Note:Governing Euqations of General 3D duct flow

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I. Mass equation

Mass consevation:

$$-ia\kappa P^a + \nabla \cdot \mathbf{U}^a = \sum_{b=-\infty}^{+\infty} (-P^{a-b}\nabla \cdot \mathbf{U}^b - \mathbf{U}^{a-b} \cdot \nabla P^b - \frac{B}{2A}iakP^bP^{a-b}) \tag{1}$$

First, derivation of eq1:

We know that:

$$h_s = 1 - \kappa r cos(\phi), h_r = 1, h_\theta = r \tag{2}$$

Then,

$$\nabla \cdot \mathbf{U}^a = \frac{1}{h_1 h_2 h_3}[\frac{\partial(v_1 h_2 h_3)}{\partial w_1} + \frac{\partial(v_2 h_3 h_1)}{\partial w_2} + \frac{\partial(v_3 h_1 h_2)}{\partial w_3}]$$

$$= \frac{1}{r(1-\kappa r cos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}] \tag{3}$$

Thus, we have the mass equation, approximate RHS by:

$$\nabla \cdot \mathbf{U}^b = ib\kappa P^b + o(M^2)$$

$$\nabla P^b = ib\kappa \mathbf{U}^b + o(M^2) \tag{4}$$

Then we have

$$-ia\kappa P^a + \frac{1}{r(1-\kappa r cos(\phi))}[\frac{\partial(U^a r)}{\partial s} + \frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}]$$

$$= \sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A}iakP^bP^{a-b}) \tag{5}$$

The fourier harmonics are expanded as follows:

$$P^a = \sum_{\beta=0}^{\infty} P_\beta^a(s)\psi_\beta(s,r,\theta)$$

$$U^a = \sum_{\beta=0}^{\infty} U_\beta^a(s)\psi_\beta(s,r,\theta)$$

$$V^a = \sum_{\beta=0}^{\infty} V_\beta^a(s)\psi_\beta(s,r,\theta) \tag{6}$$

$$W^a = \sum_{\beta=0}^{\infty} W_\beta^a(s)\psi_\beta(s,r,\theta)$$

with normalized relation:

$$\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r\,dr\,d\theta = \delta_{\alpha\beta} \tag{7}$$

Reorganize the eq5:

$$-ia\kappa P^a(1-\kappa r cos(\phi)) + \frac{1}{r}\frac{\partial(U^a r)}{\partial s} + \frac{1}{r}\frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r} + \frac{1}{r}\frac{\partial(W^a(1-\kappa r cos(\phi)))}{\partial \theta}$$

$$= (1-\kappa r cos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A}iakP^bP^{a-b}) \tag{8}$$

Intergal and insert eq 6, 7 into eq 8:

1. the first term:

$$\int_0^{2\pi}\int_0^h \psi_\alpha r[-ia\kappa(1-\kappa r cos(\phi))P^a]dr\,d\theta$$

$$= -ia\kappa\sum_\beta^\infty\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r[(1-\kappa r cos(\phi))]dr\,d\theta P_\beta^a \tag{9}$$

$$= -ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P_\beta^a$$

$$(summation convention)$$

2. the second term:

$$\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(U^a r)}{\partial s}]dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta U_\beta^a)}{\partial s}dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r\frac{\partial(U_\beta^a)}{\partial s}dr\,d\theta + \int_0^{2\pi}\int_0^h \psi_\alpha r\frac{\partial(\psi_\beta)}{\partial s}dr\,d\theta U_\beta^a$$

$$= \frac{dU_\beta^a}{ds}\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta r\,dr\,d\theta + \int_0^{2\pi}\int_0^h r\frac{\partial(\psi_\alpha\psi_\beta)}{\partial s}dr\,d\theta U_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta r\frac{\partial(\psi_\alpha)}{\partial s}dr\,d\theta U_\beta^a \tag{10}$$

$$= \frac{dU_\beta^a}{ds}\delta_{\alpha\beta} + \frac{\partial(\int_0^{2\pi}\int_0^h r\psi_\alpha\psi_\beta dr\,d\theta)}{\partial s}U_\beta^a - \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha)}{\partial s}\psi_\beta r\,dr\,d\theta U_\beta^a$$

$$= \frac{dU_\alpha^a}{ds} + 0 - \Psi_{\{\alpha\}\beta}[r]U_\beta^a$$

3. the third term:

$$\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(V^a r(1-\kappa r cos(\phi)))}{\partial r}]dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^h \psi_\alpha\frac{\partial(\psi_\beta r(1-\kappa r cos(\phi)))}{\partial r}dr\,d\theta V_\beta^a$$

$$= \int_0^{2\pi}\int_0^h \frac{\partial(\psi_\alpha\psi_\beta r(1-\kappa r cos(\phi)))}{\partial r}dr\,d\theta V_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta r(1-\kappa r cos(\phi))\frac{\partial(\psi_\alpha)}{\partial r}dr\,d\theta V_\beta^a \tag{11}$$

$$= 0(periodic) - \Psi_{[\alpha]\beta}[r(1-\kappa r cos(\phi))]V_\beta^a$$

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4. the fourth term:

$$\int_0^{2\pi}\int_0^h \psi_\alpha r[\frac{1}{r}\frac{\partial(W^a(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta$$

$$=\int_0^{2\pi}\int_0^h \psi_\alpha[\frac{\partial(\psi_\beta(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta W_\beta^a$$

$$=\int_0^{2\pi}\int_0^h [\frac{\partial(\psi_\alpha\psi_\beta(1-\kappa rcos(\phi)))}{\partial\theta}]drd\theta W_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta(1-\kappa rcos(\phi))[\frac{\partial(\psi_\alpha)}{\partial\theta}]drd\theta W_\beta^a \quad (12)$$

$$=\int_0^h [\psi_\alpha\psi_\beta(1-\kappa rcos(\phi))]_0^{2\pi} dr W_\beta^a - \int_0^{2\pi}\int_0^h \psi_\beta(1-\kappa rcos(\phi))[\frac{\partial(\psi_\alpha)}{\partial\theta}]drd\theta W_\beta^a$$

$$= 0 - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]W_\beta^a$$

5. the RHS term:

$$\int_0^{2\pi}\int_0^h \psi_\alpha r[(1-\kappa rcos(\phi))\sum_{b=-\infty}^{+\infty}(-ib\kappa P^{a-b}P^b - ib\kappa U^{a-b}U^b - ib\kappa V^{a-b}V^b - ib\kappa W^{a-b}W^b - \frac{B}{2A}iakP^b P^{a-b})]drd\theta$$

$$=\int_0^{2\pi}\int_0^h \psi_\alpha\psi_\beta\psi_\gamma r(1-\kappa rcos(\phi))drd\theta$$

$$\sum_{b=-\infty}^{+\infty}(-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa V_\beta^{a-b}V_\gamma^b - ib\kappa W_\beta^{a-b}W_\gamma^b - ia\kappa\frac{B}{2A}P_\beta^b P_\gamma^{a-b})$$

$$=\Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa V_\beta^{a-b}V_\gamma^b - ib\kappa W_\beta^{a-b}W_\gamma^b - ia\kappa\frac{B}{2A}P_\beta^b P_\gamma^{a-b})$$

$$(13)$$

Finally, we obtain the mass equation in the form of eigenfunction, the idea is same as Galerkin method:

$$\frac{dU_\alpha^a}{ds} - \Psi_{\{\alpha\}\beta}[r]U_\beta^a - ia\kappa\Psi_{\alpha\beta}[r(1-\kappa cos(\phi))]P_\beta^a - \Psi_{[\alpha]\beta}[r(1-\kappa rcos(\phi))]V_\beta^a - \Psi_{(\alpha)\beta}[(1-\kappa rcos(\phi))]W_\beta^a$$

$$= \Psi_{\alpha\beta\gamma}[r(1-\kappa rcos(\phi))]\sum_{b=-\infty}^{+\infty}(-ib\kappa P_\beta^{a-b}P_\gamma^b - ib\kappa U_\beta^{a-b}U_\gamma^b - ib\kappa V_\beta^{a-b}V_\gamma^b - ib\kappa W_\beta^{a-b}W_\gamma^b - ia\kappa\frac{B}{2A}P_\beta^b P_\gamma^{a-b})$$

$$(14)$$

II.   Momentum equation

Momentum consevation:

$$-ia\kappa\mathbf{U}^a + \nabla P^a = \sum_{b=-\infty}^{\infty}(-\mathbf{U}^{a-b}\cdot\nabla\mathbf{U}^b + P^{a-b}\nabla P^b) \quad (15)$$

First, we know that

$$\nabla P^a = \sum_i \frac{1}{h_i}\frac{\partial f}{\partial w_i}\hat{h}_i = \frac{1}{1-\kappa rcos\phi}\frac{\partial P^a}{\partial s}\hat{e}_s + \frac{\partial P^a}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial P^a}{\partial\theta}\hat{e}_\theta \quad (16)$$

The RHS term is a bit complex, with the divergence of a vector U with its gradient, with

First, we know that

$$(\mathbf{v} \cdot \nabla)\mathbf{v}^b = \begin{cases} term1 : \mathcal{D}v_1^b + \frac{v_2^b}{h_2 h_1}(v_1 \frac{\partial h_1}{\partial \xi_2} - v_2 \frac{\partial h_2}{\partial \xi_1}) + \frac{v_3^b}{h_3 h_1}(v_1 \frac{\partial h_1}{\partial \xi_3} - v_2 \frac{\partial h_3}{\partial \xi_1}) \\ term2 : \mathcal{D}v_2^b + \frac{v_3^b}{h_3 h_2}(v_2 \frac{\partial h_2}{\partial \xi_3} - v_3 \frac{\partial h_3}{\partial \xi_2}) + \frac{v_1^b}{h_1 h_2}(v_2 \frac{\partial h_2}{\partial \xi_1} - v_1 \frac{\partial h_1}{\partial \xi_2}) \\ term3 : \mathcal{D}v_3^b + \frac{v_1^b}{h_1 h_3}(v_3 \frac{\partial h_3}{\partial \xi_1} - v_1 \frac{\partial h_1}{\partial \xi_3}) + \frac{v_2^b}{h_2 h_3}(v_3 \frac{\partial h_3}{\partial \xi_2} - v_3 \frac{\partial h_2}{\partial \xi_3}) \end{cases} \tag{17}$$

Besides,

$$\mathcal{D} = \frac{v_1}{h_1}\frac{\partial}{\partial \xi_1} + \frac{v_2}{h_2}\frac{\partial}{\partial \xi_2} + \frac{v_3}{h_3}\frac{\partial}{\partial \xi_3} \tag{18}$$

Thus, we have:

$$-\mathbf{U}^{a-b} \cdot \nabla\mathbf{U}^b =$$

$$-\sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{D}1 : \frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} + V^{a-b}\frac{\partial U^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2 : \frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial V^b}{\partial s} + V^{a-b}\frac{\partial V^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3 : \frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial W^b}{\partial s} + V^{a-b}\frac{\partial W^b}{\partial r} + \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases}$$

$$-\sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{X}1 : \frac{V^b}{1-\kappa r cos\phi}(U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial r} - V^{a-b}\frac{\partial 1}{\partial s}) + \frac{W^b}{r(1-\kappa r cos\phi)}(U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial \theta} - V^{a-b}\frac{\partial r}{\partial s}) \\ term\mathcal{X}2 : \frac{W^b}{r}(V^{a-b}\frac{\partial 1}{\partial \theta} - W^{a-b}\frac{\partial r}{\partial r}) + \frac{U^b}{(1-\kappa r cos\phi)1}(V^{a-b}\frac{\partial 1}{\partial s} - U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial r}) \\ term\mathcal{X}3 : \frac{U^b}{(1-\kappa r cos\phi)r}(W^{a-b}\frac{\partial h_3}{\partial s} - U^{a-b}\frac{\partial(1-\kappa r cos\phi)}{\partial \theta}) + \frac{V^b}{1h_3}(W^{a-b}\frac{\partial h_3}{\partial r} - V^{a-b}\frac{\partial 1}{\partial \theta}) \end{cases} =$$

$$\sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{D}1 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta} \\ term\mathcal{D}2 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial V^b}{\partial s} - V^{a-b}\frac{\partial V^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial V^b}{\partial \theta} \\ term\mathcal{D}3 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial W^b}{\partial s} - V^{a-b}\frac{\partial W^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial W^b}{\partial \theta} \end{cases}$$

$$+\sum_{b=-\infty}^{\infty} \begin{cases} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b \\ term\mathcal{X}2 : \frac{W^{a-b}W^b}{r} - \frac{\kappa cos\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b \\ term\mathcal{X}3 : \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b - \frac{W^{a-b}V^b}{r} \end{cases} \tag{19}$$

Finally, we could derive the momentum conservation equation, with final term approximate by

eq 4:

$$
\begin{cases}
-ia\kappa U^a + \dfrac{1}{1-\kappa r cos\phi}\dfrac{\partial P^a}{\partial s} \\[2ex]
-ia\kappa V^a + \dfrac{\partial P^a}{\partial r} \\[2ex]
-ia\kappa W^a + \dfrac{1}{r}\dfrac{\partial P^a}{\partial \theta}
\end{cases}
$$

$$
= \sum_{b=-\infty}^{\infty}
\begin{cases}
term\mathcal{D}1 : -\dfrac{U^{a-b}}{1-\kappa r cos\phi}\dfrac{\partial U^b}{\partial s} - V^{a-b}\dfrac{\partial U^b}{\partial r} - \dfrac{W^{a-b}}{r}\dfrac{\partial U^b}{\partial \theta} \\[2ex]
term\mathcal{D}2 : -\dfrac{U^{a-b}}{1-\kappa r cos\phi}\dfrac{\partial V^b}{\partial s} - V^{a-b}\dfrac{\partial V^b}{\partial r} - \dfrac{W^{a-b}}{r}\dfrac{\partial V^b}{\partial \theta} \\[2ex]
term\mathcal{D}3 : -\dfrac{U^{a-b}}{1-\kappa r cos\phi}\dfrac{\partial W^b}{\partial s} - V^{a-b}\dfrac{\partial W^b}{\partial r} - \dfrac{W^{a-b}}{r}\dfrac{\partial W^b}{\partial \theta}
\end{cases}
\quad (20)
$$

$$
+ \sum_{b=-\infty}^{\infty}
\begin{cases}
term\mathcal{X}1 : \dfrac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \dfrac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b \\[2ex]
term\mathcal{X}2 : \dfrac{W^{a-b}W^b}{r} - \dfrac{\kappa cos\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b \\[2ex]
term\mathcal{X}3 : \dfrac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}U^b - \dfrac{W^{a-b}V^b}{r}
\end{cases}
+ \sum_{b=-\infty}^{\infty}
\begin{cases}
ib\kappa P^{a-b}U^b \\[2ex]
ib\kappa P^{a-b}V^b \\[2ex]
ib\kappa P^{a-b}W^b
\end{cases}
$$

Now, we are going to project on ψ, it may be a little complex, we will doing step by step.

First, deal with the e^s term:

$$
-ia\kappa U^a + \frac{1}{1-\kappa r cos\phi}\frac{\partial P^a}{\partial s}
$$

$$
= \sum_{b=-\infty}^{\infty} termD1 : -\frac{U^{a-b}}{1-\kappa r cos\phi}\frac{\partial U^b}{\partial s} - V^{a-b}\frac{\partial U^b}{\partial r} - \frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta}
\quad (21)
$$

$$
+ \sum_{b=-\infty}^{\infty} term\mathcal{X}1 : \frac{\kappa cos\phi}{1-\kappa r cos\phi}U^{a-b}V^b - \frac{\kappa sin\phi}{(1-\kappa r cos\phi)}U^{a-b}W^b + ib\kappa^{a-b}U^b
$$

Multiply $(1-\kappa r cos\phi)$, we have:

$$
-ia\kappa(1-\kappa r cos\phi)U^a + \frac{\partial P^a}{\partial s}
$$

$$
= termD1 : \sum_{b=-\infty}^{\infty} -U^{a-b}\frac{\partial U^b}{\partial s} - (1-\kappa r cos\phi)V^{a-b}\frac{\partial U^b}{\partial r} - (1-\kappa r cos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta}
$$

$$
+ term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b +
\quad (22)
$$

$$
term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1-\kappa r cos\phi)P^{a-b}U^b
$$

$\int\int XXr\psi_\alpha dr d\theta$, we have:

$$
RHS = \int_0^{2\pi}\int_0^h [termD1 + term\mathcal{X}1 + term\mathcal{P}1]r\psi_\alpha dr d\theta
\quad (23)
$$

1. the first $\mathcal{D}1$ tems:

We ref the wiki $https://en.wikipedia.org/wiki/Leibniz\_integral\_rule$

5

General form: Differentiation under the integral sign:

$$\frac{d}{dx}(\int_{a(x)}^{b(x)} f(x,t)dt) - f(x,b(x)) \cdot \frac{d}{dx}b(x) + f(x,a(x)) \cdot \frac{d}{dx}a(x) = \int_{a(x)}^{b(x)} \frac{d}{dx}f(x,t)dt \qquad (24)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad (25)$$

For partial difference, for a given β, the derivation of the fucntion $g(\alpha) = \int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx$ is

$$\frac{d}{d\alpha}(\int_{a(\alpha)}^{b(\beta)} f(x,\alpha)dx) - 0 + \frac{da(\alpha)}{d\alpha}f(a(\alpha),\alpha) = \int_{a(\alpha)}^{b(\beta)} \frac{\partial}{\partial\alpha}f(x,\alpha)dx \qquad (26)$$

1.1

$$\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-rU^{a-b}\frac{\partial U^b}{\partial s}]\psi_\alpha dr d\theta$$

$$= \sum_{b=-\infty}^{\infty} -\int_0^{2\pi} \int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha]dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b\frac{\partial}{\partial s}[rU^{a-b}\psi_\alpha]dr d\theta$$

$$= \sum_{b=-\infty}^{\infty} -\frac{\partial}{\partial s}\int_0^{2\pi} \int_0^h [rU^{a-b}U^b\psi_\alpha]dr d\theta + \int_0^{2\pi} \frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}d\theta \qquad (27)$$

$$+ \int_0^{2\pi} \int_0^h \frac{r\partial U^{a-b}}{\partial s}U^b\psi_\alpha dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h \frac{\partial\psi_\alpha}{\partial s}rU^{a-b}U^b dr d\theta$$

here, we gives a relationship between U^a and V^a at the boundary which to dliminate V^a tems:

$$h'U^{a-b} = (1 - \kappa h cos\phi)V^{a-b} \qquad (28)$$

1.2

$$\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-(1-\kappa r cos\phi)V^{a-b}\frac{\partial U^b}{\partial r}]r\psi_\alpha dr d\theta$$

$$= -\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h \frac{\partial(r(1-\kappa r cos\phi)V^{a-b}U^b\psi_\alpha)}{\partial r}dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1-\kappa r cos\phi)V^{a-b}\psi_\alpha)}{\partial r}dr d\theta$$

$$= -\sum_{b=-\infty}^{\infty} \int_0^{2\pi} [(r(1-\kappa r cos\phi)V^{a-b}U^b)\psi_\alpha]_0^h d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1-\kappa r cos\phi)V^{a-b})}{\partial r}\psi_\alpha dr d\theta \tag{29}$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa r cos\phi)V^{a-b}dr d\theta$$

$$= -\sum_{b=-\infty}^{\infty} \int_0^{2\pi} [(hh'U^{a-b}U^b)\psi_\alpha]_0^h d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1-\kappa r cos\phi)V^{a-b})}{\partial r}\psi_\alpha dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa r cos\phi)V^{a-b}dr d\theta$$

1.3

$$\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [-(1-\kappa r cos\phi)\frac{W^{a-b}}{r}\frac{\partial U^b}{\partial \theta}]r\psi_\alpha dr d\theta$$

$$= -\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h \frac{\partial((1-\kappa r cos\phi)W^{a-b}U^b\psi_\alpha)}{\partial \theta}dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial((1-\kappa r cos\phi)W^{a-b})}{\partial \theta}\psi_\alpha dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta}((1-\kappa r cos\phi)W^{a-b})dr d\theta$$

$$= -\sum_{b=-\infty}^{\infty} \int_0^h [(1-\kappa r cos\phi)W^{a-b}U^b\psi_\alpha]_0^{2\pi} dr \tag{30}$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial((1-\kappa r cos\phi)W^{a-b})}{\partial \theta}\psi_\alpha dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta}((1-\kappa r cos\phi)W^{a-b})dr d\theta$$

$$= 0(periodic) + \int_0^{2\pi} \int_0^h U^b \frac{\partial((1-\kappa r cos\phi)W^{a-b})}{\partial \theta}\psi_\alpha dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(\psi_\alpha)}{\partial \theta}((1-\kappa r cos\phi)W^{a-b})dr d\theta$$

Combine together:

$$\underline{\int_0^{2\pi} \int_0^h [term\mathcal{D}1] r\psi_\alpha dr d\theta = \sum_{b=-\infty}^{\infty}}$$

$$\{(\int_0^{2\pi} \int_0^h \frac{\partial \psi_\alpha}{\partial s} r U^{a-b} U^b dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial r} r(1 - \kappa r cos\phi) V^{a-b} U^b dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial \theta} ((1 - \kappa r cos\phi) W^{a-b} U^b) dr d\theta)$$

$$+ (\int_0^{2\pi} \int_0^h U^b \frac{\partial U^{a-b}}{\partial s} r \psi_\alpha dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial(r(1 - \kappa r cos\phi) V^{a-b})}{\partial r} \psi_\alpha dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h U^b \frac{\partial((1 - \kappa r cos\phi) W^{a-b})}{\partial \theta} \psi_\alpha dr d\theta)$$

$$- \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta\}$$

(31)

We apply eq 5, find that:

$$-i(a-b)\kappa r(1 - \kappa r cos(\phi)) P^{a-b} + [\frac{\partial(U^{a-b} r)}{\partial s} + \frac{\partial(V^{a-b} r(1 - \kappa r cos(\phi)))}{\partial r} + \frac{\partial(W^{a-b}(1 - \kappa r cos(\phi)))}{\partial \theta}]$$

$$= o(M^2)$$

(32)

We have the second terms in eq(31) are equal to:

$$\int_0^{2\pi} \int_0^h U^b [-i(a-b)\kappa r(1 - \kappa r cos(\phi)) P^{a-b}] \psi_\alpha dr d\theta$$

$$= i(a-b)\kappa \Psi_{\alpha\beta\gamma} [r(1 - \kappa r cos\phi)] P_\beta^{a-b} U_\gamma^b$$

(33)

And, the longitudinal derivation s can also be expand about the duct modes, with note $[r], (\theta), \{s\}$:

$$\frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r U^{a-b} U^b \psi_\alpha] dr d\theta$$

$$= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] U_\beta^{a-b} U_\gamma^b dr d\theta$$

$$= \frac{\partial}{\partial s} (\int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta) U_\beta^{a-b} U_\gamma^b$$

$$+ \frac{\partial U_\beta^{a-b} U_\gamma^b}{\partial s} \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta$$

$$= \frac{\partial}{\partial s} (\int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta) U_\beta^{a-b} U_\gamma^b$$

$$+ (\frac{dU_\beta^{a-b}}{ds} U_\gamma^b + \frac{dU_\gamma^b}{ds} U_\beta^{a-b}) \int_0^{2\pi} \int_0^h [r \psi_\beta \psi_\gamma \psi_\alpha] dr d\theta$$

(34)

2. the second $\mathcal{X}1$ tems:

$$term\mathcal{X}1 : \sum_{b=-\infty}^{\infty} \kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b$$

$$\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [\kappa cos\phi U^{a-b}V^b - \kappa sin\phi U^{a-b}W^b] r\psi_\alpha dr d\theta$$

$$= \kappa \Psi_{\alpha\beta\gamma}[r cos\phi]U_\beta^{a-b}V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma}[r sin\phi]U_\beta^{a-b}W_\gamma^b$$

(35)

3. the second $\mathcal{P}1$ tems:

$$term\mathcal{P}1 : \sum_{b=-\infty}^{\infty} ib\kappa(1 - \kappa r cos\phi)P^{a-b}U^b$$

$$\sum_{b=-\infty}^{\infty} \int_0^{2\pi} \int_0^h [ib\kappa(1 - \kappa r cos\phi)P^{a-b}U^b] r\psi_\alpha dr d\theta$$

$$= ib\kappa \Psi_{\alpha\beta\gamma}[r(1 - \kappa r cos\phi)]P_\beta^{a-b}U_\gamma^b$$

(36)

4. The LHS terms:

$$\frac{\partial P^a}{\partial s} - ia\kappa(1 - \kappa r cos\phi)U^a$$

From 1.1 as example, we know that

$$\int_0^{2\pi} \int_0^h \frac{\partial}{\partial s}[rU^{a-b}U^b\psi_\alpha] dr d\theta$$

$$= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h [rU^{a-b}U^b\psi_\alpha] dr d\theta - \frac{dh(s)}{ds}[rU^{a-b}U^b\psi_\alpha]_{r=h}$$

(37)

4.1

$$\int_0^{2\pi} \int_0^h [\frac{\partial P^a}{\partial s}] r\phi_\alpha dr d\theta$$

$$= \int_0^{2\pi} \int_0^h [\frac{\partial(P_\beta^a\psi_\beta)}{\partial s}] r\psi_\alpha dr d\theta$$

$$= \int_0^{2\pi} \int_0^h [\frac{\partial(P_\beta^a\psi_\beta)}{\partial s} r\psi_\alpha] dr d\theta - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P_\beta^a$$

$$= \frac{\partial}{\partial s} \int_0^{2\pi} \int_0^h (P_\beta^a\psi_\beta) r\psi_\alpha dr d\theta - \frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h} - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P_\beta^a$$

$$= \frac{\partial}{\partial s}(P_\beta^a\delta_{\alpha\beta}) - \int_0^{2\pi} \frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h} d\theta - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P_\beta^a$$

$$= \frac{d}{ds}P_\alpha^a - \int_0^{2\pi} \frac{dh(s)}{ds}[P_\beta^a\psi_\beta r\psi_\alpha]_{r=h} d\theta - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}] r\psi_\beta dr d\theta P_\beta^a$$

(38)

4.2

$$\int_0^{2\pi} \int_0^h [-ia\kappa(1-\kappa r cos\phi)U^a]r\phi_\alpha dr d\theta$$

$$= -ia\kappa \Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U_\beta^a \tag{39}$$

Finally, putting all together becomes:

$$\frac{d}{ds}P_\alpha^a - \int_0^{2\pi} \frac{dh(s)}{ds}[P_\beta^a \psi_\beta r\psi_\alpha]_{r=h}d\theta - \int_0^{2\pi} \int_0^h [\frac{\partial(\psi_\alpha)}{\partial s}]r\psi_\beta dr d\theta P_\beta^a - ia\kappa \Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U_\beta^a$$

$$= \frac{d}{ds}P_\alpha^a - ia\kappa \Psi_{\alpha\beta}[r(1-\kappa r cos\phi)]U_\beta^a - \int_0^{2\pi} hh'[P_\beta^a \psi_\beta \psi_\alpha]_{r=h}d\theta - \Psi_{\{\alpha\}\beta}[r]P_\beta^a$$

$$= \sum_{b=-\infty}^{\infty}$$

$$(eq31): \{(\int_0^{2\pi} \int_0^h \frac{\partial\psi_\alpha}{\partial s}rU^{a-b}U^b dr d\theta$$

$$+ \int_0^{2\pi} \int_0^h \frac{\partial(\psi_\alpha)}{\partial r}r(1-\kappa r cos\phi)V^{a-b}U^b dr d\theta$$

$$+(eq33): i(a-b)\kappa \Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}U_\gamma^b$$

$$eq(34): -\frac{\partial}{\partial s}(\int_0^{2\pi} \int_0^h [r\psi_\beta \psi_\gamma \psi_\alpha]dr d\theta)U_\beta^{a-b}U_\gamma^b$$

$$-(\frac{dU_\beta^{a-b}}{ds}U_\gamma^b + \frac{dU_\gamma^b}{ds}U_\beta^{a-b})\int_0^{2\pi} \int_0^h [r\psi_\beta \psi_\gamma \psi_\alpha]dr d\theta$$

$$+eq(35): \kappa \Psi_{\alpha\beta\gamma}[r cos\phi]U_\beta^{a-b}V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma}[r sin\phi]U_\beta^{a-b}W_\gamma^b$$

$$+eq(36): ib\kappa \Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}U_\gamma^b$$

$$= (abbreviation):$$

$$\{\Psi_{\{\alpha\}\beta\gamma}[r]U_\beta^{a-b}U_\gamma^a + \Psi_{[\alpha]\beta\gamma}[r(1-\kappa r cos\phi)]V_\beta^{a-b}U_\gamma^a + \Psi_{(\alpha)\beta\gamma}[(1-\kappa r cos\phi)]W_\beta^{a-b}U_\gamma^a$$

$$+i(a-b)\kappa \Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}U_\gamma^b$$

$$-\frac{\partial}{\partial s}(\Psi_{\alpha\beta\gamma}[r]U_\beta^{a-b}U_\gamma^b - \Psi_{\alpha\beta\gamma}[r](\frac{dU_\beta^{a-b}}{ds}U_\gamma^b + \frac{dU_\gamma^b}{ds}U_\beta^{a-b})$$

$$+\kappa \Psi_{\alpha\beta\gamma}[r cos\phi]U_\beta^{a-b}V_\gamma^b - \kappa \Psi_{\alpha\beta\gamma}[r sin\phi]U_\beta^{a-b}W_\gamma^b$$

$$+ib\kappa \Psi_{\alpha\beta\gamma}[r(1-\kappa r cos\phi)]P_\beta^{a-b}U_\gamma^b \tag{40}$$