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A convergence rate for approximate solutions of Fredholm integral equations of the first kind

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Abstract We study the convergence rate for solving Fredholm integral equations of the first kind by using the well known collocation method. By constructing an approximate interpolation neural network, we deduce the convergence rate of the approximate solution by only using continuous functions as basis functions for the Fredholm integral equations of the first kind. This convergence rate is bounded in terms of a modulus of smoothness.

Keywords Fredholm integral equations · Approximate solution · Convergence rate

Mathematics Subject Classification (2000) 65J20 · 41A15 · 41A63

1 Introduction

Many inverse problems in mathematical physics and applied mathematics have their most natural mathematical expression in terms of Fredholm integral equations of the first kind (see [6,8,16–18]). That is, equations of the form

$$f(x) = \int_Y K(x, y) g(y)\, dy, \quad x \in X, \tag{1}$$

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where X, Y are domains on which data are collected, $K(x, y)$ is defined on $X \times Y$ and represents a known (or assumed) model for a physical phenomenon, f is an observation and g is an unknown "cause" to be determined.

There are several methods for constructing numerical solutions of these equations (see [2,3,11–13] etc.). In practice, one can only get finitely many measurements of the form $f_i = f(x_i),\ x_i \in \Delta := \{x_1, \dots, x_n\}$. This suggests finding an approximate solution g which satisfies

$$f_i = \int_Y K(x_i, y) g(y)\, dy, \quad i = 1, \dots, n \tag{2}$$

and minimizes a quadratic functional $J(g)$ of the form

$$J(g) := \|R^{-1/2} g\|^2_{L^2(Y)}, \tag{3}$$

where $R^{-1/2}$ is a densely defined, unbounded linear operator on $L^2(Y)$ to be selected from a certain general class. The collocation method above has received the most favorable attention in the engineering community due to its lower computational cost in generating the coefficient matrix of the Eq. (2). We refer the readers to [4,5,9,10] and [19] etc. for more details.

In order to guarantee the feasibility of the collocation method, theoretical results for this method are needed. In particular, we are interested in deducing the convergence rate for the approximate solution obtained by the collocation method. There have been some results on this topic, such as Wahba [20], Nashed and Wahba [14], Atkinson [1], and Groetsch [7]. For example, Groetsch [7] presented a simple asymptotic convergence analysis of a degenerate kernel method for Fredholm integral equations of the first kind which is based on a quadrature of the variational equation for the Tikhonov functional. Wahba [20] gave a convergence rate of the approximate solution to Fredholm integral equation when the kernel function $K(\cdot, \cdot)$ satisfies some smoothness assumptions. They used the mesh norm and the degree of smoothness to bound the convergence rate of the approximate solution. In this paper, by introducing an approximate interpolation neural network, we develop a convergence rate analysis of the collection method without smoothness assumptions on the kernel. That is, to study the convergence rate of approximate solutions to Fredholm equations of the first kind, we only need the assumption of continuity of the basis functions.

2 Preliminaries

At first, we must introduce the reproducing kernel Hilbert space with a kernel $R(\cdot, \cdot)$, which is assumed to be fixed for the given model and is continuous, symmetric, positive definite. By the theorems of Mercer, Hilbert and Schmidt [15, p. 242], the operator R, defined by

$$(Rf)(y) := \int_Y R(y, y') f(y')\, dy', \quad f \in L^2(Y),$$

has an $L^2(Y)$ complete orthonormal system of eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ and corresponding positive eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ satisfying $\sum_{j=1}^{\infty} \lambda_j^2 < \infty$. Also, the kernel $R(\cdot, \cdot)$ has the uniformly convergent Fourier expansion

$$R(y, y') = \sum_{j=1}^{\infty} \lambda_j \phi_j(y) \phi_j(y').$$

Define

$$H_R := \left\{ g : g \in L^2(Y), \sum_{j=1}^{\infty} \frac{\hat{g}_j^2}{\lambda_j} < \infty,\ \hat{g}_j = \langle g, \phi_j \rangle_{L^2(Y)} \right\},$$

and let R_y, for fixed y, be a function on Y whose value at y' is given by

$$R_y(y') = R(y, y').$$

It is well known that H_R is a reproducing kernel Hilbert space with $R(\cdot, \cdot)$ being its reproducing kernel. Its inner product $\langle \cdot, \cdot \rangle_R$ and norm $\| \cdot \|_R$ are given by

$$\langle f, g \rangle_R = \sum_{j=1}^{\infty} \frac{\hat{f}_j \hat{g}_j}{\lambda_j},$$

and

$$\|f\|_R = \left(\sum_{j=1}^{\infty} \frac{\hat{f}_j^2}{\lambda_j} \right)^{\frac{1}{2}},$$

respectively. Moreover, it is easy to deduce that

$$\|R^{-1/2} f\|_{L^2(Y)} = \|f\|_R.$$

Thus, the approximate solution satisfying (2) and minimizing (3) is that satisfies (2) and minimizes $\|g\|_R$.

Since the approximate solution should satisfy (2) and minimize $\|g\|_R$, it depends not only on $R(\cdot, \cdot)$ but also on the kernel $K(\cdot, \cdot)$. Suppose that K has the property that the family of linear functionals $\{\Lambda_x, x \in X\}$ defined by

$$\Lambda_x g := \int_Y K(x,y)g(y)\,dy, \quad x \in X, g \in H_R$$

are all continuous in H_R, and linearly independent. Define

$$\eta_x(y) := \int_Y K(x,z)R(y,z)\,dz, \quad y \in Y,$$

then we can rewrite (2) as

$$f_i = \langle \eta_{x_i}, g\rangle_R, \quad i = 1, \ldots, n. \tag{4}$$

Let Φ_n be the (n dimensional) subspace of H_R spanned by $\{\eta_{x_i}, x_i \in \Delta\}$, and let P_{Φ_n} be the orthogonal projection in H_R onto Φ_n. The following Proposition 1 reveals an important property for the projection operator P_{Φ_n}.

Proposition 1 *If g is an arbitrary element in H_R satisfying* (4)*, then $g^* := P_{\Phi_n} g$ satisfies* (4) *also and minimizes the norm $\|g\|_R$ among all such solutions.*

Proof We only need to verify that g^* is the function with minimal norm satisfying (4). Since $g^* = P_{\Phi_n} g$, we have

$$\langle g - g^*, g^*\rangle_R = 0.$$

Using this equality, we obtain

$$\begin{aligned}\|g\|_R^2 &= \langle g^* + (g-g^*), g^* + (g-g^*)\rangle_R = \|g^*\|_R^2 + \|g-g^*\|_R^2 + 2\langle g-g^*, g^*\rangle_R \\ &= \|g^*\|_R^2 + \|g-g^*\|_R^2 \ge \|g^*\|_R^2.\end{aligned}$$

Thus, g^* is the minimum norm solution. This completes the proof of Proposition 1. □

Let $N(K)$ be the null space of K in H_R (possible the 0 function) and let $V := N(K)^\perp$ (in H_R). Then, by definition,

$$0 = \int_Y K(x,y)g(y)dy, \quad x \in X, g \in H_R \Rightarrow g \in N(K).$$

By the definition of η_x, the above equation also can be written as

$$0 = \langle \eta_x, g\rangle_R, \quad x \in X, g \in H_R \Rightarrow g \in V^\perp.$$

Thus, $\{\eta_x, x \in X\}$ spans V. Let P_V be the projection operator in H_R onto V, and define the mesh norm of Δ by

$$h_\Delta := \max_{1 \le i \le n-1} (x_{i+1} - x_i).$$

It follows that

$$\lim_{h_\Delta \to 0} \| P_V g - P_{\Phi_n} g \|_R = 0, \tag{5}$$

for any fixed $g \in H_R$. The main purpose of this paper is to establish a rate for this convergence.

Let

$$Q(x, x') := \int_Y \int_Y K(x, y) R(y, y') K(x', y') \, dy \, dy', \tag{6}$$

then we have

$$\langle \eta_{x_i}, \eta_{x_j} \rangle_R = Q(x_i, x_j). \tag{7}$$

Define operators K, K^*, and Q by

$$(Kf)(x) := \int_Y K(x, y) f(y) \, dy, \quad x \in X, \quad f \in L^2(Y),$$

$$(K^* f)(y) := \int_X K(x, y) f(x) \, dx, \quad y \in Y, \quad f \in L^2(X),$$

and

$$(Qf)(x) = \int_X Q(x, x') f(x') \, dx', \quad x \in X, \quad f \in L^2(X),$$

respectively. It is obvious that

$$Q = KRK^*$$

It is convenient to define an auxiliary Hilbert space H_Q. We let H_Q be the reproducing kernel Hilbert space with reproducing kernel $Q(x, x')$, $x, x' \in X$ defined by (6), and inner product $\langle \cdot, \cdot \rangle_Q$. Let Q_x be the element of H_Q whose value at x' is given by $Q_x(x') := Q(x, x')$. Let H_{T_n} be the subspace of H_Q spanned by the elements $\{Q_{x_i}\}_{i=1}^n$. Denote by P_{T_n} be the projection operator in H_Q onto H_{T_n}.

Finally, we need two Lemmas which will play important roles in the proof of our main result, whose proofs can be found in [20].

Lemma 1 *Given $g \in H_R$ let f be defined by*

$$f(x) := \langle \eta_x, g \rangle_R, \quad x \in X.$$

Then $f \in H_Q$ and

$$\|P_V g - P_{\Phi_n} g\|_R = \|f - P_{T_n} f\|_Q.$$

Since

$$\langle Q_x, Q_{x'} \rangle_Q = Q(x, x') = \langle \eta_x, \eta_{x'} \rangle_R, \quad x, x' \in X$$

and $\{Q_x, x \in X\}$ span H_Q, there is an isometric isomorphism between H_Q and V generated by the correspondence "$\sim$" (see [20]),

$$H_Q \ni Q_x \sim \eta_x \in V, \quad x \in X. \tag{8}$$

Now we are in a position to give the second lemma [20, Lemma 2].

Lemma 2 *Suppose g has a representation*

$$g(y) = \int_X \eta_{x'}(y) h(x')\, dx',$$

for some $h \in L^2(X)$. Then, $g \in V$ and $g \sim f$ under the correspondence "$\sim$", where

$$f(x) = \int_X Q_x(x') h(x')\, dx'. \tag{9}$$

3 Main results

In this section, we give the main result of this paper. Clearly, if g is a solution to the equation (1), then $P_V g$ is also a solution to (1). And since $V = N^{\perp}(K)$, we know that $P_V g$ is the unique solution to (1) from V. It also can be seen that P_{Φ_n} is an approximate solution to (1). In (5), we know that when the mesh norm h_Δ small enough, then the approximate $P_{\Phi_n} g$ convergence to the solution $P_V g$. So, it is natural to raise the question: what is the convergence rate of such approximation? In [20], Wahba has given an error estimate for this approximation when the kernel $Q(\cdot, \cdot)$ satisfies some smoothness properties. In this paper, we omit the smoothness properties of the kernel $Q(\cdot, \cdot)$ and use a modulus of smoothness of the kernel function $Q(\cdot, \cdot)$ to describe the convergence rate.

Let $x_1 < x_2 < \cdots < x_n$ be the samples, $X := [x_1, x_n]$, and Y be a bounded interval. Let $Q(\cdot, \cdot)$ be defined by (6). We define the modulus of smoothness of the kernel $Q(\cdot,\cdot)$ as

$$\omega(Q,t) := \sup_{|s|\le t} \max_{x,x+s,x'\in X} |Q(x',x) - Q(x',x+s)|.$$

The following theorem is the main result of this paper.

Theorem 1 *Suppose that $g \in H_R$ satisfies $P_V g \in RK^*(L^2(X))$, or equivalently,*

$$f = Kg \in KRK^*(L^2(X)) = Q(L^2(X)),$$

and suppose that $Q(\cdot,\cdot)$ is continuous on $X \times X$. Then there exists a constant C depending only on f such that

$$\|P_V g - P_{\Phi_n} g\|_R \le C\,(\omega(Q,h_\Delta))^{\frac{1}{2}}.$$

Proof By Lemmas 1 and 2 it is sufficient to prove that there exists a constant C depending only on f such that

$$\|f - P_{T_n} f\|_Q \le C\left(\omega(Q,h_\Delta)\right)^{\frac{1}{2}}, \tag{10}$$

where $f(x) = \int_X Q_x(x')h(x')\,dx'$, and $h \in L^2(X)$.

Now we construct a function $\phi \in H_{T_n}$ such that

$$\|f-\phi\|_Q^2 \le 5\left|\int\limits_X\int\limits_X h(x)h'(x)\,dx\,dx'\right|\omega(Q,h_\Delta), \tag{11}$$

Set $\sigma(t) = \frac{1}{1+e^{-t}}$. It is obvious that when $t\to\infty$, $\sigma(t)\to 1$ and when $t\to-\infty$, $\sigma(t)\to 0$. Then for $A \ge 0$, if we set

$$\delta_\sigma(A) := \sup_{t\ge A}\max(|1-\sigma(t)|, |\sigma(-t)|),$$

then $\delta_\sigma(A)$ is non-increasing, and satisfies

$$\lim_{A\to+\infty}\delta_\sigma(A) = 0. \tag{12}$$

Denote

$$c_1(x) := \sigma\left(-\frac{2A(x-x_1)}{x_2-x_1}+A\right),$$
$$c_i(x) := -\sigma\left(-\frac{2A(x-x_{i-1})}{x_i-x_{i-1}}+A\right)+\sigma\left(-\frac{2A(x-x_i)}{x_{i+1}-x_i}+A\right),\quad 2\le i\le n-1,$$
$$c_n(x) := -\sigma\left(-\frac{2A(x-x_{n-1})}{x_n-x_{n-1}}+A\right)+\sigma\left(-\frac{2A(x-x_n)}{x_n-x_{n-1}}+A\right).$$

Now we define

$$\phi(x) := \sum_{i=1}^{n} Q_{x_i}(x) \int_X c_i(x)h(x)\,dx.$$

It is obvious that $\phi \in H_{T_n}$, and hence

$$\|f - P_{T_n} f\|_Q^2 \leq \|f - \phi\|_Q^2.$$

The only thing remaining is to prove (11). Since f satisfies (9), we have

$$\langle f, f\rangle_Q = \int_X \int_X h(x)Q(x,x')h(x')\,dx\,dx',$$

$$\langle f, Q_{x_i}\rangle_Q = f(x_i) = \int_X Q_{x_i}(x)h(x)\,dx,$$

then it follows that

$$\begin{aligned}
\|f-\phi\|_Q^2 &= \langle f-\phi, f-\phi\rangle_Q \\
&= \left\langle f - \sum_{i=1}^{n} Q_{x_i} \int_X c_i(x)h(x)dx,\, f - \sum_{j=1}^{n} Q_{x_j} \int_X c_j(x')h(x')dx' \right\rangle_Q \\
&= \int_X \int_X h(x)h(x') \left\langle Q_x - \sum_{i=1}^{n} c_i(x)Q_{x_i},\, Q_{x'} - \sum_{j=1}^{n} c_j(x')Q_{x_j} \right\rangle_Q \,dx\,dx' \\
&= \int_X \int_X \Bigg(\langle Q_x, Q_{x'}\rangle_Q - \sum_{j=1}^{n} c_j(x')\langle Q_x, Q_{x_j}\rangle_Q \\
&\quad - \sum_{i=1}^{n} c_i(x) \left(\langle Q_{x_i}, Q_{x'}\rangle_Q - \sum_{j=1}^{n} c_j(x')\langle Q_{x_i}, Q_{x_j}\rangle_Q \right) \Bigg) h(x)h(x')\,dx\,dx' \\
&= \int_X \int_X \Bigg(Q_{x'}(x) - \sum_{i=1}^{n} c_i(x')Q_{x_i}(x) \\
&\quad - \sum_{j=1}^{n} c_j(x') \left(Q_{x'}(x_j) - \sum_{i=1}^{n} c_i(x')Q_{x_i}(x_j) \right) \Bigg) h(x)h(x')\,dx\,dx'.
\end{aligned}$$

Set

$$\rho(x) := Q_{x'}(x) - \sum_{i=1}^{n} c_i(x') Q_{x_i}(x).$$

Since for arbitrary $x \in X$, there exists a $j_0 \in \mathbb{N}$ such that $x \in [x_{j_0}, x_{j_0+1}]$ (without loss of generality, we assume $2 \leq j_0 \leq n-1$), we have

$$\begin{aligned}
&\rho(x) - \sum_{j=1}^{n} c_j(x)\rho(x_j) \\
&= \rho(x) - \sum_{j=1}^{n-1}(\rho(x_j) - \rho(x_{j+1}))\sigma\left(-\frac{2A(x-x_j)}{x_{j+1}-x_j} + A\right) \\
&\quad + \rho(x_n)\left(-\frac{2A(x-x_n)}{x_n - x_{n-1}} + A\right) \\
&= \sum_{j=1}^{j_0-1}(\rho(x_j) - \rho(x_{j+1}))\sigma\left(-\frac{2A(x-x_j)}{x_{j+1}-x_j} + A\right) \\
&\quad + \sum_{j=j_0+1}^{n-1}(\rho(x_j) - \rho(x_{j+1}))\left(\sigma\left(-\frac{2A(x-x_j)}{x_{j+1}-x_j} + A\right) - 1\right) \\
&\quad + \rho(x_n)\left(\sigma\left(-\frac{2A(x-x_n)}{x_n - x_{n-1}} + A\right) - 1\right) + \rho(x_{j_0+1}) - \rho(x) \\
&\quad + (\rho(x_{j_0}) - \rho(x_{j_0+1}))\sigma\left(-\frac{2A(x-x_{j_0})}{x_{j_0+1}-x_{j_0}} + A\right).
\end{aligned}$$

Since $x \in [x_{j_0}, x_{j_0+1}]$, we have

$$-\frac{2A(x-x_j)}{x_{j+1}-x_j} + A \leq -A, \quad 1 \leq j \leq j_0 - 1,$$

$$-\frac{2A(x-x_j)}{x_{j+1}-x_j} + A \geq A, \quad 1 \leq j \leq j_0 - 1, \;\; j_0 + 1 \leq j \leq n-1,$$

and

$$-\frac{2A(x-x_n)}{x_n - x_{n-1}} + A \geq A.$$

Thus by the definition of $\delta_\sigma(A)$, we have

$$\begin{aligned}\left|\rho(x)-\sum_{j=1}^{n}c_j(x)\rho(x_j)\right| &\leq \delta_\sigma(A)\sum_{j=0}^{j_0-1}|\rho(x_j)-\rho(x_{j+1})| \\ &\quad+\delta_\sigma(A)\sum_{j=j_0+1}^{n-1}\left|\rho(x_j)-\rho(x_{j+1})\right|+\delta_\sigma(A)\,|\rho(x_n)| \\ &\quad+\left|\rho(x_{j_0+1})-\rho(x)\right|+\left|(\rho(x_{j_0})-\rho(x_{j_0+1})\right| \\ &\leq \delta_\sigma(A)\left(\sum_{j=0}^{n-1}\left|\rho(x_j)-\rho(x_{j+1})\right|+|\rho(x_n)|\right) \\ &\quad+\left|\rho(x_{j_0+1})-\rho(x)\right|+\left|(\rho(x_{j_0})-\rho(x_{j_0+1})\right|,\end{aligned}$$

where in the first inequality, we use the fact

$$\frac{1}{1+e^{-t}}\leq 1,\quad t\in\mathbb{R}.$$

It follows from (12) that we can choose A large enough such that

$$\delta_\sigma(A)\leq\frac{\omega(Q,h_\Delta)}{\max_{x'\in X}\left\{\sum_{j=1}^{n-1}|\rho(x_j)-\rho(x_{j+1})|+|\rho(x_n)|\right\}}. \tag{13}$$

Since

$$\rho(x)=Q_{x'}(x)-\sum_{i=1}^{n}c_i(x')Q_{x_i}(x),$$

we have

$$\begin{aligned}&|\rho(x_{j_0+1})-\rho(x_{j_0})| \\ &\quad=\left|Q_{x'}(x_{j_0+1})-\sum_{i=1}^{n}c_i(x')Q_{x_i}(x_{j_0+1})-Q_{x'}(x_{j_0})+\sum_{i=1}^{n}c_i(x')Q_{x_i}(x_{j_0})\right| \\ &\quad\leq|Q_{x'}(x_{j_0+1})-Q_{x'}(x_{j_0})|+\max_{x'\in X}\left\{|Q_{x'}(x_{j_0+1})-Q_{x'}(x_{j_0})|\left|\sum_{i=1}^{n}c_i(x')\right|\right\}.\end{aligned}$$

Since

$$\sum_{i=1}^{n}c_i(x')=\sigma\left(-\frac{2A(x-x_n)}{x_n-x_{n-1}}+A\right), \tag{14}$$

we have

$$|\rho(x_{j_0+1}) - \rho(x_{j_0})| \leq 2\omega(Q, h_\Delta).$$

Similarly, we can prove

$$|\rho(x_{j_0+1}) - \rho(x)| \leq 2\omega(Q, h_\Delta), \quad x \in [x_{j_0}, x_{j_0+1}].$$

Combining the results above, we deduce that

$$\max_{x' \in X} \left\{ \left| \rho(x) - \sum_{j=1}^{n} c_j(x)\rho(x_j) \right| \right\} \leq 5\omega(Q, h_\Delta).$$

Thus

$$\|f - \phi\|_Q^2 \leq 5 \left| \int_X \int_X h(x)h(x')dxdx' \right| \omega(Q, h_\Delta).$$

This finishes the proof of Theorem 1. □

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