

Note:derivation of 3-31

Email: jiaqi_wang@sjtu.edu.cn

A. Substitutue pressure modes for transverse velocity modes

1. mass equation

$$\begin{aligned} & \frac{dU_\alpha^a}{ds} - \Psi_{\{\alpha\}\beta}[r]U_\beta^a - ia\kappa\Psi_{\alpha\beta}[r(1 - \kappa\cos(\phi))]P_\beta^a - \Psi_{[\alpha]\beta}[r(1 - \kappa r\cos(\phi))]V_\beta^a - \Psi_{(\alpha)\beta}[(1 - \kappa r\cos(\phi))]W_\beta^a \\ &= \Psi_{\alpha\beta\gamma}[r(1 - \kappa r\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa V_\beta^{a-b}V_\gamma^b - ib\kappa W_\beta^{a-b}W_\gamma^b - ia\kappa \frac{B}{2A}P_\beta^{a-b}P_\gamma^b) \end{aligned} \quad (1)$$

Transform:

$$\begin{aligned} & \frac{dU_\alpha^a}{ds} := \underline{\underline{I}}u_\alpha^a \\ & -\Psi_{\{\alpha\}\beta}[r]U_\beta^a := \underline{\underline{-\Psi_{\{\alpha\}\beta}[r]}}u_\beta^a \rightarrow \mathcal{G} \\ & \begin{cases} -ia\kappa\Psi_{\alpha\beta}[r(1 - \kappa\cos(\phi))]P_\beta^a := \sum_{\beta=0}^{+\infty} \underline{\underline{-ia\kappa\Psi_{\alpha\beta}[r(1 - \kappa\cos(\phi))]}p_\beta^a \rightarrow \mathcal{M}_1 \\ -\Psi_{[\alpha]\beta}[r(1 - \kappa r\cos(\phi))]V_\beta^a := \sum_{\beta=0}^{+\infty} \underline{\underline{-\Psi_{[\alpha]\delta}[r(1 - \kappa r\cos(\phi))]}V_{\delta\beta}^a p_\beta^a \rightarrow \mathcal{M}_2 \\ -\Psi_{(\alpha)\beta}[(1 - \kappa r\cos(\phi))]W_\beta^a := \sum_{\beta=0}^{+\infty} \underline{\underline{-\Psi_{(\alpha)\delta}[(1 - \kappa r\cos(\phi))]}W_{\delta\beta}^a p_\beta^a \rightarrow \mathcal{M}_3 \end{cases} \\ & \Psi_{\alpha\beta\gamma}[r(1 - \kappa r\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa P_\beta^{a-b}P_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[(1 - \kappa r\cos(\phi))]}p_\beta^{a-b}p_\gamma^b \rightarrow \mathcal{B}_2 \\ & \Psi_{\alpha\beta\gamma}[r(1 - \kappa r\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa U_\beta^{a-b}U_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[(1 - \kappa r\cos(\phi))]}u_\beta^{a-b}u_\gamma^b \rightarrow \mathcal{A}_1 \\ & \Psi_{\alpha\beta\gamma}[r(1 - \kappa r\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa V_\beta^{a-b}V_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta,\epsilon=0}^{\infty} \underline{\underline{\Psi_{\alpha\delta\epsilon}[r(1 - \kappa r\cos(\phi))]}V_{\delta\beta}^{a-b}V_{\epsilon\gamma}^b p_\beta^{a-b}p_\gamma^b \rightarrow \mathcal{B}_3 \\ & \Psi_{\alpha\beta\gamma}[r(1 - \kappa r\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ib\kappa W_\beta^{a-b}W_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ib\kappa \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta,\epsilon=0}^{\infty} \underline{\underline{\Psi_{\alpha\delta\epsilon}[r(1 - \kappa r\cos(\phi))]}W_{\delta\beta}^{a-b}W_{\epsilon\gamma}^b p_\beta^{a-b}p_\gamma^b \rightarrow \mathcal{B}_4 \\ & \Psi_{\alpha\beta\gamma}[r(1 - \kappa r\cos(\phi))] \sum_{b=-\infty}^{+\infty} (-ia\kappa \frac{B}{2A}P_\beta^{a-b}P_\gamma^b) := \sum_{b=-\infty}^{+\infty} -ia\kappa \frac{B}{2A} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma}[(1 - \kappa r\cos(\phi))]}p_\beta^{a-b}p_\gamma^b \rightarrow \mathcal{B}_1 \end{aligned} \quad (2)$$

2. momentum equation I

$$\begin{aligned}
& \frac{d}{ds} P_\alpha^a - i a \kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos \phi)] U_\beta^a - \int_0^{2\pi} h h' [\psi_\beta \psi_\alpha]_{r=h} d\theta P_\beta^a - \Psi_{\{\alpha\}\beta} [r] P_\beta^a \\
= & \sum_{b=-\infty}^{+\infty} \underline{\text{term} \mathcal{D}1} : \Psi_{\{\alpha\}\beta\gamma} [r] U_\beta^{a-b} U_\gamma^a + \Psi_{[\alpha]\beta\gamma} [r(1 - \kappa r \cos \phi)] \underline{\underline{V_\beta^{a-b} U_\gamma^a}} + \Psi_{(\alpha)\beta\gamma} [(1 - \kappa r \cos \phi)] \underline{\underline{W_\beta^{a-b} U_\gamma^a}} \\
& + \underline{\text{term} (D1 + \mathcal{P}1)} : i(a) \kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos \phi)] P_\beta^{a-b} U_\gamma^b \\
& + \underline{\text{term} \mathcal{D}1} : - \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma} [r]) U_\beta^{a-b} U_\gamma^b - \Psi_{\alpha\beta\gamma} [r] \left(\frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) \\
& + \underline{\text{term} \mathcal{X}1} : \Psi_{\alpha\beta\gamma} [r \cos \phi] U_\beta^{a-b} \underline{\underline{V_\gamma^b}} - \kappa \Psi_{\alpha\beta\gamma} [r \sin \phi] U_\beta^{a-b} \underline{\underline{W_\gamma^b}} \\
& \quad \quad \quad (3)
\end{aligned}$$

Transform:

$$\begin{aligned}
& \frac{d}{ds} P_\alpha^a := \underline{\underline{I p_\alpha^a}} \\
& - i a \kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos \phi)] U_\beta^a := \underline{\underline{-i a \kappa \Psi_{\alpha\beta} [r(1 - \kappa r \cos \phi)] u_\beta^a}} \rightarrow \mathcal{N} \\
& - \int_0^{2\pi} h h' [\psi_\beta \psi_\alpha]_{r=h} d\theta P_\beta^a - \Psi_{\{\alpha\}\beta} [r] P_\beta^a := \underline{\underline{- \int_0^{2\pi} h h' [\psi_\beta \psi_\alpha]_{r=h} d\theta p_\beta^a - \Psi_{\{\alpha\}\beta} [r] p_\beta^a}} = \underline{\underline{- \Psi_{\{\alpha\}\beta} [r] p_\beta^a}} \rightarrow \mathcal{H} \\
& \Psi_{\{\alpha\}\beta\gamma} [r] \sum_{b=-\infty}^{+\infty} U_\beta^{a-b} U_\gamma^a := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\{\alpha\}\beta\gamma} [r] u_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{D} \\
& \Psi_{[\alpha]\beta\gamma} [r(1 - \kappa r \cos \phi)] \sum_{b=-\infty}^{+\infty} \underline{\underline{V_\beta^{a-b} U_\gamma^a}} := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta, \epsilon=0}^{\infty} \underline{\underline{\Psi_{[\alpha]\delta\epsilon} [r(1 - \kappa r \cos \phi)] \mathbf{V}_{\delta\beta}^{a-b} \mathbf{I}_{\epsilon\gamma}^b p_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{C}_4 \\
& \Psi_{(\alpha)\beta\gamma} [(1 - \kappa r \cos \phi)] \sum_{b=-\infty}^{+\infty} \underline{\underline{W_\beta^{a-b} U_\gamma^a}} := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \sum_{\delta, \epsilon=0}^{\infty} \underline{\underline{\Psi_{(\alpha)\delta\epsilon} [(1 - \kappa r \cos \phi)] \mathbf{W}_{\delta\beta}^{a-b} \mathbf{I}_{\epsilon\gamma}^b p_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{C}_5 \\
& i a \kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos \phi)] \sum_{b=-\infty}^{+\infty} P_\beta^{a-b} U_\gamma^b := i a \kappa \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\alpha\beta\gamma} [r(1 - \kappa r \cos \phi)] p_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{C}_3 \\
& - \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma} [r]) \sum_{b=-\infty}^{+\infty} U_\beta^{a-b} U_\gamma^b := \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{- \frac{\partial}{\partial s} (\Psi_{\alpha\beta\gamma} [r]) u_\beta^{a-b} u_\gamma^b}} \rightarrow \mathcal{D} \\
& - \Psi_{\alpha\beta\gamma} [r] \sum_{b=-\infty}^{+\infty} \left(\frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b} \right) := \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma} [r] [\mathbf{G}, \mathbf{I}] + \Psi_{\alpha\beta\gamma} [r] [\mathbf{I}, \mathbf{G}]) u_\beta^{a-b} u_\gamma^b \rightarrow \mathcal{D} \\
& \Psi_{\alpha\beta\gamma} [r \cos \phi] U_\beta^{a-b} \underline{\underline{V_\gamma^b}} - \kappa \Psi_{\alpha\beta\gamma} [r \sin \phi] U_\beta^{a-b} \underline{\underline{W_\gamma^b}} := \left\{ \sum_{b=-\infty}^{+\infty} \Psi_{\alpha\beta\gamma} [r \cos \phi] [\mathbf{I}, \mathbf{V}] - \kappa \Psi_{\alpha\beta\gamma} [r \sin \phi] [\mathbf{I}, \mathbf{W}] \right\} u_\gamma^{a-b} p_\beta^b \rightarrow \mathcal{C}_{6,7} \\
& \quad \quad \quad (4)
\end{aligned}$$

■Proof:

$$G_{\alpha\beta}^a = -\Psi_{\{\alpha\}\beta}[r] \\ -\Psi_{\alpha\beta\gamma}[r] \sum_{b=-\infty}^{+\infty} \left(\frac{dU_{\beta}^{a-b}}{ds} U_{\gamma}^b + \frac{dU_{\gamma}^b}{ds} U_{\beta}^{a-b} \right) := \sum_{b=-\infty}^{+\infty} (\Psi_{\alpha\beta\gamma}[r][\mathbf{G}, \mathbf{I}] + \Psi_{\alpha\beta\gamma}[r][\mathbf{I}, \mathbf{G}]) u_{\beta}^{a-b} u_{\gamma}^b \quad (5)$$

3. momentum equation II

$$\begin{aligned} & -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]V_{\beta}^a \\ & + \int_0^{2\pi} [\psi_{\alpha}\psi_{\beta}r(1-\kappa r\cos\phi)]_0^h d\theta P_{\beta}^a - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)]P_{\beta}^a - \Psi_{\alpha\beta}[1-2\kappa r\cos\phi]P_{\beta}^a \\ = & \underline{termD2} : \Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_{\beta}^{a-b}V_{\gamma}^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_{\beta}^{a-b}V_{\gamma}^a \\ & + \underline{term(D2 + P2)} : i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_{\beta}^{a-b}V_{\gamma}^b \\ & + \underline{termD2} : -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_{\beta}^{a-b}V_{\gamma}^b - \Psi_{\alpha\beta\gamma}[r]\left(\frac{dU_{\beta}^{a-b}}{ds}V_{\gamma}^b + \frac{dV_{\gamma}^b}{ds}U_{\beta}^{a-b}\right) \\ & + \underline{termX2} : \Psi_{\alpha\beta\gamma}[1-\kappa r\cos\phi]W_{\beta}^{a-b}W_{\gamma}^b - \kappa\Psi_{\alpha\beta\gamma}[r\cos\phi]U_{\beta}^{a-b}U_{\gamma}^b \end{aligned} \quad (6)$$

Transform:

$$\begin{aligned} & -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]V_{\beta}^a := -N_{\alpha\beta}^a V_{\beta}^a \\ & \left\{ \int_0^{2\pi} [\psi_{\alpha}\psi_{\beta}r(1-\kappa r\cos\phi)]_0^h d\theta - \Psi_{[\alpha]\beta}[r(1-\kappa r\cos\phi)] - \Psi_{\alpha\beta}[1-2\kappa r\cos\phi] \right\} P_{\beta}^a \\ \Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^a := & \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{\{\alpha\}\delta\epsilon}[r]\mathbf{I}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b u_{\beta}^{a-b} p_{\gamma}^b}}} = \sum_{b=-\infty}^{+\infty} \Psi_{\{\alpha\}\beta\gamma}[r][\mathbf{I}, \mathbf{V}] u_{\beta}^{a-b} p_{\gamma}^b \\ \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_{\beta}^{a-b}V_{\gamma}^a := & \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{[\alpha]\delta\epsilon}[r(1-\kappa r\cos\phi)]\mathbf{V}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b p_{\beta}^{a-b} p_{\gamma}^b}}} \\ \Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_{\beta}^{a-b}V_{\gamma}^a := & \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{\Psi_{(\alpha)\delta\epsilon}[1-\kappa r\cos\phi]\mathbf{W}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b p_{\beta}^{a-b} p_{\gamma}^b}}} \quad (7) \\ i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_{\beta}^{a-b}V_{\gamma}^b := & \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{ia\kappa\Psi_{\alpha\delta\epsilon}[r(1-\kappa r\cos\phi)]\mathbf{I}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b p_{\beta}^{a-b} p_{\gamma}^b}}} \\ -\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r])U_{\beta}^{a-b}V_{\gamma}^b := & \sum_{b=-\infty}^{+\infty} \sum_{\beta=0}^{+\infty} \sum_{\gamma=0}^{+\infty} \underline{\underline{-\frac{\partial}{\partial s}(\Psi_{\alpha\delta\epsilon}[r])\mathbf{I}_{\delta\beta}^{a-b}\mathbf{V}_{\epsilon\gamma}^b u_{\beta}^{a-b} p_{\gamma}^b}}} \\ -\Psi_{\alpha\beta\gamma}[r]\left(\frac{dU_{\beta}^{a-b}}{ds}V_{\gamma}^b + \frac{dV_{\gamma}^b}{ds}U_{\beta}^{a-b}\right) := & \Psi_{\alpha\beta\gamma}[r][G, I] + \Psi_{\alpha\beta\gamma}[r][I, G] \end{aligned}$$

4. momentum equation III

$$\begin{aligned}
& -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa r\cos\phi)]W_{\beta}^a \\
& -\Psi_{(\alpha)\beta}[(1-\kappa r\cos\phi)]P_{\beta}^a - \kappa\Psi_{\alpha\beta}[r\sin\phi]P_{\beta}^a \\
= & \underline{term\mathcal{D}2}:\Psi_{\{\alpha\}\beta\gamma}[r]U_{\beta}^{a-b}W_{\gamma}^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r\cos\phi)]V_{\beta}^{a-b}W_{\gamma}^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r\cos\phi)]W_{\beta}^{a-b}W_{\gamma}^a \\
& +\underline{term(D2+\mathcal{P}2)}:i(a)\kappa\Psi_{\alpha\beta\gamma}[r(1-\kappa r\cos\phi)]P_{\beta}^{a-b}W_{\gamma}^b \\
& +\underline{term\mathcal{D}2}:-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U_{\beta}^{a-b}V_{\gamma}^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_{\beta}^{a-b}}{ds}W_{\gamma}^b + \frac{dW_{\gamma}^b}{ds}U_{\beta}^{a-b})) \\
& +\underline{term\mathcal{X}2}:\kappa\Psi_{\alpha\beta\gamma}[r\sin\phi]U_{\beta}^{a-b}U_{\gamma}^b - \Psi_{\alpha\beta\gamma}[1-\kappa r\cos\phi]W_{\beta}^{a-b}V_{\gamma}^b
\end{aligned} \tag{8}$$