{2\pi} [\psi_\alpha \psi_\beta r (1 - \kappa r \cos \phi)]_0^h d\theta P_\beta^a - \Psi_{[\alpha]\beta} [r(1 - \kappa r \cos \phi)] P_\beta^a - \Psi_{\alpha\beta} [1 - 2\kappa r \cos \phi] P_\beta^a
\end{aligned}$$

The derivation of *termD2* is identical to A, we are not prove it again. *TermP2* also could be combine with the part separated term of *termD2* with V^b . *TermX2* is also easy to derive.

Thus, we have the final equation:

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1 - \kappa r \cos \phi)]V_\beta^a \\
& \int_0^{2\pi} [\psi_\alpha \psi_\beta r (1 - \kappa r \cos \phi)]_0^h d\theta P_\beta^a - \Psi_{[\alpha]\beta} [r(1 - \kappa r \cos \phi)] P_\beta^a - \Psi_{\alpha\beta} [1 - 2\kappa r \cos \phi] P_\beta^a \\
&= \underline{termD2} : \Psi_{\{\alpha\}\beta\gamma} [r] U_\beta^{a-b} V_\gamma^a + \Psi_{[\alpha]\beta\gamma} [r(1 - \kappa r \cos \phi)] V_\beta^{a-b} V_\gamma^a + \Psi_{(\alpha)\beta\gamma} [(1 - \kappa r \cos \phi)] W_\beta^{a-b} V_\gamma^a \\
& \quad + \underline{term(D2 + P2)} : i(a)\kappa\Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos \phi)] P_\beta^{a-b} V_\gamma^b \\
& \quad + \underline{termD2} : -\frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma} [r] U_\beta^{a-b} U_\gamma^b - \Psi_{\alpha\beta\gamma} [r] (\frac{dU_\beta^{a-b}}{ds} V_\gamma^b + \frac{dV_\gamma^b}{ds} U_\beta^{a-b})) \\
& \quad + \underline{termX2} : \kappa\Psi_{\alpha\beta\gamma} [1 - \kappa r \cos \phi] W_\beta^{a-b} W_\gamma^b - \kappa\Psi_{\alpha\beta\gamma} [r \cos \phi] U_\beta^{a-b} U_\gamma^b \quad (45)
\end{aligned}$$

C. Momentum e^θ term

Third, deal with the e^θ term:

$$\begin{aligned}
& -ia\kappa W^a + \frac{1}{r} \frac{\partial P^a}{\partial \theta} \\
= & \sum_{b=-\infty}^{\infty} \text{term}\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa r \cos \phi} \frac{\partial W^b}{\partial s} - V^{a-b} \frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r} \frac{\partial W^b}{\partial \theta} \\
& + \sum_{b=-\infty}^{\infty} \text{term}\mathcal{X}3 : \frac{\kappa \sin \phi}{(1-\kappa r \cos \phi)} U^{a-b} V^b - \frac{W^{a-b} V^b}{r} \\
& + \text{term}\mathcal{P}3 : \sum_{b=-\infty}^{\infty} ib\kappa P^{a-b} W^b
\end{aligned} \tag{46}$$

$$\begin{aligned}
\text{LHS-2: } & \frac{\partial P^a}{\partial \theta} \frac{(1-\kappa r \cos \phi)}{r} \\
= & \int_0^{2\pi} \int_0^h \left[\frac{\partial P^a}{\partial \theta} \frac{(1-\kappa r \cos \phi)}{r} \right] r \psi_\alpha dr d\theta \\
= & \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\beta (1-\kappa r \cos \phi)}{\partial \theta} \psi_\alpha \right] dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\alpha (1-\kappa r \cos \phi)}{\partial \theta} \right] \psi_\beta dr d\theta P_\beta^a \\
= & 0 - \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\alpha}{\partial \theta} \right] (1-\kappa r \cos \phi) \psi_\beta dr d\theta P_\beta^a - \int_0^{2\pi} \int_0^h \left[\frac{\partial (1-\kappa r \cos \phi)}{\partial \theta} \right] \psi_\alpha \psi_\beta dr d\theta P_\beta^a \\
= & - \int_0^{2\pi} \int_0^h \left[\frac{\partial \psi_\alpha}{\partial \theta} \right] (1-\kappa r \cos \phi) \psi_\beta dr d\theta P_\beta^a + \kappa \int_0^{2\pi} \int_0^h [r \sin \phi] \psi_\alpha \psi_\beta dr d\theta P_\beta^a \\
= & -\Psi_{(\alpha)\beta} [(1-\kappa r \cos \phi)] P_\beta^a - \kappa \Psi_{\alpha\beta} [r \sin \phi] P_\beta^a
\end{aligned} \tag{47}$$

The derivation of $\text{term}\mathcal{D}3$ is identical to A, we are not prove it again. $\text{Term}\mathcal{P}3$ also could be combine with the part separated term of $\text{term}\mathcal{D}3$ with W^b . $\text{Term}\mathcal{X}3$ is also easy to derive.

Thus, we have the final equation:

$$\begin{aligned}
& -ia\kappa \Psi_{\alpha\beta} [r(1-\kappa r \cos \phi)] W_\beta^a \\
& -\Psi_{(\alpha)\beta} [(1-\kappa r \cos \phi)] P_\beta^a - \kappa \Psi_{\alpha\beta} [r \sin \phi] P_\beta^a \\
= & \underline{\text{term}\mathcal{D}2} : \Psi_{\{\alpha\}\beta\gamma} [r] U_\beta^{a-b} W_\gamma^a + \Psi_{[\alpha]\beta\gamma} [r(1-\kappa r \cos \phi)] V_\beta^{a-b} W_\gamma^a + \Psi_{(\alpha)\beta\gamma} [(1-\kappa r \cos \phi)] W_\beta^{a-b} W_\gamma^a \\
& + \underline{\text{term}(D2 + \mathcal{P}2)} : i(a)\kappa \Psi_{\alpha\beta\gamma} [r(1-\kappa r \cos \phi)] P_\beta^{a-b} W_\gamma^b \\
& + \underline{\text{term}\mathcal{D}2} : -\frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma} [r] U_\beta^{a-b} V_\gamma^b - \Psi_{\alpha\beta\gamma} [r] (\frac{dU_\beta^{a-b}}{ds} W_\gamma^b + \frac{dW_\gamma^b}{ds} U_\beta^{a-b})) \\
& + \underline{\text{term}\mathcal{X}2} : \kappa \Psi_{\alpha\beta\gamma} [r \sin \phi] U_\beta^{a-b} U_\gamma^b - \Psi_{\alpha\beta\gamma} [1-\kappa r \cos \phi] W_\beta^{a-b} V_\gamma^b
\end{aligned} \tag{48}$$

III. Merge the four equations and eliminate the V_γ^b and W_γ^b

Using the linear relationships:

$$\begin{aligned}
iakV^a &= \frac{\partial P^a}{\partial r} \\
&:= \int \int iakV_\beta^a \psi_\beta r \psi_\alpha dr d\theta = \int \int \frac{\partial P_\beta^a \psi_\beta}{\partial r} r \psi_\alpha dr d\theta \\
&= iakV_\beta^a \delta_{\alpha\beta} = \Psi_{\alpha[\beta]}[r] P_\beta^a = \int_0^{2\pi} [r \psi_\alpha \psi_\beta]_0^h d\theta P_\beta^a - \int \int \psi_\alpha \psi_\beta dr d\theta P_\beta^a - \int \int \frac{\partial \psi_\alpha}{\partial r} \psi_\beta r dr d\theta P_\beta^a
\end{aligned} \tag{49}$$

$$\begin{aligned}
iakW^a &= \frac{1}{r} \frac{\partial P^a}{\partial \theta} \\
&:= \int \int iakW_\beta^a \psi_\beta r \psi_\alpha dr d\theta = \int \int \frac{\partial P_\beta^a \psi_\beta}{\partial \theta} \frac{1}{r} r \psi_\alpha dr d\theta \\
&= iakW_\beta^a \delta_{\alpha\beta} = \Psi_{\alpha(\beta)}[r] P_\beta^a = 0 - \Psi_{(\alpha)\beta}[r] P_\beta^a
\end{aligned} \tag{50}$$

Thus, we can establish relationships between the tranverse modes and pressure modes (no summation over α)

$$V_\alpha^a = \frac{1}{iak} \left[\int_0^{2\pi} [r \psi_\alpha \psi_\beta]_0^h d\theta P_\beta^a - \Psi_{\alpha\beta} - \Psi_{[\alpha]\beta}[r] \right] P_\beta^a = V_{\alpha\beta}^a P_\beta^a \tag{51}$$

$$W_\alpha^a = -\frac{1}{iak} \Psi_{(\alpha)\beta} P_\beta^a = W_{\alpha\beta}^a P_\beta^a \tag{52}$$

We also require modal expressions for $\frac{d}{ds} V_\alpha^a$ and $\frac{d}{ds} W_\alpha^a$.

We differentiate eq49 with respect to s :

$$iakV^a = \frac{\partial P^a}{\partial r} \tag{53}$$