TIP8419 - Tensor Algebra — PPGETI/UFC

Exercise list nº 2: PARAFAC and tensor rank

Semester: 2022-1

1) By using the properties of the outer product, show that

$$\mathcal{X} = \mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2$$

is a rank-one tensor whenever $\mathbf{b}_1 = \mathbf{b}_2$ and $\mathbf{c}_1 = \mathbf{c}_2$. Is this also true in general when $\mathbf{c}_1 = \mathbf{c}_2$ but $\mathbf{b}_1 \neq \mathbf{b}_2$?

2) Show that the tensor rank is indeed a tensor property: in other words, it is invariant with respect to a multilinear transformation by nonsingular matrices, that is, if

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{A}^{(1)} \cdots \times_N \mathbf{A}^{(N)},$$

where $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times I_n}$ is nonsingular for every n, then

$$\operatorname{rank}(\mathcal{X}) = \operatorname{rank}(\mathcal{S}).$$

(Hint: write $\mathcal{S}$ as a PD with a minimal number of terms, and then use the properties of the multilinear transformation to bound the rank of $\mathcal{X}$; similarly, use the invertibility of the multilinear transformation to bound the rank of $\mathcal{S}$.) More generally, conclude that the same property holds for matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R_n}$ having linearly independent columns (and thus $R_n \leqslant I_n$).

3) Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ be given by

$$\mathcal{X} = \mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_1 + \mathbf{a}_1 \circ \mathbf{b}_2 \circ \mathbf{c}_2, \tag{1}$$

where the vectors are assumed to satisfy the following:

- $\mathbf{a}_1$ is not collinear with $\mathbf{a}_2$;
- $\mathbf{b}_1$ is not collinear with $\mathbf{b}_2$;
- $\mathbf{c}_1$ is not collinear with $\mathbf{c}_2$.

The goal of this exercise is to show that any such tensor has rank three, that is, it cannot be expressed as a sum of fewer terms. We will proceed by steps.

(i) First, show that

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C},$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix},$$

and

$$\mathbf{S}_{\cdot\cdot 1} = \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{S}_{\cdot\cdot 2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, using the result of Exercise 2), conclude that $\mathcal{X}$ and $\mathcal{S}$ have the same rank.

(ii) Hence, it suffices to show that $\operatorname{rank}(\mathcal{S}) = 3$. Suppose, for a contradiction, that $\operatorname{rank}(\mathcal{S}) = 2$. Using the properties of the PARAFAC decomposition, show that this imples the existence of matrices $\mathbf{U}, \mathbf{V}, \mathbf{D}_1, \mathbf{D}_2 \in \mathbb{C}^{2\times 2}$ such that $\mathbf{D}_1, \mathbf{D}_2$ are diagonal and

$$\mathbf{S}_{\cdot\cdot 1} = \mathbf{U}\mathbf{D}_1\mathbf{V}^\mathsf{T}, \quad \mathbf{S}_{\cdot\cdot 2} = \mathbf{U}\mathbf{D}_2\mathbf{V}^\mathsf{T}. \tag{2}$$

(iii) Now, use the fact that $\mathbf{S}_{\cdot\cdot 1} = \mathbf{I}$ to show that (2) implies that $\mathbf{S}_{\cdot\cdot 2}$ can be diagonalized by $\mathbf{U}$, that is, there exists a diagonal matrix $\mathbf{D} \in \mathbb{C}^{2\times 2}$ such that

$$\mathbf{S}_{\cdot\cdot 2} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}.$$

(iv) Conclude that this leads to a contradiction, by taking into account the Jordan form of $\mathbf{S}_{\cdot\cdot 2}$.

4) In this last exercise, we will show that, although the tensors of the form considered in the last exercise have rank 3, they are limits of sequences of rank-2 tensors. Thus, unlike happens for matrices, a sequence of rank-R tensors can converge to a rank-S tensor with $S > R$.

(i) First, show that the rank-1 tensor

$$\mathcal{Y}_m = m(\mathbf{a}_1 + m^{-1}\mathbf{a}_2) \circ (\mathbf{b}_2 + m^{-1}\mathbf{b}_1) \circ (\mathbf{c}_1 + m^{-1}\mathbf{c}_2)$$

is equal to $\mathcal{X}$ (as given by (1)) plus an $O(m)$ term $\mathcal{Z}_m$ and an $O(1/m)$ term.

(ii) Subtract the $O(m)$ term to get:

$$\mathcal{X}_m = \mathcal{Y}_m - \mathcal{Z}_m.$$

What is the rank of $\mathcal{X}_m$?

(iii) Use the expression obtained for $\mathcal{X}_m$ to conclude that

$$\lim_{m\to\infty} \mathcal{X}_m = \mathcal{X}.$$