Derivatives Test Thursday March 7 2024 Nam[redacted]

Version 2

Volume of a Cone $= \frac{1}{3}\pi r^2 h$	Surface Area of a Sphere $= 4\pi r^2$	Volume of a Sphere $= \frac{4}{3}\pi r^3$

1. Prove the product rule. In other words, prove that if $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$

/4

$p(x) = f(x)g(x)$

$$p'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) + f(x)g(x+h) - f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h}.$$

$$= \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0} g(x)\right] + \left[\lim_{h\to 0} f(x)\right]\left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right]$$

$$= f'(x)g(x) + f(x)g'(x).$$

4

2. Determine the equation of the tangent line to the curve $y = 2x^3 + 5x - 9$ at the point where $x = 4$.

/3

$y' = 6x^2 + 5$

$= 6(4)^2 + 5$

$= 101$

$y = 2(4)^3 + 5(4) - 9$

$= 128 + 20 - 9$

$= 139$

$\therefore (4, 139)$

$y = 101x + b$

sub in $(4, 139)$

$404 + b = 139$

$b = -265$ $\therefore y = 101x - 265$

3

The equation of the tangent line is $y = 101x - 265$

7

3. The curve $y = (x+3)^{\frac{3}{2}} - 3x$ has one point with a horizontal tangent. Determine the coordinates of that point. /4

$y' = \frac{3}{2}(x+3)^{\frac{1}{2}}(1) - 3$
$= 0$

$\frac{3}{2}(x+3)^{\frac{1}{2}} - 3 = 0$
$\frac{3}{2}(x+3)^{\frac{1}{2}} = 3$
$(x+3)^{\frac{1}{2}} = 2$
$x+3 = 4$
$x = 1$

sub in $x = 1$
$y = (1+3)^{\frac{3}{2}} - 3$
$= (4)^{\frac{3}{2}} - 3$
$= \sqrt{4^3} - 3$
$= 8 - 3 = 5$
$\therefore (1, 5)$

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The coordinates of the point with a horizontal tangent is $(1, 5)$

4. Given $f(x) = \left[\frac{(7x-19)^5 - 30}{x^3 - 10x + 4}\right]^3 + 3x^2$, determine an expression for the derivative $f'(x)$. You do not need to simplify that expression. Then, evaluate $f'(3)$. /5

$f'(x) = 3\left[\frac{(7x-19)^5 - 30}{x^3 - 10x + 4}\right]^2 \left[\frac{5(7x-19)^4(7)(x^3-10x+4) - (3x^2-10)(7x-19)^5 + (3x^2-10)(30)}{(x^3-10x+4)^2}\right] + 6x$

$= 3\left[\frac{(7x-19)^5 - 30}{x^3-10x+4}\right]^2 \left[\frac{35(7x-19)^4(x^3-10x+4) - (3x^2-10)(7x-19)^5 + (3x^2-10)(30)}{(x^3-10x+4)^2}\right] + 6x$

$f'(3) = 3\left[\frac{2}{1}\right]^2 \left[\frac{35 \cdot 16 \cdot 1 - 17 \cdot 32 + 17 \cdot 30}{1}\right] + 18$

$= 2 \cdot 6 \cdot (560 - 544 + 510) + 18$

$= \cancel{3}\ 6330.$

5

$f'(3) = 6330$

9

5. We know that $h(x) = f(g(x))$. Given the table shown, determine the value of $h'(3)$ ~~/3~~

$h'(x) = f'(g(x)) \cdot g'(x)$

$\therefore h'(3) = f'(g(3)) \cdot g'(3)$

$= f'(-1) \cdot 7$

$= -11 \cdot 7$

$= -77$ ✓

2/2

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	2	-5	-11	9
1	-4	10	-9	0
3	8	-1	6	7
5	2	-5	13	4
7	15	-6	-8	5

$h'(3) = -77$

6. Determine an expression for $\dfrac{d\left[(3x^2+8x+1)^3+7(3x^2+8x+1)\ \ +8\right]}{d(3x^2+8x+1)}$

Your answer should not have any variable other than x in it, but it does not need to be simplified. /2

$3(3x^2+8x+1)^2+7$

2

$\dfrac{d\left[(3x^2+8x+1)^3+7(3x^2+8x+1)\ \ +8\right]}{d(3x^2+8x+1)} = 3(3x^2+8x+1)^2+7$ ✓

(your final answer should have no variable other than x in it but you do not have to expand it)

7. The radius of a circle is expanding at a rate of 2 cm/min. At what rate is the area of the circle increasing when the area of the circle is $100\pi\ cm^2$. Include proper units of measurement in your answer.

~~/4~~

$A = \pi r^2$

$100\pi = \pi r^2$

$r = 10$

$\dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt}$

3/3

$= 2\pi(10\text{ cm})(2\text{ cm/min})$

$= 40\pi\ (cm^2/min)$ ✓

The area of the circle is expanding at a rate of $40\pi\ cm^2/min$

(include proper units of measurement in your answer)

/7

8. There are two points on the ellipse $4x^2 + 9y^2 = 36$ that have a tangent line that passes through the point (9,2). Determine the coordinates of both points. /6

$4x^2 + 9y^2 = 36$

$8x + 18y\frac{dy}{dx} = 0$

$18y\frac{dy}{dx} = -8x$

$\frac{dy}{dx} = \frac{-4x}{9y}$

$\frac{(y-2)}{(x-9)} = \frac{-4x}{9y}$

$9y(y-2) = -4x(x-9)$

$9y^2 - 18y = -4x^2 + 36x$

$36x + 18y = 4x^2 + 9y^2$

$\because 4x^2 + 9y^2 = 36$

$\therefore 36x + 18y = 36.$

$18y = -36x + 36$

$y = -2x + 2.$

$\therefore 4x^2 + 9(-2x+2)^2 = 36$

$4x^2 + 9(4x^2 + 4 - 8x) = 36$

$4x^2 + 36x^2 + 36 - 72x = 36$

$40x^2 - 72x = 0$

$5x^2 - 9x = 0$

$x(5x - 9) = 0$

$x_1 = 0 \quad x_2 = \frac{9}{5}$

when $x = 0$, $y = 2$

when $x = \frac{9}{5}$, $y = -\frac{18}{5} + \frac{10}{5}$

$= -\frac{8}{5}$

$\therefore (0,2)$ & $(\frac{9}{5}, -\frac{8}{5})$.

The two points are $(0,2)$ $(\frac{9}{5}, -\frac{8}{5})$

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