Revisiting the orthogonality of Bessel functions of the first kind on an infinite interval

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Revisiting the Orthogonality of Bessel Functions of The First Kind on an Infinite Interval

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Abstract

The rigorous proof of the orthogonality integral $\int_0^\infty \rho\, J_\nu(k\rho)\, J_\nu(k'\rho)\, d\rho = \frac{\delta(k-k')}{k}$, for $\nu \geq -1$, is laborious and requires the use of mathematical techniques that, probably, are unfamiliar to most physics students, even at the graduate level. In physics, we are used to the argument that it may be proved by the use of Hankel transforms. However, the logic of the matter is the opposite, viz., the existence of the inverse Hankel transform is a consequence of the orthogonality integral. The goal of this work is to prove this integral without circular reasoning. In this paper, using elementary properties of Bessel functions, we give a simple analytical derivation of this integral for the case where ν is an integer, zero or half-integer not less than $-1/2$. Then, using the asymptotic behavior of $J_\nu(x)$, we extend the result to any $\nu \geq -1$. This work is of pedagogical nature. Therefore, to add educational value to the discussion, we do not skip the details of the calculations.

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1 Introduction

Bessel differential equation appears in a wide variety of physical problems, which include static potentials in electromagnetism, propagation of waves in cylindrical waveguides, solutions of the radial Schrödinger equation, modes of oscillation of thin plates and membranes, heat conduction, and many others [1].

In cylindrical coordinates $(\rho,\, \phi,\, z)$ the assumption that the solution of the Laplace equation can be found as the superposition of functions of the form $\Phi(\rho,\, \phi,\, z) = F(\rho)Q(\phi)Z(z)$ leads to three ordinary differential equations, one for each function. The equation for the radial function $F(\rho)$ is given by

$$\rho^2 \frac{d^2F}{d\rho^2} + \rho \frac{dF}{d\rho} + \left(k^2\rho^2 - \nu^2\right) F = 0, \tag{1}$$

where k and ν are separation constants; k has dimensions of $(\text{length})^{-1}$ and ν is dimensionless. If the physical problem shows translational symmetry along the z-axis, then $k = 0$. Similarly, if it shows rotational symmetry about z, then $\nu = 0$.

When k is real and positive, it is customary to introduce the dimensionless variable

$$x = k\rho, \tag{2}$$

in terms of which (1) becomes

$$\frac{d^2F}{dx^2} + \frac{1}{x}\frac{dF}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) F = 0. \tag{3}$$

This is the well-known Bessel differential equation of order $\nu \geq 0$, whose general solution is customarily written as

$$F(x) = c_1 J_\nu(x) + c_2 Y_\nu(x), \tag{4}$$

where $J_\nu(x)$ and $Y_\nu(x)$ are linearly independent solutions of (3) called Bessel functions of the first kind and Bessel functions of the second kind, respectively. These functions have been thoroughly studied in the literature [2]-[4] and, thus, one could hardly add something substantial to what is already known about them.

Therefore, our interest in this note is of pedagogical nature. Here we consider the problem of finding a simple analytical derivation of the integral relation

$$\int_0^\infty \rho\, J_\nu(k\rho)\, J_\nu(k'\rho)\, d\rho = \frac{\delta(k-k')}{k}, \quad \nu \geq -1, \quad k, k' > 0, \tag{5}$$

which is the statement that Bessel functions $J_\nu(k\rho)$, with different values of k, are orthogonal on the infinite range $0 \leq \rho \leq \infty$ with respect to the weight $w(\rho) = \rho$. The proof of the orthogonality of these functions on a finite interval, $0 \leq \rho \leq a$, is a standard procedure that is widely discussed in many books and textbooks. However this is not so when the interval is infinite.

The above integral follows from Hankel's integral theorem [5]. In Section 14.4 of reference [4], Watson provides an elaborate and thorough proof of Hankel's integral theorem for $\nu \geq -1/2$. The proof is quite long, from page 456 to page 464, and is more difficult when the range of ν is extended to $\nu > -1$. Probably the complexity of the proof is the reason why the discussion, of the orthogonality of the Bessel functions $J_\nu(x)$ over an infinite interval, is avoided.

In Physics, generally we are not very much concerned with the details of the proofs of mathematical theorems. Usually we are more interested in their application for analyzing and solving physical problems. This happens, in particular, with the orthogonality relation (5). In the monograph Mathematical Methods for Physicists of Arfken and Weber [6] the integral (5) with $\nu > -1/2$ is given, without proof, in Section 11.2 (equation 11.59). Later, in the same reference (Problem 15.1.2), the inverse Hankel transform is assumed in such a way that it allows to obtain (5) in a line of calculation [7]. However, the truth of the matter is in the opposite direction: the inversion formula is a consequence of (5). Clearly, this is circular logic where the conclusion is part of the premises [8].

For several purposes it is useful when physics students, especially those of theoretical physics, have a clear notion of the mathematical foundations underlying the theory; it improves students' confidence and enhances their critical thinking. In Jackson's Classical Electrodynamics [9], in Problem 3.16(a), the verification of (5) is left as an exercise to the reader, with the hint of using Bessel's differential equation and the appropriate limiting procedures. It should be noted that despite the large number of solutions books and websites (for example [10]-[14]) for many of the problems in Jackson, most of them either do not address the problem or do it in an incorrect way.

The paper is organized as follows. In Section 2 we give a simple analytical derivation of the orthogonality integral (5) for the case where ν is an integer, zero or half-integer not less than $-1/2$. In Section 3, using the asymptotic behavior of $J_\nu(k\rho)$, by means of direct calculation we obtain the orthogonality integral (5) for an arbitrary $\nu \geq -1$. In Section 4 we give a summary and discussion.

2 Fourier-Bessel Expansion

To make the presentation self-consistent we start from the Fourier-Bessel series over a finite interval $0 \leq \rho \leq a$ and then consider the case where the interval becomes infinite ($a \to \infty$). We will show that the orthogonality integral (5) is a consequence of the fact that $\int_0^\infty J_\nu(x)\, dx = 1$.

Since $J_\nu(x)$ oscillates about 0, the equation $J_\nu(x) = 0$ has an infinite number of roots: $x_{\nu 1}$, $x_{\nu 2}$, $x_{\nu 3}$.... When a is finite these roots play an important role. First, the functions $\sqrt{\rho} J_\nu\left(x_{\nu i}\rho/a\right)$, with fixed $\nu \geq -1$ and different i, are mutually orthogonal in the interval $0 \leq \rho \leq a$, viz.,

$$\int_0^a \rho J_\nu\left(x_{\nu i}\frac{\rho}{a}\right) J_\nu\left(x_{\nu j}\frac{\rho}{a}\right) d\rho = \frac{a^2}{2} J_{\nu+1}^2\left(x_{\nu i}\right)\delta_{ij}, \quad i, j = 0,\, 1,\, 2, ... \quad \nu \geq -1. \tag{6}$$

Second, any piecewise smooth function $f(\rho)$ on the interval $0 \leq \rho \leq a$ can be expanded in a Fourier-Bessel series [15]

$$f(\rho) = \sum_{i=1}^{\infty} C_{\nu i}\, J_\nu\left(x_{\nu i}\frac{\rho}{a}\right), \quad 0 \leq \rho \leq a, \tag{7}$$

where the coefficients $C_{\nu i}$ are obtained with the help of the orthogonality condition (6) as

$$C_{\nu i} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu i})} \int_0^a \rho f(\rho) J_\nu\left(x_{\nu i}\frac{\rho}{a}\right) d\rho. \tag{8}$$

When $a \to \infty$ the discrete superposition in (7) is replaced by a continuous superposition. In this limit the chain of formal transformations

$$\begin{aligned}
k_i = \frac{x_{\nu i}}{a} \quad &\to \quad k \\
\delta_{ij} \to \delta(ak_i - ak_j) \quad &\to \quad \delta\left[a\left(k - k'\right)\right] = \frac{\delta(k-k')}{a} \\
\sum_i \quad &\to \quad \int_0^\infty a\, dk \\
a\, C_{\nu i} \quad &\to \quad C(k) \\
\frac{a J_{\nu+1}^2\left(k_i a\right)}{2} \quad &\to \quad N(k,\nu),
\end{aligned}$$

where the “normalization factor” $N(k,\nu)$ is still unknown, changes the set of equations (6), (7) and (8) into

$$\int_0^\infty \rho J_\nu\left(k\rho\right)\, J_\nu\left(k'\rho\right)\, d\rho \;=\; N(k,\nu)\,\delta(k-k') \tag{9}$$

$$f(\rho) \;=\; \int_0^\infty C_\nu(k)\, J_\nu(k\rho)\, dk \tag{10}$$

$$C_\nu(k) \;=\; \frac{1}{N(k,\nu)} \int_0^\infty \rho\, f(\rho)\, J_\nu(k\rho)\, d\rho. \tag{11}$$

Since the norm of $J_\nu\left(k_i\rho\right)$ on a finite interval depends on ν, we assume that N is also a function of ν. To check the compatibility of these equations we multiply both sides of (10) by $\rho J_\nu(k'\rho)$ and integrate over ρ form 0 to infinity. Then, using (11), we obtain

$$C(k')\, N(k',\nu) = \int_0^\infty C(k) \left[\int_0^\infty \rho J_\nu(k\rho)\, J_\nu(k'\rho)\, d\rho\right] dk,$$

which is an identity by virtue of (9).

Finally, the closure relation

$$\int_0^\infty \frac{J_\nu(k\rho)\, J_\nu(k\rho')}{N(k,\nu)}\, dk = \frac{\delta(\rho-\rho')}{\rho} \tag{12}$$

is obtained when we substitute (11) into (10).

To find the unknown normalization factor $N(k,\nu)$ we integrate both sides of (9) over k' from zero to infinity and obtain

$$N(k,\nu) = \int_0^\infty \rho J_\nu\left(k\rho\right) \left[\int_0^\infty J_\nu\left(k'\rho\right)\, dk'\right] d\rho. \tag{13}$$

To give an analytical derivation of (5), as promised in the introduction, we need to calculate these integrals manually, without the aid of computers or tables of integrals. Below we do this for the case where ν is an integer, zero or half-integer.

2.1 Evaluating the Integrals

From the recurrence relation

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2\,\frac{dJ_\nu(x)}{dx},$$

it follows that

$$\int_0^\infty J_\nu(x)\, dx - \int_0^\infty J_{\nu+2}(x)\, dx = -\, 2\, J_{\nu+1}(x)\big|_{x=0}\,, \tag{14}$$

where we have used that $J_\nu\left(\infty\right) = 0$, for an arbitrary ν. Since

$$J_{\nu+1}(x) \to \frac{1}{\Gamma\left(\nu+2\right)} \left(\frac{x}{2}\right)^{\nu+1}, \quad \text{as } x \to 0$$

the RHS of (14) is zero for $\nu > -1$. Consequently,

$$\int_0^\infty J_{\nu+2}(x)\, dx = \int_0^\infty J_\nu(x)\, dx, \quad \nu > -1. \tag{15}$$

Odd integer order: When $\nu = -1$, the RHS of (14) is equal to -2, because $J_0(0) = 1$. Thus, using that $J_{-1}(x) = -J_1(x)$, we obtain

$$\int_0^\infty J_1(x)\,dx = 1.$$

From this expression and (15) we get

$$\int_0^\infty J_{2n+1}(x)\,dx = 1, \quad n = 0,\, 1,\, 2... \tag{16}$$

Even integer order: When ν is an even number, to obtain a similar relation, we need to calculate $\int_0^\infty J_0(x)\,dx$. Since

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = 1 - \frac{x^2}{4} + \frac{x^4}{64} + \cdots, \tag{17}$$

we cannot integrate term by term. Instead, we use the Parseval integral

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos\left(x \sin\theta\right)\, d\theta, \tag{18}$$

which is easy to verify; expanding in series around $x = 0$ and integrating over θ we obtain (17).

Thus,

$$\int_0^\infty J_0(x)\,dx = \lim_{\lambda\to\infty} \frac{1}{\pi} \int_0^\lambda dx \int_0^\pi \cos\left(x\sin\theta\right)\, d\theta. \tag{19}$$

• Straightforward evaluation of $\int_0^\infty J_0(x)\,dx$: Assuming that we can change the order of integration we obtain

$$\int_0^\infty J_0(x)\,dx = \lim_{\lambda\to\infty} \frac{1}{\pi} \int_0^\pi \frac{\sin\left(\lambda\sin\theta\right)}{\sin\theta}\, d\theta. \tag{20}$$

The function under the integral, in the RHS of (20), possesses mirror symmetry with respect to $\theta = \pi/2$. It tends to λ when $\theta \to 0$, as well as when $\theta \to \pi$, in between it becomes zero $2\lfloor\mathcal{N}\rfloor$ times, where $\lfloor\mathcal{N}\rfloor = \text{floor}(\mathcal{N})$ represents the largest integer not greater than

$$\mathcal{N} = \frac{\lambda}{\pi}.$$

Therefore, in the limit $\lambda \to \infty$ this function vanishes everywhere except for the two infinitely high and infinitely narrow peaks at $\theta = 0$ and $\theta = \pi$. This suggests that, in this limit the integrand in the RHS of (20) can be expressed as a combination of Dirac delta functions.

A useful representation of the delta function comes from the theory of the Fourier integral [16]. Namely,

$$\begin{aligned}\delta(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk = \frac{1}{\pi}\int_0^\infty \cos kx\,dk = \lim_{\lambda\to\infty}\frac{1}{\pi}\int_0^\lambda \cos kx\,dk \\ &= \lim_{\lambda\to\infty} \frac{\sin\lambda x}{\pi x}. \end{aligned} \tag{21}$$

From this expression it follows that the RHS of (20) is $\int_0^\pi \delta(\sin\theta)\, d\theta$. At this point we recall that when the Dirac delta function has as argument a function $f(s)$, with simple zeros located at $s = s_i$, it reduces to

$$\delta[f(s)] = \sum_i \frac{\delta(s - s_i)}{\left|(df/ds)_{s_i}\right|}.$$

Consequently,

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{\sin(\lambda\sin\theta)}{\sin\theta} &= \delta(\sin\theta) \\ &= \delta(\theta) + \delta(\theta - \pi), \end{aligned}$$

in agreement with the above discussion. Thus, we obtain

$$\int_0^\infty J_0(x)\, dx = \int_0^\pi \delta(\sin\theta)\, d\theta = \int_0^\pi [\delta(\theta) + \delta(\theta - \pi)]\, d\theta = 1. \tag{22}$$

• Alternative evaluation of $\int_0^\infty J_0(x)\, dx$: For pedagogical reasons, it is worth mentioning that (19) can be evaluated, without the explicit use of Dirac's delta function, by solving the integral

$$I(\alpha) = \frac{1}{\pi} \int_0^\pi d\theta \int_0^\infty \cos(x\sin\theta)\, e^{-\alpha x}\, dx, \quad \alpha > 0, \tag{23}$$

which in the limit $\alpha \to 0$ yields $\int_0^\infty J_0(x)\, dx$.

The integral over x in (23) is easily done after using Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$ to express cosine in terms of exponents. The result is

$$\frac{1}{\pi} \int_0^\infty \cos(x\sin\theta)\, e^{-\alpha x}\, dx = \frac{\alpha}{\pi\left(\alpha^2 + \sin^2\theta\right)}. \tag{24}$$

Thus,

$$I(\alpha) = \frac{1}{\pi} \int_0^\pi \frac{\alpha}{\alpha^2 + \sin^2\theta}\, d\theta = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\alpha}{\alpha^2 + \cos^2\theta}\, d\theta = \frac{2}{\pi} \int_0^{\pi/2} \frac{\alpha}{\alpha^2 + \cos^2\theta}\, d\theta.$$

To evaluate the last integral we use the variable $u = \frac{\alpha}{\sqrt{\alpha^2+1}} \tan\theta$ in terms of which

$$I(\alpha) = \frac{2}{\pi\sqrt{\alpha^2+1}} \int_0^\infty \frac{du}{1+u^2} = \frac{2}{\pi\sqrt{\alpha^2+1}} \tan^{-1} u\Big|_{u=0}^\infty = \frac{1}{\sqrt{\alpha^2+1}}.$$

In the limit $\alpha \to 0$ we get $\int_0^\infty J_0(x)\, dx = 1$, in agreement with (22).

It may seem somewhat surprising that we obtain the expected result without using the Dirac delta function. However, the delta function appears in the limit of $\alpha \to 0$ since

$$\delta(x) = \frac{1}{\pi} \lim_{\alpha\to 0} \frac{\alpha}{\alpha^2 + x^2},$$

and the RHS of (24) becomes $\delta(\sin\theta)$. Thus, in this limit (23) reduces to (22).

In conclusion, we have shown that $\int_0^\infty J_0(x)\, dx = 1$. Then from (15) we get

$$\int_0^\infty J_{2n}(x)\, dx = 1, \quad n = 0, 1, 2... \tag{25}$$

Half-integer order: Bessel functions of half-integer order are elementary functions. For $\nu = 1/2$ we have

$$\int_0^\infty J_{1/2}(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx. \tag{26}$$

To evaluate this integral we first introduce the variable

$$x = u^2,$$

in terms of which (26) becomes

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \sin u^2\,du = 2\sqrt{\frac{2}{\pi}}\,\mathrm{Im}\left(\int_0^\infty e^{iu^2}\,du\right), \tag{27}$$

where we have used Euler's formula, and $\mathrm{Im}(\alpha)$ means the complex part of the complex number α.

Next, we make a second change of variables, viz., $i\,u^2 = -v^2$ so that $u = e^{i\pi/4}\,v$. In terms of this new variable, (27) becomes

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = 2\sqrt{\frac{2}{\pi}}\,\mathrm{Im}\left(e^{i\pi/4}\int_0^\infty e^{-v^2}\,dv\right). \tag{28}$$

Since

$$\int_0^\infty e^{-v^2}\,dv = \frac{\sqrt{\pi}}{2},$$

we obtain

$$\int_0^\infty J_{1/2}(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = 1. \tag{29}$$

In a similar way we get

$$\int_0^\infty J_{-1/2}(x)\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos x}{x}\,dx = 2\sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left(e^{i\pi/4}\int_0^\infty e^{-v^2}\,dv\right) = 1, \tag{30}$$

where $\mathrm{Re}(\alpha)$ means the real part of the complex number α.

Consequently, using (15) we get

$$\int_0^\infty J_{n-1/2}(x)\,dx = 1, \quad n = 0,\, 1,\, 2, \ldots \tag{31}$$

- Collecting the results given by (16), (25) and (31), we have shown that

$$\int_0^\infty J_\nu(x)\,dx = 1, \tag{32}$$

for any integer, zero, or half-integer $\nu \geq -1/2$.

To extend this result to an arbitrary ν we would have to use an integral representation for $J_\nu(x)$ valid for all values of ν. However, this is beyond the scope of the present note.

2.2 Finding the Factor $N(k, \nu)$

As mentioned earlier (2), in physical problems $x = k\rho$. Therefore, when x varies from 0 to infinity either k or ρ should be kept fixed. If k remains fixed and $0 \le \rho < \infty$, then (32) yields

$$\int_0^\infty J_\nu\left(k\rho\right)\, d\rho = \frac{1}{k}. \tag{33}$$

The same way, if ρ remains fixed and $0 \le k < \infty$, then

$$\int_0^\infty J_\nu\left(k\rho\right)\, dk = \frac{1}{\rho}. \tag{34}$$

With these two integrals at hand, the task of finding of $N(k, \nu)$ is straightforward. Namely, from (13) we get

$$N(k, \nu) = \frac{1}{k}. \tag{35}$$

Thus, N is independent of ν.

As a consequence of (35) the orthogonality integral (9) and the closure integral (12) become completely symmetric under the change $\rho \leftrightarrow k$, viz.,

$$\begin{aligned} \int_0^\infty \rho J_\nu\left(k\rho\right)\, J_\nu\left(k'\rho\right)\, d\rho &= \frac{\delta(k-k')}{k} \\ \int_0^\infty k J_\nu\left(k\rho\right)\, J_\nu\left(k\rho'\right)\, dk &= \frac{\delta(\rho-\rho')}{\rho}. \end{aligned} \tag{36}$$

Similarly, the Fourier-Bessel transform is also symmetric if we change $C_\nu(k) = kA_\nu(k)$, namely

$$f(\rho) = \int_0^\infty k\, A_\nu(k) J_\nu(k\rho)\, dk \tag{37}$$

$$A_\nu(k) = \int_0^\infty \rho f(\rho) J_\nu(k\rho)\, d\rho. \tag{38}$$

In this notation, $A_\nu(k)$ is the ith order Hankel transform of $f(\rho)$. Reciprocally, $f(\rho)$ is the ith order Hankel transform of $A_\nu(k)$; in short it is called the inverse Hankel transform.

To finish this Section, we would like to remark that we are perfectly aware that the integrals considered here are tabulated and can be found, for example, in Ref.[17]. If we use that table of integrals, or some computational software program, we can readily verify the validity of (5), for any ν. However, although fast and simple, such a verification would have no, or little, educational value because it teaches nothing about the Bessel functions nor develops technical or analytical skills [18].

3 Generalization to an Arbitrary $\nu \ge -1$

In this Section we use the asymptotic behavior of Bessel functions $J_\nu(x)$ to demonstrate that $N(k) = 1/k$, for all $\nu \ge -1$. In addition we explain why this factor is independent of ν.

To this end we consider the limit

$$\lim_{a\to\infty} \int_0^a \rho J_\nu(k\rho)\, J_\nu(k'\rho)\, d\rho = \int_0^\infty \rho J_\nu(k\rho)\, J_\nu(k'\rho)\, d\rho, \tag{39}$$

where the integral in the LHS is given by [19]

$$\int_0^a \rho J_\nu(k\rho)\, J_\nu(k'\rho)\, d\rho = \left. \frac{\rho\left[k\, J_\nu\left(k'\rho\right) J_{\nu+1}\left(k\rho\right) - k'\, J_\nu\left(k\rho\right) J_{\nu+1}\left(k'\rho\right)\right]}{k^2 - k'^2} \right|_{\rho=0}^{a}. \tag{40}$$

• To evaluate this expression at $\rho = 0$ we use the asymptotic expression of $J_\nu(k\rho)$ for small values of ρ

$$J_\nu(k\rho) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{k\rho}{2}\right)^\nu, \quad J_{-1}(k\rho) = -\frac{k\rho}{2}.$$

Substituting into (40) we get

$$\rho \to 0 : \text{RHS of } (40) \to \begin{cases} \frac{1}{2\,(\nu+1)\Gamma^2(\nu+1)} \left(\frac{kk'}{4}\right)^\nu \rho^{2(\nu+1)} & \text{for } \nu \neq -1 \\ \frac{lk}{16}\, \rho^4 & \text{for } \nu = -1 \end{cases} \tag{41}$$

Consequently, for any $\nu \geq -1$ the lower limit of (40) is zero.

• To compute the limit $a \to \infty$ we first evaluate (40) for $a \gg 1$. To this end we use the large ρ behavior of $J_\nu(k\rho)$, viz.,

$$J_\nu\left(k\rho\right) \to \sqrt{\frac{2}{\pi k\rho}} \cos\left[k\rho - \frac{(2\nu+1)\,\pi}{4}\right]. \tag{42}$$

Substituting this into the RHS of (40) we find

$$\begin{aligned} \text{RHS of } (40) \quad & \longrightarrow \frac{1}{\pi\sqrt{kk'}} \left\{ \frac{\sin\left[a\left(k-k'\right)\right]}{k-k'} - \right. \\ & \left. \frac{\sin \pi\nu \sin\left[a\left(k+k'\right)\right]}{k+k'} - \frac{\cos \pi\nu \cos\left[a\left(k+k'\right)\right]}{k+k'} \right\}. \end{aligned} \tag{43}$$

We note that the denominator in the last two expressions never vanishes because k and k' are positive.

Next, in the limit $a \to \infty$ we have

$$\lim_{a\to\infty} \frac{\sin a\left(k \pm k'\right)}{\pi\left(k \pm k'\right)} = \delta(k \pm k'), \tag{44}$$

$$\lim_{a\to\infty} \frac{\cos a\left(k + k'\right)}{\pi\left(k + k'\right)} \propto \delta(k + k'). \tag{45}$$

The first expression follows from (**??**). To verify (45) we take the limit in two steps. First, we introduce the small parameter ξ as follows

$$\cos a\left(k+k'\right) = \sin\left[a\left(k+k'\right) + \pi/2\right] = \sin\left[a\left(k+k'+\pi\xi/2\right)\right], \quad \xi = \frac{1}{a},$$

then we expand in series around $\xi = 0$, and rearrange terms to obtain

$$\left(1 - \frac{\pi}{2} + \frac{\pi^3}{48} - \cdots\right) \frac{\cos a\left(k+k'\right)}{\pi\left(k+k'\right)} = \left(1 - \frac{\pi^2}{8} + \frac{\pi^4}{384} - \cdots\right) \frac{\sin a\left(k+k'\right)}{\pi\left(k+k'\right)}, \quad a \gg 1.$$

Finally, in the limit $a \to \infty$ the RHS yields $\delta(k+k')$, which justifies (45).

Thus, in the limit $a \to \infty$ the second and third terms in (43) are proportional to $\delta(k+k')$, which is nonzero only when $(k+k')=0$. Given that k and k' are positive, in this limit their contribution to (39) is zero. The only term that survives is the first one, which is proportional to $\delta(k-k')$. Consequently, from (9) and (39) we get

$$N(k) = \frac{1}{k}, \quad \nu \geq -1. \tag{46}$$

3.1 Role of Phase Shift

To elucidate why $N(k)$ is independent of ν we write (42) as

$$J_\nu(k\rho) \to \sqrt{\frac{2}{\pi k\rho}} \cos(k\rho - \phi_\nu),$$

where the explicit form of the phase $\phi_\nu = \phi(\nu)$ is irrelevant. Using this we obtain

$$\begin{aligned} \text{RHS of (40)} \qquad \longrightarrow & \frac{1}{\pi\sqrt{kk'}} \left\{ \sin\delta \, \frac{\sin[a(k-k')]}{k-k'} + \right. \\ & \left. \frac{\sin\sigma \sin[a(k+k')]}{k+k'} + \frac{\cos\sigma\cos[a(k+k')]}{k+k'} + \frac{\cos\delta\cos[a(k-k')]}{k+k'} \right\}, \end{aligned}$$

where

$$\begin{aligned} \delta &= \phi_{\nu+1} - \phi_\nu \\ \sigma &= \phi_{\nu+1} + \phi_\nu. \end{aligned}$$

In the limit $a \to \infty$ the second and third term are proportional to $\delta(k+k')$ and, therefore, they do not contribute to (39) regardless of σ. Meanwhile, the first and fourth terms are proportional to $\delta(k-k')$. The fourth term vanishes when $\delta = \pi/2$. Consequently, the normalizing factor (46) turns out to be independent of ν because, for large arguments, $J_\nu(x)$ and $J_{\nu+1}(x)$ are shifted by a phase $\delta = \phi_{\nu+1} - \phi_\nu = \pi/2$, i.e., $J_\nu(k\rho) \to \sqrt{2/\pi k\rho}\cos(k\rho - \phi_\nu)$ and $J_{\nu+1}(k\rho) \to Y_\nu(k\rho) = \sqrt{2/\pi k\rho}\sin(k\rho - \phi_\nu)$.

4 Summary and Discussion

The rigorous proof of the orthogonality integral (5) is laborious and requires the use of mathematical techniques that, probably, are unfamiliar to most physics students, even at the graduate level.

In physics, to make calculations more feasible, we often reduce the problems by making simplifying hypothesis, which have reasonable physical meaning although not always well supported mathematically. This is epitomized by Problem 15.1.2 of Ref.[6]. In that problem the orthogonality integral (5) is obtained, in one line of calculation, from the Hankel transform and its inverse [7].

However, this is an exercise of circular reasoning [8] which does not prove the orthogonality integral. In practice the technique of orthogonal functions and expansions works in the opposite way: first we identify an appropriate set of orthonormal functions and then use it as a basis to expand other functions.

It should be noted that the orthogonality relationship (5) discussed here is between Bessel functions of the same order ν, which have the same functional form, but have distinct constants in the arguments. This is different from other common orthogonal functions used in Physics (e.g., Legendre/Hermite/Laguerre polynomials) where the orthogonality is between functions of different order - i.e., corresponding to different eigenvalues - which have different functional forms, but have the same argument.

In Section 2, we used elementary properties of Bessel functions $J_\nu(x)$ to compute $\int_0^\infty J_\nu(x)\,dx$. For ν integer, zero or half-integer $\geq -1/2$ we obtained $\int_0^\infty J_\nu(x)\,dx = 1$. From this we found the normalization factor $N(k) = 1/k$. As a consequence, the orthogonality integral and the closure integral (36) become symmetrical under the change $\rho \leftrightarrow k$. Also, the integrals (10)-(11) turn into Hankel transforms. The merit of this approach is its simplicity. The shortcoming is that our calculation is limited to the case where ν is an integer, zero, or a half-integer no less than $-1/2$.

In Section 3, using the asymptotic behavior of $J_\nu(k\rho)$, we obtained the factor $N(k)$ by means of direct calculation. The advantage of this approach, which is probably the one suggested by Jackson in his Problem 3.16, is that it works for any $\nu \geq -1$. Besides, it allows us to understand why this factor is independent of ν. Perhaps, the disadvantage could be the level of mathematical rigor used to evaluate the limit $a \to \infty$ in (43). Apart of this, one is left with the false sense that has succeeded in proving Hankel's integral theorem, when the truth is that much of the details of the demonstration are skipped when using the asymptotic form (42).

Although the techniques used in Sections 2 and 3 appear to be mutually different, they are complementary; the bridge between them is attained at a higher mathematical level when using contour integral representations of $J_\nu(x)$, which is beyond the scope of this work. From a pedagogical point of view they serve, not only, as a source of validation of Hankel's integral theorem, but also to develop valuable technical and analytical skills that can be applied to broader issues.

References

[1] B.G. Korenev, *Bessel Functions and Their Applications*, CRC Press LLC, 2002.

[2] A. Gray and G. B. Mathews, *A Treatise on Bessel Functions and Their Applications to Physics*, Dover Publications, New York, 1966.

[3] F. E. Relton, *Applied Bessel Functions*, Dover Publications, New York, 1965.

[4] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, U.K., 1966.

[5] The statement of Hankel's integral theorem, in Watson's notation (Ref.[4], Sec. 14.4) is: Let $F(R)$ be an arbitrary function of the real variable R subject to the condition that $\int_0^\infty F(R)\sqrt{R}\,dR$ exists and is absolutely convergent; and let the order ν of Bessel functions be not less than $-1/2$. Then

$$\int_0^\infty u\,du \int_0^\infty F(R) J_\nu(uR) J_\nu(ur)\,R\,dR = \frac{1}{2}\left[F(r+0) + F(r-0)\right],$$

provided that the positive number r lies inside an interval in which $F(R)$ has limited total fluctuation.

[6] George B. Arfken and Hans J. Weber, *Mathematical Methods for Physicists*, Sixth Edition. Elsevier Academic Press, 2005.

[7] For the sake of clarity and self-consistency, the Problem 15.1.2 of Ref.[6] is: Assuming the validity of Hankel transform-inverse transformation transform pair of equations

$$\begin{aligned} g(\alpha) &= \int_0^\infty f(t)\,J_n(\alpha\,t)\,t\,dt, \\ f(t) &= \int_0^\infty g(\alpha)\,J_n(\alpha t)\,\alpha\,d\alpha, \end{aligned}$$

show that the Dirac delta function has a Bessel integral representation

$$\delta(t-t') = t \int_0^\infty J_n(\alpha t)\ J_n(\alpha t')\,\alpha\, d\alpha.$$

[8] Circular Reasoning: Logical fallacy in which the reasoner begins with what they are trying to end with. ⟨http://en.wikipedia.org/wiki/Circular_reasoning⟩.

[9] John David Jackson, *Classical Electrodynamics*. Third Edition. Wiley, 1999.

[10] *Jackson Physics Problem Solutions*, ⟨http://www-personal.umich.edu/ pran/jackson/⟩.

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[12] Axel Velasco Chavez, *Classical Electrodynamics 3rd Ed J.D. Jackson - Solutions - 214 Pg*, ⟨http://www.scribd.com/doc/31752818/Classical-Electrodynamics-3rd-Ed-J-D-Jackson-Solutions-214-Pg⟩.

[13] Rudolph J. Magyar, *Solutions to classical electrodynamics 3rd edition by J.D. Jackson*, ⟨http://www.slideshare.net/PedroPrez19/solutions-to-classical-electrodynamics-3rd-edition-by-jd-jackson⟩.

[14] Homer Reid, *Solutions to Problems in Jackson, Classical - Homer Reid*, ⟨http://www.excelhonour.org/read/solutions-to-problems-in-jackson-classical-homer-reid-1589504/⟩.

[15] The word "any" assumes that the set of functions $\sqrt{\rho} J_\nu\left(x_{\nu i}\rho/a\right)$ - with ν fixed and $i = 1, 2, 3, \ldots$ - is complete.

[16] The straightforward demonstration of $\int_{-\infty}^{\infty} \frac{\sin\lambda x}{\pi x}\,dx = 1$ requires techniques of contour integration in the complex plane (see, e.g., A. Papoulis, *The Fourier integral and its applications*, McGraw-Hill, New York. 1962, p. 301). However, this can be demonstrated "the Feynman way" using nothing but elementary calculus by evaluating the integral $I(\alpha) = \int_0^\infty \frac{\sin\lambda x}{\pi x}\,e^{-\alpha x}\,dx$, with $\alpha > 0$ and the condition that $I(\infty) = 0$ ⟨http://ocw.mit.edu/courses/mathematics/18-304-undergraduate-seminar-in-discrete-mathematics-spring-2006/projects/integratnfeynman.pdf⟩. Assuming that we can differentiate under the integral we get $\frac{dI}{d\alpha} = -\int_0^\infty \frac{\sin\pi x}{\pi}\,e^{-\alpha x}\,dx = -\frac{\lambda}{\pi(\alpha^2+\lambda^2)}$. Consequently, $I(\alpha) = -\frac{1}{\pi}\,\tan^{-1}\left(\frac{\alpha}{\lambda}\right) + C$, where C is a constant of integration. Since $I(\infty) = 0$ we find $C = 1/2$. Thus $I(\alpha) = \frac{1}{2} - \frac{1}{\pi}\,\tan^{-1}\left(\frac{\alpha}{\lambda}\right)$. In the limit $\alpha \to 0$ we get $\int_0^\infty \frac{\sin\lambda x}{\pi x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin\lambda x}{\pi x}\,dx = \frac{1}{2}$.

[17] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*. Fifth Edition. Academic Press, 1994.

[18] This is author's opinion, as theoretical physicist. However, in modern times, the concept of what has or does not have educational value can be controversial. It all depends on the objectives of education, which in turn rely upon market requirements.

[19] The recipe for this equation is as follows: (i) Multiply (1), with $J_\nu(k\rho)$ replacing $F(\rho)$, by $J_\nu(k'\rho)$; (ii) In the equation thus obtained change $k \leftrightarrow k'$; (iii) Subtract the later from the former to obtain

$$\left(k^2 - k'^2\right)\ \rho\, J_\nu(k\rho)\, J_\nu(k'\rho) = \frac{d}{d\rho}\left[\rho\,\left(J_\nu(k\rho)\frac{dJ_\nu(k'\rho)}{d\rho} - J_\nu(k'\rho)\frac{dJ_\nu(k\rho)}{d\rho}\right)\right];$$

(iv) Use the recurrence relation $\frac{dJ_\nu(x)}{dx} = \frac{\nu}{x}\,J_\nu(x) - J_{\nu+1}(x)$; (v) Integrate from $\rho = 0$ to $\rho = a$.