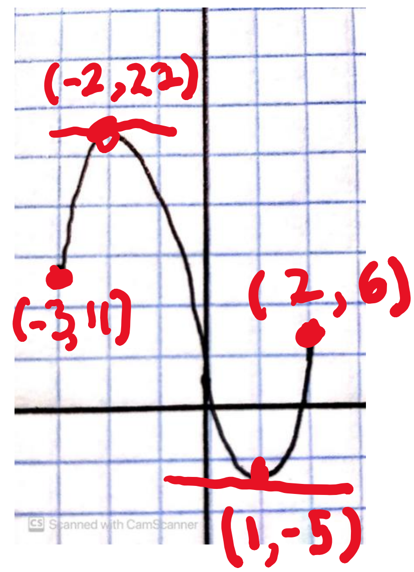
PDF 3.020 Optimization

Maximum and Minimum Values of a Function on an Interval (Extreme Values)

Consider the graph of the function $f(x) = 2x^3 + 3x^2 - 12x + 2, \quad -3 \le x \le 2$.

The point where $x = -2$ is a local maximum and the point where $x = 1$ is a local minimum.



Both of those points have horizontal tangents.

This means that the derivative at both of those points is equal to 0

Therefore we can come up with an algorithm for determining maximum and minimum points on the interval $a \le x \le b$

Both minimum and maximum values are called optimum values.

The process of determining optimum values is called optimization.

Optimization Algorithm

- Solve for x where $f'(x) = 0$
- Evaluate $f(x)$ for all x values determined in the above step, provided that $a \le x \le b$ (i.e., for all x values in the interval where f'(x) = 0)
- Evaluate $f(a)$ and $f(b)$; in other words, determine the y-values of the endpoints.
- The maximum value of f on this interval is the largest of these $f(x)$ values and the minimum value of f on this interval is the least of these $f(x)$ values.
- We have to check the endpoint y-values because, as we will see in later examples, it is possible for the maximum or minimum x or y values to be the x or y values of one of these points.

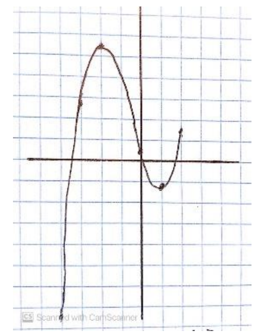
We can tell by observation that the maximum value on the interval is 22 and the minimum value on the interval is -5.

However, suppose we wanted to see a formal solution to the question "Determine the maximum and minimum values of the function
$f(x) = 2x^3 + 3x^2 - 12x + 2$ over the interval $-3 \le x \le 2$

Example 1

Determine the maximum and minimum values of the function

$f(x) = 2x^3 + 3x^2 - 12x + 2$ over the interval $-4 \leq x \leq 2$



Optimization

Optimization may involve finding either the maximum or minimum value of the function over a set of values (domain).

Algorithm for Solving Optimization Problems

- Develop a formula with only one independent variable.
- Diagrams are sometimes helpful.
- Determine the relevant domain of the independent variable (the question might include constraints on the variable).
- Use the algorithm to optimize

- Some convenient equations include
  - Revenue = (number of items sold) x (price of one unit)
  - Cost = (number of items produced) x (average cost to produce each unit)
  - Profit = Revenue – Cost
    - The best guide is common sense. The formula is usually relatively simple to apply if you slow down and think it through.

In this section, the question will sometimes tell you that you can use decimal approximations for your answers.

Similarly, sometimes a question will tell you that you can skip the domain analysis. In these questions, we just need to set the derivative to 0 and solve for the independent variable, then put the value(s) that we find back into the original formula and answer the question asked.

Example 4

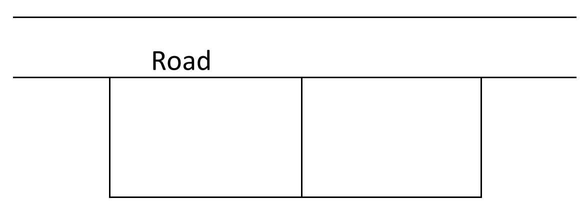
Two positive numbers add to 24. The sum of the cube of one and the square of the other is a minimum. What are the numbers? (You may round your answers to two decimal places)

Example 5

A farmer has 600 m of fencing and wishes to enclose a rectangular field. One side of the field is against a country road that is already fenced, so the farmer needs to fence only the remaining sides of the field. The farmer wants to enclose the maximum possible area and to use all the fencing. What dimensions will achieve this goal?

Example 6

A woman wants to create a rectangular fenced in field as shown. She wants to divide it in half with a fence parallel to one of the sides. She has 600 metres of fence with which to do the job. Determine the maximum possible area.



Example 7

A piece of sheet metal, 60m by 30m, is to be used to make a rectangular box with an open top. Determine the dimensions that will give the box the largest volume. You may round your answer to two decimal places.

Example 8

A man wants to create a fenced in area of 1000 m^2. Three of the sides of the rectangle will use fencing that costs \$50/m, and one of the sides will use fencing that costs \$10/m. What is the cheapest price possible?

Example 9

A rectangular poster (which includes a rectangular printing area and margins all around) has an area of 1014 m^2. The top and bottom margins are each 2 metres and the side margins are each 3 metres. Find the dimensions of the poster that give the maximum printing area.

Example 10

Todd's house is 30 km west of Eleanor's ho use. At noon, Todd leaves his house and jogs 5km/h directly north. At the same time, Eleanor leaves her house and jogs directly west at a speed of 7 km/h. They each run for six hours before stopping. At what time, rounded to the nearest minute, are they closest together?

Example 11

Skippy is flying an airplane that is currently right over his house. His altitude is 4000 m, and his velocity is 60 km/h, due north. At the same moment, Charmaine is 20 km due west of Skippy's house. Charmaine's altitude is 2000 m, and her velocity is 30 km/h due east. Charmaine is climbing at 50 m/min. At what time t will the planes be closest together, and what will be that distance?

Example 12

Joey is on an island that is 1000 m from a straight shoreline. There is a grocery store 2000 m down the straight shoreline from the island. Joey can swim 3 m/s and he can walk 5 m/s. What is the minimum length of time that it will take Joey to get to the store?

Example 13

A wire of length ten metres is divided into two parts. One forms a right isosceles triangle, and the other forms a rectangle with length twice as long as width. How should the wire be cut so that the total area is

a) a minimum   b) a maximum

Example 14

A window has a perimeter of 20 m, and it consists of a rectangular window surmounted on one end by a semicircle. The rectangular portion allows twice as much sunlight in as the semicircular (tinted) portion. Determine the length and width of the rectangle that will allow the maximum sunlight in.

Example 15

A cylindrical shaped aluminum can must have a capacity of 500 cm^3. Determine the dimensions that will create the can but use the least aluminum possible.

Example 16

A cylindrical chemical storage tank with a capacity of 1000 m^3 is going to be constructed in a warehouse that is 12 m by 15 m, with a height of 11 m. the specifications call for the base to be made of sheet steel that costs \$100/m^2, the top to be made of sheet steel that costs \$50/m^2, and the wall to be made of sheet steel that costs \$80/m^2.

a) Determine whether it is possible for a tank of this capacity to fit in the warehouse. If it *is* possible, state the restrictions on the radius.
b) If fitting the tank in the warehouse is possible, determine the proportions that meet the conditions and that minimize the cost of the steel for construction.

Example 17

Determine the maximum area of a rectangle that can be inscribed inside of an ellipse with the equation $4x^2 + 25y^2 = 100$