Chapter 8

Applications of Differentiation

Ex 8.1

Velocity and Acceleration (s in metres and t in seconds)

1. For the following position function $s = 2t^3 - 15t^2 + 36t - 22,\ t \geq 0.$
   a) Find the velocity and acceleration at time t.
   b) Find the average velocity and average acceleration from t = 2 to t = 4.
   c) When is the particle at rest?
   d) When does the particle moving in the positive direction?
   e) What is the total distance traveled during the first 5 seconds?

2. For the position function $s = 3t^3 - 7t^2 - 2t,\ t \geq 0$, choose the correct answer for t = 2.
   a) Is s increasing or decreasing?
   b) Is v increasing or decreasing?
   c) Is the particle moving in the positive or negative direction?
   d) Is the object moving away or towards the origin?
   e) Is the object speeding up or slowing down?

3. A particle moves along a line with position function $s = 2t^3 - 27t^2 + 108t - 83,\ t \geq 0.$
   a) Find the average velocity form t = 2 to t = 3.
   b) Find the time at which it is at rest.
   c) At what time is the acceleration positive?

4 An object is thrown upward from the roof of a building with $s = -5t^2 + 20t + 25,\ t \geq 0$ where s is the height of the object above ground in metres and t in seconds.
   a) What is the height of the building?
   b) What is the initial velocity/
   c) When does it reach the maximum height?
   d) What is the maximum height?
   e) When does it hit the ground?
   f) With what velocity does it hit the ground?

5. A tower is 500 m tall. A ball is thrown downward at a initial velocity of 2 m/s. So $s = 500 - 2t - 5t^2,\ t \geq 0$ where s is the height of the ball above the ground after t seconds.
   a) Find the average velocity of the ball from t = 1 to t = 3 seconds.
   b) Find the velocity after 3 seconds.
   c) Find when will it hit the ground.
   d) What is the velocity when it hits the ground?

6. A man drives a car at a speed of 48 m/s sees a stop sign 130 m away. He presses the brake and his position function is $s = 48t - t^3$. Will he go beyond the stop sign?

7. Given a position function $s = t^3 - 15t^2 + 63t - 49,\ \ t \geq 0.$
   a) Find the velocity and acceleration functions at time t.
   b) Find the intervals of time in which the object is moving away or towards origin.
   c) Find the intervals of time in which it is speeding up or slowing down.

Related Rates Problems

Ex 8.2

Basic

1. If V is the volume of a cube and x the length of an edge.
   Express $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$. What is $\frac{dV}{dt}$ when x is 5 and $\frac{dx}{dt}=2$?

2. If V is the volume of a sphere and r is the radius.
   Express $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$. What is $\frac{dV}{dt}$ when $r=4$ and $\frac{dr}{dt}=2$?

3. If V is the volume of a cone and h is the height and r is the radius.
   Express $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if r is $\frac{1}{2}$ of h. What is $\frac{dh}{dt}$ when $\frac{dV}{dt}=5$ and $h=2$?

Cube

4. If the side of a cube increases at a rate of 1 cm/s, find the rate of increase of the total surface area of the cube when the side of the cube is 20 cm.

5. The side of a cube increases at 1 cm/s. How fast is the diagonal of the cube changing when the side is 1 cm?

Circle

6. A pebble is thrown into water and causes circular ripple to spread outward at a rate of $2\ m/s$. Find the rate of change of the area in terms of π when
   a) 3 seconds after the pebble falls into the water.
   b) the area of the ripple is $9\pi\ m^2/s$.

7. Radius of a circle varies as time t in m/s according to the following rule $r=t^3+2t$. Find the rate of change of the area at $t=2$.

Triangles

8. The sides of a equilateral triangle increase at a rate of 2 cm/s. Find the rate of increase of the area when each side is 50 cm.

Rectangles

9. The side of a square is increasing at a rate of 1 cm/s. Find the rate of change of area when the area is 100 cm^2.

10. The width of a rectangle is decreasing at a rate of 1 cm/s and the length is increasing at a rate of 2 cm/s. Find the rate of change of the area at the moment when both the width and the length are 10 cm.

Rectangular tank

11. The base of a rectangular tank is 3 m by 4 m and is 10 m high. Water is added at a rate of 8 m^3/min. Find the rate of change of water level when the water is 5 m deep.

12. The dimensions $WxLxH$ of a rectangular box is increasing at rates of 1 cm/s, 2 cm/s, and 3 cm/s respectively. Find the rate of change of the volume at the instant when $W=1$ cm, $L=2$ cm, and $H=3$ cm.

Cylinder

13. Water is added to a cylindrical tank of radius 5 m and height 10 m at a rate of 100 L/min. Find the rate of change of the water level when the water is 6 m deep. ($1\ \text{L} = 1000\ \text{cm}^3$)

14. The radius of a cylinder is decreasing at a rate of 1 cm/min. The height remains the same as 20 cm. How fast is the volume changing when the radius is 12 cm?

Sphere

15. A sphere is growing at a rate of 1 cm^3/s. Find the rate at which the radius is changing when the radius is 2 cm.

16. A snowball melts at a rate proportional to its surface area. Show that its radius decreases at a constant rate.

17. A balloon is rising at a rate of 200m/min. For every 500 m increase in height the air pressure causes the radius to increase 2 cm. Find the rate of increase of the volume when the radius is 10 m.

18. A spherical balloon is inflated at a rate of $4\pi\ \text{cm}^3$ /s .At the moment the radius is 5 cm, find the rate of change of
    a) the radius   b) the surface area

Cone

19. Sand is poured by a conveyor belt at a rate of $2m^3$ / min onto the ground which forms a conical pile in which the radius is always twice the height. How fast is the height rising when the height is 4 m ?

20. The dimensions of a conical tank is of radius 3 m and height 6 m. Water is added to it at a rate of $\pi\ \text{m}^3$ /min. Find the rate of change of the water level when the height is 3 m.

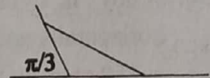
Ladder

21. A ladder is 10 m long and leans against a vertical wall. If the foot of the ladder is moving away at a rate of 2 m/s and the foot is 6 m away from the wall, find the rate of change
    a) of the top of the ladder that is moving down   b) of the slope of the ladder

22. An extension ladder with the top resting on a vertical wall is being extended at a rate of 10 m/min. The base is 5 m from the wall. Find the rate of sliding up of the top of the ladder when the ladder is 13 m long.

23. A ladder is resting on a vertical wall. The foot of the ladder is sliding away at a rate of 4 m /s and is 12 m from the wall. At this moment the top of the ladder is moving down at a rate of 3 m/s. Find the length of the ladder.

24. A wall is inclined 60 degree to the ground. A ladder $4\sqrt{3}$ m is resting on the wall. The foot of the ladder is moving away at a rate of 2 m/s. How fast is the top sliding down at the instant when the foot of the ladder is 4 m away from the foot of the wall?



Intersection

25. Two objects are moving away from the same location at the same time. Object A is moving due east at a rate of 3 cm/s. Object B is moving due north at a rate of 4 cm/s. Find the rate of change of distance between the two objects 2 seconds later.

26. Ship A is 25 km south of ship B and is moving due north at a rate of 5 km/h. Ship B is moving due east at a rate of 10 km/h. Find the rate of change of the distance between the two ships 2 hours later.

27. Object A is 15 m east of object B and is moving due north at a rate of 2 m/s while object B is moving due south at a rate of 3 m/s. Find the rate of change of the distance between the two objects 4 seconds later.

28. At noon ship A leaves a location and is moving due east at the rate of 10 km/h. One hour later another ship B starts to leave the same location and is moving at a rate of 15 km/h in the direction N30°E. Find the rate of change of distance between the two ships at 3:00 p.m.

29. A small snake (height negligible) moves along a straight path at a speed of 2 m/s passes a tree (10 m tall) 4 m away. How fast is the distance between the top of the tree and the snake changing 2 seconds later?

30. A dog runs across a bridge at a rate of 4 m/s. The bridge is 5 m above the water. A fish swims right underneath the dog at a rate of 3 m/s in a direction perpendicular to the path of the dog. Find the rate of change of the distance between the two animals 1 second later.

31. At noon ship A is 10 km west of ship B and is moving due east at a speed of 10 km/h. Ship B is moving due south at a rate of 5 km/h. When is the distance between them stop changing?

Trough

32. A water trough is 10 m long and has a cross section of an equilateral triangle. Water is added at a rate of 2 m^3/min. Find the rate of change of the water level when the water is $\sqrt{3}$ m deep.

33. A trough is 10 m long and its ends are isosceles trapezoids with base 2 m, top 3 m, and height 4 m. Water is added at a rate of 5 m^3/min. Find the rate of change of the water level when the water is 2 m deep.

Shadow

34. A man 2 m tall is walking away from a lamppost which is 6 m tall at a rate of 2 m/s. Find the rate of change of
a) the tip of his shadow b) the length of his shadow

35. A spot light on the ground is shining on a vertical building which is 20 m from the spot light. A man 2 m tall is walking away from the light and is walking directly towards the building at a rate of 4 m/s. Find the rate of change of the length of his shadow on the building when he is 5 m from the light.

Kite

36. A boy is flying a kite. The string of the kite is being paid out at a rate of 10 m/min. The kite is moving horizontally and is at a height of 150 m above the ground. Find the rate at which the kite is moving when it is 250 m away from the boy.

37. A boat which is 3 m below the dock is being pull in by a cable. When the boat is 4 m away horizontally from the dock it is approaching the dock horizontally at a rate of 5 m/s. How fast is the cable being pull in?

38. A fisherman is on a bridge 3 m above the water. He reels in his line at a rate of 1 m/s thinking that he will get a big fish. But actually it is a piece of wood. (So it remains on the surface of the water). How fast is the piece of wood approaching the bridge when it is 4 m from the foot of the bridge?

39 A pulley is 15 m above the ground and a box is located right under the pulley. A rope 30 m long passes over the pulley and fastened to the box. The other end of the rope moves horizontally away at a rate of 5 m/s. How fast is the box rising when it is 10 m high?

Moving along curves

40. Point P(x,y) moves along the parabola $y=x^2$ from left to right at a rate of 5 unit /s. Find dy/dt at the moment it passes the point (4, 16).

41. A point moves along the curve $y=x^3$ form left to right with at a rate of 2 unit/s. Find the rate of change of the slope of the curve at x = 4.

42. Area bounded by $y=x^3$, the x-axis and the line x = a (a >0) is given by $A=\frac{a^4}{4}$. If the line x = a is moving to the right at a rate of 2 units/s. Find the rate of change of area at the moment x = 2.

Miscellaneous

43. There is a spherical marble of radius 1 cm in the middle of a hemispherical bowl which is of radius 2 cm. Water is added at a rate of π cm^3/s. Find the rate of change of the water level when the water is 1 cm deep.

(Note: Skip this problem if you do not know the formula $\frac{dV}{dt}=(A)\left(\frac{dh}{dt}\right)$ or you may apply the formula $V=\pi rh^2-\frac{1}{3}\pi h^3$)

44. The intensity of illumination I of a point r unit from a light source is inversely proportional to the square of the distance r unit from the light source. i.e. $I=\frac{k}{r^2}$. Given I = 100 when r = 1.

An object is moving away from the light source at a rate of 4 unit/s. Find the rate of change of the intensity of illumination when it is 2 unit from the light source.

45. A plane 2 km high is flying at a rate of 120 km/h due west sees an oncoming car. The distance between the plane and the car is 4 km and is decreasing at a rate of 160 km/h. Find the speed of the car at this moment.

Maximum and Minimum Problems

Ex 8.3

Numbers

1. The sum of two positive numbers is 20. Find the two numbers such that
   a) the sum of the square is minimum. b) the product of one and the square of the other is maximum.
   c) the product of the square of one and the cube of the other is maximum.

2. Product of two positive numbers is 16. Find the two numbers such that
   a) the sum is minimum b) the sum of one and the square of the other is minimum.

3. If the sum of two positive number is k. Show that. the sum of the squares of the two number will be greater than or equal to $\frac{1}{2}k^2$.

4. The sum of two positive numbers is k. Show that the sum of one and the square of the other is at least $k-\frac{1}{4}$.

5. Sum of two positive numbers is 8. Find the minimum value of the sum of the square of one and the cube of the other.

6. Find the minimum value of the sum of a positive number and its reciprocal.

7. Sum of two positive numbers is 12. Find the two numbers such that the product of one and the cube of the other is maximum.

8. Find a positive number which exceeds its square by a maximum amount.

Area

9. A rectangle is inscribed in a circle $x^2 + y^2 = 9$. Find the dimensions of the largest rectangle. What is it's area?

10. A rectangle is inscribed in a ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. Find the area of the largest rectangle.

11. A rectangle is inside a parabola $y=12-x^2$ with its base lying on the x-axis and the other two vertices lie on the parabola. Find the dimension of the largest rectangle.

12. An isosceles trapezoid ABCD, with AB // DC, is inscribed in a semicircle of radius 2. If AB is the diameter of the semicircle, find the length of CD at which the area of the trapezoid is maximum.

13. Find the area of the largest isosceles triangle ABC, AB = AC, that can be inscribed in a ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, with vertex $A=(0,2)$.

14. The volume of an open top cylinder is V. Find the ratio of height to radius that gives the minimum surface area.

15. Show that of all the rectangles with a given perimeter P, the one with the greatest area is a square.

16. Three sides of a trapezoid are 10 cm each. Find the length of the fourth side such that the area of the trapezoid is largest.

17. In constructing a cylindrical can of given volume, nothing is wasted in making the side of the can. But the top and bottom are cut from square sheets and the remainder are wasted. Find the ratio of h to r so that the material used is a minimum.

18. A rectangle is inscribed in an isosceles right angled triangle of two sides each equals to 10 cm. One side of the rectangle rests on the hypotenuse and the other two vertices on the two shorter sides. Find the dimensions of the largest rectangle.

Poster

19. A rectangular poster is of area 6912 cm^2. Side margin is 8 cm each and top and bottom margin are each 6 cm. Find the dimensions of the poster that give the maximum printing area.

20. A poster contain 200 cm^2 of printing area. The top and bottom margins are both 8 cm and the side margins are both 4 cm. What are the dimensions of the poster of minimum area?

Wire

21. A wire of length 8m is divided into two parts. One forms a circle and the other forms a square. How should the wire be cut so that the total area is a) minimum b) maximum?

22. A wire of length 8m is divided into two parts. One forms a rectangle with length equals to twice the width and the other part forms an equilateral triangle. How should the wire be cut so that the total area is a) maximum b) minimum?

Fence

23. A man has 600 m of fence. He wants to enclose a rectangular field with one side along a wall. Find the dimensions of the rectangular field of maximum area.

24. A man wants to fence an rectangular field of area 3750 m^2 and divide it in half with a fence parallel to one of the sides of the rectangular field. What are the dimensions of the field that can minimize the cost of the fence?

25. A rectangular field is to have 100 m^2 in area. It was enclosed by fence. The north-south sides costs $20/m, east-west sides cost $5/ m. What are the dimensions so that the cost is minimum?

Window

26. A window of fixed perimeter 8 m consist of a rectangle surmounted by a semicircle. What is the ratio of height to radius that will admit most sunlight?

27. A window consists of a rectangle surmounted by a semicircle. The perimeter is 8 m. Suppose rectangular part admit twice as much sunlight as the circular part. Find the width of the window which admit most sunlight.

28. Suppose there is a semicircle below the rectangle as well as above it. Find the dimensions of the window which admit most sunlight for a given perimeter 8 m.

29. A window of perimeter 8 m consists of a rectangle surmounted by an equilateral triangle. Find the width of the window that admit most sunlight.

Volume

30. Squares of equal length are cut from the 4 corners of a square sheet of side 12 cm. The four sides are then folded to form an open top box. Find the length of the side of the squares cut that give a maximum volume.

31. A closed container is made with a hemisphere on top of a cylinder. The height and radius of the cylinder are h and r respectively. Find h:r so that the volume is maximum if the total surface area is a constant A.

32. The total surface area of a squared base open top rectangular box is 12 square units. Find the dimensions of the box such that the volume is maximum.

33 Of all the circular cone of slant height 8 cm, find the dimensions of the cone of largest volume.

34. A cylinder is inscribed in a fixed cone. Find the ratio of the volume of the largest cylinder to the volume of the cone.

35. For a circular cylinder of given volume, find the ratio of height to radius so that the total surface is minimum.

36. Find the volume of the largest cylinder that can be inscribed in a sphere of radius R.

37. Find the volume of the cone of maximum value that can be inscribed in a sphere of radius R.

38. A trough is to be made by bending a long rectangular piece of tin 3 units wide. The cross section of the trough is a isosceles trapezoid with sides making angles of 120 degree with the base. Find the length of one of the sides that is bent which give a maximum capacity.

Distance and shortest time

39. A man in an island is 4 km from the shore. He wants to go to a pub which is 8 km down the shore. He can row at 3 km/h and walk at 5 km/h. Where should he land if he want to reach the pub as soon as possible.

40. The cost of laying power line underwater is 3 times that of underground. An island is 4 km from the shore and a power station is at distance 8 km from the point on the shore which is closest to the island. How should the power line be laid so that the cost is minimum

41. Two vertical posts 7 m apart are of lengths 3 and 4 m. A wire is to run from the top of a post, reaches the ground and then goes to the top of another post. Find the minimum length of the wire.

42. A and B are points on the opposite side of a straight line. P and Q are points on line such that AP and BQ are perpendicular to the line. $AP = 3$, $BQ = 4$, $PQ = 7$. R is a point on the line. At what point should R be located so that the total distance form A to B through R is minimum?

Closest Distance

43. Find a point on the line $3x + 4y - 25 = 0$ which is closest to the origin.

44. Find the minimum distance from the point (4,2) to $y^2 = 8x$.

Revenue, profit and cost

45. A shop can sell 30 radio at \$20 each per week. If the price is increased, for each dollar increase there will be a lost of one sale per week. The cost of radio is \$10 each. What is the price that will give the maximum profit?

46. In a certain type of soil, if 20 apple trees are planted, each will yield 100 apples. But if more trees are planted, for each addition tree planted, the yield will be reduced by 2 apples. How many trees should be planted so that the total yield is maximum?

47. A open top rectangular box is of volume 250 cm^3. The width of the box is 5 cm. The cost is $2/$cm^2$ for the base and $1/$cm^2$ for the side. What is minimum cost for making the box?

<u>Miscellaneous</u>

48. A rectangular beam is made from a right circular cylindrical rod of radius 10 cm. If the strength of the beam is proportional to the product of the width and the square of the depth, find the dimensions of the strongest beam.

49. A ship A leaves a dock at noon and travels due south at 20 km/h. Another ship B has been heading due east at 30 km/h and reach the same dock at 3:00 p.m . At what time will the two ship be closest?

50. The area of a rectangle is 4 cm^2. Find the dimension of the rectangle so that the distance from one vertex to the midpoint of a non-adjacent side is minimum.

51 A line passes (1, 1) cuts the x- and y- axis at A and B respectively. Find the minimum value of the length of AB.

52. A line passes (3,4) in the first quadrant cut the x-axis at A and y-axis at B where O is the origin. Find the minimum value of the area of the triangle OAB.

53. An isosceles triangle has one of its vertex at origin. Its base is parallel to and above the x-axis. The other two vertices lies on the curve $y=12-x^2$. Find the largest possible area of the triangle.

54. Given a triangle of sides 5, 12, 13 units. A rectangle is inscribed in the triangle such that two sides of the rectangle lie along the two shorter sides of the triangle. The remaining vertex lies on the hypotenuse. Find the area of the largest rectangle.

55. Find the minimum vertical distance between the two curves $y=x^2$ and $y=-(x-2)^2$.

56. A, B are two light source 100 m apart. A is twice as strong as B. P is a point between A, B. Intensity of illumination varies inversely as the square of the distance from the light source. Find the length of AP so that the point P is darkest.

57. A wall 27 m high is parallel to a tall building which is 8 m from the wall. Find the length of the shortest ladder that will reach from the ground across the top of the wall to reach the building.

58. A metal rod is being carried down a corridor 4 m wide. At the end of the corridor there is a right angle turn to another corridor of 3 m wide. What is the longest rod that can be carried around the corner horizontally?

Chapter 9 Curve Sketching

Increasing and Decreasing, Local Maximum and Minimum

Ex 9.1

For the following curves, find a) the intervals of increase and decrease
b) the local maximum/minimum points
c) sketch the graph

(Note: You may not be able to sketch the graphs accurately. You may need the information of concavity)

1. $y=x^2(x-3)$
2. $y=2x^3-9x^2+9$
3. $y=x^3-5x^2+14x-9$
4. $y=x^4-12x^3+52x^2-96x+59$
5. $y=x+\frac{1}{x}$
6. $y=\frac{(x+1)^2}{x^2+2x+5}$
7. $y=x^{\frac{3}{5}}(8-x)$
8. $y=(x+1)x^{\frac{2}{5}}$

Concavity and Points of Inflection

Ex 9.2

For the following curves, find a) the intervals of increase and decrease
b) the local minimum and minimum points
c) the intervals of concavity
d) points of inflection
e) sketch the graph

1. $y=-x^3+6x^2-9x$
2. $y=x^3-9x^2+24x-15$
3. $y=x^3-9x^2+27x-26$
4. $y=x^4-6x^3+12x^2-8x$
5. $y=5x^3-x^5$
6. $y=x^{\frac{4}{3}}-1$
7. $y=2-(x-2)^{\frac{1}{3}}$

Sketching Graphs of Polynomial Functions

Ex 9.3

Fully analysis the following curves

1. $y=2x^3-3x^2+1$
2. $y=2x^3-9x^2+12x-4$
3. $y=x^3-3x^2+3x-9$
4. $y=3x^4-28x^3+84x^2-96x+42$
5. $y=x^4+2x^2+8x+2$
6. $y=-x^4+8x^3-18x^2+16x-5$
7. $y=x^5-5x^4$

Sketching Graphs of Rational Functions

Ex 9.4

Fully analysis the following curves

1. $y=\dfrac{1}{x^2+1}$
2. $y=\dfrac{x}{x^2+1}$
3. $y=\dfrac{x^2}{x^2+1}$
4. $y=\dfrac{x^3}{x^2+1}$
5. $y=\dfrac{1}{x-1}$
6. $y=\dfrac{x+2}{x-1}$
7. $y=\dfrac{(x+1)^2}{x-1}$
8. $y=\dfrac{1}{x^2-1}$
9. $y=\dfrac{x}{x^2-1}$
10. $y=\dfrac{x^2-4}{x^2-1}$
11. $y=\dfrac{x^3}{x^2-1}$
12. $y=\dfrac{1}{x^3-3x}$
13. $y=\dfrac{x}{(x+1)^2}$
14. $y=\dfrac{x^2-x+2}{x-2}$
15. $y=x^2+\dfrac{1}{x^2}$
16. $y=x^2+\dfrac{16}{x}$
17. $y=\dfrac{x^2-x-1}{x-1}$