LinSys

TTK 4115 Assignment 1

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Problem 1

I) $\ddot{y}(t) + 2\dot{y}(t) = \dot{u}(t) + 4u(t)$

a) $\underline{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) - u(t) \end{bmatrix}$

$\underline{\dot{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) - \dot{u}(t) \end{bmatrix}$

from I) $\ddot{y}(t) - \dot{u}(t) = -2\dot{y}(t) + 4u(t)$

$\underline{\dot{x}}(t) = \begin{bmatrix} \dot{y}(t) \\ -2\dot{y}(t) + 4u(t) \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -2x_2 + 2u \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$, $\underline{\underline{A = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}}$

$y(t) = C\underline{x}(t) + D\,u(t)$

$= [1 \; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0\,u(t)$ $\underline{\underline{C = [1 \; 0], \; D = 0.}}$

b) $\hat{g}(s) = C(sI - A)^{-1} B + D$

$= [1 \; 0] \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0$

$= [1 \; 0] \frac{1}{s(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{s^2+2s} [1 \; 0] \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$= \frac{1}{s^2+2s} [s+2 \quad 1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{s^2+2s} [s+2 \; +2] = \underline{\underline{\frac{s+4}{s^2+2s}}}$

c) $\ddot{y}(t) + 2\dot{y}(t) = \dot{u}(t) + 4u(t)$ (Notasjon $\mathcal{L}\{f(t)\} = F(s)$)

$\mathcal{L}\updownarrow$

$$s^2 y(s) - s y(0) - \dot{y}(0) + 2s y(s) - 2y(0) = s u(s) - u(0) + 4u(s)$$

zero initial condition: $y(0) = \dot{y}(0) = u(0) = 0$

$$\Rightarrow s^2 y(s) + 2s y(s) = s u(s) + 4u(s)$$

$$(s^2 + 2s)\, y(s) = (s+4)\, u(s)$$

$$\frac{y(s)}{u(s)} = \frac{s+4}{s^2+2s} = \hat{g}(s), \text{ same as in b)}$$

d) $$\frac{s+4}{s(s+2)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2}$$

$$s + 4 = \alpha_1(s+2) + \alpha_2 s$$

$$s + 4 = (\alpha_1 + \alpha_2)s + 2\alpha_1$$

I $\alpha_1 + \alpha_2 = 1$

II $2\alpha_1 = 4$

$\alpha_1 = 2$

I. $\alpha_1 + \alpha_2 = 2 + \alpha_2 = 1$

$\alpha_2 = -1$

$$\frac{s+4}{s(s+2)} = -\frac{1}{s} + \frac{2}{s+2}$$

$$\mathcal{L}^{-1}\left\{+\frac{2}{s} - \frac{1}{s+2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \underline{\underline{2 - e^{-2t}}}$$

Problem 2

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix},\ B = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$y(t) = \underline{C}\,\underline{x}(t) + D\,u(t) \qquad C = [3\ 0] \quad D = 5$$

a) $$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+3 \end{bmatrix}^{-1} = \frac{1}{s(s+3)} \begin{bmatrix} s+3 & 1 \\ 0 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cancel{(s+3)}}{s\cancel{(s+3)}} & \frac{1}{s(s+3)} \\ 0 & \frac{\cancel{s}}{\cancel{s}(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \right\} = \begin{bmatrix} \mathcal{L}^{-1}\{\frac{1}{s}\} & \mathcal{L}^{-1}\{\frac{1}{s(s+3)}\} \\ \mathcal{L}^{-1}\{0\} & \mathcal{L}^{-1}\{\frac{1}{s+3}\} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \mathcal{L}\{\frac{\alpha_1}{s} + \frac{\alpha_2}{s+3}\} \\ 0 & e^{-3t} \end{bmatrix}$$

$$\frac{\alpha_1}{s} + \frac{\alpha_2}{s+3} = \frac{1}{s(s+3)}$$

$$\alpha_1(s+3) + \alpha_2 s = 1$$

$$(\alpha_1 + \alpha_2)s + 3\alpha_1 = 1 \qquad \alpha_1 + \alpha_2 = 0,\ \alpha_1 = -\alpha_2$$

$$3\alpha_1 = 1,\ \alpha_1 = 1/3,\ \alpha_2 = -1/3$$

$$\mathcal{L}^{-1}\left\{\frac{\alpha_1}{s} + \frac{\alpha_2}{s+3}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{3} - \frac{1}{3}e^{-3t}$$

$$\boxed{e^{At} = \begin{bmatrix} 1 & \frac{1}{3}(1 - e^{-3t}) \\ 0 & e^{-3t} \end{bmatrix}}$$

b)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda+3 \end{vmatrix} = \lambda(\lambda+3) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = -3$$

$$[\lambda_1 I - A] = \begin{bmatrix} 0 & -1 \\ 0 & 3 \end{bmatrix} \cdot -\tfrac{1}{3} \sim \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \qquad \underline{\lambda_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 I - A = \begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$$

$$3x_1 + x_2 = 0$$

$$3x_1 = -x_2$$

$$x_1 = -\tfrac{1}{3}x_2 \qquad \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} x_2 \sim \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = -3$

Eigenvectors: $\underline{\lambda_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\underline{\lambda_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

c) As learned in linear algebra:

$$\underline{Q} = \begin{bmatrix} \underline{\lambda_1} & \underline{\lambda_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \qquad \underline{\hat{A}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$$

d)

$$e^{At} = \underline{Q} e^{\hat{A}t} \underline{Q}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{3}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -e^{-3t} \\ 0 & 3e^{-3t} \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & 1 - e^{-3t} \\ 0 & 3e^{-3t} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3}(1-e^{-3t}) \\ 0 & e^{-3t} \end{bmatrix}$$

Same as in a)

e) $u(t) = 1$ for all t.

$$\hat{g}(s) = C(sI-A)^{-1}B + D \quad , \quad \hat{g}(s) = \frac{y(s)}{u(s)}$$

$$y(s) = C(sI-A)^{-1}[Bu + X(0)] + D\,u(s)$$

$$y(s) = [3\ 0]\begin{bmatrix} s & -1 \\ 0 & s+3 \end{bmatrix}^{-1}\left[\begin{bmatrix} -2 \\ 6 \end{bmatrix}u(s) + \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\right] + 5u(s)$$

$$= [3,0]\frac{1}{s(s+3)}\begin{bmatrix} s+3 & 1 \\ 0 & s \end{bmatrix}\left(\begin{bmatrix} -2 \\ 6 \end{bmatrix}\frac{1}{s} + \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\right) + \frac{5}{s}$$

$$= \frac{1}{s(s+3)}\begin{bmatrix} 3(s+3) & 3 \end{bmatrix}\left(\begin{bmatrix} -2 \\ 6 \end{bmatrix}\frac{1}{s} + \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}\right) + \frac{5}{s}$$

$$= \frac{1}{s(s+3)}\left(\left(\frac{1}{s}(-6(s+3)+18)\right) + 3(s+3)x_{10} + 3x_{20}\right) + \frac{5}{s}$$

$$= \frac{1}{s^2(s+3)}(-6(s+3)+18) + \frac{1}{s(s+3)}(3(s+3)x_{10} + 3x_{20}) + \frac{5}{s}$$

$$= -\frac{6}{s^2} + \frac{18}{s^2(s+3)} + \frac{3}{s}x_{10} + \frac{3}{s(s+3)}x_{20} + \frac{5}{s} = y(s)$$

$$\frac{18}{s^2(s+3)} = \frac{6}{s^2} + \frac{2}{s+3} - \frac{2}{s}$$

$$\frac{3x_{20}}{s(s+3)} = \frac{x_{20}}{s} - \frac{x_2}{s+3}$$

$$y(s) = \cancel{-\frac{6}{s^2}} + \left(\cancel{\frac{6}{s^2}} + \frac{2}{s+3} - \frac{2}{s}\right) + \frac{3x_{10}}{s} + \left(\frac{x_{20}}{s} - \frac{x_{20}}{s+3}\right) + \frac{5}{s}$$

$$= \frac{3 + x_{20} + 3x_{10}}{s} + \frac{2 - x_{20}}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3 + 3x_{10} + x_{20}}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{2 - x_{20}}{s+3}\right\}$$

$$= 3 + 3x_{10} + x_{20} + e^{-3t}(2 - x_{20})$$

f) $x_{10} = -1/3$, $x_{20} = 2$

$$y(1) = 3 + 3x_{10} + x_{20} + e^{-3\cdot 1}(2 - x_{20})$$
$$= 3 + 3(-1/3) + 2 + e^{-3}(2-2) = 3 - 1 + 2 = \underline{\underline{4}}$$

$$y(2) = 3 + 3x_{10} + x_{20} + e^{-3\cdot 2}(2 - x_{20})$$
$$= 3 + 3(-1/3) + 2 + e^{-3\cdot 2}(2-2) = 3 - 1 + 2 = \underline{\underline{4}}$$

With $\underline{x}(0) = \begin{bmatrix} -1/3 \\ 2 \end{bmatrix}$, both $y(1)$ and $y(2)$ (and for any t), equals 4.

Problem 3

a)

$$I\ \ddot{r}(t) + \frac{2u_1}{\sqrt{r(t)^2+9}}\dot{r}(t) + 3r(t) + 4 = \frac{r(t)}{\sqrt{r(t)^2+9}}u_1(t) + u_2(t)$$

$$x_1 = r \qquad \dot{x}_1 = \dot{r} = x_2$$

$$x_2 = \dot{r} \qquad \dot{x}_2 = \ddot{r}$$

$$I\ \dot{x}_2 + \frac{2u_1}{\sqrt{x_1^2+3^2}}x_2 + 3x_1 + 4 = \frac{x_1}{\sqrt{x_1^2+3^2}}u_1 + u_2$$

$$\dot{x}_2 = \frac{x_1}{\sqrt{x_1^2+3^2}}u_1 + u_2 - \frac{2u_1}{\sqrt{x_1^2+3^2}}x_2 - 3x_1 - 4$$

$$\dot{x}(t) = \begin{bmatrix} x_2 \\ \frac{x_1}{\sqrt{x_1^2+3^2}}u_1 + u_2 - \frac{2u_1}{\sqrt{x_1^2+3^2}}x_2 + 3x_1 - 4 \end{bmatrix}$$

$$y(t) = 5\sqrt{x_1^2+3^2}$$

b) $x_0 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $u_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$$h_1 = x_2$$

$$h_2 = x_1\left[-3 + \frac{u_1}{\sqrt{x_1^2+9}}\right] - x_2\frac{2u_1}{\sqrt{x_1^2+9}} + u_2 - 4$$

$$A = \begin{bmatrix} \frac{\partial h_1}{\partial x_1}\Big|_p & \frac{\partial h_1}{\partial x_2}\Big|_p \\ \frac{\partial h_2}{\partial x_1}\Big|_p & \frac{\partial h_2}{\partial x_2}\Big|_p \end{bmatrix} \qquad (p \Rightarrow x = x_0,\ u = u_0)$$

- $\frac{\partial h_1}{\partial x_1}\Big|_p = 0$, • $\frac{\partial h_1}{\partial x_2}\Big|_p = 1$

- $$\frac{\partial h_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(-3x_1 + \frac{a_1 x_1}{\sqrt{x_1^2+9}} - \frac{2a_1 x_2}{\sqrt{x_1^2+9}} \cancel{+ x_2} \cancel{+ 4}\right)$$

$$= 3 + a_1 \frac{\partial}{\partial x} \frac{x_1}{\sqrt{x_1^2+9}} - 2a_1 x_2 \frac{\partial}{\partial x} \frac{1}{\sqrt{x_1^2+9}}$$

$$\left[\quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} \qquad (x_1^2+9)^{-1/2} \quad \frac{d}{dx} f(g(x)) = g'(x) \cdot f'(g(x)) \right]$$

$$= -3 + a_1 \frac{9}{(x_1^2+9)^{3/2}} + 2a_1 x_2 \frac{x_1}{(x_1^2+9)^{3/2}}$$

$$\left.\frac{\partial h_2}{\partial x_1}\right|_p = 3 + (5) \cdot \frac{9}{((4)^2+9)^{3/2}} + 2(5)(2) \frac{(4)}{((4)^2+9)^{3/2}} =$$

$$-3 + \frac{9}{25} + \frac{16}{25} = \underline{\underline{-2}}$$

- $$\frac{\partial h_2}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\cancel{-3x_1} \cancel{+ \frac{a_1 x_1}{\sqrt{x_1^2+9}}} - \frac{2a_1 x_2}{\sqrt{x_1^2+9}} \cancel{+ x_2} \cancel{+ 4}\right)$$

$$= -\frac{2a_1}{\sqrt{x_1^2+9}}$$

$$\left.\frac{\partial h_2}{\partial x_2}\right|_p = -\frac{2(5)}{\sqrt{16+9}} = -\frac{2 \cdot 5}{\sqrt{25}} = -2 \cdot \frac{5}{5} = \underline{\underline{-2}}$$

$$A = \begin{bmatrix} \left.\frac{\partial h_1}{\partial x_1}\right|_p & \left.\frac{\partial h_1}{\partial x_2}\right|_p \\ \left.\frac{\partial h_2}{\partial x_1}\right|_p & \left.\frac{\partial h_2}{\partial x_2}\right|_p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial h_1}{\partial u_1}\Big|_P & \frac{\partial h_1}{\partial u_2}\Big|_P \\ \frac{\partial h_2}{\partial u_1}\Big|_P & \frac{\partial h_2}{\partial u_2}\Big|_P \end{bmatrix}$$

- $\frac{\partial h_1}{\partial u_1} = 0$, • $\frac{\partial h_1}{\partial u_2} = 0$

- $\frac{\partial h_2}{\partial u_1} = \frac{\partial}{\partial u_1}\left[\cancel{-3x_1} + u_1 \frac{x_1}{\sqrt{x_1^2+9}} - 2u_1 \frac{x_2}{\sqrt{x_1^2+9}} \cancel{+u_2+4}\right)$

$$= \frac{x_1}{\sqrt{x_1^2+9}} - 2\frac{x_2}{\sqrt{x_1^2+9}} = \frac{4}{\sqrt{25}} - \frac{2 \cdot 2}{\sqrt{25}} = 0$$

- $\frac{\partial h_2}{\partial u_2} = \frac{\partial}{\partial u_2}\left[\cancel{-3x_1} + \cancel{u_1 \frac{x_1}{\sqrt{x_1^2+9}}} - \cancel{2u_1 \frac{x_2}{\sqrt{x_2^2+9}}} + u_2 \cancel{+4}\right]$

$$= 1$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f = 5\sqrt{x_1^2+9}$$

$$C = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix}$$

- $\frac{\partial f}{\partial x_1} = 5\frac{\partial}{\partial x_1}(x_1^2+9)^{1/2} = 5 \cdot 2x_1 \cdot \frac{1}{2}(x_1^2+9)^{-1/2} = 5 \cdot \frac{x_1}{\sqrt{x_1^2+9}}$

$$\frac{\partial f}{\partial x_1}\Big|_P = 5 \cdot \frac{4}{\sqrt{25}} = \underline{\underline{4}}$$

- $\frac{\partial f}{\partial x_2} = 0$

$$\underline{C = \begin{bmatrix} 4 & 0 \end{bmatrix}}$$

$$D = \left[\frac{\partial f}{\partial u_1}\Big|_p \quad \frac{\partial f}{\partial u_2}\Big|_p \right]$$

No u_1 or u_2 in our defined f.

$$D = [0 \; 0].$$

Problem 4

$$A = \begin{bmatrix} 0 & -9 \\ 1 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad C = [1 \; -7] \quad D = 2$$

a) $\det(\lambda I - A) = \begin{vmatrix} \lambda & 9 \\ -1 & \lambda+6 \end{vmatrix} = \lambda(\lambda+6) + 9 = \lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)^2 = 0, \quad \lambda_1 = \lambda_2 = -3$

$\lambda I - A = \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad -x_1 + 3x_2 = 0$

$x_1 = 3x_2$

$\Delta_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Eigenvalue(s) $= -3$

Eigenvectors(s) $= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

b) No, we will get a matrix $Q = \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}$ which is not invertable, and then the similarity transform will be impossible.

c) The jordan form of the matrix is $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}.$