RESEARCH STATEMENT

JOE WEBSTER

1. Introduction

My research lies in the intersection of number theory, combinatorics, and mathematical physics related to the *generalized Mehta integral*

$$\mathcal{Z}_N^{\rho}(K, \boldsymbol{s}) := \int_{K^N} \rho(\boldsymbol{x}) \prod_{i<j} |x_i - x_j|^{s_{ij}} \, d\boldsymbol{x} \ .$$

Here K is a local field with canonical absolute value $|\cdot|$, $d\boldsymbol{x}$ is an additive Haar measure on K^N, and $\boldsymbol{s} = (s_{ij})_{i<j}$ is a tuple of complex numbers. Integrals of this form are important in several areas of mathematics:

Random matrix theory. Suppose $K = \mathbb{R}$ and $\rho(\boldsymbol{x}) = \exp(-\frac{1}{2}\|\boldsymbol{x}\|^2)$, with $d\boldsymbol{x}$ and $\|\boldsymbol{x}\|$ the Lebesgue measure and Euclidean norm on $\mathbb{R}^N$. If $\beta > 0$ and $s_{ij} = \beta$ for all $i < j$, then $\mathcal{Z}_N^{\rho}(K, \boldsymbol{s})$ becomes the classical Mehta integral, which is the normalization for the probability density

$$P_{\beta}(\boldsymbol{x}) := \frac{1}{\mathcal{Z}_N^{\rho}(K, \boldsymbol{s})} e^{-\frac{1}{2}\|\boldsymbol{x}\|^2} \prod_{i<j} |x_i - x_j|^{\beta} \ .$$

In the 1960's, Mehta and Dyson showed that the eigenvalues $x_1, \ldots, x_N$ of the Gaussian orthogonal, unitary, and symplectic ensembles are respectively distributed according to P_1, P_2, and P_4 [DM63].

Statistical mechanics. A 1-dimensional *log-Coulomb gas* in K is a system of N charged particles with locations $x_1, \ldots, x_N \in K$, where each pair of particles has an associated *log-Coulomb potential* $-\log|x_i - x_j|$. The log-Coulomb potential arises naturally as a fundamental solution to a Laplace equation when $K = \mathbb{R}$ and statistical models of log-Coulomb gases in $\mathbb{R}$ have been studied extensively [Ser15, For10, Sin12, RSX13]. Torba and Zúñiga-Galindo showed that log-Coulomb potentials arise in essentially the same way for nonarchimedean K (such as $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$), initiating a very recent and active study of nonarchimedean log-Coulomb gases [ZnGT20, ZZL20, Sin20]. In all cases, the domain and formula for the function $\boldsymbol{s} \mapsto \mathcal{Z}_N^{\rho}(K, \boldsymbol{s})$ play key roles in the study of log-Coulomb gases, but few are known explicitly.

Local zeta functions. Suppose $K = \mathbb{Q}_p$ and ρ is the indicator function for $\mathbb{Z}_p^N$. In this case $\mathcal{Z}_N^{\rho}(K, \boldsymbol{s})$ is an example of a multivariate *local zeta function*. It is closely related to the Poincaré series that enumerates the sequence $N_m(\boldsymbol{d}) := \#\{\boldsymbol{x} \in (\mathbb{Z}/p^m\mathbb{Z})^N : \prod_{i<j}(x_i - x_j)^{d_{ij}} \equiv 0 \mod p^m\}$, where $\boldsymbol{d} = (d_{ij})_{i<j}$ is any tuple of nonnegative integers [Igu74, Igu75]. Loeser's multivariate generalization of Igusa's Theorem implies $\mathcal{Z}_N^{\rho}(K, \boldsymbol{s})$ is rational in the variables $p^{-s_{ij}}$ for all $i < j$, and it follows that the Poincaré series is a finite sum of geometric series in scaled powers of the same variables [Loe89]. Loeser's result holds for a very wide class of polynomials (in place of $\{x_i - x_j\}_{i<j}$) and any finite extension $K/\mathbb{Q}_p$ in place of $\mathbb{Q}_p$ (with a similar choice of ρ), and for such K it provides a semi-explicit formula for $\mathcal{Z}_N^{\rho}(K, \boldsymbol{s})$ in terms of a resolution of singularities. However, this method does not generalize easily to local fields with $\mathrm{char}(K) > 0$, as such resolutions are only guaranteed in characteristic 0 [Hir64].

2. Splitting chains and a uniform explicit formula

My thesis work establishes and examines an explicit combinatorial formula for the generalized Mehta integral $\mathcal{Z}_N^\rho(K, \boldsymbol{s})$ for all nonarchimedean local fields K at once, provided $\rho(\boldsymbol{x}) = \rho(\|\boldsymbol{x}\|)$ depends only on $\|\boldsymbol{x}\|$ and has mild growth and decay conditions as $\|\boldsymbol{x}\| \to 0$ and $\|\boldsymbol{x}\| \to \infty$. Henceforth we will assume K is such a field and that ρ has these properties (for instance, $K = \mathbb{Q}_p$ or $K = \mathbb{F}_p(\!(t)\!)$ and $\rho = \mathbf{1}_{[0,1]}$ or $\rho(t) = e^{-t^\alpha}$ with $\alpha > 0$). My explicit formula is built on a collection of chains in the lattice of partitions of $[N] = \{1, 2, \ldots, N\}$ ordered by refinement $\leq$.

Definition 2.1. *A* splitting chain *of order $N \geq 2$ is a finite tuple $\pitchfork = (\pitchfork_0, \pitchfork_1, \ldots, \pitchfork_L)$ of partitions of $[N]$ satisfying the proper refinement condition*

$$[N] = \pitchfork_0 > \pitchfork_1 > \cdots > \pitchfork_L = \{1\}\{2\}\ldots\{N\}\ .$$

Let $\mathcal{S}_N$ be the set of all splitting chains of order N.

In [Web20] we attach an explicit convex open region $\Omega_N \subset \mathbb{C}^{\binom{N}{2}}$ and an explicit family of rational expressions $\{J_{\pitchfork,t}(\boldsymbol{s})\}_{\pitchfork \in \mathcal{S}_N} \subset \mathbb{Q}(t, t^{s_{12}}, t^{s_{13}}, \ldots, t^{s_{(N-1)N}})$ to the finite set $\mathcal{S}_N$ and prove the following:

Result 2.2. *Suppose K is a nonarchimedean local field and let q be the cardinality of its residue field. The region Ω_N is the largest open domain on which the integral $\mathcal{Z}_N^\rho(K, \boldsymbol{s})$ converges absolutely, and on each compact subset of Ω_N it converges uniformly to*

$$\mathcal{Z}_N^\rho(K, \boldsymbol{s}) = \frac{1 - q^{-z}}{1 - q^{-(z-1)}} \sum_{m \in \mathbb{Z}} \rho(q^m) q^{mz} \cdot \sum_{\pitchfork \in \mathcal{S}_N} J_{\pitchfork,q}(\boldsymbol{s})$$

where $z = N + \sum_{i<j} s_{ij}$. In particular, $\mathcal{Z}_N^\rho(K, \boldsymbol{s})$ is holomorphic in Ω_N, and the right-hand side determines its meromorphic continuation.

A key feature of this result is that Ω_N and $\{J_{\pitchfork,t}(\boldsymbol{s})\}_{\pitchfork \in \mathcal{S}_N}$ are independent of ρ and K, and thus may be attributed directly to the family of polynomials $\{x_i - x_j\}_{i<j}$. This suggests that Result 2.2 belongs in the context of *motivic integration*, a "field independent" theory of measures and integration appearing in many recent and diverse publications [SVPV07, HK09, JKU19]. This theory was introduced by Kontsevich and subsequently developed by Denef and Loeser [DL99, DL02], and is deeply connected to algebraic geometry and model theory [Hal05, Hal18]. The uniformity of motivic specializations to $K = \mathbb{Q}_p$ often relies on an assumption that p is "sufficiently large", and a similar phenomenon appears in Result 2.2: There always exist $\pitchfork \in \mathcal{S}_N$ such that $J_{\pitchfork,t}(\boldsymbol{s})$ vanishes when evaluated at $t = p$, unless $p \geq N$. I plan to investigate the following problem:

Problem 2.3. *Determine the precise motivic interpretation of $\{J_{\pitchfork,t}(\boldsymbol{s})\}_{\pitchfork \in \mathcal{S}_N}$. In particular, determine the geometric and/or model-theoretic significance of the vanishing of certain $J_{\pitchfork,p}(\boldsymbol{s})$ at primes $p < N$.*

3. Thermodynamic limits and joint moments

In [ZZL20], Cardenal, Zambrano-Luna, and Zúñiga-Galindo defined the dimensionless free energy per particle for a log-Coulomb gas in K attached to the star graph corresponding to the potentials $\{-\log|x_1 - x_j| : 2 \leq j \leq N\}$, in which $N/2$ particles have charge 1 and $N/2$ have charge -1. They computed its thermodynamic limit and used it to determine the gas' phase transition temperature and mean energy per particle. I am working on generalizing these results to the complete graph corresponding to the full set of potentials $\{-\log|x_i - x_j| : 1 \leq i < j \leq N\}$. The crux of this generalization is the following problem:

Problem 3.1. *Fix a particle density $d > 0$ and an inverse temperature $\beta > 0$. Suppose N is a positive even number of particles such that $N/2$ have charge 1 and $N/2$ have charge -1, and define $\rho_N := \mathbf{1}_{[0,N/d]}$ and $s_{ij} = e_i e_j \beta$ where e_i and e_j are the charges of the ith and jth particles. Determine the asymptotic behavior of $\mathcal{Z}_N^{\rho_N}(K, \boldsymbol{s})$ as $N \to \infty$.*

I suspect that those $J_{\pitchfork,q}(\boldsymbol{s})$ with "highly symmetric" splitting chains dominate the asymptotic behavior of $\mathcal{Z}_N^{\rho_N}(K,\boldsymbol{s})$. The challenge in proving this lies in the fact that both the size of the family $\{J_{\pitchfork,t}(\boldsymbol{s})\}_{\pitchfork\in\mathcal{S}_N}$ and the complexity of its elements grow rapidly with N, which is also a central matter in the following problem:

Problem 3.2. *Fix $\rho = \mathbf{1}_{[0,1]}$, an inverse temperature $\beta > 0$, and a fugacity parameter f, and let $s_{ij} = \beta$ for all $i < j$. Find the explicit formula for the* grand canonical partition function

$$\mathcal{Z}(\beta, f) = \sum_{N=0}^{\infty} \mathcal{Z}_N^{\rho}(K,\boldsymbol{s})\frac{f^N}{N!} \ .$$

The recent paper [Sin20] provides the high and low temperature limits of $\mathcal{Z}(\beta,f)$, namely

$$\lim_{\beta\to 0}\mathcal{Z}(\beta,f) = e^f \qquad \text{and} \qquad \lim_{\beta\to\infty}\mathcal{Z}(\beta,f) = \left(1+\frac{f}{q}\right)^q \ ,$$

and the solution to Problem 3.2 will explain precisely how these are interpolated by $\beta\in(0,\infty)$.

Simple modifications of the integral $\mathcal{Z}_N^{\rho}(K,\boldsymbol{s})$ and the family $\{J_{\pitchfork,t}(\boldsymbol{s})\}_{\pitchfork\in\mathcal{S}_N}$ give a generalization of Result 2.2 that provides the explicit joint moments of a log-Coulomb gas' diameter $\max_{i<j}|x_i - x_j|$ and minimum particle spacing $\min_{i<j}|x_i-x_j|$. More precisely, in [Web20] we attach to each $\pitchfork\in\mathcal{S}_N$ and $b\in\mathbb{C}$ an explicit rational expression $J_{\pitchfork,t}(b,\boldsymbol{s})\in\mathbb{Q}(t,t^b,t^{s_{12}},t^{s_{13}},\ldots,t^{s_{(N-1)N}})$ that specializes to $J_{\pitchfork,t}(0,\boldsymbol{s}) = J_{\pitchfork,t}(\boldsymbol{s})$, and we prove the following:

Result 3.3. *Suppose K is a nonarchimedean local field and let q be the cardinality of its residue field. If $a,b\in\mathbb{C}$, ρ is nonnegative and not identically zero, and $s_{ij} = e_ie_j\beta$ with arbitrary charges $e_1,\ldots,e_N>0$ and sufficiently large inverse temperature $\beta>0$, then*

$$\mathbb{E}\left[\big(\max_{i<j}|x_i-x_j|\big)^a\big(\min_{i<j}|x_i-x_j|\big)^b\right] = \frac{\frac{1-q^{-(a+b+z)}}{1-q^{-(a+b+z-1)}}\sum_{m\in\mathbb{Z}}\rho(q^m)q^{m(a+b+z)}}{\frac{1-q^{-z}}{1-q^{-(z-1)}}\sum_{m\in\mathbb{Z}}\rho(q^m)q^{mz}}\cdot\frac{\sum_{\pitchfork\in\mathcal{S}_N}J_{\pitchfork,q}(b,\boldsymbol{s})}{\sum_{\pitchfork\in\mathcal{S}_N}J_{\pitchfork,q}(\boldsymbol{s})}$$

where $z = N+\sum_{i<j}e_ie_j\beta$. Moreover, if ρ has bounded support, then this expected value converges to the product of an integer power of q^{a+b} and a finite weighted average of integer powers of q^{-b} in the low-temperature limit ($\beta\to\infty$). This weighted average is identically 1 if $q\geq N$.

4. Combinatorics and functional equations

Motivated by Problems 2.3-3.2, I am currently investigating the combinatorial structure of splitting chains in their own right. Given $N\geq 2$, each $\pitchfork = (\pitchfork_0,\ldots,\pitchfork_L)\in\mathcal{S}_N$ has an associated *length* $L(\pitchfork) := L$ and set of *branches* $\mathcal{B}(\pitchfork) := (\pitchfork_0\cup\cdots\cup\pitchfork_{L-1})\setminus\pitchfork_L$ comprised of the distinct non-singleton parts λ appearing in the partitions $\pitchfork_0,\ldots,\pitchfork_{L-1}$. For instance, the three splitting chains in $\mathcal{S}_4$ that share the branch set $\{\{1,2,3,4\},\{1,2\},\{3,4\}\}$ can be written vertically,

{1,2,3,4}	{1,2,3,4}	{1,2,3,4}
{1,2}{3,4}	{1,2}{3,4}	{1,2}{3,4}
{1,2}{3}{4}	{1}{2}{3,4}	{1}{2}{3}{4}
{1}{2}{3}{4}	{1}{2}{3}{4}	

and their lengths are respectively 3, 3, and 2. The rightmost, say $\pitchfork^*$, is an example of a *reduced* splitting chain. That is, each branch $\lambda\in\mathcal{B}(\pitchfork^*)$ appears in exactly one of the partitions $\pitchfork_\ell^*$. For general $N\geq 2$, the set $\mathcal{R}_N$ of all reduced splitting chains of order N corresponds in an obvious way to the collection of branch sets $\{\mathcal{B}(\pitchfork) : \pitchfork\in\mathcal{S}_N\}$. Thus $\mathcal{S}_N$ partitions into $\simeq$-equivalence classes, where $\pitchfork\simeq\pitchfork' \iff \mathcal{B}(\pitchfork) = \mathcal{B}(\pitchfork')$, and each class contains a unique $\pitchfork^*\in\mathcal{R}_N$. The three examples above form one such class in $\mathcal{S}_4$. The relationship between the members of a $\simeq$-equivalence class is reflected by the expressions $J_{\pitchfork,t}(b,\boldsymbol{s})$ in a remarkable way [Web20].

Result 4.1. *Define $I_{\pitchfork^*,t}(b,\boldsymbol{s}) := \sum_{\pitchfork\simeq\pitchfork^*} J_{\pitchfork,t}(b,\boldsymbol{s})$ for $\pitchfork^* \in \mathcal{R}_N$ and $b \in \mathbb{C}$. When $b=0$ and $t=q$ is any prime power, the sum collapses into the simple product*

$$I_{\pitchfork^*,q}(0,\boldsymbol{s}) = \frac{M_{\pitchfork^*}(q)}{q^{N-1}} \prod_{\lambda\in\mathcal{B}(\pitchfork^*)\setminus\{[N]\}} \frac{1}{q^{e_\lambda(\boldsymbol{s})}-1}$$

where $M_{\pitchfork^*}(t) \in \mathbb{Z}[t]$ is a product of falling factorials and $e_\lambda(\boldsymbol{s}) = \#\lambda - 1 + \sum_{i,j\in\lambda:i<j} s_{ij}$. In particular, for $N \geq 4$ the formula in Result 2.2 is simplified via $\sum_{\pitchfork\in\mathcal{S}_N} J_{\pitchfork,q}(\boldsymbol{s}) = \sum_{\pitchfork^*\in\mathcal{R}_N} I_{\pitchfork^*,q}(0,\boldsymbol{s})$.

We obtained the formula for $I_{\pitchfork^*,t}(0,\boldsymbol{s})$ in Result 4.1 for $t=q$ by recognizing it as a summand of $\mathcal{Z}_N^\rho(K,\boldsymbol{s})$. However, the formula holds for arbitrary t in place of q in all examples we've checked by hand, which strongly suggests the existence of a purely combinatorial proof. Finding this proof will clarify the role of $\min_{i<j}|x_i - x_j|$ in Result 3.3 and precisely describe which poles of the meromorphic functions $\boldsymbol{s} \mapsto I_{\pitchfork^*,q}(b,\boldsymbol{s})$ cancel when b becomes 0.

Problem 4.2. *Use properties of $\simeq$-equivalence classes to extend Result 4.1 to arbitrary t directly, without reference to the integral $\mathcal{Z}_N^\rho(K,\boldsymbol{s})$. Adapt this proof to extend Result 4.1 to general $b \in \mathbb{C}$.*

It is not hard to show that the families of auxiliary expressions $F_{\pitchfork,t}(b,\boldsymbol{s}) = t^{N-1}/M_{\pitchfork}(t)\cdot J_{\pitchfork,t}(\boldsymbol{s})$ attached to $\pitchfork \in \mathcal{S}_N$ and $f_{\pitchfork^*,t}(\boldsymbol{s}) = t^{N-1}/M_{\pitchfork^*}(t)\cdot I_{\pitchfork^*,t}(0,\boldsymbol{s})$ attached to $\pitchfork^* \in \mathcal{R}_N$ satisfy

$$F_{\pitchfork,t^{-1}}(b,\boldsymbol{s}) = (-1)^{L(\pitchfork)-1}\sum_{\pitchfork'\leq_{\mathcal{S}}\pitchfork} F_{\pitchfork',t}(b,\boldsymbol{s}) \qquad \text{and} \qquad f_{\pitchfork^*,t^{-1}}(b,\boldsymbol{s}) = (-1)^{\#\mathcal{B}(\pitchfork^*)-1}\sum_{\pitchfork'\leq_{\mathcal{R}}\pitchfork^*} f_{\pitchfork',t}(b,\boldsymbol{s})\,,$$

where $\pitchfork' \leq_{\mathcal{S}} \pitchfork$ means $\{\pitchfork'_0,\dots,\pitchfork'_{L(\pitchfork')}\} \subset \{\pitchfork_0,\dots,\pitchfork_{L(\pitchfork)}\}$ and $\pitchfork' \leq_{\mathcal{R}} \pitchfork^*$ means $\mathcal{B}(\pitchfork') \subset \mathcal{B}(\pitchfork^*)$. Both are examples of the *inversion property* described by Voll and connected to Ehrhart-MacDonald-Stanley reciprocity in [Vol10a]. A fascinating paper of Klopsch and Voll combines the inversion property, Coxeter groups, and flags over $\mathbb{F}_q$ to explain the functional equations for certain local zeta functions via the inversion $q \mapsto q^{-1}$ [KV09]. It cannot be a coincidence that $\mathcal{Z}_N^\rho(K,\boldsymbol{s})$ involves a linear combination of functions satisfying the same inversion property with coefficients defined by falling factorials.

Problem 4.3. *Find the connection between the family of coefficients $\{M_\pitchfork(q)/q^{N-1}\}_{\pitchfork\in\mathcal{S}_N}$, Coxeter groups, and flags over $\mathbb{F}_q$. Follow* [KV09] *to establish a functional equation for $\mathcal{Z}_N^\rho(K,\boldsymbol{s})$ under the inversion $q \mapsto q^{-1}$.*

5. Future goal: The Adèlic perspective

The theory of local zeta functions is closely related to and partly motivated by analysis on the adèles. In his celebrated thesis [Tat67], Tate defined the adèle ring $\mathbb{A}_k$ for a number field k, proved that $\mathbb{A}_k$ is locally compact and self-Pontrjagin dual, and showed that $\mathbb{A}_k$ contains k as a discrete, cocompact, self-annihilating subgroup. In the spirit of Riemann and Hecke, this allowed him to define an adèlic theta function, establish its functional equation via Poisson summation over k, and use an idèlic version of the Mellin transform of the theta function to prove the functional equation for the Hecke L-function attached to k.

A key idea in Tate's thesis is that an Euler product over the primes of k (such as Hecke's L-function) can be realized as a product of local zeta functions, where each is an integral over a completion K of k. It is natural to ask the following question:

Problem 5.1. *Can the local zeta functions $\{\mathcal{Z}_N^\rho(K,\boldsymbol{s})\}_K$ (one for each completion K of k), be "stitched together" to form a convergent integral over the adèles? If so, does the adèlic integral have a statistical mechanical interpretation?*

Manin pointed out that some questions in mathematical physics can and should be brought into the adèlic setting [Man89], and this idea has since been successfully applied to string theory [Dra04, Vol10b]. Torba and Zúñiga-Galindo also proved that the adèle ring is metrizable and complete with respect to an explicit metric, showed that the metric is compatible with Fourier analysis and distribution theory on the adèles, and found a heat kernel that solves an adèlic version of the heat equation [TZnG13]. This convinces me that the notion of log-Coulomb gas might also lift to the adèlic setting, where I believe fruitful relationships between $\mathcal{Z}_N^\rho(\mathbb{Q}_p, \boldsymbol{s})$ and $\mathcal{Z}_N^\rho(\mathbb{R}, \boldsymbol{s})$ will be found.

References

[DL99] Jan Denef and François Loeser, *Germs of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232. MR 1664700

[DL02] ———, *Motivic integration, quotient singularities and the McKay correspondence*, Compositio Math. **131** (2002), no. 3, 267–290. MR 1905024

[DM63] Freeman J. Dyson and Madan Lal Mehta, *Statistical theory of the energy levels of complex systems. IV*, J. Mathematical Phys. **4** (1963), 701–712. MR 151231

[Dra04] Branko Dragovich, *Non-Archimedean geometry and physics on adelic spaces*, Contemporary geometry and related topics, World Sci. Publ., River Edge, NJ, 2004, pp. 141–158. MR 2070869

[For10] P. J. Forrester, *Log-gases and random matrices*, London Mathematical Society Monographs Series, vol. 34, Princeton University Press, Princeton, NJ, 2010. MR 2641363

[Hal05] Thomas C. Hales, *What is motivic measure?*, Bull. Amer. Math. Soc. (N.S.) **42** (2005), no. 2, 119–135. MR 2133307

[Hal18] Immanuel Halupczok, *An introduction to motivic integration*, Lectures in model theory, Münst. Lect. Math., Eur. Math. Soc., Zürich, 2018, pp. 181–202. MR 3888980

[Hir64] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR 0199184

[HK09] Ehud Hrushovski and David Kazhdan, *Motivic Poisson summation*, Mosc. Math. J. **9** (2009), no. 3, 569–623, back matter. MR 2562794

[Igu74] Jun-ichi Igusa, *Complex powers and asymptotic expansions. I. Functions of certain types*, J. Reine Angew. Math. **268/269** (1974), 110–130, Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, II. MR 0347753

[Igu75] ———, *Complex powers and asymptotic expansions. II. Asymptotic expansions*, J. Reine Angew. Math. **278/279** (1975), 307–321. MR 0404215

[JKU19] David Jensen, Max Kutler, and Jeremy Usatine, *The motivic zeta functions of a matroid*, 2019.

[KV09] Benjamin Klopsch and Christopher Voll, *Igusa-type functions associated to finite formed spaces and their functional equations*, Trans. Amer. Math. Soc. **361** (2009), no. 8, 4405–4436. MR 2500892

[Loe89] François Loeser, *Fonctions zêta locales d'igusa à plusieurs variables, intégration dans les fibres, et discriminants*, Annales scientifiques de l'École Normale Supérieure **4e série, 22** (1989), no. 3, 435–471 (fr). MR 90m:11194

[Man89] Yu. I. Manin, *Reflections on arithmetical physics*, Conformal invariance and string theory (Poiana Braşov, 1987), Perspect. Phys., Academic Press, Boston, MA, 1989, pp. 293–303. MR 1010662

[RSX13] Brian Rider, Christopher D. Sinclair, and Yuan Xu, *A solvable mixed charge ensemble on the line: global results*, Probab. Theory Related Fields **155** (2013), no. 1-2, 127–164. MR 3010395

[Ser15] Sylvia Serfaty, *Coulomb gases and Ginzburg-Landau vortices*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2015. MR 3309890

[Sin12] Christopher D. Sinclair, *The partition function of multicomponent log-gases*, J. Phys. A **45** (2012), no. 16, 165002, 18. MR 2910493

[Sin20] Christopher D. Sinclair, *Non-Archimedean Electrostatics*, arXiv e-prints (2020), arXiv:2002.07121.

[SVPV07] Dirk Segers, Lise Van Proeyen, and Willem Veys, *The motivic zeta function and its smallest poles*, J. Algebra **317** (2007), no. 2, 851–866. MR 3155337

[Tat67] J. T. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 305–347. MR 0217026

[TZnG13] S. M. Torba and W. A. Zúñiga Galindo, *Parabolic type equations and Markov stochastic processes on adeles*, J. Fourier Anal. Appl. **19** (2013), no. 4, 792–835. MR 3089424

[Vol10a] Christopher Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218. MR 2680489

[Vol10b] Igor V. Volovich, *Number theory as the ultimate physical theory*, p-Adic Numbers Ultrametric Anal. Appl. **2** (2010), no. 1, 77–87. MR 2594440

[Web20] Joe Webster, *log-coulomb gas with norm-density in p-fields*, 2020.

[ZnGT20] W. A. Zúñiga Galindo and Sergii M. Torba, *Non-Archimedean Coulomb gases*, J. Math. Phys. **61** (2020), no. 1, 013504, 16. MR 4048290

[ZZL20] W. A. Zúñiga-Galindo, B. A. Zambrano-Luna, and Edwin León-Cardenal, *Graphs, local zeta functions, Log-Coulomb Gases, and phase transitions at finite temperature*, arXiv e-prints (2020), arXiv:2003.08532.