The p-adic Mehta Integral

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A statistical model of electrostatics on a line: Setup

1. Consider a system of N labeled point charges with random locations $x_1, \ldots, x_N \in \mathbb{R}$. Call each tuple $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ a **microstate**.

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4. Fix the unique constant $k > 0$ that makes $\frac{E(\boldsymbol{x})}{kT}$ dimensionless and define the **inverse temperature parameter** $\beta = \frac{1}{kT}$.

A statistical model of electrostatics on a line: Key idea

The energy E induces a probability distribution on the microstates:

$$d\mathbb{P}_\beta(\boldsymbol{x}) = \frac{1}{\mathcal{Z}_N(\beta)} e^{-\beta E(\boldsymbol{x})} d\boldsymbol{x} \quad \text{where} \quad \mathcal{Z}_N(\beta) = \int_{\mathbb{R}^N} e^{-\beta E(\boldsymbol{x})} \, d\boldsymbol{x} \ .$$

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- **Practical use:** Taking expectations with $d\mathbb{P}_\beta$ for various β reveals the system's observable/macroscopic behavior.
- **Important task:** Determine the domain and explicit form of the **canonical partition function** $\mathcal{Z}_N$.

Example: log-Coulomb gas in a harmonic well

The **Mehta integral** is

$$\mathcal{Z}_N(\beta) = \int_{\mathbb{R}^N} e^{-\frac{1}{2}\|\boldsymbol{x}\|^2} \prod_{i<j} |x_i - x_j|^\beta \, d\boldsymbol{x}.$$

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  - *harmonic potential* energies $\frac{1}{2\beta}x_i^2$ for $i = 1, 2, \ldots, N$ and
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Conjecture (Mehta and Dyson, early 1960's)

$$\mathcal{Z}_N(\beta) = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)} \quad \textit{if} \quad \mathrm{Re}(\beta) > -2/N \ .$$

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Theorem (Bombieri, late 1970's)

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p-adic log-Coulomb gas

- Suppose the charges have random locations $x_1, \ldots, x_N \in \mathbb{Q}_p$ instead.

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- Now $\mathbb{Q}_p^N$ is the space of microstates $\boldsymbol{x} = (x_1, \ldots, x_N)$ with standard norm $\| \cdot \|_p$ and Haar measure $d\boldsymbol{x}$ defined by

$$\|\boldsymbol{x}\|_p = \max_{1 \le i \le N} |x_i|_p \qquad \text{and} \qquad \int_{\mathbb{Z}_p^N} d\boldsymbol{x} = 1 \ ,$$

where $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \le 1\}$ is the ring of p-adic integers.

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- Choose an analogue $V(\boldsymbol{x})$ of the total harmonic potential, so that $e^{-\beta V(\boldsymbol{x})} = \rho(\|\boldsymbol{x}\|_p)$ is "nice" (like $e^{-\frac{1}{2}\|\boldsymbol{x}\|^2}$ for $\boldsymbol{x} \in \mathbb{R}^N$) and define

$$E(\boldsymbol{x}) = V(\boldsymbol{x}) - \sum_{i<j} \log |x_i - x_j|_p \ .$$

The p-adic Mehta integral

Main question:

$$\mathcal{Z}_N(\beta) = \int_{\mathbb{Q}_p^N} \rho(\|\boldsymbol{x}\|_p) \prod_{i<j} |x_i - x_j|_p^{\beta} \, d\boldsymbol{x} = ???$$

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- **Nice fact 1:** It suffices to compute $\int_{\mathbb{Z}_p^N} \prod_{i<j} |x_i - x_j|_p^\beta \, d\mathbf{x}$ because

$$\mathcal{Z}_N(\beta) = \left[\left(1 - p^{-(N+\binom{N}{2}\beta)}\right) \sum_{m\in\mathbb{Z}} \rho(p^m) p^{m(N+\binom{N}{2}\beta)} \right] \cdot \int_{\mathbb{Z}_p^N} \prod_{i<j} |x_i - x_j|_p^\beta \, d\mathbf{x}$$

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- **Nice fact 2:** $V_0 := \{\mathbf{x} \in \mathbb{Z}_p^N : x_i = x_j \text{ for some } i < j\}$ has measure 0, so we only need to do the integral over $\mathbb{Z}_p^N \setminus V_0$.

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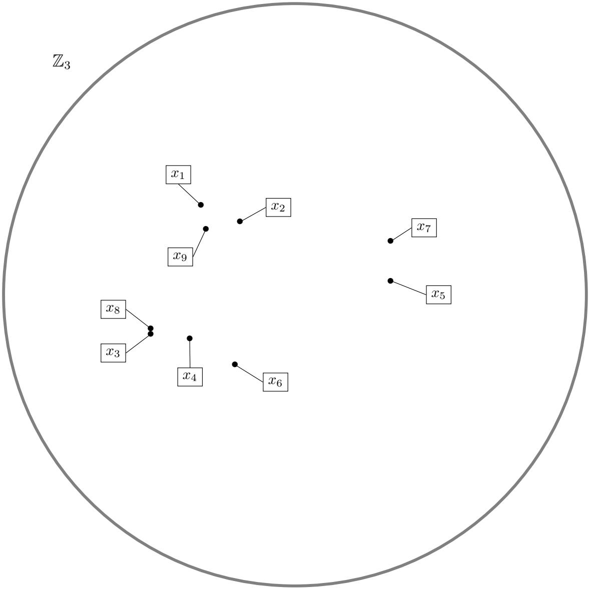
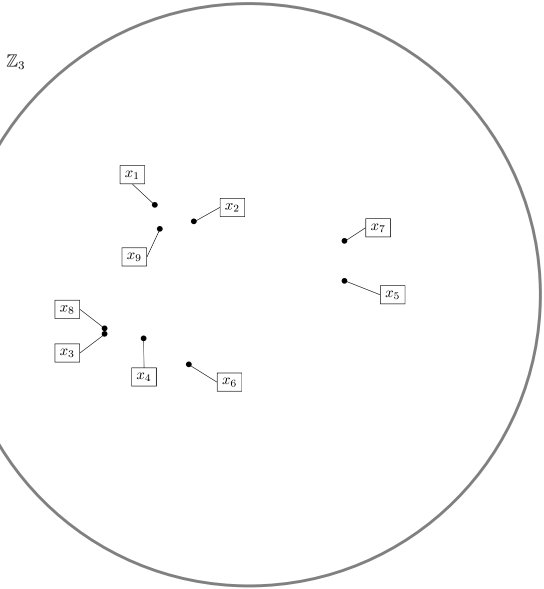
- **Question:** What do $\boldsymbol{x} \in \mathbb{Z}_p^N \setminus V_0$ look like?

What a microstate $\boldsymbol{x} \in \mathbb{Z}_3^9 \setminus V_0$ looks like

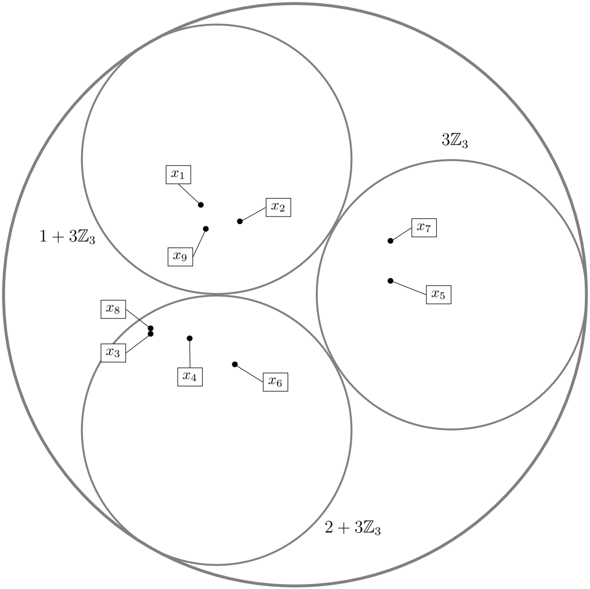


What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^0



$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^1



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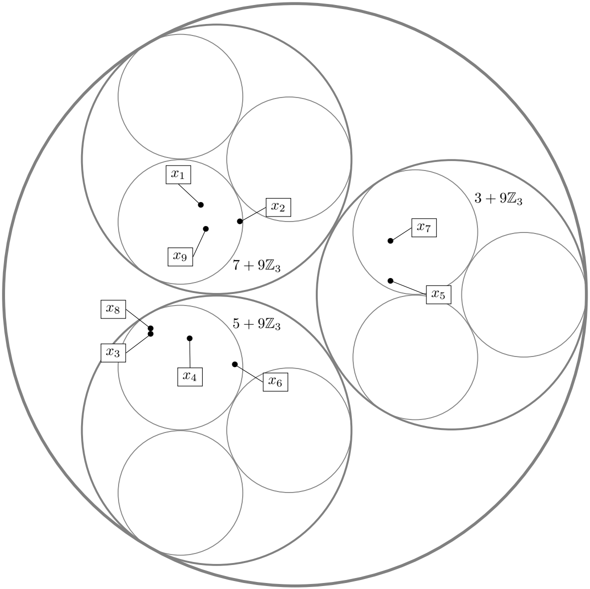

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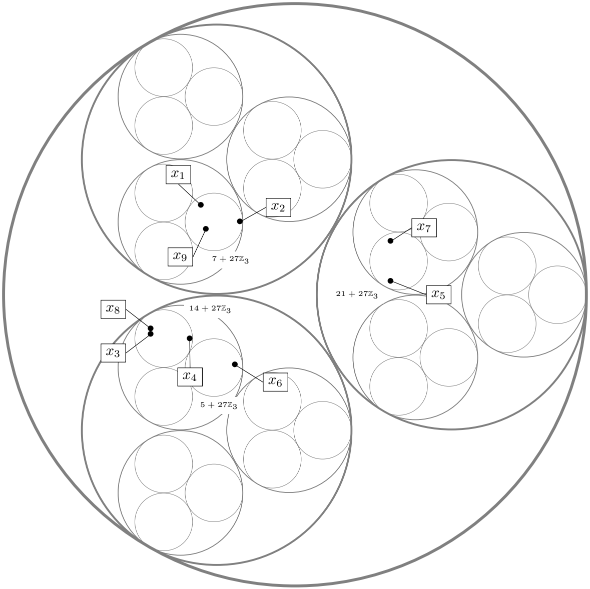
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^3



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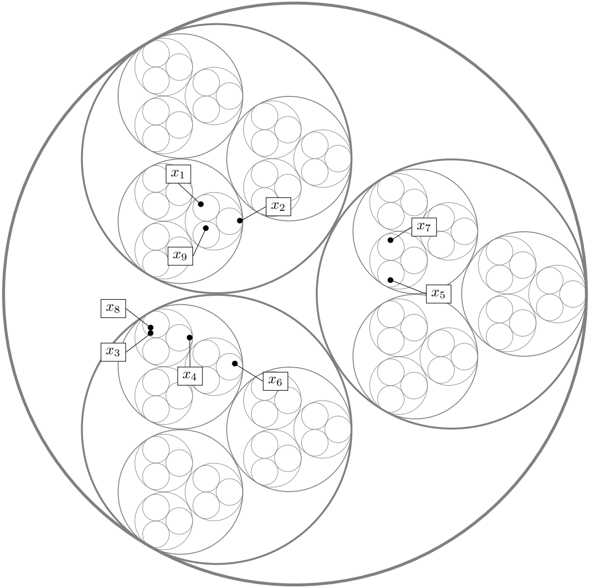
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^4



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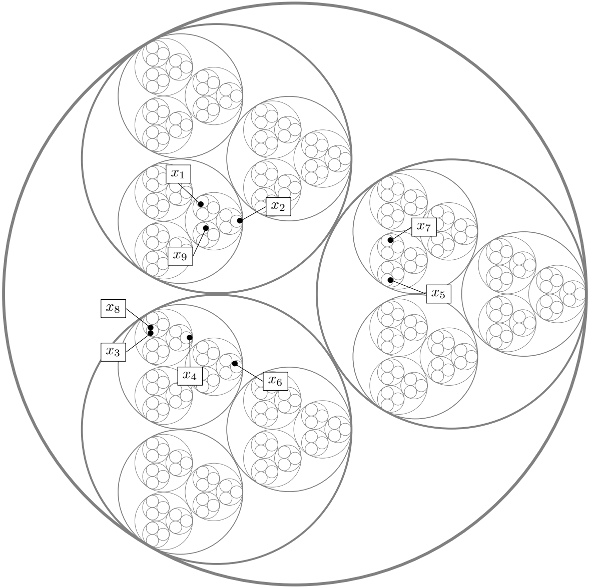
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like ...mod 3^5



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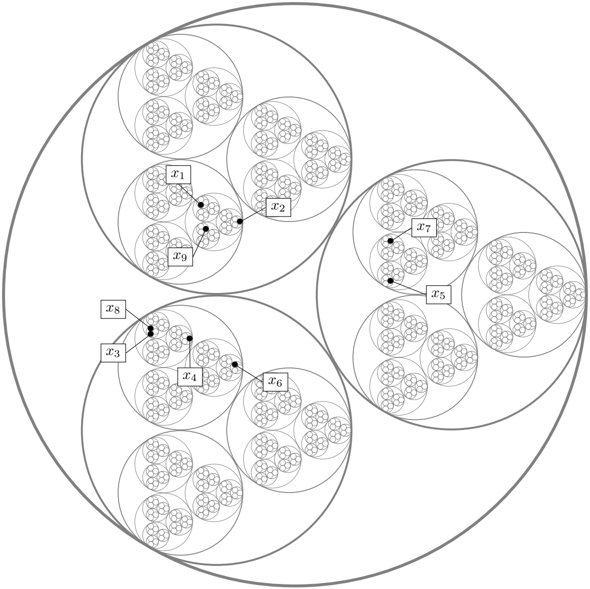
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like... mod 3^6



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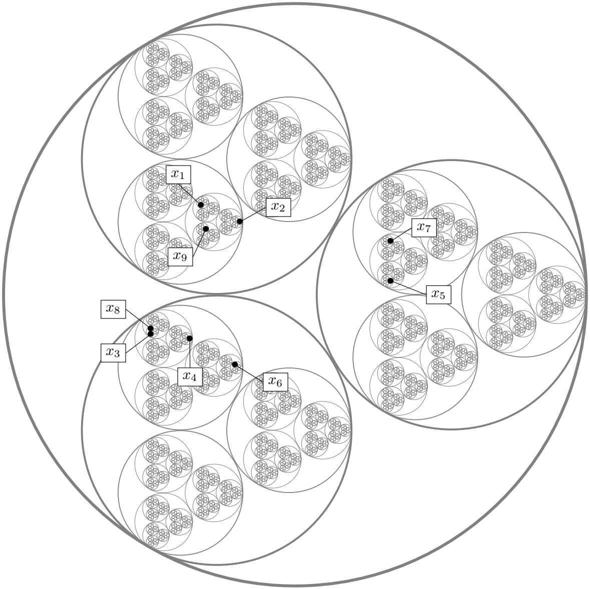
What a microstate $x \in \mathbb{Z}_3^9 \setminus V_0$ looks like... mod 3^7



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$\vdots$

What does the diagram tell us?

- The microstate $\boldsymbol{x} \in \mathbb{Z}_3^9 \setminus V_0$ determines a pair of tuples

$$\pitchfork = (\pitchfork_0, \pitchfork_1, \pitchfork_2, \pitchfork_3, \pitchfork_4) \qquad \text{and} \qquad \boldsymbol{n} = (n_0, n_1, n_2, n_3) :$$

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\end{aligned}$$

- **Note:** $p = 3$ is not special here. Many $\boldsymbol{x}$ in $\mathbb{Z}_5^9$, $\mathbb{Z}_7^9$, $\mathbb{Z}_{11}^9$, ..., etc., determine the same pair $(\boldsymbol{\pitchfork}, \boldsymbol{n})$ in the same way.

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$\pitchfork_3 = \{7\}\{5\}\{2\}\{1\}\{9\}\{6\}\{4\}\{3,8\}$	appeared $n_3 = 2$ times
$\pitchfork_4 = \{7\}\{5\}\{2\}\{1\}\{9\}\{6\}\{4\}\{8\}\{3\}$	appeared forever after.

- **Note:** $p = 3$ is not special here. Many $\boldsymbol{x}$ in $\mathbb{Z}_5^9$, $\mathbb{Z}_7^9$, $\mathbb{Z}_{11}^9$, ..., etc., determine the same pair $(\boldsymbol{\pitchfork}, \boldsymbol{n})$ in the same way.

- For any p, let $\mathcal{T}_p(\boldsymbol{\pitchfork}, \boldsymbol{n})$ be the set of all $\boldsymbol{x} \in \mathbb{Z}_p^9$ that determine $(\boldsymbol{\pitchfork}, \boldsymbol{n})$.

The value of $\prod_{i<j} |x_i - x_j|_p$ on $\mathcal{T}_p(\pitchfork, \boldsymbol{n})$

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- If $\mathcal{T}_p(\pitchfork, \boldsymbol{n}) \neq \varnothing$, then for every $\boldsymbol{x} \in \mathcal{T}_p(\pitchfork, \boldsymbol{n})$ we have

$$|x_i - x_j|_p = p^{1-(n_0+n_1+\cdots+n_{\ell_{ij}})} \quad \text{for } 1 \leq i < j \leq 9,$$

where $\ell_{ij} = \max\{\ell : i \text{ and } j \text{ are in a common } \lambda \in \pitchfork_\ell\}$.

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- This means any function that factors through $\boldsymbol{x} \mapsto (|x_i - x_j|_p)_{i<j}$ is constant on $\mathcal{T}_p(\pitchfork, \boldsymbol{n})$, with value explicitly determined by $(\pitchfork, \boldsymbol{n})$!

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- In particular, the product of the factors $|x_i - x_j|_p$ has a nice form:

Key Fact 1:

Every $\boldsymbol{x} \in \mathcal{T}_p(\pitchfork, \boldsymbol{n})$ satisfies

$$\prod_{i<j} |x_i - x_j|_p = p^{\binom{9}{2}} \prod_{\ell=0}^{3} p^{-\left[\sum_{\lambda \in \pitchfork_\ell} \binom{\#\lambda}{2}\right] n_\ell} = p^{-29}$$

The measure of $\mathcal{T}_p(\pitchfork, \boldsymbol{n})$

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Key Fact 2:

The set $\mathcal{T}_p(\pitchfork, \boldsymbol{n})$ is compact and open with Haar measure

$$M_\pitchfork(p) \cdot \prod_{\ell=0}^{3} p^{-\mathrm{rank}(\pitchfork_\ell) n_\ell} = (p-1)^6(p-2)^2 \cdot p^{-27}$$

Putting the Key Facts together

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For each partition $\pitchfork$ and $\beta \in \mathbb{C}$ it is convenient to define

$$E_{\pitchfork}(\beta) := \mathrm{rank}(\pitchfork) + \sum_{\lambda \in \pitchfork} \binom{\#\lambda}{2} \beta,$$

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Key Fact 1 + Key Fact 2 $\Longrightarrow$

$$\sum_{\boldsymbol{n} \in \mathbb{Z}_{>0}^4} \int_{\mathcal{T}_p(\pitchfork, \boldsymbol{n})} \prod_{i<j} |x_i - x_j|_p^\beta \, d\boldsymbol{x} = \sum_{\boldsymbol{n} \in \mathbb{Z}_{>0}^4} p^{\binom{9}{2}\beta} \cdot M_{\pitchfork}(p) \cdot \prod_{\ell=0}^{3} p^{-E_{\pitchfork_\ell}(\beta) n_\ell}$$

$$= p^{\binom{9}{2}\beta} \cdot M_{\pitchfork}(p) \cdot \prod_{\ell=0}^{3} \frac{1}{p^{E_{\pitchfork_\ell}(\beta)} - 1}$$

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***Punchline:** Summing over all possible $\pitchfork$ gives $\int_{\mathbb{Z}_p^9} \prod_{i<j} |x_i - x_j|_p^{\beta} \, d\boldsymbol{x}$!

Definition: Splitting chains

A tuple $\boldsymbol{\pitchfork} = (\pitchfork_0, \ldots, \pitchfork_L)$ of partitions of $\{1, 2, \ldots, N\}$ is called a **splitting chain** of order N and length $L(\boldsymbol{\pitchfork}) = L$ if

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- an associated rational expression

$$J_{\boldsymbol{\pitchfork},t}(\beta) := \frac{M_{\boldsymbol{\pitchfork}}(t)}{t^{N-1}} \prod_{\ell=1}^{L(\boldsymbol{\pitchfork})-1} \frac{1}{t^{E_{\pitchfork_\ell}(\beta)} - 1} \in \mathbb{Q}(t, t^{\beta}) \ .$$

The value of the p-adic Mehta integral

Theorem (W., 2020)

The p-adic Mehta integral converges for $\mathrm{Re}(\beta) > -2/N$ *with value*

$$\mathcal{Z}_N(\beta) = \left[\frac{1 - p^{-(N+\binom{N}{2}\beta)}}{1 - p^{-(N-1+\binom{N}{2}\beta)}} \sum_{m\in\mathbb{Z}} \rho(p^m) p^{m(N+\binom{N}{2}\beta)} \right] \sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,p}(\beta)$$

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Note: This is a corollary of a more general formula for

$$\int_{K^N} \rho(\|\boldsymbol{x}\|)\Big(\max_{i<j} |x_i - x_j|\Big)^a \Big(\min_{i<j} |x_i - x_j|\Big)^b \prod_{i<j} |x_i - x_j|^{s_{ij}}\, d\boldsymbol{x}$$

where K is an arbitrary nonarchimedean local field. This provides the canonical partition function for mixed-charge gases and joint moments of the diameter $\max_{i<j} |x_i - x_j|$ and minimal particle spacing $\min_{i<j} |x_i - x_j|$.

Examples: $N = 2$ and $N = 3$

$$\mathcal{Z}_2(\beta) = \frac{1 - p^{-(2+\beta)}}{1 - p^{-(1+\beta)}} \cdot \sum_{m \in \mathbb{Z}} \rho(p^m) p^{m(2+\beta)} \cdot \frac{p-1}{p}$$

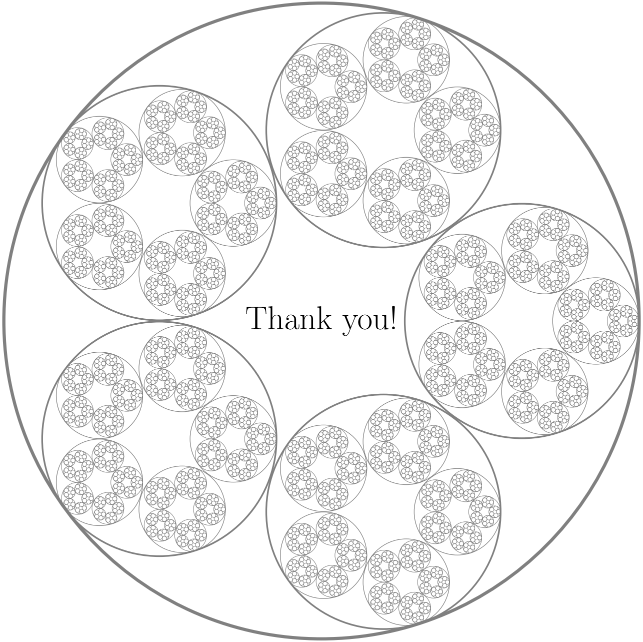
$$\begin{aligned} Z_3(\beta) = \frac{1 - p^{-(3+3\beta)}}{1 - p^{-(2+3\beta)}} &\cdot \sum_{m \in \mathbb{Z}} \rho(p^m) p^{m(3+3\beta)} \\ &\cdot \frac{1}{p^2} \left((p-1)(p-2) + 3(p-1)^2 \cdot \frac{1}{p^{1+\beta} - 1} \right) \end{aligned}$$

Example: $N = 4$

$$Z_4(\beta) = \frac{1 - p^{-(4+6\beta)}}{1 - p^{-(3+6\beta)}} \cdot \sum_{m \in \mathbb{Z}} \rho(p^m) p^{m(4+6\beta)}$$

$$\cdot \frac{1}{p^3} \Bigg\{ (p-1)(p-2)(p-3) + (p-1)^2(p-2) \left[4 \cdot \frac{1}{p^{2+3\beta}-1} + 6 \cdot \frac{1}{p^{1+\beta}-1} \right]$$

$$+ 3 \cdot (p-1)^3 \left[\frac{1}{p^{2+2\beta}-1} \left(1 + 2 \cdot \frac{1}{p^{1+\beta}-1} \right) + 4 \cdot \frac{1}{p^{2+3\beta}-1} \cdot \frac{1}{p^{1+\beta}-1} \right] \Bigg\}$$


Thank you!