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Multilinear Algebra
PARAFAC and Tensor Rank

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1. We know that if we have the tensor defined as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{1}$$

then if we have $\boldsymbol{b}_1 = \boldsymbol{b}_2$ and $\boldsymbol{c}_1 = \boldsymbol{c}_2$ we can guarantee that $\mathcal{X}$ is rank one. We can begin this proof by using the associativity property of the outer product to write tensor $\mathcal{X}$ as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{2}$$
$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1, \tag{3}$$
$$\mathcal{X} = (\boldsymbol{a}_1 + \boldsymbol{a}_2) \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1, \tag{4}$$

and by inspecting above expression we can observe that independent of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ being collinear we will have a rank one tensor defined as

$$\mathcal{X} = \boldsymbol{a}_3 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1, \tag{5}$$

where $\boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2$. In a similar fashion, if we have $\boldsymbol{b}_1 \neq \boldsymbol{b}_2$ and $\boldsymbol{c}_1 = \boldsymbol{c}_2$ then we can rewrite the tensor as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{6}$$
$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_1, \tag{7}$$
$$\mathcal{X} = (\boldsymbol{a}_1 \circ \boldsymbol{b}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2) \circ \boldsymbol{c}_1, \tag{8}$$

but since we can guarantee that the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are not collinear then we know that the sum in above expression cannot be further reduced. Thus, we will have a tensor composed of the sum of two subtensors of rank one meaning that our tensor $\mathcal{X}$ will be a rank two tensor.

2. To show that the tensor rank is a property we will begin by writting the following multilinear transformations

$$\mathcal{X} = \mathcal{S} \times_1 \boldsymbol{A}^{(1)} \cdots \times_3 \boldsymbol{A}^{(3)}, \tag{9}$$
$$\mathcal{S} = \mathcal{X} \times_1 \boldsymbol{A}^{(1)^{\mathrm{H}}} \cdots \times_3 \boldsymbol{A}^{(3)^{\mathrm{H}}}. \tag{10}$$

First we can begin by defining the core tensor $\mathcal{S}$ by its PARAFAC Decomposition as

$$\mathcal{S} = \sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(3)}, \tag{11}$$

and by returning to the original transformation we can rewrite it as

$$\mathcal{X} = \left(\sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(N)} \right) \times_1 \boldsymbol{A}^{(1)} \cdots \times_3 \boldsymbol{A}^{(3)}, \tag{12}$$
$$\mathcal{X} = \left(\sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(N)} \times_1 \boldsymbol{A}^{(1)} \cdots \times_3 \boldsymbol{A}^{(3)} \right), \tag{13}$$
$$\mathcal{X} = \sum_{r=1}^{R} \boldsymbol{A}^{(1)} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{A}^{(N)} \boldsymbol{s}_r^{(N)}. \tag{14}$$

Now considering the inverse multilinear transformation and that ${\boldsymbol{A}^{(n)}}^{\mathrm{H}}\boldsymbol{A}^{(n)} = \boldsymbol{I}, \forall n \in \{1, \cdots, N\}$ we can rewrite tensor $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{X} \times_1 {\boldsymbol{A}^{(1)}}^{\mathrm{H}} \cdots \times_3 {\boldsymbol{A}^{(3)}}^{\mathrm{H}}, \tag{15}$$

$$\mathcal{S} = \left(\sum_{r=1}^{R} \boldsymbol{A}^{(1)}\boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{A}^{(N)}\boldsymbol{s}_r^{(N)} \right) \times_1 {\boldsymbol{A}^{(1)}}^{\mathrm{H}} \cdots \times_3 {\boldsymbol{A}^{(3)}}^{\mathrm{H}}, \tag{16}$$

$$\mathcal{S} = \left(\sum_{r=1}^{R} \boldsymbol{A}^{(1)}\boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{A}^{(N)}\boldsymbol{s}_r^{(N)} \times_1 {\boldsymbol{A}^{(1)}}^{\mathrm{H}} \cdots \times_3 {\boldsymbol{A}^{(3)}}^{\mathrm{H}} \right), \tag{17}$$

$$\mathcal{S} = \sum_{r=1}^{R} {\boldsymbol{A}^{(1)}}^{\mathrm{H}}\boldsymbol{A}^{(1)}\boldsymbol{s}_r^{(1)} \circ \cdots \circ {\boldsymbol{A}^{(N)}}^{\mathrm{H}}\boldsymbol{A}^{(N)}\boldsymbol{s}_r^{(N)}, \tag{18}$$

$$\mathcal{S} = \sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(3)} \tag{19}$$

and since the tensor rank is defined by the minimum number of rank one tensors that together compose the original tensor then by analysing above expressions we can observe that we have bounded tensors $\mathcal{X}$ and $\mathcal{S}$ to the same number of elements that compose them. Thus, the tensor rank is indeed a property and

$$\operatorname{rank}(\mathcal{X}) = \operatorname{rank}(\mathcal{S}). \tag{20}$$

3. (a)

(b)

(c)

(d)

4. (a)

(b)

(c)