PDF 7.040 Dot Product (aka Scalar Product)

The dot product of the vectors $\vec{a}$ and $\vec{b}$ is equal to the magnitude of $\vec{a}$ times the magnitude of $\vec{b}$ times the cosine of the angle between them.

In other words,

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

Example 1

Given the magnitude of $\vec{a}$ is 7, the magnitude of $\vec{b}$ is 4, and that there is an angle of 60° between $\vec{a}$ and $\vec{b}$, evaluate $\vec{a} \cdot \vec{b}$.

Example 2

Given the magnitude of $\vec{a}$ is 9, magnitude of $\vec{b}$ is 5, and that there is an angle of 140° between $\vec{a}$ and $\vec{b}$, evaluate $\vec{a} \cdot \vec{b}$.

Relationship Between Dot Product and Angle Between Vectors

Assume that neither $\vec{a}$ nor $\vec{b}$ have a magnitude of 0, meaning that each of the magnitudes is positive.

Next, recall the following:

1. $\cos\theta > 0$ if $0^0 \leq \theta < 90^0$
2. $\cos\theta = 0$ if $\theta = 90^0$
3. $\cos\theta < 0$ if $90^0 < \theta \leq 180^0$

Since $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, therefore:

1. $\vec{a} \cdot \vec{b}$ is positive if θ is acute
2. $\vec{a} \cdot \vec{b} = 0$ if $\theta = 90^0$
3. $\vec{a} \cdot \vec{b}$ is negative if θ is obtuse

* A significant detail of this course is that two vectors are perpendicular if and only if the dot product is 0

Example 3

Determine the value of $\vec{a} \cdot \vec{a}$ given that $|\vec{a}| = 7$

Properties of the Dot Product

Commutative Property: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Associative Property: $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b})$ For example, $(3\vec{a}) \cdot \vec{b} = 3(\vec{a} \cdot \vec{b})$

Distributive Property (2) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$$

Example 4

Suppose the vectors $\vec{a} + 3\vec{b}$ and $4\vec{a} - \vec{b}$ are perpendicular and suppose that $|\vec{a}| = 2|\vec{b}|$. Determine the angle between the vectors $\vec{a}$ and $\vec{b}$.

Example 5

If $|\vec{x} + \vec{y}| = |\vec{x} - \vec{y}|$, then prove that the non-zero vectors $\vec{x}$ and $\vec{y}$ are perpendicular.