

TIP8419 - Tensor Algebra — PPGETI/UFC

Exercise list n° 2: PARAFAC and tensor rank

Semester: 2022-1

1) By using the properties of the outer product, show that

$$\mathcal{X} = \mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_2$$

is a rank-one tensor whenever $\mathbf{b}_1 = \mathbf{b}_2$ and $\mathbf{c}_1 = \mathbf{c}_2$. Is this also true in general when $\mathbf{c}_1 = \mathbf{c}_2$ but $\mathbf{b}_1 \neq \mathbf{b}_2$?

2) Show that the tensor rank is indeed a tensor property: in other words, it is invariant with respect to a multilinear transformation by nonsingular matrices, that is, if

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{A}^{(1)} \cdots \times_N \mathbf{A}^{(N)},$$

where $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times I_n}$ is nonsingular for every n , then

$$\text{rank}(\mathcal{X}) = \text{rank}(\mathcal{S}).$$

(Hint: write \mathcal{S} as a PD with a minimal number of terms, and then use the properties of the multilinear transformation to bound the rank of \mathcal{X} ; similarly, use the invertibility of the multilinear transformation to bound the rank of \mathcal{S} .) More generally, conclude that the same property holds for matrices $\mathbf{A}^{(n)} \in \mathbb{C}^{I_n \times R_n}$ having linearly independent columns (and thus $R_n \leq I_n$).

3) Let $\mathcal{X} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ be given by

$$\mathcal{X} = \mathbf{a}_1 \circ \mathbf{b}_1 \circ \mathbf{c}_1 + \mathbf{a}_2 \circ \mathbf{b}_2 \circ \mathbf{c}_1 + \mathbf{a}_1 \circ \mathbf{b}_2 \circ \mathbf{c}_2, \quad (1)$$

where the vectors are assumed to satisfy the following:

- \mathbf{a}_1 is not collinear with \mathbf{a}_2 ;
- \mathbf{b}_1 is not collinear with \mathbf{b}_2 ;
- \mathbf{c}_1 is not collinear with \mathbf{c}_2 .

The goal of this exercise is to show that any such tensor has rank three, that is, it cannot be expressed as a sum of fewer terms. We will proceed by steps.

- (i) First, show that

$$\mathcal{X} = \mathcal{S} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C},$$

where

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2], \quad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2], \quad \mathbf{C} = [\mathbf{c}_1 \quad \mathbf{c}_2],$$

and

$$\mathbf{S}_{..1} = \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{S}_{..2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, using the result of Exercise 2), conclude that \mathcal{X} and \mathcal{S} have the same rank.

- (ii) Hence, it suffices to show that $\text{rank}(\mathcal{S}) = 3$. Suppose, for a contradiction, that $\text{rank}(\mathcal{S}) = 2$. Using the properties of the PARAFAC decomposition, show that this implies the existence of matrices $\mathbf{U}, \mathbf{V}, \mathbf{D}_1, \mathbf{D}_2 \in \mathbb{C}^{2 \times 2}$ such that $\mathbf{D}_1, \mathbf{D}_2$ are diagonal and

$$\mathbf{S}_{..1} = \mathbf{U} \mathbf{D}_1 \mathbf{V}^\top, \quad \mathbf{S}_{..2} = \mathbf{U} \mathbf{D}_2 \mathbf{V}^\top. \quad (2)$$

- (iii) Now, use the fact that $\mathbf{X}_{..1} = \mathbf{I}$ to show that (2) implies that $\mathbf{S}_{..2}$ can be diagonalized by \mathbf{U} , that is, there exists a diagonal matrix $\mathbf{D} \in \mathbb{C}^{2 \times 2}$ such that

$$\mathbf{S} = \mathbf{U} \mathbf{D} \mathbf{U}^{-1}.$$

- (iv) Conclude that this leads to a contradiction, by taking into account the Jordan form of $\mathbf{S}_{..2}$.

4) In this last exercise, we will show that, although the tensors of the form considered in the last exercise have rank 3, they are limits of sequences of rank-2 tensors. Thus, unlike happens for matrices, a sequence of rank- R tensors can converge to a rank- S tensor with $S > R$.

- (i) First, show that the rank-1 tensor

$$\mathcal{Y}_m = m(\mathbf{a}_1 + m^{-1}\mathbf{a}_2) \circ (\mathbf{b}_2 + m^{-1}\mathbf{b}_1) \circ (\mathbf{c}_1 + m^{-1}\mathbf{c}_2)$$

is equal to \mathcal{X} (as given by (1)) plus an $O(m)$ term \mathcal{Z}_m and an $O(1/m)$ term.

- (ii) Subtract the $O(m)$ term to get:

$$\mathcal{X}_m = \mathcal{Y}_m - \mathcal{Z}_m.$$

What is the rank of \mathcal{X}_m ?

- (iii) Use the expression obtained for \mathcal{X}_m to conclude that

$$\lim_{m \rightarrow \infty} \mathcal{X}_m = \mathcal{X}.$$