PDF 9.040 Intersection of Three Planes

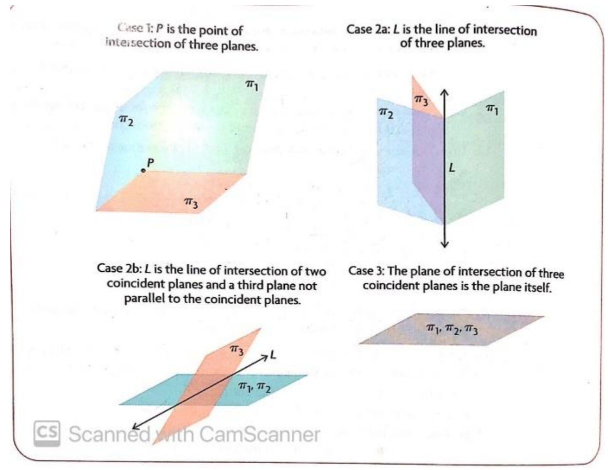
In talking about the intersection of three planes, we will be talking about three equations with three unknowns.

Recall from earlier that when discussing systems of equations, we can talk about consistent and inconsistent systems.

Consistent Systems

A consistent system is a system of equations with at least one solution that satisfies each equation. In the case of the intersection of three planes, this means that there is at least one point of intersection. We will see that three planes can intersect at a single point, along a line of intersection, or at the whole plane of intersection.

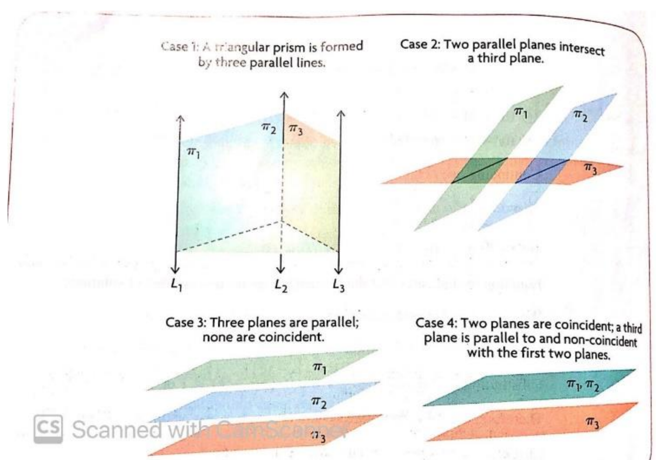
- Three planes can intersect at a single point.
- Three non-parallel planes can intersect along a line.
- Two coincident planes can intersect a third plane along a line (much like two non-parallel planes intersecting along a line).
- Three coincident planes.



Inconsistent Systems

An inconsistent system is a system of equations in which there is no one solution that satisfies each equation. In the case of the intersection of three planes, this means that there is no one point of intersection.

- Three non-parallel planes can intersect along three parallel lines of intersection (forming a triangular prism).
- Two parallel, non-coincident planes each intersect a third non-parallel plane.
- Three parallel, non-coincident planes.
- Two parallel, coincident planes and another parallel, non-coincident plane.

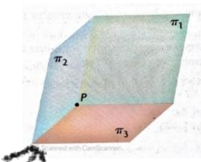


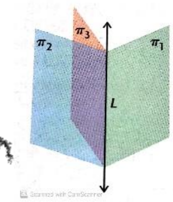
Steps to Determine which of the Eight Scenarios we Have

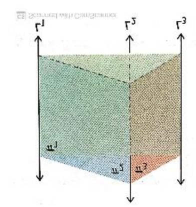
****** The first step is to determine if any of the normal vectors are parallel ******

1. Convert each of the planes to Cartesian form (if necessary).
2. Determine whether any of the normal vectors are parallel.

3. **If none of the normal vectors are parallel**, then we have either one single point of intersection, or three non parallel planes that intersect along a line, or a triangular prism with 0 points of intersection.
4. We then build a matrix and solve to determine which scenario we have
5. If we get a single point of intersection, we have the first scenario pictured. We then determine the point of intersection
6. If we get a final row of 0 0 0 0, then we have infinite points of intersection, and we are dealing with the second scenario picture on the right. We can ignore the third row of the matrix and determine the equation of the line of intersection by focusing only on the top two rows of the matrix.
7. If we get a final row of 0 0 0 non-zero, then we have zero points of intersection and we have the third scenario shown







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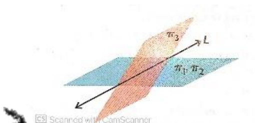
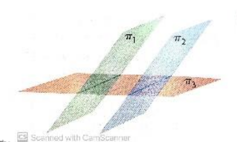
8. **If two of the normal vectors are parallel**, then determine whether the two planes with parallel normal vectors are coincident or non-coincident.
9. If the planes are coincident, then the third non-parallel plane intersects the two coincident planes along a line (fourth picture on the right). To determine the equation of the plane of intersection, we can discard one of the two coincident planes. Then we build a matrix and determine the equation of the line of intersection between the remaining plane of the two parallel planes and the third non-parallel plane.
10. If the two planes are parallel and non-coincident, then the third non-parallel plane intersects each of them along a separate line of intersection (fifth picture on the right). The system is inconsistent, and no further work needs to be done.





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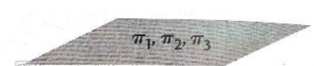
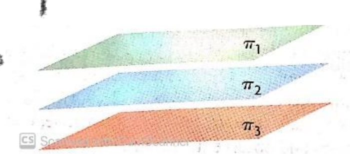
11. **If all three of the normal vectors are parallel**, then the planes are parallel and you should determine whether any of the planes are also coincident.
12. If all three planes are coincident, then we have a consistent system (sixth diagram on the right). The points of intersection are every point on the plane. There is no further work to be done.
13. If two of the planes are coincident and the third is non-coincident, then there are no points of intersection (seventh diagram on the right). It is an inconsistent system, no further work needs be done.
14. If all three planes are non-coincident, then there are no points of intersection. It is an inconsistent system. No further work needs to be done (last diagram on the right).





Example 1

Solve the following system of equations.

$x + y - 3z + 23 = 0$

$5x - 4y + 2z - 14 = 0$

$2x + 3y - z + 19 = 0$

Example 2

Solve the following system of equations

$3x - 4y + 10z = -22$

$x + 2y = 16$

$x - 8y + 10z = -54$

Example 3

Solve the following system of equations

$4x + 2y - z = 12$

$x - 3y + 5z = 11$

$19x + 5y + 2z = 0$

Example 4

Solve the following system of equations

$3x + 2y + 2z = 22$

$x - 4y + 6z = 15$

$6x + 4y + 4z = 44$

Example 5

Solve the system of equations

$5x - 11y + 17z - 90 = 0$

$15x - 33y + 51z - 200 = 0$

$3x - 8y + 2z - 1 = 0$

Example 6

Solve the system of equations

$2x - 9y + 4z - 15 = 0$

$4x - 18y + 8z - 30 = 0$

$8x - 36y + 16z - 60 = 0$

Example 7

Solve the system of equations

$5x - y - z - 15 = 0$

$10x - 2y - 2z - 30 = 0$

$15x - 3y - 3z - 50 = 0$

Example 8

Solve the system of equations

$x + 4y - 4z - 13 = 0$

$10x + 40y - 40z - 40 = 0$

$3x + 12y - 12z - 18 = 0$