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Multilinear Algebra
PARAFAC and Tensor Rank

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1. We know that if we have the tensor defined as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{1}$$

then if we have $\boldsymbol{b}_1 = \boldsymbol{b}_2$ and $\boldsymbol{c}_1 = \boldsymbol{c}_2$ we can guarantee that $\mathcal{X}$ is rank one. We can begin this proof by using the associativity property of the outer product to write tensor $\mathcal{X}$ as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{2}$$
$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1, \tag{3}$$
$$\mathcal{X} = (\boldsymbol{a}_1 + \boldsymbol{a}_2) \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1. \tag{4}$$

By inspection of the expression above we can see that the elements that compose vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ acts as weights in a linear combination of the matrix $\boldsymbol{b}_1 \circ \boldsymbol{c}_1$. Thus, if we assume that the tensor is not rank one then we should at least one of the vectors equals to zero so we can obtain a sum of linearly independent terms that leads to a tensor rank greater than one. However, this does not makes sense because if we have one of these vectors equals to zero then we would not have a sum at all. Thus, by contradiction, we know that in the proposed scenario the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ must be collinear meaning that the tensor is indeed rank one. In a similar fashion we can obtain a conclusion for the case where we have $\boldsymbol{b}_1 \neq \boldsymbol{b}_2$ and $\boldsymbol{c}_1 = \boldsymbol{c}_2$ by writting the tensor as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{5}$$
$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_1, \tag{6}$$
$$\mathcal{X} = (\boldsymbol{a}_1 \circ \boldsymbol{b}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2) \circ \boldsymbol{c}_1. \tag{7}$$

By inspecting the expression above by the same procedure as before we can once again reach the same conclusion that the only way to the tensor $\mathcal{X}$ to have a rank greater than one in this scenario is if one of the vectors is zero, but that would be contradictory. Thus, we can guarantee once more that the tensor $\mathcal{X}$ will be rank one in this case.

2.

3. (a)

(b)

(c)

(d)

4. (a)

(b)

(c)