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Multilinear Algebra
PARAFAC and Tensor Rank

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1. We know that if we have the tensor defined as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{1}$$

then if we have $\boldsymbol{b}_1 = \boldsymbol{b}_2$ and $\boldsymbol{c}_1 = \boldsymbol{c}_2$ we can guarantee that $\mathcal{X}$ is rank one. We can begin this proof by using the associativity property of the outer product to write tensor $\mathcal{X}$ as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{2}$$
$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1, \tag{3}$$
$$\mathcal{X} = (\boldsymbol{a}_1 + \boldsymbol{a}_2) \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1, \tag{4}$$

and by inspecting above expression we can observe that independent of the vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ being collinear we will have a rank one tensor defined as

$$\mathcal{X} = \boldsymbol{a}_3 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1, \tag{5}$$

where $\boldsymbol{a}_3 = \boldsymbol{a}_1 + \boldsymbol{a}_2$. In a similar fashion, if we have $\boldsymbol{b}_1 \neq \boldsymbol{b}_2$ and $\boldsymbol{c}_1 = \boldsymbol{c}_2$ then we can rewrite the tensor as

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2, \tag{6}$$
$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_1, \tag{7}$$
$$\mathcal{X} = (\boldsymbol{a}_1 \circ \boldsymbol{b}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2) \circ \boldsymbol{c}_1, \tag{8}$$

but since we can guarantee that the vectors $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are not collinear then we know that the sum in above expression cannot be further reduced. Thus, we will have a tensor composed of the sum of two subtensors of rank one meaning that our tensor $\mathcal{X}$ will be a rank two tensor.

2. To show that the tensor rank is a property we will begin by writting the following multilinear transformations

$$\mathcal{X} = \mathcal{S} \times_1 \boldsymbol{A}^{(1)} \cdots \times_3 \boldsymbol{A}^{(3)} \in \mathbb{C}^{I_1 \times \cdots \times I_N}, \tag{9}$$
$$\mathcal{S} = \mathcal{X} \times_1 \boldsymbol{A}^{(1)^{\mathrm{H}}} \cdots \times_3 \boldsymbol{A}^{(3)^{\mathrm{H}}} \in \mathbb{C}^{I_1 \times \cdots \times I_N}. \tag{10}$$

First we can begin by defining the core tensor $\mathcal{S}$ by its PARAFAC Decomposition as

$$\mathcal{S} = \sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(N)}, \tag{11}$$

and by returning to the original transformation we can rewrite it as

$$\mathcal{X} = \left(\sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(N)} \right) \times_1 \boldsymbol{A}^{(1)} \cdots \times_3 \boldsymbol{A}^{(3)}, \tag{12}$$
$$\mathcal{X} = \left(\sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(N)} \times_1 \boldsymbol{A}^{(1)} \cdots \times_3 \boldsymbol{A}^{(3)} \right), \tag{13}$$
$$\mathcal{X} = \sum_{r=1}^{R} \boldsymbol{A}^{(1)} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{A}^{(N)} \boldsymbol{s}_r^{(N)}. \tag{14}$$

Now considering the inverse multilinear transformation and that $\boldsymbol{A}^{(n)^{\mathrm{H}}}\boldsymbol{A}^{(n)} = \boldsymbol{I}, \forall n \in \{1, \cdots, N\}$ we can rewrite tensor $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{X} \times_1 \boldsymbol{A}^{(1)^{\mathrm{H}}} \cdots \times_3 \boldsymbol{A}^{(3)^{\mathrm{H}}}, \tag{15}$$

$$\mathcal{S} = \left(\sum_{r=1}^{R} \boldsymbol{A}^{(1)}\boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{A}^{(N)}\boldsymbol{s}_r^{(N)}\right) \times_1 \boldsymbol{A}^{(1)^{\mathrm{H}}} \cdots \times_3 \boldsymbol{A}^{(3)^{\mathrm{H}}}, \tag{16}$$

$$\mathcal{S} = \left(\sum_{r=1}^{R} \boldsymbol{A}^{(1)}\boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{A}^{(N)}\boldsymbol{s}_r^{(N)} \times_1 \boldsymbol{A}^{(1)^{\mathrm{H}}} \cdots \times_3 \boldsymbol{A}^{(3)^{\mathrm{H}}}\right), \tag{17}$$

$$\mathcal{S} = \sum_{r=1}^{R} \boldsymbol{A}^{(1)^{\mathrm{H}}}\boldsymbol{A}^{(1)}\boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{A}^{(N)^{\mathrm{H}}}\boldsymbol{A}^{(N)}\boldsymbol{s}_r^{(N)}, \tag{18}$$

$$\mathcal{S} = \sum_{r=1}^{R} \boldsymbol{s}_r^{(1)} \circ \cdots \circ \boldsymbol{s}_r^{(3)} \tag{19}$$

and since the tensor rank is defined by the minimum number of rank one tensors that together compose the original tensor then by analysing above expressions we can observe that we have bounded tensors $\mathcal{X}$ and $\mathcal{S}$ to the same number of elements that compose them. Thus, the tensor rank is indeed a property and

$$\operatorname{rank}(\mathcal{X}) = \operatorname{rank}(\mathcal{S}). \tag{20}$$

Moreover, if we consider the case of dimension reduction where $\mathcal{S} \in \mathbb{C}^{R_1 \times \cdots \times R_N}$ with $R_n << I_n, \forall n \in \{1, \cdots, N\}$ the if we can guarantee that $\boldsymbol{A}^{(n)^{\dagger}}\boldsymbol{A}^{(n)} = \boldsymbol{I}, \forall n \in \{1, \cdots, N\}$ then we should once more have the same tensor rank for the case where the factor matrices are defined as $\boldsymbol{A}^{n} \in \mathbb{C}^{I_n \times R_n}, \forall \{1, \cdots, N\}$.

3. Proof that the following tensor has rank three if all the vector pairs are non collinear

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_1 + \boldsymbol{a}_1 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2 \in \mathbb{C}^{I_1 \times I_2 \times I_3}, \tag{21}$$

First we need to show that above tensor have a multilinear transformation defined as

$$\mathcal{X} = \mathcal{S} \times_1 \boldsymbol{A} \times_2 \boldsymbol{B} \times_3 \boldsymbol{C} \in \mathbb{C}^{I_1 \times I_2 \times I_N}, \tag{22}$$

where the unfoldings of the core tensor $\mathcal{S}$

$$[\boldsymbol{S}]_{(1)} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, [\boldsymbol{S}]_{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, [\boldsymbol{S}]_{(3)} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{23}$$

thus we have that the tensor $\mathcal{X}$ can be written

$$[\boldsymbol{X}]_{(1)} = \boldsymbol{A}\,[\boldsymbol{S}]_{(1)}\,(\boldsymbol{C} \otimes \boldsymbol{B})^{\mathrm{T}}, \tag{24}$$

$$[\boldsymbol{X}]_{(1)} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} (\boldsymbol{c}_1 \otimes \boldsymbol{b}_1)^{\mathrm{T}} & (\boldsymbol{c}_1 \otimes \boldsymbol{b}_2)^{\mathrm{T}} & (\boldsymbol{c}_2 \otimes \boldsymbol{b}_1)^{\mathrm{T}} & (\boldsymbol{c}_2 \otimes \boldsymbol{b}_2)^{\mathrm{T}} \end{bmatrix}, \tag{25}$$

$$[\boldsymbol{X}]_{(1)} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} \begin{bmatrix} (\boldsymbol{c}_1 \otimes \boldsymbol{b}_1)^{\mathrm{T}} + (\boldsymbol{c}_2 \otimes \boldsymbol{b}_2)^{\mathrm{T}} \\ (\boldsymbol{c}_1 \otimes \boldsymbol{b}_2)^{\mathrm{T}} \end{bmatrix}, \tag{26}$$

$$[\boldsymbol{X}]_{(1)} = \boldsymbol{a}_1(\boldsymbol{c}_1 \otimes \boldsymbol{b}_1)^{\mathrm{T}} + \boldsymbol{a}_1(\boldsymbol{c}_2 \otimes \boldsymbol{b}_2)^{\mathrm{T}} + \boldsymbol{a}_2(\boldsymbol{c}_1 \otimes \boldsymbol{b}_2)^{\mathrm{T}}, \tag{27}$$

and by doing the folding we get

$$\mathcal{X} = \boldsymbol{a}_1(\boldsymbol{c}_1 \otimes \boldsymbol{b}_1)^{\mathrm{T}} + \boldsymbol{a}_1(\boldsymbol{c}_2 \otimes \boldsymbol{b}_2)^{\mathrm{T}} + \boldsymbol{a}_2(\boldsymbol{c}_1 \otimes \boldsymbol{b}_2)^{\mathrm{T}}, \tag{28}$$

$$\mathcal{X} = \boldsymbol{a}_1 \circ \boldsymbol{b}_1 \circ \boldsymbol{c}_1 + \boldsymbol{a}_1 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_2 + \boldsymbol{a}_2 \circ \boldsymbol{b}_2 \circ \boldsymbol{c}_1. \tag{29}$$

We also know from the previous problem that the tensor rank is indeed a property. Thus, the tensor $\mathcal{X}$ and the core tensor $\mathcal{S}$ of its multilinear transformation will have the same tensor rank. Now, we only need to proof that either one of them are a tensor with rank three. The easiest way to do this is by working with the core tensor of the multilinear transformation, the tensor $\mathcal{S}$, and start a proof by contradiction defining that $rank(\mathcal{S}) = 2$. This means that we can write the PARAFAC Decomposition of this tensor assunto

$$\mathcal{S} = \sum_{r=1}^{2} \boldsymbol{s}_r^{(1)} \circ \boldsymbol{s}_r^{(2)} \circ \boldsymbol{s}_r^{(3)}, \tag{30}$$

where it is possible to rewrite the core tensor using the frontal slices notations for the PARAFAC model

$$\mathcal{S}_{\dots 1} = \sum_{r=1}^{2} s_{1,r}^{(3)} \boldsymbol{s}_r^{(1)} \boldsymbol{s}_r^{(2)^{\mathrm{T}}} = \boldsymbol{S}^{(1)} \mathrm{D}_1 \left(\boldsymbol{S}^{(3)}\right) \boldsymbol{S}^{(2)^{\mathrm{T}}}, \tag{31}$$

$$\mathcal{S}_{\dots 2} = \sum_{r=1}^{2} s_{2,r}^{(3)} \boldsymbol{s}_r^{(1)} \boldsymbol{s}_r^{(2)^{\mathrm{T}}} = \boldsymbol{S}^{(1)} \mathrm{D}_2 \left(\boldsymbol{S}^{(3)}\right) \boldsymbol{S}^{(2)^{\mathrm{T}}}, \tag{32}$$

and if we supose that $\mathcal{S}_{\dots 1} = \boldsymbol{I}$ and then rewrite it with the SVD

$$\mathcal{S}_{\dots 1} = \boldsymbol{U} \boldsymbol{\Sigma} \boldsymbol{V}^{\mathrm{T}} = \boldsymbol{S}^{(1)} \mathrm{D}_1 \left(\boldsymbol{S}^{(3)}\right) \boldsymbol{S}^{(2)^{\mathrm{T}}} = \boldsymbol{I}, \tag{33}$$

$$\tag{34}$$

and we know that the SVD of the identity matrix leads to $\boldsymbol{U} = \boldsymbol{\Sigma} = \boldsymbol{V}^{\mathrm{T}} = \boldsymbol{I}$. By inspecting above expression it also possible to write the following relations

$$\boldsymbol{S}^{(1)} = \boldsymbol{U} = \boldsymbol{I}, \tag{35}$$

$$\boldsymbol{S}^{(2)} = \boldsymbol{V} = \boldsymbol{I}, \tag{36}$$

$$\mathrm{D}_1 \left(\boldsymbol{S}^{(3)}\right) = \boldsymbol{\Sigma} = \boldsymbol{I}, \tag{37}$$

and since $\boldsymbol{S}^{(1)} = \boldsymbol{S}^{(2)} = \boldsymbol{I}$ then it is possible to write

$$\mathcal{S}_{\dots 2} = \boldsymbol{S}^{(1)} \mathrm{D}_2 \left(\boldsymbol{S}^{(3)}\right) \boldsymbol{S}^{(2)^{\mathrm{T}}}, \tag{38}$$

$$\mathcal{S}_{\dots 2} = \boldsymbol{S}^{(1)} \boldsymbol{S}^{(1)} \mathrm{D}_2 \left(\boldsymbol{S}^{(3)}\right) \boldsymbol{S}^{(2)^{\mathrm{T}}} \boldsymbol{S}^{(1)^{-1}}, \tag{39}$$

$$\mathcal{S}_{\dots 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathrm{D}_2 \left(\boldsymbol{S}^{(3)}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{40}$$

$$\mathcal{S}_{\dots 2} = \mathrm{D}_2 \left(\boldsymbol{S}^{(3)}\right) = \begin{bmatrix} s_{21} & 0 \\ 0 & s_{22} \end{bmatrix}, \tag{41}$$

and thus it is possible to diagonalized the second frontal slice $\mathcal{S}_{..2}$ with matrix $\boldsymbol{S}^{(1)}$. However, this similarity transformation leads to an invalid Jordan matrix as defined by $\mathcal{S}_{..2}$. Thus, it is not possible to the core tensor $\mathcal{S}$ be rank two and it must be in fact rank three.

4. (a)

(b)

(c)