

Full length article

# A general radial quasi-interpolation operator on the sphere<sup>☆</sup>

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Received 17 May 2011; received in revised form 10 June 2012; accepted 16 July 2012

Available online 22 July 2012

Communicated by Feng Dai

## Abstract

Geodetic and meteorological data, collected via satellites for example, are genuinely scattered and not confined to any special set of points. In order to learn geodetic and meteorological rules, one needs to use these scattered data only to construct an approximant or interpolant. In this paper, we introduce a general distance generated from the scattered data, and, using this, construct a general radial quasi-interpolation operator on the sphere, and we study the convergence rate of this operator. We also show some potential applications of the results obtained here in satellite geodesy.

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**Keywords:** Scattered data fitting; Quasi-interpolation; Convergence rate; Sphere

## 1. Introduction

Many recent applications require the modeling and analysis of signals whose domain is the surface of a sphere. Examples include studies of seismic signals, gravitational phenomena, hydrogen atoms, the solar corona, medical imaging of the brain, etc. Geodetic and meteorological

<sup>☆</sup> The research was supported by the Key Program of the National Natural Science Foundation of China (No. 11131006), and the National Natural Science Foundation of China (Nos. 61075054, 90818020, 60873206).

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data collected via satellites, for example, are genuinely scattered and not confined to any special sets of points on a sphere. Thus the problem of effectively representing an underlying function based on its scattered sampled values on the sphere is both important and useful.

A popular way to tackle this problem is to exactly interpolate the samples by using spherical basis function (SBF), i.e. a positive definite radial basis function on the unit sphere. The method of exact interpolation by SBF is to find shifts of an SBF  $\phi$ ,  $N_\phi$ , such that

$$N_\phi(x_i) = f_i, \quad i = 1, \dots, n,$$

where  $X := \{x_1, \dots, x_n\} \subset \mathbf{S}^d$  is the set of scattered data (or interpolation knots in the interpolation terminology),  $\{(x_i, f_i) : x_i \in X, f_i \in \mathbf{R}\}_{i=1}^n$  denotes the set of interpolation samples and  $\mathbf{S}^d$  denotes the unit sphere embedded into the  $(d+1)$ -dimensional Euclidean space.

There have been several research studies on this topic. In [11], by using the general theory of Golomb and Weinberger and the well known norming set method, Jetter et al. deduced an error estimate for exact interpolation by SBF. They used the mesh norm of  $X$  to describe the error between the interpolant and the target function. Three years later, Morton and Neamtu [15] gave a slight improvement of the result of [11] by using the norming set method again. It can be seen that the results of [11,15] only applied to functions in the native space associated with the SBF. If  $\phi$  is smooth, then the associated native space is small in the sense that it is composed of smooth functions. Narcowich and Ward [17] first made this observation and dealt with it by finding a spherical polynomial which can exactly interpolate the samples and nearly best approximate the target function. They showed that one can use SBF shifts to interpolate the target function out of the native space of the SBF. More recently, Narcowich et al. [16] got a Sobolev error estimate for this topic by using a similar method. They also gave the inverse theorem for SBF interpolation. More results for exact interpolation by SBF can also be found in [4,18,1] and references therein.

It is well known that to fulfill the exact interpolation, one needs to solve the following system of equations:

$$\sum_{i=1}^n c_i \phi(x_i \cdot x_j) = f_j, \quad j = 1, \dots, n.$$

When the number of data is large, this process cannot be carried out easily. Furthermore, the samples that we collect using satellites or other tools are usually noised, and the method of exact interpolation cannot deal with noised data. All above urges us to find a tool which takes account of the noise and can be easily implemented. Thus one turns to constructing a quasi-interpolant on the sphere.

There have been some studies devoting to constructing radial quasi-interpolation operators under various assumptions. For equiangular grid points on  $\mathbf{S}^2$ , Gomes et al. [10] constructed a radial quasi-interpolation operator by using the relation between the continuous Fourier transform and discrete Fourier transform. But the method in [10] cannot be extended to scattered data fitting easily. For scattered data, the quasi-interpolation operator has already been studied in [15, Lemma 9]. But there are three points that need be improved in [15, Lemma 9]. Firstly, the quasi-interpolation operator is not constructive, and it cannot be implemented directly. Secondly, there are some restrictions on the set of interpolation knots  $X$ . Thirdly, the quasi-interpolation operator is indeed a spherical polynomial, which usually possesses a bad space localized property (see [7]). For more references on quasi-interpolation on the sphere, we refer the readers to [2,12–14].

The main purpose of the present paper is to construct a quasi-interpolation operator, which is applicable to scattered data on the sphere without any restrictions and can be implemented easily. In order to construct the quasi-interpolation operator, we should introduce a general distance  $\bar{d}(\cdot, \cdot)$  with the help of a rearrangement of the scattered data. The constructed general radial quasi-interpolation operator can be mathematically represented as

$$G_n(x) := \sum_{i=1}^n f_i g(\bar{d}(x_i, x)), \quad x \in \mathbf{S}^d, \quad (1)$$

where  $g$  is a univariate function. In the rest of this paper, we will study the quasi-interpolation operator (1) in the following two directions. On one hand, we will deduce an upper bound estimate of the approximation by the quasi-interpolation (1). Our results will show that the convergence rate of the constructed quasi-interpolation operator depends not only on the mesh norm of  $X$ , but also on the mesh ratio of  $X$ . On the other hand, we will study some applications of the interpolant (1). We will construct the basis function  $g$ , and using this we will deduce some superb properties of (1), similar to those of the well known  $B$ -spline. From these properties, we find that the constructed quasi-interpolation operator has some merit in solving spherical Fredholm integral equations of the first kind.

The rest of this paper is organized as follows. In the next section, some preliminaries together with the general distance will be introduced. In Section 3, we will construct the general radial quasi-interpolation operator. The spline property of the constructed operator as well as its application will also be given in this section. In Section 4, we will give some proofs.

## 2. Preliminaries

In this section, we give some preliminaries for this paper. First we introduce three quantities associated with the scattered data set  $X$ . The mesh norm of  $X$  is defined by

$$h_X := \max_{x \in \mathbf{S}^d} \min_j d(x, x_j),$$

where  $d(x, y)$  is the geodesic (great circle) distance between the points  $x$  and  $y$  on  $\mathbf{S}^d$ . The mesh norm measures the maximum distance that any point on  $\mathbf{S}^d$  can be from  $X$ . The separation radius is defined via

$$q_X := \frac{1}{2} \min_{j \neq k} d(x_j, x_k).$$

This is half of the smallest geodesic distance between any two distinct points in  $X$ . It is easy to see that  $h_X \geq q_X$ ; equality can hold only for a uniform distribution of point on  $\mathbf{S}^1$ , the circle. The mesh ratio

$$\tau_X := \frac{h_X}{q_X} \geq 1$$

provides a measure of how uniformly points in  $X$  are distributed on  $\mathbf{S}^d$ . From the definitions, we can easily deduce the following two lemmas, which describe the relation between the distribution and number of the scattered data. Denote by  $D(x_0, \gamma)$  the spherical cap with center  $x_0$  and angle  $\gamma$ , i.e.,

$$D(x_0, \gamma) := \{x \in \mathbf{S}^d : x \cdot x_0 \geq \cos \gamma\},$$

and by  $D(\gamma)$  the volume of  $D(x, \gamma)$ , i.e.,

$$D(\gamma) := \int_0^\gamma \Omega_{d-1} \sin^{d-1} \theta d\theta,$$

where  $\Omega_{d-1}$  denotes the volume of the  $(d-1)$ -dimensional sphere  $\mathbf{S}^{d-1}$ .

**Lemma 1.** Let  $h_X$  be the mesh norm of  $X$ , then

$$\mathbf{S}^d \subset \bigcup_{i=1}^n D(x_i, h_X).$$

This lemma shows that the sphere can be covered by spherical caps whose angles are mesh norms and whose centers belong to  $X$ . To cover the whole sphere, it is obvious that every spherical cap  $D(x_i, h_X)$  ( $i = 1, \dots, n$ ) must contain at least one point. In the next lemma, we show that this number can be controlled by the mesh ratio of  $X$ . Let

$$s_{c_i} := |\{x \in X, x \in D(x_i, h_X)\}|, \quad \text{and} \quad s_c := \max_{x_i \in X} s_{c_i},$$

where  $|A|$  denotes the cardinal norm of the set  $A$ .

**Lemma 2.** Let  $X$  be the set of scattered data with mesh ratio  $\tau_X$ , then we have

$$s_c \leq 2\pi^{d-1} \tau_X^d. \quad (2)$$

Now we introduce a general distance corresponding to  $X$  on the sphere. First we rearrange the points in  $X$  to obey the following three rules:

(A1)  $x_1$  can be chosen arbitrarily.

(A2)  $d(x_k, x_{k+1}) \leq 4h_X, k = 1, 2, \dots, n-1$ .

(A3) For  $j \neq k, \overline{x_k x_{k+1}} \cap \overline{x_j x_{j+1}} = \begin{cases} x_{k+1}, & j = k+1, \\ x_k, & j = k-1, \\ \emptyset, & \text{otherwise,} \end{cases}$

where  $\overline{x_k x_{k+1}}$  is the minor arc of the great circle from  $x_k$  to  $x_{k+1}$ . It is obvious that there must be more than one arrangement satisfying (A1)–(A3), and we choose just one of them arbitrarily. For the sake of completeness, we will give an example for the above rearrangement in the [Appendix](#).

Since  $\mathbf{S}^d \subset \bigcup_{i=1}^n D(x_i, h_X)$ , for arbitrary  $x \in \mathbf{S}^d$  there exists at least one point such that  $x \in D(x_k, h_X)$ . If we set

$$k := \min\{j : x \in D(x_j, h_X)\}, \quad (3)$$

then for arbitrary  $x \in \mathbf{S}^d$ , there exists a unique  $k$  satisfying (3) such that  $x \in D(x_k, h_X)$ . For arbitrary points  $x, y \in \mathbf{S}^d$ , we define a general distance between  $x \in D(x_{k_0}, h_X)$  and  $y \in D(x_{j_0}, h_X)$  as

$$\bar{d}(x, y) := \begin{cases} d(x, y), & k_0 = j_0, \\ \sum_{i=j_0}^{k_0} d(x_i, x_{i+1}) + d(x_{k_0}, x) + d(x_{j_0}, y), & j_0 < k_0, \\ \sum_{i=k_0}^{j_0} d(x_i, x_{i+1}) + d(x_{k_0}, x) + d(x_{j_0}, y), & k_0 < j_0. \end{cases} \quad (4)$$

Then we prove that  $\bar{d}(x, y)$  defined in (4) is a distance between  $x$  and  $y$ . It is obvious that  $\bar{d}(x, y) \geq 0$ , where the equality holds if and only if  $x = y$ . Furthermore, it is obvious that

$d(x, y) = d(y, x)$ . The only thing remaining is to prove that for arbitrary  $x, y, z \in \mathbf{S}^d$ , it holds that

$$\bar{d}(x, z) \leq \bar{d}(x, y) + \bar{d}(y, z). \quad (5)$$

It is obvious that there exist unique  $k_0, j_0$  and  $l_0$  satisfying (3) such that  $x \in D(x_{k_0}, h_X), y \in D(x_{j_0}, h_X), z \in D(x_{l_0}, h_X)$ . Without loss of generality, we assume that  $k_0 < j_0 < l_0$ . It follows from (4) that

$$\bar{d}(x, y) = \sum_{i=k_0}^{j_0} d(x_i, x_{i+1}) + d(x_{k_0}, x) + d(x_{j_0}, y),$$

$$\bar{d}(x, z) = \sum_{i=k_0}^{l_0} d(x_i, x_{i+1}) + d(x_{k_0}, x) + d(x_{l_0}, z),$$

$$\bar{d}(z, y) = \sum_{i=j_0}^{l_0} d(x_i, x_{i+1}) + d(x_{j_0}, y) + d(x_{l_0}, z),$$

which implies (5) easily.

### 3. The general radial quasi-interpolation operator

In this section, we construct two general radial quasi-interpolation operators and study some properties of them. Let  $\sigma$  be the Heaviside function, i.e.

$$\sigma(t) := \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

For  $x_i \in X$   $i = 1, \dots, n$ , and  $x \in \mathbf{S}^d$ , define

$$c_1(x) := 1 - \sigma(\bar{d}(x_1, x)) = 0,$$

$$c_i(x) := \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_{i-1})) - \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_i)), \quad 2 \leq i \leq n-1,$$

$$c_n(x) := \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_{n-1})).$$

Then we can construct the general radial quasi-interpolation operator as

$$G_n(x) := \sum_{i=1}^n f_i c_i(x). \quad (6)$$

In order to study the approximation property of  $G_n$ , we need to introduce a modulus of smoothness on the sphere. Let  $SO(d+1)$  be the (compact) group of rotations on  $\mathbf{S}^d$ . For  $\rho \in SO(d+1)$ , the modulus of smoothness on  $\mathbf{S}^d$  is defined as

$$\omega(f, t) := \sup_{\rho \in O_t} \max_{x \in \mathbf{S}^d} |f(\rho x) - f(x)|,$$

where

$$O_t := \left\{ \rho \in SO(d+1) : \max_{x \in \mathbf{S}^{d-1}} \arccos x \cdot \rho x \leq t \right\}.$$

For more details of the modulus of smoothness  $\omega(f, t)$ , we refer the readers to [3].

The following theorem, [Theorem 1](#), gives a Jackson type error estimate for the general radial quasi-interpolation operator  $G_n$ .

**Theorem 1.** Let  $G_n$  be defined in (6). Let  $X$  be the set of scattered data with mesh norm  $h_X$  and mesh ratio  $\tau_X$  and  $\{(x_i, f_i)\}_{i=1}^n$  be the interpolation samples. If  $f \in C(\mathbf{S}^d)$  is the target function satisfying  $f(x_i) = f_i$ ,  $i = 1, \dots, n$ , then

$$|f(x) - G_n(x)| \leq 17\pi^{d-1}\tau_X^d\omega(f, h_X), \quad x \in \mathbf{S}^d. \quad (7)$$

**Remark 1.** It is obvious that the modulus of smoothness  $\omega(f, t)$  describes the smoothness of the target function  $f$ . If  $f$  satisfies some smoothness assumptions, then we can obtain the convergence rate of  $G_n$ . A popular assumption on the target function is that  $f$  satisfies the following Lipschitz condition:

$$|f(x) - f(y)| \leq C(d(x, y))^s,$$

where  $0 < s \leq 1$ . Thus, under the condition of [Theorem 1](#), it follows from the definition of  $\omega(f, t)$  that

$$\max_{x \in \mathbf{S}^d} |f(x) - G_n(x)| \leq C\tau_X^d h_X^s.$$

Suppose further that  $X$  is a  $\tau$ -uniform set (see [16]), i.e. there exists an absolute constant  $\tau$  such that  $\tau_X \leq \tau$  then we have

$$\max_{x \in \mathbf{S}^d} |f(x) - G_n(x)| \leq Cn^{-s/d},$$

where  $C$  is a constant depending only on  $\tau$  and  $d$ .

**Remark 2.** From the definition of the Heaviside function, we can deduce that the values of  $c_i(x)$ ,  $i = 1, \dots, n$ , are either 0 or 1. Furthermore, it follows from the definition of  $c_i(x)$  that

$$\sum_{i=1}^n c_i(x) = 1. \quad (8)$$

This implies that there exists a unique  $j_0$  such that  $c_{j_0}(x) = 1$  and  $c_i(x) = 0$  for all  $i \neq j_0$ .

**Remark 3.** From [Lemma 1](#) we know that for arbitrary  $x \in \mathbf{S}^d$ , there exists a unique point  $x_k$  satisfying (3) such that  $x \in D(x_k, h_X)$ . Then by the definition of  $\omega(f, t)$ , we have

$$|f(x) - f(x_k)| \leq \omega(f, h_X).$$

It is obvious that determining the index  $k$  for arbitrary fixed  $x \in \mathbf{S}^d$  is very difficult if we do not know the localization of  $x$ . The quasi-interpolation operator  $G_n$  is indeed a site function, which provides a method for determining  $k$ .

**Remark 4.** We assume that a system of functions  $\{g_i\}_{i=1}^m$  possesses a spline property if it holds that

$$\left| f(x) - \sum_{i=1}^m f(x_i)g_i(x) \right| \leq \varepsilon_m(f),$$

and

$$\sum_{i=1}^m |g_i(x)| \leq \kappa,$$

where  $\{\varepsilon_m(f)\}$  is a sequence of positive real numbers such that  $\varepsilon_m(f) \rightarrow 0$  as  $m \rightarrow \infty$  and  $\kappa > 0$  is an absolute constant. It follows from [Theorem 1](#) and [Remark 2](#) that the system of functions  $\{c_i(\cdot)\}_{i=1}^n$  possesses the spline property with  $\varepsilon_m(f) = 17\pi^{d-1}\tau_X^d\omega(f, h_X)$  and  $\kappa = 1$ . The spline property of  $\{c_i\}_{i=1}^n$  will play a crucial role for the applications of  $G_n$ .

In the following, we will construct another general radial quasi-interpolation operator. Let  $\phi$  be a sigmoidal function, i.e.

$$\phi(t) = \begin{cases} 1, & t \rightarrow \infty, \\ 0, & t \rightarrow -\infty. \end{cases}$$

For  $A \geq 0$ , if we set

$$\delta_\phi(A) := \sup_{t \geq A} \max(|1 - \phi(t)|, |\phi(-t)|),$$

then  $\delta_\phi(A)$  is non-increasing, and satisfies

$$\lim_{A \rightarrow +\infty} \delta_\phi(A) = 0. \quad (9)$$

Define

$$\begin{aligned} b_1(x) &:= \phi\left(-\frac{2A\bar{d}(x_1, x)}{\bar{d}(x_1, x_2)} + A\right), \\ b_i(x) &:= -\phi\left(-\frac{2A(\bar{d}(x_1, x) - \bar{d}(x_1, x_{i-1}))}{\bar{d}(x_1, x_i) - \bar{d}(x_1, x_{i-1})} + A\right) \\ &\quad + \phi\left(-\frac{2A(\bar{d}(x_1, x) - \bar{d}(x_1, x_i))}{\bar{d}(x_1, x_{i+1}) - \bar{d}(x_1, x_i)} + A\right), \end{aligned}$$

where  $2 \leq i \leq n-1$ , and

$$\begin{aligned} b_n(x) &:= -\phi\left(-\frac{2A(\bar{d}(x_1, x) - \bar{d}(x_1, x_{n-1}))}{\bar{d}(x_1, x_n) - \bar{d}(x_1, x_{n-1})} + A\right) \\ &\quad + \phi\left(-\frac{2A(\bar{d}(x_1, x) - \bar{d}(x_1, x_n))}{\bar{d}(x_1, x_n) - \bar{d}(x_1, x_{n-1})} + A\right). \end{aligned}$$

Then we can construct a general radial quasi-interpolation operator as

$$\Phi_n(x) := \sum_{i=1}^n f_i b_i(x). \quad (10)$$

The following theorem, [Theorem 2](#), gives an upper bound error estimate for this operator.

**Theorem 2.** Let  $d \geq 2$ , and  $\phi$  be a bounded sigmoidal function. Let  $X$  be the set of scattered data with mesh norm  $h_X$  and mesh ratio  $\tau_X$  and  $\{(x_i, f_i)\}_{i=1}^n$  be the interpolation samples. If

$f \in C(\mathbf{S}^d)$  is the target function satisfying  $f(x_i) = f_i$ ,  $i = 1, \dots, n$ , and  $\Phi_n$  is defined by (10), then

$$|f(x) - \Phi_n(x)| \leq \delta_\sigma(A) \left( \sum_{j=1}^{n-1} |f_j - f_{j+1}| + |f_n| \right) + (9 + 8\|\phi\|)\pi^{d-1}\tau^d\omega(f, h_X), \quad x \in \mathbf{S}^d, \quad (11)$$

where  $\|\phi\| := \max_{t \in \mathbf{R}} |\phi(t)|$ .

In the rest of this section, we illustrate some potential applications of the quasi-interpolation operators  $G_n$  and  $\Phi_n$ . The primary aim of satellite missions such as CHAMP and GOCE (see [5,6,9,8]) is to provide a unique model of the Earth's gravitational field and of the equipotential reference surface of the geoid on a global scale with high spatial resolution. It is well known that the mathematical model of satellite gravity gradiometry and satellite to satellite tracking is the ill-posed Fredholm integral equations of the first kind on the sphere (see [6,9,8]), which can be mathematically represented as

$$\int_{\mathbf{S}^d} k(x, y) f(y) d\omega(y) = g(x), \quad x \in \mathbf{S}^d, \quad (12)$$

where the non-degenerate kernel  $k(\cdot, \cdot)$  and  $g$  are assumed to be continuous functions.

The collocation scheme for solving Eq. (12) is determined by the set of collection points  $\Delta_n := \{t_i\}_{i=1}^n$  and by operators  $\mathcal{T}_{\Delta_n} : C(\mathbf{S}^d) \rightarrow \mathbf{R}^n$  such that

$$\mathcal{T}_{\Delta_n} f = (f(t_1), f(t_2), \dots, f(t_n)), \quad \forall f \in C(\mathbf{S}^d).$$

Then the original Fredholm integral equation (12) is replaced by a operator equation in  $\mathbf{R}^n$ , which can be written abstractly as

$$K_n f = \mathcal{T}_{\Delta_n} g, \quad (13)$$

where  $K_n := \mathcal{T}_{\Delta_n} K$ , and  $K$  is the integral operator defined by

$$(Kf)(x) = \int_{\mathbf{S}^d} k(x, y) f(y) d\omega(y).$$

For the ill-posed Eq. (13), a usual approach is Tikhonov regularization using perturbed values of  $g$  at the collocation points  $\{t_i\}$ . We assume further that the measurements of  $g$  are made at the points  $\{s_i\}_{i=1}^m$ , usually in the presence of some noise. This means that the measurement data are

$$g_j^\xi = g(s_j) + \xi_j, \quad j = 1, 2, \dots, m,$$

where  $\xi_j$  denotes an error of the  $j$ th measurement satisfying  $|\xi_j| \leq \xi$  for all  $j = 1, 2, \dots, m$ , and  $\xi$  is a constant. In the previous work, one usually took the measurement points as the collocation points; therefore the number of collocation points was interpreted as the amount of measurement points.

By using the operator (6), we can take arbitrary set of points as the collocation points. In other words, by the superb properties of the operator (6), we can produce an arbitrary amount of perturbed collocation data from a fixed amount of noisy measurement points, which is different from the classical approach. Indeed, from Theorem 1 and Remark 4, one can calculate  $\{g^\delta(t_i)\}_{i=1}^n$



from  $\{g^\xi(s_j)\}_{j=1}^m$  by using

$$g^\delta(t_i) = \sum_{j=1}^m g^\xi(s_j) c_j(t_i), \quad i = 1, \dots, n,$$

where  $\{c_j(\cdot)\}_{j=1}^m$  is defined in (6). Then, we have

$$\begin{aligned} |g(t_i) - g^\delta(t_i)| &\leq \left| g(t_i) - \sum_{j=1}^m g(s_j) c_j(t_i) \right| + \sum_{j=1}^m |g^\xi(s_j) - g(s_j)| |c_j(t_i)| \\ &\leq 17\pi^{d-1} \tau_X^d \omega(g, h_X) + \xi, \end{aligned} \quad (14)$$

where  $h_X$  and  $\tau_X$  are the mesh norm and mesh ratio of the set of measurement points  $\{s_j\}_{j=1}^m$ , respectively. The estimate (14) shows that it is reasonable to choose the set of measurement points satisfying  $17\pi^{d-1} \tau_X^d \omega(g, h_X) = \xi$ . Furthermore, it also follows from (14) that the level of collocation data noise  $\delta = \max\{|g(t_i) - g^\delta(t_i)|, i = 1, 2, \dots, n\}$  depends on the interplay between the level of the measurement errors  $\xi$  and the distribution of the measurement points, but it does not depend on the number of collocation points. Thus, to implement the regularized collocation method for Fredholm integral equations of the first kind, we can take arbitrary data as the collocation points and reduce the number of the measurements greatly.

#### 4. Proofs

In this section, some proofs are given.

**Proof of Theorem 1.** From Lemma 1 it follows that for arbitrary  $x \in \mathbf{S}^d$  there exists a unique  $k_0$  satisfying (3) such that  $x \in D(x_{k_0}, h_X)$ . Without loss of generality, we assume  $k_0 \geq \lceil 2\pi^{d-1} \tau_X^d \rceil$ , where  $\lceil a \rceil$  is the smallest integer larger than  $a$ . From (2), there exist at most  $\lceil 2\pi^{d-1} \tau_X^d \rceil$  points from  $X$  belonging to  $D(x_k, h_X)$ ,  $k = 1, \dots, n$ , then it follows from the definition of  $\bar{d}(\cdot, \cdot)$  that

$$\bar{d}(x_1, x_k) - \bar{d}(x_1, x) \geq h_X, \quad k \geq k_0 + \lceil 2\pi^{d-1} \tau_X^d \rceil, \quad (15)$$

$$\bar{d}(x_1, x_k) - \bar{d}(x_1, x) \leq -h_X, \quad k \leq k_0 - \lceil 2\pi^{d-1} \tau_X^d \rceil, \quad (16)$$

$$\begin{aligned} |\bar{d}(x_1, x) - \bar{d}(x_1, x_k)| &\leq 4\pi^{d-1} \tau_X^d h_X, \\ k_0 - \lceil 2\pi^{d-1} \tau_X^d \rceil &< k < k_0 + \lceil 2\pi^{d-1} \tau_X^d \rceil. \end{aligned} \quad (17)$$

Therefore, it follows from (15), (16) and the definition of  $\sigma$  that

$$\begin{aligned} f(x) - G_n(x) &= f(x) - \sum_{i=1}^n f_i c_i(x) \\ &= f(x) - \left( f_1 + \sum_{i=1}^{n-1} (f_{i+1} - f_i) \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_i)) \right) \\ &= f(x) - f_1 - \sum_{i=1}^{k_0 - \lceil 2\pi^{d-1} \tau_X^d \rceil - 1} (f_{i+1} - f_i) (\sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_i)) - 1) \\ &\quad + f_1 - f_{k_0 - \lceil 2\pi^{d-1} \tau_X^d \rceil} - \sum_{i=k_0 - \lceil 2\pi^{d-1} \tau_X^d \rceil}^{k_0 + \lceil 2\pi^{d-1} \tau_X^d \rceil} (f_{i+1} - f_i) \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_i)) \end{aligned}$$

$$\begin{aligned}
& - \sum_{i=k_0+\lceil 2\pi^{d-1}\tau_X^d \rceil+1}^{n-1} (f_{i+1} - f_i) \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_i)) \\
& = f(x) - f(x_{k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil}) \\
& \quad - \sum_{i=k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil}^{k_0+\lceil 2\pi^{d-1}\tau_X^d \rceil} (f(x_{i+1}) - f(x_i)) \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_i)).
\end{aligned}$$

Since  $\arccos(x \cdot x_{k_0}) \leq h_X$  and  $\arccos(x_{k_0} \cdot x_{k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil}) \leq 4\lceil 2\pi^{d-1}\tau_X^d \rceil$ , it follows from the definition of  $\omega(f, t)$  and (17) that

$$\begin{aligned}
|f(x) - f(x_{k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil})| & \leq \max_{\arccos(\rho x \cdot x) \leq (1+4\lceil 2\pi^{d-1}\tau_X^d \rceil)h_X} |f(\rho x) - f(x)| \\
& \leq (2 + 4\lceil 2\pi^{d-1}\tau_X^d \rceil) \max_{\arccos(\rho x \cdot x) \leq h_X} |f(\rho x) - f(x)| \\
& \leq 9\pi^{d-1}\tau_X^d \omega(f, h_X),
\end{aligned}$$

where the proof of the second inequality can be found in [3, Theorem 2.3]. Furthermore,

$$\begin{aligned}
& \left| \sum_{i=k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil}^{k_0+\lceil 2\pi^{d-1}\tau_X^d \rceil} (f(x_{i+1}) - f(x_i)) \sigma(\bar{d}(x_1, x) - \bar{d}(x_1, x_i)) \right| \\
& \leq 8\pi^{d-1}\tau_X^d \max_{\arccos(\rho x \cdot x) \leq h_X} |f(\rho x) - f(x)| \leq 8\pi^{d-1}\tau_X^d \omega(f, h_X).
\end{aligned}$$

Thus for any  $x \in \mathbf{S}^d$ , it holds that

$$\left| f(x) - \sum_{i=1}^n f_i c_i(x) \right| \leq 17\pi^{d-1}\tau_X^d \omega(f, h_X).$$

This finishes the proof of Theorem 1.  $\square$

**Proof of Theorem 2.** The proof of Theorem 2 is the same as that of Theorem 1, so we only sketch it. From the definition of  $\phi$ , (15)–(17) we have

$$\begin{aligned}
f(x) - \Phi_n(x) & = \sum_{i=1}^{k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil} (f_i - f_{i+1}) \phi \left( -\frac{2A(\bar{d}(x_1, x) - \bar{d}(x_1, x_i))}{\bar{d}(x_1, x_{i+1}) - \bar{d}(x_1, x_i)} + A \right) \\
& \quad + \sum_{i=k_0+\lceil 2\pi^{d-1}\tau_X^d \rceil+1}^{n-1} (f_i - f_{i+1}) \\
& \quad \times \left( \phi \left( -\frac{2A(\bar{d}(x_1, x) - \bar{d}(x_1, x_i))}{\bar{d}(x_1, x_{i+1}) - \bar{d}(x_1, x_i)} + A \right) - 1 \right) \\
& \quad + f(x) - f_{k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil} - \sum_{i=k_0-\lceil 2\pi^{d-1}\tau_X^d \rceil}^{k_0+\lceil 2\pi^{d-1}\tau_X^d \rceil} (f_i - f_{i+1}) \phi \\
& \quad \times \left( -\frac{2A(\bar{d}(x_1, x) - \bar{d}(x_1, x_i))}{\bar{d}(x_1, x_{i+1}) - \bar{d}(x_1, x_i)} + A \right).
\end{aligned}$$

By the definition of  $\delta_\phi(A)$ , we have

$$|f(x) - \Phi_n(x)| \leq \delta_\phi(A) \left( \sum_{j=1}^{n-1} |f_j - f_{j+1}| + |f_n| \right) + (9 + 8\|\phi\|)\pi^{d-1}\tau_X^d \omega(f, h_X).$$

This completes the proof of Theorem 2.  $\square$

## Acknowledgments

The authors wish to thank the anonymous referees for their careful reviews and helpful suggestions.

## Appendix

In this Appendix, we give a method for rearranging scattered data such that (A1)–(A3) hold. Without loss of generality, we only consider the case  $d = 2$ . Denote by  $B(x_0, \alpha, \beta)$  the spherical band with center  $x_0$  and angle from  $\alpha$  to  $\beta$ , i.e.,

$$B(x_0, \alpha, \beta) := \{x \in \mathbf{S}^2 : \cos \alpha < x \cdot x_0 \leq \cos \beta\}.$$

Set  $x^* := (0, 0, 1)$  and  $x_0^* = (0, 0, -1)$ , then

$$\mathbf{S}^2 = D(x^*, 2h_X) \cup \left( \bigcup_{1 \leq k \leq \pi/h_X} B(x^*, 2kh_X, 4kh_X) \right) \cup D(x_0^*, h_0),$$

where  $h_0 \leq 2h_X$ . Now, we give a partition for every spherical band  $B(x^*, 2kh_X, 4kh_X)$ . Since the width of the band equals  $2h_X$ , there is at least one spherical cap  $D(z, h_X)$  with  $z \in E_k := \{x \in \mathbf{S}^2 : x^* \cdot x = \cos 3kh_X\}$  belonging to  $B(x^*, 2kh_X, 4kh_X)$ . Therefore, we can give a partition of  $B(x^*, 2kh_X, 4kh_X)$  as

$$B(x^*, 2kh_X, 4kh_X) = \bigcup_{1 \leq j \leq \mathcal{L}_k} S_k^j, \quad S_k^j \cap S_k^i = \emptyset, j \neq i,$$

where  $S_k^j \subset B(x^*, 2kh_X, 4kh_X)$  satisfies that there is a  $z^j \in E_k$  such that

$$D(z^j, h_X) \subset S_k^j \subset D(z^j, 2h_X),$$

and  $\mathcal{L}_k$  is the number of the  $S_k^j$  based on the above partition. Therefore,

$$\mathbf{S}^2 = D(x^*, 2h_X) \cup \left( \bigcup_{1 \leq k \leq \pi/h_X} \bigcup_{1 \leq j \leq \mathcal{L}_k} S_k^j \right) \cup D(x_0^*, h_0).$$

Since for arbitrary  $S_k^j$ , there is a spherical cap  $D(z^j, h_X) \subset S_k^j$ , then there is at least one point in  $X$  belonging to  $S_k^j$ .

For arbitrary  $x \in \mathbf{S}^2$ , denote by  $\text{lat}(x)$  and  $\text{lon}(x)$  the latitude and longitude of  $x$ , respectively. We define  $x < y$  if one of the following two relations holds: (i)  $\text{lat}(x) < \text{lat}(y)$ , (ii)  $\text{lat}(x) = \text{lat}(y)$ ,  $\text{lon}(x) < \text{lon}(y)$ . Now, we give a rearrangement of the scattered data based on the above partition. Without loss of generality, we assume that  $x^* \in X$  and there are  $b_1$  points

of  $X$  belonging to  $D(x^*, 2h_X)$ ,  $b_2$  points belonging to  $D(x_0^*, h_0)$ , and  $b_{k,j}$  points belonging to  $S_k^j$ .

Step 1: Set  $y_1 = x^*$ .

Step 2: Rearrange the points in  $D(x^*, 2h_X)$  such that

$$y_1 < y_2 < \cdots < y_{b_1}.$$

Then, it is obvious that (A1)–(A3) hold for  $j = 1, \dots, b_1$ .

Step 3: Set  $a_{s,t}(x)$  as the point satisfying  $\text{lat}(a_{s,t}(x)) = \text{lat}(x) + s$ ,  $\text{lon}(a_{s,t}(x)) = \text{lon}(x) + t$ . Then there exists a  $j_0$  such that  $S_1^{j_0} \subset B(x^*, 2h_X, 4h_X)$  satisfying  $a_{2h_X,0}(y_{b_1}) \in S_1^{j_0}$ . For the sake of brevity, we rewrite  $S_1^{j_0}$  as  $S_1^1$ . Now we rearrange the points in  $S_1^1$  as

$$y_{b_1+1} < y_{b_1+2} < \cdots < y_{b_1+b_{1,1}}.$$

Since there exists a  $z_{1,1} \in E_1$  such that  $S_1^1 \subset D(z_{1,1}, 2h_X)$ , we have

$$d(y_{b_1}, y_{b_1+1}) \leq d(y_{j_1}, a_{2h_X,0}(y_{b_1})) + d(a_{2h_X,0}(y_{b_1}), y_{b_1+1}) \leq 2h_X + 2h_X = 4h_X.$$

Thus for all  $j = 1, \dots, b_1 + b_{1,1}$ , (A1)–(A3) hold.

Step 4: Since for every  $S_1^j$ ,  $j = 1, \dots, \mathcal{L}_1$ , there is at least one point  $z_{1,j} \in E_1$  such that  $D(z_{1,j}, h_X) \subset S_1^j \subset D(z_{1,j}, 2h_X)$ , it follows from  $S_1^j \cap S_1^i = \emptyset$ ,  $i \neq j$ , that we can choose  $\mathcal{L}_1$  points  $z_{1,j} \in E_1$ ,  $j = 1, \dots, \mathcal{L}_1$ , to label  $S_{1,j}$ . If we rearrange  $\{z_{1,1}, z_{1,2}, \dots, z_{1,\mathcal{L}_1}\}$  clockwise, then we can find  $S_1^2$  and rearrange the points in  $S_1^2$  as

$$y_{b_1+b_{1,1}+1} < y_{b_1+b_{1,1}+2} < \cdots < y_{b_1+b_{1,1}+b_{1,2}}.$$

Since  $S_1^1 \subset D(z_{1,1}, 2h_X)$ ,  $S_1^2 \subset D(z_{1,2}, 2h_X)$ , there exists a point  $z^*$  such that

$$S_1^1 \cup S_1^2 \subset D(z^*, 4h_X).$$

Thus,  $d(y_{b_1+b_{1,1}}, y_{b_1+b_{1,1}+1}) \leq 4h_X$ . Then for all  $j = 1, \dots, b_1 + b_{1,1} + b_{1,2}$ , (A1)–(A3) hold.

Step 5: Repeating the method in Step 4  $\mathcal{L}_1$  times, we can prove that for all  $j = 1, \dots, b_1 + b_{1,1} + \cdots + b_{1,\mathcal{L}_1}$ , (A1)–(A3) hold.

Step 6: Repeating the method in Step 3, we can find  $S_2^1$  and  $\{y_i\}_{i=b_1+b_{1,1}+\cdots+b_{1,\mathcal{L}_1}+1}^{b_1+b_{1,1}+\cdots+b_{1,\mathcal{L}_1}+b_{2,1}}$  and prove that (A1)–(A3) hold for  $j = 1, \dots, b_1 + b_{1,1} + \cdots + b_{1,\mathcal{L}_1} + b_{2,1}$ .

Step 7: Repeating the method in Step 4  $\mathcal{L}_2$  times, we can prove that for all  $j = 1, \dots, b_1 + b_{1,1} + \cdots + b_{2,\mathcal{L}_2}$ , (A1)–(A3) hold.

Step 8: Repeating Step 3 to Step 7 until all scattered data have been chosen we can give a rearrangement of  $X$  such that (A1)–(A3) hold.

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