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Multilinear Algebra
Computational Homeworks

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Course Multilinear Algebra - TIP8419

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Homework 0
Kronecker Product Run Time

Run Time Perfomance of Sequential Kronecker Products

In here I will briefly analyze the run time performance of the inverse operator while also using the Kronecker Product. In the first case the number of products is fixed while the number of columns is varying. In the second case we have a varying number of products for a fixed number of columns. In both cases is possible to see that is preferable to first invert the matrices before applying the Kronecker operator.

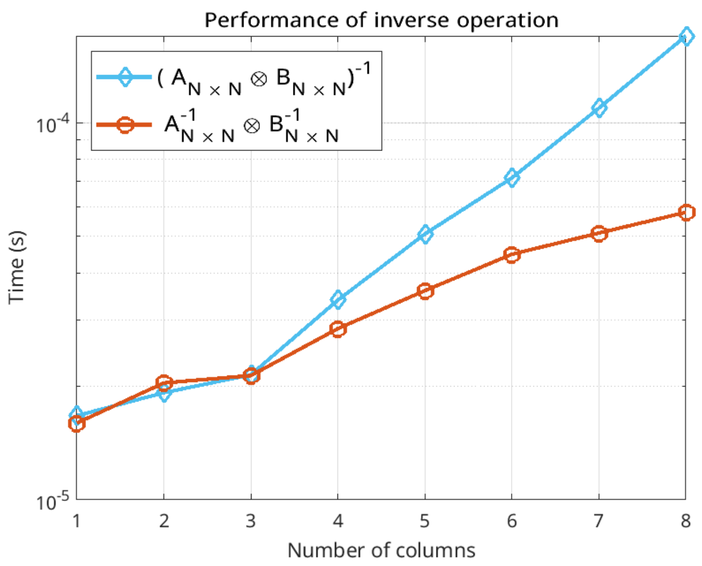


Figure 1: Monter Carlo Experiment with 5000.

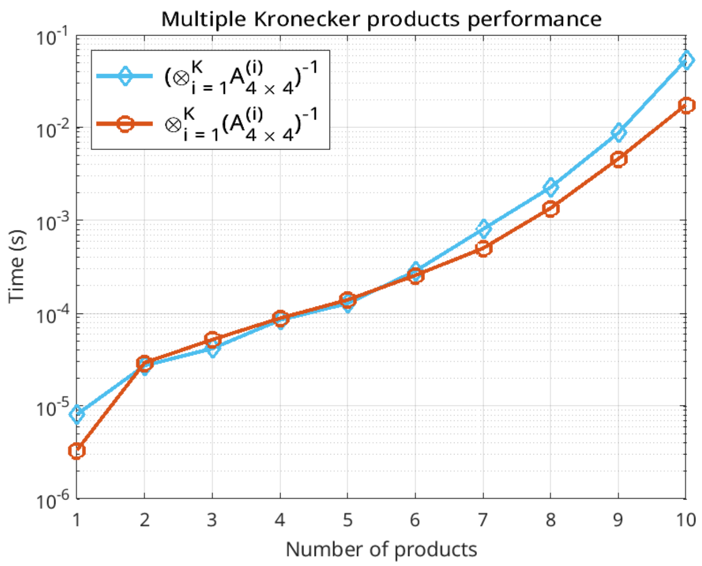


Figure 2: Monter Carlo Experiment with 10000 runs.

Show that $\mathbf{eig}(\boldsymbol{A} \otimes \boldsymbol{B}) = \mathbf{eig}(\boldsymbol{A}) \otimes \mathbf{eig}(\boldsymbol{B})$

By using the eigenvalue decomposition (eig) of two matrices and apply the Kronecker Product to them it is possible to reach the intended result

$$\boldsymbol{A} \otimes \boldsymbol{B} = (\boldsymbol{C}_1 \boldsymbol{\Lambda_1} \boldsymbol{C}_1^{-1}) \otimes (\boldsymbol{C}_2 \boldsymbol{\Lambda_2} \boldsymbol{C}_2^{-1}), \tag{1}$$

$$\boldsymbol{A} \otimes \boldsymbol{B} = (\boldsymbol{C}_1 \boldsymbol{\Lambda_1} \otimes \boldsymbol{C}_2 \boldsymbol{\Lambda_2})(\boldsymbol{C}_2^{-1} \otimes \boldsymbol{C}_2^{-1}), \tag{2}$$

$$\boldsymbol{A} \otimes \boldsymbol{B} = (\boldsymbol{C}_1 \otimes \boldsymbol{C}_2)(\boldsymbol{\Lambda_1} \otimes \boldsymbol{\Lambda_2})(\boldsymbol{C}_2^{-1} \otimes \boldsymbol{C}_2^{-1}), \tag{3}$$

$$\mathrm{eig}(\boldsymbol{A} \otimes \boldsymbol{B}) = (\boldsymbol{\Lambda_1} \otimes \boldsymbol{\Lambda_2}) = \mathrm{eig}(\boldsymbol{A}) \otimes \mathrm{eig}(\boldsymbol{B}) \tag{4}$$

Produced Algorithm

```
%% ----- Homework 0 ----- %%
clc;
clear;
close all;
N = [1 2 3 4 5 6 7 8];
time1 = zeros(length(N),1);
time2 = zeros(length(N),1);
for nn = 1:length(N)
    for mc = 1:5000
        A = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        B = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        tic;
        inv(tensor.mtx_prod_kron(A,B));
        aux = toc;
        time1(nn,1) = time1(nn,1) + aux;
        tic;
        tensor.mtx_prod_kron(inv(A),inv(B));
        aux = toc;
        time2(nn,1) = time2(nn,1) + aux;
    end
end
time1 = time1/5000;
time2 = time2/5000;
figure
txt = ['(\bf A_{N \times N} \otimes B_{N \times N})^{-1}'];
semilogy(N,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['\bf A^{-1}_{N \times N} \otimes B^{-1}_{N \times N}'];
semilogy(N,time2,'-o','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['Performance of inverse operation'])
xlabel('Number of columns')
ylabel('Time (s)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northwest',"fontsize", 12)
grid on;
saveas(gcf,'hw0a1.png')
N = 2;
```

```
K = [1 2 3 4 5 6 7 8 9 10];
time1 = zeros(length(K),1);
time2 = zeros(length(K),1);
for kk = 1:length(K)
    for mc = 1:1000
        tic;
        for ii = 1:K(kk)
            if ii == 1
                A1 = randn(N,N) + 1j*randn(N,N);
                continue
            else
                A2 = randn(N,N) + 1j*randn(N,N);
                A1 = tensor.mtx_prod_kron(A1,A2);
            end
        end
        inv(A1);
        aux = toc;
        time1(kk,1) = time1(kk,1) + aux;

        tic;
        for ii = 1:K(kk)
            if ii == 1
                A1 = randn(N,N) + 1j*randn(N,N);
                continue
            else
                A2 = randn(N,N) + 1j*randn(N,N);
                A1 = tensor.mtx_prod_kron(inv(A1),inv(A2));
            end
        end
        aux = toc;
        time2(kk,1) = time2(kk,1) + aux;
    end
end
time1 = time1/1000;
time2 = time2/1000;
figure
txt = ['\bf (\otimes^{K}_{i = 1}A^{(i)}_{4 \times 4})^{-1}'];
semilogy(K,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['\bf \otimes^{K}_{i = 1}(A^{(i)}_{4 \times 4})^{-1}'];
semilogy(K,time2,'-o','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['Multiple Kronecker products performance'])
xlabel('Number of products')
ylabel('Time (s)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northwest',"fontsize", 12)
grid on;
saveas(gcf,'hw0a2.png')
```

Homework 1
Hadamard, Kronecker and Khatri-Rao Products

In Figure 3, 4 and 5 I compare my implemention of Hadarmard, Kronecker and Khatri-Rao Products with the ones availables in the TensorLab package. As we can the only case where my function have shown an almost equal performance to the benchmark was the one where I do the Kronecker Product. The other two functions are far from the ideal performance offered by the TensorLab package. This could probably explained because I did not take full advantage of the vectorization that MATLAB offers when I was creating those two algorithms.

Run Time Perfomance of Hadamard Product

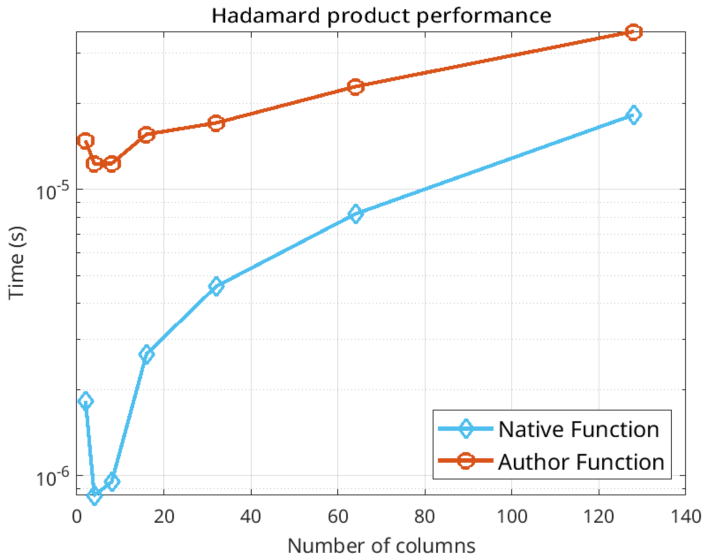


Figure 3: Monter Carlo Experiment with 1000 runs.

Run Time Perfomance of Kronecker Product

Run Time Perfomance of Khatri-Rao Product

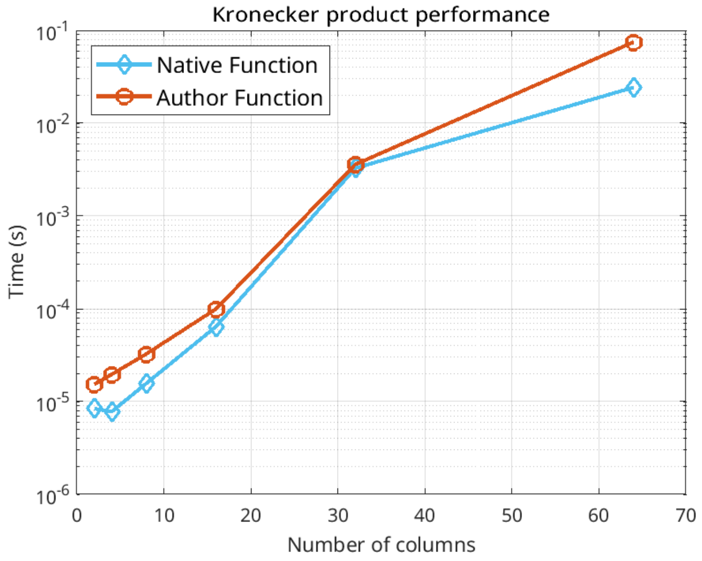


Figure 4: Monter Carlo Experiment with 1000 runs.

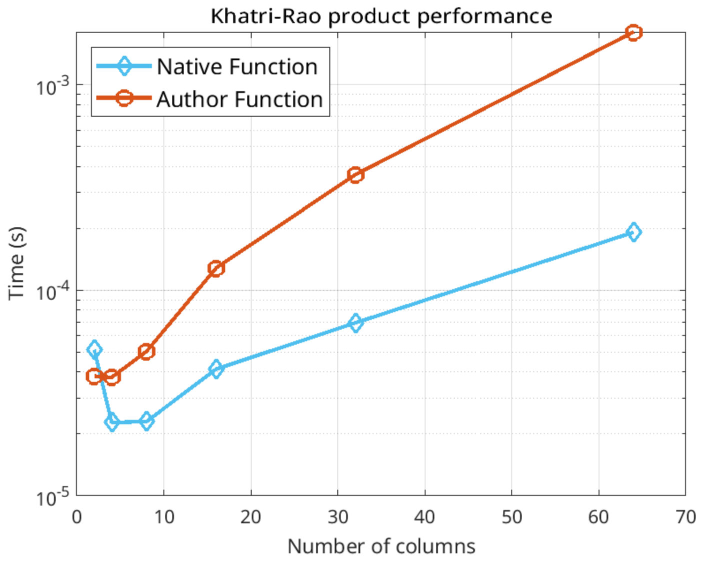


Figure 5: Monter Carlo Experiment with 1000 runs.

Produced Algorithm

```
%% Hadamard Product

% This function computes the Hadarmard Product of two given matrices.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: January 2022

function C = mtx_prod_had(A,B)

    [ia,ja] = size(A);
```

```
    [ib,jb] = size(B);

    if (ia ~= ib) || (ja~=jb)
        disp('Invalid Matrices!')
        return;
    else
        C = A.*B;
        %C = zeros(ia,ja);
        %for i = 1:ia
            %for j = 1:ja
                %C(i,j) = A(i,j)*B(i,j);
            %end
        %end
    end
end

%% Kronecker Product

% This function computes the Kronecker Product of two given matrices.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: January 2022

function C = mtx_prod_kron(A,B)

    [ia,ja] = size(A);
    [ib,jb] = size(B);

    A = repelem(A,ib,jb);
    B = repmat(B,[ia ja]);
    C = A.*B;

end

%% Khatri-Rao Product

% This function computes the Khatri-Rao Product of two given matrices.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: January 2022

function C = mtx_prod_kr(A,B)
    [ia,ja] = size(A);
    [ib,jb] = size(B);

    if (ja~=jb)
        disp('Invalid Matrices!')
        return;
    else
        C = zeros(ia*ib,ja);
        for j = 1:ja
            C(:,j) = tensor.mtx_prod_kron(A(:,j),B(:,j));
        end
    end
end
```

```
%% ----- Homework 1 ----- %%
clc;
clear;
close all;

N = [2 4 8 16 32 64 128];
time1 = zeros(length(N),1);
time2 = zeros(length(N),1);
for nn = 1:length(N)
    for mc = 1:1000
        A = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        B = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        tic;
        A.*B;
        aux = toc;
        time1(nn,1) = time1(nn,1) + aux;
        tic;
        tensor.mtx_prod_had(A,B);
        aux = toc;
        time2(nn,1) = time2(nn,1) + aux;
    end
end
time1 = time1/1000;
time2 = time2/1000;

figure
txt = ['Native Function'];
semilogy(N,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['Author Function'];
semilogy(N,time2,'-o','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['Hadamard product performance'])
xlabel('Number of columns')
ylabel('Time (s)')
legend_copy = legend("location", "southeast");
set(legend_copy,'Interpreter','tex','location','southeast',"fontsize", 12)
grid on;
saveas(gcf,'hw1a1.png')

N = [2 4 8 16 32 64];
time1 = zeros(length(N),1);
time2 = zeros(length(N),1);
for nn = 1:length(N)
    for mc = 1:1000
        A = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        B = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        tic;
        kron(A,B);
        aux = toc;
```

```
        time1(nn,1) = time1(nn,1) + aux;
        tic;
        tensor.mtx_prod_kron(A,B);
        aux = toc;
        time2(nn,1) = time2(nn,1) + aux;
    end
end
time1 = time1/1000;
time2 = time2/1000;

figure
txt = ['Native Function'];
semilogy(N,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['Author Function'];
semilogy(N,time2,'-o','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['Kronecker product performance'])
xlabel('Number of columns')
ylabel('Time (s)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northwest',"fontsize", 12)
grid on;
saveas(gcf,'hw1a2.png')

N = [2 4 8 16 32 64];
time1 = zeros(length(N),1);
time2 = zeros(length(N),1);
for nn = 1:length(N)
    for mc = 1:1000
        A = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        B = randn(N(nn),N(nn)) + 1j*randn(N(nn),N(nn));
        tic;
        kr(A,B);
        aux = toc;
        time1(nn,1) = time1(nn,1) + aux;
        tic;
        tensor.mtx_prod_kr(A,B);
        aux = toc;
        time2(nn,1) = time2(nn,1) + aux;
    end
end
time1 = time1/1000;
time2 = time2/1000;

figure
txt = ['Native Function'];
semilogy(N,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['Author Function'];
```

```
semilogy(N,time2,'-o','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['Khatri-Rao product performance'])
xlabel('Number of columns')
ylabel('Time (s)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northwest',"fontsize", 12)
grid on;
saveas(gcf,'hw1a3.png')
```

Homework 2
Khatri-Rao Product Run Time

Run Time Performance of Khatri-Rao Product for Different Implementations

In Figures 6 and 7 we can see the processing time for the considered methods to compute the pseudoinverse of a Khatri-Rao product for the cases where we have two and four columns in each matrix. To draw the curves it was implemented a Monte Carlo Experiment with only 250 runs for each value N of rows with $N \in \{2, 4, 8, 16, 32, 64\}$. For both cases we can see a clear advantage in using the third method as the dimmensions of the matrices increases. In a similar maner, we can also observe that the second method is a bit better than the first however not as good as the third. To see a better behavior for these methods it should be necessary to increase the number of rows to better analyse the advantages of each one, however due to technical constraints I could not increase as much as I wanted. In the sequence we can see the definition of each implemented method.

$$(\boldsymbol{A} \diamond \boldsymbol{B})^{\dagger} = \mathrm{pinv}(\boldsymbol{A} \diamond \boldsymbol{B}), \tag{5}$$

$$(\boldsymbol{A} \diamond \boldsymbol{B})^{\dagger} = [(\boldsymbol{A} \diamond \boldsymbol{B})^{\mathrm{T}}(\boldsymbol{A} \diamond \boldsymbol{B})]^{-1}(\boldsymbol{A} \diamond \boldsymbol{B})^{\mathrm{T}}, \tag{6}$$

$$(\boldsymbol{A} \diamond \boldsymbol{B})^{\dagger} = [(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})(\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B})]^{-1}(\boldsymbol{A} \diamond \boldsymbol{B})^{\mathrm{T}}, \tag{7}$$

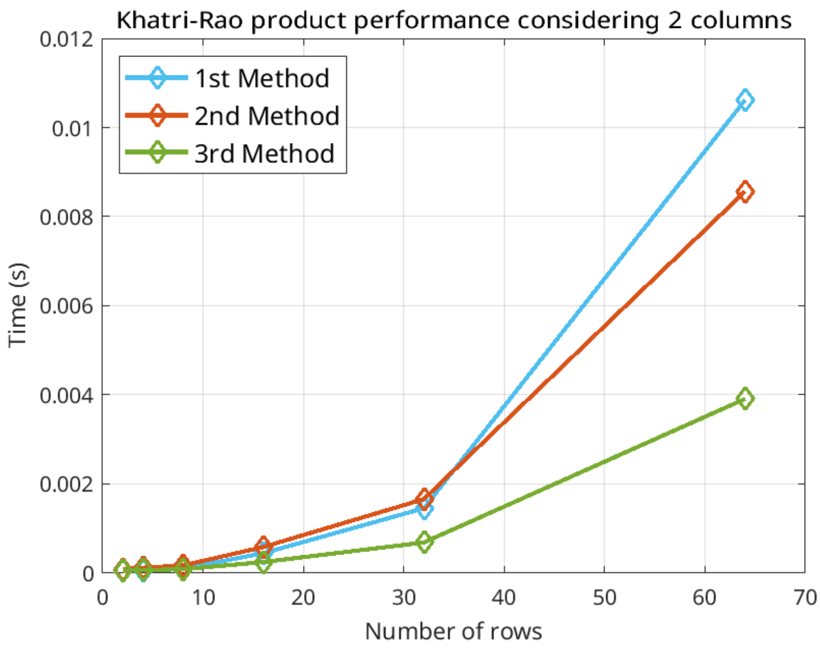


Figure 6: Monter Carlo Experiment with 250 runs and R = 2.

Run Time Perfomance of Sequential Khatri-Rao Products

In Figure 8 we analyze the behavior of the Khatri-Rao product when sequential products are taken. As we can see it is an exponencial curve since after every product we will have matrices with analyse increasing dimmensions. Thus, when we do need to use consecutive Khatri-Rao products we should seek a way to simplify the operation by using algebraic manipulations.

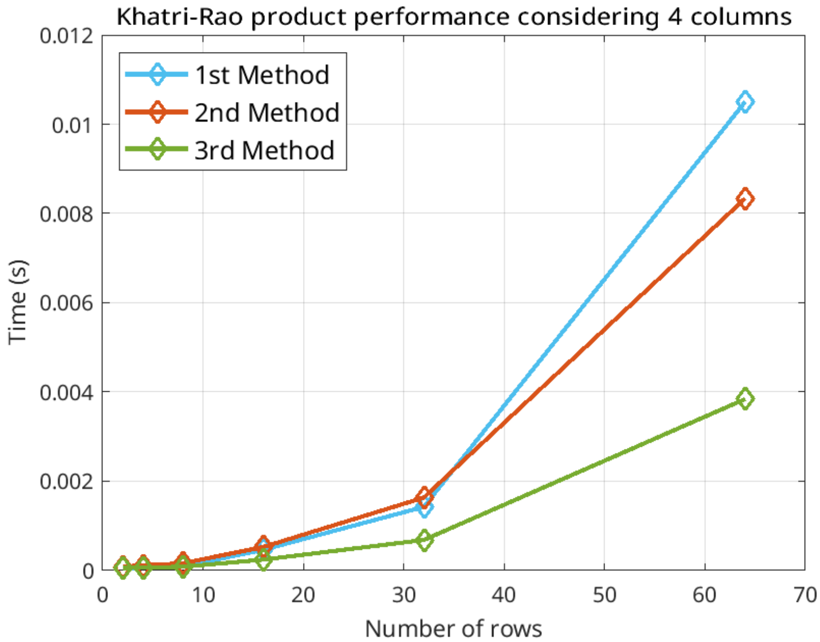


Figure 7: Monter Carlo Experiment with 250 runs and R = 4.

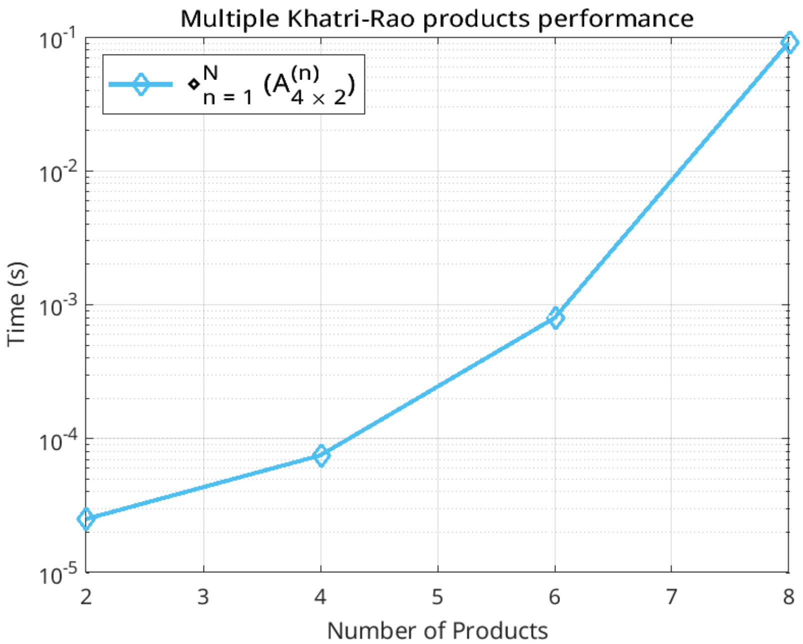


Figure 8: Monter Carlo Experiment with 250 runs.

Produced Algorithm

```
%% ----- Homework 2 ----- %%
clc;
clear;
close all;
```

```
R = 2;
I = [2 4 8 16 32 64];
time1 = zeros(length(I),1);
time2 = zeros(length(I),1);
time3 = zeros(length(I),1);
for ii = 1:length(I)
    for mc = 1:250
        A = randn(I(ii),I(ii)) + 1j*randn(I(ii),I(ii));
        B = randn(I(ii),I(ii)) + 1j*randn(I(ii),I(ii));
        tic;
        pinv(tensor.mtx_prod_kr(A,B));
        aux = toc;
        time1(ii,1) = time1(ii,1) + aux;
        tic;
        (tensor.mtx_prod_kr(A,B).'*tensor.mtx_prod_kr(A,B))...
            \(tensor.mtx_prod_kr(A,B).');
        aux = toc;
        time2(ii,1) = time2(ii,1) + aux;
        tic;
        tensor.mtx_prod_had((A.'*A),(B.'*B))\(tensor.mtx_prod_kr(A,B).');
        aux = toc;
        time3(ii,1) = time3(ii,1) + aux;
    end
end
time1 = time1/250;
time2 = time2/250;
time3 = time3/250;

figure
txt = ['1st Method'];
plot(I,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['2nd Method'];
plot(I,time2,'-d','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['3rd Method'];
plot(I,time3,'-d','color', [0.4660 0.6740 0.1880], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['Khatri-Rao product performance considering 2 columns'])
xlabel('Number of rows')
ylabel('Time (s)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northwest',"fontsize", 12)
grid on;
saveas(gcf,'hw2a1.png')

R = 4;
I = [2 4 8 16 32 64];
time1 = zeros(length(I),1);
```

```
time2 = zeros(length(I),1);
time3 = zeros(length(I),1);
for ii = 1:length(I)
    for mc = 1:250
        A = randn(I(ii),I(ii)) + 1j*randn(I(ii),I(ii));
        B = randn(I(ii),I(ii)) + 1j*randn(I(ii),I(ii));
        tic;
        pinv(tensor.mtx_prod_kr(A,B));
        aux = toc;
        time1(ii,1) = time1(ii,1) + aux;
        tic;
        (tensor.mtx_prod_kr(A,B).'*tensor.mtx_prod_kr(A,B))...
            \(tensor.mtx_prod_kr(A,B).');
        aux = toc;
        time2(ii,1) = time2(ii,1) + aux;
        tic;
        tensor.mtx_prod_had((A.'*A),(B.'*B))\(tensor.mtx_prod_kr(A,B).');
        aux = toc;
        time3(ii,1) = time3(ii,1) + aux;
    end
end
time1 = time1/250;
time2 = time2/250;
time3 = time3/250;

figure
txt = ['1st Method'];
plot(I,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['2nd Method'];
plot(I,time2,'-d','color', [0.8500 0.3250 0.0980], "linewidth",...
    2, "markersize", 8, "DisplayName", txt);
hold on;
txt = ['3rd Method'];
plot(I,time3,'-d','color', [0.4660 0.6740 0.1880], "linewidth",...
    2, "markersize", 8, "DisplayName", txt);
hold off;
title(['Khatri-Rao product performance considering 4 columns'])
xlabel('Number of rows')
ylabel('Time (s)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northwest',"fontsize", 12)
grid on;
saveas(gcf,'hw2a2.png')

I = 4;
R = 2;
N = [2 4 6 8];
time1 = zeros(length(N),1);
time2 = zeros(length(N),1);
for nn = 1:length(N)
    for mc = 1:250
```

```
        tic;
        for ii = 1:N(nn)
            if ii == 1
                A1 = randn(I,R) + 1j*randn(I,R);
                continue
            else
                A2 = randn(I,R) + 1j*randn(I,R);
                A1 = tensor.mtx_prod_kron(A1,A2);
            end
        end
        aux = toc;
        time1(nn,1) = time1(nn,1) + aux;
    end
end
time1 = time1/250;

figure
txt = ['\bf \diamond^{N}_{n = 1} (A^{(n)}_{4 \times 2})'];
semilogy(N,time1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
title(['Multiple Khatri-Rao products performance'])
xlabel('Number of Products')
ylabel('Time (s)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northwest',"fontsize", 12)
grid on;
saveas(gcf,'hw2a3.png')
```

Homework 3
Least-Squares Khatri-Rao Factorization (LSKRF)

Implementation LSKRF

The LSKRF algorithm aims to solve the following estimation problem

$$\left(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}\right) = \min_{\boldsymbol{A},\boldsymbol{B}} ||\boldsymbol{X} - \boldsymbol{A} \diamond \boldsymbol{B}||_{\mathrm{F}}^2 , \tag{8}$$

and by implementing the algorithm it was possible to reach the following table of NMSE (dB) values using the validation files

NMSE($\boldsymbol{X}, \hat{\boldsymbol{X}}$)	NMSE($\boldsymbol{A}, \hat{\boldsymbol{A}}$)	NMSE($\boldsymbol{B}, \hat{\boldsymbol{B}}$)
-623.4093	+11.5658	+7.8479

as we can see the estimation of the matrix $\boldsymbol{X}$ is perfect, but the estimated factor matrices are far from the original ones. This is simply explained by the scale ambiguity intrinsic to the LSKRF. This could be eliminated if we assume previous knowledge of either the structure of the factor matrices or some of their elements.

Monte Carlo Experiment

In Figure 9 the Monte Carlo Experiment is used to draw the curves for two different scenarios: $(I, J, R) = (10, 10, 4)$ and $(I, J, R) = (30, 10, 4)$. As we can observe the best performance is the one where the algorithm have fewer elements to estimate. This is to be expected since the noise will have a lot less components that could potentially harm the performance of the LSKRF.

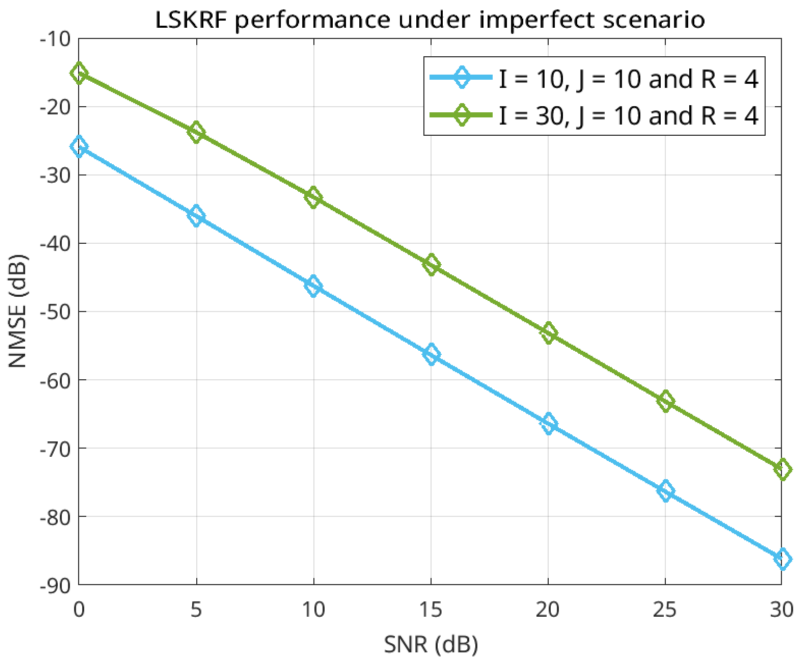


Figure 9: Monter Carlo Experiment with 1000 runs for LSKRF algorithm.

Produced Algorithm

```
%% Least-Squares Khatri-Rao Factorization (LSKRF)

% This function computes the LSKRF of a given matrix.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: February 2022

function [Ahat,Bhat] = LSKRF(C,ia,ib)
    [~, jc] = size(C);

    Ahat = complex(zeros(ia,jc),0);
    Bhat = complex(zeros(ib,jc),0);

    for j = 1:jc
        Cp = C(:,j);
        Cp = reshape(Cp, [ib ia]);
        [U,S,V] = svd(Cp);
        Ahat(:,j) = sqrt(S(1,1)).*conj(V(:,1));
        Bhat(:,j) = sqrt(S(1,1)).*U(:,1);
    end
end

%% ----- Homework 3 ----- %%
clc;
clear;
close all;

A = randn(4,2) + 1j*randn(4,2);
B = randn(6,2) + 1j*randn(6,2);
X = tensor.mtx_prod_kr(A,B);
[Ahat,Bhat] = tensor.LSKRF(X,4,6);
Xhat = tensor.mtx_prod_kr(Ahat,Bhat);

disp('Checking the NMSE (dB) between the original matrix X and its'...
    'reconstruction with LSKRF:')
nmsex = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
nmsex = 20*log10(nmsex)
disp('Checking the NMSE (dB) between the original matrix A and its'...
    'estimation:')
nmsea = (norm(A- Ahat,'fro')^2)/(norm(A,'fro')^2);
nmsea = 20*log10(nmsea)
disp('Checking the NMSE (dB) between the original matrix B and its'...
    'estimation:')
nmseb = (norm(B- Bhat,'fro')^2)/(norm(B,'fro')^2);
nmseb = 20*log10(nmseb)

I = 10;
J = 10;
R =  4;
SNR = [0 5 10 15 20 25 30];
nmse = zeros(length(SNR),1);
for snr = 1:length(SNR)
```

```
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(I*J,R) + 1j*randn(I*J,R));

        A = randn(I,R) + 1j*randn(I,R);
        B = randn(J,R) + 1j*randn(J,R);
        X = tensor.mtx_prod_kr(A,B);
        X_noisy = X + noise;

        [Ahat,Bhat] = tensor.LSKRF(X_noisy,I,J);
        Xhat = tensor.mtx_prod_kr(Ahat,Bhat);
        aux = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse(snr,1) = nmse(snr,1) + 20*log10(aux);
    end
end
nmse  = nmse/1000;

figure
txt = ['I = ' num2str(I), ', J = ' num2str(J), ' and R = ' num2str(R)];
plot(SNR,nmse,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;

I = 30;
J = 10;
R =  4;
SNR = [0 5 10 15 20 25 30];
nmse = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(I*J,R) + 1j*randn(I*J,R));

        A = randn(I,R) + 1j*randn(I,R);
        B = randn(J,R) + 1j*randn(J,R);
        X = tensor.mtx_prod_kr(A,B);
        X = X + noise;

        [Ahat,Bhat] = tensor.LSKRF(X,I,J);
        Xhat = tensor.mtx_prod_kr(Ahat,Bhat);
        aux = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse(snr,1) = nmse(snr,1) + 20*log10(aux);
    end
end
nmse  = nmse/1000;

txt = ['I = ' num2str(I), ', J = ' num2str(J), ' and R = ' num2str(R)];
plot(SNR,nmse,'-d','color', [0.4660 0.6740 0.1880], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['LSKRF performance under imperfect scenario'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
```

```
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw3.png')
```

Homework 4
Least Squares Kronecker Product Factorization (LSKronF)

Implementation LSKronF

The LSKronF algorithm aims to solve the following estimation problem

$$\left(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}\right) = \min_{\boldsymbol{A},\boldsymbol{B}} ||\boldsymbol{X} - \boldsymbol{A} \otimes \boldsymbol{B}||_{\mathrm{F}}^2 \,, \tag{9}$$

and by implementing the algorithm it was possible to reach the following table of NMSE (dB) values using the validation files

NMSE($\boldsymbol{X}, \hat{\boldsymbol{X}}$)	NMSE($\boldsymbol{A}, \hat{\boldsymbol{A}}$)	NMSE($\boldsymbol{B}, \hat{\boldsymbol{B}}$)
-619.2196	+13.5472	+9.5922

as we can see the estimation of the matrix $\boldsymbol{X}$ is perfect, but the estimated factor matrices are far from the original ones. This is simply explained by the scale ambiguity intrinsic to the LSKronF. This could be eliminated if we assume previous knowledge of either the structure of the factor matrices or some of their elements.

Monte Carlo Experiment

In Figure 10 the Monte Carlo Experiment is used to draw the curves for two different scenarios: $(I, J, P, Q) = (2, 4, 3, 5)$ and $(I, J, P, Q) = (4, 8, 3, 5)$. As we can observe the best performance is the one where the algorithm have fewer elements to estimate. This is to be expected since the noise will have a lot less components that could potentially harm the performance of the LSKronF.

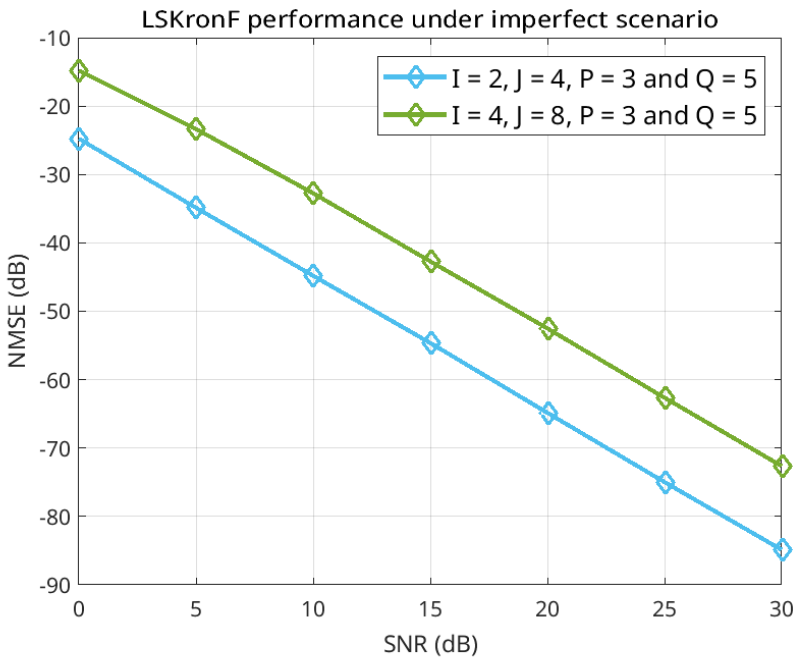


Figure 10: Monter Carlo Experiment with 1000 runs for LSKronf algorithm.

Produced Algorithm

```
%% Least-Square Kronecker Product Factorization (LSKronF)

% This function computes the LSKronF of a given matrix.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: February 2022

function [Ahat,Bhat] = LSKronF(C,ia,ja,ib,jb)
    [ic,jc] = size(C);

    I = (ic/ia) + zeros(1,ia);
    J = (jc/ja) + zeros(1,ja);
    blocks_of_C = mat2cell(C,I,J);

    k = 1;
    Chat = complex(zeros(ib*jb,ia*ja),0);
    for j = 1:ja
        for i = 1:ia
            vec_of_block = cell2mat(blocks_of_C(i,j));
            vec_of_block = vec_of_block(:);
            Chat(:,k) = vec_of_block;
            k = k + 1;
        end
    end

    [U,S,V] = svd(Chat);
    ahat = sqrt(S(1,1)).*conj(V(:,1));
    bhat = sqrt(S(1,1)).*U(:,1);
    Ahat = reshape(ahat,[ia ja]);
    Bhat = reshape(bhat, [ib jb]);
end

%% ----- Homework 4 ----- %%
clc;
clear;
close all;

A = randn(4,2) + 1j*randn(4,2);
B = randn(6,3) + 1j*randn(6,3);
X = tensor.mtx_prod_kron(A,B);
[Ahat,Bhat] = tensor.LSKronF(X,4,2,6,3);
Xhat = tensor.mtx_prod_kron(Ahat,Bhat);

disp('Checking the NMSE (dB) between the original matrix X and its'...
    'reconstruction with LSKronF:')
nmsex = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
nmsex = 20*log10(nmsex)
disp('Checking the NMSE (dB) between the original matrix A and its'...
    'estimation:')
nmsea = (norm(A- Ahat,'fro')^2)/(norm(A,'fro')^2);
nmsea = 20*log10(nmsea)
disp('Checking the NMSE (dB) between the original matrix B and its'...
```

```
    'estimation:')
nmseb = (norm(B- Bhat,'fro')^2)/(norm(B,'fro')^2);
nmseb = 20*log10(nmseb)

I = 2;
J = 4;
P = 3;
Q = 5;
SNR = [0 5 10 15 20 25 30];
nmse = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(I*J,P*Q) + 1j*randn(I*J,P*Q));

        A = randn(I,P) + 1j*randn(I,P);
        B = randn(J,Q) + 1j*randn(J,Q);
        X = tensor.mtx_prod_kron(A,B);
        X_noisy = X + noise;

        [Ahat,Bhat] = tensor.LSKronF(X_noisy,I,P,J,Q);
        Xhat = tensor.mtx_prod_kron(Ahat,Bhat);
        aux = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse(snr,1) = nmse(snr,1) + 20*log10(aux);
    end
end
nmse  = nmse/1000;

figure
txt = ['I = ' num2str(I), ', J = ' num2str(J), ', P = ' num2str(P),...
    ' and Q = ' num2str(Q)];
plot(SNR,nmse,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;

I = 4;
J = 8;
P = 3;
Q = 5;
SNR = [0 5 10 15 20 25 30];
nmse = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(I*J,P*Q) + 1j*randn(I*J,P*Q));

        A = randn(I,P) + 1j*randn(I,P);
        B = randn(J,Q) + 1j*randn(J,Q);
        X = tensor.mtx_prod_kron(A,B);
        X = X + noise;

        [Ahat,Bhat] = tensor.LSKronF(X,I,P,J,Q);
        Xhat = tensor.mtx_prod_kron(Ahat,Bhat);
```

```
        aux = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse(snr,1) = nmse(snr,1) + 20*log10(aux);
    end
end
nmse  = nmse/1000;

txt = ['I = ' num2str(I), ', J = ' num2str(J), ', P = ' num2str(P),...
    ' and Q = ' num2str(Q)];
plot(SNR,nmse,'-d','color', [0.4660 0.6740 0.1880], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['LSKronF performance under imperfect scenario'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw4.png')
```

Homework 5
Kronecker Product Singular Value Decomposition (KPSVD)

Implementation and Validation of KPSVD

The KPSVD algorithm aims to solve the following estimation problem

$$\boldsymbol{X} = \sum_{k=1}^{r_{kp}} \sigma_k \boldsymbol{U}_k \otimes \boldsymbol{V}_k, \tag{10}$$

Since a validation file was not available I decided to create my own validation by generating a full-rank matrix with $R = 9$ and in sequence two approximations using my KPSVD implementation are provided: One using the full-rank approximation and other using a r-rank approximation with $r \in \{1, 3, 5, 7\}$. The NMSE (dB) between the original matriz $\boldsymbol{X}$ and its low-rank approximations are shown in the following table

r = 9	r = 7	r = 5	r = 3	r = 1
-604.8023	-51.3722	-26.7589	-16.9054	-6.3068

and as we can observe the low-rank approximations are not good enought mainly because we are working with a full-rank matrix. Thus, it would be expected to obtain a better performance for low-rank approximations of the matrix $\boldsymbol{X}$ if the matrix would to be rank defficient.

Produced Algorithm

```
%% Kronecker Product Single Value Decomposition (KPSVD)

% This function computes the LSKRF of a given matrix.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: March 2022

function [U,S,V,rkp] = KPSVD(X,ia,ja,ib,jb)
    [ix,jx] = size(X);

    I = (ix/ia) + zeros(1,ia);
    J = (jx/ja) + zeros(1,ja);
    blocks_of_X = mat2cell(X,I,J);

    k = 1;
    Xhat = complex(zeros(ib*jb,ia*ja),0);
    for j = 1:ja
        for i = 1:ia
            vec_of_block = cell2mat(blocks_of_X(i,j));
            vec_of_block = vec_of_block(:);
            Xhat(:,k) = vec_of_block;
            k = k + 1;
        end
    end
    [U,S,V] = svd(Xhat');
    rkp = sum(sum(S>0));
end
```

```
%% ----- Homework 5 ----- %%
clc;        % This function computes the LSKRF of a given matrix.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: March 2022

function [U,S,V,rkp] = KPSVD(X,ia,ja,ib,jb)
    [ix,jx] = size(X);

    I = (ix/ia) + zeros(1,ia);
    J = (jx/ja) + zeros(1,ja);
    blocks_of_X = mat2cell(X,I,J);

    k = 1;
    Xhat = complex(zeros(ib*jb,ia*ja),0);
    for j = 1:ja
        for i = 1:ia
            vec_of_block = cell2mat(blocks_of_X(i,j));
            vec_of_block = vec_of_block(:);
            Xhat(:,k) = vec_of_block;
            k = k + 1;
        end
    end
    [U,S,V] = svd(Xhat');
    rkp = sum(sum(S>0));
end

r1 = 3;
c1 = 3;
r2 = 3;
c2 = 3;
rank_of_A = 3;

% Using the outer product between vectors to control the rank of a matrix.
A = complex(zeros(r1*r2,c1*c2),0);
for i = 1:rank_of_A
    aux1 = randn(r1*r2,1)+ 1i*randn(r1*r2,1);
    aux2 = randn(1,c1*c2)+ 1i*randn(1,c1*c2);
    A = A + aux1*aux2;
end
[U,S,V,rkp] = tensor.KPSVD(A,r1,c1,r2,c2);

% Reconstructing A with full rank.
Ahat = complex(zeros(r1*r2,c1*c2),0);
for r = 1:rkp
    aux1 = U(:,r);
    aux2 = conj(V(:,r));
    Uk{r} = reshape(aux1,[r1 c1]);
    Vk{r}  = reshape(aux2, [r2 c2]);
    Ahat = Ahat + S(r,r)*tensor.mtx_prod_kron(Uk{r},Vk{r});
end
disp('Checking the NMSE (dB) between the original matrix A and its'...
    'reconstruction with full rank:')
Ahat = conj(Ahat);
```

```
nmse = (norm(A - Ahat,'fro')^2)/(norm(A,'fro')^2);
nmse = 20*log10(nmse)

% Reconstructing A with deficient rank.
Ahat = complex(zeros(r1*r2,c1*c2),0);
for r = 1:3
    aux1 = U(:,r);
    aux2 = conj(V(:,r));
    Uk{r} = reshape(aux1,[r1 c1]);
    Vk{r}  = reshape(aux2, [r2 c2]);
    Ahat = Ahat + S(r,r)*tensor.mtx_prod_kron(Uk{r},Vk{r});
end
disp('Checking the NMSE (dB) between the original matrix A and its'...
    'reconstruction with KPSVD using deficient rank:')
Ahat = conj(Ahat);
nmse = (norm(A - Ahat,'fro')^2)/(norm(A,'fro')^2);
nmse = 20*log10(nmse)
```

Homework 6
Unfolding, folding, and n-mode product

Implementation and Validation of unfolding, folding and n-mode product

The unfolding, folding and n-mode product operations were implemented according the following prototype

$$
\begin{aligned}
[\mathcal{X}]_n &= \text{unfold}(\mathcal{X}, [I_1 \cdots I_N], n) \in \mathbb{C}^{I_n \times I_1 \cdots I_{n-1} I_{n+1} \cdots I_N}, && (11)\\
\mathcal{X} &= \text{fold}([\mathcal{X}]_n, [I_1 \cdots I_N], n) \in \mathbb{C}^{I_1 \times \cdots \times I_N}, && (12)\\
\mathcal{Y} &= \mathcal{X} \times_1 \boldsymbol{U}_1 \times_2 \cdots \times_N \boldsymbol{U}_N, && (13)
\end{aligned}
$$

and in sequence the provided files were used to validade the algorithms. As it will be possible to see by running the corresponde homework file ('homerwork6.m') it was possible to validade all the algorithms. Nonentheless, all the three algorithms are generalized for Nth order tensors and TensorLab was used to verify that.

Produced Algorithm

```
%% Unfolding

% This function computes the unfolding of a given tensor in its matrix.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: April 2022

function [A] = unfold(ten,mode)
    dim = size(ten);
    order = 1:numel(dim);
    order(mode) = [];
    order = [mode order];
    A = reshape(permute(ten,order), dim(mode), prod(dim)/dim(mode));
end

%% Folding

% This function computes the folding of a given matrix into its tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: April 2022

% It's interesting to see how the dimmensions get swapped by the unfolding
% so the understanding of the code is clear.
function [ten] = fold(A,dim,mode)
    order = 1:numel(dim);
    order(mode) = [];
    order = [mode order];
    dim = dim(order);
    ten = reshape(A,dim);

    if mode == 1
        ten = permute(ten,order);
    else
        order = 1:numel(dim);
```

```
        for i = 2:mode
            order([i-1 i]) = order([i i-1]);
        end
        ten = permute(ten,order);
    end
end

%% N-mode Product

% This function computes n-mode product of a set of matrices and a tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: April 2022

function [ten] = n_mod_prod(ten,matrices,modes)
    dim = size(ten);
    number = numel(matrices);
    if nargin < 3
    modes = 1:number;
    end

    for i = modes
        ten = cell2mat(matrices(i))*tensor.unfold(ten,i);
        [aux,~] = size(cell2mat(matrices(i)));
        dim(i) = aux;
        ten = tensor.fold(ten,[dim],i);
    end
end

%% ----- Homework 6 ----- %%
clc;
clear;
close all;

% Tensor for testing
disp('Generic Tensor:');
load('homework6_unfolding_folding.mat')
dimension = size(tenX);

% Kolda example for testing
% load('homework6_kolda_example.mat')
% tenX = xxx;
% dimension = size(tenX);

% Unfolding
disp('X1./unfold(tenX,1):');
tenX_1 = tensor.unfold(tenX,1);
X1./tensor.unfold(tenX,1)
disp('X2./unfold(tenX,2):');
tenX_2 = tensor.unfold(tenX,2);
X2./tensor.unfold(tenX,2)
disp('X3./unfold(tenX,3):');
tenX_3 = tensor.unfold(tenX,3);
X3./tensor.unfold(tenX,3)
```

```
% Folding
disp('tenX./fold(unfold(tenX,1),1):');
tenX./tensor.fold(tenX_1,dimension,1)
disp('tenX./fold(unfold(tenX,2),2):');
tenX./tensor.fold(tenX_2,dimension,2)
disp('tenX./fold(unfold(tenX,3),3):');
tenX./tensor.fold(tenX_3,dimension,3)

% N-mode product
load('homework6_n_mode.mat')
tenY_test = tensor.n_mod_prod(tenX,{Z},[1]);
disp('Checking the NMSE (dB) between the original tensor Y and the one'...
    'after the N-mode product:')
nmsey = (norm(tensor.unfold(tenY- tenY_test,1),'fro')^2)...
    /(norm(tensor.unfold(tenY,1),'fro')^2);
nmsey = 20*log10(nmsey)
```

Homework 7
High Order Singular Value Decomposition (HOSVD)

Implementation HOSVD

NMSE($\mathcal{X}, \hat{\mathcal{X}}$)	NMSE($\mathcal{S}, \hat{\mathcal{S}}$)	NMSE($\boldsymbol{U}_1, \hat{\boldsymbol{U}}_1$)	NMSE($\boldsymbol{U}_2, \hat{\boldsymbol{U}}_2$)	NMSE($\boldsymbol{U}_3, \hat{\boldsymbol{U}}_3$)
-611.2162	+7.7656	+2.6667	+2.0000	+1.6063

Validation of HOSVD

The multilinear rank advantage is maximized when it comes to process sparce tensors since the dimmensions can be greatly reduced without losing of too much relevant information by analysing the profile of its multiples unfoldings.

$$\mathcal{X} \in \mathbb{C}^{8\times 4\times 10} \to \hat{\mathcal{X}} \in \mathbb{C}^{R_1\times R_2\times R_3}, \tag{14}$$

$$\mathcal{X} \in \mathbb{C}^{5\times 5\times 5} \to \hat{\mathcal{Y}} \in \mathbb{C}^{P_1\times P_2\times P_3}, \tag{15}$$

Produced Algorithm

```
%% Full High Order Single Value Decomposition (HOSVD)

% This function computes the Truncated HOSVD of a given tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: April 2022

function [S,U] = HOSVD_full(ten)
    number = numel(size(ten));
    for i = 1:number
    [aux,~,~] = svd(tensor.unfold(ten,i));
    %[aux,~,~] = svd(tens2mat(ten,i));
    U{i} = aux;
    end
    % Core tensor uses the hermitian operator.
    Ut = cellfun(@(x) x', U,'UniformOutput',false);
    S = tensor.n_mod_prod(ten,Ut);
    % The normal factors should be transposed.
    U = cellfun(@(x) x, U,'UniformOutput',false);
end

%% Truncated High Order Single Value Decomposition (HOSVD)

% This function computes the Truncated HOSVD of a given tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: April 2022

function [S,U] = HOSVD_truncated(ten,ranks)
    if nargin < 2
        number = numel(size(ten));
        for i = 1:number
        [aux,eig,~] = svd(tensor.unfold(ten,i));
        [aa,~] = size(eig(eig > eps));
        aux = aux(:,1:aa);
        U{i} = aux;
        end
        % Core tensor uses the hermitian operator.
        Ut = cellfun(@(x) x', U,'UniformOutput',false);
        S = tensor.n_mod_prod(ten,Ut);
        % The normal factors should be transposed.
        U = cellfun(@(x) x, U,'UniformOutput',false);
    else
        number = numel(size(ten));
        for i = 1:number
        [aux,~,~] = svd(tensor.unfold(ten,i));
        aux = aux(:,1:ranks(i));
        U{i} = aux;
        end
        % Core tensor uses the hermitian operator.
        Ut = cellfun(@(x) x',U,'UniformOutput',false);
        S = tensor.n_mod_prod(ten,Ut);
        % The normal factors should be transposed.
```

```
        U = cellfun(@(x) x,U,'UniformOutput',false);
    end
end

%% ----- Homework 7 ----- %%
clc;
clear;
close all;

% Full HOSVD
load('homework7_HOSVD.mat');
[tenS_hat,U_hat] = tensor.HOSVD_full(tenX);
tenX_hat = tensor.n_mod_prod(tenS_hat,U_hat);

% Checking the orthogonality
disp('Checking the orthogonality propertie between the subtensors'...
    'formed from the core tensor:')
reshape(tenS_hat(:,:,1),[],1)'*reshape(tenS_hat(:,:,4),[],1)

disp('Checking the NMSE (dB) between the original tensor X and its'... '
    reconstruction:')
nmsex = (norm(tensor.unfold(tenX - tenX_hat,1),'fro')^2)...
    /(norm(tensor.unfold(tenX,1),'fro')^2);
nmsex = 20*log10(nmsex)
disp('Checking the NMSE (dB) between the original core tensor S and its'...
    'reconstruction:')
nmses = (norm(tensor.unfold(tenS - tenS_hat,1),'fro')^2)...
    /(norm(tensor.unfold(tenS,1),'fro')^2);
nmses = 20*log10(nmses)

disp('Checking the NMSE (dB) between the original matrices and its'...
    'estimations:')
nmseU1 = (norm(U1 - U_hat{1},'fro')^2)/(norm(U1,'fro')^2)
nmseU2 = (norm(U2 - U_hat{2},'fro')^2)/(norm(U2,'fro')^2)
nmseU3 = (norm(U3 - U_hat{3},'fro')^2)/(norm(U3,'fro')^2)

% Truncated HOSVD and Denoising
X = randn(8,4,10) + 1i*randn(8,4,10);
Y = randn(5,5,5) + 1i*randn(5,5,5);

[S1,U1] = tensor.HOSVD_truncated(X);
multilinear_rank1 = size(S1);
[S2,U2] = tensor.HOSVD_truncated(Y);
multilinear_rank2 = size(S2);

Xhat = tensor.n_mod_prod(S1,U1);
Yhat = tensor.n_mod_prod(S2,U2);

disp('Checking the NMSE (dB) between the original noisy tensor X and'...
    'its reconstruction:')
nmsex = (norm(tensor.unfold(X- Xhat,1),'fro')^2)...
    /(norm(tensor.unfold(X,1),'fro')^2);
nmsex = 20*log10(nmsex)
```

```
disp('Checking the NMSE (dB) between the original noisy tensor Y'...
    'and its reconstruction:')
nmsey = (norm(tensor.unfold(Y- Yhat,1),'fro')^2)...
    /(norm(tensor.unfold(Y,1),'fro')^2);
nmsey = 20*log10(nmsey)

disp('The multilinear rank for the tensor X is:')
size(S1)
disp('The multilinear rank for the tensor Y is:')
size(S2)
```

Homework 8
High Order Order Orthogonal Iteration (HOOI)

Implementation HOOI

NMSE($\mathcal{X}, \hat{\mathcal{X}}$)	NMSE($\mathcal{S}, \hat{\mathcal{S}}$)	NMSE($\boldsymbol{U}_1, \hat{\boldsymbol{U}}_1$)	NMSE($\boldsymbol{U}_2, \hat{\boldsymbol{U}}_2$)	NMSE($\boldsymbol{U}_3, \hat{\boldsymbol{U}}_3$)
-607.9515	+7.2483	+2.6667	+3.9622	+1.16160

Validation of HOOI

The multilinear rank advantage is maximized when it comes to process sparce tensors since the dimmensions can be greatly reduced without losing of too much relevant information by analysing the profile of its multiples unfoldings.

$$\mathcal{X} \in \mathbb{C}^{8\times 4\times 10} \rightarrow \hat{\mathcal{X}} \in \mathbb{C}^{R_1\times R_2\times R_3}, \tag{16}$$

$$\mathcal{X} \in \mathbb{C}^{5\times 5\times 5} \rightarrow \hat{\mathcal{Y}} \in \mathbb{C}^{P_1\times P_2\times P_3}, \tag{17}$$

Produced Algorithm

```
%% High Order Orthogonal Iteration (HOOI)

% This function computes the HOOI of a given tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: May 2022

function [S,U,k] = HOOI_full(ten)
    max_iter = 10;
    [~, U] = tensor.HOSVD_full(ten);
    number = numel(size(ten));
    for k = 1:max_iter
        for i = 1:number
            modes = 1:number;
            modes(i) = []; % It will skip this mode in the n_mod_prod.
            Un = tensor.n_mod_prod(ten,U,modes);
            [aux,~,~] = svd(tensor.unfold(Un,i));
            U{i} = aux;
        end
    end
    % The conjugate transpose
    Ut = cellfun(@(x) x', U ,'UniformOutput',false);
    S = tensor.n_mod_prod(ten,Ut);
end

%% Truncated High Order Orthogonal Iteration (HOOI)

% This function computes the Truncated HOOI of a given tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: May 2022

function [S,U] = HOOI_truncated(ten,ranks)
    if nargin < 2
        max_iter = 10;
        [~, U] = tensor.HOSVD_full(ten);
        number = numel(size(ten));
        for k = 1:max_iter
            for i = 1:number
                modes = 1:number;
                modes(i) = []; % It will skip this mode in the n_mod_prod.
                Un = tensor.n_mod_prod(ten,U,modes);
                [aux,eig,~] = svd(tensor.unfold(Un,i));
                [aa,~] = size(eig(eig > eps));
                U{i} = aux(:,1:aa);
            end
        end
        % The conjugate transpose
        Ut = cellfun(@(x) x', U ,'UniformOutput',false);
        S = tensor.n_mod_prod(ten,Ut);
    else
        max_iter = 10;
        [~, U] = tensor.HOSVD_full(ten);
```

```
        number = numel(size(ten));
        for k = 1:max_iter
            for i = 1:number
                modes = 1:number;
                modes(i) = []; % It will skip this mode in the n_mod_prod.
                Un = tensor.n_mod_prod(ten,U,modes);
                [aux,~,~] = svd(tensor.unfold(Un,i));
                U{i} = aux(:,1:ranks(i));
            end
        end
        % The conjugate transpose
        Ut = cellfun(@(x) x', U ,'UniformOutput',false);
        S = tensor.n_mod_prod(ten,Ut);
    end
end

%% ----- Homework 8 ----- %%
clc;
clear;
close all;

% Full HOSVD
load('homework8_HOOI.mat');
[tenS_hat,U_hat] = tensor.HOOI_full(tenX);
tenX_hat = tensor.n_mod_prod(tenS_hat,U_hat);

% Checking the orthogonality
disp('Checking the orthogonality propertie between the subtensors'...
    'formed from the core tensor:')
reshape(tenS_hat(:,:,1),[],1)'*reshape(tenS_hat(:,:,4),[],1)

disp('Checking the NMSE (dB) between the original tensor X and'...
    'its reconstruction:')
nmsex = (norm(tensor.unfold(tenX - tenX_hat,1),'fro')^2)...
    /(norm(tensor.unfold(tenX,1),'fro')^2);
nmsex = 20*log10(nmsex)
disp('Checking the NMSE (dB) between the original core tensor S'...
    'and its reconstruction:')
nmses = (norm(tensor.unfold(tenS - tenS_hat,1),'fro')^2)...
    /(norm(tensor.unfold(tenS,1),'fro')^2);
nmses = 20*log10(nmses)

disp('Checking the NMSE (dB) between the original matrices and'...
    'its estimations:')
nmseU1 = (norm(U1 - U_hat{1},'fro')^2)/(norm(U1,'fro')^2)
nmseU2 = (norm(U2 - U_hat{2},'fro')^2)/(norm(U2,'fro')^2)
nmseU3 = (norm(U3 - U_hat{3},'fro')^2)/(norm(U3,'fro')^2)

% Truncated HOSVD and Denoising
X = randn(8,4,10) + 1i*randn(8,4,10);
Y = randn(5,5,5) + 1i*randn(5,5,5);

[S1,U1] = tensor.HOOI_truncated(X);
```

```
multilinear_rank1 = size(S1);
[S2,U2] = tensor.HOOI_truncated(Y);
multilinear_rank2 = size(S2);

Xhat = tensor.n_mod_prod(S1,U1);
Yhat = tensor.n_mod_prod(S2,U2);

disp('Checking the NMSE (dB) between the original noisy tensor X and'...
    its reconstruction:')
nmsex = (norm(tensor.unfold(X- Xhat,1),'fro')^2)...
    /(norm(tensor.unfold(X,1),'fro')^2);
nmsex = 20*log10(nmsex)
disp('Checking the NMSE (dB) between the original noisy tensor Y'...
    'and its reconstruction:')
nmsey = (norm(tensor.unfold(Y- Yhat,1),'fro')^2)...
    /(norm(tensor.unfold(Y,1),'fro')^2);
nmsey = 20*log10(nmsey)

disp('The multilinear rank for the tensor X is:')
size(S1)
disp('The multilinear rank for the tensor Y is:')
size(S2)
```

Homework 9
Multidimensional Least-Squares Khatri-Rao Factorization (MLS-KRF)

Implementation MLS-KRF

The MLS-KRF algorithm aims to solve the following estimation problem

$$\left(\hat{\boldsymbol{A}}^{(1)}, \cdots, \hat{\boldsymbol{A}}^{(N)}\right) = \min_{\boldsymbol{A}^{(1)},\cdots,\boldsymbol{A}^{(N)}} \left|\left|\boldsymbol{X} - \boldsymbol{A}^{(1)} \diamond \cdots \diamond \boldsymbol{A}^{(N)}\right|\right|_{\mathrm{F}}^{2}, \tag{18}$$

and by implementing the algorithm it was possible to reach the following table of NMSE (dB) values using the validation files

NMSE($\boldsymbol{X}, \hat{\boldsymbol{X}}$)	NMSE($\boldsymbol{A}_1, \hat{\boldsymbol{A}}_1$)	NMSE($\boldsymbol{A}_2, \hat{\boldsymbol{A}}_2$)	NMSE($\boldsymbol{A}_3, \hat{\boldsymbol{A}}_3$)
-606.2255	+3.1432	+5.0165	+4.7297

as we can see the estimation of the matrix $\boldsymbol{X}$ is perfect, but the estimated factor matrices are far from the original ones. This is simply explained by the scale ambiguity intrinsic to the MLS-KRF. This could be eliminated if we assume previous knowledge of either the structure of the factor matrices or some of their elements. It is also important to say that the produced algorithm was generalized for a random number of factors.

Monte Carlo Experiment

In Figure 11 the Monte Carlo Experiment is used to draw the curve for the scenario $(I_1, I_2, I_3, R) = (2, 3, 4, 4)$. As we can observe in this simple channel model the MLSKRF reaches a decent performance. Of course that this is only possible because the original matrix $\boldsymbol{X}$ has a strong Khatri-Rao structure, but if by some reason the channel or the noise could disrupt this structure (as in the low-snr scenario that we can analyze in the same figure) then the performance could be an impediment for certain applications.

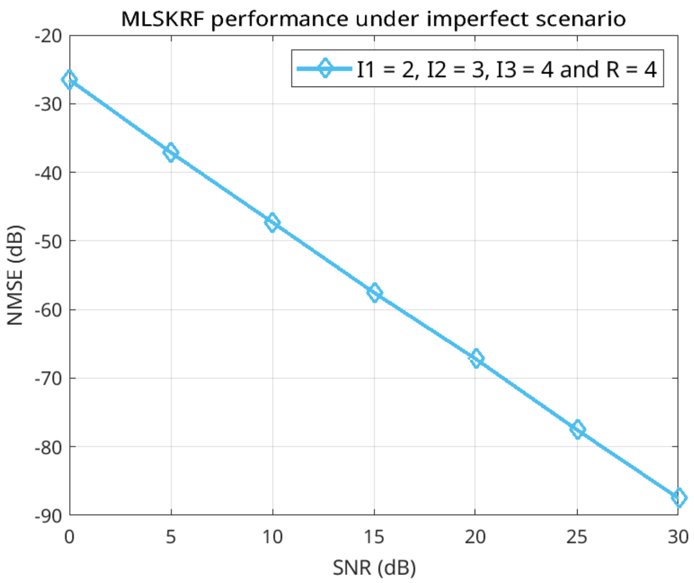


Figure 11: Monter Carlo Experiment with 1000 runs for MLS-KRF algorithm.

Produced Algorithm

```
%% Multidimensional Least-Squares Khatri-Rao Factorization (MLS-KRF)

% This function computes the MLS-KRF of a given matrix.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: March 2022

% The dimensions should be inserted in the order that the products are
% performed.
function [A] = MLSKRF(X,N,dim)
    [~,R] = size(X);
    for r = 1:R
        xr = X(:,r);
        tenXr = reshape(xr,flip(dim));

        % Aplicar SVDs consecutivas em estrategia recursiva? Como lidar com
        % o nd  array nesse caso?
        [Sr,Ur] = tensor.HOSVD_full(tenXr);
        for n = 1:N
            Ar{r,n} = (Sr(1)^(1/N))*Ur{N - n + 1}(:,1);
        end
    end

    for n = 1:N
    aux = cell2mat(Ar(:,n));
    A{n} = reshape(aux,[dim(n) R]);
    end
end

%% ----- Homework 9 ----- %%
clc;
clear;
close all;

load('homework9_MLSKRF.mat')

N = 3;
dim = [5 4 8];
[Matrices] = tensor.MLSKRF(X,N,dim);
Xhat = tensor.mtx_prod_kr(Matrices{1},Matrices{2});
Xhat = tensor.mtx_prod_kr(Xhat,Matrices{3});

disp(['Checking the NMSE (dB) between the original matrix X and its'...
    'reconstruction with MLSKRF:'])
nmsex = (norm(X - Xhat,'fro')^2)/(norm(X,'fro')^2);
nmsex = 20*log10(nmsex)
disp(['Checking the NMSE (dB) between the original matrix A and its'...
    'estimation:'])
nmsea = (norm(A - Matrices{1},'fro')^2)/(norm(A,'fro')^2);
nmsea = 20*log10(nmsea)
disp(['Checking the NMSE (dB) between the original matrix B and its'...
    'estimation:'])
```

```
nmseb = (norm(B - Matrices{2},'fro')^2)/(norm(B,'fro')^2);
nmseb = 20*log10(nmseb)
disp(['Checking the NMSE (dB) between the original matrix C and its'...
    'estimation:'])
nmsec = (norm(C - Matrices{3},'fro')^2)/(norm(C,'fro')^2);
nmsec = 20*log10(nmsec)

%% Monte Carlo Experiment
N = 3;
R  = 4;
I = 2;
J = 3;
K = 4;
dim = [I J K];
SNR = [0 5 10 15 20 25 30];
nmse = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(I*J*K,R) + 1j*randn(I*J*K,R));

        A = randn(I,R) + 1j*randn(I,R);
        B = randn(J,R) + 1j*randn(J,R);
        C = randn(K,R) + 1j*randn(K,R);
        X = tensor.mtx_prod_kr(A,B);
        X = tensor.mtx_prod_kr(X,C);
        X_noisy = X + noise;

        [Matrices] = tensor.MLSKRF(X_noisy,N,dim);
        Xhat = tensor.mtx_prod_kr(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kr(Xhat,Matrices{3});
        aux = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse(snr,1) = nmse(snr,1) + 20*log10(aux);
    end
end
nmse  = nmse/1000;

figure
txt = ['I1 = ' num2str(I), ', I2 = ' num2str(J), ', I3 = ' num2str(K),...
    ' and R = ' num2str(R)];
plot(SNR,nmse,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
title(['MLSKRF performance under imperfect scenario'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw9.png')
```

Homework 10
Multidimensional Least-Squares Kronecker Factorization (MLS-KronF)

Implementation MLS-KronF

The MLS-KronF algorithm aims to solve the following estimation problem

$$\left(\hat{\boldsymbol{A}}^{(1)}, \cdots, \hat{\boldsymbol{A}}^{(N)}\right) = \min_{\boldsymbol{A}^{(1)}, \cdots, \boldsymbol{A}^{(N)}} \left|\left|\boldsymbol{X} - \boldsymbol{A}^{(1)} \otimes \cdots \otimes \boldsymbol{A}^{(N)}\right|\right|_{\mathrm{F}}^{2}, \tag{19}$$

and by implementing the algorithm it was possible to reach the following table of NMSE (dB) values using the validation files for the HOSVD and HOOI initialization, respectively.

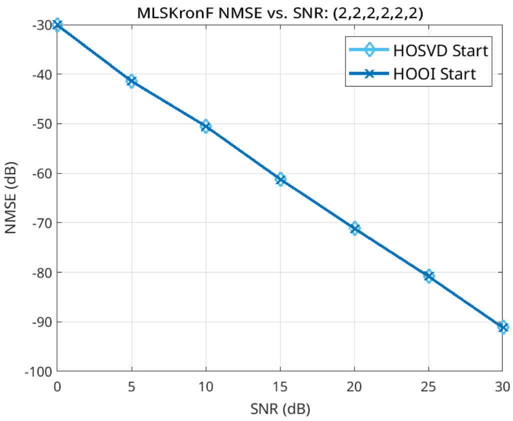
NNMSE($\boldsymbol{X}, \hat{\boldsymbol{X}}$)	NMSE($\boldsymbol{A}_1, \hat{\boldsymbol{A}}_1$)	NMSE($\boldsymbol{A}_2, \hat{\boldsymbol{A}}_2$)	NMSE($\boldsymbol{A}_3, \hat{\boldsymbol{A}}_3$)
-605.1941	+11.9214	+11.5548	+6.0950

NNMSE($\boldsymbol{X}, \hat{\boldsymbol{X}}$)	NMSE($\boldsymbol{A}_1, \hat{\boldsymbol{A}}_1$)	NMSE($\boldsymbol{A}_2, \hat{\boldsymbol{A}}_2$)	NMSE($\boldsymbol{A}_3, \hat{\boldsymbol{A}}_3$)
-608.4705	+5.4597	+15.0478	+5.8977

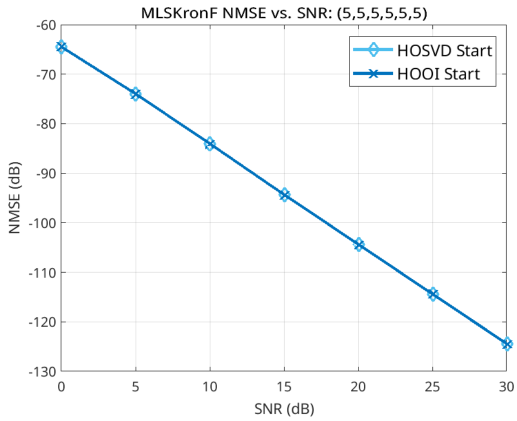
as we can see the estimation of the matrix $\boldsymbol{X}$ is perfect, but the estimated factor matrices are far from the original ones for both cases. This is simply explained by the scale ambiguity intrinsic to the MLS-KronF. This could be eliminated if we assume previous knowledge of either the structure of the factor matrices or some of their elements. It is also important to say that the produced algorithm was generalized for a random number of factors.

Monte Carlo Experiment

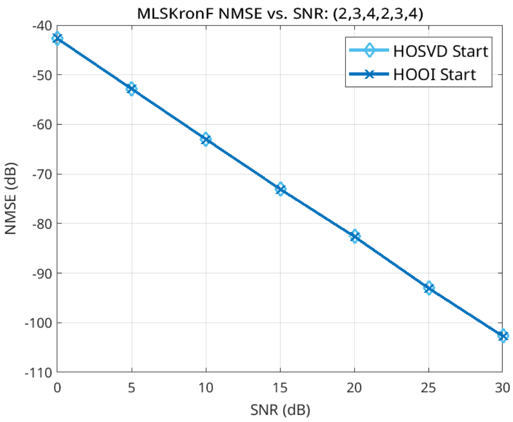
In Figures 12a, 12b, 12c and 12d the Monte Carlo Experiment is used to draw the curves for the scenarios $(I_1, I_2, I_3) = (J_1, J_2, J_3) = \{(2, 2, 2), (5, 5, 5), (2, 3, 4)\}$ and $(I_1, I_2, I_3) = (2, 3, 4), (J_1, J_2, J_3) = (5, 6, 7)$ for both the HOSVD and HOOI initializations. As we can observe in this simple channel model the MLSKronF reaches a decent performance. Of course that this is only possible because the original matrix $\boldsymbol{X}$ has a strong Kronecker structure, but if by some reason the channel or the noise could disrupt this structure (as it can be seen in the low-snr scenarios) then the performance could be an impediment for certain applications. Its also interesting to note that all for all the scenarios it was not possible to verify any significant difference between the two methods of initialization of the MLSKronF.



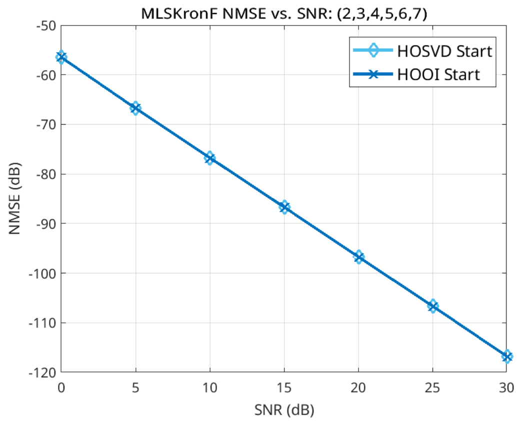
(a) Scenario 1



(b) Scenario 2



(c) Scenario 3



(d) Scenario 4

Figure 12: Monter Carlo Experiment with 1000 runs for MLS-KronF algorithm.

Produced Algorithm

```
%% Multidimensional Least-Squares Kronecker Factorization (MLS-KronF)

% This function computes the MLS-KronF of a given matrix.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: March 2022

% It is interesting to note that this process could easily be applied to a
% random number of matrices in a iterative form considering groups of 3 objects.
% init controls the initialization: 1 for HOSVD and 2 for HOOI.
function [Ahat] = MLSKronF(X,rows,columns,init)
    % 3rd structure
    %[ix,jx] = size(X);

    % 2nd structure
    I = rows(2)*rows(3) + zeros(1,rows(1));
    J = columns(2)*columns(3) + zeros(1,columns(1));
    blocks_of_X = mat2cell(X,I,J);

    % 1st structure
    Z = 1;
    for j = 1:columns(1)
        for i = 1:rows(1)
        aux = cell2mat(blocks_of_X(i,j));
        I = rows(3) + zeros(1,rows(2));
        J = columns(3) + zeros(1,columns(2));
        blocks_of_aux = mat2cell(aux,I,J);
        for jj = 1:columns(2)
                for ii = 1:rows(2)
                    vec_of_block = cell2mat(blocks_of_aux(ii,jj));
                    vec_of_block = vec_of_block(:);
                    mtx_1st(:,ii,jj) = vec_of_block;
                end
        end
            Xhat(:,Z) = reshape(mtx_1st,[],1);
            Z = Z + 1;
        end
    end

    tenXhat = reshape(Xhat,[rows(3)*columns(3), rows(2)*columns(2), rows(1)*columns(1)]);
    if init == '1'
        [S,U] = tensor.HOSVD_full(tenXhat);
    elseif init == '2'
        [S,U] = tensor.HOOI_full(tenXhat);
    end
    rows = flip(rows);
    columns = flip(columns);
    for u = 1:length(U)
        index = length(U) - u + 1;
        aux = (S(1)^(1/length(U)))*U{u}(:,1);
        Ahat{index} = reshape(aux,[rows(u) columns(u)]);
    end
```

```
end

%% ----- Homework 10 ----- %%
clc;
clear;
close all;

load('homework_10_MLSKronF.mat')

rows = [4 4 6];
columns = [3 2 5];

%% Initialization by HOSVD
[Matrices] = tensor.MLSKronF(X,rows,columns,'1');
aux1 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
Xhat = tensor.mtx_prod_kron(aux1,Matrices{3});
disp(['Checking the NMSE (dB) between the original matrix X and its'...
    'reconstruction with MLSKronF:'])
nmse1 = (norm(X - Xhat,'fro')^2)/(norm(X,'fro')^2);
nmse1 = 20*log10(nmse1)
disp(['Checking the NMSE (dB) between the original matrix A and its'...
    'estimation:'])
nmsea = (norm(A - Matrices{1},'fro')^2)/(norm(A,'fro')^2);
nmsea = 20*log10(nmsea)
disp(['Checking the NMSE (dB) between the original matrix B and its'...
    'estimation:'])
nmseb = (norm(B - Matrices{2},'fro')^2)/(norm(B,'fro')^2);
nmseb = 20*log10(nmseb)
disp(['Checking the NMSE (dB) between the original matrix C and its'...
    'estimation:'])
nmsec = (norm(C - Matrices{3},'fro')^2)/(norm(C,'fro')^2);
nmsec = 20*log10(nmsec)

%% Initialization by HOOI
[Matrices] = tensor.MLSKronF(X,rows,columns,'2');
aux1 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
Xhat = tensor.mtx_prod_kron(aux1,Matrices{3});
disp(['Checking the NMSE (dB) between the original matrix X and its'...
    'reconstruction with MLSKronF:'])
nmse1 = (norm(X - Xhat,'fro')^2)/(norm(X,'fro')^2);
nmse1 = 20*log10(nmse1)
disp(['Checking the NMSE (dB) between the original matrix A and its'...
    'estimation:'])
nmsea = (norm(A - Matrices{1},'fro')^2)/(norm(A,'fro')^2);
nmsea = 20*log10(nmsea)
disp(['Checking the NMSE (dB) between the original matrix B and its'...
    'estimation:'])
nmseb = (norm(B - Matrices{2},'fro')^2)/(norm(B,'fro')^2);
nmseb = 20*log10(nmseb)
disp(['Checking the NMSE (dB) between the original matrix C and its'...
    'estimation:'])
nmsec = (norm(C - Matrices{3},'fro')^2)/(norm(C,'fro')^2);
nmsec = 20*log10(nmsec)
```

```
%% Monte Carlo Experiment
ia = 2;
ib = 2;
ic = 2;
ja = 2;
jb = 2;
jc = 2;
rows = [ia ib ic];
columns = [ja jb jc];
SNR = [0 5 10 15 20 25 30];
nmse1 = zeros(length(SNR),1);
nmse2 = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(ia*ib*ic,ja*jb*jc) ...
            + 1j*randn(ia*ib*ic,ja*jb*jc));

        A = randn(ia,ja) + 1i*randn(ia,ja);
        B = randn(ib,jb) + 1i*randn(ib,jb);
        C = randn(ic,jc) + 1i*randn(ic,jc);
        X = tensor.mtx_prod_kron(A,B);
        X = tensor.mtx_prod_kron(X,C);
        X_noisy = X + noise;

        % HOSVD
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'1');
        aux1 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux1,Matrices{3});
        aux1 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse1(snr,1) = nmse1(snr,1) + 20*log10(aux1);
        % HOOI
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'2');
        aux2 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux2,Matrices{3});
        aux2 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse2(snr,1) = nmse2(snr,1) + 20*log10(aux2);
    end
end
nmse1  = nmse1/1000;
nmse2  = nmse2/1000;

figure
txt = ['HOSVD Start'];
plot(SNR,nmse1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['HOOI Start'];
plot(SNR,nmse2,'-x','color', [0 0.4470 0.7410], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['MLSKronF NMSE vs. SNR: (2,2,2,2,2,2)'])
```

```matlab
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw10a1.png')

ia = 5;
ib = 5;
ic = 5;
ja = 5;
jb = 5;
jc = 5;
rows = [ia ib ic];
columns = [ja jb jc];
SNR = [0 5 10 15 20 25 30];
nmse1 = zeros(length(SNR),1);
nmse2 = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(ia*ib*ic,ja*jb*jc) ...
            + 1j*randn(ia*ib*ic,ja*jb*jc));

        A = randn(ia,ja) + 1i*randn(ia,ja);
        B = randn(ib,jb) + 1i*randn(ib,jb);
        C = randn(ic,jc) + 1i*randn(ic,jc);
        X = tensor.mtx_prod_kron(A,B);
        X = tensor.mtx_prod_kron(X,C);
        X_noisy = X + noise;

        % HOSVD
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'1');
        aux1 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux1,Matrices{3});
        aux1 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse1(snr,1) = nmse1(snr,1) + 20*log10(aux1);
        % HOOI
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'2');
        aux2 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux2,Matrices{3});
        aux2 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse2(snr,1) = nmse2(snr,1) + 20*log10(aux2);
    end
end
nmse1  = nmse1/1000;
nmse2  = nmse2/1000;

figure
txt = ['HOSVD Start'];
plot(SNR,nmse1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
```

```
txt = ['HOOI Start'];
plot(SNR,nmse2,'-x','color', [0 0.4470 0.7410], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['MLSKronF NMSE vs. SNR: (5,5,5,5,5,5)'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw10a2.png')

ia = 2;
ib = 3;
ic = 4;
ja = 2;
jb = 3;
jc = 4;
rows = [ia ib ic];
columns = [ja jb jc];
SNR = [0 5 10 15 20 25 30];
nmse1 = zeros(length(SNR),1);
nmse2 = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(ia*ib*ic,ja*jb*jc) ...
            + 1j*randn(ia*ib*ic,ja*jb*jc));

        A = randn(ia,ja) + 1i*randn(ia,ja);
        B = randn(ib,jb) + 1i*randn(ib,jb);
        C = randn(ic,jc) + 1i*randn(ic,jc);
        X = tensor.mtx_prod_kron(A,B);
        X = tensor.mtx_prod_kron(X,C);
        X_noisy = X + noise;

        % HOSVD
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'1');
        aux1 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux1,Matrices{3});
        aux1 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse1(snr,1) = nmse1(snr,1) + 20*log10(aux1);
        % HOOI
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'2');
        aux2 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux2,Matrices{3});
        aux2 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse2(snr,1) = nmse2(snr,1) + 20*log10(aux2);
    end
end
nmse1  = nmse1/1000;
nmse2  = nmse2/1000;
```

```
figure
txt = ['HOSVD Start'];
plot(SNR,nmse1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['HOOI Start'];
plot(SNR,nmse2,'-x','color', [0 0.4470 0.7410], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['MLSKronF NMSE vs. SNR: (2,3,4,2,3,4)'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw10a3.png')

ia = 2;
ib = 3;
ic = 4;
ja = 5;
jb = 6;
jc = 7;
rows = [ia ib ic];
columns = [ja jb jc];
SNR = [0 5 10 15 20 25 30];
nmse1 = zeros(length(SNR),1);
nmse2 = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(ia*ib*ic,ja*jb*jc) ...
            + 1j*randn(ia*ib*ic,ja*jb*jc));

        A = randn(ia,ja) + 1i*randn(ia,ja);
        B = randn(ib,jb) + 1i*randn(ib,jb);
        C = randn(ic,jc) + 1i*randn(ic,jc);
        X = tensor.mtx_prod_kron(A,B);
        X = tensor.mtx_prod_kron(X,C);
        X_noisy = X + noise;

        % HOSVD
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'1');
        aux1 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux1,Matrices{3});
        aux1 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse1(snr,1) = nmse1(snr,1) + 20*log10(aux1);
        % HOOI
        [Matrices] = tensor.MLSKronF(X_noisy,rows,columns,'2');
        aux2 = tensor.mtx_prod_kron(Matrices{1},Matrices{2});
        Xhat = tensor.mtx_prod_kron(aux2,Matrices{3});
        aux2 = (norm(X- Xhat,'fro')^2)/(norm(X,'fro')^2);
        nmse2(snr,1) = nmse2(snr,1) + 20*log10(aux2);
```

```
    end
end
nmse1  = nmse1/1000;
nmse2  = nmse2/1000;

figure
txt = ['HOSVD Start'];
plot(SNR,nmse1,'-d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['HOOI Start'];
plot(SNR,nmse2,'-x','color', [0 0.4470 0.7410], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['MLSKronF NMSE vs. SNR: (2,3,4,5,6,7)'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "northwest");
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw10a4.png')
```

Homework 11
Alternating Least Squares (ALS) Algorithm

Implementation of ALS

$$\left(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{C}}\right) = \min_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}} \left\| \mathcal{X} - \sum_{r=1}^{R} \boldsymbol{a}_r \circ \boldsymbol{b}_r \circ \boldsymbol{c}_r \right\|_{\mathrm{F}}^2, \tag{20}$$

Monte Carlo Experiment

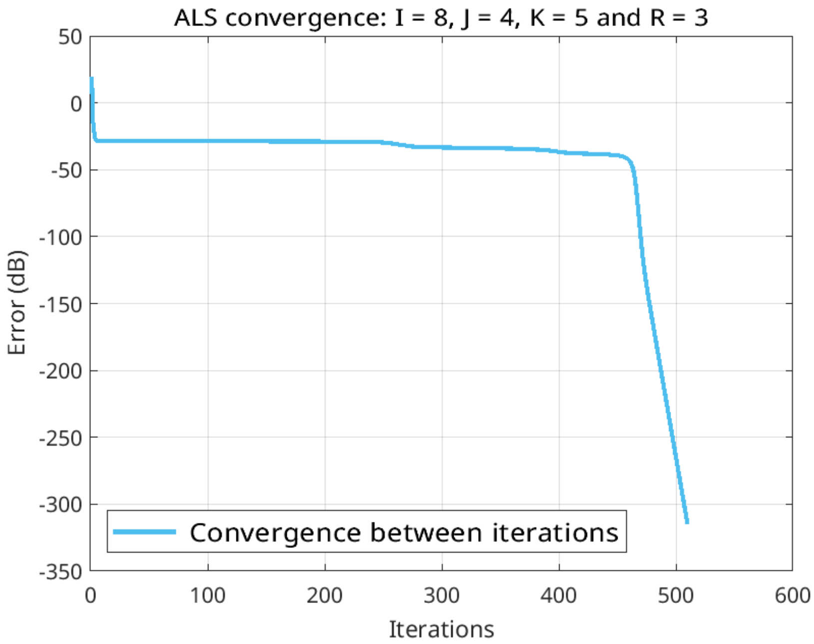


Figure 13: Convergence behavior of ALS algorithm.

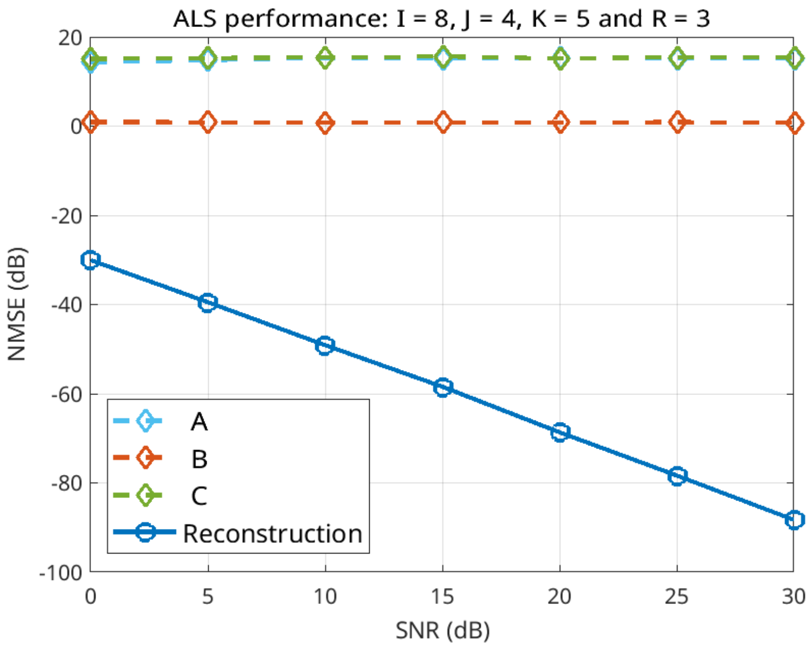


Figure 14: Monter Carlo Experimento with 1000 runs for ALS algorithm.

Produced Algorithm

```
%% Alternate Least-Square (ALS)

% This function computes the ALS of a given tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: April 2022

function [Ahat,Bhat,Chat,error] = ALS(X,R)
    I = zeros(R,R,R);
    for i = 1:R
        I(i,i,i) = 1;
    end
    [ia,ib,ic] = size(X);

    mode_1 = tensor.unfold(X,1);
    mode_2 = tensor.unfold(X,2);
    mode_3 = tensor.unfold(X,3);

    Ahat = randn(ia,R) + 1j*randn(ia,R);
    Bhat = randn(ib,R) + 1j*randn(ib,R);
    Chat = randn(ic,R) + 1j*randn(ic,R);

    aux = 1000;
    error = zeros(1,aux);
    error(1) = ((norm((mode_1 - Ahat*(tensor.mtx_prod_kr(Chat,Bhat).')),'fro'))^2)/((norm(mo
    for i = 2:aux
        Bhat = mode_2*pinv((tensor.mtx_prod_kr(Chat,Ahat)).');
        Chat = mode_3*pinv((tensor.mtx_prod_kr(Bhat,Ahat)).');
```

```
        Ahat = mode_1*pinv((tensor.mtx_prod_kr(Chat,Bhat)).');
        error(i) = ((norm((mode_1 - Ahat*(tensor.mtx_prod_kr(Chat,Bhat).')),'fro'))^2)/((nor
        if abs(error(i) - error(i-1)) < eps
            error = error(1:i);
            break;
        else
            continue;
        end
    end
end

%% ----- Homework 11 ----- %%
clc;
clear;
close all;

% Random tensor example
% I = 8;
% J = 4;
% K = 5;
% R = 3;
% A = randn(I,R) + 1j*randn(I,R);
% B = randn(J,R) + 1j*randn(J,R);
% C = randn(K,R) + 1j*randn(K,R);
% X = tensor.fold(A*(tensor.mtx_prod_kr(C,B).'),[I J K],1);

load('homework_11_CPD.mat')
X = tenX;
R = 3;

[ia,ib,ic] = size(X);
mode_1 = tensor.unfold(X,1);
mode_2 = tensor.unfold(X,2);
mode_3 = tensor.unfold(X,3);

Ahat = randn(ia,R) + 1j*randn(ia,R);
Bhat = randn(ib,R) + 1j*randn(ib,R);
Chat = randn(ic,R) + 1j*randn(ic,R);

aux = 10000;
error = zeros(1,aux);
error(1) = ((norm((mode_1 - Ahat*(tensor.mtx_prod_kr(Chat,Bhat).'))...
    ,'fro'))^2)/((norm(mode_1,'fro')^2));
for i = 2:aux
    Bhat = mode_2*pinv((tensor.mtx_prod_kr(Chat,Ahat)).');
    Chat = mode_3*pinv((tensor.mtx_prod_kr(Bhat,Ahat)).');
    Ahat = mode_1*pinv((tensor.mtx_prod_kr(Chat,Bhat)).');
    error(i) = ((norm((mode_1 - Ahat*(tensor.mtx_prod_kr(Chat,Bhat).'))...
        ,'fro'))^2)/((norm(mode_1,'fro')^2));
    if abs(error(i) - error(i-1)) < eps
        error = error(1:i);
        break;
    else
```

```
        continue;
    end
end

disp('NMSE (dB) between the original tensor X and its'...
    'reconstruction with MLSKRF:')
Xhat = tensor.fold(Ahat*(tensor.mtx_prod_kr(Chat,Bhat).'),[ia ib ic],1);
nmsex = (norm(tensor.unfold(X- Xhat,1),'fro')^2)...
    /(norm(tensor.unfold(X,1),'fro')^2);
nmsex = 20*log10(nmsex)
disp('NMSE (dB) between the original matrix A and its estimation:')
nmsea = (norm(A - Ahat,'fro')^2)/(norm(A,'fro')^2);
nmsea = 20*log10(nmsea)
disp('NMSE (dB) between the original matrix B and its estimation:')
nmseb = (norm(B - Bhat,'fro')^2)/(norm(B,[0.3010 0.7450 0.9330]'fro')^2);
nmseb = 20*log10(nmseb)
disp('NMSE (dB) between the original matrix C and its estimation:')
nmsec = (norm(C - Chat,'fro')^2)/(norm(C,'fro')^2);
nmsec = 20*log10(nmsec)

figure
txt = ['\bf Convergence between iterations'];
plot(1:i,20*log10(error),'-','color', [0.3010 0.7450 0.9330],...
    "linewidth", 2, "markersize", 8, "DisplayName", txt);
title(['ALS convergence: I = ' num2str(ia), ', J = ' num2str(ib),...
    ', K = ' num2str(ic), ' and R = ' num2str(R)])
xlabel('Iterations')
ylabel('Error (dB)')
legend_copy = legend("location", "southwest");
set(legend_copy,'Interpreter','tex','location','southwest',"fontsize", 12)
grid on;
saveas(gcf,'hw11a1.png')

%% ALS in the presence of noisy signals

I = 8;
J = 4;
K = 5;
R = 3;
SNR = [0 5 10 15 20 25 30];
nmse1 = zeros(length(SNR),1);
nmse2 = zeros(length(SNR),1);
nmse3 = zeros(length(SNR),1);
nmse4 = zeros(length(SNR),1);
for snr = 1:length(SNR)
    for mc = 1:1000
        A = randn(I,R) + 1j*randn(I,R);
        B = randn(J,R) + 1j*randn(J,R);
        C = randn(K,R) + 1j*randn(K,R);
        X = tensor.fold(A*(tensor.mtx_prod_kr(C,B).'),[I J K],1);

        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise/2)*(randn(I,J,K) + 1j*randn(I,J,K));
```

```
        X = X + noise;

        [ia,ib,ic] = size(X);
        mode_1 = tensor.unfold(X,1);
        mode_2 = tensor.unfold(X,2);
        mode_3 = tensor.unfold(X,3);
        Ahat = randn(ia,R) + 1j*randn(ia,R);
        Bhat = randn(ib,R) + 1j*randn(ib,R);
        Chat = randn(ic,R) + 1j*randn(ic,R);
        aux = 1000;
        error = zeros(1,aux);
        error(1) = ((norm((mode_1 - Ahat*(tensor.mtx_prod_kr(Chat...
            ,Bhat).'')),'fro'))^2)/((norm(mode_1,'fro')^2));
        for i = 2:aux
            Bhat = mode_2*pinv((tensor.mtx_prod_kr(Chat,Ahat)).');
            Chat = mode_3*pinv((tensor.mtx_prod_kr(Bhat,Ahat)).');
            Ahat = mode_1*pinv((tensor.mtx_prod_kr(Chat,Bhat)).');
            error(i) = ((norm((mode_1 - Ahat*(tensor.mtx_prod_kr(Chat...
                ,Bhat).'')),'fro'))^2)/((norm(mode_1,'fro')^2));
            if abs(error(i) - error(i-1)) < 1e-6
                error = error(1:i);
                break;
            else
                continue;
            end
        end
        Xhat = tensor.fold(Ahat*(tensor.mtx_prod_kr(Chat,Bhat).')...
            ,[I J K],1);

        aux1 = (norm(A- Ahat,'fro')^2)/(norm(A,'fro')^2);
        nmse1(snr,1) = nmse1(snr,1) + 20*log10(aux1);
        aux2 = (norm(B- Bhat,'fro')^2)/(norm(B,'fro')^2);
        nmse2(snr,1) = nmse2(snr,1) + 20*log10(aux2);
        aux3 = (norm(C- Chat,'fro')^2)/(norm(C,'fro')^2);
        nmse3(snr,1) = nmse3(snr,1) + 20*log10(aux3);
        aux4 = (norm(tensor.unfold(X- Xhat,1),'fro')^2)...
            /(norm(tensor.unfold(X,1),'fro')^2);
        nmse4(snr,1) = nmse4(snr,1) + 20*log10(aux4);
    end
end
nmse1  = nmse1/1000;
nmse2  = nmse2/1000;
nmse3  = nmse3/1000;
nmse4  = nmse4/1000;

figure
txt = ['\bf A'];
plot(SNR,nmse1,'--d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['\bf B'];
plot(SNR,nmse2,'--d','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
```

```
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['\bf C'];
plot(SNR,nmse3,'--d','color', [0.4660 0.6740 0.1880], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['Reconstruction'];
plot(SNR,nmse4,'-o','color', [0 0.4470 0.7410], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['ALS performance: I = ' num2str(I), ', J = ' num2str(J),...
    ', K = ' num2str(K), ' and R = ' num2str(R)])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "southwest");
set(legend_copy,'Interpreter','tex','location','southwest',"fontsize", 12)
grid on;
saveas(gcf,'hw11a2.png')
```

Homework 12
Tensor Kronecker Product SVD (TKPSVD)

Implementation of TKPSVD

$$\mathcal{X} = \sum_{j=1}^{R} \sigma_i \mathcal{A}_j^{(d)} \otimes \mathcal{A}_j^{(d-1)} \otimes \cdots \otimes \mathcal{A}_j^{(1)}, \tag{21}$$

NMSE($\mathcal{X}, \hat{\mathcal{X}}$)	NMSE($\mathcal{A}^{(1)}, \hat{\mathcal{A}}^{(1)}$)	NMSE($\mathcal{A}^{(2)}, \hat{\mathcal{A}}^{(2)}$)	NMSE($\mathcal{A}^{(3)}, \hat{\mathcal{A}}^{(3)}$)
-625.5192	+37.0803	+0.1706	+32.5731

Monte Carlo Experiment

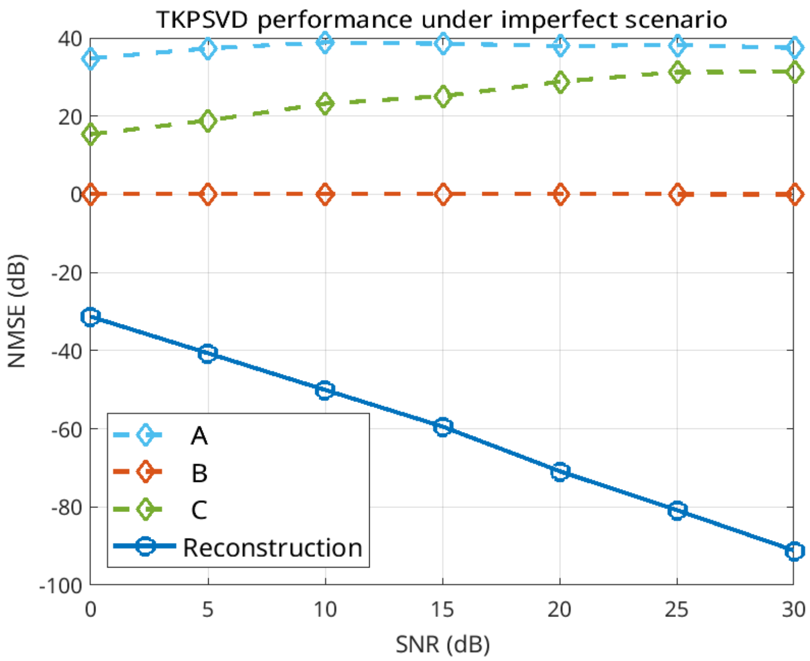


Figure 15: Monter Carlo Experiment with 1000 runs for TKPSVD algorithm.

Produced Algorithm

```
%% Tensor Kronecker Product

% This function computes the Tensor Kronecker Product of two given tensors.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: June 2022

function C = ten_prod_kron(A,B)

    aux1 = num2cell([size(A)]);
    aux2 = num2cell([size(B)]);

    A = repelem(A,aux2{:});
    B = repmat(B,aux1{:});
    C = A.*B;

end

%% Tensor Kronecker Product Single Value Decomposition (TKPSVD)

% This function computes the TKPSVD of a tensor.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: June 2022

function [Ahat,Bhat,Chat] = TKPSVD(tenX,tenSize,tenDim,N,R)
    % First reorder
    var = 1:length(tenSize);
    for n = 1:N
        tenXhatreorder{n} = cell2mat(tenSize(var(n:N:length(tenSize))));
    end
    tenXhatreorder = cell2mat(tenXhatreorder);
    tenXhat = reshape(tenX,tenXhatreorder);
    % Second reorder
    var = 1:length(tenSize);
    for n = 1:N
        tenXhatpermute{n} = var(n:N:length(tenSize));
    end
    tenXhatpermute = cell2mat(tenXhatpermute);
    tenXhat = permute(tenXhat,tenXhatpermute);
    % Third reorder
    tenXhat = reshape(tenXhat,tenDim{:});
    % ALS estimation
    [Ahat,Bhat,Chat,~] = tensor.ALS(tenXhat,R);
end

%% ----- Homework 12 ----- %%
clc;
clear;
close all;

%% TKPSVD
R = 1;
```

```
N = 3;

A = zeros(80,R);
B = zeros(20,R);
C = zeros(8,R);
tenX = zeros(100,16,8);
for r = 1:R
    tenA = randn(10,4,2);
    %tenA = tenA./frob(tenA);
    A(:,r)= tenA(:);
    tenB = randn(5,2,2);
    %tenB = tenB./frob(tenB);
    B(:,r)= tenB(:);
    tenC = randn(2,2,2);
    %tenC = tenC./frob(tenC);
    C(:,r)= tenC(:);
    varr = tensor.ten_prod_kron(tenC,tenB);
    tenX = tenX + tensor.ten_prod_kron(varr,tenA);
end

aux{1} = num2cell(size(tenA));
aux{2} = num2cell(size(tenB));
aux{3} = num2cell(size(tenC));
tenSize = horzcat(aux{:});
tenDim = cellfun(@(aux) prod(cell2mat(aux)), aux,'UniformOutput',false);

tenX = tensor.ten_prod_kron(tenC,tenB);
tenX = tensor.ten_prod_kron(tenX,tenA);

[Ahat,Bhat,Chat] = tensor.TKPSVD(tenX,tenSize,tenDim,N,R);

aux1 = aux{1};
aux2 = aux{2};
aux3 = aux{3};
tenXhat = zeros(100,16,8);
for r = 1:R
    tenAhat_r = reshape(Ahat(:,r),aux1{:});
    %tenAhat_r = tenAhat_r./frob(tenAhat_r);
    tenBhat_r = reshape(Bhat(:,r),aux2{:});
    %tenBhat_r = tenBhat_r./frob(tenBhat_r);
    tenChat_r = reshape(Chat(:,r),aux3{:});
    %tenChat_r = tenChat_r./frob(tenChat_r);
    varr = tensor.ten_prod_kron(tenChat_r,tenBhat_r);
    tenXhat = tenXhat + tensor.ten_prod_kron(varr,tenAhat_r);
end

disp('Checking the NMSE (dB) between the original tensor X and its'...
    'reconstruction with TKPSVD:')
nmsex = (norm(tensor.unfold(tenX- tenXhat,1),'fro')^2)...
    /(norm(tensor.unfold(tenX,1),'fro')^2);
nmsex = 20*log10(nmsex)
disp('Checking the NMSE (dB) between the original matrix A and its'...
    'reconstruction with TKPSVD:')
```

```
nmsea = 20*log10((norm(A- Ahat,'fro')^2)/(norm(A,'fro')^2))
disp('Checking the NMSE (dB) between the original matrix B and its'...
    'reconstruction with TKPSVD:')
nmseb = 20*log10((norm(B- Bhat,'fro')^2)/(norm(B,'fro')^2))
disp('Checking the NMSE (dB) between the original matrix C and its'...
    'reconstruction with TKPSVD:')
nmsec = 20*log10((norm(C- Chat,'fro')^2)/(norm(C,'fro')^2))

%% Monte Carlo Simulation

R = 2;
N = 3;
SNR = [0 5 10 15 20 25 30];
nmse1 = zeros(length(SNR),1);
nmse2 = zeros(length(SNR),1);
nmse3 = zeros(length(SNR),1);
nmse4 = zeros(length(SNR),1);
for snr = 1:length(SNR)
    snr
    for mc = 1:1000
        A = zeros(80,R);
        B = zeros(20,R);
        C = zeros(8,R);
        tenX = zeros(100,16,8);
        for r = 1:R
            tenA = randn(10,4,2);
            %tenA = tenA./frob(tenA);
            A(:,r)= tenA(:);
            tenB = randn(5,2,2);
            %tenB = tenB./frob(tenB);
            B(:,r)= tenB(:);
            tenC = randn(2,2,2);
            %tenC = tenC./frob(tenC);
            C(:,r)= tenC(:);
            varr = tensor.ten_prod_kron(tenC,tenB);
            tenX = tenX + tensor.ten_prod_kron(varr,tenA);
        end

        aux{1} = num2cell(size(tenA));
        aux{2} = num2cell(size(tenB));
        aux{3} = num2cell(size(tenC));
        tenSize = horzcat(aux{:});
        tenDim = cellfun(@(aux) prod(cell2mat(aux)), aux,...
            'UniformOutput',false);

        tenX = tensor.ten_prod_kron(tenC,tenB);
        tenX = tensor.ten_prod_kron(tenX,tenA);

        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise)*(randn(size(tenX)));
        tenX_noisy = tenX + noise;

        [Ahat,Bhat,Chat] = tensor.TKPSVD(tenX_noisy,tenSize,tenDim,N,R);
```

```
        aux1 = aux{1};
        aux2 = aux{2};
        aux3 = aux{3};
        tenXhat = zeros(100,16,8);
        for r = 1:R
            tenAhat_r = reshape(Ahat(:,r),aux1{:});
            %tenAhat_r = tenAhat_r./frob(tenAhat_r);
            tenBhat_r = reshape(Bhat(:,r),aux2{:});
            %tenBhat_r = tenBhat_r./frob(tenBhat_r);
            tenChat_r = reshape(Chat(:,r),aux3{:});
            %tenChat_r = tenChat_r./frob(tenChat_r);
            varr = tensor.ten_prod_kron(tenChat_r,tenBhat_r);
            tenXhat = tenXhat + tensor.ten_prod_kron(varr,tenAhat_r);
        end

        var1 = (norm(A - Ahat,'fro')^2)/(norm(A,'fro')^2);
        nmse1(snr,1) = nmse1(snr,1) + 20*log10(var1);
        var2 = (norm(B - Bhat,'fro')^2)/(norm(B,'fro')^2);
        nmse2(snr,1) = nmse2(snr,1) + 20*log10(var2);
        var3 = (norm(C - Chat,'fro')^2)/(norm(C,'fro')^2);
        nmse3(snr,1) = nmse3(snr,1) + 20*log10(var3);
        nmse4(snr,1) = nmse4(snr,1) + 20*log10((norm(tensor.unfold(tenX...
            - tenXhat,1),'fro')^2)/(norm(tensor.unfold(tenX,1),'fro')^2));

    end
end
nmse1  = nmse1/1000;
nmse2  = nmse2/1000;
nmse3  = nmse3/1000;
nmse4  = nmse4/1000;

figure
txt = ['\bf A'];
plot(SNR,nmse1,'--d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['\bf B'];
plot(SNR,nmse2,'--d','color', [0.8500 0.3250 0.0980], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['\bf C'];
plot(SNR,nmse3,'--d','color', [0.4660 0.6740 0.1880], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['Reconstruction'];
plot(SNR,nmse4,'-o','color', [0 0.4470 0.7410], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
title(['TKPSVD performance under imperfect scenario'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "southwest");
```

```
set(legend_copy,'Interpreter','tex','location','northeast',"fontsize", 12)
grid on;
saveas(gcf,'hw12.png')
```

Homework 13
Tensor Train Single Value Decomposition (TTSVD)

Implementation of TTSVD

$$\mathcal{X} = \boldsymbol{G}_1 \bullet_2^1 \mathcal{G}_2 \bullet_3^1 \mathcal{G}_3 \bullet_4^1 \boldsymbol{G}_4, \tag{22}$$

$$\left(\hat{\boldsymbol{G}}_1, \hat{\mathcal{G}}_2, \hat{\mathcal{G}}_3, \hat{\boldsymbol{G}}_4\right) = \min_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}} \left|\left|\mathcal{X} - \boldsymbol{G}_1 \bullet_2^1 \mathcal{G}_2 \bullet_3^1 \mathcal{G}_3 \bullet_4^1 \boldsymbol{G}_4\right|\right|_{\mathrm{F}}, \tag{23}$$

NMSE($\mathcal{X}, \hat{\mathcal{X}}$) with $R = (3, 3, 3)$	NMSE($\mathcal{X}, \hat{\mathcal{X}}$) with $R = (2, 2, 2)$
-606.0326	-25.1084

Monte Carlo Experiment

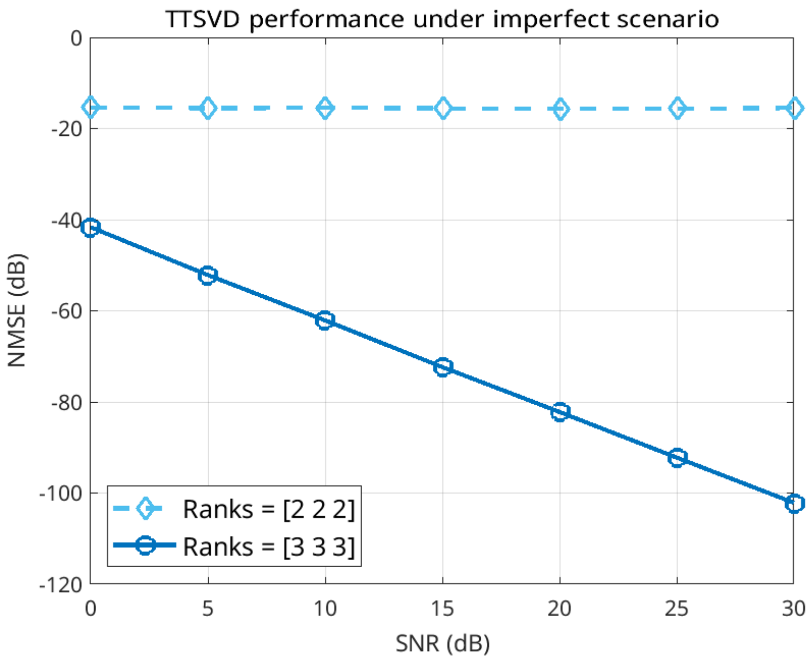


Figure 16: Monter Carlo Experiment with 1000 runs for TTSVD algorithm.

Produced Algorithm

```
%% Tensor Contraction

% This function computes the tensor contraction between a tensor and a matrix
% or between two tensors.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: June 2022

function [tenZ] = contraction(tenX,n1,tenY,n2)
    % Obtaining the unfolds
    tenX_n = tensor.unfold(tenX,n1);
    tenY_n = tensor.unfold(tenY,n2);
    % Obtaining the unfold of the contracted tensor
    tenZ_n = (tenX_n.')*tenY_n;
    % Obtaining the fold of the contracted tensor
    dim_x = size(tenX);
    dim_x(n1) = [];
    dim_y = size(tenY);
    dim_y(n2) = [];
    dim_z = [dim_x dim_y];
    tenZ   = tensor.fold(tenZ_n,dim_z,1);
end

%% Tensor Train Single Value Decomposition (TT-SVD)

% This function computes the TT-SVD of a fourth order tensor but can be extended later.
% Author: Kenneth B. dos A. Benicio <kenneth@gtel.ufc.br>
% Created: June 2022

function [G] = TTSVD(tenX,Ranks)
    X_size = size(tenX);
    % Step 1
    X1 = tensor.unfold(tenX,1);
    [U1,S1,V1] = svd(X1,'econ');
    G1 = U1*sqrt(S1);
    G1 = G1(:,1:Ranks(1));
    G{1} = G1;
    % Step 2
    V1 = sqrt(S1)*V1';
    V1 = V1(1:Ranks(1),:);
    X2 = reshape(V1,[Ranks(1)*X_size(2) X_size(3)*X_size(4)]);
    [U2,S2,V2] = svd(X2,'econ');
    G2 = U2*sqrt(S2);
    G2 = G2(:,1:Ranks(2));
    G{2} = reshape(G2, [Ranks(1) X_size(2) Ranks(2)]);
    % Step 3
    V2 = sqrt(S2)*V2';
    V2 = V2(1:Ranks(2),:);
    X3 = reshape(V2,[Ranks(2)*X_size(3) X_size(4)]);
    [U3,S3,V3] = svd(X3,'econ');
    G3 = U3*sqrt(S3);
    G3 = G3(:,1:Ranks(3));
```

```
    G{3} = reshape(G3, [Ranks(2) X_size(3) Ranks(3)]);
    V3 = sqrt(S3)*V3';
    V3 = V3(1:Ranks(3),:);
    G{4} = V3;

    %% ----- Homework 13 ----- %%
clc;
clear;
close all;

%% Tensor Train SVD for a fourth order tensor
I1 = 5;
I2 = 5;
I3 = 5;
I4 = 5;
Ranks = [3 3 3];

G1 = randn(I1,Ranks(1));
G2 = randn(Ranks(1),I2,Ranks(2));
G3 = randn(Ranks(2),I3,Ranks(3));
G4 = randn(Ranks(3),I4);

tenX = tensor.contraction(G1,2,G2,1);
tenX = tensor.contraction(tenX,3,G3,1);
tenX = tensor.contraction(tenX,4,G4,1);

[G] = tensor.TTSVD(tenX,Ranks);

tenXhat = tensor.contraction(G{1},2,G{2},1);
tenXhat = tensor.contraction(tenXhat,3,G{3},1);
tenXhat = tensor.contraction(tenXhat,4,G{4},1);

disp('Checking the TTSVD NMSE (dB) from reconstruction using'...
    'ranks = [3 3 3]:')
nmsex = (norm(tensor.unfold(tenX- tenXhat,1),'fro')^2)...
    /(norm(tensor.unfold(tenX,1),'fro')^2);
nmsex = 20*log10(nmsex)

Ranks = [2 2 2];
[G] = tensor.TTSVD(tenX,Ranks);

tenXhat = tensor.contraction(G{1},2,G{2},1);
tenXhat = tensor.contraction(tenXhat,3,G{3},1);
tenXhat = tensor.contraction(tenXhat,4,G{4},1);
disp('Checking the TTSVD NMSE (dB) from reconstruction using'...
    'ranks = [2 2 2]:')
nmsex = (norm(tensor.unfold(tenX- tenXhat,1),'fro')^2)...
    /(norm(tensor.unfold(tenX,1),'fro')^2);
nmsex = 20*log10(nmsex)

%% Monte Carlo Simulation
I1 = 5;
I2 = 5;
```

```
I3 = 5;
I4 = 5;
Ranks  = [3 3 3];
Ranks1 = [2 2 2];
Ranks2 = [3 3 3];

SNR = [0 5 10 15 20 25 30];
nmse1 = zeros(length(SNR),1);
nmse2 = zeros(length(SNR),1);
for snr = 1:length(SNR)
    snr
    for mc = 1:1000
        G1 = randn(I1,Ranks(1));
        G2 = randn(Ranks(1),I2,Ranks(2));
        G3 = randn(Ranks(2),I3,Ranks(3));
        G4 = randn(Ranks(3),I4);

        tenX = tensor.contraction(G1,2,G2,1);
        tenX = tensor.contraction(tenX,3,G3,1);
        tenX = tensor.contraction(tenX,4,G4,1);

        var_noise = 1/(10^(SNR(snr)/10));
        noise = sqrt(var_noise)*(randn(size(tenX)));
        tenX_noisy = tenX + noise;

        [GR1] = tensor.TTSVD(tenX_noisy,Ranks1);
        tenXhat = tensor.contraction(GR1{1},2,GR1{2},1);
        tenXhat = tensor.contraction(tenXhat,3,GR1{3},1);
        tenXhat = tensor.contraction(tenXhat,4,GR1{4},1);
        nmse1(snr,1) = nmse1(snr,1) + 20*log10((norm(tensor.unfold(tenX...
            - tenXhat,1),'fro')^2)/(norm(tensor.unfold(tenX,1),'fro')^2));

        [GR2] = tensor.TTSVD(tenX_noisy,Ranks2);
        tenXhat = tensor.contraction(GR2{1},2,GR2{2},1);
        tenXhat = tensor.contraction(tenXhat,3,GR2{3},1);
        tenXhat = tensor.contraction(tenXhat,4,GR2{4},1);
        nmse2(snr,1) = nmse2(snr,1) + 20*log10((norm(tensor.unfold(tenX...
            - tenXhat,1),'fro')^2)/(norm(tensor.unfold(tenX,1),'fro')^2));

    end
end
nmse1  = nmse1/1000;
nmse2  = nmse2/1000;

figure
txt = ['Ranks = [2 2 2]'];
plot(SNR,nmse1,'--d','color', [0.3010 0.7450 0.9330], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold on;
txt = ['Ranks = [3 3 3]'];
plot(SNR,nmse2,'-o','color', [0 0.4470 0.7410], "linewidth", 2,...
    "markersize", 8, "DisplayName", txt);
hold off;
```

```
title(['TTSVD performance under imperfect scenario'])
xlabel('SNR (dB)')
ylabel('NMSE (dB)')
legend_copy = legend("location", "southwest");
set(legend_copy,'Interpreter','tex',"fontsize", 12)
grid on;
saveas(gcf,'hw13.png')
```