Chapter 6 Limits

Indeterminate Forms

Ex 6.1 A Evaluate the following limits:

1. $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$
2. $\lim_{x\to 1}\dfrac{x^2+3x-4}{x-1}$
3. $\lim_{x\to -2}\dfrac{x+2}{x^3+8}$
4. $\lim_{x\to 2}\dfrac{x^4-16}{x-2}$
5. $\lim_{x\to 2}\dfrac{x^2+x-6}{x^2-4}$
6. $\lim_{x\to -\frac{1}{2}}\dfrac{2x^3+3x^2-17x-9}{2x+1}$
7. $\lim_{x\to -2}\dfrac{x^4+x^3-6x^2-4x+8}{2x^3+3x^2-17x-30}$
8. If $\lim_{x\to -1}\dfrac{x^3-ax^2-x+4}{x+1}$ exists, find a

Ex 6.1 B Evaluate the following limits:

1. $\lim_{x\to 1}\dfrac{\sqrt{x}-1}{x-1}$
2. $\lim_{x\to 1}\dfrac{x-1}{\sqrt{10-x}-3}$
3. $\lim_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$
4. $\lim_{x\to 4}\dfrac{6-\sqrt{x^2+20}}{x^2-16}$
5. $\lim_{x\to 3}\dfrac{\sqrt{x^2+7}-\sqrt{x+13}}{x-3}$
6. $\lim_{x\to 2}\dfrac{\sqrt{x+2}-\sqrt{3x-2}}{\sqrt{5x-1}-\sqrt{4x+1}}$
7. $\lim_{x\to 1}\dfrac{\sqrt[3]{x}-1}{x-1}$
8. $\lim_{x\to 64}\dfrac{\sqrt[3]{x}-4}{\sqrt{x}-8}$
9. $\lim_{x\to 0}\dfrac{1-\sqrt[3]{x+1}}{x}$

One Side Limits

Ex 6.2 Evaluate the following limits:

1. $\lim_{x\to 1^-}\dfrac{1}{x-1}$
2. $\lim_{x\to 1^+}\dfrac{1}{x-1}$
3. $\lim_{x\to 1}\dfrac{1}{x-1}$
4. $\lim_{x\to 1^-}\dfrac{x}{(x-1)^2}$
5. $\lim_{x\to 1^+}\dfrac{x}{(x-1)^2}$
6. $\lim_{x\to 1}\dfrac{x}{(x-1)^2}$
7. $\lim_{x\to 1^-}\dfrac{x^2-1}{(x-1)^2}$
8. $\lim_{x\to 1^+}\dfrac{x^2-1}{(x-1)^2}$
9. $\lim_{x\to 1}\dfrac{x^2-1}{(x-1)^2}$
10. $\lim_{x\to 3^-}\sqrt{x-3}$
11. $\lim_{x\to 3^+}\sqrt{x-3}$
12. $\lim_{x\to 3}\sqrt{x-3}$
13. $\lim_{x\to 2^-}\dfrac{|x-2|}{x-2}$
14. $\lim_{x\to 2^+}\dfrac{|x-2|}{x-2}$
15. $\lim_{x\to 2}\dfrac{|x-2|}{x-2}$

Limits as x Approaches to Infinity

Ex. 6.3 Evaluate the following limits

1. $\lim_{x\to\infty}\dfrac{3}{x}$
2. $\lim_{x\to\infty}\dfrac{3}{x^2+1}$
3. $\lim_{x\to\infty}\dfrac{3x}{4x}$
4. $\lim_{x\to\infty}\dfrac{3x}{x^2+1}$.
5. $\lim_{x\to\infty}\dfrac{x^3+6x^2}{2x^2+5x+6}$
6. $\lim_{x\to\infty}\dfrac{x^3+6x^2}{2x^3+5x^2+6x}$
7. $\lim_{x\to\infty}2^x$
8. $\lim_{x\to-\infty}2^x$
9. $\lim_{x\to\infty}\dfrac{3^x}{4^x}$
10. $\lim_{x\to\infty}\dfrac{4^x}{3^x}$
11. $\lim_{x\to\infty}\dfrac{2x+3}{\sqrt{x^2+7}}$
12. $\lim_{x\to-\infty}\dfrac{2x+3}{\sqrt{x^2+7}}$
13. $\lim_{x\to\infty}\left(\sqrt{x+8}-\sqrt{x}\right)$
14. $\lim_{x\to\infty}\left(\sqrt{x^2+8}-\sqrt{x}\right)$
15. $\lim_{x\to\infty}\left(\dfrac{x}{x+1}-\dfrac{x}{x-1}\right)$

Chapter 7 Differentiation of Algebraic Functions

First Principles

Ex 7.1 Find the derivative from first principles, given y equals

1. 3
2. $4x$
3. $2x^3$
4. x^3+2x^2+1
5. $\sqrt{ax^2+b}$
6. $\frac{1}{\sqrt{3x}}$
7. $\frac{x+1}{x-1}$
8. $\frac{1}{x^3}$
9. $2x+\frac{1}{x^2}$

Rules of Power, Sum and Difference

Ex 7.2 In Problems 1 - 9, find the derivative given y equals

1. $3x+7$
2. $3x^7-3x+10$
3. $\frac{1}{4}x^4-\frac{1}{3}x^3+\frac{1}{2}x^2+10$
4. $4x^{-4}+5x^{-5}+6x^{-6}$
5. $(3x+2)(2x-3)$
6. $\left(\frac{2}{3}x^2\right)^2+\left(4x^3\right)^3$
7. $1-\frac{1}{x}+\frac{2}{x^2}$
8. $\frac{3x^3+4x^4-5x^5}{x^2}$
9. $\frac{5}{x}+\sqrt[5]{x}+\sqrt{5x}$

10. Find the point on the curve $y = x^2 + 2x + 6$ such that the tangent is horizontal.

11. Find the equation of tangent and normal to the curve $y=x^3+3x$ at (1, 4).

12. A cubic polynomial f(x) is given. If $f'(0)=0$, $f'(1)=5$, $f'(2)=16$, *find* $f'(3)$.

13. Find the quadratic curve which passes (1, 4) and touches the line $y = x$ at (2, 2).

14. Given $y = 4x^3+12x^2+9x+7$ and (-3/2, 7) is a point on the curve. Find the equation of tangent line to the curve at the given point. Find the other intersecting point that the tangent cut with the curve again.

15. Show that $y=x^3-3x^2+4x-2$ *and* $y=2x^2-3x+1$ touch each other. Find the point of contact.

16. Show that the two curves $y=2x^3+6x^2+6x+3$ *and* $y=3x^2+6x+3$ touch each other. Find the common tangent.

17. Show that there is no tangent to $y=x^2-4x$ which passes (2,1).

18. Find the tangent to the curve $y=x^2-8x+14$ which slope is –2. Show that this tangent touches the curve $y=-x^2-4x+4$. Find the point of contact.

19. Find the area of the triangle formed by the three points where the slopes of the tangent lines to the curve $y=x^4-2x^2+4$ are zero at those points.

20. Find the equations of tangents from (3,0) to $y=x^2-x+3$.

21. Find the equations of tangents from (3, 18) to $y=-x^2+6x+5$.

22. Find all tangents on $y=\frac{x^3}{3}-\frac{1}{3}$ which make 45° with the positive x-axis. (*hint*: $\tan 45^\circ = 1$)

23. Find the area of the triangle bounded by the axes and the tangent to $y=\frac{1}{x}$ at point $(1,1)$

24. Find a, and b such the $y=x^2+ax+b$ has slope -2 at the point $(0,2)$

25. Find a quadratic function which passes $(4, 4)$ and has slope -1 when $x=2$ and slope 11 when $x=8$.

26. Show that $y=x^2$ and $2y+x-3=0$ cut at right angles at one of their intersecting points.

Product Rule and Quotient Rule

Ex 7.3A Product Rule. In Problems 1 - 9, find the derivatives given y equals

1. $(x^2+1)(x^2+2)$
2. $5x(x^2+3)$
3. $\sqrt[3]{x^2}\left(x^4-5(\sqrt[3]{x^5})\right)$
4. $(x^2-3x+4)(2x^2+4x)$
5. $(2+x^3)\left(5-\frac{1}{x^2}\right)$
6. $(2x^2+3x^3)^2$
7. $(x^2+1)(x^2+2)(x^2+3)$
8. $(4x+1)(x+2)$ *at* $x=-2$
9. $(3x^2+6x+2)(4-x)$ *at* $x=-1$
10. Find the slope of the tangent to the curve $y=(10-\sqrt{x})(5+\sqrt{x})$ *at* $x=25$
11. If $f(2)=f'(2)=g'(2)=g(2)=2$, find $(fg)'(2)$

Ex 7.3 B Quotient Rule. In Problems 1 - 10, find the derivatives where y equals

1. $\frac{1}{x}$
2. $\frac{4x-1}{x+9}$
3. $\frac{1}{x^2-x+1}$
4. $\frac{x^2+3x+3}{x+1}$
5. $\frac{x+3}{4-x^2}$
6. $\frac{3x^2+9x+4}{x^2+3x+1}$
7. $\frac{3x^3+2x^2-x+6}{x^3}$
8. $\frac{7\sqrt{x}-10}{1-\sqrt{x}}$
9. $\frac{x^2+3x}{x^2-5x+1}$
10. $\frac{(2x+5)(3x+5)}{x+1}$

11. Find the equation of the tangent to the curve $y=\frac{4x+8}{x+1}$ *which is parallel to* $4x+y=7$.

12. Given $f(x)=\frac{ax}{x^2+b}$ Find a an b such that $f'(1)=0$ *and* $f'(0)=1$

13. Find the equation of the tangent and normal to the curve $y=\frac{x^2+2}{x^2-2}$ at the point (1, -3)

14. If $f(1)=2$, $f'(1)=3$, $g(1)=4$, $g'(1)=5$, evaluate $\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx}$ at $x=1$.

The Chain Rule

Ex 7.4 In problems 1 - 12, find the derivatives

1. $y=(1+x)^{10}$
2. $y=(7x^3+6x)^5$
3. $y=\sqrt{x^2-1}$
4. $y=(x^3+2)^{-2}$
5. $y=(2x^3+1)^4(4x^2+7)^2$
6. $y=\sqrt{5+\sqrt{x}}$
7. $y=\left(\frac{x+7}{x+5}\right)^5$
8. $y=\frac{7x}{\sqrt{5-2x}}$
9. $y=(x+1)\sqrt{x^2+2x+2}$
10. $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$
11. $y=[x^5+(3x^2-2)^4]^6$
12. $\sqrt[3]{(1+4x)^4+x^5}$
13. If $h(x)=f(g(x))$, *and* $f(3)=f'(3)=g'(3)=g(3)=3$, *find* $h'(3)$.
14. $y=\frac{u+3}{u-1}$, and $u=x^{\frac{1}{3}}$, find $\frac{dy}{dx}$ at $x=8$.
15. Find the equation of the tangent line to the curve $y=(x^3-1)^{10}$ at the point $(1,0)$
16. Find the equation of the tangent line to the curve $y=(4+3x^2)^{\frac{1}{4}}$ at $x=2$.
17. Find the tangent and normal to the curve $y=\sqrt[3]{x+64}$ at the point where the curve cut the y-axis.
18. *Given* $x^3f(x)+(f(x))^3+f(x^3)=3$ *and* $f(1)=2$, *find* $f'(1)$.
19. Find the area of the triangle formed by the three points on the curve $y=\left(\frac{x^2-1}{x^2+1}\right)^2$ where the slopes of the tangent lines to the curve at those points are zero.
20. Find the equations of the tangent and normal to the curve $y=\sqrt[3]{(x^3-2x+4)^2}$ *at* $x=2$.
21. Given $f(x^3-1)=4x^2$. *Find* $f'(0)$.
22. Find $f'(x)$ in terms of g *and* g' in the following cases
    a) $f(x)=g(x^n)$ *b)* $f(x)=(g(x))^n$ *c)* $f(x)=x^n g(x^n)$ *d)* $f(x)=g(g(x))$
23. Differentiate $(x^2+1)^4$ *with respect to* *a)* $(x^2+1)^4$ *b)* $(x^2+1)^2$ *c)* x^2 *d)* x

Implicit Differentiation

Ex 7.5 In Problems 1 - 9, find the derivatives dy/dx

1. $y^2+x=0$
2. $x^2+4y^2=36$
3. $x^3y^4=3$
4. $xy+y^3=x^2+x+1$
5. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$
6. $\sqrt{x}-\sqrt{y}=\sqrt{a}$
7. $y\sqrt{x}+x\sqrt{y}=10$
8. $y^2=\frac{x+y}{x-y}$
9. $\frac{x^3}{y^3}+\frac{y^3}{x^3}=1$
10. Find the equation of tangent to $y^2+2y+10x+11=0$ which to parallel to 2y+x = 0.
11. Find the tangent and normal to the curve $x^3-6xy-2y^3=0$ at $(4,2)$.
12. Find the tangent and normal to the curve $x^2=y^2x+2$ *at* $x=-1$.
13. Find the points on $\frac{x^2}{16}-\frac{y^2}{9}=-1$ whose tangents are either vertical or horizontal.
14. Find the equation of the tangent to $\sqrt{x}+\sqrt{y}=2\sqrt{a}$ which is parallel to $x+y=2$

15. *If* $x+y=\frac{x}{y}+\frac{y}{x}$, *find* y' *when* $x=1$ *and* $y=1$.

16. Find the equations of tangents to the curve $x^3-x^2-2x+y-4=0$ with slope equal to 1.

17. Show that $xy=1$ and $x^2-y^2=1$ cut at right angles.
(Note: do not try to find the intersecting points, it may be tedious)

Higher Derivatives

Ex 7.6 In problems 1 - 11, find the second derivatives of y

1. $y=3x$
2. $y=3x^3+x^4$
3. $y=\frac{1}{x^2}$
4. $y^2=2x$
5. $y=\frac{1}{3x+2}$
6. $y=\frac{2x+1}{x+2}$
7. $y=\sqrt{x^2-1}$
8. $y=\sqrt[3]{x+2}$
9. $x^2+y^2=25$
10. $4x^2-y^2=4$
11. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
12. $y=ax^2+2x+b$, Show $y''-\frac{1}{x}y'+\frac{2}{x}=0$
13. If $y=\frac{1}{1-x}$, find y', y'', y''' and guess a formula for $\frac{d^n y}{dx^n}$
14. If $x^2+y^2=2y$, show $y''=\frac{1}{(1-y)^3}$
15. If $y=x^n$ and $x^2y''+2xy'-2y=0$. find n.
16. *If* $x=y^3+y$, *Show that* $y''=\frac{-6y}{(1+3y^2)^3}$
17. f(x) is a cubic polynomial such that $f(0)=5$, $f'(1)=13$, $f''(1)=12$, $f'''(1)=6$, find $f(1)$