MCV4UP UNIT 1 TEST: LIMITS AND CONTINUITY

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Name:________________________

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SHOW YOUR WORK WHERE POSSIBLE.

1. Evaluate the following limits, if they exist. If the limit does not exist, explain why. (Note: If $f(x) \to \pm\infty$ as $x \to c$, please write $\lim_{x\to c} f(x) = \pm\infty$ instead of $\lim_{x\to c} f(x)$ does not exist.) The variables $x \in R$ and $n \in N$.

[1,1] a) $\lim_{x\to 4} \left(|x-4| - \sqrt{x} + \sqrt{3}\right)$

b) $\lim_{x\to 0^+} \dfrac{2+\frac{1}{x}}{1-\frac{1}{x}}$

[3,3] c) $\lim_{x\to -2} \dfrac{3x^2+5x-2}{x^2+x-2}$

d) $\lim_{x\to 5} \dfrac{x^2-25}{125-x^3}$

[3,2] e) $\lim_{x\to 0} \dfrac{\sqrt{7-x}-\sqrt{7}}{x}$

f) $\lim_{x\to 3} \left(\frac{1}{x}-\frac{1}{3}\right)\left(\frac{1}{x-3}\right)$

[2,2] g) $\lim_{x\to\infty} \left[\left(\frac{2}{5}\right)^x + \dfrac{x^3(1-x)^4}{(x+3)^6}\right]$

h) $\lim_{x\to\infty} \dfrac{(588x^7+4)(60x^3-4)}{(7056x^5+13x^4-11x)^2}$

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1. . . . continued from the first page

[3,3] i) $\lim_{n\to\infty} (-1)^{n+1}\dfrac{\pi+n}{n}$

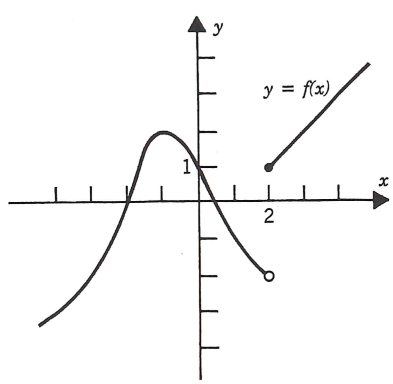
j) $\lim_{x\to 1} \dfrac{|1-x|}{x-1}$

/2 **2.** The following limit represents the slope of the tangent line to some function f at some number "a." State f and the value of a in the space provided.

$\lim_{h\to 0} \dfrac{5^{2-h}-25}{h}$ $f(x) =$ ________ $a =$ ________

/2 **3.** Complete the following definition using the notion of a limit: A function $f(x)$ defined on the open interval (a, b) is said to be continuous at $c \in (a, b)$ if and only if

/5 **4.** Refer to the graph of $f(x)$ provided on the right.



a) State the following limits.

(i) $\lim_{x\to 2^+} f(x)$ (iv) $\lim_{x\to 0^+} f(x)$

(ii) $\lim_{x\to 2^-} f(x)$ (v) $\lim_{x\to 0^-} f(x)$

(iii) $\lim_{x\to 2} f(x)$ (vi) $\lim_{x\to 0} f(x)$

b) At what values of x, if any, is $f(x)$ discontinuous. Explain using your definition in question 3.

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/2 **5.** Let $f(x) = \begin{cases} x^2, & \text{if } x < 1 \\ cx - 3, & \text{if } x \geq 1 \end{cases}$. Find the value of c given that $f(x)$ is continuous at $x = 1$.

/4 **6.** Let $f(x) = \begin{cases} x+2, & \text{if } x \leq -4 \\ x^2 - 2, & \text{if } -4 < x < 1 \\ 3 - x^3, & \text{if } x \geq 1 \end{cases}$.

a) State the following limits without graphing.

(i) $\lim\limits_{x \to -4^-} f(x)$ (ii) $\lim\limits_{x \to -4^+} f(x)$ (iii) $\lim\limits_{x \to -4} f(x)$

(iv) $\lim\limits_{x \to 1^-} f(x)$ (v) $\lim\limits_{x \to 1^+} f(x)$ (vi) $\lim\limits_{x \to 1} f(x)$

b) At what values of x, if any, is $f(x)$ discontinuous. You do not need to justify your answer.

/5 **7.** Using first principles (i.e., the definition of the derivative), find the equation, in the form $Ax + By + C = 0$, of the tangent line to the curve $f(x) = \dfrac{x-1}{x+1}$ at $x = 1$.

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MCV4UP UNIT 2 TEST: DIFFERENTIATION

Name:________________________

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1. Find $\frac{dy}{dx}$ for each of the following. Do not simplify (i.e., one-line solutions are requested).

/3 a) $y = 6\sqrt{x^5} - 2\sqrt[5]{x^3} - \frac{1}{\sqrt[6]{x^{11}}} + \left(\sqrt{2\pi}\right)^3$

/4 b) $y = \frac{1}{\sqrt[5]{5x^{-1} + \sqrt{x - \sqrt{3x}}}}$ (Do not use the quotient rule.)

/4 c) $y = \sqrt[3]{9 - 2x}\left(x^2 + \frac{2}{x^3}\right)^3$ (Use the product rule.)

/4 d) $y = \frac{(3x-2)^2}{\sqrt{1-5x^2}}$ (Use the quotient rule.)

/3 **2.** Using the chain rule, find an expression for the derivative of $y = \sqrt{f\left([f(x^{-4})]^3\right)}$ with respect to x, where f is some differentiable function. (A one-line solution is requested.)

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/5 **3.** Differentiate $y = x^3\left(\frac{x+1}{1-x}\right)^2$ with respect to x. Express your final answer in simplified, factored form.

/4 **4.** If $y = \frac{u^3}{u^2+1}$ and $u = 3x - x^2$, find $\left.\frac{dy}{dx}\right]_{x=2}$ using the chain rule.

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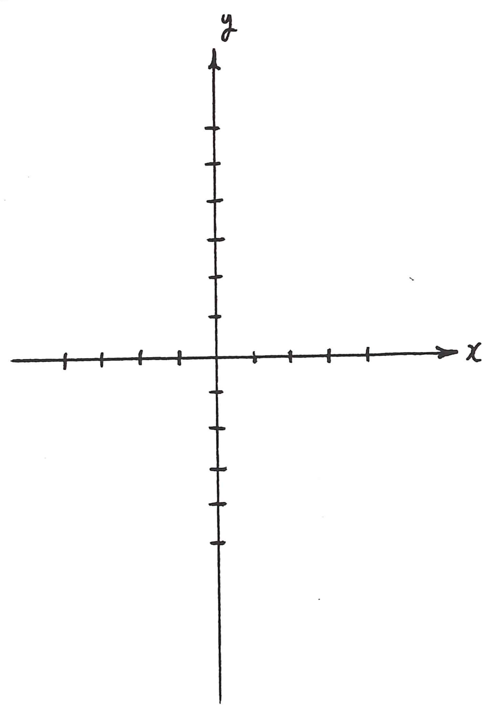
/5 **5.** Find the coordinates of the point(s) on the curve $y = \dfrac{3x}{1-x}$ where the tangent is perpendicular to the line $12x + y - 13 = 0$.

/6 **6.** Find the slopes of the two tangents to $f(x) = -x^2 + 5$ that pass through $P(2, 2)$. Provide a clear sketch of the function, the two tangents, and include the points of contact.



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/6 **7.** Find the equation (in the form $Ax + By + C = 0$) of the tangent to the curve defined by $(x+y)^3 = x^3 + y^3$ at $(-1, 1)$.

/6 **8.** Determine a simplified expression in terms of x and y for $\frac{d^2y}{dx^2} = y''$ if $x^4 + y^4 = 16$.

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MCV4UP UNIT 3 TEST

Name:______________________________

1. Evaluate the following limits using the fundamental limits studied in class. Show only the major steps.

[2, 2] a) $\lim_{x \to 0} \frac{\sin^2 3x}{\tan^2 5x}$

b) $\lim_{x \to 0} \frac{\cos x - 1}{\sin x}$

[2, 3] c) $\lim_{x \to 0} \frac{1 - \cos 2x}{x \sin x}$

d) $\lim_{x \to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\tan x}$

[3, 3] e) $\lim_{x \to 1} \frac{\sin(x-1)}{x^2 + x - 2}$

f) $\lim_{x \to -\frac{\pi}{2}} \frac{4x + 2\pi}{5\cos(-x)}$

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/2 **2.** Complete the following:

a) If $\lim_{x\to c} \pi \ln(x^2 - x) = -\infty$, then $c =$ _______.

b) If $\lim_{x\to c} \sqrt{2}\, e^{-x} = 0$, then $c =$ _______.

/3 **3.** Determine how long (to two decimal places of accuracy) it would take to double an investment (in dollars) at an interest rate of 5% per annum compounded continuously.

4. Differentiate with respect to x. Do not simplify (i.e., one-line solutions are requested).

/4 a) $y = 3\sin^{-5}(4x^2 - 3) + 4\cos^6(1-3x)^{-4}$

/4 b) $y = -4\csc^4\left(\frac{1}{x^3}\right) + 7\sec^{-2}\sqrt{x^3+9}$

/5 c) $y = e^{\cot\sqrt{x}} \log_5(\tan x)$

/4 d) $y = \dfrac{5\sqrt[3]{x^2}}{\ln\left|\frac{10}{x}\right|}$

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5. Find y' in fully-simplified, factored form (if possible).

[4, 4] a) $\ln(2y+4x^2)=4+\ln y$ b) $\sin(xy)=xy$

/4 **6.** Express $y=\left(\sqrt{x^3}\right)^{4x^2}$ in the form $y=x^{ax^b}$, and then differentiate with respect to x, writing your final answer strictly as a function of x.

/5 **7.** If $x=\tan(\ln y)$, find the equation, in the form $y=mx+b$, of the tangent line at $y=1$.

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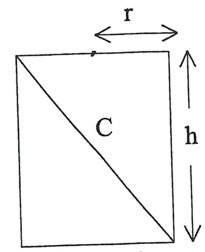
MCV4UP UNIT 4 TEST: RELATED RATES

Name:____________________________________

All answers must be exact and fully reduced. Note that the following formulas will prove useful:

$$V=\frac{1}{3}\pi r^2 h,\ V=\frac{4}{3}\pi r^3,\ V=\pi r^2 h,\ A=4\pi r^2,\ A_{trapezoid}=\frac{h}{2}(a+b),\ \sin 2\theta = 2\sin\theta\cos\theta$$

/4 **1.** A cylinder changes from a radius of 5 cm and a height of 24 cm to a radius of 15 cm and a height of 16 cm. Find the average rate of change of the distance C with respect to the volume, V, as shown in the cross-sectional diagram below.



cross section
of cylinder

/5 **2.** Find the instantaneous rate of change of the volume of a sphere, V, with respect to the diameter of the sphere, x, when the surface area of the sphere is 100π m^2.

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/5 **3.** The voltage V (measured in volts), current I (in amperes) and resistance R (in ohms) of an electric circuit are related by the equation $V = IR$. If V is increasing at the rate of 1 volt/s, and I is decreasing at the rate of $\frac{1}{3}$ amp/s, find the rate at which R is changing when $V = 12$ volts and $I = 2$ amps.

/6 **4.** Sand falls from a conveyor belt at the rate of 32π m^3/min, forming a conical pile. The height of the pile is always three-eighths of the base diameter. How fast is the height changing when the radius of the base is 4 m?

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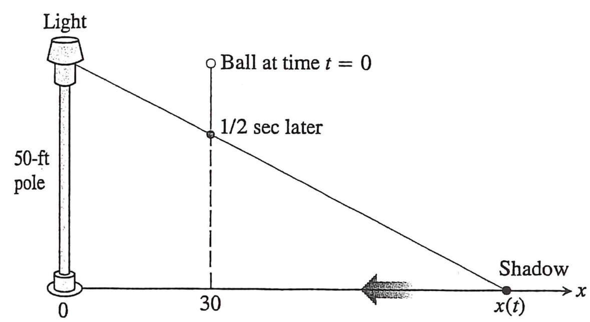
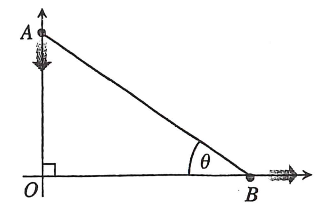
/6 **5.** A trough is 15 ft long and 4 ft across the top. Its ends are isosceles triangles with height 3 ft. Water runs into the trough at the rate of 2.5 ft^3/min. How fast is the water level rising when it is 2 ft deep?

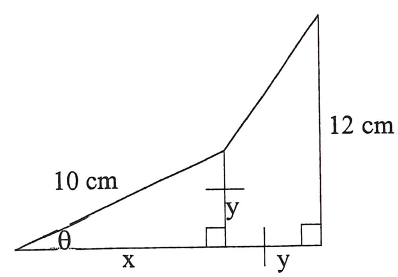
/6 **6.** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light as shown in the diagram below. If the ball falls a distance of $s = 16t^2$ feet in t seconds, how fast is the ball's shadow moving along the ground 0.5 second later?



/6 **7.** "A" and B are walking on straight streets that meet at right angles. "A" approaches the intersection at 2 m/s and B moves away from the intersection at 1 m/s, as shown in the figure below. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection?



/6 **8.** A 10-cm line is rotating clockwise at $\frac{1}{4}$ rad/s, producing a composite figure that is always composed of a right triangle joined to a trapezoid with one of its parallel sides equal to its height and the other parallel side having a length of 12 cm, as shown. How quickly is the combined area changing when $\theta = \frac{\pi}{3}$ rad?



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MCV4UP UNIT 4 TEST: POSITION, VELOCITY, ACCELERATION & ABSOLUTE MAXIMUMS AND MINIMUMS

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Name:______________________________

1. Typically, time t is thought of as a real number greater than or equal to zero; however, it is possible to consider negative time as well. For example, if 12:00 midnight this Saturday were to be defined as $t = 0$, then any event prior to this arbitrary frame of reference could be thought of as occurring when $t < 0$. Consequently, any event after 12:00 midnight this Saturday could be thought of as occurring when $t > 0$.

Keeping the above in mind, it would now make sense if the motion of an object along the x–axis were to be described by the position function $s(t) = -t(t-1)^4$ for the domain $t \in [-1, 2]$, where t is measurerd in seconds and $s(t)$ in metres. Now suppose that the velocity function of this object was found to be $v(t) = (1-5t)(t-1)^3$ and the acceleration, $a(t) = 4(2-5t)(t-1)^2$. (Accept the given derivations of $v(t)$ and $a(t)$ without verifying on your own.)

/1 a) Construct a signed time-velocity number line below.

$t \longleftrightarrow$

$v(t) = (1-5t)(t-1)^3$

/1 b) Construct a signed time-acceleration number line below.

$t \longleftrightarrow$

$a(t) = 4(2-5t)(t-1)^2$

/6 c) Combine the two signed number lines from parts "a" and "b" into one time-velocity-acceleration number line; hence, complete the blanks below using the "less than" symbol (for example, $5 < t < 8$ could be an answer for some function).

$t \longleftrightarrow$

$v(t) = (1-5t)(t-1)^3$

$a(t) = 4(2-5t)(t-1)^2$

The object is moving to the left and slowing down when ____________.

The object is moving to the left and speeding up when ____________.

The object is moving to the right and slowing down when ____________.

The object is moving to the right and speeding up when ____________.

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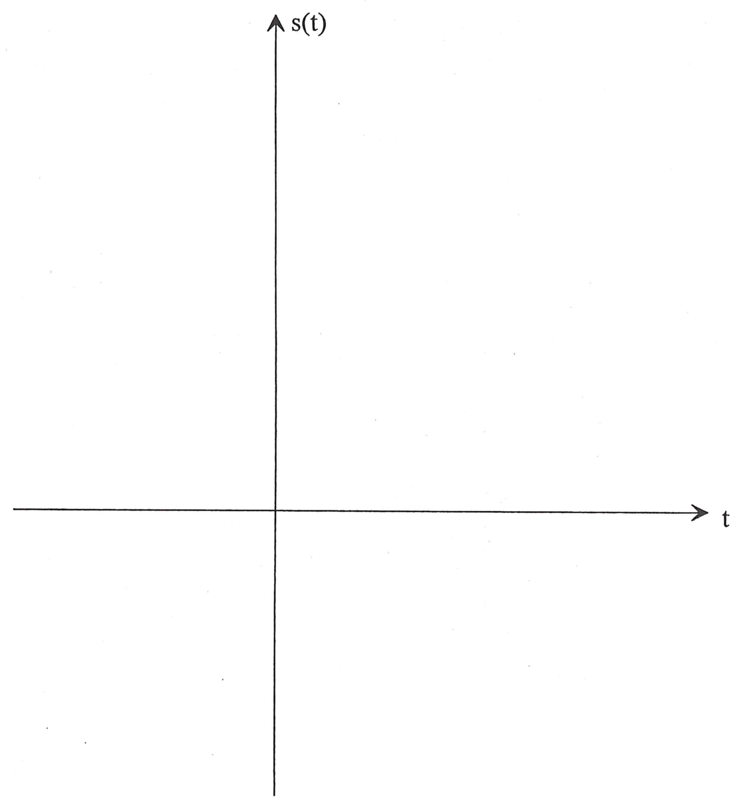
/6 d) Sketch a neat s versus t graph for the motion of the object, labelling the coordinates of all local maximum and minimum points, inflection points, endpoints and the s–intercept. Do not attempt to scale the axes nor insert any "tick" marks on the axes. (Any calculations may be shown on the left side below, if desired.)



/2 e) Calculate the total distance travelled by the object.

/8 **2.** Determine the coordinates of the absolute maximum and minimum points of

$f(x) = \frac{1}{2}e^{2x} - e^{x} - 2x$ for $x \in [0, \ln 3]$.

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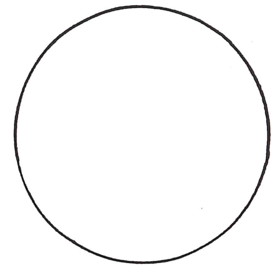
MCV4UP UNIT 4 TEST: OPTIMIZATION PROBLEMS

Name:________________________

INSTRUCTIONS: Provide full solutions, with all final answers expressed as exact, fully-reduced numbers. Do not verify that a maximum or minimum occurs except for question 5 .

/6 **1.** Determine the area of the largest rectangle that can be inscribed in a circle of radius 20 cm.



/5 **2.** Find the coordinates of the point on the parabola $y^2 = 12x$ that is nearest to $(6, 1)$. Express your answer in the form $\left(\sqrt[3]{a}, k\sqrt[3]{b}\right)$, where $k, a, b \in \mathrm{N}$, $1 < k, a, b < 10$, and $a \neq b$.

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/7 **3.** A wire that is 120 cm long is cut into two pieces. One part is bent to form a circle, and the other is bent to form a square. Determine the lengths of the two pieces if the sum of their areas is a minimum.

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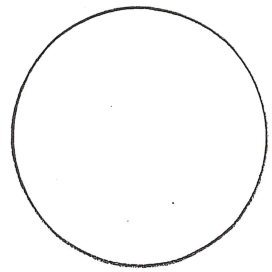
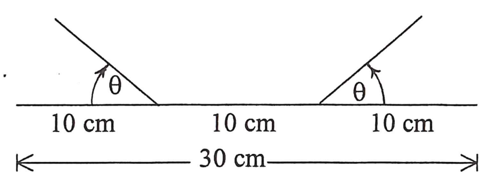
/6 **4.** Determine the volume of the largest right circular cone that can be inscribed in a sphere of radius 24 cm. (The volume of a cone is $\frac{1}{3}\pi r^2 h$.)



/7 **5.** A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle of θ as shown in the cross-sectional diagram. How should θ be chosen so that the gutter will carry the maximum amount of water? Verify that a maximum does indeed occur using the Second Derivative Test.



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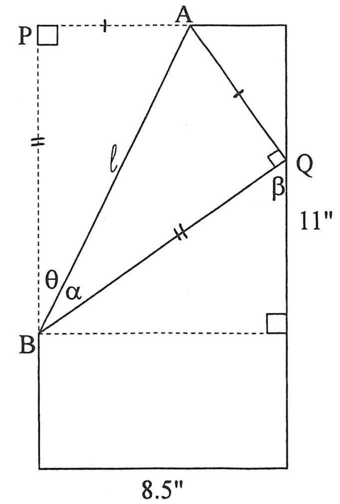
/8 **6.** A letter size piece of paper is folded along AB so that the upper left-hand corner at P meets the right edge at Q as shown. Using trigonometry, determine the minimum length, ℓ, of the fold. (Hint: Begin by expressing α and β in terms of θ and then express ℓ in terms of θ in a two-step process. A line through B parallel to the edge of length 8.5 inches has been constructed for you as an aid.)



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MCV4UP UNIT 5 TEST

Name:______________________

INSTRUCTIONS: PROVIDE NEAT, FULLY-LABELLED SKETCHES OF EACH OF THE FOLLOWING FUNCTIONS. ENSURE THAT ALL INTERCEPTS, ASYMPTOTES, CRITICAL POINTS AND INFLECTION POINTS ARE LABELLED AS EXACT NUMBERS, UNLESS INSTRUCTED OTHERWISE. SKETCHES NEED NOT BE TO SCALE, AND CALCULATORS ARE NOT ALLOWED.

/10 **1.** $f(x) = x^{\frac{5}{3}} + 5x^{\frac{2}{3}}$

Note that $f'(x) = \dfrac{5x+10}{3\sqrt[3]{x}}$, $f''(x) = \dfrac{10x-10}{9\sqrt[3]{x^4}}$ and $f(-2) \approx 4.7$.

/10 **2.** $f(x) = \dfrac{x}{x^2 - 4x + 4} = \dfrac{x}{(x-2)^2}$

/10 **3.** $f(x) = \dfrac{x^2 - 3x + 2}{x - 3}$

Note that $f(1.6) \approx 0.2$ and $f(4.4) \approx 5.8$. As per instructions, remember to express the x-coordinates of any critical points or points of inflection as exact numbers, though the y-coordinates may be expressed to one decimal place of accuracy.

/10 **4.** $f(x) = \frac{x^2}{e^x}$

Note that $f'(x) = \frac{2x - x^2}{e^x}$, $f''(x) = \frac{x^2 - 4x + 2}{e^x}$, $f(0.6) \approx 0.2$, $f(2) \approx 0.5$ and $f(3.4) \approx 0.4$. Once again, remember to express the x-coordinates of any critical points or points of inflection as exact numbers, though the y-coordinates may be expressed to one decimal place of accuracy.

/6 **5.** Given $f(x) = \frac{x^2}{e^x}$ from question # 4, verify that $f'(x) = \frac{2x - x^2}{e^x}$ and $f''(x) = \frac{x^2 - 4x + 2}{e^x}$.

/2 **6.** Suppose $f(x)$ is some continuous function. If $f''(c) = 0$ and $f'''(c) > 0$, where c is some real value in the domain of $f(x)$, what can be concluded about the graph of $f(x)$? A one-sentence response is sufficient.