PDF 5.060 Logarithmic Differentiation

We know how to find the derivative of a power of x. For example, if we are given the function $y = 3x^2$, we see that x is in the base of the exponent and we know that we can use the power rule.

We also know how to find the derivative of an exponential function, where x is in the exponent. For example, if we are given the function $y = 3^{2x-1}$, we see that x is in the exponent and we can use exponential differentiation.

But what do we do if we have a function where x is in the base of the exponent and in the exponent. For example, how would we determine the derivative of the function $y = x^x$.

If we have a power with x in both the base of the exponent and in the exponent itself, we can determine the derivative by taking the natural logarithm (i.e., ln) of both sides, then using previously learned rules of derivatives such as implicit differentiation and the power rule, among others, to isolate y' and thus determine the derivative.

Example 1

Determine the derivative with respect to x of $y = x^x$

Example 2

Determine the derivative with respect to x of $y = (3x + 4)^{x^4 - 2x}$

Example 3

Determine the derivative with respect to x of $y = (x^2 + 3)^x$

Sometimes logarithmic differentiation can be used to answer other questions where it might not necessarily be applicable.

We will prove the power rule using logarithmic differentiation in a moment, and then we will do an example where logarithmic differentiation is beneficial to a question that would otherwise be rather complicated.

Example 4 (Cool Example)

Using logarithmic differentiation, prove the power rule; i.e. prove that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Example 5

Use logarithmic differentiation to evaluate $\frac{dy}{dx}$ at $x = -1$ given that

$y = \frac{(x^4+1)\sqrt{x+2}}{2x^2+2x+1}$

Another time when we use logarithmic differentiation is when we are given a logarithmic function in a base other than e (i.e., when we are not using the natural logarithm ln).

We will see that in these cases, we convert our logarithmic function equation to the equivalent exponential equation, and then use the natural logarithm.

Example 6

Determine $\frac{dy}{dx}$ given that $y = \log_3(4x^2 - 5x + 1)$