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A COMPARISON OF STOCHASTIC AND DETERMINISTIC TRAFFIC ASSIGNMENT OVER CONGESTED NETWORKS

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Abstract—The similarity between link flows obtained from deterministic and stochastic equilibrium traffic assignment models is investigated at different levels of congestion. A probit-based stochastic assignment is used (over a congested network) where the conditions for equilibrium are those given by Daganzo and Sheffi (1977). Stochastic equilibrium flows are generated using an iterative procedure with predetermined step sizes, and the resulting assignment is validated on the basis of the equilibrium criteria. The procedure is intended to assist in the choice of the most appropriate assignment algorithm for a given level of congestion.

Traffic assignment models may be classified on the basis of whether or not they account for two phenomena: (a) congestion effects in link travel times, and (b) errors in user's perceptions of actual link times. The well known all-or-nothing assignment model does not include either of these effects. Virtually all other assignment algorithms currently in use consider only one of the two phenomena, and therefore can be distinguished as either deterministic equilibrium or stochastic assignment methods. Efficient algorithms now exist for solving the deterministic equilibrium problem (e.g. LeBlanc, 1975; Nguyen, 1974), producing a set of flows which satisfy what is commonly known as Wardrop's (1952) first principle.

Stochastic assignment models account for errors in user's perceptions by representing link travel times as random variables distributed across the population of trip makers. The most widespread stochastic assignment procedure is Dial's (1971) STOCH algorithm which is based on exponential (logit) trip diversion formulae. This and most other stochastic assignment models do not account for the dependency of travel times on link flows.† Van Vliet (1976b) presented one attempt at finding equilibrium flows using a stochastic assignment procedure, but his results were inconclusive.

This paper addresses some of the issues associated with models of stochastic traffic assignment over congested networks, i.e. equilibrium stochastic assignment models. The flow pattern which satisfies the equilibrium conditions (presented in Section 2) for this problem is termed stochastic user equilibrium (SUE) flows to differentiate it from deterministic user equilibrium (UE) models. Based on the arguments presented by Daganzo and Sheffi (1977) for uncongested networks we use a probit based assignment, thereby avoiding some of the drawbacks of the logit based formulations.‡ As such, this work represents an extension of their paper to congested newtorks. The issues that arose in this context are concerned primarily with the choice of the most appropriate assignment procedure for given levels of congestion. Daganzo (1977) and Van Vliet (1976a) have argued for using stochastic assignment for relatively uncongested networks and deterministic UE assignment for congested networks. While equilibrium stochastic models may be more realistic in terms of modelling behavior, the associated computational requirements are invariably larger, as compared with UE methods. Furthermore, the differences between UE and SUE flow patterns grow smaller§ as congestion increases.

At the moment, there do not exist guidelines for choosing between the two approaches on the basis of the level of congestion. This stems primarily from the lack of an algorithm that solves the SUE problem. This paper presents such an algorithm which is heuristic in nature since a

†See for example, Burrell (1976), Daganzo and Sheffi (1977), Gunnarson (1972), Tobin (1977), Von Falkenhausen (1966) and Wildermuth (1972).

‡See for example Burrell (1976), Schneider (1973), Florian and Fox (1976), Tobin (1977), and Daganzo and Sheffi (1977).

§In the limit, the change in travel times produced by an individual switching paths will be so large that the *direction* of the change is likely to be perceived correctly.

proof of its convergence does not yet exist. However, the numerical results presented are very encouraging. The paper also includes comparisons between several algorithmic approaches, all of which are evaluated on the basis of their ability to satisfy a set of equilibrium conditions. The numerical experiments were conducted over a small test network to allow for a very high degree of accuracy without excessive computational expenses. Since the algorithm is intended as a research tool rather than a production assignment package, computational efficiency considerations are not felt to be an issue.

The intent of this paper is to investigate the convergence of SUE to UE flow patterns as the level of congestion increases. Section 1 presents the stochastic equilibrium conditions and the different algorithms that were considered for the problem. Section 2 discusses convergence criteria and illustrates the results of consistency (with the equilibrium conditions) tests for each algorithm. Finally, Section 3 presents the experimental comparisons of UE and SUE flow patterns for the test network as a function of the level of congestion, and Section 4 concludes the paper.

1. STOCHASTIC EQUILIBRIUM ASSIGNMENT

The theory underlying our model of user path choice has been developed by Daganzo and Sheffi (1977), and is presented briefly here for the sake of completeness. The equilibrium definition suggested by these authors simply states that no user can reduce his *perceived* travel time by unilaterally changing routes.

To state the problem formally, let the (directed) graph representation of the network be given by a set of nodes, N, and a set of arcs, A. The set of centroids (i.e. sources and sinks) is denoted by C where $C \in N$. Let X_a denote the arc flow and X_k^{rs} the flow on path k between origin r and destination s ($r,s \in C$ and $k \in K^{rs}$ is the set of all paths from r to s). Let $T_a = T_a(X_a)$ denote the travel time on link a and let T_k^{rs} be the travel time on path k between origin r and destination s. The usual link-path incidence relationships, $X_a = \sum_{r,s} \sum_k X_k^{rs} \delta_{a,k}^{rs}$, and $T_k^{rs} = \sum_a T_a \delta_{a,k}^{rs}$, are assumed to hold, where $\delta_{a,k}^{rs} = 1$ if link a belongs to path k from r to s, and $\delta_{a,k}^{rs} = 0$ otherwise. Throughout this paper the summation over r,s is assumed to include all elements of C, the summation over a includes all elements in A and the summation over k includes all elements in K^{rs}. Let q^{rs} denote the $r \to s$ flow rate† and let P_k^{rs} denote the probability that an individual randomly chosen from the population (q^{rs}) would choose route k. Naturally $P_k^{rs} = P_k^{rs}(T^{rs})$ where $T^{rs} = (\ldots, T_k^{rs}, \ldots)$, i.e. the path choice probability is a function of the travel times over all paths.

In order to establish the probit route choice model we assume:

$$T_a \sim N(t_a, \theta t_a) \tag{1}$$

where t_a is the actual travel time on link a‡ and θ is a constant.

Using the aforementioned incidence relationships and the characteristics of the normal distribution, it is easy to see that the vector of path travel times, T_{rs}, is multivariate normally distributed with moments:

$$E[T_k^{rs}] = t_k^{rs} = \sum_a t_a \delta_{a,k}^{rs} \quad ; \quad \forall k \in K^{rs}, \forall r,s \in C, \tag{2a}$$

and

$$\operatorname{cov}[T_k^{rs}, T_l^{rs}] = \theta \sum_a t_a \delta_{a,k}^{rs} \delta_{al}^{rs} \quad ; \quad \forall k,l \in k^{rs}, \forall r,s \in C \tag{2b}$$

Thus the path choice probability can be expressed as:

$$P_k^{rs} = Pr[T_k^{rs} = \min_{l \in K^{rs}} \{T_l^{rs}\}] \quad ; \quad \forall k \in K^{rs}, r,s \in C. \tag{3}$$

†In this paper we exclusively discuss the non-elastic demand problem. Note that there is no loss of generality here since elastic demand assignment problems can be posed as non-elastic (fixed) demand problems by network representation as shown by Dantzig (1976), Gartner (1978) and Sheffi and Daganzo (1979).

‡The variance can be specified as $\theta t_{a,0}$, where $t_{a,0}$ is the free flow travel time if it is felt at the variance is more closely related to distance and fixed link characteristics. This would also serve to improve convergence of the algorithm.

For a given network, the weak law of large numbers implies that *on the average* path flows, X_k^{rs} would satisfy:

$$P_k^{rs} = \frac{X_k^{rs}}{q^{rs}} \qquad \forall k, r, s. \tag{4}$$

It should be emphasized that all the flow variables represent mean flow. This point is discussed further in the Appendix.

Since the path choice probabilities are functions of the path flow, eqns (1) and (4), in conjunction with the incidence relationships, flow non-negativity and the conservation constraints (i.e. $\sum_k X_k^{rs} = q^{rs}; \forall r,s$) define a set of stochastic user equilibrium conditons.

Thus, in terms of a demand-supply equilibrium the SUE conditions are:

$$\text{Demand:} \quad X_k^{rs} = q^{rs} \cdot P_k^{rs}(t^{rs}), \qquad \forall k,r,s \tag{5a}$$

and,

$$\text{Supply:} \quad t_k^{rs} = \sum_a t_a(X_a)\delta_{a,k}^{rs} \qquad \forall k,r,s. \tag{5b}$$

The basic algorithmic procedure discussed in this paper is the following:

Step 0. INITIALIZATION. Determine a set of initial link flows $X^0 = (\ldots, X_a^0, \ldots)$ and a corresponding set of initial link times $t^0 = (\ldots, t_a^0, \ldots)$. Set iteration counter $n = 1$.

Step 1. NETWORK LOADING. Determine path choice probabilities $\{P_k^{rs}[t^n]\}$ using eqs. (1)–(3) and obtain a flow vector $Y^n = \{Y_a^n\}$ using eqn (4), i.e.

$$Y_a^n = \sum_{rs} q^{rs} \sum_k P_k^{rs}(t^n) \cdot \delta_{a,k}^{rs}. \tag{6}$$

Step 2. STEP SIZE. Determine a value for α, $0 \le \alpha \le 1$, for finding an appropriate convex combination of $X^n = (\ldots, X_a^n, \ldots)$ and $Y^n = (\ldots, Y_a^n, \ldots)$.

Step 3. UPDATE. Update flows and travel times:

$$X_a^{n+1} = X_a^n + \alpha(Y_a^n - X_a^n) \tag{7a}$$

$$t_a^{n+1} = t_a(X_a^{n+1}). \tag{7b}$$

Step 4. Test for convergence. If the test fails, set $n = n + 1$ and go to STEP 1.

When this procedure is used to solve the UE problem, Step 1 is an all-or-nothing assignment and is a descent direction for minimizing the objective function of the program (PUE):

$$\text{(PUE):} \quad \min_X \sum_a \int_0^{X_a} t_a(\omega)\,\mathrm{d}\omega \tag{8}$$

Subject to flow conservation and non-negativity constraints. In solving (PUE) with the abovementioned procedure, the step size is determined from a one-dimensional search to minimize eqn (8) [with the argument $X + \alpha\,(Y - X)$] with respect to α.

In the case of (probit based) stochastic user equilibrium there does not exist an objective function which can be efficiently solved to yield flow patterns satisfying the equilibrium conditions mentioned in the first part of this section. Thus Step 2 is not a "descent" direction but rather defined in a fashion analogous to the UE solution procedure as an assignment *given* the link travel times. Step 2 can be accomplished either with an approximate analytical method [i.e. the Clark (1961) approximation discussed by Daganzo, Bouthelier and Sheffi (1977)] or by a simulation procedure. This issue is discussed further in the Appendix.

In every iteration of the algorithm, the set of flows $\{Y_a\}$ is consistent with the demand relationship (eqn 5a) and the set of flows $\{X_a\}$ is consistent with the supply relationship (eqn 5b). Both sets of flows are non-negative and comply with the conservation constraints. Thus the algorithm can be thought of as minimizing the difference between $\{X_k^{rs}\}^n$ and $\{q^{rs}P_k^{rs}(t^n)\}$, and a step size, α, could be chosen as the value that solves the following objective function:

$$\min_{\alpha} \sum_{r,s} \sum_{k} \{q^{rs} P_k^{rs}(t_\alpha^n) - [(X_k^{rs})^n + \alpha((Y_k^{rs})^n - (X_k^{rs})^n)]\}, \tag{9}$$

where

$$t_\alpha^n = t[X^n + \alpha(Y^n - X^n)].$$

The difficulty in using this approach is that aside from the path formulation of eqn (9), it requires the computation of the path choice probabilities throughout the network for each value of α that is tested. This is an expensive proposition even for small test networks.

To avoid the problem of developing and using an associated objective function to determine α, the predetermined sequence $\alpha = 1/n$ was used. This implies that the vector X^n represents an average of the sequence of "descent" vectors Y^n. For this reason the algorithm is termed the method of successive averages. Powell and Sheffi (1979) give a historical development of the procedure and present proofs of convergence of the method for the deterministic equilibrium and logit-based stochastic equilibrium problems. Since no proof could be provided for a probit based assignment, which is being used here, the validity of the algorithm must be based on numerical arguments. This is accomplished by comparing the equilibrium path flow $\{X_k^{rs}\}^n$ to the expected path flow given the equilibrium path times, $\{Y_k^{rs}\}^n$. Such an approach, although extremely sensitive to small differences between the calculated equilibrium flows and the exact ones, serves to directly verify the equilibrium conditions 5(a) and (5b).

As mentioned in the previous section, two other heuristic algorithms for the SUE problem are evaluated numerically. These differ from the successive averaging in the execution of Step 2 (the definition of α) only. The first one is based on the UE problem. In other words, we use the step size α that minimizes the UE objective function PUE-(eqn 8). Thus this algorithm is similar to LeBlanc's (1975) adaptation of the Frank–Wolfe (1956) algorithm with the exception that the flows $\{Y_a^n\}$ in Step 1 are determined from eqn (6) rather than from an all-or-nothing assignment.

The second approach simply utilizes α = constant, parallelling the well known capacity-restraint assignment technique. This technique was used by Burrell (1968) and Van Vliet (1976b) for equilibrium stochastic assignment.

The following section discusses the stopping criterion used for all three algorithms and the comparisons between the performance of the successive averages algorithm and the other two algorithms, for the network example used here.

2. CONVERGENCE CRITERIA AND EVALUATION OF EACH ALGORITHM

Since the successive averaging algorithm is not based on solving the equilibrium equations directly or solving an associated mathematical program, our convergence criterion is based on the decision variables (the flows) directly†. In order to emphasize it we refer to the criterion used as a stopping rule (rather than as a convergence criterion).

To test directly for the equilibrium conditon, a stopping rule can be based on comparing Y_a^n to X_a^n at every iteration n. This comparison, however, is extremely sensitive to small deviations from the exact solution, especially when a small value for t (see eqn 1) is used and as the network becomes more congested. Note that in the limit as $\theta \to 0$ (i.e. deterministic assignment) the use of Y_a^n is completely meaningless. A more meaningful stopping rule should be based on the sequence of link flows itself.

The rule used is based on the change in the average link flows over several iterations on account of two considerations: (a) due to the heuristic nature of the algorithm it is not expected to generate a monotonically decreasing series of link flow differences (even in the aggregate), and (b) the method used in Step 1 (simulation or Clark's approximation) is subject to stochastic

†Note that even in cases where an associated mathematical program is available, convergence criteria are often based on the flows (e.g. LeBlanc *et al.*, 1975) since these are the quantities of interest in the assignment problem.

variations (i.e. the resulting flows are random variables) and the averaging is used to smooth out these variations.

Thus the stopping rule statistic is taken to be the sum of the standard deviations of link flows (over all links) over the last I iterations divided by the sum of link flows, producing the dimensionless statistic S_n,

$$S_n = \frac{\sum_a [\frac{1}{I} \sum_{l=n-I+1}^{n} (X_a^l - \bar{X}_a^{n,I})^2]^{1/2}}{\sum_a \bar{X}_a^{n,I}}; \quad n \geq I \tag{10}$$

where $\bar{X}^{n,I}$ is defined as the average link flow over the last I iterations computed at the n-th iteration, i.e.

$$\bar{X}_a^{n,I} = \frac{1}{I} \sum_{l=n-I+1}^{n} X_a^l. \tag{11}$$

A value of $I = 7$ iterations was used for the numerical examples in this paper.

The stopping rule terminates the procedure whenever S_n is less than a predetermined positive constant. However, since with the successive averages method the stopping rule is bound to indicate "convergence" due to the decreasing step size, a consistency test is used to check if the demand condition (5a) is being met. Since this test is not a part of the algorithmic procedure, and in order not to mask it by a summary statistic, the path flows and travel times at convergence are graphically compared with the flows and times generated from a posterior application of the loading algorithm using the equilibrium travel times. Such comparative plots are shown towards the end of this section.

Our test network is shown in Fig. 1. The size of the network is limited as a result of two considerations: (a) the path formulation of the consistency test makes the use of larger examples prohibitively expensive and (b) the small network enables one to achieve extremely accurate assignments by using a large number of iterations, and a very accurate loading algorithm. A large number of equilibrium iterations were used for the deterministic (UE) assignment (which was compared to the SUE problem) as well as the SUE problem, since the convex combinations algorithm (LeBlanc *et al.*, 1975) used for the UE problem is relatively slow to converge.

The link congestion functions used for our test network were the U.S.B.P.R. (1966) type volume-delay curves of the form:

$$t_a = a_a + b_a X_a^4. \tag{12}$$

The values of a_a, b_a used are given in Table 1. Two origin destination pairs (nodes 1→12 and 4→9) were assumed, each associated with a demand rate of λ trips/hr, where λ was varied, in the different experiments, between $\lambda = 1$ and $\lambda = 20$. A value of $\lambda = 10$ trips/hr was found to produce moderate levels of congestion.

In the remainder of this section we discuss the comparisons between the above mentioned

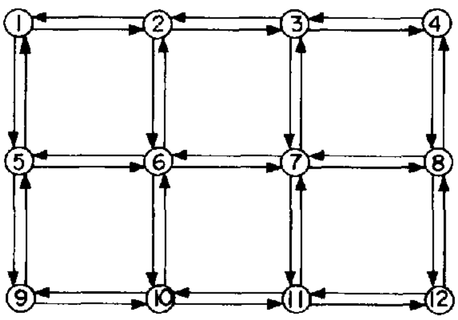

Fig. 1. Test network.

Table 1. List of links and cost parameters*

From node	To node	a	b
1	2	20	.008
1	5	18	.008
2	1	20	.008
2	6	19	.008
2	3	23	.008
3	2	23	.008
3	7	16	.008
3	4	17	.008
4	3	17	.008
4	8	22	.008
5	1	18	.008
5	6	14	.008
5	9	24	.008
6	2	19	.008
6	5	14	.008
6	7	17	.008
6	10	20	.008
7	3	16	.008
7	6	17	.008
7	8	13	.008
7	11	26	.008
8	4	22	.008
8	7	13	.008
8	12	19	.008
9	5	24	.008
9	10	7	.008
10	9	7	.008
10	6	20	.008
10	11	18	.008
11	10	18	.008
11	7	26	.008
11	12	17	.008
12	8	19	.008
12	11	17	.008

algorithms (i.e. the capacity restraint, minimizing the UE objective function, and the successive averaging method) and demonstrate the consistency tests.

Figure 2 shows the series of iterations generated by the capacity restraint method (using $\alpha = 0.2$) by showing S_n versus the iteration number. Evidently, this assignment method shows no signs of reaching a stable flow pattern (in accordance with the results obtained by Burrell (1968) and Van Vliet (1976b) for a similar algorithm). The use of different values for α did not improve convergence.

The two other algorithms, however, appear to reach a stable flow pattern almost monotonically for both demand levels, $\lambda = 10, 20$, as shown in Fig. 3 for the UE objective function

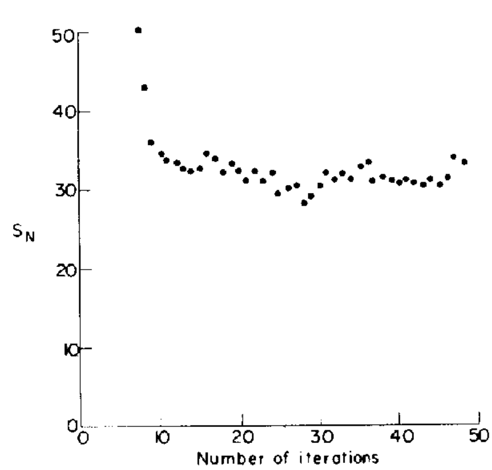

Fig. 2. Convergence of capacity restraint method.

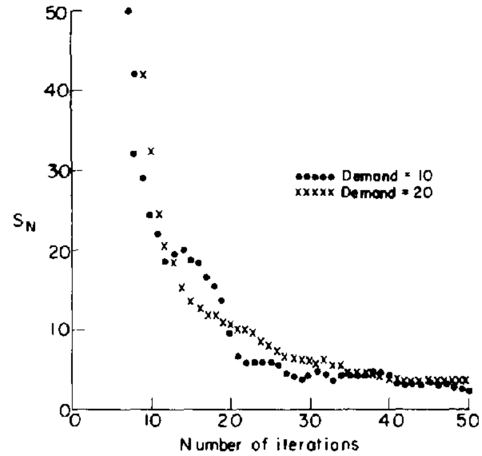

Fig. 3. Convergence of assignment by minimization of U-E objective function.

minimizing algorithms, and in Fig. 4 for the successive averaging algorithm. Note that the rate of "convergence" of the successive averaging algorithm is better than the UE-based algorithm for $\lambda = 10$ trips/hr and slower for $\lambda = 20$ trips/hr. If both algorithms achieve a correct flow pattern this is to be expected since, as mentioned in our first section, the UE problem is a good approximation for the SUE problem at high congestion levels (high values of λ, in our case). For $\lambda = 20$ trips/hr the SUE and UE problems are similar and thus a minimization algorithm would outperform the successive averaging one, as found by Fisk (1980).

However, at lower congestion levels the stochastic effects are more pronounced than the equilibrium effects and the successive averaging method (not being guided by a wrong objective function) would perform better.

Naturally, as mentioned earlier, a mere stopping rule does not prove that the resulting flow pattern is the one sought. In order to establish the similarity between the equilibrium conditions and the results of the algorithm, Figure 5 shows the fraction of flow using each path vs. the path's travel time for each path, for the successive averaging algorithm. Also shown are each path's choice probability at equilibrium. Since these probabilities were computed by simulation, confidence intervals are plotted as well to account for random errors produced by the simulation. These intervals were found by modelling the choice of any given path as a binomial process. Thus if P_k is the estimated probability of choosing path K, the 95% confidence limits would be given approximately by $\pm 2\sqrt{P_k(1-P_k)/n}$, where n is the number of times the simulation was repeated. In this case, $n = 1500$ trials was used to achieve the desired level of

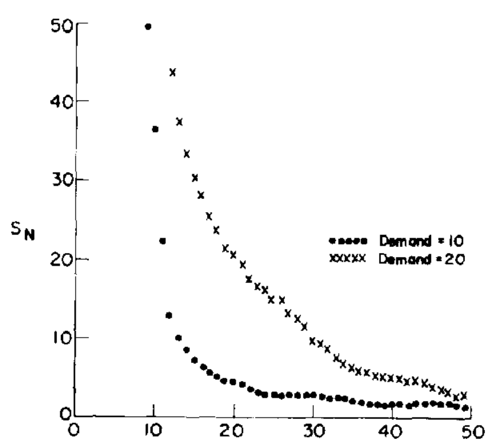

Fig. 4. Convergence of successive averaging assignment.

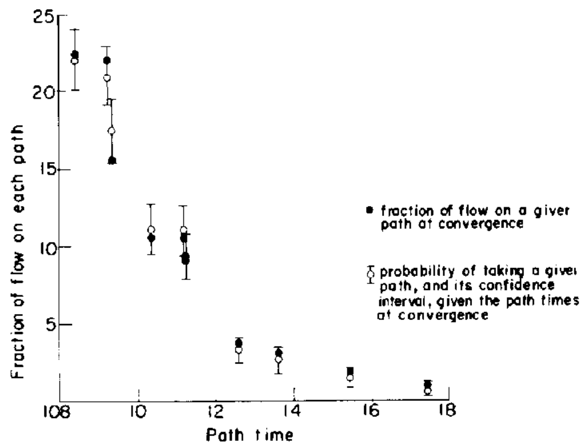

Fig. 5. Distribution of path times (demand rate, $\lambda = 10$).

accuracy. It should be noted, of course, that these confidence limits ignore the jointness between the paths and therefore serve only as an indication of the degree of variability in the estimates.

Thus, we conclude that for our example, the flow pattern produced by the successive approximation method is in satisfactory agreement with the equilibrium conditions. In comparing the successive averaging algorithm to the UE-based one, note that even though the results are quite similar, the successive averaging algorithm gives better results at the limit for any demand level in terms of satisfying the equilibrium conditions. This is due to the premature convergence of the step size, in the UE-based algorithm, to zero.

3. STOCHASTIC AND DETERMINISTIC EQUILIBRIUM ASSIGNMENT

The stochastic user equilibrium is defined by eqn (5) presented in Section 1 of this paper. Equation (5b) is the equilibrium part while eqn (5a) represents the stochastic choice part. If we replace the choice probability definition, eqn (3), with the deterministic rule:

$$P_k^{rs} = \begin{cases} 1 \text{ if } T_k^{rs} = \min\limits_{l \in K^{rs}} \{T_l^{rs}\} \\ 0 \text{ otherwise} \end{cases} \tag{13}$$

the resulting equilibrium corresponds to Wardrop's UE rule. Alternatively, this is equivalent to setting $\theta = 0$ in eqn (1). Thus as $\theta \to 0$, at a given congestion level, the equilibrium SUE flows would become more and more similar to the equilibrium UE flows.

In this section we investigate the rate at which SUE flows would resemble the UE flows, for a fixed value of θ, as the network becomes more congested. Figure 6 gives the mean path time

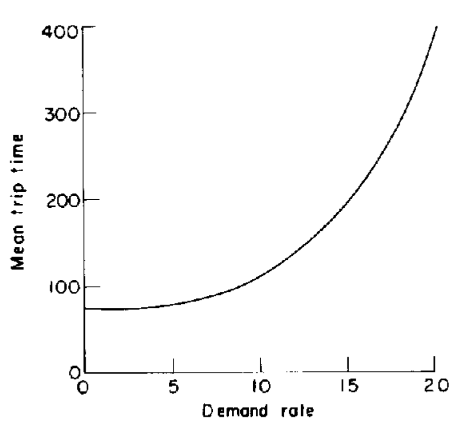

Fig. 6. Mean equilibrium trip time vs demand rate λ.

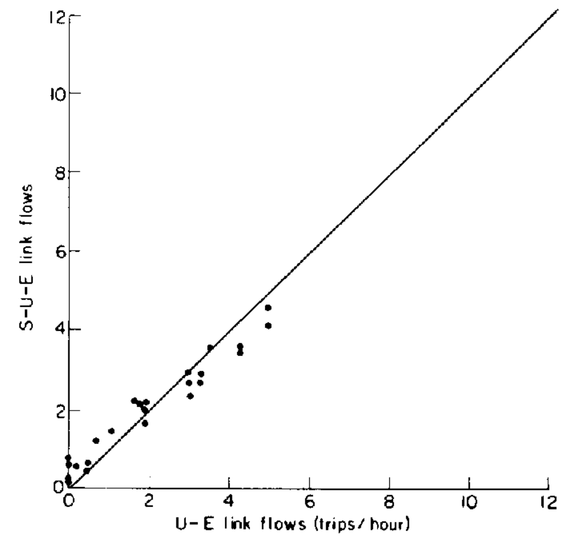

Fig. 7. Scatter plot of U–E and S–U–E link flows for demand rate $\lambda = 5$.

at equilibrium for various demand levels λ to indicate the degrees of congestion that are being dealt with.

In all the following experiments, the successive averaging algorithm was used for determining the SUE flows and the convex combination algorithm for the UE flows. A large number of iterations was used in running both algorithms, ensuring that consistency of the SUE flows and convergence of the UE algorithm was achieved with a high degree of accuracy. The variance to mean ratio (for the SUE problem) was set at $\theta = 0.3$.

Figures 7–9 compare the link flows obtained from UE and SUE assignments for demand levels of $\lambda = 5$, 10, and 20, respectively. As expected, the differences between the flow patterns tend to diminish as the demand level is increased. This is evident from the tighter cluster of the points along the 45° diagonal which represents identical flows.

It is interesting to note that at low demand levels (Fig. 7) the UE flow pattern tends to underestimate the flow on low volume links. This, of course, can be readily explained if one considers the limiting case of no congestion at all. In this case the links along the minimum path would be assigned flows according to the UE algorithm and the flow would be zero for the other links. The SUE algorithm would tend to spread the flow and thus assign less to the links on the

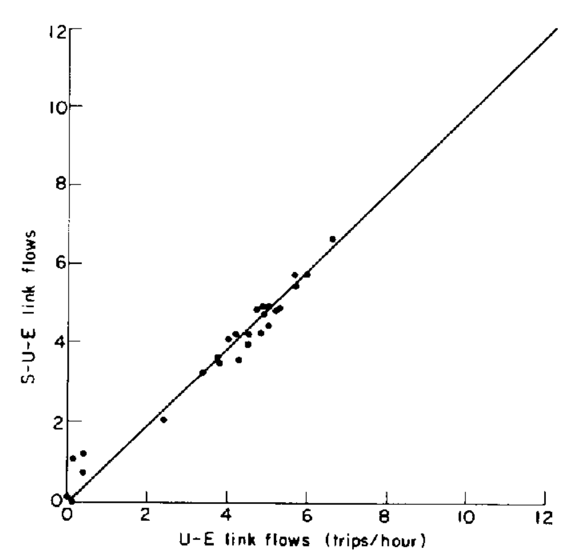

Fig. 8. Scatter plot of U–E and S–U–E link flows for demand rate $\lambda = 10$.

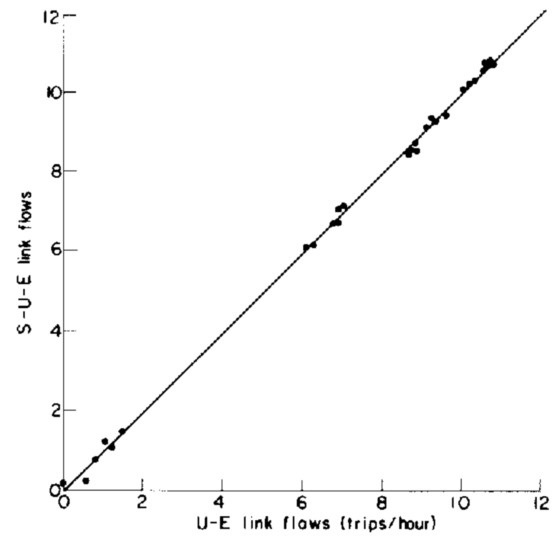

Fig. 9. Scatter plot of U–E and S–U–E link flows for demand rate $\lambda = 20$.

minimum paths and non-zero flow to the other links. However, this discrepancy disappears very quickly with higher demand rates, as evident from Figures 8 and 9. In other words, the flows seem to be quite similar at relatively moderate levels of congestion; at $\lambda = 10$ trips/hr equilibrium travel times are only 20% greater than those for $\lambda = 0$ and the abovementioned bias in the UE/SUE flows becomes relatively small, as evident from Fig. 8.

To quantify the similarity between SUE and UE flow patterns the following dimensionless statistic is defined:

$$S(\lambda) = \frac{\sum_a |X_a^{\mathrm{SUE}} - X_a^{\mathrm{UE}}|}{\sum_a X_a^{\mathrm{SUE}}} 100 \tag{14}$$

where X_a^{SUE} refers to the equilibrium flow on link a under SUE (resulting from the successive averaging method), and X_a^{UE} refers to the equilibrium flow under UE.

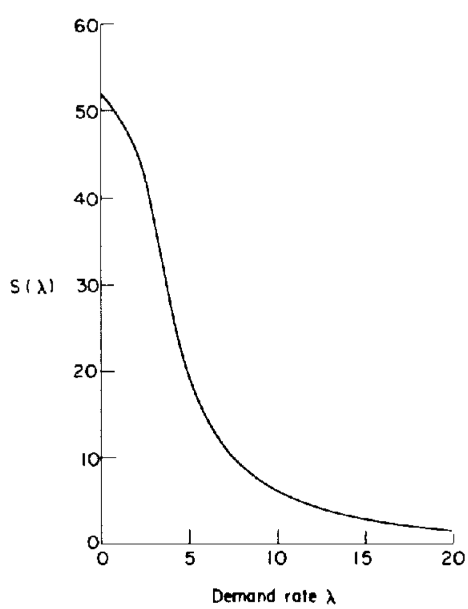

Fig. 10. Convergence of S–U–E to U–E as a function of demand level.

$S(\lambda)$ is shown in Fig. 10 and illustrates the convergence of SUE to the UE flow pattern as a function of λ. For the network tested, this convergence appears to be quite rapid at moderately low congestion levels, suggesting that deterministic assignment may be quite accurate for all but the least congested networks. However, further research on larger, more realistic networks is required before such a pattern can be established with any certainty. More specifically, networks with links of widely varying capacities (reflected by different values of the link parameters b_a) may exhibit much slower rates of convergence of deterministic flow pattern to a stochastic one.

4. DISCUSSION

The stochastic approximation method presented in this paper offers promise as a research tool for investigating stochastic network equilibrium. While lacking a rigorous theoretical proof of convergence, the numerical results based on a small test network are very encouraging. The validation of the flow patterns produced by the algorithm represents the first successful attempt at obtaining a network loading that can be justified on theoretical grounds. Of course, what the procedure gains in accuracy, it loses in computational efficiency. It is felt, then, that the algorithm could be best used to investigate the relative accuracy of stochastic assignment over uncongested networks versus deterministic equilibration over congested networks for different levels of congestion. Guidelines could then be developed for choosing the most appropriate assignment methodology for a given situation.

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REFERENCES

Burrell J. (1968) Multipath route assignment and its application to capacity restraint. *Proc. 4th Int. Symp. on the Theory of Traffic Flow* Karlsruhe, Germany, pp. 229–239.

Burrell J. (1976) Multipath route assignment: A comparison of two methods. *Traffic Equilibrium Methods: Lecture Notes in Economics and Mathematical Systems* (Edited by M. Florian), Vol. 118, pp. 210–229. Springer-Verlag, New York.

Clark C. E. (1961) The greatest of a finite set of random variables. *Opns. Res.* **9**, 145–162.

Daganzo C. (1977) Some Remarks on Traffic Assignment Methodology Selection. Working Paper 7703, Stochastic Traffic Assignment Project, ITS, U. of CA., Berkeley, CA.

Daganzo C. and Sheffi Y. (1977) On stochastic models of traffic assignment. *Transpn Sci.* **11**, 253–274.

Dantzig G., Maier F. and Lansdowne Z. (1976) *The Application of Decomposition to Transportation Network Analysis*, Report DOT-TSC-76-26. Control Analysis Corporation.

Dial R. (1971) A probabilistic traffic assignment model which obviates path enumeration. *Transpn Res.* **5**, 83–111.

Fisk C. (1980) Some developments in equilibrium traffic assignment methodology. *Transpn Res.* (in press).

Florian M. and Fox B. (1976) On the probabilistic origin of Dial's multipath assignment model. *Transpn Res.* **10**, 339–341.

Gartner N. H. (1979) Optimal traffic assignment with elastic demands: A review—Part II: Algorithmic approaches. Paper presented at the Joint TIMS/ORSA Meeting NYC, May 1978.

Gunnarson S. (1972) An algorithm for multipath traffic assignment. *Proc. PTRC, Urban Traffic Model Research Seminar*, London, England.

LeBlanc L., Morlok E. and Pierskalla W. (1975) An efficient approach to solving the road network equilibrium traffic assignment problem. *Transpn Sci.* **9**, 209–218.

Nguyen S. (1974) An algorithm for the traffic assignment problem. *Transpn Sci.* **8**, 203–216.

Powell W. B. and Sheffi Y. (1979) The convergence of equilibrium algorithms with predicted step sizes. *Transpn Sci.* (To be published).

Schneider M. (1973) Probability maximization in networks. *Proc., Int. Conf. on Trans. Res.* Bruges, Belgium, pp. 748–755.

Sheffi Y. and Daganzo C. (1979) Hypernetworks and supply demand equilibrium with disaggregate demand models. *Trans. Res. Rec.* **673**, 1, 13–120.

Tobin R. (1977) An extension of Dial's algorithm utilizing a model of tripmakers' preceptions. *Transpn Res.* **11** (5), 337–342.

U. S. B. P. R. (1964) *Traffic Assignment Manual*. Washington, D.C.

Van Vliet D. (1976a) Road Assignment—I. Principles and parameters of model formulation. *Transpn Res.* **10**(2), 137–143.

Van Vliet D. (1976b) Road Assignment—III. Comparative tests of stochastic methods. *Transpn Res.* **10**, 151–157.

Von Falkenhausen H. (1966) Traffic Assignments by a Stochastic Model. *Proc. 4th Int. Conf. on Operational Science*, 415–421.

Wardrop J. G. (1952) Some theoretical aspects of road traffic research. *Proc. Inst. of Civil Engng. Part II*, Vol. 1, pp. 325–378.

Wildermuth B. (1972) The use of a multiple routing technique for one pass capacity restraint assignment. *Traffic Quart.* **26**, 247–265.

APPENDIX

The network loading algorithm

The (stochastic) network loading problem seeks to determine link flows from a known set of O-D trip rates for a *given* set of link times. This is the problem encountered in step 1 of the algorithm for determining the flow vector Y^n (see Section 1). In the following discussion we present the algorithm used to solve this problem.

For a given vector of link travel times $t^n = (\ldots, t_a^n, \ldots)$, the path flows are multinomially distributed random variables (with the parameter vector given by the set of path choice probabilities, P_k^{rs}), with the link flow vectors therefore being random variables as well. The network loading procedure is needed to provide an estimator for the mean link flows. The problem is solved by estimating the path choice probabilities based on travel times which are characterized by a multivariate normal distribution. Determining the probabilities, therefore, requires evaluation of a multinomial probit choice function. This can be accomplished using the numerical approximation procedure suggested by Daganzo, Bouthelier and Sheffi (1977) (see also Daganzo and Sheffi, 1977). However, this approximation introduces errors which cannot be controlled for. Since our comparisons are based on numerical experiments, a Monte Carlo simulation was used for computing path choice probabilities with a high degree of accuracy (the simulation error can be made smaller than any predetermined tolerance by enough repetitions).

The actual algorithm used for the stochastic network loading problem is as follows:

STEP 0: Set iteration counter $m = 1$; $\bar{Y}_a^{n,(0)} = 0$, $\bar{\bar{Y}}_a^{n,(0)} = 0 \forall a$.

STEP 1: Sample once for each link a perceived travel time obtaining $T_a \forall a$. [$T_a \sim n(t_a^n, \theta t_a^n)$ where t_a is the current mean travel time on link a.]

STEP 2: Assign "All or Nothing" between all 0-D pairs, using $\{T_a\}$ as the set of link travel times. This yields a flow vector $\{Y_a^{n,(m)}\}$.

STEP 3: Compute $\bar{Y}_a^{n,(m)} = [(m-1)\bar{Y}^{n,(m)} + Y_a^{n,(m)}]/m$, $\bar{\bar{Y}}_a^{n,(m)} = [(m-1)\bar{\bar{Y}}_a^{n,(m-1)} + (Y_a^{n,(m)})^2]/m$ and $\sigma_a^{n,(m)} = \{[\bar{\bar{Y}}_a^{n,(m)} - (\bar{Y}_a^{n,(m)})^2]/(m-1)\}^{1/2}$.

STEP 4: If convergence criterion met, terminate: $\bar{Y}_a^n = \bar{Y}_a^{n,(m)}$; otherwise, set $m = m + 1$ and go to STEP 1.

Note that the resulting flow is a simple average over all (simulation) iterations, which is an estimator to the abovementioned flow. The convergence criterion is based on the sum of link standard deviations normalized by the sum of mean link flows, i.e. we terminate if:

$$\frac{\sum_a \sigma_a^{n,(m)}}{\sum_a Y_a^{n,(m)}} < \varepsilon$$

where ε is a predetermined tolerance ($\varepsilon = 0.03$ was used in the experiments presented in this paper).