PDF 6.010 Introduction to Vectors

A vector is a quantity that requires both a magnitude and a direction for a complete description. Examples of vectors are weight, velocity and friction. On the other hand, a scalar is a magnitude that can be completely specified by just one number. Examples of scalars include age, volume, area, speed, mass and temperature.

A vector can be represented by a directed line segment. The magnitude of the vector is indicated by the length of the line segment, and the direction of the vector is indicated by an arrowhead on the end of the segment.

Speed is a scalar quantity. We can describe the speed of an airplane as 200 km/h.

Velocity is a vector quantity. We can refer to the velocity of one airplane as 400 km/h in a southwesterly direction (represented by the vector $\vec{a}$ in the diagram) and the velocity of another airplane as 100 km/h in a northerly direction (represented by $\vec{v}$)

We can indicate movement in a direction from one location to another using vectors. For example, the vector $\overrightarrow{AB}$ is a line segment running from A to B with its tail at A and its head at B. Its actual size, or magnitude, is denoted by $|\overrightarrow{AB}|$, and is represented by the length of the line segment. The magnitude of a vector is always non-negative. The direction of the arrow represents the direction of the airplane, and its length represents the speed.

Equal Vectors

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal if and only if

1. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel to each other, and the direction from A to B is the same as the direction from C to D.

   and

2. The magnitude of $\overrightarrow{AB}$ equals the magnitude of $\overrightarrow{CD}$. In other words, $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

Opposite Vectors

Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are opposite if and only if

1. $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel to each other, and the direction from A to B is the opposite of the direction from C to D. Another way of putting this is that the direction from A to B is the same as the direction from D to C.

   and

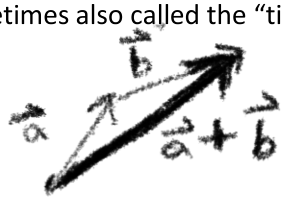
2. The magnitude of $\overrightarrow{AB}$ equals the magnitude of $\overrightarrow{CD}$. In other words, $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

Vector Addition

Triangle Law of Addition

You can determine the sum $\vec{a} + \vec{b}$ by putting the tail of $\vec{b}$ on the tip of $\vec{a}$ and then drawing a vector straight from the tail of $\vec{a}$ to the tip of $\vec{b}$. The resulting vector is called the resultant.

This method is sometimes also called the "tip to tail" method



A way of thinking about the sum of two vectors is as follows:

If you start at Point A and walk to Point B, then walk from Point B to Point C, the net result is as if you walked from A directly to C.

Therefore, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Example 1

Suppose that an airplane is traveling with a component velocity of 500 km/h N when it encounters a wind blowing with a velocity of 100 km/h E. What is the resultant velocity?

Angle Between Two Vectors

When you are told the angle between two vectors, it is always the angle between those vectors if they are drawn tail to tail.

Note, however, that often when we add two vectors, the angle that we use for our sin law or cos law calculations will not be that same value.

Example 2

Example: Two unit vectors $\vec{a}$ and $\vec{b}$ have an angle of 30^0 between them. Determine the magnitude and direction of $2\vec{a} + \vec{b}$

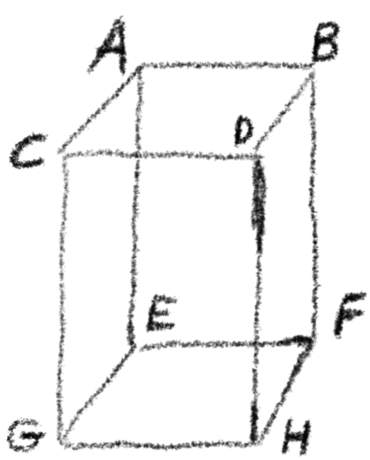
Example 3

Suppose that $|\vec{u}| = 4$ and $|\vec{v}| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is 120°. Determine $\vec{u} + \vec{v}$.

Example 4

In the picture below of a rectangular prism, we know that $\overrightarrow{AB} = \vec{a}, \overrightarrow{AC} = \vec{b}, \overrightarrow{AE} = \vec{c}$

Determine an expression in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$ equal to each of the following



a) $\overrightarrow{CH}$

b) $\overrightarrow{FG}$

c) $\overrightarrow{CF}$

Representing Vectors on a Cartesian Plane

Components

We can represent a two-dimensional vector on a Cartesian plane.

For example:

- a vector that moves 1 unit to the right and 2 units up could be represented by the vector $\overrightarrow{(1,2)}$
- a vector that moves 3 units to the left and 5 units down could be represented by the vector $\overrightarrow{(-3,-5)}$
- a vector that only moves 10 units down could be represented by the vector $\overrightarrow{(0,-10)}$

Unit Vectors on the Cartesian Plane

A unit vector is a vector with a magnitude of 1.

A vector that moves 1 unit to the right is $\overrightarrow{(1,0)}$ and a vector that moves 1 unit up is $\overrightarrow{(0,1)}$. These vectors are so significant that they are referred to as $\vec{\imath}$ and $\vec{\jmath}$ respectively. In other words, $\vec{\imath} = \overrightarrow{(1,0)}$ and $\vec{\jmath} = \overrightarrow{(0,1)}$.

All vectors represented with components on the two-dimensional Cartesian plane can be represented as a sum or difference of these unit vectors; for example,

$$\overrightarrow{(3,-4)} = 3\overrightarrow{(1,0)} - 4\overrightarrow{(0,1)} = 3\vec{\imath} - 4\vec{\jmath}$$

Distributive Property With Vectors

Component Form

Example 5

Simplify $3\left(2\vec{a} - 5\vec{b} + \vec{c}\right) - 2\left(\vec{a} - 4\vec{b} + 6\vec{c}\right)$

Example 6

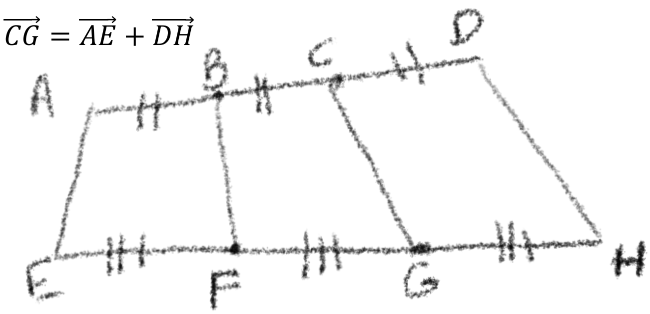
Given that $\vec{u} = (3,-1), \vec{v} = (-4,-7), \vec{w} = (10,1)$, state the components of the vector $4\vec{u} + 2\vec{v} - 7\vec{w}$

Taking Different Paths Questions

Another type of question is the "taking different paths" question. For some questions, you have to try getting from one place to another in a variety of ways until a solution to your problem presents itself.

Example 7

Prove that $\overrightarrow{BF} + \overrightarrow{CG} = \overrightarrow{AE} + \overrightarrow{DH}$



Example 8

You are given that $\overrightarrow{AD} = \frac{3}{5}\overrightarrow{BC}$. Prove that $\overrightarrow{AE} = \frac{5}{8}\overrightarrow{AD} + \frac{3}{8}\overrightarrow{AB}$