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Chapter 1 • Polynomial Functions

Review of Prerequisite Skills

2. **g.** $(x+n)^2-9$

$=(x+n+3)(x+n-3)$

h. $49u^2-(x-y)^2$

$=(7u+x-y)(7u-x+y)$

i. x^4-16

$=(x^2+4)(x^2-4)$

$=(x^2+4)(x+2)(x-2)$

3. **c.** h^3+h^2+h+1

$=h^2(h+1)+(h+1)$

$=(h+1)(h^2+1)$

e. $4y^2+4yz+z^2-1$

$=(2y+z)^2-1$

$=(2y+z-1)(2y+z+1)$

f. $x^2-y^2+z^2-2xz$

$=x^2-2xz+z^2-y^2$

$=(x-z)^2-y^2$

$=(x-z-y)(x-z+y)$

4. **f.** y^3+y^2-5y-5

$=y^2(y+1)-5(y+1)$

$=(y+1)(y^2-5)$

g. $60y^2+10y-120$

$=10(6y^2-y+12)$

$=10(3y+4)(2y-3)$

5. $36(2x-y)^2-25(u-2y)^2$

$=[6(2x-y)]^2-[5(u-2y)]^2$

$=[6(2x-y)-5(u-2y)][6(2x-y)+5(u-2y)]$

$=[12x-6y-5u+10y][12x-6y+5u-10y]$

$=(12x+4y-5u)(12x-16y+5u)$

c. $y^5-y^4+y^3-y^2+y-1$

$=y^4(y-1)+y^2(y-1)+(y-1)$

$=(y-1)(y^4+y^2+1)$

e. $9(x+2y+z)^2-16(x-2y+z)^2$

$=[3(x+2y+z)-4(x-2y+z)][3(x+2y+z)+4(x-2y+z$

$=[3x+6y+3z-4x+8y-4z][3x+6y+3z+4x-8y+4z]$

$=[-x+14y-z][7x-2y+7z]$

g. $p^2-2p+1-y^2-2yz-z^2$

$=(p-1)^2-(y+z)^2$

$=(p-1+y+z)(p-1-y-z)$

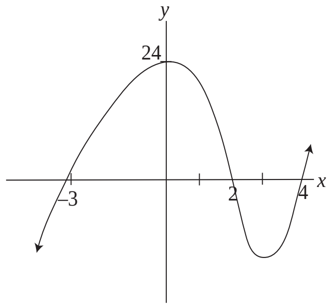
Section 1.1

Investigation 1: Cubic Functions

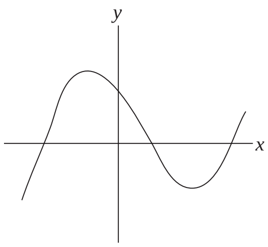
2. There can be 1, or 3 real roots of a cubic equation.

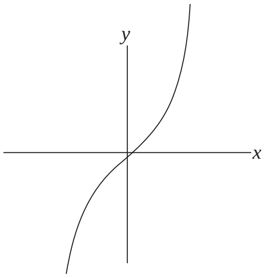
3. **a.** Find the *x*-intercepts, i.e., the zeros of the function $x=2$, $x=-3$, $x=4$, $y=24$, and the *y*-intercepts. Since the cubic term has a positive coefficient, start at the lower left, i.e., the third quadrant, crossing the *x*-axis at –3, then again at 2 and at 4, ending in the upper right of the first quadrant.

b.



4.





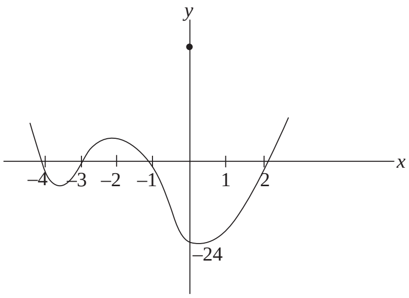
5. When the coefficient of x^3 is negative, the graph moves from the second quadrant to the fourth.

Investigation 2: Quartic Functions

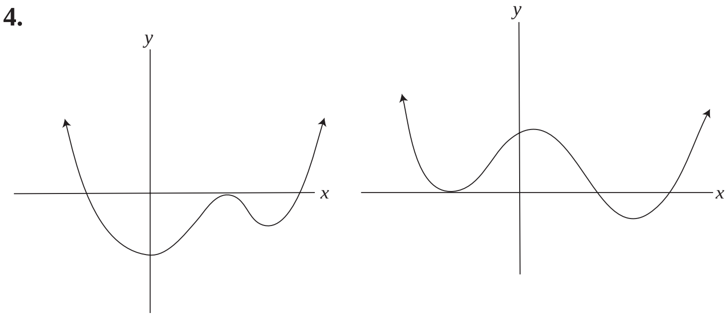
2. There can be 0, 2, or 4 real roots for a quartic equation.

3. **a.** Find the x-intercepts at the function, i.e., $x – 3$, 2, – 1, – 4. Find the y-intercept, i.e., $y = –24$. Begin in the second quadrant crossing the x-axis at – 4, – 3, – 1, and 2 and end in the first quadrant; draw a smooth curve through intercepts.

b.



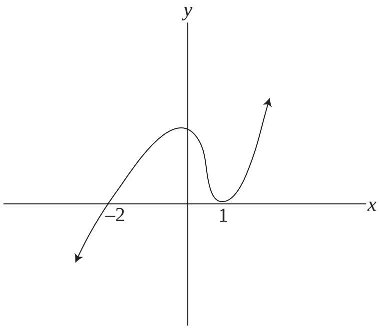
4.



5. If the coefficient of x^4 is negative, the quartic function is a reflection of quartic with a positive coefficient of x^4, i.e. the graph moves from the second to the fourth quadrant, passing through the x-axis a maximum of four times.

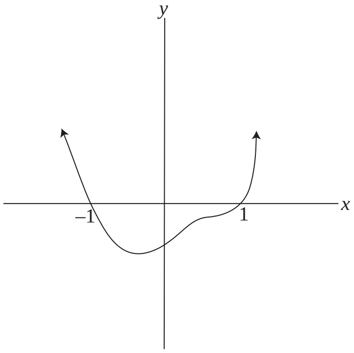
Investigation 3

1.



$y = (x+2)(x-1)^2$

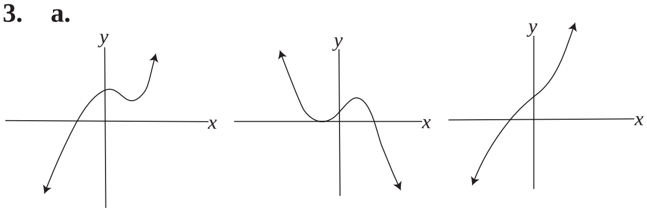
2.



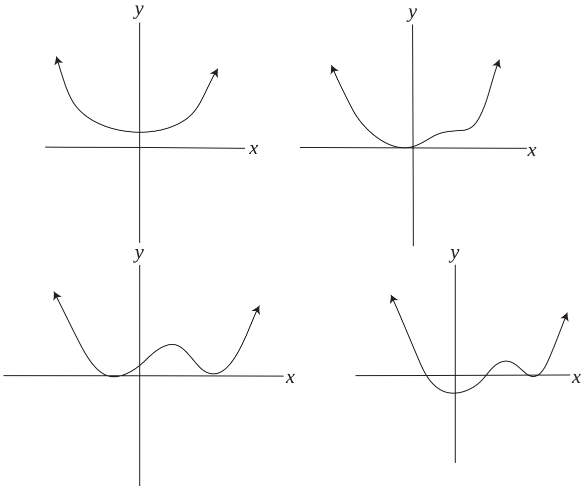
$y = (x+1)(x-1)^3$

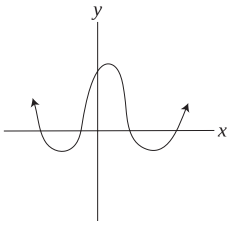
Exercise 1.1

3. **a.**



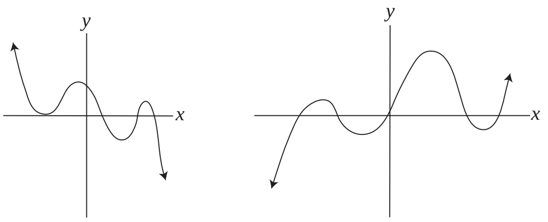
b.



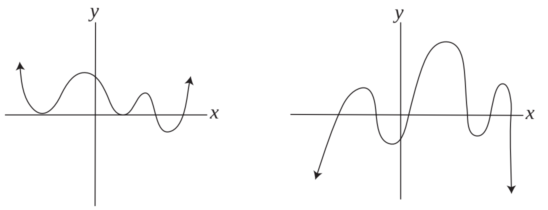


Also includes the reflections of all these graphs in the x-axis.

4. **a.**



b.



Section 1.2

Investigation 1: Cubic Functions

2.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^2 f(x)$
1	1	$8-1=7$	$19-7=12$	$18-12=6$
2	8	$27-8=19$	$37-19=18$	$24-18=6$
3	27	$64-27=37$	$61-37=24$	$30-24=6$
$m-2$	$(m-2)^3$	$(m-1)^3-(m-2)^3=3m^2-9m+7$	$(3m^2-3m+1)-(3m^2-9m+7)=6m-6$	$6m-(6m-6)=6$
$m-1$	$(m-1)^3$	$m^3-(m-1)^3=3m^2-3m+1$	$(3m^2+3m+1)-(3m^2-3m+1)=6m$	$(6m+6)-(6m)=6$
m	m^3	$(m+1)^3-m^3=3m^2+3m+1$	$(3m^2+9m+7)-(3m^2+3m+1)=6m+6$	$(6m+12)-(6m+6)=6$
$m+1$	$(m+1)^3$	$(m+2)^3-(m+1)^3=3m^2+9m+7$	$(3m^2+15m+19)-(3m^2+9m+7)=6m+12$	$(6m+18)-(6m+12)=6$
$m+2$	$(m+2)^3$	$(m+3)^3-(m+2)^3=3m^2+15m+19$	$(3m^2+21m+17)-(3m^2+15m+19)=6m+18$	$(6m+24)-(6m+18)=6$

For quadratic functions, the second finite differences are constant.
For cubic functions, the third finite differences are constant.
It appears that for a polynomial function, a constant finite difference occurs at that difference that is the same as the degree of the polynomial.

Exercise 1.2

1. $(1,0),\ (2,-2),\ (3,-2),\ (4,0),\ (5,4),\ (6,10)$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	0	$-2-0=-2$	$0-(-2)=2$
2	-2	$-2-(-2)=0$	$2-0=2$
3	-2	$0-(-2)=2$	$4-2=2$
4	0	$4-0=4$	$6-4=2$
5	4	$10-4=6$	
6	10		

Since $\Delta^2 f(x)$ for any x, then the polynomial function is a quadratic of the form $f(x)=ax^2+bx+c$.

Substituting the given ordered pairs, we get

$f(1)=a+b+c=0$ …(1)
$f(2)=4a+2b+c=-2$ …(2)
$f(3)=9a+3b+c=-2$ …(3)

Solving these equations, we have

$(2)-(1)$ $3a+b=-2$ …(4)
$(3)-(2)$ $5a+b=0$ …(5)
$(5)-(4)$ $2a=2$
$a=1$

Substituting into (4), $3(1)+b=-2$

$b=-5.$

Substituting into (1), $(1)+(-5)+c=0$

$c=4.$

Therefore, the function is $f(x)=x^2-5x+4.$

2. $(1,-1),\ (2,2),\ (3,5),\ (4,8),\ (5,11),\ (6,14)$

x	$f(x)$	$\Delta f(x)$
1	-1	$2-(-1)=3$
2	2	$5-2=3$
3	5	$8-5=3$
4	8	$11-8=3$
5	11	$14-11=3$
6	14	⋮

Since $\Delta f(x)=3$ for any x, then the function is linear of the form $y=mx+b$.

Substituting the given ordered pairs, we get

$f(1)=m+b=-1$ …(1)

$f(2)=2m+b=2$ …(2).

Solving these equations, we get

$(2)-(1)$ $m=3.$

Substituting into …(1)

$3+b=-1$

$b=-4.$

Therefore, the function is $f(x)=3x-4.$

3. $(1,4),\ (2,15),\ (3,30),\ (4,49),\ (5,72),\ (6,99)$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	4	$15-4=11$	$15-11=4$
2	15	$30-15=15$	$19-15=4$
3	30	$49-30=19$	$23-19=4$
4	49	$72-49=23$	$27-23=4$
5	72	$99-72=27$	
6	99		

Since $\Delta^2 f(x)$ is constant, the function is of the form $f(x)=ax^2+bx+c.$

Substituting the given ordered pairs,

$f(1)=a+b+c=4$ …(1)

$f(2)=4a+2b+c=15$ …(2)

$f(3)=9a+3b+c=30$ …(3).

Solving these equations, we have

$(2)-(1)$ $3a+b=11$ …(4)

$(3)-(2)$ $5a+b=15$ …(5)

$(5)-(4)$ $2a=4$

$a=2.$

Substituting into (5) $5(2)+b=15$

$b=5.$

Substituting into (1) $2+5+c=4$

$c=-3.$

Therefore, the function is $f(x)=2x^2+5x-3.$

4. (1, –9), (2, –10), (3, –7), (4, 0), (5, 11), (6, 26)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	– 9	– 10 – (– 9) = 1	3 – (– 1) = 4
2	– 10	– 7 – (– 10) = 3	7 – 3 = 4
3	– 7	0 – (– 7) = 7	11 – 7 = 4
4	0	11 – 0 = 11	15 – 11 = 4
5	11	26 – 11 = 15	
6	26		

Since $\Delta^2 f(x)$ is constant, the function is of the form $f(x)=ax^2+bx+c$.

Substituting the given ordered pairs,

$f(1)=a+b+c=0$ …(1)

$f(2)=4a+2b+c=-2$ …(2)

$f(3)=9a+3b+c=-2$ …(3).

Solving these equations,

$(2)-(1)$ $3a+b=-2$ …(4)

$(3)-(2)$ $5a+b=0$ …(5)

$(5)-(4)$ $2a=2$

$a=1.$

Substituting into (4), $3(1)+b=-2$

$b=-5.$

Substituting into (1), $\quad (1)+(-5)+c=0$

$$c=4.$$

Therefore, the function is $f(x)=x^2-5x+4.$

5. $(1,\ 12),\ (2,\ -10),\ (3,\ -18),\ (4,\ 0),\ (5,\ 56),\ (6,\ 162)$

x	***f(x)***	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	12	$(-10)-(-12)=-22$	$-8-(-22)=14$	$26-14=12$
2	–10	$(-18)-(-10)=-8$	$18-(-8)=26$	$38-26=12$
3	–18	$0-(-18)=18$	$56-18=38$	$50-38=12$
4	0	$56-0=56$	$106-56=50$	
5	56	$162-56=106$		
6	162			

Since Δf(x) is constant, the function is of the form $f(x)=ax^3+bx^2+cx+d.$

Substituting the given ordered pairs,

$f(1)=a+b+c+d=12 \quad \ldots(1)$

$f(2)=8a+4b+2c+d=-10 \quad \ldots(2)$

$f(3)=27a+9b+3c+d=-18 \quad \ldots(3)$

$f(4)=64a+16b+4c+d=0 \quad \ldots(4).$

Solving the equations,

$(2)-(1) \quad 7a+3b+c=-22 \quad \ldots(5)$

$(3)-(2) \quad 19a+5b+c=-8 \quad \ldots(6)$

$(4)-(3) \quad 37a+7b+c=18 \quad \ldots(7)$

$(6)-(5) \quad 12a+2b=14 \quad \ldots(8)$

$(7)-(6) \quad 18a+2b=26 \quad \ldots(9)$

$(9)-(8) \quad 6a=12$

$$a=2.$$

Substituting into (8), $\quad 12(2)+2b=14$

$$b=-5.$$

Substituting into (5), $\quad 7(2)+3(-5)+c=-23$

$$c=-21.$$

Substituting into (1), $\quad 2-5-21+d=12$

$$d=36.$$

Therefore, the function is
$f(x)=2x^3-5x^2-21x+36.$

6. $(1,-34),\ (2,-42),\ (3,-38),\ (4,-16),\ (5,30),\ (6,106)$
Using differences we obtain the following.

x	***f(x)***	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	–34	–8	12	6
2	–42	4	18	6
3	–38	22	24	6
4	–16	46	30	⋮
5	30	76	⋮	⋮
6	106	⋮	⋮	⋮

Since is $\Delta^3 f(x)$ is constant, the function is of the form $f(x) = ax^3 + bx^2 + cx + d.$

Substituting the given ordered pairs,

$f(1)=a+b+c+d=-34 \quad \ldots(1)$

$f(2)=8a+4b+2c+d=-42 \quad \ldots(2)$

$f(3)=27a+9b+3c+d=-38 \quad \ldots(3)$

$f(4)=64a+16b+4c+d=-16 \quad \ldots(4).$

Solving the equations,

$(2)-(1) \quad 7a+3b+c=-8 \quad \ldots(5)$

$(3)-(2) \quad 19a+5b+c=4 \quad \ldots(6)$

$(4)-(3) \quad 37a+7b+c=22 \quad \ldots(7)$

$(6)-(5) \quad 12a+2b=12 \quad \ldots(8)$

$(7)-(6) \quad 18a+2b=18 \quad \ldots(9)$

$(9)-(8) \quad 6a=6$

$$a=1.$$

Substituting into (8), $\quad 12(1)+2b=12$

$$b=0.$$

Substituting into (5), $\quad 7(1)+3(0)+c=-18$

$$c=-15.$$

Substituting into (1), $\quad 1+0-15+d=-34$

$$d=-20.$$

Therefore, the function is $f(x)=x^3-15x-20.$

7. $(1,10),\ (2,0),\ (3,0),\ (4,16),\ (5,54),\ (6,120),\ (7,220)$
Using differences, we obtain the following.

x	*f(x)*	*Δf(x)*	*Δ²f(x)*	*Δ³f(x)*
1	10	– 10	10	6
2	0	0	16	6
3	0	16	22	6
4	16	38	28	
5	54	66	34	
6	120	100		
7	220			

Since $\Delta^3 f(x)$ is constant, the function is of the form $f(x) = ax^3 + bx^2 + cx + d$.

Substituting the given ordered pairs,

$f(1) = a + b + c + d = 10$ …(1)

$f(2) = 8a + 4b + 2c + d = 0$ …(2)

$f(3) = 27a + 9b + 3c + d = 0$ …(3)

$f(4) = 64a + 16b + 4c + d = 16$ …(4).

Solving the equations,

$(2) - (1)$ $7a + 3b + c = -10$ …(5)

$(3) - (2)$ $19a + 5b + c = 0$ …(6)

$(4) - (3)$ $37a + 7b + c = 16$ …(7)

$(6) - (5)$ $12a + 2b = 10$ …(8)

$(7) - (6)$ $18a + 2b = 16$ …(9)

$(9) - (8)$ $6a = 6$

$a = 1.$

Substituting into (8), $12(1) + 2b = 10$

$b = -1.$

Substituting into (5), $7(1) + 3(-1) + c = -10$

$c = -14.$

Substituting into (1), $1 - 1 - 14 + d = 10$

$d = 24.$

Therefore, the function is $f(x) = x^3 - x^2 - 14x + 24$.

8. $(1,\ -4),\ (2,\ 0),\ (3,\ 30),\ (4,\ 98),\ (5,\ 216),\ (6,\ 396)$
Using differences, we obtain the following.

x	*f(x)*	*Δf(x)*	*Δ²f(x)*	*Δ³f(x)*
1	– 4	4	26	12
2	0	30	38	12
3	30	68	50	12
4	98	118	62	
5	216	180		
6	396			

Since $\Delta^3 f(x)$ is constant, the function is of the form $f(x) = ax^3 + bx^2 + cx + d$.

Substituting the given ordered pairs,

$f(1) = a + b + c + d = -4$ …(1)

$f(2) = 8a + 4b + 2c + d = 0$ …(2)

$f(3) = 27a + 9b + 3c + d = 30$ …(3)

$f(4) = 64a + 16b + 4c + d = 98$ …(4).

Solving the equations,

$(2) - (1)$ $7a + 3b + c = 4$. …(5)

$(3) - (2)$ $19a + 5b + c = 30$ …(6)

$(4) - (3)$ $37a + 7b + c = 68$ …(7)

$(6) - (5)$ $12a + 2b = 26$ …(8)

$(7) - (6)$ $18a + 2b = 38$ …(9)

$(9) - (8)$ $6a = 12$

$a = 2.$

Substituting into (8), $12(2) + 2b = 26$

$b = 1.$

Substituting into (5), $7(2) + 3(1) + c = 4$

$c = -13.$

Substituting into (1), $2 + 1 - 13 + d = -4$

$d = 6.$

Therefore, the function is $f(x) = 2x^3 + x^2 - 13x + 6.$

9. $(1,-2),\ (2,-4),\ (3,-6),\ (4,-8),\ (5,14),\ (6,108),\ (7,346)$
Using differences, we obtain the following:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	– 2	– 2	0	0	24
2	– 4	– 2	0	24	24
3	– 6	– 2	24	48	24
4	– 8	22	72	72	
5	14	94	144		
6	108	238			
7	346				

Since $\Delta^4 f(x)$ is constant, the function is of the form $f(x) = ax^4 + bx^3 + dx + e$.

Substituting the given ordered pairs,

$f(1) = a + b + c + d + e = -2$ …(1)
$f(2) = 16a + 8b + 4c + 2d + e = -4$ …(2)
$f(3) = 81a + 27b + 9c + 3d + e = -6$ …(3)
$f(4) = 256a + 64b + 16c + 4d + e = -8$ …(4).

Solving the equations,

$(2)-(1)$ $15a + 7b + 3c + d = -2$ …(6)
$(3)-(2)$ $65a + 19b + 5c + d = -2$ …(7)
$(4)-(3)$ $175a + 37b + 7c + d = -2$ …(8)
$(5)-(4)$ $369a + 61b + 9c + d = 22$ …(9)
$(7)-(6)$ $50a + 12b + 2c = 0$ …(10)
$(8)-(7)$ $110a + 18b + 2c = 0$ …(11)
$(9)-(8)$ $194a + 24b + 2c = 24$ …(12)
$(11)-(10)$ $60a + 6b = 0$ …(13)
$(12)-(11)$ $84a + 6b = 24$ …(14)
$(14)-(13)$ $24a = 24$
$a = 1.$

Substituting into (13), $60(1) + 6b = 0$
$b = -10.$

Substituting into (10), $50(1) + 12(-10) + 2c = 0$
$c = 35.$

Substituting into (6), $15(1) + 7(-10) + 3(35) + d = -2$
$d = -52.$

Substituting into (1), $1 - 10 + 35 - 52 + e = -2$
$e = 24.$

Therefore, the function is
$f(x) = x^4 - 10x^3 + 35x^2 - 52x + 24.$

10. $(1,1),\ (2,2),\ (3,4),\ (4,8),\ (5,16),\ (6,32),\ (7,64)$
Using differences, we obtain the following:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1	1	1	1	1	1	1
2	2	2	2	2	2	2
3	4	4	4	4	4	4
4	8	8	8	8	8	
5	16	16	16	16		
6	32	32	32			
7	64					

As there is no constant difference, this will not be defined as a polynomial function. This is $f(x) = 2^{n-1}$, by inspection.

11. **a.** Using the $\boxed{\text{STAT}}$ function, the function is
$V = -0.0374x^3 + 0.1522x^2 + 0.1729x.$

b. The maximum volume of air during the cycle is 0.8863 and occurs after 3.2.

12. **a.**

t	$f(t)$	$\Delta f(t)$	$\Delta^2 f(t)$	$\Delta^3 f(t)$
1	4031	–23	–48	6
2	4008	–71	–42	6
3	3937	–113	–36	
4	3824	–149		
5	3675	–179		
6	3496			

Since the third differences are constant, it forms a cubic function. Using the $\boxed{\text{STAT}}$ mode on the graphing calculator, $f(t) = t^3 - 30t^2 + 60t + 4000.$

b. From the graph of $f(t)$, it seems that the population began to increase 9 years ago, in 1971.

c. For the year 2030, $t = 50$.

$$f(50) = 50^3 - 30(50)^2 + 60(50) + 4000$$
$$= 57\ 000$$

So, if the function continues to describe the population after 2002, in the year 2030, it will be about 57 000.

Exercise 1.3

7. **b.**

$$
\begin{array}{r}
x^2+5x+2 \\
x-1\overline{)x^3+4x^2-3x-2} \\
\underline{x^3-x^2} \qquad\qquad \\
5x^2-3x \qquad \\
\underline{5x^2-5x} \qquad \\
2x-2 \\
\underline{2x-2} \\
0
\end{array}
$$

Since the remainder is 0, $x-1$ is a factor of x^3+4x^2-3x-2. The other factor is x^2+5x+2.
$x^3+4x^2-3x-2 = (x-1)(x^2+5x+2)$

c.

$$
\begin{array}{r}
2x^2+2x+3 \\
x-3\overline{)2x^3-4x^2-3x+5} \\
\underline{2x^3-6x^2} \qquad\qquad \\
2x^2-3x \qquad \\
\underline{2x^2-6x} \qquad \\
3x+5 \\
\underline{3x-9} \\
14
\end{array}
$$

Since the remainder $r(x)=14$ is of a degree less than that of the divisor, the division is complete. So, $2x^3-4x^2-3x+5 = (x-3)(2x^2+2x+3)+14$.

g.

$$
\begin{array}{r}
2x^2-3 \\
2x+3\overline{)4x^3+6x^2-6x-9} \\
\underline{4x^3+6x^2} \qquad\qquad \\
-6x-9 \\
\underline{-6x-9} \\
0
\end{array}
$$

Since the remainder is 0, 2*x* + 3 is a factor of $4x^3+6x^2-6x-9$. Therefore, $4x^3+6x^2-6x-9 = (2x+3)(2x^2-3)$.

h.

$$
\begin{array}{r}
x^2-3x+5 \\
3x-2\overline{)3x^3-11x^2+21x-7} \\
\underline{3x^3-2x^2} \qquad\qquad\qquad \\
-9x^2+21x \qquad \\
\underline{-9x^2+6x} \qquad \\
15x-7 \\
\underline{15x-10} \\
+3
\end{array}
$$

Since the remainder, $r(x)=3$ is of a degree less than that of the divisor, the division is complete. So, $3x^3-11x^2+21x-7 = (3x-2)(x^2-3x+5)+3$.

9. **b.**

$$
\begin{array}{r}
2x^3-2x^2-x+1 \\
x+1\overline{)2x^4+0x-3x^2+1} \\
\underline{2x^4+2x^3} \qquad\qquad\qquad \\
-2x^3-3x^2 \qquad\qquad \\
\underline{-2x^3-2x^2} \qquad\qquad \\
-x^2 \qquad\quad \\
\underline{-x^2-x} \qquad \\
x+1 \\
\underline{x+1} \\
0
\end{array}
$$

c.

$$\begin{array}{r}
4x^2 - 8x + 16 \\
x+2\overline{)4x^3 + 0x^2 + 0x + 3^2 + 32} \\
\underline{4x^3 + 8x^2} \qquad\qquad\qquad\quad \\
-8x^2 \qquad\qquad\qquad \\
\underline{-8x^2 - 16x} \qquad\qquad \\
16x + 32 \\
\underline{16x + 32} \\
0
\end{array}$$

d.

$$\begin{array}{r}
x^4 + x^3 + x^2 + x + 1 \\
x-1\overline{)x^5 + 0x^4 + 0x^3 + 0x^2 + 0x^1 + 0x - 1} \\
\underline{x^5 - x^4} \qquad\qquad\qquad\qquad\qquad\qquad \\
x^4 \qquad\qquad\qquad\qquad\qquad \\
\underline{x^4 - x^3} \qquad\qquad\qquad\qquad \\
x^3 \qquad\qquad\qquad\quad \\
\underline{x^3 - x^2} \qquad\qquad\qquad \\
x^2 \qquad\qquad \\
\underline{x^2 - x} \qquad\quad \\
x - 1 \\
\underline{x - 1} \\
0
\end{array}$$

12. Dividing *f(x)* by *d(x)*.

$$\begin{array}{r}
x^2 - x \\
x^2 + 2x + 1\overline{)x^4 + x^3 - x^2 - x} \\
\underline{x^4 + 2x^3 + x^2} \qquad \\
-x^3 - 2x^2 - x \\
\underline{-x^3 - 2x^2 - x} \\
0
\end{array}$$

16. $x = yq + r$ where $y \le x$ and $x, y \in N$

a. If y is a factor of x, it will divide into x without leaving a remainder. So, $r = 0$.

b. The value of the remainder must be less than that of the divisor if the division is complete, and y is not a factor of $9x$, so if $y = 5$, the values of r are 1, 2, 3, or 4. If $y = 7$, $r = 1, 2, 3, 4, 5, 6$, and if $r = n$, $r = 1, 2, 3, \ldots, n - 1$.

17. a.

$$\begin{array}{r}
x^2 + 6x + 7 \\
x-2\overline{)x^3 + 4x^2 - 5x - 9} \\
\underline{x^3 - 2x^2} \qquad\qquad \\
6x^2 - 5x \qquad \\
\underline{6x^2 - 12x} \qquad \\
7x - 9 \\
\underline{7x - 14} \\
5
\end{array}$$

So, $x^3 + 4x^2 - 5x - 9 = (x-2)(x^2 + 6x + 7) + 5$ where $q(x) = x^2 + 6x + 7$ and $r = 5$.

$$\begin{array}{r}
x + 5 \\
x+1\overline{)x^2 + 6x + 7} \\
\underline{x^2 + x} \qquad \\
5x + 7 \\
\underline{5x + 5} \\
2
\end{array}$$

So, $x^2 + 6x + 7 = (x+1)(x+5) + 2$, where $Q(x) = x + 5$ and $r_2 = 2$.

b. If $f(x)$ is divided by $(x-2)(x+1)$, the quotient is the $Q(x)$ obtained in **a.** Since

$x^3 + 4x^2 - 5x - 9 = (x-2)(x^2 + 6x + 7) = 5,$

by substituting,

$= (x-2)[(x+1)(x+5) + 2] + 5$

$= (x-2)[(x+1)(x+5)] + (x-2)[(2)] + 5$

and simplifying,

$= (x-2)(x+1)(x+5) + 2(x-2) + 5$

$= (x-2)(x+1)(x+5) + 2x + 1.$

Therefore, when *f*(*x*) is divided by $(x-2)(x+1)$, the quotient is $(x+5)$ and the remainder is $2\,(x-2) + 5$ or $2x + 1$.

Exercise 1.4

2. **b.** When $f(x)$ is divided by $x+1$, the remainder is $f(-1)$.

$$\begin{aligned} r &= f(-1) \\ &= (-1)^3 - 4(-1)^2 + 2(-1) - 6 \\ &= -13 \end{aligned}$$

c. When $f(x)$ is divided by $2x-1$, the remainder is $f\left(\frac{1}{2}\right)$.

$$\begin{aligned} r &= f\left(\frac{1}{2}\right) \\ &= \left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 6 \\ &= -\frac{47}{8} \text{ or } -5.875 \end{aligned}$$

d. When $f(x)$ is divided by $2x+3$, the remainder is $f\left(-\frac{3}{2}\right)$.

$$\begin{aligned} r &= f\left(-\frac{3}{2}\right) \\ &= \left(-\frac{3}{2}\right)^3 - 4\left(-\frac{3}{2}\right)^2 + 2\left(-\frac{3}{2}\right) - 6 \\ &= -\frac{171}{8} \text{ or } -21.375 \end{aligned}$$

3. **c.** Let $f(x) = 2x^3 + 4x - 1$.

The remainder when is divided by $x+2$ is

$$\begin{aligned} r &= f(-2) \\ &= 2(-2)^3 + 4(-2) - 1 \\ &= -25. \end{aligned}$$

f. Let $f(x) = -2x^4 + 3x^2 - x + 2$.

When $f(x)$ is divided by $x+2$, the remainder is

$$\begin{aligned} r &= f(-2) \\ &= -2(-2)^4 + 3(-2)^2 - (-2) + 2 \\ &= -2(16) + 3(4) + 2 + 2 \\ &= -16. \end{aligned}$$

4. **f.** The remainder is

$$\begin{aligned} r &= f\left(\frac{1}{2}\right) \\ &= 4\left(\frac{1}{2}\right)^3 + 9\left(\frac{1}{2}\right) - 10 \\ &= -5. \end{aligned}$$

5. **a.** Since the remainder is 1 when the divisor is $x+2$, then $f(-2) = 1$ by the Remainder Theorem.

$$\begin{aligned} (-2)^3 + k(-2)^2 + 2(-2) - 3 &= 1 \\ -8 + 4k - 4 - 3 &= 1 \\ 4k &= 16 \end{aligned}$$

b. Since the remainder is 16 when the divisor is $x-3$, then $f(3) = 16$ by the Remainder Theorem.

$$\begin{aligned} (3)^4 - k(3)^3 - 2(3)^2 + (3) + 4 &= 16 \\ -27k &= -54 \\ k &= 2 \end{aligned}$$

c. Since the remainder is 1 when the divisor is $2x-1$, then

$f\left(\frac{1}{2}\right) = 1$ by the Remainder Theorem.

$$\begin{aligned} 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + k\left(\frac{1}{2}\right) - 1 &= 1 \\ \frac{1}{4} - \frac{3}{4} + \frac{1}{2}k - 1 &= 1 \\ \frac{1}{2}k &= \frac{5}{2} \\ k &= 5 \end{aligned}$$

6. $f(x) = mx^3 + gx^2 - x + 3$

When the divisor is $x+1$, the remainder is 3.

By the Remainder Theorem, $f(-1) = 3$

$$\begin{aligned} m(-1)^3 + g(-1)^2 - (-1) + 3 &= 3 \\ -m + g &= -1. \qquad (1) \end{aligned}$$

When the divisor is $x+2$, the remainder is -7.

Therefore, $f(-2) = -7$

$$\begin{aligned} m(-2)^3 + g(-2)^2 - (-2) + 3 &= -7 \\ -8m + 4g &= -12 \\ \text{or} \qquad 2m - g &= 3. \qquad (2) \end{aligned}$$

Solving the resulting linear equations,

$(2)+(1) \qquad m = 2.$

Substituting into (1), $\qquad g = 1.$

7. $f(x) = mx^3 + gx^2 - x + 3$

When the divisor is $x - 1$, the remainder is 3.

By the Remainder Theorem, $f(1) = 3$.

$$\begin{aligned} m + g - 1 + 3 &= 3 \\ m + g &= 1 \qquad (1) \end{aligned}$$

When the divisor is $x + 3$, the remainder is –1.

So, $f(-3) = -1$.

$$\begin{aligned} m(-3)^3 + g(-3)^2 - (-3) + 3 &= -1 \\ -27m + 9g &= -7 \qquad (2) \end{aligned}$$

$$9\times(1) \qquad \begin{aligned} \underline{9m + 9g = 9} \\ -36m = -16 \\ m = \frac{4}{9} \end{aligned}$$

Substituting into (1), $\qquad g = \frac{5}{9}.$

8. *Solution 1: Using the Remainder Theorem*

Let $f(x) = x^3 + 3x^2 - x - 2. \qquad (1)$

Then, $f(x) = (x+3)(x+5)q(x) + r(x)$

where x is a linear expression.

Let $r(x) = Ax + B.$

So, $f(x) = (x+3)(x+5)q(x) + (Ax + B). \qquad (2)$

From (2), $\quad f(-3) = (0)(2)q(x) + (-3A + B)$
$= -3A + B.$

From (1), $\quad f(-3) = (-3)^3 + 3(-3)^2 - (-3) - 2$
$= 1.$

Therefore, $-3A + B = 1. \qquad (3)$

Similarly, $f(-5) = (-2)(0)q(x) + (-5A + B)$
$= -5A + B$

and $\qquad f(-5) = (-5)^3 + 3(-5)^2 - (-5) - 2$
$= -47.$

Therefore, $-5A + B = -47. \qquad (4)$

Solving (3) and (4),

$$(3)-(4) \qquad \begin{aligned} 2A &= 48 \\ A &= 24. \end{aligned}$$

Substituting in (4), B = 73.

Since $r(x) = Ax + B$
$= 24x + 73.$

The remainder is $24 + 73.$

Solution 2: Using Long Division

Expanding $(x+3)(x+5) = x^2 + 8x + 15$

$$\begin{array}{r}
x-5\\
x^2+8x+15\overline{)\,x^3+3x^2-x-2}\\
\underline{x^3+8x+15x}\\
-5x^2-16x-2\\
\underline{-5x^2-40x-75}\\
24x+73
\end{array}$$

The remainder is $24x + 73.$

9. *Solution 1: Using Long Division*

Expanding the divisor $(x-1)(x+2) = x^2 + x - 2$

$$\begin{array}{r}
3x^3-3x^2+9x-20\\
x^2+x-2\,\overline{)\,3x^5-5x^2+4x+1}\\
\underline{3x^5+3x^4-6x^3}\\
-3x^4+6x^3-5x^2\\
\underline{-3x^4-3x^3+6x^2}\\
9x^3-11x^2+4x\\
\underline{9x^3+9x^2-18x}\\
-20x^2+22x+1\\
\underline{-20x^2-20x+40}\\
42x-39
\end{array}$$

The remainder is $42x - 39$.

Solution 2: Using the Remainder Theorem

Let $f(x) = 3x^5 - 5x^2 + 4x + 1.$ $\quad(1)$

So, $f(x) = (x-1)(x+2)q(x) + Ax + B$ $\quad(2)$

since $r(x)$ is at most a linear expression.

Since $f(1) = 0(3)q(x) + A + B$ from (1)

and $f(1) = 3(1)^5 - 5(1)^2 + 4(1) + 1$ from (2)

$= 3.$

So $A + B = 3.$ (3)

Similarly, $f(-2) = (-3)(0)q(x) + A(-2) + B$ from (1)

and $f(-2) = 3(-2)^5 - 5(-2)^2 + 4(-2) + 1$ from (2)

So $-2A + B = -123.$

Solving (3) and (4) by subtracting,

and $3A = 126$

$A = 42$

$B = -39.$

The remainder is $42x - 39.$

10. If the remainder is 3 when $x + 2$ is divided into $f(x)$, then $f(-2) = 3$.

a. Since the remainder is a constant, adding 1 to $f(x)$, increases the remainder by 1. So, the remainder is 3 + 1 = 4.

b. Since $(x + 2)$ is divisible exactly by the divisor $x + 2$, there is no remainder for that division. So, the remainder for $f(x) + x + 2$ is the same as that for $f(x)$, i.e., the remainder is 3.

c. The remainder of $f(x)$ divided by $x + 2$ is 3. By the Remainder Theorem, the remainder of $(4x + 7)$ divided by $x + 2$ is $4(-2)+7=-1$.
Therefore, the remainder of $f(x) + 4x + 7$ is the remainder of $f(x)$ plus the remainder of $4x + 7$, that is, $3 - 1 = 2.$

d. The remainder of $f(x)$ divided by $x + 2$ is 3. Hence, the remainder of $2f(x)$ divided by $x + 2$ is $2(3) = 6.$ The remainder of -7 divided by $x + 2$ is -7. So, the remainder of $2f(x) - 7$ is $6 - 7 = -1$.

e. If $[f(x)]^2$ is divided by $x + 2$, the division statement becomes $f(x)f(x) = (x+2)q(x) + r.$

Let $x = -2,$ then $f(-2)f(-2) = 0q(x) + r$

$(3)(3) = r$

$9 = r.$

The remainder is 9.

11. In order to have a multiple of $(x + 5)$, there must be no remainder after division by $x + 5$. The remainder for $f(x)$ is $x + 3$. The first multiple for the remainder is $x + 5$, or $(x + 3) + 2$. So, the first multiple greater than $f(x)$ is $f(x) + 2$.

12. Factoring by completing a square:

a.
$$\begin{aligned} & x^4 + 5x + 9 \\ &= x^4 + 6x^2 + 9 - x^2 \\ &= (x^2 + 3)^2 - x^2 \\ &= (x^2 + 3 + x)(x^2 + 3 - x) \\ &= (x^2 + x + 3)(x^2 - x - 3) \end{aligned}$$

b.
$$\begin{aligned} & 9y^4 + 8y^2 + 4 \\ &= 9y^4 + 12y^2 + 4 - 4y^2 \\ &= (3y^2 + 2)^2 - 4y^2 \\ &= (3y^2 + 2y + 2)\,(3y^2 - 2y + 2) \end{aligned}$$

c.
$$\begin{aligned} & x^4 + 6x^2 + 25 \\ &= x^4 + 10x^2 + 25 - 4x^2 \\ &= (x^2 + 5)^2 - 4x^2 \\ &= (x^2 + 2x + 5)(x^2 - 2x + 5) \end{aligned}$$

d.
$$\begin{aligned} & 4x^4 + 8x^2 + 9 \\ &= 4x^4 + 12x^2 + 9 - 4x^2 \\ &= (2x^2 + 3)^2 - 4x^2 \\ &= (2x^2 + 2x + 3)(2x^2 - 2x + 3). \end{aligned}$$

Review Exercise

2. a.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
–1	–27	16	– 10	6
0	–11	6	– 4	6
1	–5	2	2	6
2	–3	4	8	
3	1	12		
4	13			

Since $\Delta^3 f(x)$ is constant, $f(x)$ is of the form $f(x) = ax^3 + bx^2 = cx + d.$

Substituting the given ordered pairs,

$f(0) = d = -11$

$f(1) = a + b + c + d = -5.$

Substituting for d, $a + b + c = 6$ (1)

$f(2) = 8a + 4b + 2c + d = -3$

$8a + 4b + 2c = 8$

$4a + 2b + c = 4$ (2)

$f(3) = 27a + 9b + 3c + d = 1$

$27a + 9b + 3c = 12$

$9a + 3b + c = 4$ (3)

Solving,

(2) – (1) $3a + b = -2$ (4)

(3) – (2) $5a + b = 0$ (5)

(5) – (4) $-2a = -2$

$a = 1.$

Substituting into (5), $b = -5$.
Substituting into (1), $c = 10$.

Therefore, the function is
$f(x) = x^3 - 5x^2 + 10x - 11.$

b.

x	***f(x)***	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	4	11	6	12
1	15	17	18	12
2	32	35	30	12
3	67	65	42	
4	132	107		
5	239			

Since is $\Delta^3 f(x)$ constant, $f(x)$ is of the form
$f(x) = ax^3 + bx^2 + cx + d.$

Substituting the given ordered pairs,

$f(0) = d = 4$

$f(1) = a + b + c + 4 = 15$

$a + b + c = 11$ (1)

$f(2) = 8a + 4b + 2c + 4 = 32$

$8a + 4b + 2c = 28$

$4a + 2b + c = 14$ (2)

$f(3) = 27a + 9b + 3c + 4 = 67$

$27a + 9b + 3c = 63$

$9a + 3b + c = 21$ (3)

Solving

(2) – (1) $3a + b = 3$ (4)

(3) – (2) $5a + b = 7$ (5)

(5) – (4) $2a = 4$

$a = 2$

Substituting into (5), $b = -3$.
Substituting into (1), $c = 12$.
Therefore, the function is $f(x) = 2x^3 - 3x^2 + 12x + 4.$

c.

x	***f(x)***	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	– 9	– 22	22	60	24
2	– 31	0	82	84	24
3	– 31	82	166	108	
4	51	248	274		
5	299	522			
6	821				

Since $\Delta^4 f(x)$ is constant,
$f(x) = ax^4 + bx^3 + cx^2 + dx + e.$
Substituting the given ordered pairs,
$f(1) = a + b + c + d + e = -9$ (1)
$f(2) = 16a + 8b + 4c + 2d + 2 = -31$ (2)
$f(3) = 81a + 27b + 9c + 3d + e = -31$ (3)
$f(4) = 256a + 64b + 16c + 4d + e = 51$ (4)
$f(5) = 625a + 125b + 25c + 5d + e = 299$ (5)

Solving the equations,

(2) – (1)	$15a + 7b + 3c + d = -22$	(6)
(3) – (2)	$65a + 19b + 5c + d = 0$	(7)
(4) – (3)	$175a + 37b + 7c + d = 82$	(8)
(5) – (4)	$369a + 61b + 9c + d = 228$	(9)
(7) – (6)	$50a + 12b + 2c = 22$	(10)
(8) – (7)	$110a + 18b + 2c = 0$	(11)
(9) – (8)	$194a + 24b + 2c = 166$	(12)
(11) – (10)	$60a + 6b = 60$	(13)
(12) – (11)	$84a + 6b = 84$	(14)
(14) – (13)	$24a = 24$	
	$a = 1.$	

Substituting, $b = 0$

$c = -14$

$d = 5$

$e = -1.$

Therefore, $f(x) = x^4 - 14x^2 + 5x - 1$.

d.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	1	1	2	6
2	2	3	8	
3	5	11		
4	16			

There is not enough information to find a constant finite difference.

e.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
–2	75	–86	76	–72
–1	–11	–10	4	24
0	–21	–6	–20	
1	–27	–26		
2	–53			

There is not enough information to establish the function.

3. c.

$$\begin{array}{r}
2x^2+3x-2 \\
2x+1\overline{\big)4x^3+8x^2-x+1} \\
\underline{4x^3+2x^2} \\
6x^2-x \\
\underline{6x^2+3x} \\
-4x+1 \\
\underline{-4x-2} \\
3
\end{array}$$

$$4x^3+8x^2-x+1=(2x+1)(2x^2+3-2)+3$$

d.

$$\begin{array}{r}
x^2-5x+10 \\
x^2+x-2\overline{\big)x^4-4x^3+3x^2-3} \\
\underline{x^4+x^3-2x^2} \\
-5x^3+5x^2 \\
\underline{-5x^3-5x^2+10x} \\
10x^2-10x-3 \\
\underline{10x^2+10x-20} \\
-20x+17
\end{array}$$

$$x^4-4x^3+3x^2-3=(x^2+x-2)(x^2-5x+10)-20x+17$$

4. c. Let $f(x)=x^3-5x^2+2x-1$.

The remainder is

$$\begin{aligned} f(-2) &= (-2)^3-5(-2)^2+2(-2)-1 \\ &= -8-20-4-1 \\ &= -33. \end{aligned}$$

e. Let $f(x)=3x^3+x+2.$

The remainder is $f\left(\frac{1}{3}\right)=3\left(\frac{1}{3}\right)^3+\frac{1}{3}+2$

$$=\frac{22}{9}.$$

5. a.

$$\begin{array}{r}
x^2+3x+2 \\
x-1\overline{\big)x^3+2x^2-x-2} \\
\underline{x^3-x^2} \\
3x^2-x \\
\underline{3x^2-3x} \\
2x-2 \\
\underline{2x-2} \\
0
\end{array}$$

$$\begin{aligned} x^3+2x^2-x-2 &= (x-1)(x^2+3x+2) \\ &= (x-1)(x+1)(x+2) \end{aligned}$$

c.

$$\begin{array}{rl}
 & 3x^2+11x-4 \\
2x+3 & \overline{\smash{)}6x^3+31x^2+25x-12} \\
 & \underline{6x^3+\ 9x^2} \\
 & 22x+25x \\
 & \underline{22x+33x} \\
 & -8x-12 \\
 & \underline{-8x-12} \\
 & 0
\end{array}$$

$$\begin{aligned}6x^3+31x^2+25x-12 &= (2x+3)\left(3x^2+(x-4)\right)\\ &= (2x+3)(3x-1)(x+4)\end{aligned}$$

6. **a.** Let $f(x)=x^3-3kx^2+x+5.$
When the divisor is x – 2, the remainder is f(2) = 9.

$$\begin{aligned}(2)^3-3k(2)^2+2+5&=9\\ 8-12k+2+5&=9\\ k&=\frac{1}{2}\quad\text{or}\quad 0.5\end{aligned}$$

b. Let $f(x)=rx^3+gx^2+4x+1.$
When the divisor is x – 1, the remainder is f(1) = 12.

$$\begin{aligned}r(1)^3+g(1)^2+4(1)+1&=12\\ r+g&=7 \qquad (1)\end{aligned}$$

When the divisor is x + 3, the remainder is f(3–) = –20.

$$\begin{aligned}r-3^3+g(-3)^2+4(-3)+1&=-20\\ -27r+9g&=-9\\ 3r-g&=1 \qquad (2)\end{aligned}$$

Solving $(1)+(2)$, $\quad 4r=8$

$r=2.$

Substituting into (1), $g=5.$

Chapter 1 Test

1. **a.**

$$\begin{aligned}&18x^2-50y^2\\ &=2\left(9x^2-25y^2\right)\\ &=2(3x-5y)(3x+5y)\end{aligned}$$

b.

$$\begin{aligned}&pm^3+m^2+pm+1\\ &=m^2(pm+1)+(pm+1)\\ &=(pm+1)\left(m^2+1\right)\end{aligned}$$

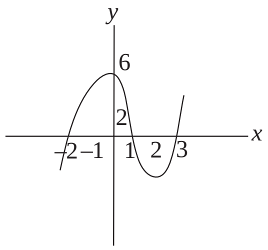
c.

$$\begin{aligned}&12x^2-26x+12\\ &=2\left(6x^2-13x+6\right)\\ &=2(3x-2)(2x-3)\end{aligned}$$

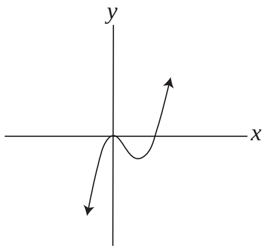
d.

$$\begin{aligned}&x^2+6y-y^2-9\\ &=x^2-\left(y^2-6y+9\right)\\ &=x^2-(y-3)^2\\ &=(x+y-3)(x-y+3)\end{aligned}$$

2. **a.** $y=(x+2)(x-1)(x-3)$
The x-intercepts are –2, 1, and 3. The y-intercept is 6.



b. $y=x^2(x-2)$
The x-intercepts are 0 and 2.
The y-intercept is 0.



3. **a.**

$$\begin{array}{r} x^2-7x+20 \\ x+2\overline{\big)x^3-5x^2+6x-4} \\ \underline{x^3+2x^2} \qquad\qquad \\ -7x^2+6x \qquad \\ \underline{-7x^2-14x} \qquad \\ 20x-4 \\ \underline{20x+40} \\ -44 \end{array}$$

The quotient is $q(x)=x^2-7x+20.$

The remainder is $r(x)=-44.$

b.

$$\begin{array}{r} x^2+3x+3 \\ x-3\overline{\big)x^3-6x+2} \\ \underline{x^3-3x^2} \qquad\qquad \\ 3x^2-6x \qquad \\ \underline{3x^2-9x} \qquad \\ 3x+2 \\ \underline{3x-9} \\ 11 \end{array}$$

The quotient is $q(x)=x^2+3x+3.$
The remainder is 11.

4. Since when $f(x)$ is divided by $(x-1)$, $f(1)$ is the remainder and $f(1)=0$, then the remainder is 0. When the remainder is 0, the divisor $(x-1)$ is a factor.

5. Let $f(x)=x^3-6x^2+5x+2.$ When dividing by $(x+2)$, the remainder is $f(-2).$

$$\begin{aligned} r &= f(-2) \\ &= (-2)^3-6(-2)^2+5(-2)+2 \\ &= -40 \end{aligned}$$

6. Let $f(x)=x^3-3x^2+4x+k.$
When $f(x)$ is divided by $(x-2)$, the remainder is $f(2)$.

$$\begin{aligned} f(2) &= 7 \\ (2)^3-3(2)^2+4(2)+k &= 7 \\ k &= 3 \end{aligned}$$

7. **a.**

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
1	–1	0	2
2	–1	2	2
3	1	4	
4	5		

Since the second differences are constant, the points lie on a graph of a quadratic function.

b. Using the graphing calculator, the cubic function is given as $f(x)=2x^3-3x^2=5x-8.$ Since for the function $f(1)=-4,$ $f(2)=6,$ $f(3)=34$, and $f(4)=92$, it is the simplest polynomial function.

8. Let $f(x)=x^3+cx+d.$

When $f(x)$ is divided by $x-1$, the remainder is 3.

$$\begin{aligned} f(-1) &= 3 \\ (-1)^3+c(-1)+d &= 3 \\ -c+d &= 4 \qquad\qquad (1) \end{aligned}$$

When $f(x)$ is divided by $x-2$, the remainder is –3.

$$\begin{aligned} f(-2) &= -3 \\ (-2)^3+c(-2)+d &= -3 \\ -2c+d &= 5 \qquad\qquad (2) \end{aligned}$$

Solving the resulting equation,
(2) – (1) $c=1$
$d=3.$

9. By dividing $x^3-2x^2-9x+18=(x-2)(x^2-9).$

So, the other factors are $(x-3)$ and $(x+3)$.

Chapter 2 • Polynomial Equations and Inequalities

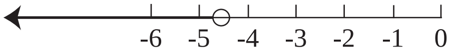
Review of Prerequisite Skills

1. **b.**
$$\begin{aligned} 3(x-2)+7&=3(x-7)\\ 3x-6+7&=3x-21\\ 3x+1&=3x-21\\ 0x&=-22 \end{aligned}$$

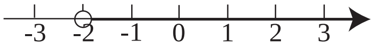
There is no solution.

2. **c.**
$$\begin{aligned} 4x-5&\le 2(x-7)\\ 4x-5&\le 2x-14\\ 2x&\le -9\\ x&\le -\frac{9}{2} \end{aligned}$$

or $x \le -4.5$



d.
$$\begin{aligned} 4x+7&<9x+17\\ -5x&<10\\ x&>-2 \end{aligned}$$



3. $f(x)=2x^2-3x+1$

b.
$$\begin{aligned} f(-2)&=2(-2)^2-3(-2)+1\\ &=15 \end{aligned}$$

d.
$$\begin{aligned} f\left(\frac{1}{2}\right)&=2\left(\frac{1}{2}\right)^2-3\left(\frac{1}{2}\right)+1\\ &=0 \end{aligned}$$

4. $f(x)=x^3-2x^2+4x+5$

c.
$$\begin{aligned} f(-3)&=(-3)^3-2(-3)^2+4(-3)+5\\ &=-52 \end{aligned}$$

d.
$$\begin{aligned} f\left(\frac{1}{2}\right)&=\left(\frac{1}{2}\right)^3-2\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)+5\\ &=\frac{53}{8} \end{aligned}$$

5. **d.**
$$\begin{aligned} &3x^3-75x\\ =&3x(x^2-25)\\ =&3x(x-5)(x+5) \end{aligned}$$

f.
$$\begin{aligned} &x^3+x^2-56x\\ =&x(x^2+x-56)\\ =&x(x+8)(x-7) \end{aligned}$$

h.
$$\begin{aligned} &3x^3-12x\\ =&3x(x^2-4)\\ =&3x(x-2)(x+2) \end{aligned}$$

6. **e.**
$$\begin{aligned} &x^2-2x-15=0\\ &(x-5)(x+3)=0\\ &x-5=0 \quad \text{or} \quad x+3=0\\ &x=5 \quad \text{or} \quad x=-3 \end{aligned}$$

f.
$$\begin{aligned} &7x^2+3x-4=0\\ &(7x-4)(x+1)=0\\ &7x-4=0 \quad \text{or} \quad x+1=0\\ &x=\frac{4}{7} \quad \text{or} \quad x=-1 \end{aligned}$$

h.
$$\begin{aligned} x^3-9x&=0\\ x(x^2-9)&=0\\ x(x-3)(x+3)&=0 \end{aligned}$$
$$x=0 \quad \text{or} \quad x-3=0 \quad \text{or} \quad x+3=0$$
$$x=0 \quad \text{or} \quad x=3 \quad \text{or} \quad x=-3$$

7. **b.** $3y^2-5y-4=0$
$$\begin{aligned} y&=\frac{5\pm\sqrt{(-5)^2-4(3)(-4)}}{2(3)}\\ &=\frac{5\pm\sqrt{73}}{6}\\ &\doteq 2.3 \quad \text{or} \quad -0.6 \end{aligned}$$

c. $3x^2 + x + 3 = 0$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{41}}{2}$$

$$\doteq 5.7 \quad \text{or} \quad -0.7$$

e. $2x^2 - 5x - 3 = 0$

Solution 1

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{49}}{4}$$

$$= 3 \quad \text{or} \quad -0.5$$

Solution 2

$$(2x+1)(x-3) = 0$$

$$2x + 1 = 0 \quad \text{or} \quad x - 3 = 0$$

$$2x = -1 \quad \text{or} \quad x = 3$$

$$x = -\frac{1}{2} \quad \text{or} \quad x = 3$$

g. $2p^2 - 3p + 5 = 0$

$$p = \frac{3 \pm \sqrt{(-3)^2 - 4(2)(5)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{-31}}{4}$$

$$= \frac{3 \pm i\sqrt{31}}{4}$$

i. $2x(x-5) = (x+2)(x-3)$

$$2x^2 - 10x = x^2 - x - 6$$

$$x^2 - 9x + 6 = 0$$

$$x = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{9 \pm \sqrt{57}}{2}$$

$$\doteq 8.3 \quad \text{or} \quad 0.7$$

Exercise 2.1

2. **b.** The other factors can be found by dividing $x - 5$ into $f(x)$ then checking factors for the quotient, either by inspection or using the Factor Theorem.

3. If $f(x) = x^3 + 2x^2 - 5x - 6$ and $f(-1) = f(2) = f(-3) = 0$.

then the factors are $(x+1), (x-2)$, and $(x+3)$. This is true since $f(a)$ is the remainder, and in this case, all remainders are zero, giving division that is complete. Also, this is the Factor Theorem.

4. **a.** $x - 1$ is a factor of $f(x) = x^2 - 7x + 6$ only if $f(1) = 0$.

Since $f(1) = 1^2 - 7(1) + 6 = 0$, then $x - 1$ is a factor.

d. $f(x) = x^3 + 6x^2 - 2x + 3$

$$f(3) = 3^3 + 6(3^2) - 2(3) + 3$$

$$\neq 0$$

Therefore, $(x-3)$ is not a factor of $f(x)$.

f. $f(x) = 4x^3 - 6x^2 + 8x - 3$

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right) - 3$$

$$= \frac{1}{2} - \frac{3}{2} + 4 - 3$$

$$= 0$$

Therefore, $(2x - 1)$ is a factor of (x).

5. $f(x)=x^3-2x^2-2x-3$

a. $f(3)=3^3-2(3)^2-2(3)-3$
$=27-18-6-3$
$=0$

b. $x-3$ is a linear factor of $f(x)$.

c.
$$\begin{array}{r} x^2+x+1 \\ x-3\overline{)\,x^3-2x^2-2x-3} \\ \underline{x^3-3x^2} \\ x^2-2x \\ \underline{x^2-3x} \\ x-3 \\ \underline{x-3} \\ 0 \end{array}$$

The quadratic factor is x^2+x+1.

6. $g(x)=x^3-2x^2-5x+6$

a. $g(-2)=(-2)^3-2(-2)^2-5(-2)+6$
$=0$

b. $x+2$ is the linear factor of $f(x)$.

c. $x^3-2x^2-5x+6=(x+2)(x^2+kx+1)$
$=x^3+(k+2)x^2+(k+1)x+2$

By comparing coefficients, $k+2=-2$
$k=-4.$

$\therefore$ the quadratic factor is x^2-4x+1.

7. **a.** Let $f(x)=x^3-4x+3.$
$f(1)=1^3-4(1)+3$
$=0$

$\therefore$ $(x-1)$ is a factor of $f(x)$.
$x^3-4x+3=(x-1)(x^2+kx-3)$
$=x^3+(k-1)x^2+(-k-3)x+3$

Comparing coefficients, $k-1=0$
$k=1.$
$\therefore x^3-4x+3=(x-1)(x^2+x-3).$

b. Let $f(x)=x^3+2x^2-x-2.$
$f(1)=(1)^3+2(1)^2-(1)-2$
$=0$
$\therefore$ $(x-1)$ is a factor of $f(x)$.
So, $x^3+2x^2-x-2=(x-1)(x^2+kx+1)$
$=x^3+(k-1)x^2+\ldots$
Comparing coefficients, $k=3$
$\therefore x^3+x^2+x+1=(x-1)(x^2+3x+2)$
$=(x-1)(x+2)(x+1).$

e. Let $f(y)=y^3-y^2-y-2$
$f(2)=(2)^3-(2)^2-(2)-2$
$=0.$
$\therefore$ $(y-2)$ is a factor of $f(y)$.
By dividing, $y^3-y^2-y-2=(y-2)(y^2+y+1)$.

g. $f(x)=x^4-8x^3+3x^2+40x-12$
Because the function is quartic and the constant is –12, which presents many possibilities, we use the graphing calculator in $\boxed{\text{VALUE}}$ mode in the $\boxed{\text{CALC}}$ function to establish $f(-2)=f(3)=0$. Therefore, both $(x+2)$ and $(x-3)$ are factors of $f(x)$.
Using the method of comparing coefficients to factor,
$x^4-8x^3+3x^2+40x-12$
$=(x+2)(x-3)(x^2+kx+2)$
$=(x^2-x-6)(x^2+kx+2)$
$=x^4+(k-1)x^3+\ldots$
Since $k-1=-8$
$k=-7.$
$\therefore x^4-8x^3+3x^2+40x-12$
$=(x+2)(x-3)(x^2-7x+2).$

h. Let $f(x) = x^4 - 6x^3 - 15x^2 - 6x - 16.$

By graphing, it appears x-intercepts are –2 and 8.

Checking,

$$f(-2) = (-2)^4 - 6(-2)^3 - 15(-2)^2 - 6(-2) - 16 = 0$$

and $f(8) = (8)^4 - 6(8)^3 - 15(8)^2 - 6(8) - 16 = 0.$

Therefore, both and $(x-8)$ are factors of $f(x)$.

$$\therefore x^4 - 6x^3 - 15x^2 - 6x - 16$$
$$= (x+2)(x-8)(x^2 + kx + 1)$$
$$= (x^2 - 6x - 16)(x^2 + kx + 1)$$
$$= x^4 + (k-6)x^3 + \ldots$$

Comparing coefficients, $k - 6 = -6$

$$\therefore k = 0.$$

So, $x^4 - 6x^3 - 15x^2 - 6x - 16$
$$= (x+2)(x-8)(x^2+1).$$

9. If $x^3 + 4x^2 + kx - 5$ is divisible by $(x+2)$,
then $f(-2) = 0$,

or $(-2)^3 + 4(-2)^2 + k(-2) - 5 = 0$
$$-8 + 16 - 2k - 5 = 0$$
$$-2k = -3$$
$$k = 1.5.$$

10. **c.** $125u^3 - 64r^3 = (5u)^3 - (4r)^3$
$$= (5u - 4r)(25u^2 + 20ur + 16r^2)$$

d. $2000w^3 + 2y^3 = 2(1000w^3 + y^3)$
$$= 2(10w + y)(100w - 10wy + y^2)$$

e.

$(x+y)^3 - u^3z^3 = (x+y)^3 - (uz)^3$
$$= (x + y - uz)\left[(x+y)^2 + (x+y)uz + u^2z^2\right]$$
$$= (x + y - uz)\left[x^2 + 2xy + y^2 + xuz + yuz + u^2z^2\right]$$

f.

$5u^3 - 40(x+y)^3$
$$= 5\left[u^3 - 8(x+y)^3\right]$$
$$= 5\left[u^3 - (2(x+y))^3\right]$$
$$= 5(u - 2[2x+y])(u^2 + 2u(2x+y) + 4(2x+y)^2)$$
$$= 5(u - 4x - 2y)(u^2 + 4ux + 2uy + 4(4x^2 + 4xy + y^2))$$
$$= 5(u - 4x - 2y)(u^2 + 4ux + 2uy + 16x^2 + 16xy + 4y^2)$$

11. Let $f(x) = x^3 - 6x^2 + 3x + 10.$

Since $x^2 - x - 2 = (x-2)(x+1).$

If $f(x)$ is divisible by $x^2 - x - 2$, it must be divisible by both $(x-2)$ and $(x+1)$, that is, $f(2) = f(-1) = 0$.

Substituting for x, $f(2) = 2^3 - 6(2)^2 + 3(2) + 10$
$$= 8 - 24 + 6 + 10 = 0$$

and $f(-1) = (-1)^3 - 6(-1)^2 + 3(-1) + 10$
$$= -1 - 6 - 3 + 10 = 0.$$

Therefore, $x^3 - 6x^2 + 3x + 10$ is divisible by $x^2 - x - 2$.

12. **a.** Let $f(x) = x^4y^4$
$$f(y) = y^4 - y^4 = 0$$
$\therefore (x - y)$ is a factor of $x^4 - y^4$.

b. By division, the other factor is $x^3 + x^2y + xy^2 + y^3$.

$$
\begin{array}{rl}
 & x^3 + x^2y + xy^2 + y^3 \\
x - y\big) & x^4 \qquad\qquad\qquad - y^4 \\
 & \underline{x^4 - x^3y} \\
 & \qquad x^3y \\
 & \qquad \underline{x^3y - x^2y^2} \\
 & \qquad\qquad x^2y^2 \\
 & \qquad\qquad \underline{x^2y^2 - xy^3} \\
 & \qquad\qquad\qquad xy^3 - y^4 \\
 & \qquad\qquad\qquad \underline{xy^3 - y^4} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}
$$

c. From the pattern of **2. b**.
$$x^4 - 81 = x^4 - (3)^4$$
$$= (x-3)(x^3 + x^2(3) + x(3)^2 + (3)^3)$$
$$= (x-3)(x^3 + 3x^2 + 9x + 27)$$

13. **a.** Let $f(x) = x^5 - y^5$
$$f(y) = y^5 - y^5 = 0$$
$\therefore (x - y)$ is a factor of $x^5 - y^5$.

b. By dividing,

$$\begin{array}{r l}
 & x^4+x^3y+x^2y^2+xy^3+y^4 \\
x-y\big) & x^5 \qquad\qquad\qquad -y^5 \\
 & \underline{x^5-x^4y} \\
 & \qquad x^4y \\
 & \qquad \underline{x^4y-x^3y^2} \\
 & \qquad\qquad x^3y^2 \\
 & \qquad\qquad \underline{x^3y^2-x^2y^3} \\
 & \qquad\qquad\qquad x^2y^3 \\
 & \qquad\qquad\qquad \underline{x^2y^3-xy^4} \\
 & \qquad\qquad\qquad\qquad xy^4-y^5 \\
 & \qquad\qquad\qquad\qquad \underline{xy^4-y^5} \\
 & \qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

$$x^5-y^5=(x-y)\left(x^4+x^3y+x^2y^2+xy^3+y^4\right)$$

c.

$$\begin{aligned}
x^5-32&=x^5-2^5\\
&=(x-2)\left(x^4+x^3(-2)+x^2(-2)^2+x(-2)^3+(-2)^4\right)\\
&=(x-2)\left(x^4-2x^3+4x^2-8x+16\right)
\end{aligned}$$

14. **a.** Let $f(x)=x^n-y^n$.
Since $f(y)=y^n-y^n=0$, then $(x-y)$ is a factor of x^n-y^n by the Factor Theorem.

b. From the factoring pattern developed in questions 2 and 3, the other factor is
$x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots xy^{n-2}+y^{n-1}$.

15. Let $f(x)=(x+a)^5+(x+c)^5+(a-c)^5$

$$\begin{aligned}
f(-a)&=(a-a)^5+(-a+c)^5+(a-c)^5\\
&=0+[(-1)(a-c)]^5+(a-c)^5\\
&=(-1)^5(a-c)^5+(a-c)^5\\
&=-(a-c)^5+(a-c)^5\\
&=0
\end{aligned}$$

$\therefore (x+a)$ is a factor of $f(x)$.

16. Let $f(x)=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc.$

$$\begin{aligned}
f(a)&=a^3-(a+b+c)a^2+(ab+bc+ca)a-abc\\
&=a^3-a^3-a^2b-a^2c+a^2b+abc+a^2c-abc\\
&=0
\end{aligned}$$

$\therefore (x-a)$ is a factor of $f(x)$.

17. If $n\in N, (x+y)$ will be a factor of $f(x)=x^n+y^n$ if and only if n is an odd number. If n is an odd number, then

$$\begin{aligned}
f(-y)&=(-y)^n+y^n\\
&=y^n+y^n\\
&=0.
\end{aligned}$$

However, if n is an even number, then

$$\begin{aligned}
(f(-y)&=(-y)^n+y^n\\
&=y^n+y^n\\
&\neq 0,
\end{aligned}$$

and in order for $(x+y)$ to be a factor, $f(-y)=0$.

18. Let $f(x)=x^5+y^5.$

Since $f(y)=(-y)^5+y^5$

$$\begin{aligned}
&=-y^5+y^5\\
&=0
\end{aligned}$$

then $(x+y)$ is a factor of $f(x)$.

By dividing,

$x^5+y^5=(x+y)\left(x^4-x^3y+x^2y^2-xy^3+y^4\right).$

19. $f(x)=x^3+2x^2+5x+12$

Since $f(x)$ is a cubic function, it could have at least one factor of the form $(x-p)$ where p is negative. Possible values for p are $\pm1, \pm2, \pm3, \pm4, \pm6,$ and $\pm12.$ Using the graphing calculator, the function has no value for p. We cannot find a rational number for p.

Exercise 2.2

1. **a.** $f(x)=2x^2+9x-5$

For factors with integer coefficients, the first terms must be either $2x$ or x. Since the only factors of 5 are 5and 1, the possible values of $\dfrac{p}{q}$ are

$\pm\dfrac{1}{2}, \pm\dfrac{5}{2}, \pm1,$ and $\pm5.$

b. $f(x) = 3x^3 - 4x^2 + 7 + 8$

For factors, the first terms must be 3*x* and *x*, and the second terms must be $\pm 1, \pm 2, \pm 4$, and ± 8.

A graph of the function shows $k = \frac{p}{q}$ can be between 0 and –1. Since *p* divides 3 and *q* divides 8, we try $-\frac{1}{3}$, and $-\frac{2}{3}$.

c. $f(x) = 4x^3 + 3x^2 - 11x + 2$

The first terms of the possible factors must be 4*x*, 2*x*, or *x*. The second terms must be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$, or ± 12.

Graphing gives possible values for $\frac{p}{q}$ as between –2 and –3. Therefore, there are no possible values for $\frac{p}{q}$.

d. $f(x) = 8x^3 - 7x^2 + 23x - 4$

The first terms of the possible factors are 8*x*, 4*x*, 2*x*, or *x*.

The second terms could be $\pm 1, \pm 2, \pm 3$, or ± 4.
By graphing, we see possible values for *k* are between 0 and 1, closer to 0.

Possible values for $\frac{p}{q}$ are then $\frac{1}{8}, \frac{1}{4}$, or $\frac{3}{8}$.

e. $f(x) = 6x^3 - 7x^2 + 4x + 3$

The first terms could be $6x, 3x, 2x$, or *x*. (*q*)

The second terms could be $\pm 1, \pm 2, \pm 3$, or ± 6. (*p*)

By graphing, we see possible values for *k* are between 0 and –1.

Possible values for $\frac{p}{q}$ are $-\frac{1}{3}$ and $-\frac{1}{2}$.

2. If $f\left(\frac{3}{2}\right) = 0$, then $(2x - 3)$ is a factor of *f*(*x*). Since $(x - 2)$ is also a factor and *f*(*x*) is cubic, then $f(x) = (x - 2)(2x - 3)(ax + b)$ where $a, b \in I, a \neq 0$.

But $f(4) = 50$

$$\begin{aligned} \therefore (4-2)(2(4)-3)(4a+b) &= 50 \\ 2(5)(4a+b) &= 50 \\ 4a+b &= 5 \\ \text{or} \qquad b &= 5-4a. \end{aligned}$$

Since there are many values that satisfy this equation, we select one possibility, i.e., $a = 1$, $b = 1$. One possibility is $f(x) = (x - 2)(2x - 3)(x + 1)$.

3. If $g(3) = 0$, then $(x - 3)$ is a factor of $g(x)$.

If $g\left(-\frac{3}{4}\right) = 0$, then $(4x + 3)$ is a factor.

Since $(x + 2)$ is a given factor as well, then the quartic function is $g(x) = (x + 2)(x - 3)(4x + 3)(ax + b)$, where *a, b.* $\in I$ $a \neq$ 0.

$$\begin{aligned} \text{Since} \qquad g(1) &= -84 \\ (1+2)(1-3)(4+3)(a+b) &= -84 \\ (3)(-2)(7)(a+b) &= -84 \\ a+b &= 2. \end{aligned}$$

Let *a* = 1, then *b* = 1.
The function is
$g(x) = (x + 2)(x - 3)(4x + 3)(x + 1)$.

4. **a.** $f(x) = 2x^3 + x^2 + x - 1$

From the graph and the possibilities for $\frac{p}{q}$, we see possible values for *k* is $\frac{1}{2}$.

Using the $\boxed{\text{CALC}}$ function, we have $x = \frac{1}{2}, y = 0$.

Therefore, $f\left(\frac{1}{2}\right)=0$, so $\left(x-\frac{1}{2}\right)$ or $(2x-1)$ is a factor.

$$\begin{aligned} f(x) &= 2x^3+x^2+x-1 \\ &= (2x-1)(x^2+kx+1), \text{ where } k\in I \\ &= 2x^3+(2k-1)x^2+\dots \end{aligned}$$

Comparing coefficients, $2k-1=1$.

$$\begin{aligned} 2k &= 2 \\ k &= 1 \end{aligned}$$

$\therefore 2x^3+x^2+x-1=(2x-1)(x^2+x+1).$

c. $f(x)=6x^3-17x^2+11x-2$

From the graph, we see possible values for k are between 0 and 1, and at 2. Checking the $\boxed{\text{CALC}}$ and $\boxed{\text{VALUE}}$ functions,

$f(2)=0$ and $f\left(\frac{1}{2}\right)=0.$

So, $(x-2)$ and $(2x-1)$ are factors of $f(x)$.

$\therefore 6x^3-17x^2+11x-2=(2x-1)(x-2)(3x-1).$

e. $f(x)=5x^4+x^3-22x^2-4x+8$

From the graph, we see that there are four factors. Possible values for $k=\frac{p}{q}$ are –2, between –1 and 0, between 0 and 1, and 2. Testing, $f(-2)=0$ and $f(+2)=0$. So, $(x+2)$ and $(x-2)$ are factors.

$$\begin{aligned} &\therefore 5x^4+x^3-22x^2-4x+8 \\ &= (x+2)(x-2)(5x^2+kx-2) \\ &= (x^2-4)(5x^2+kx-2) \\ &= 5x^4+kx^3+\dots \end{aligned}$$

Comparing coefficients, $k=1$.

$\therefore 5x^4+x^3-22x^2-4x+8=(x+2)(x-2)(5x^2+x-2).$

f. $f(x)=18x^3-15x^2-x+2$

From the graph, we see that there are three factors, one between –1 and 0, and two between 0 and 1. The first terms could be $1x, 2x, 3x, 6x, 9x,$ and $18x$. (q)
The second terms could be ±1, or ±2. (p)
$(3x+1)$ may be a factor.

$$\begin{aligned} \text{Testing, } f\left(-\frac{1}{3}\right) &= 18\left(-\frac{1}{3}\right)^3-15\left(-\frac{1}{3}\right)^2-\left(-\frac{1}{3}\right)+2 \\ &= -\frac{18}{27}-\frac{15}{9}+\frac{1}{3}+2 \\ &= 0. \end{aligned}$$

$\therefore (3x+1)$ is a factor.

$$\begin{aligned} &\text{So, } 18x^3-15x^2-x+2 \\ &= (3x+1)(6x^2+kx+2) \\ &= 18x^3+(3k+6)x^2+\dots \end{aligned}$$

Comparing coefficients, $3k+6=-15$

$$\begin{aligned} 3k &= -21 \\ k &= -7. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } 18x^3-15x^3-x+2 &= (3x+1)(6x^2-7x+2) \\ &= (3x+1)(3x-2)(2x-1). \end{aligned}$$

g. $f(x)=3x^4-5x^3-x^2-4x+4$

There are two possible values for $k=\frac{p}{q}$ at 2 and between 0 and 1. Using the $\boxed{\text{CALC}}$ function and testing $\boxed{\text{VALUE}}$ of $x=2$ and $\frac{2}{3}$, we find

$f(2)=0$ and $f\left(\frac{2}{3}\right)=0.$

So, the two factors are $(x-2)$ and $(3x-2)$.

$$\begin{aligned} &3x^4-5x^3-x^2-4x+4 \\ &= (x-2)(3x-2)(x^2+kx+1) \\ &= (3x^2-8x+4)(x^2+kx+1) \\ &= 3x^4+(3k-8)x^3+\dots \end{aligned}$$

Comparing coefficients, $3k-8=-5$

$$\begin{aligned} 3k &= 3 \\ k &= 1. \end{aligned}$$

$\therefore 3x^4-5x^3-x^2-4x+4=(x-2)(3x-2)(x^2+x+1).$

5. **a.** $f(x) = px^3 + (p-q)x^2 + (-2p-q)x + 2q$

The first term can be p or 1.

The second term can be $\pm 2, \pm q$, or $\pm 2q$.

We try $x = -2$:

$$\begin{aligned} f(-2) &= p(-2)^3 + (p-q)(-2)^2 + (-2p-q)(-2) + 2 \\ &= -8p + 4p - 4q + 4p + 2q + 2q \\ &= 0. \end{aligned}$$

So, $(x+2)$ is a factor of $f(x)$.

$$\begin{aligned} \therefore\ & px^3 + (p-q)x^2 + (-2p-q)x + 2q \\ &= (x+2)(px^2 + kx + q) \\ &= px^3 + (k+2p)x^2 + \ldots \end{aligned}$$

By comparing coefficients, we have

$$\begin{aligned} k + 2p &= p - q \\ k &= -p - q. \end{aligned}$$

$$\begin{aligned} \therefore\ & px^3 + (p-q)x^2 + (-2p-q)x + 2q \\ &= (x+2)(px^2 - (p+q)x + q). \end{aligned}$$

b. $f(x) = abx^3 + (a-2b-ab)x^2 + (2b-a-2)x + 2$

If the factors are integer values, the first term can be ax, bx, abx, or x and the second term can be ± 1 or ± 2.

So, $K = \dfrac{p}{q}$ can be ± 1, ± 2, $\pm\dfrac{1}{a}$, $\pm\dfrac{1}{b}$, $\pm\dfrac{1}{ab}$, etc.

$$\begin{aligned} f(1) &= ab + a - 2b - ab + 2b - a - 2 + 2 \\ &= 0 \end{aligned}$$

$\therefore (x-1)$ is a factor.

$$\begin{aligned} \text{So,}\quad & abx^3 + (a-2b-ab)x^2 + (2b-a-2)x + 2 \\ &= (x-1)(abx^2 + kx - 2) \\ &= abx^3 + (k-ab)x^2 + (-2-k)x + 2. \end{aligned}$$

Comparing coefficients, $k - ab = a - 2b - ab$

$$k = a - 2b.$$

$$\begin{aligned} \therefore\ & abx^3 + (a-2b-ab)x^2 + (2b-a-2)x + 2 \\ &= (x-1)(abx^2 + (a-2b)x - 2) \\ &= (x-1)(ax-2)(bx+1). \end{aligned}$$

Exercise 2.3

3. **a.** Since the x-intercepts are –3, 0, and 2, $(x+3)$, x, and $(x-2)$ must be factors of the cubic function. Therefore, $f(x) = k(x)(x-2)(x+3)$, where k is a constant, represents the family of cubic functions.

b. If $(-1, 12)$ lies on the graph of one member of the family, then $(-1, 12)$ must satisfy the equation.

$$\begin{aligned} \text{Substituting, } 12 &= k(-1)(-1-2)(-1+3) \\ 12 &= 6k \\ k &= 2. \end{aligned}$$

So, the particular member is $f(x) = 2x(x-2)(x+3)$.

4. **a.** Since the x-intercepts are –2, –1, and 1, then $(x+2)$, $(x+1)$, and $(x-1)$ are factors of $f(x)$.

$\therefore f(x) = k(x-1)(x+1)(x+2)$, where k is a constant.

6. For roots $1, 2$, and $\dfrac{3}{5}$, the factors must be $(x-1), (x-2)$, and $(5x-3)$. A polynomial equation with these roots is $(x-1)(x-2)(5x-3) = 0$.

7. If 2 is a root of the equation, substituting $x = 2$ will satisfy the equation. Then,

$$\begin{aligned} 2(2)^3 - 5k(2)^2 + 7(2) + 10 &= 0 \\ 16 - 20k + 14 + 10 &= 0 \\ -20k &= -40 \\ k &= 2. \end{aligned}$$

8. **b.** $x^2 + 2x + 10 = 0$

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2(1)} \\ &= \frac{-2 \pm 6i}{2} \\ &= -1 \pm 3i \end{aligned}$$

e.

$$\begin{aligned} x^3 &= x \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x-1)(x+1) &= 0 \end{aligned}$$

$x = 0$ or $x - 1 = 0$ or $x + 1 = 0$

$x = 0$ or $x = 1$ or $x = -1$

f.

$$\begin{aligned} x^4 - 1 &= 0 \\ (x^2 + 1)(x^2 - 1) &= 0 \end{aligned}$$

$x^2 + 1 = 0$ or $x^2 - 1 = 0$

$x^2 = -1$ or $x^2 = 1$

$x = \pm i$ or $x = \pm 1$

h.

$$\begin{aligned} 8x^3 - 27 &= 0 \\ (2x)^3 - 3^3 &= 0 \\ (2x - 3)(4x^2 + 6x + 9) &= 0 \end{aligned}$$

$2x - 3 = 0$ or $4x^2 + 6x + 9 = 0$

$2x = 3$ or $x = \dfrac{-6 \pm \sqrt{36 - 4(4)(9)}}{2(4)}$

$x = \dfrac{3}{2}$ or $x = \dfrac{-6 \pm \sqrt{-108}}{8}$

$$= \frac{-6 \pm 6i\sqrt{3}}{8}$$

$$= \frac{-3 \pm 3i\sqrt{3}}{4}$$

i. Let $f(x) = x^3 - 3x^2 - 4x + 12.$

$$\begin{aligned} f(2) &= 2^3 - 3(2)^2 - 4(2) + 12 \\ &= 0 \end{aligned}$$

$\therefore (x - 2)$ is a factor of $f(x)$.

By dividing, the other factor is $x^2 - x - 6$.

$$\begin{array}{r} x^2 - x - 6 \\ x - 2 \overline{) x^3 - 3x^2 - 4x + 12} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ -x^2 - 4x \qquad \\ \underline{-x^2 + 2x} \qquad \\ -6x + 12 \\ \underline{-6x + 12} \\ 0 \end{array}$$

$$\begin{aligned} x^3 - 3x^2 - 4x + 12 &= 0 \\ (x - 2)(x^2 - x - 6) &= 0 \\ (x - 2)(x - 3)(x + 2) &= 0 \end{aligned}$$

$x - 2 = 0$ or $x - 3 = 0$ or $x + 2 = 0$

$x = 2$ or $x = 3$ or $x = -2$

j. Let $f(x) = x^3 - 9x^2 + 26x - 24.$

$$\begin{aligned} f(2) &= 2^3 - 9(2)^2 + 26(2) - 24 \\ &= 8 - 36 + 52 - 24 \\ &= 0 \end{aligned}$$

$\therefore (x - 2)$ is a factor of $f(x)$.

So, $x^3 - 9x^2 + 26x = 24$

$x^3 - 9x^2 + 26x - 24 = 0$

$$\left.\begin{aligned} (x - 2)(x^2 - 7x + 12) &= 0 \\ (x - 2)(x - 4)(x - 3) &= 0 \end{aligned}\right\} \text{by comparing coefficients or by division}$$

$x - 2 = 0$ or $x - 4 = 0$ or $x - 3 = 0$

$x = 2$ or $x = 4$ or $x = 3$

l.

$$\left.\begin{aligned} \text{Let } f(x) &= x^3 - 2x^2 - 15x + 36 \\ f(3) &= 27 - 2(9) - 15(3) + 36 \end{aligned}\right\} \text{To find the zeros, use this } \pm 1, \pm 2, \pm 3, \ldots \text{ all factors of 36.}$$

$$\begin{aligned} &= 27 - 18 - 45 + 36 \\ &= 0 \end{aligned}$$

$\therefore (x - 3)$ is a factor of $f(x)$.

$x^3 - 2x^2 - 15x + 36 = 0$

$(x - 3)(x^2 + x - 12) = 0$ by inspection

$(x - 3)(x + 4)(x - 3) = 0$

$x - 3 = 0$ or $x + 4 = 0$ or $x - 3 = 0$

$x = 3$ or $x = -4$ or $x = 3$

Then $x = 3$ or -4.

m. $x^3 + 8x + 10 = 7x^2$

$x^3 - 7x^2 + 8x + 10 = 0$

Let $f(x) = x^3 - 7x^2 + 8x + 10.$

To find the zeros, we try $\pm1, \pm2, \pm5, \ldots$ all factors of 10.

$$f(5) = 5^3 - 7(5)^2 + 8(5) + 10$$
$$= 0$$

$\therefore (x-5)$ is a factor of $f(x)$.

So, $x^3 - 7x + 8x + 10 = 0$

$(x-5)(x^2 - 2x - 2) = 0$ by inspection

$x - 5 = 0$ or $x^2 - 2x - 2 = 0$

$$x = 5 \quad \text{or} \quad x = \frac{2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{12}}{2}$$
$$= \frac{2 \pm 2\sqrt{3}}{2}$$
$$= 1 \pm \sqrt{3}$$

n. $x^3 - 3x^2 + 16 = 6x$

$x^3 - 3x^2 - 6x + 16 = 0$

To find x, such that $f(x) = 0$, we try the factors of 16, i.e., $\pm1, \pm2$, etc.

$$f(2) = 2^3 - 3(2)^2 - 6(2) + 16$$
$$= 0$$

$\therefore (x-2)$ is a factor of $f(x)$.

$x^3 - 3x^2 - 6x + 16 = 0$

$(x-2)(x^2 - x - 8) = 0$

$x - 2 = 0$ or $x^2 - x - 8 = 0$

$$x = 2 \quad \text{or} \quad x = \frac{1 \pm \sqrt{1 - 4(1)(-8)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{33}}{2}$$

9. b. $4x^3 + 19x^2 + 11x - 4 = 0$

By graphing $f(x) = 4x^3 + 19x^2 + 11x - 4,$ it appears that the x-intercepts are –4, –1, and 0.25.

By using $\boxed{\text{VALUE}}$ selection in $\boxed{\text{CALC}}$ mode, we see $f(-4) = f(-1) = f(0.25) = 0.$

$\therefore (x+4)$ and $(x+1)$ and $(4x+1)$ are factors of $f(x)$.

By taking the product, we can verify this.

$$(x+4)(x+1)(4x+1)$$
$$= (x^2 + 5x + 4)(4x - 1)$$
$$= 4x^2 - x^2 + 20x^2 - 5x + 16x - 4$$
$$= 4x^2 + 19x^2 + 11x - 4$$
$$\therefore x = -4 \quad \text{or} \quad -1 \quad \text{or} \quad 0.25.$$

d. $4x^4 - 2x^3 - 16x^2 + 8x = 0$

$x(4x^3 - 2x^2 - 16x + 8) = 0$

$x[2x^2(2x-1) - 8(2x-1)] = 0$

$x[(2x-1)(2x^2 - 8)] = 0$

$x = 0$ or $2x - 1 = 0$ or $2x^2 - 8 = 0$

or $x = \dfrac{1}{2}$ or $2x^2 = 8$

$x^2 = 4$

$x = \pm2$

f. $x^4 - 7 = 6x^2$

$x^4 - 6x^2 - 7 = 0$

$(x^2 - 7)(x^2 + 1) = 0$

$x^2 - 7 = 0$ or $x^2 + 1 = 0$

$x^2 = 7 \qquad x^2 = -1$

$x = \pm\sqrt{7} \qquad x = \pm i$

h. $(x+1)(x+5)(x+3) = -3$

$$(x^2+6x+5)(x+3) = -3$$
$$x^3+3x^2+6x^2+18x+5x+15 = -3$$
$$x^3+9x^2+23x+18 = 0$$

Let $f(x) = x^3+9x^2+23x+18\ldots$

Try $x = \pm 1, \pm 2, \pm 3$.

$$f(-2) = (-2)^3 + 9(-2)^2 + 23(-2) + 18$$
$$= -8+36-46+18 = 0$$

$\therefore (x+2)$ is a factor of $x+2$

By division,

$$x^3+9x^2+23x+18 = (x+2)(x^2+7x+9) = 0$$
$$x+2 = 0 \quad \text{or} \quad x^2+7x+9 = 0$$
$$x = -2 \qquad x = \frac{-7 \pm \sqrt{49-4(1)(9)}}{2(1)}$$
$$= \frac{-7 \pm \sqrt{13}}{2}.$$

10. a. $x^8 - 10x^4 + 9 = 0$

$$(x^4-9)(x^4-1) = 0$$
$$x^4 - 9 = 0 \quad \text{or} \quad x^4 - 1 = 0$$
$$(x^2-3)(x^2+3) = 0 \quad \text{or} \quad (x^2-1)(x^2+1) = 0$$
$$x^2-3=0 \quad \text{or} \quad x^2+3=0 \quad \text{or} \quad x^2-1=0 \quad \text{or}\ x^2+1=0$$
$$x = \pm\sqrt{3} \quad \text{or} \quad x = \pm i\sqrt{3} \quad \text{or} \quad x = \pm 1 \quad \text{or} \quad x = \pm i$$

b. $x^6 - 7x^3 - 8 = 0$

Let $x^3 = a$.

$$a^2 - 7a - 8 = 0$$
$$(a-8)(a+1) = 0$$
$$a-8 = 0 \quad \text{or} \quad a+1 = 0$$

But $a = x^3$; substituting,

$$x^3 - 8 = 0 \quad \text{or} \quad x^3 + 1 = 0$$
$$(x-2)(x^2+2x+4) = 0 \quad \text{or} \quad (x+1)(x^2-x+1) = 0$$
$$x-2 \ \text{or}\ x^2+2x+4 = 0 \ \text{or}\ x+1 = 0 \ \text{or}\ x^2-x+1 = 0$$
$$x = 2 \quad \text{or} \quad x = \frac{-2 \pm \sqrt{4-4(1)(4)}}{2(1)}$$
$$\text{or} \quad x = -1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{1-4(1)(1)}}{2}$$
$$x = 2 \ \text{or} \ x = \frac{-2 \pm \sqrt{-12}}{2} \quad \text{or} \quad x = -1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{-3}}{2}$$
$$x = 2 \ \text{or} \ x = -1 \pm 3i \quad \text{or} \quad x = -1 \quad \text{or} \quad x = \frac{1 \pm \sqrt{3}}{2}$$

c. $(x^2-x)^2 - 8(x^2-x) + 12 = 0$

Let $a = x^2 - x$.

Then substituting,

$$a^2 - 8a + 12 = 0$$
$$(a-6)(a-2) = 0$$
$$a-6 = 0 \quad \text{or} \quad a-2 = 0.$$

But $a = x^2 - x$

$$\therefore x^2-x-6 = 0 \quad \text{or} \quad x^2-x-2 = 0$$
$$(x-3)(x+2) = 0 \qquad (x-2)(x+1) = 0$$
$$x-3=0 \ \text{or}\ x+2=0 \qquad x-2=0 \ \text{or}\ x+1=0$$
$$x = 3 \ \text{or}\ x = -2 \qquad x = 2 \ \text{or}\ x = -1.$$

d. $\left(x - \frac{1}{x}\right)^2 - \frac{77}{12}\left(x - \frac{1}{x}\right) + 10 = 0$

Let $a = x - \frac{1}{x}$.

$$a^2 - \frac{77}{12}a + 10 = 0$$
$$12a^2 - 77a + 120 = 0$$

Since there are so many possible integers to try, we use the quadratic formula.

$$a = \frac{77 \pm \sqrt{(-77)^2 - 4(12)(120)}}{2(12)}$$
$$= \frac{77 \pm 13}{24}$$
$$= \frac{15}{4} \quad \text{or} \quad \frac{8}{3}$$

But $a = x - \frac{1}{x}$

$$\therefore x - \frac{1}{x} = \frac{15}{4} \quad \text{or} \quad x - \frac{1}{x} = \frac{8}{3}.$$

Since $x \neq 0$

$$x^2 - 1 = \frac{15x}{4} \quad \text{or} \quad 3x^2 - 8x - 3 = 0$$
$$4x^2 - 15x - 4 = 0$$
$$(4x+1)(x-4) = 0 \quad \text{or} \quad (3x+1)(x-3) = 0$$
$$4x+1 = 0 \quad \text{or} \quad x-4 = 0 \quad \text{or} \quad 3x+1 = 0 \quad \text{or}\ x-3 = 0$$
$$x = -\frac{1}{4} \quad \text{or} \quad x = 4 \quad \text{or} \quad x = -\frac{1}{3} \quad \text{or}\ x = 3.$$

e. $(3x-5)(3x+1)^2(3x+7)+68=0$

Let $a=3x+1$.

Then $3x-5=a-6$ and $3x+7=a+6$.

Substituting,

$$(a-6)(a)^2(a+6)+68=0$$
$$a^2(a^2-36)+68=0$$
$$a^4-36a^2+68=0$$
$$(a^2-34)(a^2-2)=0$$
$$a^2-34=0 \text{ or } a^2-2=0.$$

But $a=3x+1$

$\therefore (3x+1)^2-34=0$ or $(3x+1)^2-2=0$

$$9x^2+6x-33=0 \qquad\qquad 9x^2+6x-1=0$$
$$3x^2+2x-11=0 \qquad\qquad x=\frac{-6\pm\sqrt{6^2-4(9)(-1)}}{2(9)}$$
$$x=\frac{-2\pm\sqrt{2^2-4(3)(-11)}}{2(3)} \qquad\qquad =\frac{-6\pm\sqrt{72}}{18}$$
$$x=\frac{-2\pm\sqrt{136}}{6} \qquad\qquad =\frac{-6\pm6\sqrt{2}}{18}$$
$$=\frac{-1\pm\sqrt{34}}{3} \qquad\qquad =\frac{-1\pm\sqrt{2}}{3}.$$

f. $(x^2+6x+6)(x^2+6x+8)=528$

Let $a=x^2+6x+6$.

Then substituting,

$$a(a+2)=528$$
$$a^2+2a-528=0$$
$$(a+24)(a-22)=0$$
$$a+24=0 \text{ or } a-22=0$$

But $a=x^2+6x+6$

$$\therefore x^2+6x=6+24=0 \text{ or } x^2=6x+6-22=0$$

$$x^2+6x+30\ =0 \qquad\qquad x^2+6x-16=0$$
$$x=\frac{-6\pm\sqrt{6^2-4(1)(30)}}{2(1)} \qquad\qquad (x+8)(x-2)=0$$
$$=\frac{-6\pm\sqrt{-84}}{2} \qquad\qquad x+8=0 \text{ or } x-2=0$$
$$=-3\pm i\sqrt{21} \qquad\qquad x=-8 \text{ or } x=2$$

11. The volume of ice is given by $y=8x^3+36x^2+54x$.

If the volume of ice is 2170 cm^3,

$$8x^3+36x^2+54x=2170$$
$$8x^3+36x^2+54x-2170=0$$
$$4x^3+18x^2+27x-1085=0.$$

By graphing $y=4x^3+18x^2+27x-1085$,

we find $y=0$ when $x=5$.

Since x represents the thickness of ice that gives a specific volume, there is only one value, i.e., the thickness of ice is 5 cm.

12. b. $x^3-2x^2-8x+13=0$

Graphing $y=x^3-2x^2-8x+13$, we find the roots using $\boxed{\text{CALC}}$ mode and $\boxed{\text{ZERO}}$, locating roots between –3 and –2, and 1, 2, 3 and 4. The roots are $x \doteq -2.714, 1.483$, and 3.231.

c. $2x^3-6x^2+4=0$

Graphing $y=2x^3-6x^2+4$, the roots lie between –1 and 0, between 2 and 3, and exactly 1. Using $\boxed{\text{ZERO}}$ option in $\boxed{\text{CALC}}$ mode, we find roots at –0.732 and 2.732.

$\therefore$ the roots are 1, –0.732, and 2.732.

13. Let the dimensions of the box have a height of x cm, a width of $(x+1)$ cm, and a length of $(x+2)$ cm. The volume of the rectangular box is $V_0=x(x+1)(x+2)$ where volume, V, is in cm^3. The new dimensions are $2x$, $x+2$, and $x+3$.

$\therefore$ the new volume is $V_1=2x(x+2)(x+3)$.

The increase in volume is

$$V_1-V_0=120$$
$$2x(x+2)(x+3)-x(x+1)(x+2)=120$$
$$2x(x^2+5x+6)-x(x^2+3x+2)=120$$
$$2x^3+10x^2+12x-x^3-3x^2-2x=120$$
$$x^3+7x^2+10x-120=0$$

Let $f(x)=x^3+7x^2+10x-120$.

Since $f(3)=3^3+7(3)^2+10(3)-120$
$=0.$

then $(x-3)$ is a factor.

By dividing, $x^3+7x^2+10x-120=0$

becomes $(x-3)(x^2+10x+40)=0$

$x-3=0$ or $x^2+10x+40=0$

$$x=3 \text{ or } x=\frac{-10\pm\sqrt{10^2-4(1)(40)}}{2}$$

$$=\frac{-10\pm\sqrt{-60}}{2}$$

$$=-5\pm i\sqrt{15}.$$

But $x>0, x\in R$ since it represents the height of the box; $\therefore x=3$

$\therefore$ the dimensions are 3 cm by 4 cm by 5 cm.

14. The volume of the silo is to be 2000 m^3. Let r cm be the radius of the main section.

$$\underset{\text{(main section)}}{\underset{\uparrow}{V=\pi r^2h+\frac{1}{2}}} \quad \underset{\text{(roof volume)}}{\underset{\uparrow}{\left[\frac{4}{3}\pi r^3\right]}}$$

$$V=10\pi r^2+\frac{2}{3}\pi r^3$$

But $V=2000$

$$\therefore \frac{2}{3}\pi r^3+10\pi r^2-2000=0$$

Graphing,

$$y=\frac{2}{3}\pi r^3+10\pi r^2-2000,$$

we find one real root at $x=3.6859$. Therefore, the radius should be about 3.69 m.

15. $a=0.6t+2 \quad (1)$

$v=0.3t^2+2t+4 \quad (2)$

$s=0.1t^3+t^2+4t \quad (3)$

where a is acceleration in km/s^2 v is the velocity in km/s and s is the displacement in km.

If the displacement is 25 km, then

$25=0.1t^3+t^2+4t$ where $t>0, t\in R$

or $0.1t^3+t^2+4t-25=0$.

Using the graph of $f(t)=0.1t^3+t^2+4t-25$, we find one real root at $t=3.100833$.

Therefore, after 3.1 the rocket will have travelled 25 km.

Section 2.4 Investigation

Equation	a	b	c	Roots of Roots	Sum	Product of Roots
$x^2-5x+6=0$	1	–5	6	3, 2	5	6
$x^2+3x-28=0$	1	3	–28	–7, 4	–3	–28
$3x^2+19x+6=0$	3	19	6	$-\frac{1}{3},\ -6$	$-\frac{19}{3}$	2
$x^2-4x+1=0$	1	-4	1	$2\pm\sqrt{3}$	4	1
$2x^2-17x+2=0$	2	–17	2	$\frac{17\pm\sqrt{273}}{4}$	$\frac{17}{2}$	1
$5x^2+x+2=0$	5	1	2	$\frac{-1\pm i\sqrt{39}}{10}$	$-\frac{1}{5}$	$\frac{2}{5}$

1. The sum of the roots of a quadratic equation is the opposite of the coefficient of the linear term divided by the coefficient of the quadratic term, that is, $x_1+x_2=-\frac{b}{a}\cdot$

2. The product of the roots of a quadratic equation is the quotient of the constant term divided by the coefficient of the quadratic term, that is, $(x_1)(x_2)=\frac{c}{a}$.

Exercise 2.4

2. The quadratic equation is
x^2 – (sum of the roots) × (product of the roots) = 0.

a. The equation is $x^2-3x+7=0$.

b. The equation is $x^2+6x+4=0$.

c. The equation is

$$x^2-\frac{1}{5}x-\frac{2}{25}=0$$

or $25x^2-5x-2=0.$

d. The equation is

$$x^2+\frac{13}{12}x+\frac{1}{4}=0$$

or $12x^2+13x+3=0.$

e. The equation is

$$x^2+11x-\frac{2}{3}=0$$

or $3x^2+33x-2=0.$

3. **b.** $x_1 + x_2 = -5 + 8$ and $x_1 x_2 = (-5)(8)$

$= 3$ $= -40$

The equation is $x^2 - 3x - 40 = 0$.

c. $x_1 + x_2 = 3 + \frac{1}{3}$ and $x_1 x_2 = (3)\left(\frac{1}{3}\right)$

$= \frac{10}{3}$ $= 1$

The equation is $x^2 - \frac{10}{3}x + 1 = 0$

$$3x^2 - 10x + 3 = 0.$$

e. $x_1 + x_2 = -\frac{4}{5} + \frac{3}{25}$ and $(x_1)(x_2) = \left(-\frac{4}{5}\right)\left(\frac{3}{25}\right)$

$= -\frac{17}{25}$ $= -\frac{12}{125}$

The equation is

$$x^2 + \frac{17}{25}x - \frac{12}{125} = 0$$

or $125x^2 + 85x - 12 = 0.$

f.

$x_1 + x_2 = (2 + i)(2 - i)$ and $(x_1)(x_2) = (2 + i)(2 - i)$

$= 4$ $= 4 - i^2$

$= 5$

The equation is $x^2 - 4x + 5 = 0$.

4. *Solution 1*

Since 5 is a root of $2x^2 + kx - 20 = 0$, it must satisfy the equation. Therefore,

$$2(5)^2 + k(5) - 20 = 0$$
$$50 + k(5) - 20 = 0$$
$$5k = -30$$
$$k = -6.$$

Solution 2

Let h represent the second root of $2x^2 + kx - 20 = 0$.

The sum of the roots is $h + 5 = -\frac{k}{2}$ (1)

and the product is $5h = -\frac{20}{2}$

$5h = -10$

$h = -2$ (2)

Substituting into (1) $-2 + 5 = -\frac{k}{2}$

$3 = -\frac{k}{2}$

$k = -6$.

5. Let h represent the other root of $x^2 + x - 2k = 0$.

The sum of the roots is $h - 7 = -1$ or $h = 6$.

The product of the roots is $(-7)(h) = -2k$.

But $h = 6$,

$\therefore (-7)(6) = -2k$

$-42 = -2k$

$k = 21.$

The other root is 6, and k = 21.

6. Let x_1 and x_2 represent the roots of the given equations,

$x^2 + 8x - 1 = 0.$

$\therefore x_1 + x_2 = -8$ and $(x_1)(x_2) = -1.$

The roots of the required equation are $x_1 + 6$ and $x_2 + 6$.

For the sum of the new equation, the sum of the roots is

$$(x_1 + 6) + (x_2 + 6) = x_1 + x_2 + 12.$$

But $x_1 + x_2 = -8.$

Therefore, the sum of the roots of the new equation is $-8 + 12$ or 4.

For the new equation, the product of the roots is

$$(x_1 + 6)(x_2 + 6)$$
$$= x_1x_2 + 6x_1 + 6x_2 + 36$$
$$= x_1x_2 + 6(x_1 + x_2) + 36$$
$$= (-1) + 6(-8) + 36$$
$$= -13.$$

So, the new equation is $x^2 - 4x - 13 = 0$.

7. Let x_1 and x_2 represent the roots of the given equation.

$x_1 + x_2 = \frac{17}{2}$ and $(x_1 x_2) = 1$

The roots of the required equation are $x_1 + 5$ and $x_2 + 5$.

For the new equation, the sum of the roots is

$$\begin{aligned}&(x_1+5)+(x_2+5)\\ &=(x_1+x_2)+10\\ &=\frac{17}{2}+10\\ &=\frac{37}{2}.\end{aligned}$$

For the new equation, the product of the roots is

$$\begin{aligned}&(x_1+5)(x_2+5)\\ &=x_1x_2+5(x_1+x_2)+25\\ &=1+5\left(\frac{17}{2}\right)+25\\ &=\frac{137}{2}.\end{aligned}$$

So, the new equation is $x^2 - \frac{37}{2}x + \frac{237}{2} = 0$

or $2x^2 - 37x + 137 = 0$.

8. Let x_1 and x_2 be the roots of $3x^2 + 7x + 3 = 0$,

$x_1 + x_2 = -\frac{7}{3}$ and $(x_1 x x_2) = 1$. The roots of the required equation are $3x_1$ and $3x_2$. For the new equation, the sum of the roots is

$$\begin{aligned}&3x_1+3x_2\\ &=3(x_1+x_2)\\ &=3\left(-\frac{7}{3}\right)\\ &=-7.\end{aligned}$$

For the new equation, the product of the roots is

$$\begin{aligned}&(3x_1)(3x_2)\\ &=9x_1x_2\\ &=9(1)\\ &=9.\end{aligned}$$

Therefore, the new equation is $x^2 + 7x + 9 = 0$.

9. Let the roots of $4x^2 - 9x - 2 = 0$ be represented by x_1 and x_2.

$x_1 + x_2 = \frac{9}{4}$ and $x_1x_2 = -\frac{2}{4} = -\frac{1}{2}$

The roots of the required equation are x_1^2 and x_2^2.

For the new equation, the sum of the roots is $x_1^2 + x_2^2$.

But $(x_1+x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$

$$\begin{aligned}\left(\frac{9}{4}\right)^2 &= \left(x_1^2+x_2^2\right)+2(x_1x_2)\\ \frac{81}{16} &= \left(x_1^2+x_2^2\right)+2\left(-\frac{1}{2}\right)\\ \therefore x_1^2+x_2^2 &= \frac{81}{16}+1\\ &=\frac{97}{16}\end{aligned}$$

For the new equation, the product of the roots is

$$\begin{aligned}x_1^2x_2^2 &= (x_1x_2)^2\\ &=\left(-\frac{1}{2}\right)^2\\ &=\frac{1}{4}.\end{aligned}$$

So, the required equation is

$$\begin{aligned}x^2-\left(x_1^2+x_2^2\right)x+x_1^2x_2^2&=0\\ x^2-\frac{97}{16}x+\frac{1}{4}&=0\end{aligned}$$

or $16x^2 - 97x + 4 = 0$.

10. Let x_1 and x_2 be the roots of $5x^2 + 10x + 1 = 0$.

Then $x_1 + x_2 = -\frac{10}{5} = -2$ and $x_1x_2 = \frac{1}{5}$.

Since the roots of the required equation are their reciprocals, and the new roots are $\frac{1}{x_1}$ and $\frac{1}{x_2}$.

The sum of the new roots is $\frac{1}{x_1}+\frac{1}{x_2}$

$$= \frac{x_2+x_1}{x_1x_2}$$

$$= -\frac{2}{\frac{1}{5}}$$

$$= -10.$$

The product of the new roots is $\left(\frac{1}{x_1}\right)\left(\frac{1}{x_2}\right)$

$$= \frac{1}{\frac{1}{5}}$$

$$= 5.$$

So, the required equation is $x^2+10x+5=0$.

11. Let the roots of the given equation be x_1 and x_2.

For $x^2+6x-2=0$, $x_1+x_2=-6$ and $x_1x_2=-2$.

The roots of the required equation are $\left(\frac{1}{x_1}\right)^2$ and $\left(\frac{1}{x_2}\right)^2$.

The sum of the new roots is $\frac{1}{x_1{}^2}+\frac{1}{x_2{}^2}$

$$= \frac{x_2{}^2+x_1{}^2}{x_1{}^2x_2{}^2}$$

$$= \frac{x_1{}^2+x_2{}^2}{(x_1x_2)^2}.$$

Now, $(x_1+x_2)^2 = x_1{}^2+2x+x_2+x_2{}^2$

$(-6)^2 = x_1{}^2+2(-2)+x_2{}^2.$

So, $x_1{}^2+x_2{}^2 = 36+4$

$= 40.$

and $x_1{}^2x_2{}^2 = -2$

so, $(x_1x_2)^2 = 4.$

Therefore, the sum of the new roots is

$$\frac{x_1{}^2+x_2{}^2}{(x_1x_2)^2} = \frac{40}{4} = 10.$$

The product of the new roots is $\left(\frac{1}{x_1}\right)^2\left(\frac{1}{x_2}\right)^2$

$$= \frac{1}{x_1{}^2x_2{}^2}$$

$$= \frac{1}{(x_1x_2)^2}$$

$$= \frac{1}{(-2)^2}$$

$$= \frac{1}{4}.$$

The required equation is
x^2 − (sum of roots)x + (product of roots) = 0.

$$x^2-10x+\frac{1}{4}=0$$

$$4x^2-40x+1=0$$

12. Let x_1 and x_2 be the roots of $2x^2+4x+1=0$.

$x_1+x_2 = -\frac{4}{2}$ and $(x_1)(x_2)=\frac{1}{2}$

$= -2$

The roots of the new equation are $x_1{}^3$ and $x_2{}^3$.

The product of the new roots is $x_1{}^3x_2{}^3 = (x_1x_2)^3$

$$= \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{8}.$$

The sum of the new roots is $x_1{}^3+x_2{}^3$.

But $(x_1+x_2)^3 = x_1{}^3+3x_1{}^2x_2+3x_1x_2{}^2+x_2{}^3$

$$(-2)^3 = x_1{}^3+3\left(x_1{}^2x_2+3x_1x_2{}^2\right)+x_2{}^3$$

$$-8 = x_1{}^3+x_2{}^3+3x_1x_2(x_1+x_2)$$

$$-8 = x_1{}^3+x_2+{}^3+3\left(\frac{1}{2}\right)(-2).$$

So, $x_1{}^3+x_2{}^3 = -5.$

Therefore, the new equation is $x^2+5x+\frac{1}{8}=0$

$$8x^2+40x+1=0.$$

13. A cubic equation with roots x_1, x_2, x_3 may be written as

$$(x - x_1)(x - x_2)(x - x_3) = 0 \qquad (1)$$

Expanding, $\quad (x - x_1)(x^2 - (x_2 + x_3)x + x_2x_3) = 0$

$$x^3 - (x_2 + x_3)x^2 + x_2x_3x$$
$$-(x_1)x^2 + (x_1x_2 + x_1x_3)x - x_1x_2x_3 = 0$$
$$x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_2x_3 + x_1x_3)x - x_1x_2x_3 = 0$$

Comparing the coefficients of this expanded expression with the general cubic equation

$$ax^3 + bx^2 + cx + d = 0, \qquad (2)$$

we note that the cubic term must be 1 in order to compare these equations, therefore, dividing (2) by a, we get

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0\,.$$

Now, $\quad x_1 + x_2 + x_3 = -\frac{b}{a}$

$$x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a}$$

and $\quad x_1x_2x_3 = -\frac{d}{a}.$

14. Given the roots of a cubic equation are $\frac{1}{2}$, 2, and 4,

$$x_1 + x_2 + x_3 = \frac{1}{2} + 2 + 4$$
$$= \frac{13}{2}$$

$$x_1x_2 + x_1x_3 + x_2x_3 = \left(\frac{1}{2}\right)(2) + \left(\frac{1}{2}\right)(4) + (2)(4)$$
$$= 11$$
$$x_1x_2x_3 = \left(\frac{1}{2}\right)(2)(4)$$
$$= 4.$$

Therefore, the cubic equation is

$$x^3 - \frac{13}{2}x^2 + 11x - 4 = 0$$

or $\quad 2x^3 - 13x^2 + 22x - 8 = 0.$

15. Let the roots of $x^3 - 4x^2 - 2 = 0$ be represented by x_1, x_2, x_3. From the solution to question 13,

$$x_1 + x_2 + x_3 = \frac{4}{1} = 4$$

$$x_1\ x_2 + x_1\ x_3 + x_2\ x_3 = 3$$
$$x_1\ x_2\ x_3 = 3.$$

For the required equation, the roots are $x_1 + 2$, $x_2 + 2$, and $x_3 + 2$.

(i) $(x_1 + 2) + (x_2 + 2) + (x_3 + 2) = (x_1 + x_2 + x_3\ ) + 6$
$$= 4 + 6$$
$$= 10$$

(ii) $(x_1 + 2)(x_2 + 2)(x_1 + 2)(x_3 + 2)(x_2 + 2)\ (x_3 + 2)$

$$= \left[x_1x_2 + 2(x_1 + x_2) + 4\right] + \left[x_1x_3 + 2(x_3 + x_1) + 4\right]$$
$$+ \left[x_2x_3 + 2(x_3 + x_2) + 4\right]$$

$$= x_1x_2 + x_1x_3 + x_2x_3 + 2$$
$$(x_1 + x_2 + x_3 + x_1 + x_3 + x_2) + 4 + 4 + 4$$

$$= (x_1x_2 + x_1x_3 + x_2x_3) + 4(x_1 + x_2 + x_3) + 12$$
$$= 3 + 4(4) + 12$$
$$= 31$$

(iii) $(x_1 + 2)(x_2 + 2)(x_3 + 2)$
$$= (x_1 + 2)(x_2x_3 + 2(x_3 + x_2) + 4)$$
$$= x_1x_2x_3 + 2(x_1x_3 + x_1x_2) + 4(x_1 + 2x_2x_3 + 4x_3 + 4x_2) + 8$$
$$= x_1x_2x_3 + 2(x_1x_2 + x_1x_3 + x_2x_3) + 4(x_1 + x_2 + x_3) + 8$$
$$= 2 + 2(3) + 4(4) + 8$$
$$= 32$$

The required equation is $x^3 - 10x^2 + 31x - 32 = 0.$

16. A quartic equation with roots x_1, x_2, x_3, and x_4 may be written as $(x-x_1)(x-x_2)(x-x_3)(x-x_4)=0$.

Expanding, we have

$$\left(x^2-(x_1+x_2)x+x_1x_2\right)\left(x^2-(x_3+x_4)x+x_3x_4\right)=0$$

$$x^4-(x_3+x_4)x^3+x_3x_4x^2-(x_1+x_2)x^3-(x_1+x_2)(x_3+x_4)x^2-(x_1+x_2)x_3x_4x+(x_1x_2)x^2-(x_3+x_4)x_1x_2x+x_1x_2x_3x_4=0$$

$$x^4-(x_1+x_2+x_3+x_4)x^3+(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)x^2-(x_1x_3x_4+x_2x_3x_4+x_1x_2x_3+x_1x_2x_4)x+x_1x_2x_3x_4x=0$$

Comparing coefficients with the general quartic equation of $ax^4+bx^3+cx^2+dx+e=0$

or $x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0$.

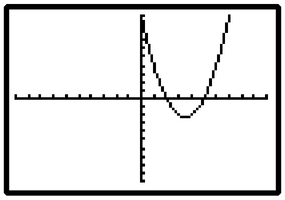
We have

$$\begin{aligned}x_1+x_2+x_3+x_4&=-\frac{b}{a}\\ x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4&=\frac{c}{a}\\ x_1x_2x_3+x_1x_3x_4+x_2x_3x_4+x_1x_2x_4&=-\frac{d}{a}\\ x_1x_2x_3x_4&=\frac{e}{a}.\end{aligned}$$

Exercise 2.5

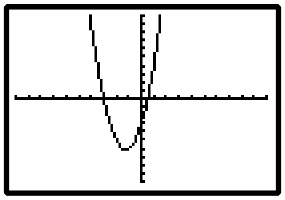
2. **c.** $x^2-7x+10\le 0$

From the graph of $y=x^2-7x+10$, it appears that y = 0 if x = 2 or 5. By substituting into the function, we see $y=0$ if $x=2$ or 5. So, the intercepts are 2 and 5. For $x^2-7x+10\le 0$, the graph is below or on the x-axis. Therefore, the solution is $2\le x\le 5$.



d. $2x^2+5x-3>0$

From the graph of $f(x)=2x^2+5x-3$, it appears that the intercepts are –3 and 0.5. Using the $\boxed{\text{VALUE}}$ mode in the $\boxed{\text{CALC}}$ function or by substituting, we find $f(-3)=f(0.5)=0$. The solution to $2x^2+5x-3>0$ is the set of values for x for which $f(x)$ is above the x-axis, i.e., $x<-3$ or $x>0.5$.

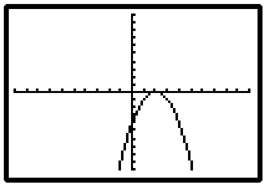


e. $-x^2 + 4x - 4 \geq 0$

For $y = f(x) = -x^2 + 4x - 4$, the intercept appears to be 2.

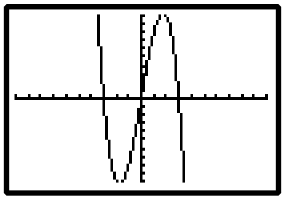
$f(2) = 0$

So, the solution to $-x + 4x - 4 \geq 0$ is the set of values for x where y is on or above the x-axis. But there is only one point that satisfies the condition, (2, 0), so the solution is $x = 2$.



f. $-x^3 + 9x \geq 0$

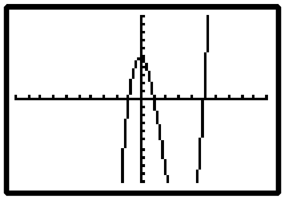
From the graph of $y = f(x) = -x^3 + 9x$, it appears the x-intercepts are –3, 0, and 3. Verifying this from $f(-3) = f(0) = f(3) = 0$, then the solution to $-x^3 + 9x \geq 0$ is the set of values for x where y is on or above the x-axis, i.e., $x \leq -3$ or $0 \leq x \leq 3$.



g. $x^3 - 5x^2 < x - 5$

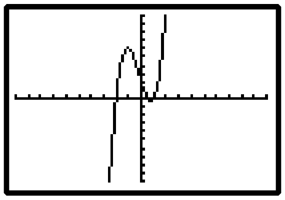
$x^3 - 5x^2 - x + 5 < 0$

The graph of $f(x) = x^3 - 5x^2 - x + 5$ is shown. We can verify intercepts at –1, 1, and 5 by using substitution or the $\boxed{\text{CALC}}$ function in $\boxed{\text{VALUE}}$ mode. The solution of $x^3 - 5x^2 - x + 5 < 0$ is the set of values for which $f(x)$ is below the x-axis, i.e., $x \leq -1$ or $1 \leq x \leq 5$.



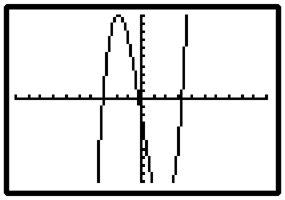
h. $2x^3 + x^2 - 5x + 2 \leq 0$

The graph of $f(x) = 2x^3 + x^2 - 5x + 2$ is shown. The intercepts appear to be –2, and between 0 and 1. By using the $\boxed{\text{CALC}}$ function and $\boxed{\text{VALUE}}$ and $\boxed{\text{ZERO}}$ modes, we find intercepts at –2, 0.5, and 1. The solution to $2x^3 + x^2 - 5x + 2 \leq 0$ is the set of values for x for which $f(x)$ is on or below the x-axis. The solution is $x \leq -2$ or $0.5 \leq x \leq 1$.



i. $x^3 - 10x - 2 \geq 0$

The graph of $f(x) = x^3 - 10x - 2$ is shown. The intercepts appear to be close to –3, 0, and 3. Using the $\boxed{\text{ZERO}}$ mode of the $\boxed{\text{CALC}}$ function, we find approximate x-intercepts at $x = -3.057$, -0.201, and 3.258. The solution will be those values for x for which $f(x)$ is on or above the x-axis. Then, for accuracy to one decimal place, the solution is $-3.1 \leq x \leq -0.2$ or $x \geq 3.3$.



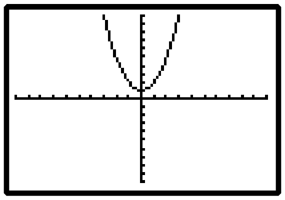
j. $x^2 + 1 > 0$

Solution 1

For all real values of x, $x^2 \geq 0$, so $x^2 + 1 \geq 0$. The solution is $x \in R$.

Solution 2

The graph of $f(x) = x^2 + 1$ shows all is above the x-axis. Therefore, the solution is R.



3. $v = -t^3 + 9t^2 - 27t + 21$

b. The intercept of the graph $v = -t^3 + 9t^2 - 27t + 21$ can be found to be $x \doteq 1.183$. For $v > 0$, $t < 1.183$ or t less than 60.3ºC.

c. Using the $\boxed{\text{TRACE}}$ function to find values for *t* that give values of *v* close to –20, and further refining the answer using the $\boxed{\text{VALUE}}$ mode in the $\boxed{\text{CALC}}$ function, we find $t \doteq 5.45$ when $v \doteq -20$. So, for $v < -20$, $t > 5.45$. The value of *t* is greater than $5.45\left(50^\circ\right)$ or 272.5ºC.

4. Graph $f\left(t\right) = 30t - 4.9t^2$ on your graphing calculator. Use the $\boxed{\text{TRACE}}$ function to find values for *t* that give values of *h* = 40. Using the $\boxed{\text{VALUE}}$ function in $\boxed{\text{CALC}}$ modes, we can refine our answers to give answers closer to 40. The projectile will be above 40 m between 2.0 s and 4.1 s after it is shot upwards.

5. Let the width of the base be *x* cm. The length is 2*x* cm, and the height is *h* cm. The total amount of wire is

$$\begin{aligned} 4(x) + 4(2x) + 4(h) &= 40 \\ 4x + 8x + 4h &= 40 \\ h &= \frac{40 - 12x}{4} \\ &= 10 - 3x. \end{aligned}$$

The volume of the solid is

$$\begin{aligned} v &= (x)(2x)(h) \\ &= (x)(2x)(10 - 3x) \\ &= -6x^3 + 20x^2. \end{aligned}$$

Graphing, $v = f\left(x\right) = -6x^3 + 20x^2$.

Use the $\boxed{\text{TRACE}}$ function to find values for *x* that give values of *f*(*x*) to be close to 2 and 4.

$x \doteq 0.34, v \doteq 2.08$
$x \doteq 0.51, v \doteq 4.42$
and $x \doteq 3.23, v \doteq 4.5$
$x \doteq 3.32, v \doteq 2.1.$

We can use the $\boxed{\text{VALUE}}$ mode in the $\boxed{\text{CALC}}$ function to find closer approximations. We investigate the larger values, since the total amount of wire is 40 cm.

When $x = 3.30, v = 2.18$
and $x = 3.27, v = 4.06.$

So, to have the solid with a volume of approximately between 2 cm^3 and 4 cm^3, the width of the box must be between 3.27 cm and 3.3 cm.

Exercise 2.6

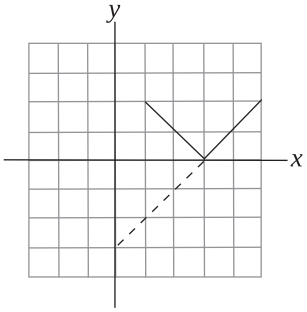
1. **a.** $\left|-3-7\right| = \left|-10\right|$
$= 10$

c. $\left|3\right| - \left|-5\right| + \left|3-9\right|$
$= 3 - (5) + \left|-6\right|$
$= 3 - 5 + 6$
$= 4$

d. $\left|9-3\right| + 5\left|-3\right| - 3\left|7-12\right|$
$= \left|6\right| + 5(3) - 3\left|-5\right|$
$= 6 + 15 - 3(5)$
$= 6$

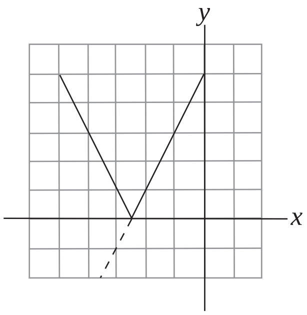
3. **a.** $f\left(x\right) = \left|x-3\right|,\ x \in R$

First, graph the line $f\left(x\right) = x - 3$. Then, reflect the portion of the graph that is below the *x*-axis in the *x*-axis so that *f*(*x*) is not negative.



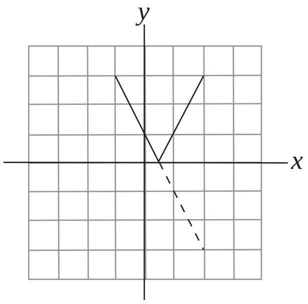
c. $h(x) = |2x + 5|$

Graph $h(x) = 2x + 5$. Then, reflect that portion of the graph below the *x*-axis in the *x*-axis.



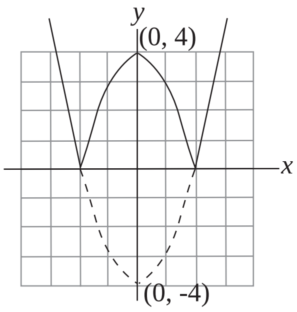
f. $g(x) = |1 - 2x|$

Graph $g(x) = 1 - 2x$. For the portion of the graph below the *x*-axis, reflect each point in the *x*-axis.



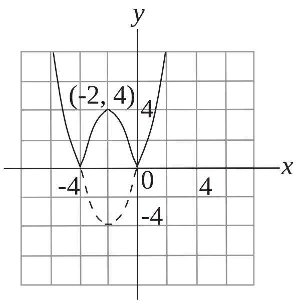
4. **a.** $y = |x^2 - 4|$

Graph the parabola $y = x^2 - 4$. Then, reflect the portion of the graph that is below the *x*-axis in the *x*-axis.



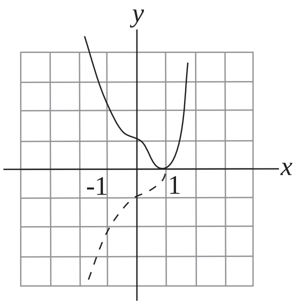
d. $y = |x^2 + 4x|$

First, graph the parabola $y = x^2 + 4x$. Then, reflect the portion of the graph where *y* is negative in the *x*-axis.

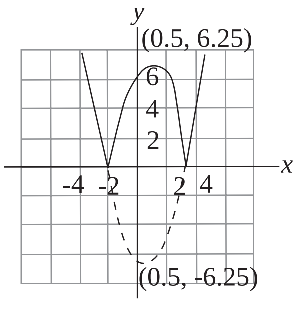


e. $y = |x^3 - 1|$

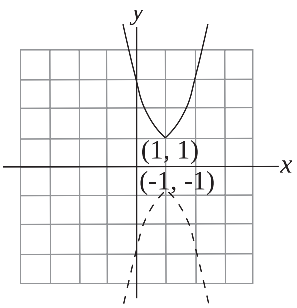
First, graph the cubic $y = x^3 - 1$. Then, reflect the portion of the graph where *y* is negative in the *x*-axis.



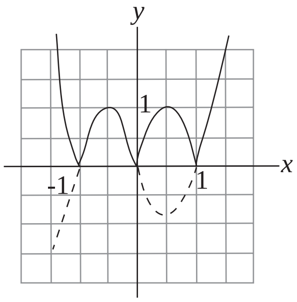
6. **a.** $y = |x^2 - x - 6|$



b. $y = |-2x^2 + 4x - 3|$



c. $y = \left| x^3 - x \right|$



7. **a.** $\left| 2x - 1 \right| = 7$

Since $(2x-1)$ is 7 units from the origin,

either $2x - 1 = 7$ or $2x - 1 = -7$

$2x = 8$ or $2x = -6$

$x = 4$ or $x = -3.$

b. $\left| 3x + 2 \right| = 6$

Since $3x + 2$ is 6 units from the origin,

either $3x + 2 = 6$ or $3x + 2 = -6$

$3x = 4$ or $3x = -8$

$x = \frac{4}{3}$ or $x = -\frac{8}{3}.$

c. $\left| x - 3 \right| \le 9$

If $\left| x - 3 \right| \le 9$, then $(x-3)$ lies between –9 and 9 on the number line:

$$-9 \le x - 3 \le 9.$$

Add 3: $-6 \le x \le 12.$

d. $\left| x + 4 \right| \ge 5$

$(x+4)$ lies beyond 5 and –5 on the number line,

so either $x + 4 \ge 5$ or $x + 4 \le -5$

$x \ge 1$ or $x \le -9.$

e. $\left| 2x - 3 \right| < 4$

Then $-4 < 2x - 3 < 4.$

Add 3: $-1 < 2x < 7$

Divide by 2: $-\frac{1}{2} < x < \frac{7}{2}$

f. $\left| x \right| = -5$

Since $\left| x \right|$ is always a positive number, there is no value of x for which $\left| x \right|$. Therefore, there is no solution.

8. **a.** $\left| x \right| = 3x + 4$

By definition $\left| x \right| = \begin{cases} x \text{ if } x \ge 0 \\ -x \text{ if } x < 0 \end{cases}$

Therefore, if $x \ge 0$,

then $x = 3x + 4$

$-2x = 4$

$x = -2,$

but only if $x \ge 0$, $\therefore x \ne -2,$

and if $x < 0$,

then $-x = 3x + 4$

$-4x = 4$

$x = -1.$

Therefore, the solution is $x = -1$.

b. $\left| x - 5 \right| = 4x + 1$

Solution 1

By definition, if $x - 5 \ge 0$, then $x - 5 = 4x + 1$

if $x \ge 5$ $-3x = 6$

$x = -2.$

But $x \ge 5$, $\therefore x \ne -2.$

Also, if $x - 3 < 0$, then $-(x-5) = 4x + 1$

$-5x = -4$

$x = \frac{4}{5}$ or 0.8.

Solution 2

First graph $y = \left| x - 5 \right|$

and $y = 4x + 1.$

The point of intersection is $(0.8, 4.2)$

so $\left| x - 5 \right| = 4x + 1$ when $x = 0.8.$

c. $|4x-8|=2x$

If $4x-8\ge 0,$ then $4x-8=2x$

$4x\ge 8$ $\quad$ $2x=8$

$x\ge 2$ $\quad$ $x=4$

If $4x-8<0,$ then $4x-8=-2x$

$4x<8$ $\quad$ $6x=8$

$x<2$ $\quad$ $x=\frac{8}{6}=\frac{4}{3}.$

The solution $x=4$ or $x=\frac{4}{3}$ can be verified by graphing $y=|4x-8|$ and $y=2x$ and checking that the points of intersection occur when $x=4,\ x=\frac{4}{3}.$

d. $|x-1|<x$

Solution 1

Graphing $y_1=|x-1|$ and $y_2=x$ yields the following angle. We need to find the values for x for which $y_1<y_2$. Since the point of intersection is

$\left(\frac{1}{2},\frac{1}{2}\right),$ $y_1<y_2$ when $x>\frac{1}{2}.$

Solution 2

Consider $|x-1|=x$

Then either $x-1=x$ or $x-1=-x$

$0x=4$ or $2x=1$

No solution. $\quad$ $x=\frac{1}{2}.$

Since $|x-1|=x$ when $x=\frac{1}{2}$, we test points on either side of $\frac{1}{2}$ on the number line to find for what values of x does $|x-1|<x.$

Test: $x=0$

L.S. $=|0-1|$ $\quad$ R.S. $=0$

$=1$

Since L.S. $\not<$ R.S., $x\ne 0$

Test: $x=1$

L.S. $=|1-1|$ $\quad$ R.S. $=1$

$=0$

Since L.S. < R.S. $x=1$.

Therefore, the solution set is $\left\{x|x>\frac{1}{2}\right\}.$

Solution 3

By definition of absolute value,

if $x-1\ge 0,$ then $x-1<x$

$x\ge 1$ $\quad$ $0x<1.$

This is true for all x, $x\in R$.

And if $x-1<0,$ then $-x+1<x$

$x<1$ $\quad$ $-2x<-1$

$x>\frac{1}{2}.$

Therefore, the solution set is $\left\{x|x>\frac{1}{2}\right\}.$

e. $|2x+4|\ge 12x$

Consider $|2x+4|=12x.$

Then, either $2x+4=12x$ or $2x+4=-12x$

$-10x=-4$ or $14x=-4$

$x=\frac{4}{10}=\frac{2}{5}$ or $x=-\frac{4}{14}=-\frac{2}{7}.$

But substituting each value into the equation gives only one solution, that is $x=\frac{2}{5}.$

Test values for x on either side of $x=\frac{2}{5}.$

Let $x=0$

L.S. $=|2(0)+4|$

$=4$

R.S. $=12(0)$

$=0$

Since L.S. > R.S., $x=0$

Let $x=1$

L.S. $=|2(1)+4$

$=6$

R.S. $=12(1)$

$=12$

Since L.S. $\not>$ R.S, $x\ne 1$.

Therefore, the solution set is $\left\{x|x\le\frac{2}{5}\right\}.$

This solution can be verified by graphing $y_1=|2x+4|$ and $y_2=|12x|$ and noting that $y_1\ge y_2$ when $x\le\frac{2}{5}.$

f. $|3x-1| \le 5|3x-1|-16$

Consider $|3x-1| = 5|3x-1|-16.$

Either $(3x-1) = 5(3x-1)-16$ or $-(3x-1) = 5(-3x+1)-16$

$$3x-1 = 15x-5-16 \qquad -3x+1 = -15x+5-16$$

$$-12x = -20 \qquad 12x = -12$$

$$x = \frac{20}{12} = \frac{5}{x} \qquad x = -1.$$

Both answers verify when substituted into the equation. Now, to find which values satisfy the inequality, we can use test values between and beyond –1 and $\frac{5}{3}$.

Test: $x = 0$

$\text{L.S.} = |3(0)-1|$ $\text{R.S.} = |3(0)-1|-16$

$= 1$ $= -11$

Since L.S. $\not\le$ R.S. , then $x = 0$

Test: $x = -2$

$\text{L.S.} = |3(-2)-1|$ $\text{R.S.} = 5|3(-2)-1|-16$

$= 7$ $= 19$

Since L.S. > R.S., then $x = -2$.

Test: $x = 2$

$\text{L.S.} = |3(2)-1|$ $\text{R.S.} = 5|3(2)-1|-16$

$= 5$ $= 9$

Since L.S. < R.S., then $x = 2$.

So, the solution set is $\left\{x \middle| x \le -1 \text{ or } x \ge \frac{5}{3}\right\}$.

Or,
we can graph $y_1 = |3x-1|$ and $y_2 = 5|3x-1|-16$ and, using the values of x found earlier, locate those values of x for which $y_1 \le y_2$.

g. $|x-2|+|x| = 6$

Solution 1

Graph $y_1 = |x-2|+|x|$ and $y_2 = 6$. The points of intersection are the points where $y_1 = y_2$, $\therefore x = -2$ or 4.

Solution 2

Since we need to concern ourselves when $|f(x)| = f(x)$ or $-f(x)$, we use the cases where
$x < 0$, $0 < x < 2$, and $x > 2$.

Case 1:

If $x < 0$

then $|x-2|+|x| = 6$

becomes $-x+2-x = 6$

$-2x = 4$

$x = 2.$

$\therefore x = -2$ or $x = 4$

Case 2:

If $0 < x < 2$

then $|x-2|+|x| = 6$

becomes $-x+2+x = 6$

$0x = 4$

No solution.

Case 3:

If $x > 0$,

then $|x-2|+|x| = 6$

becomes $x-2+x = 6$

$2x = 8$

$x = 4.$

h. $|x+4| - |x-1| = 3$

Graph $y_1 = |x+4| - |x-1|$ and $y_2 = 3$.
Since the point of intersection is (0, 3), $y_1 = y_2$ when $x = 0$. Therefore, the solution is $x = 0$.

9. Since $|x-|x||$ is always positive, then $\frac{|x-|x||}{x}$ is positive when $x > 0$.

Since $x > 0$, $|x| \ge 0$

$$\therefore |x-|x||$$

$$= |x-x|$$

$$= 0.$$

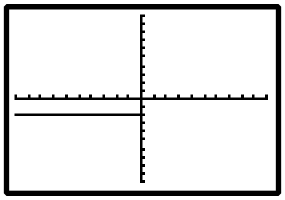
But $\frac{0}{x} = 0$

Therefore, there are no values for x for $\frac{|x-|x||}{x}$ which is a positive integer.

10. *Solution 1*

Using a table of values, we find

x	$f(x)$
–3	$\frac{\lvert -3-\lvert -3\rvert\rvert}{-3}=-2$
–2	$\frac{\lvert -2-\lvert -2\rvert\rvert}{-2}=-2$
–1	$\frac{\lvert -1-\lvert -1\rvert\rvert}{-1}=-2$
0	undefined
1	$\frac{\lvert 1-\lvert 1\rvert\rvert}{1}=0$
2	$\frac{\lvert 2-\lvert 2\rvert\rvert}{2}=0$



Solution 2

Use a graphing calculator to find the graph. Using the CALC mode and the VALUE function, we see $x=0$ gives no answer and is not included in the graph.

Review Exercise

2. **a.** If the x-intercepts are 4, 1, and –2, then $(x-4)$, $(x-1)$, and $(x+2)$ are factors of the cubic function. Therefore, $y=a(x-4)(x-1)(x+2)$, where a is a constant, represents the family of cubic functions.

3. **a.** Let $f(x)=x^5-4x^3+x^2-3$

$$f(-2)=(-2)^5-4(-2)^3+(-2)^2-3$$
$$=-32+32+4-3$$
$$=1$$

Since $f(-2)\neq 0$, $x+2$ is not a factor.

4. Let $f(x)=x^3-6x^2+6x-5$

$$f(5)=5^3-6(5)^2+6(5)-5$$
$$=0.$$

Therefore, $(x-5)$ is a factor of $f(x)$.

By division, x^3-6x^2+6x-5
$$=(x-5)(x^2-x+1).$$

5. **a.** Since $(x-1)$ is a factor of $x^3-3x^2+4kx-1$, then $f(1)=0$.

Substituting, $1^3-3(1)^2+4k(1)-1=0$
$$1-3+4k-1=0$$
$$4k=3$$
$$k=\frac{3}{4}.$$

6. **a.** Let $f(x)=x^3-2x^2+2x-1$

$$f(1)=1^3-2(1)^2+2(1)-1$$
$$=1-2+2-1$$
$$=0.$$

Therefore, $(x-1)$ is a factor of $f(x)$.

By dividing, x^3-2x^2+2x-1
$$=(x-1)(x^2-x+1).$$

b. Let $f(x)=x^3-6x^2+11x-6$

$$f(1)=1^3-6(1)^2+11(1)-6$$
$$=0.$$

Therefore, $(x-1)$ is a factor of $f(x)$.

By dividing, $x^3-6x^2+11x-6$
$$=(x-1)(x^2-5x+6)$$
$$=(x-1)(x-2)(x-3).$$

7. Since x^2-4x+3
$$=(x-3)(x-1),$$

both $f(3)$ and $f(1)$ must be equal to 0 in order to have x^2-4x+3 be a factor of $f(x)=x^5-5x^4+7x^3-2x^2-4x+3$.

$$f(1)=1^5-5(1)^4+7(1)^3-2(1)^2-4(1)+3$$
$$=1-5+7-2-4+3$$
$$=0$$

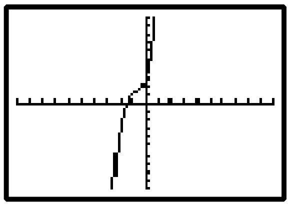
Also, $f(3) = 3^5 - 5(3)^4 + 7(3)^3 - 2(3)^2 - 4(3) + 3$

$$= 243 - 405 + 189 - 18 - 12 + 3$$
$$= 0$$

Therefore, $(x-1)$ and $(x-3)$ are factors of $f(x)$, and so $(x-1)(x-3)$ or $x^2 - 4x + 3$ is a factor of $f(x)$.

8. **a.** Graphing $y = 2x^3 + 5x^2 + 5x + 3$ yields the graph below. Using the $\boxed{\text{CALC}}$ mode and the $\boxed{\text{VALUE}}$ function, we find when $x = -1.5$, $y = 0$.

Therefore, $f\left(-\dfrac{3}{2}\right)$ is 0. So, $(2x+3)$ is a factor of $f(x)$.

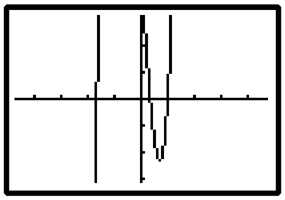


Therefore, $2x^3 + 5x^2 = 5x + 3$

$$= (2x+3)(x^2 + x + 1)$$

by division.

b. Graphing $y = 9x^3 + 3x^2 - 17x + 5$ yields the graph below. We can see the x-intercept is between –2 and –1 and perhaps 1. $x = 1$ can be verified by using the $\boxed{\text{VALUE}}$ function in the $\boxed{\text{CALC}}$ mode. Therefore, $x - 1$ is a factor of $f(x)$.



By dividing, $9x^3 + 3x^2 - 17x + 5$

$$= (x-1)(9x^2 + 12x - 5)$$
$$= (x-1)(3x+5)(3x-1)$$

The other factors can be tested in the same way as $x = 1$, i.e., let $x = -\dfrac{5}{3}$ and $x = \dfrac{1}{3}$.

9. For $f(x) = 5x^4 - 2x^3 + 7x^2 - 4x + 8$,

$f\left(\dfrac{p}{q}\right) = 0$ if q divides into 5 and p into 8.

a. If $\dfrac{p}{q} = \dfrac{5}{4}$, since 5 divides into 5 and 5 into 8,

then it is possible for $f\left(\dfrac{5}{4}\right)$ to be 0.

b. If $\dfrac{p}{q} = \dfrac{4}{5}$, since 4 does not divide into 5, then it is

not possible for $f\left(\dfrac{4}{5}\right) = 0$.

10. **a.** Let $f(x) = 3x^3 - 4x^2 + 4x - 1$

Try $f(1) = 3(1)^3 - 4(1)^2 + 4(1) - 1$

$$= 3 - 4 + 4 - 1$$
$$\neq 0$$

$$f(-1) = 3(-1)^3 - 4(-1)^2 + 4(-1) - 1$$
$$= -3 - 4 - 4 - 1$$
$$\neq 0.$$

Therefore, the only binomial factor with integer coefficients must be either $(3x-1)$ or $(3x+1)$. From the graph, we see an x-intercept between 0 and 1, so $(3x – 1)$ is a possible factor.

By division, $3x^3 - 4x^2 + 4x - 1$

$$= (3x-1)(x^2 - x + 1).$$

b. First, graph $y = 2x^3 + x^2 - 13x - 5$ on your calculator.

We see intercepts $k = \dfrac{p}{q}$ between –3 and –2, –1 and 0, 2, and 3. Where q divides into 2 and p divides into 5, we try $k = \dfrac{5}{2}$, $f\left(\dfrac{5}{2}\right) = 0$.

Therefore, $(2x-5)$ is a factor of $f(x)$.

By division, $2x^3 + x^2 - 13x - 5$

$$= (2x-5)(x^2 + kx + 1).$$

$$= 2x^3 + (-5+2k)x^2 + \ldots$$

By comparing coefficients,
$$\begin{aligned} -5 + 2k &= 1 \\ +2k &= 6 \\ k &= 3 \end{aligned}$$

Therefore,
$$\begin{aligned} &2x^3 + x^2 - 13x - 5 \\ &= (2x-5)(x^2+3x+1). \end{aligned}$$

c. Graphing $y = 30x^3 - 31x^2 + 10x - 1$ on your calculator, it can be seen that there is only one value for $k = \dfrac{p}{q}$ and it lies between 0 and 1. Since q divides into 30 and p into 1, we try $k = \dfrac{1}{5}$, $f(0.2) = 0$. Therefore, $(5x - 1)$ is a factor of $f(x)$.

By dividing,
$$\begin{aligned} &30x^3 - 31x^2 + 10x - 1 \\ &= (5x-1)(6x^2+kx+1) \\ &= 30x^3 + (-6+5k)x^2 + \ldots \end{aligned}$$

Comparing coefficients, we have
$$\begin{aligned} -6 + 5k &= -31 \\ 5k &= -25 \\ k &= -5. \end{aligned}$$

Therefore,
$$\begin{aligned} &30x^3 - 31x^2 + 10x - 1 \\ &= (5x-1)(6x^2-5x+1) \\ &= (5x-1)(3x-1)(2x-1). \end{aligned}$$

11. c.
$$\begin{aligned} x^3 + 8 &= 0 \\ (x+2)(x^2-2x+4) &= 0 \end{aligned}$$
$x + 2 = 0$ or $x^2 - 2x + 4 = 0$

$x = 0$ or $x = -2$ or $x + 5 = 0$

$x = -2$ or $x = \dfrac{2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$
$$\begin{aligned} &= \frac{2 \pm \sqrt{-12}}{2} \\ &= \frac{2 \pm 2i\sqrt{3}}{2} \\ &= 1 \pm i\sqrt{3} \end{aligned}$$

Solution set is $\left\{-2,\ 1 \pm i\sqrt{3}\right\}$.

d.
$$\begin{aligned} x^3 - x^2 - 9x + 9 &= 0 \\ x^2(x-1) - 9(x-1) &= 0 \\ (x-1)(x^2-9) &= 0 \\ (x-1)(x-3)(x+3) &= 0 \end{aligned}$$
$x - 1 = 0$ or $x - 3 = 0$ or $x + 3 = 0$

$x = 1$ or $x = 3$ or $x = -3$

e.
$$\begin{aligned} x^4 - 12x^2 - 64 &= 0 \\ (x^2-16)(x^2+4) &= 0 \\ (x-4)(x+4)(x^2+4) &= 0 \end{aligned}$$
$x - 4 = 0$ or $x + 4 = 0$ or $x^2 + 4 = 0$

$x = 4$ or $x = -4$ or $x^2 = -4$
$$\begin{aligned} x &= \pm\sqrt{-4} \\ &= \pm 2i \end{aligned}$$

f. $x^3 - 4x^2 + 3 = 0$

Let $f(x) = x^3 - 4x^2 + 3$
$$\begin{aligned} f(1) &= 1^3 - 4(1)^2 + 3 \\ &= 0. \end{aligned}$$

Therefore, $(x-1)$ is a factor of $f(x)$.

So,
$$\begin{aligned} x^3 - 4x^2 + 3 &= 0 \\ (x-1)(x^2-3x-3) &= 0 \text{, by dividing} \end{aligned}$$
$x - 1 = 0$ or $x^2 - 3x - 3 = 0$

$x = 1$ or $x = \dfrac{3 \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2(1)}$
$$= \frac{3 \pm \sqrt{21}}{2}.$$

g. $x^3 - 3x^2 + 3x - 2 = 0$

Let $f(x) = x^3 - 3x^2 + 3x - 2$
$$\begin{aligned} f(2) &= 2^3 - 3(2)^2 + 3(2) - 2 \\ &= 8 - 12 + 6 - 2 \\ &= 0. \end{aligned}$$

Therefore, $(x-2)$ is a factor of $f(x)$.
Note: To select which integer factor to try, first graph $y = f(x)$ and note where the x-intercept lies.

Dividing to find the other factor, we find

$$(x-2)(x^2-x+1)=0$$

$$x-2=0 \quad \text{or} \quad x^2-x+1=0$$

$$x=2 \quad \text{or} \quad x=\frac{1\pm\sqrt{1^2-4(1)(1)}}{2(1)}$$

$$=\frac{1\pm i\sqrt{3}}{2}$$

h. $$x^6-26x^3-27=0$$

$$(x^3-27)(x^3+1)=0$$

$$(x-3)(x^2+3x+9)(x+1)(x^2-x+1)=0$$

$$x-3=0 \quad \text{or} \quad x^2+3x+9=0$$

$$x=3 \quad \text{or} \quad x=\frac{-3\pm\sqrt{9-4(9)}}{2}$$

$$=\frac{-3\pm 3i\sqrt{3}}{2}$$

$$\text{or} \quad x+1=0 \quad \text{or} \quad x^2-x+1=0$$

$$\text{or} \quad x=-1 \quad \text{or} \quad x=\frac{1\pm\sqrt{1-4(1)}}{2}$$

$$=\frac{1\pm i\sqrt{3}}{2}$$

The solution set is $\left\{3,\ -1,\ \frac{-3\pm 3i\sqrt{3}}{2},\ \frac{1\pm i\sqrt{3}}{2}\right\}$.

i. $$(x^2+2x)^2-(x^2+2x)-12=0$$

Let $a=x^2+2x$

$$a^2-a-12=0$$

$$(a-4)(a+3)=0$$

$$a-4=0 \quad \text{or} \quad a+3=0.$$

But $a=x^2+2x$

Therefore, $x^2+2x-4=0$ or $x^2+2x+3=0$

$$x=\frac{-2\pm\sqrt{2^2-4(1)(-4)}}{2(1)} \qquad x=\frac{-2\pm\sqrt{2^2-4(1)(3)}}{2(1)}$$

$$=\frac{-2\pm\sqrt{20}}{2} \qquad =\frac{-2\pm\sqrt{-8}}{2}$$

$$=-1\pm\sqrt{5} \qquad =\frac{-2\pm 2i\sqrt{2}}{2}$$

$$=-1\pm i\sqrt{2}.$$

Note: Graphing $y=(x^2+2x)^2-(x^2+2x)-12$ confirms the existence of only 2 real roots.

12. c. $x^3-x^2-4x-1=0$

The graph of $y=x^3-x^2-4x-1$ shows 3 real roots between –2 and –1, –1 and 0, and 2 and 3. Using the $\boxed{\text{ZERO}}$ function in $\boxed{\text{CALC}}$ mode, we find $x\doteq-1.377$, $x\doteq-0.274$, $x\doteq 2.651$.

13. If –2 is a root of $x^2+kx-6=0$, where $f(x)=x^2+kx-6$, it means $f(-2)=0$. Substituting to find k, we get

$$(-2)^2+k(-2)-6=0$$

$$4-2k-6=0$$

$$-2k=2$$

$$k=-1$$

$$\therefore x^2-x-6=0.$$

Also, $(x+2)$ is a factor of $f(x)=x^2-x-6$.

By dividing, the other factor is $(x-3)$.

$$\therefore x-3=0$$

$$x=3$$

So, $k=-1$, and the other root is 3.

14. Let r_1, r_2 be the roots of $2x^2+5x+1=0$. Therefore,

$r_1+r_2=-\frac{5}{2}$ and $r_1r_2=\frac{1}{2}$. The roots of the required equation are $x_1=\frac{1}{r_1}$ and $x_2=\frac{1}{r_2}$. The sum of the new roots is

$$x_1+x_2=\frac{1}{r_1}+\frac{1}{r_2}$$

$$=\frac{r_2+r_1}{r_1r_2}$$

$$=\frac{-\frac{5}{2}}{\frac{1}{2}}=-5.$$

The product of the new roots is

$$x_1+x_2=\frac{1}{r_1r_2}$$

$$=2.$$

Therefore, the new equation is

$$x^2-(x_1+x_2)x+x_1x_2=0$$

or $$x^2-(-5)x+2=0$$

$$x^2+5x+2=0.$$

15. a. Since

$x^2 -$ (sum of roots) x + (product of roots) = 0,

for $2x^2-x+4=0$, the sum of the roots

is $\frac{1}{2}$, and the product is $\frac{4}{2}$ or 2.

b. Let x_1, and x_2 be the roots of the quadratic equation

$x_1+x_2=\frac{1}{15}$ and $x_1+x_2=-\frac{2}{15}$. The equation is

$x^2-\frac{1}{15}x-\frac{2}{15}=0$ or $15x^2-x-2=0$.

c. Let the roots of the quadratic equation be x_1 and x_2.

$$\begin{aligned}x_1+x_2&=(3+2i)+(3-2i)\\&=6\\x_1x_2&=(3+2i)(3-2i)\\&=9-4i^2\\&=9+4\\&=13\end{aligned}$$

The required equation is $x^2-6x+13=0$.

d.

Solution 1

$3x^2+4kx-4=0$ where $f(x)=3x^2+4kx-4$

If 2 is one root, then $f(2)=0$.

Substituting, we have

$$\begin{aligned}3(2)^2+4k(2)-4&=0\\12+8k-4&=0\\8k&=-8\\k&=-1.\end{aligned}$$

Therefore, the equation can be written as $3x^2-4x-4=0$. If 2 is one root, then $(x-2)$ is a factor of the function $f(x)$; therefore, $3x^2-4x-4=0$ becomes $(x-2)(3x+2)=0.$

The other root can be found from

$$\begin{aligned}3x+2&=0\\x&=-\frac{2}{3}.\end{aligned}$$

Therefore, the other root is $-\frac{2}{3}$ and $k=-1$.

Solution 2

Let h represent the other root of $3x^2+4kx-4=0$.

The sum of the roots is $h+2=-\frac{4k}{3}.$ $\quad(1)$

The product of the roots is $2h=-\frac{4}{3}.$ $\quad(2)$

Therefore, $h=-\frac{2}{3}.$

Substituting into (1) to find k,

$$\begin{aligned}-\frac{2}{3}+2&=-\frac{4k}{3}\\-2+6&=-4k\\4k&=-4\\k&=-1\end{aligned}$$

e. Let x_1 and x_2 represent the roots of

$x^2-5x+2=0.$

$x_1+x_2=5$ and $x_1x_2=2.$

The roots of the required equation are x_1-3 and x_2-3.

For the new equation, the sum of the roots is

$$\begin{aligned}&(x_1-3)+(x_2-3)\\&=x_1+x_2-6\\&=5-6\\&=-1.\end{aligned}$$

The product of the new roots is

$$\begin{aligned}&(x_1-3)(x_2-3)\\&=x_1x_2-3(x_1+x_2)+9\\&=2-3(5)+9\\&=-4.\end{aligned}$$

The required equation is $x^2+x-4=0.$

f. Let x_1 and x_2 represent the roots of $2x^2 + x - 4 = 0$.

$$x_1 + x_2 = -\frac{1}{2} \text{ and } x_1x_2 = -\frac{4}{2} = -2$$

The roots of the required equation are $\frac{1}{x_1}$ and $\frac{1}{x_2}$.

For the new equation, the sum of the roots is

$$\begin{aligned}\frac{1}{x_1} + \frac{1}{x_2} &= \frac{x_2 + x_1}{x_1x_2} \\ &= \frac{-\frac{1}{2}}{-2} \\ &= \frac{1}{4}.\end{aligned}$$

and the product is

$$\begin{aligned}\left(\frac{1}{x_1}\right)\left(\frac{1}{x_2}\right) &= \frac{1}{x_1x_2} \\ &= \frac{1}{-2}.\end{aligned}$$

The required equation is $x^2 - \frac{1}{4}x - \frac{1}{2} = 0$

or $4x^2 - x - 2 = 0$.

16. a. $(x-2)(x+4) < 0$

Solution 1

Graph $y = (x-2)(x+4)$.

Since y is below the x-axis between –4 and 2, therefore, the solution is x such that $-4 < x < 2$.

Solution 2

Consider $(x-2)(x+4) = 0$.

Therefore, $x = 2$ or $x = -4$.

Test:

$x = -5$	$x = 0$	$x = 3$
$\text{L.S.} = (-5-2)(-5+4)$	$\text{L.S.} = (-5-2)(-5+4)$	$\text{L.S.} = (3-2)(3+4)$
$=7$	$= -6$	$=7$
But, $\text{L.S.} \nless 0$	$\text{L.S.} < 0$	$\text{L.S.} \nless 0$
$\therefore x \neq -5$	$\therefore x = 0$	$\therefore x \neq 3$

The solution set is $\{x \mid -4 < x < 2\}$.

Solution 3

For the product $(x-2)(x+4)$ to be negative, there are two cases.

Case 1: $x - 2 > 0$ and $x + 4 < 0$
$x > 0$ and $x < -4$

No solution.

Case 2: $x - 2 < 0$ and $x + 4 > 0$
$x < 2$ and $x < -4$

The solution is $-4 < x < 2$.

b. $x^2 + x - 2 \geq 0$
$(x+2)(x-1) \geq 0$

Consider the graph of $y = (x+2)(x-1)$. The values that satisfy the inequality are the values for x for which the y values are on or above the x-axis. The solution is $x \leq -2$ or $x \geq 1$.

c. $x^3 + 3x \leq 0$
$x(x^2 + 3) \leq 0$

Consider the graph of $y = x^3 + 3x$. The solution is those values for x where y is below or on the x-axis, i.e., for $x \leq 0$.

d. $x^3 - 2x^2 - x + 2 > 0$

The graph of $y = x^3 - 2x^2 - x + 2$ is shown with x-intercepts at –1, 1, and 2 as confirmed by using the $\boxed{\text{CALC}}$ mode and $\boxed{\text{VALUE}}$ function. The solution to the inequality is those values for x where y is above the x-axis, that is $-1 < x < 1$ or $x > 2$.

e. $x^4 \leq 0$

Since x^4 always returns a positive or zero for any value of x, the only solution is $x = 0$. This can be verified graphically by noting that the graph of $y = x^4$ is never below the x-axis.

f. $x^4 + 5x^2 + 2 \geq 0$

Solution 1

From the graph of $y = x^4 + 5x^2 + 2$, we see that *y* is always above the *x*-axis.

Solution 2

Since x^4 and x^2 are always positive, $x^4 + 5x^2 + 2$ is always greater than zero. The solution set is *R*.

17. a. $|3x - 1| = 1$

$$\text{Either } 3x - 1 = 11 \quad \text{or} \quad 3x - 1 = -11$$
$$3x = 12 \qquad\qquad 3x = -10$$
$$x = 4 \qquad\qquad x = -\frac{10}{3}.$$

By substituting into the equation, we can verify both answers are correct.

18. The dimensions of the open box are $8 - 2x$, $6 - 2x$, and *x*. The volume is 16 cm^3 or

$$\begin{aligned} x(8-2x)(6-2x) &= 16 \\ 48x - 28x^2 + 4x^3 &= 16 \\ 4x^3 - 28x^2 + 48x - 16 &= 0 \\ x^3 - 7x^2 + 12x - 4 &= 0. \end{aligned}$$

By graphing $y = x^3 - 7x^2 + 12x - 4$, we find only one real root at $x \doteq 5.11$, but this is an inadmissable root as $x < 3$. Therefore, it is impossible to make a box from this rectangular sheet.

Chapter 2 Test

1. Let $f(x) = x^3 - 5x^2 + 9x - 3$

$$\begin{aligned} f(-3) &= (-3)^3 - 5(-3)^2 + 9(-3) - 3 \\ &= -27 - 45 - 27 - 3 \\ &\neq 0. \end{aligned}$$

$\therefore (x+3)$ is not a factor of $f(x)$.

2. a. $x^3 + 3x^2 - 2x - 2$

Let $f(x) = x^3 + 3x^2 - 2x - 2$

$$\begin{aligned} f(1) &= 1^3 + 3(1)^2 - 2(1) - 2 \\ &= 0. \end{aligned}$$

$\therefore (x-1)$ is a factor of *f*(*x*).

By dividing, $x^3 + 3x^2 - 2x - 2 = (x-1)(x^2 + 4x + 2)$.

b. $2x^3 - 7x^2 + 9$

Let $f(x) = 2x^3 - 7x^2 + 9$

$$\begin{aligned} f(-1) &= 2(-1)^3 - 7(-1)^2 + 9 \\ &= -2 - 7 + 9 \\ &= 0. \end{aligned}$$

$\therefore (x+1)$ is a factor of *f*(*x*).

By dividing, we find

$$\begin{aligned} &2x^3 - 7x^2 + 9 \\ &= (x+1)(2x^2 - 9x + 9) \\ &= (x+1)(2x-3)(x-3). \end{aligned}$$

c. $x^4 - 2x^3 + 2x - 1$

Let $f(x) = x^4 - 2x^3 + 2x - 1$

$$\begin{aligned} f(1) &= 1 - 2 + 2 - 1 \\ &= 0. \end{aligned}$$

Also, $f(-1) = (1) + (2) - 2 - 1$
$= 0.$

$\therefore (x-1), (x+1)$ are factors of $x^4 - 2x^3 + 2x - 1$.

By dividing,
$$\begin{aligned} &x^4 - 2x^3 + 2x - 1 \\ &= (x^2 - 1)(x^2 - 2x + 1) \\ &= (x-1)(x+1)(x-1)(x-1) \\ &= (x+1)(x-1)^3. \end{aligned}$$

3. Graphing $y = 3x^3 + 4x^2 + 2x - 4$ shows one *x*-intercept between 0 and 1. So, $k = \dfrac{q}{p}$ where *p* is a divisor of 3 and *q* is a divisor of 4. Trying $k = \dfrac{q}{3}$ with the $\boxed{\text{VALUE}}$ function in $\boxed{\text{CALC}}$ mode gives $y = 0$.

$\therefore (3x - 2)$ is a factor of $3x^3 + 4x^2 + 2x - 4$.

By dividing,
$$\begin{aligned} &3x^3 + 4x^2 + 2x - 4 \\ &= (3x-2)(x^2 + kx + 2) \\ &= 3x^3 + (-2 + 3k)x^2 + \cdots \end{aligned}$$

Comparing coefficients,
$$\begin{aligned} -2 + 3k &= 4 \\ 3k &= 6 \\ k &= 2. \end{aligned}$$

$$\begin{aligned} \therefore\ &3x^3 + 4x^2 + 2x - 4 \\ &= (3x-2)(x^2 + 2x + 2) \end{aligned}$$

4. **a.**

$$
\begin{aligned}
2x^3 - 54 &= 0\\
2\left(x^3 - 27\right) &= 0\\
2(x-3)\left(x^2+3x+9\right) &= 0
\end{aligned}
$$

$x - 3 = 0$ or $x^2 + 3x + 9 = 0$

$$
\begin{aligned}
x = 3 \quad \text{or} \quad x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2(1)}\\
&= \frac{-3 \pm \sqrt{-27}}{2}\\
&= \frac{-3 \pm 3i\sqrt{3}}{2}
\end{aligned}
$$

b. $x^3 - 4x^2 + 6x - 3 = 0$

Let $f(x) = x^3 - 4x^2 + 6x - 3$

$$
\begin{aligned}
f(1) &= 1 - 4 + 6 - 3\\
&= 0.
\end{aligned}
$$

$\therefore (x-1)$ is a factor of $f(x)$.

$$
\begin{aligned}
x^3 - 4x^2 + 6x - 3 &= 0\\
(x-1)\left(x^2 - 3x + 3\right) &= 0
\end{aligned}
$$

$x - 1 = 0$ or $x^2 - 3x + 3 = 0$

$$
\begin{aligned}
x = 1 \quad \text{or} \quad x &= \frac{3 \pm \sqrt{9 - 4(1)(3)}}{2(1)}\\
&= \frac{3 \pm \sqrt{-3}}{2}\\
&= \frac{3 \pm i\sqrt{3}}{2}
\end{aligned}
$$

c. $2x^3 - 7x^2 + 3x = 0$

$$
\begin{aligned}
x\left(2x^2 - 7x + 3\right) &= 0\\
x(2x-1)(x-3) &= 0
\end{aligned}
$$

$x = 0$ or $2x^2 - 1 = 0$ or $x - 3 = 0$

$x = 0$ or $x = \frac{1}{2}$ or $x = 3$

d.

$$
\begin{aligned}
x^4 - 5x^2 + 4 &= 0\\
\left(x^2 - 4\right)\left(x^2 - 1\right) &= 0\\
(x-2)(x+2)(x-1)(x+1) &= 0
\end{aligned}
$$

$x - 2 = 0$ or $x + 2 = 0$ or $x - 1 = 0$

or $x + 1 = 0$

$x = 2$ or $x = -2$ or $x = 1$ or $x = -1$

5. Let the roots of $x^2 - 2x + 5 = 0$ be x_1 and x_2. The sum is 2 and the product is 5. The roots of the required equation are $x_1 + 3$ and $x_2 + 3$. The sum of the new roots is

$$
\begin{aligned}
&(x_1 + 3) + (x_2 + 3)\\
&= (x_1 + x_2) + 6\\
&= 2 + 6\\
&= 8.
\end{aligned}
$$

The product of the new roots is

$$
\begin{aligned}
&(x_1 + 3)(x_2 + 3)\\
&x_1x_2 + 3(x_1 + x_2) + 9\\
&= 5 + 3(2) + 9\\
&= 20.
\end{aligned}
$$

The required quadratic equation is $x^2 - 8x + 20 = 0$.

7. **a.** $(x - 3)(x + 2)^2 < 0$

From the graph of $y = (x-3)(x+2)^2$, the x-intercepts are 3 and –2. y is below the x-axis only for $x < -2$, $-3 < x < 3$, but not for $x = -2$.

b. $x^3 - 4x \geq 0$

Either $2x - 3 = 7$ or $2x - 3 = -7$

$2x = 10$ $\quad$ $2x = -4$

$x = 5$ $\quad$ $x = -2$

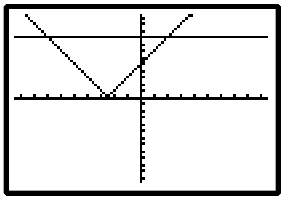
c. $\left|2x + 5\right| > 9$

Solution 1

Graph $y_1 = |2x + 5$

$y_2 = 9$

The graph of $y_1 > y_2$ for values of x less than –7 and for values of x greater than 9. The solution is $x < -7$ or $x > 2$.



Solution 2

Consider $|2x+5|=9$

$$\begin{array}{lll} \text{Either } 2x+5=9 & \text{or} & 2x+5=-9 \\ 2x=4 & & 2x=-14 \\ x=2 & & x=-7. \end{array}$$

Take regions to find the solution to $|2x+5|>9$.

$x=-8$	$x=0$	$x=0$
$\text{L.S.}=\lvert+2(-8)+5\rvert$	$\text{L.S.}=\lvert 2(0)+5\rvert$	$\text{L.S.}=\lvert 2(0)+5\rvert$
$=11$	$=5$	$=5$
$\text{L.S.}>9$	$\text{L.S.}\ngtr 9$	$\text{L.S.}\ngtr 9$
$\therefore x=-8$	$x\neq 0$	$x\neq 0$

Write the answer as $x < -7$ or $x > 2$.

8. **a.** The graph shows 3 real zeros at $x=-2$, $x\doteq 1.5$, and $x\doteq 3.5$. The leading coefficient is positive, and the polynomial function is at least cubic, i.e., of degree 3.

b. The graph shows 2 zeros. Since the graph appears to begin in quadrant 2 and to end in quadrant 1, we deduce that the leading coefficient is positive. The shape seems to show a quarter polynomial, i.e., of degree 4.

c. The graph shows 3 zeros. Since the graph appears to begin in quadrant 2 and end in quadrant 4, it will be a cubic function of degree 3 and has a negative leading coefficient.

9. $C=0.0002x^3-0.005x^2+0.5x$

a. Let $x=95$

$$\begin{aligned} C &= 0.0002(95)^3-0.005(95)^2+0.5(95) \\ &=173.85. \end{aligned}$$

When a diver who weighs 95 kg stands on the board, it will dip 173.9 cm.

b. If the diving board dips 40 cm, $C = 40$.
Substituting, $40 = 0.0002x^3-0.005x^2+0.5x$
or $0.0002x^3-0.005x^2+0.5x-40=0$
Graphing $y=0.0002x^3-0.005x^2+0.5x-40$ and using the $\boxed{\text{ZERO}}$ function in $\boxed{\text{CALC}}$ mode, we find $x\doteq 51.6$. So, the diver has a mass of about 52 kg.

Chapter 3 • Introduction to Calculus

Review of Prerequisite Skills

2. **e.** The slope of line is

$$m = \frac{12-6}{4-(-1)}$$
$$= \frac{6}{5}$$

The equation of the line is in the form $y - y_1 = m(x - x_1)$. The point is $(-1, 6)$ and $m = \frac{6}{5}$.

The equation of the line is $y - 6 = \frac{6}{5}(x+1)$ or $6x - 5y + 36 = 0$.

4. $$f(x) = \begin{cases} \sqrt{3-x} & \text{if } x < 0 \\ \sqrt{3+x} & \text{if } x \geq 0 \end{cases}$$

a. $f(-33) = 6$

b. $f(0) = \sqrt{3}$

c. $f(78) = 9$

6. **b.** $$\frac{6+\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}+\sqrt{6}}{3}$$

d. $$\frac{1}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$$
$$= \frac{3-\sqrt{3}}{9-3}$$
$$= \frac{3-\sqrt{3}}{6}$$

g. $$\frac{5\sqrt{3}}{2\sqrt{3}+4} \times \frac{2\sqrt{3}-4}{2\sqrt{3}-4}$$
$$= \frac{30-20\sqrt{3}}{12-16}$$
$$= \frac{30-20\sqrt{3}}{-4}$$
$$= \frac{10\sqrt{3}-15}{2}$$

7. **b.** $$\frac{\sqrt{3}}{6+\sqrt{2}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3}{6\sqrt{3}+\sqrt{6}}$$

c. $$\frac{\sqrt{7}-4}{5} \times \frac{\sqrt{7}+4}{\sqrt{7}+4}$$
$$= \frac{7-16}{5\sqrt{7}+20} = \frac{-9}{5\sqrt{7}+20}$$

d. $$\frac{2\sqrt{3}-5}{3\sqrt{2}} \times \frac{2\sqrt{3}+5}{2\sqrt{3}+5}$$
$$= \frac{12-25}{6\sqrt{6}+15\sqrt{2}} = \frac{-13}{6\sqrt{6}+15\sqrt{2}}$$

f. $$\frac{2\sqrt{3}+\sqrt{7}}{5} \times \frac{2\sqrt{3}-\sqrt{7}}{2\sqrt{3}-\sqrt{7}}$$
$$= \frac{12-7}{10\sqrt{3}-5\sqrt{7}} = \frac{5}{10\sqrt{3}-5\sqrt{7}}$$

8. **h.** $$x^3 - 2x^2 + 3x - 6 = x^2(x-2) + 3(x-2)$$
$$= (x-2)(x^2+3)$$

i. $$2x^3 - x^2 - 7x + 6$$
$$f(1) = 2 - 1 - 7 + 6$$
$$= 0$$

Therefore, $x - 1$ is a factor.
By division, the other factor is $2x^2 + x - 6$.
Therefore, $2x^3 - x^2 - 7x + 6 = (x-1)(2x^2 + x - 6)$
$= (x-1)(2x-3)(x+2)$.

9. **j.** $$y = \frac{7}{x^2 - 3x - 4}$$
$$= \frac{7}{(x-4)(x+1)}$$

The domain is $x \in R, x \neq 4$, or -1.

k. $$y = \frac{6x}{2x^2 - 5x - 3}$$
$$= \frac{6x}{(2x+1)(x-3)}$$

The domain is $x \in R, x = -\frac{1}{2}$, and 3.

Section 3.1

Investigation 1

1. $y = x^2$

a. $Q_1(3.5, 12.25)$ **b.** $Q_2(3.1, 9.61)$

c. $Q_3(3.01, 9.0601)$ **d.** $Q_4(3.001, 9.00600)$

2. Slope of secant $P(3, 9)$, Q_i:

$PQ_1 \to 6.5$
$PQ_2 \to 6.1$
$PQ_3 \to 6.01$
$PQ_4 \to 6.001$

3. **a.** $Q_5(2.5, 6.25)$ **b.** $Q_6(2.9, 8.41)$

c. $Q_7(2.99, 8.9401)$ **d.** $Q_8(2.999, 8.994001)$

$PQ_5 \to 5.5$
$PQ_6 \to 5.9$
$PQ_7 \to 5.99$
$PQ_8 \to 5.999$

4. Slope of the tangent at $P(3, 9)$ seems to be 6.

Investigation 2

3. $P(1,\ 1)$, $Q_1(1.5,\ f(1.5))$

Slope $PQ_1 \to 2.5$

Slope $PQ_2(1.1,\ f(1.1)) = 2.1$

Slope $PQ_3(1.01,\ f(1.01)) = 2.01$

Slope $PQ_4(1.001,\ f(1.001)) = 2.001$

Slope $PQ_5(1.0001,\ f(1.0001)) = 2.0001$

5. Slope of the tangent at $P(1, 1)$ is 2.

Investigation 3

1. $P(3,9)$ $Q\left(3+h,\ (3+h)^2\right)$

Slope of PQ
$$\begin{aligned}&= \frac{(3+h)^2-9}{3+h-3}\\ &= \frac{9+6h+h^2-9}{h}\\ &= 6+h, \quad h \neq 0\end{aligned}$$

2. Substitute $h = 0$ in the above slope.

Exercise 3.1

4. **b.**
$$\begin{aligned}\frac{(5+h)^3-125}{h} &= \frac{(5+h-5)\left((5+h)^2+5(5+h)+25\right)}{h}\\ &= \frac{h\left(75+15h+h^2\right)}{h}\\ &= 75+15h+h^2\end{aligned}$$

c.
$$\begin{aligned}\frac{(3+h)^4-81}{h} &= \frac{\left((3+h)^2-9\right)\left((3+h)^2+9\right)}{h}\\ &= \frac{\left(9+6h+h^2-9\right)\left(9+6h+h^2+9\right)}{h}\\ &= (6+h)\left(18+6h+h^2\right)\\ &= 108+54h+12h^2+h^3\end{aligned}$$

d.
$$\frac{\frac{1}{1+h}-1}{h} = \frac{1-1-h}{h(1+h)} = -\frac{1}{1+h}$$

e.
$$\begin{aligned}\frac{3(1+h)^2-3}{h} &= \frac{3\left((1+h)^2-1\right)}{h}\\ &= \frac{3\left(1+2h+h^2-1\right)}{h}\\ &= \frac{3\left(2h+h^2\right)}{h}\\ &= 6+3h\end{aligned}$$

f.
$$\begin{aligned}\frac{(2+h)^3-8}{h} &= \frac{(2+h-2)\left((2+h)^2+2(2+h)+4\right)}{h}\\ &= 12+6h+h^2\end{aligned}$$

g.
$$\begin{aligned}\frac{\frac{3}{4+h}-\frac{3}{4}}{h} &= \frac{\frac{12-12-3h}{4(4h)}}{h}\\ &= \frac{-3}{4(4+h)}\end{aligned}$$

5. **a.**
$$\frac{\sqrt{16+h}-4}{h} = \frac{16+h-16}{h\left(\sqrt{16+h}+4\right)} = \frac{1}{\sqrt{16+h}+4}$$

b.
$$\begin{aligned}\frac{\sqrt{h^2+5h+4}-2}{h} &= \frac{h^2+5h+4-4}{h\left(\sqrt{h^2+5h+4}+2\right)}\\ &= \frac{h+5}{\sqrt{h^2+5h+4}+2}\end{aligned}$$

c. $\dfrac{\sqrt{5+h}-\sqrt{5}}{h}=\dfrac{5+h-5}{h\left(\sqrt{5+h}+\sqrt{5}\right)}$

$$=\frac{1}{\sqrt{5+h}+\sqrt{5}}$$

6. **a.** $P(1,3),\ Q(1+h,\ f(1+h)),\ f(x)=3x^2$

$$m=\frac{3(1+h)^2-3}{h}$$
$$=6+3h$$

b. $R(1,3),\ S\left(1+h,\ (1+h)^3+2\right)$

$$m=\frac{(1+h)^3+2-3}{h}$$
$$=\frac{1+3h+3h^2+h^3-1}{h}$$
$$=3+3h+h^2$$

c. $T(9,\ 3),\ U\left(9+h,\ \sqrt{9+h}\right)$

$$m=\frac{\sqrt{9+h}-3}{h}\bullet\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}$$
$$=\frac{1}{\sqrt{9+h}+3}$$

7. **a.**

P	**Q**	**Slope**
$(2,8)$	$(3,27)$	19
	$(2.5, 15.625)$	15.25
	$(2.1, 9.261)$	12.61
	$(2.01, 8.120601)$	12.0601
	$(1, 1)$	7
	$(1.5, 3.375)$	9.25
	$(1.9, 6.859)$	11.41
	$(1.99, 7.880599)$	11.9401

8. **a.** $y=3x^2,\ (-2,12)$

$$m=\lim_{h\to 0}\frac{3(-2+h)^2-12}{h}$$
$$=\lim_{h\to 0}\frac{12-12h+3h^2-12}{h}$$
$$=\lim_{h\to 0}(-12+3h)$$
$$=-12$$

b. $y=x^2-x$ at $x=3,\ y=6$

$$m=\lim_{h\to 0}\frac{(3+h)^2-(3+h)-6}{h}$$
$$=\lim_{h\to 0}\frac{9+6h+h^2-3-h-6}{h}$$
$$=\lim_{h\to 0}(5+h)$$
$$=5$$

c. $y=x^3$ at $x=-2,\ y=-8$

$$m=\lim_{h\to 0}\frac{(-2+h)^3+8}{h}$$
$$=\lim_{h\to 0}\frac{-8+12h-6h^2+h^3+8}{h}$$
$$=\lim_{h\to 0}(12-6h+h^2)$$
$$=12$$

9. **a.** $y=\sqrt{x-2}$; (3, 1)

$$m=\lim_{h\to 0}\frac{\sqrt{3+h-2}-1}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}\times\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}$$
$$=\frac{1}{2}$$

b. $y=\sqrt{x-5}$ at $x=9,\ y=2$

$$m=\lim_{h\to 0}\frac{\sqrt{9+h-5}-2}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\times\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$
$$=\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}$$
$$=\frac{1}{4}$$

c. $y=\sqrt{5x-1}$ at $x=2,\ y=3$

$$m=\lim_{h\to 0}\frac{\sqrt{10+5h-1}-3}{h}$$
$$=\lim_{h\to 0}\frac{\sqrt{9+5h}-3}{h}\times\frac{\sqrt{9+5h}+3}{\sqrt{9+5h}+3}$$
$$=\lim_{h\to 0}\frac{5}{\sqrt{9+5h}+3}$$
$$=\frac{5}{6}$$

10. a. $y=\frac{8}{x}$ at (2, 4)

$$\lim_{x\to 5}\frac{|x-5|}{x-5}\ m=\lim_{h\to 0}\frac{\frac{8}{2+h}-4}{h}$$
$$=\lim_{h\to 0}\frac{-4}{2+h}$$
$$=-2$$

b. $y=\frac{8}{3+x}$ at $x=1;\ y=2$

$$m=\lim_{h\to 0}\frac{\frac{8}{4+h}-2}{h}$$
$$=\lim_{h\to 0}\frac{-2}{4+h}$$
$$=-\frac{1}{2}$$

c. $y=\frac{1}{x+2}$ at $x=3;\ y=\frac{1}{5}$

$$m=\lim_{h\to 0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h}$$
$$=\lim_{h\to 0}\frac{-1}{5(5+h)}$$
$$=-\frac{1}{10}$$

11. a. $y=x^2-3x,\ (2,-2);\ y'=2x-3,\ m=1$

b. $f(x)(-2),-2;=\frac{4}{x},\ \ y'=-\frac{4}{x^2},\ m=-1$

c. $y=3x^3$ at $x=1;\ y'=9x^2,\ m=9$

d. $y=\sqrt{x-7}$ at $x=16;\ y'=\frac{1}{2}(x-7)^{-\frac{1}{2}},\ m=\frac{1}{6}$

e. $f(x)=\sqrt{16-x},\ y=5;\ x=-9,\ y'=1-\frac{1}{2}(16-x)^{-\frac{1}{2}},$

$m=-\frac{1}{10}$

f. $y=\sqrt{25-x^2},\ (3,4);\ y'=\frac{1-2x}{2\sqrt{25-x^2}},\ m=-\frac{3}{4}$

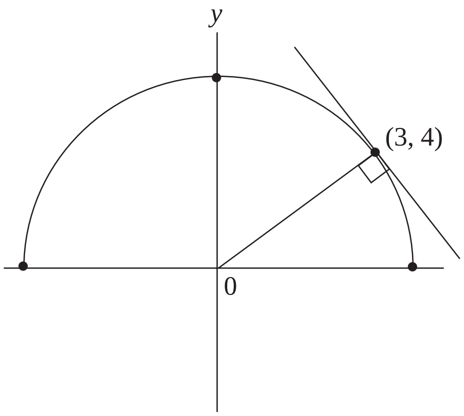
g. $y=\frac{4+x}{x-2}$ at $x=8;\ y=2$

$$m=\lim_{h\to 0}\frac{\frac{12+h}{6+h}-2}{h}$$
$$=\lim_{h\to 0}\frac{-h}{h(6+h)}$$
$$=-\frac{1}{6}$$

h. $y=\frac{8}{\sqrt{x+11}}$ at $x=5;\ y=2$

$$m=\lim_{h\to 0}\frac{\frac{8}{\sqrt{16+h}}-2}{h}$$
$$=\lim_{h\to 0}\frac{8-2\sqrt{16+h}}{h\sqrt{16+h}}\times\frac{4+\sqrt{16+h}}{4+\sqrt{16+h}}$$
$$=2\lim_{h\to 0}\frac{16-16-h}{h\sqrt{16+h}\left(4+\sqrt{16+h}\right)}$$
$$=2\bullet\frac{-1}{4(8)}$$
$$=-\frac{1}{16}$$

12.



$y=\sqrt{25-x^2} \rightarrow$ Semi-circle centre (0, 0) rad 5, $y \geq 0$

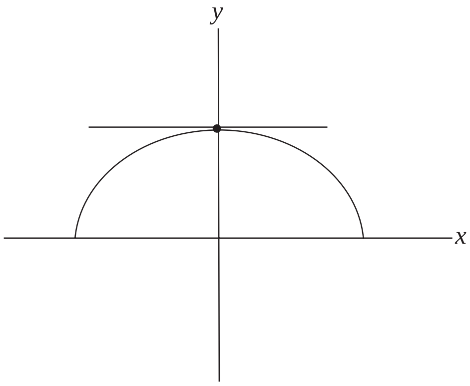
OA is a radius.

The slope of OA is $\frac{4}{3}$.

The slope of the tangent is $-\frac{3}{4}$.

13. Take values of x close to the point, then determine $\frac{\Delta y}{\Delta x}$.

14.



Since the tangent is horizontal, the slope is 0.

16. $D(p)=\frac{20}{\sqrt{p-1}}$, $p>1$ at (5, 10)

$$m=\lim_{h\to0}\frac{\frac{20}{\sqrt{4+h}}-10}{h}$$

$$=10\lim_{h\to0}\frac{2-\sqrt{4+h}}{h\sqrt{4+h}}\times\frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}$$

$$=10\lim_{h\to0}\frac{4-4-h}{h\sqrt{4+h}\left(2+\sqrt{4+h}\right)}$$

$$=-\frac{10}{8}$$

$$=-\frac{5}{4}$$

17. $C(t)=100t^2+400t+5000$

Slope at $t=6$

$C'(t)=200t+400$

$C'(6)=1200+400=1600$

Increasing at a rate of 1600 papers per month.

18. Point on $f(x)=3x^2-4x$ tangent parallel to $y=8x$. Therefore, tangent line has slope 8.

$$\therefore m=\lim_{h\to0}\frac{3(h+a)^2-4(h+a)-3(a^2+4a)}{h}=8$$

$$\lim_{h\to0}\frac{3h^2+6ah-4h}{h}=8$$

$$\therefore 6a-4=8$$

$$a=2$$

The point has coordinates (2, 4).

19. $y=\frac{1}{3}x^3-5x-\frac{4}{x}$

$$\frac{1}{3}(a+h)^3-\frac{1}{3}a^3$$

$$=a^2h+ah^2+\frac{1}{3}h^3$$

$$\lim_{h\to0}\left(a^2+ah+\frac{1}{3}h^3\right)=a^2$$

$$5\lim_{h\to0}-\frac{(a+h)-(-a)}{h}=-5$$

$$-\frac{4}{a+h}+\frac{4}{a}=-\frac{4a+4a+4h}{a(a+h)}$$

$$\lim_{h\to0}\frac{4}{a(a+h)}=\frac{4}{a^2}$$

$$m=a^2-5+\frac{4}{a^2}=0$$

$$a^4-5a^2+4=0$$

$$(a^2-4)(a^2-1)=0$$

$$a=\pm2,\ a=\pm1$$

Points on the graph for horizontal tangents are:

$$\left(-2,\frac{28}{3}\right),\ \left(-1,\frac{26}{3}\right),\ \left(1,-\frac{26}{3}\right),\ \left(2,-\frac{28}{3}\right).$$

20. $y = x^2$ and $y = \frac{1}{2} - x^2$

$$x^2 = \frac{1}{2} - x^2$$

$$x^2 = \frac{1}{4}$$

$$x = \frac{1}{2} \text{ or } x = -\frac{1}{2}$$

The points of intersection are

$$P\left(\frac{1}{2}, \frac{1}{4}\right),\ Q\left(-\frac{1}{2}, \frac{1}{4}\right).$$

Tangent to $y = x$:

$$\begin{aligned} m &= \lim_{h\to0} \frac{(a+h)^2 - a^2}{h} \\ &= \lim_{h\to0} \frac{2ah + h^2}{h} \\ &= 2a \end{aligned}$$

The slope of the tangent at $a = \frac{1}{2}$ is $1 = m_p$,

at $a = -\frac{1}{2}$ is $n - 1 = m_q$.

Tangents to $y = \frac{1}{2} - x^2$:

$$\begin{aligned} m &= \lim_{h\to0} \frac{\left[\frac{1}{2} - (a+h)^2\right] - \left[\frac{1}{2} - a^2\right]}{h} \\ &= \lim_{h\to0} \frac{-2ah - h^2}{h} \\ &= -2a. \end{aligned}$$

The slope of the tangents at $a = \frac{1}{2}$ is $-1 = M_p$;

at $a = -\frac{1}{2}$ is $1 = M_q$

$m_p M_p = -1$ and $m_q M_q = -1$

Therefore, the tangents are perpendicular at the points of intersection.

Exercise 3.2

2. **a.** $\frac{s(9) - s(2)}{7}$. Slope of the secant between the points $(2, s(2))$ and $(9, s(9))$.

b. $\lim_{h\to0} \frac{s(6+h) - s(6)}{h}$. Slope of the tangent at the point $(6, s(6))$.

3. $\lim_{h\to0} \frac{\sqrt{4+h} - 2}{h}$. Slope of the tangent to the function with equation $y = \sqrt{x}$ at the point (4, 2).

7. $s(t) = 5t^2, 0 \le t \le 8$

a. Average velocity during the first second:

$$\frac{s(1) - s(0)}{1} = 5 \text{ m/s};$$

third second:

$$\frac{s(3) - s(2)}{1} = \frac{45 - 20}{1} = 25 \text{ m/s};$$

eighth second:

$$\frac{s(8) - s(7)}{1} = \frac{320 - 245}{1} = 75 \text{ m/s}.$$

b. Average velocity $3 \le t \le 8$

$$\begin{aligned} \frac{s(8) - s(3)}{8 - 3} &= \frac{320 - 45}{5} \\ &= \frac{275}{5} \\ &= 55 \text{ m/s} \end{aligned}$$

c. $s(t) = 320 - 5t^2$

$$\begin{aligned} v(t) &= \lim_{h\to0} \frac{-5(2+h)^2 + 5(2)^2}{h} \\ &= 5\lim_{h\to0} \frac{-4h + h^2}{h} \\ &= -20 \end{aligned}$$

Velocity at $t = 2$ is 20 m/s downward.

8. $s(t) = 8t(t+2), 0 \le t \le 5$

$s(t) = \lim_{1}\ t - \text{hours}$

a. i) from $t = 3$ to $t = 4$

Average velocity $\dfrac{s(4) - s(3)}{1}$

$$= 32(6) - 24(5)$$
$$= 24(8 - 5)$$
$$= 72 \text{ km/h}$$

ii) from to $t = 3.1$

$$\frac{s(3.1) - s(3)}{0.1}$$
$$= \frac{126.48 - 120}{0.1}$$
$$= 64.8 \text{ km/h}$$

iii) $3 \le t \le 3.01$

$$\frac{s(3.01) - s(3)}{0.01}$$
$$= 64.08 \text{ km/h}$$

b. Instantaneous velocity is approximately 64 km/h.

c. At $t = 3$.

$$s(t) = 8t^2 + 16t$$
$$v(t) = 16t + 16$$
$$v(3) = 48 + 16$$
$$= 64 \text{ km/h}$$

9. a. $N(t) = 20t - t^2$

$$\frac{N(3) - N(2)}{1}$$
$$= \frac{51 - 36}{1}$$
$$= 15$$

15 terms are learned between $t = 2$ and $t = 3$.

b. $N'(t) = 20 - 2t$

$N'(2) = 20 - 4 = 16$

At $t = 2$, the student is learning at a rate of 16 terms/hour.

10. a. M in mg in 1 mL of blood t hours after the injection.

$$M(t) = -\frac{1}{3}t^2 + t,\ 0 \le t \le 3$$
$$M(t) = -\frac{2}{3}t + 1$$
$$M(2) = -\frac{4}{3} + 1 = -\frac{1}{3}$$

Rate of change is $-\dfrac{1}{3}$ mg/h.

b. Amount of medicine in 1 mL of blood is being dissipated throughout the system.

11. $t = \sqrt{\dfrac{s}{5}}$

$$t' = \frac{1}{2}\left(\frac{s}{5}\right)^{-\frac{1}{2}} \bullet \frac{1}{5}$$
$$= \frac{1}{10} \bullet \left(\sqrt{\frac{s}{5}}\right)^{-1}$$
$$s = 125,\ t' = \frac{1}{10} \bullet \frac{1}{5} = \frac{1}{50}$$

At $s = 125$, rate of change of time with respect to height is $\dfrac{1}{50}$ s/m.

12. $T(h) = \dfrac{60}{h+2}$

$$T'(h) = -(60)(h+2)^{-2}$$
$$= -\frac{60}{(h+2)^2}$$
$$T'(3) = -\frac{60}{25}$$
$$= -\frac{12}{5}$$

Temperature is decreasing at $\dfrac{12^{\circ}}{5}$ C/km.

13. $h = 25t^2 - 100t + 100$

$h(t)' = 50t - 100$

When $h = 0$, $25t^2 - 100t + 100 = 0$

$$t^2 - 4t + 4 = 0$$

$$(t-2)^2 = 0$$

$$t = 2.$$

$h'(2) = 0$

It hit the ground in 2 s at a speed of 0 m/s.

14. Sale of x balls per week:

$P(x) = 160x - x^2$ dollars.

a. $P(40) = 160(40) - (40)^2$

$= 4800$

Profit on the sale of 40 balls is $4800.

b. $P'(x) = 160 - 2x$

$P'(40) = 160 - 80$

$= 80$

Rate of change of profit is $80 per ball.

c. $160 - 2x > 0$

$-2x > 160$

$x < 80$

Rate of change of profit is positive when the sales level is less than 80.

15. a. $f(x) = -x^2 + 2x + 3;\ (-2, -5)$

$$\lim_{x\to -2} \frac{f(x) - f(-2)}{x+2}$$

$$= \lim_{x\to -2} \frac{-x^2 + 2x + 3 + 5}{x+2}$$

$$= \lim_{x\to -2} \frac{-(x^2 - 2x - 8)}{x+2}$$

$$= -\lim_{x\to -2} \frac{(x-4)(x+2)}{x+2}$$

$$= -\lim_{x\to -2}(x-4)$$

$$= 6$$

b. $f(x) = \dfrac{x}{x-1}$, $x = 2$

$$\lim_{x\to 2} \frac{\frac{x}{x-1} - 2}{x-2}$$

$$= \lim_{x\to 2} \frac{x - 2x + 2}{(x-1)(x-2)}$$

$$= \lim_{x\to 2} \frac{-(x-2)}{(x-1)(x-2)}$$

$$= -1$$

c. $f(x) = \sqrt{x+1}$, $x = 24$

$$\lim_{x\to 24} \frac{f(x) - f(24)}{x-24}$$

$$= \lim_{x\to 24} \frac{\sqrt{x+1} - 5}{x-24} \bullet \frac{\sqrt{x+1} + 5}{\sqrt{x+1} + 5}$$

$$= \lim_{x\to 24} \frac{x-24}{(x-24)\left(\sqrt{x+1} + 5\right)}$$

$$= \frac{1}{10}$$

17. $C(x) = F + V(x)$

$C(x+h) = F + V(x+h)$

Rate of change of cost is

$$\lim_{x\to R} \frac{C(x+h) - C(x)}{h}$$

$$= \lim_{x\to h} \frac{V(x+h) - V(x)}{} h,$$

which is independent of F – (fixed costs).

18. $P(r) = \pi r^2$

Rate of change of area is

$$\lim_{h\to 0} \frac{A(r+h) - A(r)}{h}$$

$$= \lim_{h\to 0} \frac{\pi(r+h)^2 - \pi r^2}{h}$$

$$= \pi \lim_{h\to 0} \frac{(r+h-r)(r+h+r)}{h}$$

$$= 2\pi r$$

$r = 100$ m

Rate is 200π m^2/m.

19. Cube of dimension x by x by x has volume $V = x^3$. Surface area is $6x^2$.

$$V'(x) = 3x^2 = \frac{1}{2} \text{ surface area.}$$

Exercise 3.3

13. $f(x) = mx + b$

$$\lim_{x\to1} f(x) = -2 \quad \therefore m + b = -2$$

$$\lim_{x\to-1} f(x) = 4 \quad \therefore -m + b = 4$$

$$2b = 2$$

$$b = 1, \quad m = -3$$

14. $f(x) = ax^2 + bx + c, \ a \neq 0$

$$f(0) = 0 \quad \therefore c = 0$$

$$\lim_{x\to1} f(x) = 5 \quad \therefore a + b = 5$$

$$\lim_{x\to-2} f(x) = 8 \quad \therefore 4a - 2b = 8$$

$$6a = 18$$

$$a = 3, \ b = 2$$

Therefore, the values are $a = 3, \ b = 2,$ and $c = 0$.

Exercise 3.4

7. **a.** $$\lim_{x\to2} \frac{4 - x^2}{2 - x} = \lim_{x\to2} \frac{(2 - x)(2 + x)}{(2 - x)}$$
$$= \lim_{x\to2}(2 + x)$$
$$= 4$$

b. $$\lim_{x\to-2} \frac{4 - x^2}{2 + x} = \lim_{x\to-2} \frac{(2 - x)(2 + x)}{(2 + x)}$$
$$= 4$$

c. $$\lim_{x\to0} \frac{7x - x^2}{x} = \lim_{x\to0} \frac{x(7 - x)}{x} = 7$$

d. $$\lim_{x\to-1} \frac{2x^2 + 5x + 3}{x + 1} = \lim_{x\to-1} \frac{(x + 1)(2x + 3)}{x + 1} = 5$$

e. $$\lim_{x\to-\frac{4}{3}} \frac{3x^2 + x - 4}{3x + 4} = \lim_{x\to-\frac{4}{3}} \frac{(3x + 4)(x - 1)}{3x + 4} = -\frac{7}{3}$$

f. $$\lim_{x\to3} \frac{x^3 - 27}{x - 3} = \lim_{x\to3} \frac{(x - 3)(x^2 + 3x + 9)}{x - 3}$$
$$= 9 + 9 + 9 = 27$$

g. $$x^3 + 2x^2 - 4x - 8 = x^2(x + 2) - 4(x + 2)$$
$$= (x - 2)(x + 2)(x + 2)$$
$$\lim_{x\to-2} \frac{x^3 + 2x^2 - 4x - 8}{x + 2} = \lim_{x\to-2} \frac{(x - 2)(x + 2)(x + 2)}{(x + 2)}$$
$$= 0$$

h. $$2x^3 - 5x^2 + 3x - 2 = (x - 2)(2x^2 - x + 1)$$
$$\lim_{x\to2} \frac{2x^3 - 5x^2 + 3x - 2}{2(x - 2)} = \frac{7}{2}$$

i. $$\lim_{x\to0} \frac{\sqrt{x+1} - 1}{x} \times \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} = \lim_{x\to0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{2}$$

j. $$\lim_{x\to0} \frac{2 - \sqrt{4+x}}{x} \times \frac{2 + \sqrt{4+x}}{2 + \sqrt{4+x}}$$
$$= \lim_{x\to0} \frac{-1}{2 + \sqrt{4+x}} = -\frac{1}{4}$$

k. $$\lim_{x\to4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x\to4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)}$$
$$= \frac{1}{4}$$

l. $$\lim_{x\to0} \frac{\sqrt{7-x} - \sqrt{7+x}}{x} \times \frac{\sqrt{7-x} + \sqrt{7+x}}{\sqrt{7-x} + \sqrt{7+x}}$$
$$= \lim_{x\to0} \frac{7 - x - 7 - x}{x(\sqrt{7-x} + \sqrt{7+x})}$$
$$= -\frac{1}{\sqrt{7}}$$

m. $$\lim_{x\to1}\frac{\sqrt{5-x}-\sqrt{3+x}}{x-1}\times\frac{\sqrt{5-x}+\sqrt{3+x}}{\sqrt{5-x}+\sqrt{3+x}}$$

$$=\lim_{x\to1}\frac{5-x-3-x}{(x-1)\left(\sqrt{5-x}+\sqrt{3+x}\right)}$$

$$=\lim_{x\to1}\frac{-2(x-1)}{(x-1)\left(\sqrt{5-x}+\sqrt{3+x}\right)}$$

$$=-\frac{2}{4}$$

n. $$\lim_{x\to4}\frac{2-\sqrt{x}}{3-\sqrt{2x+1}}\bullet\frac{3+\sqrt{2x+1}}{3+\sqrt{2x+1}}$$

$$=\lim_{x\to4}\frac{\left(2-\sqrt{2}\right)\left(3+\sqrt{2x+1}\right)}{9-2x-1}\bullet\frac{2+\sqrt{x}}{2+\sqrt{x}}$$

$$=\lim_{x\to4}\frac{(2-x)\left(3+\sqrt{2x+1}\right)}{4(2-x)\left(2+\sqrt{x}\right)}$$

$$=\frac{6}{16}$$

$$=\frac{3}{8}$$

o. $$\lim_{x\to0}\frac{2^{2x}-2^{x}}{2^{x}-1}$$

$$=\lim_{x\to0}\frac{2^{x}\left(2^{x}-1\right)}{2^{x}-1}$$

$$=1$$

8. **a.** $$\lim_{x\to8}\frac{\sqrt[3]{x}-2}{x-8}$$

Let $u=\sqrt[3]{x}$. Therefore, $u^3=x$ as $x\to8$, $u\to2$.

Here, $\lim_{u\to2}\frac{u-2}{u^3-8}=\lim_{u\to2}\frac{1}{u^2+2u+4}$

$$=\frac{1}{12}$$

b. $$\lim_{x\to27}\frac{27-x}{x^{\frac{1}{3}}-3}$$

Let $x^{\frac{1}{3}}=u$

$x=u^3$

$x\to27,\ u\to3.$

$$=\lim_{u\to3}\frac{u^3-27}{u-3}$$

$$=-\lim_{u\to3}\frac{(u-3)\left(u^2+3u+9\right)}{u-3}$$

$$=-(9+9+9)$$

$$=-27$$

c. $$\lim_{x\to1}\frac{x^{\frac{1}{6}}-1}{x-1}$$

$x^{\frac{1}{6}}=u,\ x=u^6$

$x\to1,\ u\to1$

$$=\lim_{u\to1}\frac{u-1}{u^6-1}$$

$$=\lim_{u\to1}\frac{(u-1)}{(u-1)\left(u^5+u^4+u^3+u^2+u+1\right)}$$

$$=\frac{1}{6}$$

d. $$\lim_{x\to1}\frac{x^{\frac{1}{6}}-1}{x^{\frac{1}{3}}-1}$$

Let $x^{\frac{1}{6}}=u$

$u^6=x$

$x^{\frac{1}{3}}=u^2$

As $x\to1,\ u\to1$

$$=\lim_{u\to1}\frac{u-1}{u^2-1}$$

$$=\lim_{u\to1}\frac{u-1}{(u-1)(u+1)}$$

$$=\frac{1}{2}$$

e. $$\lim_{x\to4}\frac{\sqrt{x}-2}{\sqrt{x^3}-8}$$

Let $x^{\frac{1}{2}}=u$

$x^{\frac{3}{2}}=u^3$

$x\to4,\ u\to2.$

$$=\lim_{u\to2}\frac{u-2}{u^3-8}$$

$$=\lim_{u\to2}\frac{u-2}{(u-2)\left(u^2+2u+4\right)}$$

$$=\frac{1}{12}$$

f. $$\lim_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{x}$$

$$=\lim_{u\to 2}\frac{u-2}{u^3-8}$$

$$=\frac{1}{12}$$

Let $(x+8)^{\frac{1}{3}}=u$

$x+8=u^3$

$x=u^3-8$

$x\to 0,\ u\to 2.$

9. **c.** $$\lim_{x\to 1}\frac{x^3+x^2-5x+3}{x^2-2x+1}$$

$$=\lim_{x\to 1}\frac{(x-1)(x-1)(x+3)}{(x-1)(x-1)}$$

$$=\lim_{x\to 1}(x+3)=4$$

d. $$\lim_{x\to -1}\frac{x^2+x}{x+1}=\lim_{x\to -1}x\,\frac{x(x+1)}{x+1}$$

$$=-1$$

e. $$\lim_{x\to 6^+}\frac{\sqrt{x^2-5x-6}}{x-3}=0$$

f. $$\lim_{x\to 0}\frac{(2x+1)^{\frac{1}{3}}-1}{x}$$

$$=\lim_{u\to +1}\frac{2(u-1)}{u^3-1}$$

$$=\lim_{u\to 1}\frac{2}{u^2+u+1}$$

$$=\frac{2}{3}$$

Let $(2x+1)^{\frac{1}{3}}=u$

$2x+1=u^3$

$x=\dfrac{u^3-1}{2}$

$x\to 0,\ u\to 1.$

g. $$\lim_{x\to 2}\frac{x^2-4}{\left(\frac{1}{x}\right)-\frac{1}{2}}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{2-x}\bullet 2x$$

$$=\lim_{x\to 2}-2x(x+2)$$

$$=-16$$

h. $$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{\sqrt{x+1}-2}{(x+1)-4}$$

$$=\lim_{x\to 3}\frac{\sqrt{x+1}-2}{\left(\sqrt{x+1}-2\right)\left(\sqrt{x+1}+2\right)}$$

$$=\lim_{x\to 3}\frac{1}{\sqrt{x+1}+2}$$

$$=\frac{1}{4}$$

i. $$\lim_{x\to 0}\frac{x^2-9x}{5x^3+6x}=\lim_{x\to 0}\frac{x-9}{5x^2+6}=-\frac{3}{2}$$

j. $$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x+1-1}$$

$$=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}$$

$$=\frac{1}{2}$$

k. $$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$=2x$$

l. $$\lim_{x\to 1}\left(\frac{1}{x-1}\right)\left(\frac{1}{x+3}-\frac{2}{3x+5}\right)$$

$$=\lim_{x\to 1}\left(\frac{1}{x-1}\right)\left(\frac{3x+5-2x-6}{(x+3)(3x+5)}\right)$$

$$=\lim_{x\to 1}\frac{1}{(x+3)(3x+5)}$$

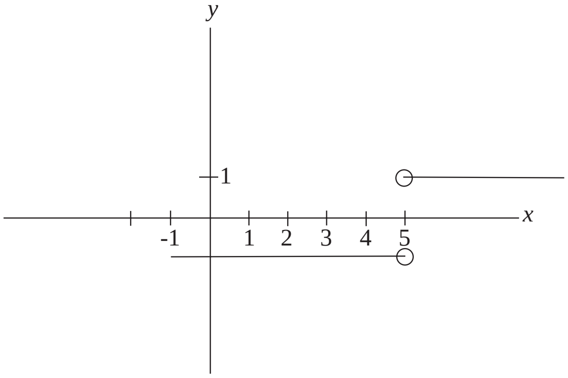
$$=\frac{1}{4(8)}$$

$$=\frac{1}{32}$$

10. a. $\lim\limits_{x\to5}\dfrac{|x-5|}{x-5}$ does not exist.

$$\lim_{x\to5^+}\frac{|x-5|}{x-5}=\lim_{x\to5^+}\frac{x-5}{x-5}$$
$$=1$$

$$\lim_{x\to5^-}\frac{|x-5|}{x-5}=\lim_{x\to5^-}-\left(\frac{x-5}{x-5}\right)$$
$$=-1$$



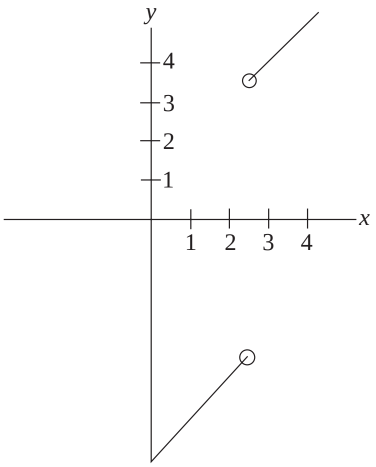
b. $\lim\limits_{x\to\frac{5}{2}}\dfrac{|2x-5|(x+1)}{2x-5}$ does not exist.

$$|2x-5|=2x-5,\quad x\geq\frac{5}{2}$$

$$\lim_{x\to\frac{5^+}{2}}\frac{(2x-5)(x+1)}{2x-5}=x+1$$

$$|2x-5|=-(2x-5),\ x<\frac{5}{2}$$

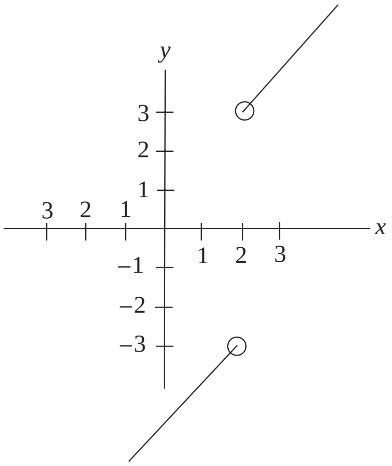
$$\lim_{x\to\frac{5^-}{2}}\frac{-(2x-5)(x+1)}{2x-5}=-(x+1)$$



c. $$\lim_{x\to2}\frac{x^2-x-2}{|x-2|}=\lim_{x\to2}\frac{(x-2)(x+1)}{|x-2|}$$

$$\lim_{x\to2^+}\frac{(x-2)(x+1)}{|x-2|}=\lim_{x\to2^+}\frac{(x-2)(x+1)}{x-2}=\lim_{x\to2^+}x+1$$
$$=3$$

$$\lim_{x\to2^-}\frac{(x-2)(x+1)}{|x-2|}=\lim_{x\to2^-}-\frac{(x-2)(x+1)}{(x-2)}=\lim_{x\to2^-}-(x+1)$$
$$=-3$$

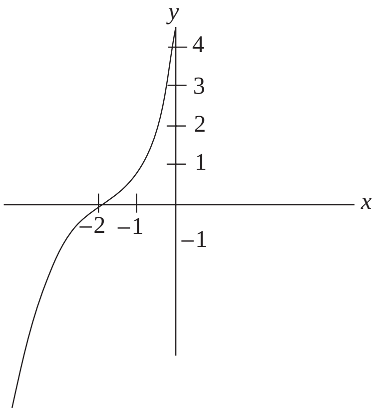


d. $|x+2|=x+2$ if $x>-2$

$=-(x+2)$ if $x<-2$

$$\lim_{x\to-2^+}\frac{(x+2)(x+2)^2}{x+2}=\lim_{x\to-2^+}(x+2)^2=0$$

$$\lim_{x\to-2^-}\frac{(x+2)(x+2)^2}{-(x+2)}=0$$



11. a.

ΔT	T	V	ΔV
	–40	19.1482	
20	–20	20.7908	1.6426
20	0	22.4334	1.6426
	20	24.0760	1.6426
	40	25.7186	1.6426
	60	27.3612	1.6426
	80	29.0038	1.6426

ΔV is constant, therefore T and V form a linear relationship.

b.
$$V = \frac{\Delta V}{\Delta T} \bullet T + K$$
$$\frac{\Delta V}{\Delta T} = \frac{1.6426}{20} = 0.08213$$
$$V = 0.08213T + K$$
$$T = 0 \qquad V = 22.4334$$

Therefore, $k = 22.4334$
and $V = 0.08213T + 22.4334$.

c. $T = \dfrac{V - 22.4334}{0.08213}$

d. $\lim\limits_{v \to 0} T = –273.145$

12.
$$\lim_{x \to 5} \frac{x^2 - 4}{f(x)}$$
$$= \frac{\lim\limits_{x \to 5}(x^2 - 4)}{\lim\limits_{x \to 5} f(x)}$$
$$= \frac{21}{3}$$
$$= 7$$

13. $\lim\limits_{x \to 4} f(x) = 3$

a. $\lim\limits_{x \to 4}[f(x)]^3 = 3^3 = 27$

b.
$$\lim_{x \to 4} \frac{[f(x)]^2 - x^2}{f(x) + x} = \lim_{x \to 4} \frac{(f(x) - x)(f(x) + x)}{f(x) + x}$$
$$= \lim_{x \to 4} (f(x) - x)$$
$$= 3 - 4$$
$$= -1$$

c.
$$\lim_{x \to 4} \sqrt{3f(x) - 2x} = \sqrt{3 \times 3 - 2} \times 4$$
$$= 1$$

14. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$

a. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{f(x)}{x} \times x = 0$

b. $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \dfrac{x}{g(x)} \dfrac{f(x)}{x} = 0$

15. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$ and $\lim\limits_{x \to 0} \dfrac{g(x)}{x} = 2$

a. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x\left(\dfrac{f(x)}{x}\right) = 0$

b.
$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x\left(\frac{g(x)}{x}\right) = 0 \times 2$$
$$= 0$$

c.
$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\frac{f(x)}{x}}{\frac{g(x)}{x}} = \frac{1}{2}$$

16.
$$\lim_{x \to 0} \frac{\sqrt{x+1} - \sqrt{2x+1}}{\sqrt{3x+4} - \sqrt{2x+4}}$$
$$= \lim_{x \to 0} \frac{\sqrt{x+1} - \sqrt{2x+1}}{\sqrt{x+1} + \sqrt{2x+1}} \times \frac{\sqrt{x+1} + \sqrt{2x+1}}{\sqrt{3x+4} - \sqrt{2x+4}} \times \frac{\sqrt{3x+4} + \sqrt{2x+4}}{\sqrt{3x+4} + \sqrt{2x+4}}$$
$$= \lim_{x \to 0} \frac{(x + 1 - 2x - 1)}{(3x + 4 - 2x - 4)} \times \frac{\sqrt{3x+4} + \sqrt{2x+4}}{\sqrt{x+1} + \sqrt{2x+1}}$$
$$= \frac{2+2}{1+1}$$
$$= 2$$

17. $$\lim_{x\to 1}\frac{x^2+|x-1|-1}{|x-1|}$$

$$x\to 1^+ \quad |x-1|=x-1$$

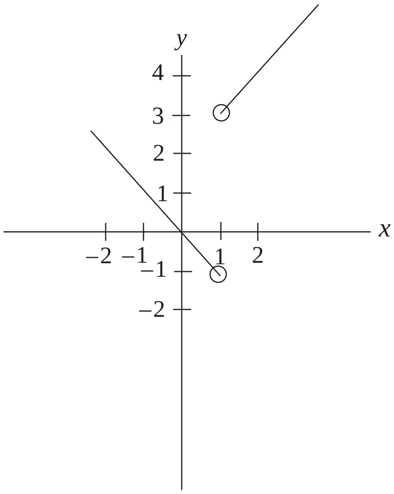
$$\therefore \quad \frac{x^2+x-2}{x-1}=\frac{(x+2)(x-1)}{x-1}$$

$$\lim_{x\to 1^+}\frac{x^2+|x-1|-1}{|x-1|}=3$$

$$x\to 1^- \quad |x-1|=-x+1$$

$$\lim_{x\to 1^-}\frac{x^2-x}{-x+1}=\lim_{x\to 1^-}\frac{x(x-1)}{-x+1}$$
$$=-1$$

Therefore, this function does not exist.



18. $$\lim_{x\to 1}\frac{x^2+bx-3}{x-1}$$

$$x^2+bx-3=(x-1)(x+3)$$
$$=x^2+2x-3$$

$$b=2 \quad \lim_{x\to 1}\frac{x^2+2x-3}{x-1}=\lim_{x\to 1}(x+3)$$
$$=4$$

Exists for b = 2.

19. $$\lim_{x\to 0}\frac{\sqrt{mx+b}-3}{x}=1$$

$$\lim_{x\to 0}\frac{\left(\sqrt{mx+b}-3\right)\left(\sqrt{mx+b}+3\right)}{x\left(\sqrt{mx+b}+3\right)}=1$$

$$\lim_{x\to 0}\frac{mx+b-9}{x\sqrt{mx+b}+3}=1$$
$$b=9$$

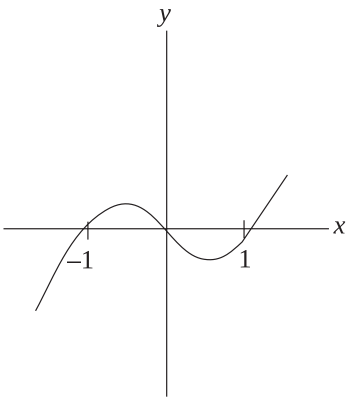
$$\lim_{x\to 0}\frac{m}{3+3}=1$$
$$\therefore m=6$$

$$\lim_{x\to 0}\frac{6x}{x\sqrt{6x+9}+3}=\lim_{x\to 0}\frac{6}{\sqrt{6x+9}+3}=1$$

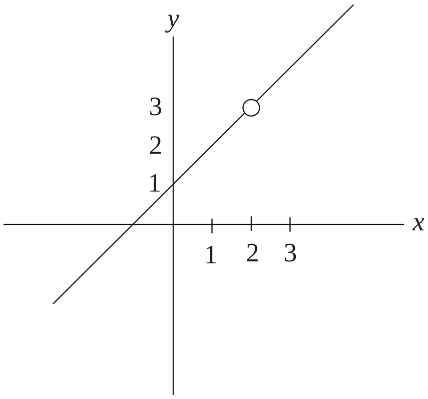
$m=6,\ b=9$

Section 3.5

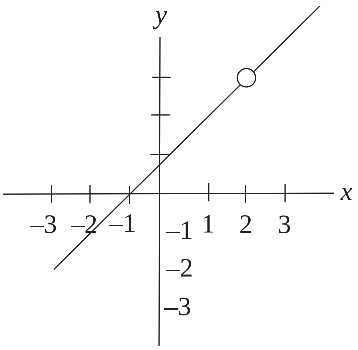
Investigation



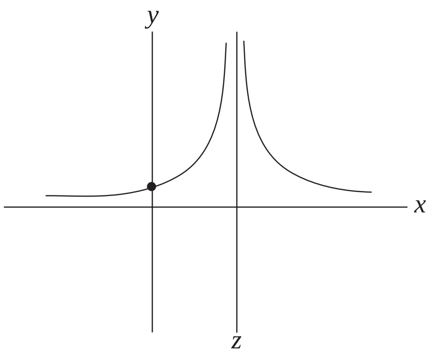
b.



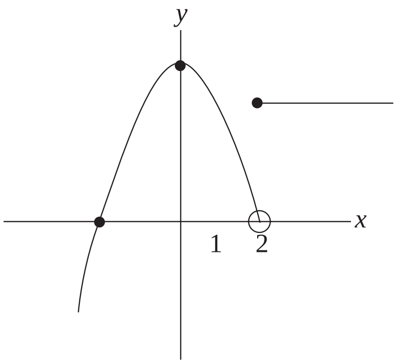
c.



d.



e.



2. **a.** and **c**. are continuous. **b**. contains a hole. **e.** has a jump. **d.** has a vertical asymptote.

3. Window may be too small.

4. Not defined when $x^2 + 300x = 0$ or $x(x+300)=0.$

$x = 0$ or $x = -300$

Continuous for $x < -300$

$-300 < x < 0$

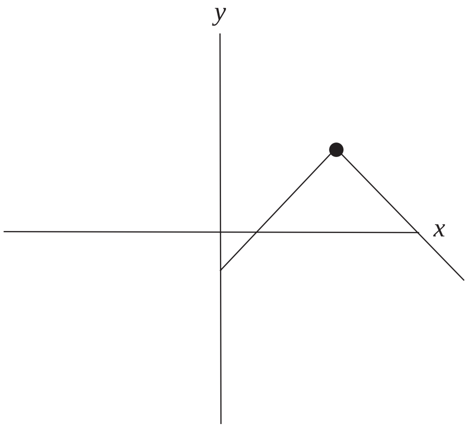
$x > 0$

Exercise 3.5

4. **e.** Discontinuous when

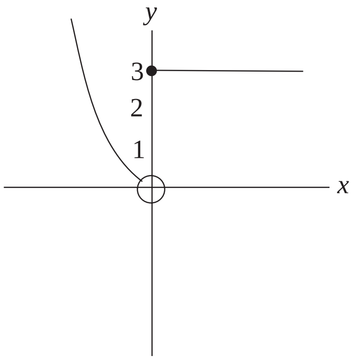
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3 \text{ or } x = 2.$$

7.



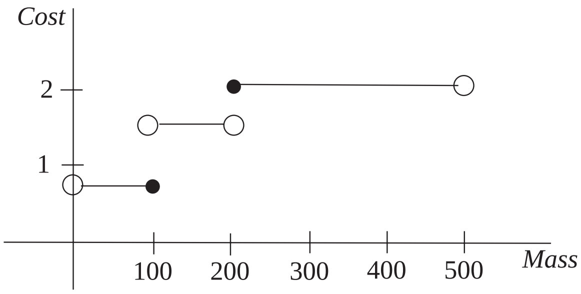
Continuous everywhere.

8.



Discontinuous.

9.

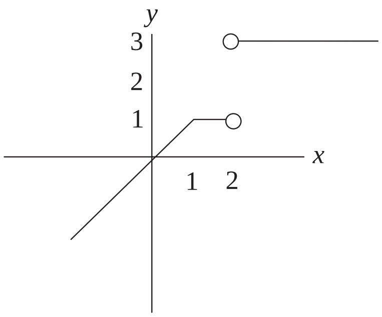


Discontinuous at 0, 100, 200, and 500.

10.
$$\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x^2 - x - 6}{x - 3}$$
$$= \lim_{x\to 3}\frac{(x-3)(x+2)}{x-3}$$
$$= 5$$

Function is discontinuous at $x = 3$.

11.



Discontinuous at x = 3.0

12. $g(x)=\begin{cases} x+3, & x\neq 3\\ 2+\sqrt{k}, & x=3\end{cases}$

$g(x)$ is continuous.

$$\therefore 2+\sqrt{k}=6$$

$$\sqrt{k}=4,\ k=16$$

13. $f(x)=\begin{cases} -x, -3\le x\le -2\\ ax^2+b,\ -2<x<0\\ 6,\ x=0\end{cases}$

at $x=-2,\ 4a+b=2$

at $x=0,\quad b=6$

$\therefore a=-1$

$$f(x)=\begin{cases} -x, -3\le x\le -2\\ -x^2+b,\ -2<x<0\\ 6,\ x=0\end{cases}$$

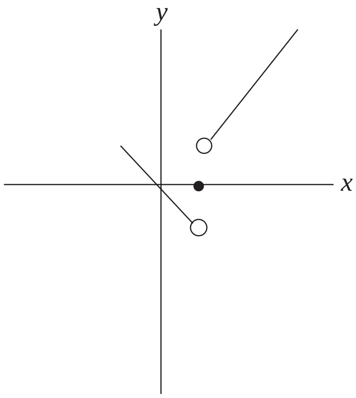
If $a=-1$, $b=6$. $f(x)$ is continuous.

14. $g(x)=\begin{cases} \dfrac{x|x-1|}{x-1},\ x\neq 1\\ 0,\ x=1\end{cases}$

a. $\left.\begin{aligned}\lim_{x\to 1^-} g(x)&=-1\\ \lim_{x\to 1^+} g(x)&=1\end{aligned}\right\}\lim_{x\to 1} g(x)$

$\lim_{x\to 1} g(x)$ does not exist.

b.



Review Exercise

2. **a.** $f(x)=\dfrac{3}{x+1},\ P(2,1)$

$$m=\lim_{h\to 0}\frac{\frac{3}{3+h}-1}{h}$$

$$=\lim_{h\to 0}-\frac{1}{3+h}$$

$$=-\frac{1}{3}$$

b. $g(x)=\sqrt{x+2},\ \ x=-1$

$$m=\lim_{h\to 0}\frac{\sqrt{-1+h+2}-1}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{h+1}-1}{x}\times\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{h+1}+1}$$

$$=\frac{1}{2}$$

c. $h(x)=\dfrac{2}{\sqrt{x+5}},\ \ x=4$

$$m=\lim_{h\to 0}\frac{\frac{2}{\sqrt{4+h+5}}-\frac{2}{3}}{h}$$

$$=2\lim_{h\to 0}\frac{3-\sqrt{9+h}}{3h\sqrt{9+h}}\times\frac{3+\sqrt{9+h}}{3+\sqrt{9+h}}$$

$$=2\lim_{h\to 0}-\frac{1}{3\sqrt{9+h}\left(3+\sqrt{9+h}\right)}$$

$$=-\frac{2}{9(6)}$$

$$=-\frac{1}{27}$$

d. $f(x)=\dfrac{5}{x-2}$, $x=4$

$$m=\lim_{h\to 0}\frac{\frac{5}{4+h-2}-\frac{5}{2}}{h}$$

$$=\lim_{h\to 0}\frac{10-5(2+h)}{h(2+h)(2)}$$

$$=\lim_{h\to 0}-\frac{-5h}{h(2+h)(2)}$$

$$=-\frac{5}{4}$$

3. $f(x)=\begin{cases}4-x^2, & x\le 1\\ 2x+1, & x>1\end{cases}$

a. Slope at $(-1,\ 3)$ $f(x)=4-x^2$

$$m=\lim_{h\to 0}\frac{4-(-1+h)^2-3}{h}$$

$$=\lim_{h\to 0}\frac{4-1+2h-h^2-3}{h}$$

$$=\lim_{h\to 0}(2-h)$$

$$=2$$

Slope of the graph at P (–1, 3) is 2.

b. Slope at P(2, 0.5)

$\therefore f(x)=2x+1$

$$f(2+h)-f(2)=2(2+h)+1-5$$
$$=2h$$

$$m=\lim_{h\to 0}\frac{2h}{h}=2$$

Slope of the graph at P(2, 0.5) is 2.

4. $s(t)=-5t^2+180$

a. $s(0)=180$, $s(1)=175$, $s(2)=160$

Average velocity during the first second is

$$\frac{s(1)-s(0)}{1}=-5\text{ m/s.}$$

Average velocity during the second second is

$$\frac{s(2)-s(1)}{1}=-15\text{ m/s.}$$

b. At $t=4$:

$$s(4+h)-s(4)$$
$$=-5(4+h)^2+180-(-5(16)+180)$$
$$=-80-40h-5h^2+180+80-180$$

$$\frac{s(4+h)-s(4)}{h}=\frac{-40h-5h^2}{h}$$

$$v(4)=\lim_{h\to 0}(-40-5h)=-40$$

Velocity is at –40 m/s.

c. Time to reach ground is when $s(t)=0$.
Therefore, $-5t^2+180=0$
$t^2=36$
$t=6$, $t>0$.

Velocity at $t=6$:

$$s(6+h)=-5(36+12h+h^2)+180$$
$$=-60h-5h^2$$
$$s(6)=0$$

Therefore, $v(6)=\lim_{h\to 0}(-60-5h)=-60.$

5. $M(t)=t^2$ mass in grams

a. Growth during $3\le t\le 3.01$

$$M(3.01)=(3.01)^2=9.0601$$
$$M(3)=3^2\qquad=9$$

Grew 0.0601g during this time interval.

b. Average rate of growth is

$$\frac{0.0601}{0.01}=6.01\text{ g/s}.$$

c.
$$s(3+h)=9+6h+h^2$$
$$s(3)=9$$

$$\frac{s(3+h)-s(3)}{h}=\frac{6h+h^2}{h}$$

Rate of growth is $\lim_{h\to 0}(6+h)=6$ g/s.

6. $Q(t)=10^4(t^2+15t+70)$ tonnes of waste, $0\le t\le 10$

a. At $t=0$,

$$Q(t)=70\times 10^4$$
$$=700\ 000.$$

700 000 t have accumulated up to now.

b. Over the next three years, the average rate of change:

$$Q(3) = 10^4(9+45+70)$$

$$= 124 \times 10^4$$

$$Q(0) = 70 \times 10^4$$

$$\frac{Q(3)-Q(0)}{3} = \frac{5^4 \times 16^4}{3}$$

$$= 18 \times 10^4 \text{ t per year.}$$

c. Present rate of change:

$$Q(h) = 10^4(h^2 + 15h + 70)$$

$$Q(0) = 10^4 + 70$$

$$\lim_{h\to 0}\frac{Q(h)-Q(0)}{h} = \lim_{h\to 0} 10^4(h+15)$$

$$= 15 \times 10^4 \text{ t per year.}$$

d. $Q(a+h) = 10^4\left[a^2 + 2ah + h^2 + 15a + 15h + 70\right]$

$$Q(a) = 10^4\left[a^2 + 15a + 70\right]$$

$$\frac{Q(a+h)-Q(a)}{h} = \frac{10^4\left[2ah + h^2 + 15h\right]}{h}$$

$$\lim_{h\to 0}\frac{Q(a+h)-Q(a)}{h} = \lim_{h\to 0} 10^4(2a+h+15)$$

$$= (2a+15)10^4$$

Now,

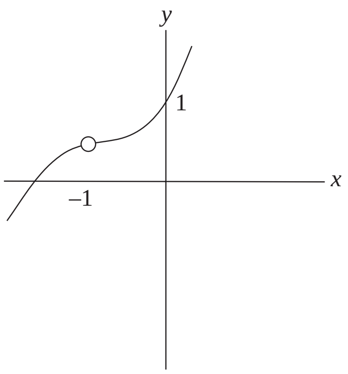
$$(2a+15)10^4 = 3 \times 10^5$$

$$2a + 15 = 30$$

$$a = 7.5.$$

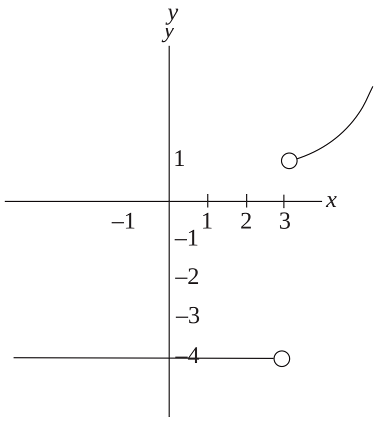
It will take 7.5 years to reach a rate of 3.0×10^5 t per year.

8. **a.** $\lim_{x\to -1} f(x) = 0.5$, f is discontinuous at $x = -1$



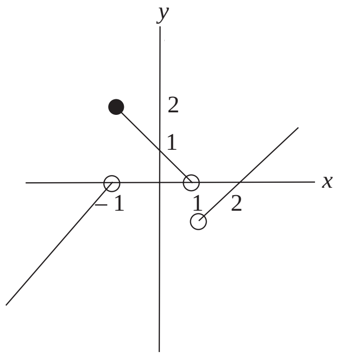
b. $f(x) = -4$ if $x < 3$; f is increasing for $x > 3$

$$\lim_{x\to 3^+} f(x) = 1$$



9. $f(x) = \begin{cases} x+1, & x < -1 \\ -x+1, & -1 \le x < 1 \\ x-2, & x > 1 \end{cases}$

a.



Discontinuous at $x = -1$ and $x = 1$.

b. They do not exist.

10. $f(x) = \dfrac{x^2 - x - 6}{x-3}$

$$= \frac{(x-3)(x+2)}{(x-3)}$$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3}(x+2)$$

$$= 5$$

$f(x)$ is not continuous at $x = 3$.

11. $$f(x)=\frac{2x-2}{x^2+x-2}$$

$$=\frac{2(x-1)}{(x-1)(x+2)}$$

a. f is discontinuous at $x=1$ and $x=-2$.

b. $$\lim_{x\to1}f(x)=\lim_{x\to1}\frac{2}{x+2}$$
$$=\frac{2}{3}$$

$$\lim_{x\to-2}f(x)\text{: }\lim_{x\to-2^+}\frac{2}{x+2}=+\infty$$

$$\lim_{x\to-2^-}\frac{2}{x+2}=-\infty$$

$\lim_{x\to-2}f(x)$ does not exist.

12. a. $f(x)=\frac{1}{x^2}$, $\lim_{x\to0}f(x)$ does not exist.

b. $g(x)=x(x-5)$, $\lim_{x\to0}g(x)=0$

c. $$h(x)=\frac{x^3-27}{x^2-9},$$

$$\lim_{x\to4}h(x)=\frac{37}{7}=5.2857$$

$\lim_{x\to-3}h(x)$ does not exist.

15. a. $$f(x)=\frac{\sqrt{x+2}-2}{x-2}$$

x	2.1	2.01	2.001
$f(x)$	0.24846	0.24984	0.24998

x = 2.0001
$f(x)$ – 0.25

b. $\lim_{x\to2}f(x)=0.25$

c. $$\lim_{x\to2}\frac{\sqrt{x+2}-2}{x-2}\times\frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$$
$$=\lim_{x\to2}\frac{1}{\sqrt{x+2}+2}$$
$$=\frac{1}{4}=0.25$$

16. a. $$\lim_{h\to0}\frac{(5+h)^2-25}{h}$$
$$=\lim_{h\to0}(10+h)$$
$$=10.$$

Slope of the tangent to $y=x^2$ at $x=5$ is 10.

b. $$\lim_{h\to0}\frac{\sqrt{4+h}-2}{h}=\lim_{h\to0}\frac{\sqrt{4+h}-2}{4+h-4}$$
$$=\lim_{h\to0}\frac{1}{\sqrt{4+h}+2}$$
$$=\frac{1}{4}$$

Slope of the tangent to $y=\sqrt{x}$ at $x=4$ is $\frac{1}{4}$.

c. $$\lim_{h\to0}\frac{\frac{1}{4+h}-\frac{1}{4}}{h}=\lim_{h\to0}\frac{4-4h}{4(4+h)(h)}$$
$$=\lim_{h\to0}-\frac{1}{4(4+h)}$$
$$=-\frac{1}{16}$$

Slope of the tangent to $y=\frac{1}{x}$ at (x = 4) is $-\frac{1}{16}$.

d. $$\lim_{h\to0}\frac{(343+h)^{\frac{1}{3}}-7}{h}=\lim_{h\to0}\frac{(343+h)^{\frac{1}{3}}-7}{343+h-343}$$
$$=\lim_{h\to0}\frac{(343+h)^{\frac{1}{3}}-7}{\left((343+h)^{\frac{1}{3}}-7\right)\left((343+h)^{\frac{2}{3}}+7(343+h)^{\frac{1}{3}}+49\right)}$$
$$=\frac{1}{49+49+49}$$
$$=\frac{1}{147}$$

Slope of the tangent to $y=x^{\frac{1}{3}}$ at $x=343$ is $\frac{1}{147}$.

17. h. $$\lim_{x\to a}\frac{(x+4a)^2-25a^2}{x-a}=\lim_{x\to a}\frac{(x-a)(x+9a)}{x-a}$$
$$=10a$$

o. $$\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5-x}}{x}\times\frac{\sqrt{x+5}+\sqrt{5-x}}{\sqrt{x+5}+\sqrt{5-x}}$$

$$=\lim_{x\to 0}\frac{x+5-5+x}{x\left(\sqrt{x+5}+\sqrt{5-x}\right)}$$

$$=\frac{1}{2\sqrt{5}}$$

q. $$\lim_{x\to -2}\frac{x^3+x^2-8x-12}{x+2}$$

$$=\lim_{x\to -2}\frac{(x+2)(x^2-1x-6)}{(x+2)}$$

$$=4+2-6$$

$$=0$$

r. $$\lim_{x\to 2}\frac{x^3+x^2-12}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+3x+6)}{x-2}$$

$$=4+6+6$$

$$=16$$

t. $$\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{2+x}-\frac{1}{2}\right)$$

$$=\lim_{x\to 0}\frac{1}{x}\times-\frac{x}{2(2+x)}$$

$$=\lim_{x\to 0}-\frac{1}{2(2+x)}$$

$$=-\frac{1}{4}$$

u. $$\lim_{x\to -1}\frac{108(x^2+2x)(x+1)(x+1)(x+1)}{(x+1)^3(x^2-x+1)^3(x-1)}$$

$$=\lim_{x\to -1}\frac{108(x^2+2x)}{(x^2-x+1)^3(x-1)}$$

$$=-\frac{108}{27(-2)}$$

$$=2$$

18. d. $$\lim_{x\to 0}\frac{|x|}{x}$$

$x\to 0^-$ $|x|=-x$

$$\lim_{x\to 0^-}\frac{|x|}{x}=-1$$

$$\lim_{x\to 0^+}\frac{|x|}{x}=1$$

$$\lim_{x\to 0^+}\frac{|x|}{x}\neq\lim_{x\to 0^-}\frac{|x|}{x}$$

e. $$f(x)=\begin{cases}-5, & x<1\\ 2, & x\geq 1\end{cases}$$

$$\lim_{x\to 1^-}f(x)=-5\neq\lim_{x\to 1^+}f(x)=2$$

f. $$f(x)=\begin{cases}5x^2, & x<-1\\ 2x+1, & x\geq -1\end{cases}$$

$$\lim_{x\to -1^+}f(x)=-1$$

$$\lim_{x\to -1^-}f(x)=5$$

$$\lim_{x\to -1^+}f(x)\neq\lim_{x\to -1^-}f(x)$$

Therefore, $\lim_{x\to -1}f(x)$ does not exist.

Chapter 3 Test

3. $\lim_{x\to 1}\frac{1}{x-1}$ does not exist since

$$\lim_{x\to 1^+}\frac{1}{x-1}=+\infty\neq\lim_{x\to 1^-}\frac{1}{x-1}=-\infty.$$

4. $$f(x)=\frac{x}{x-3},\quad g(x)=\frac{3}{x-3}$$

$\lim_{x\to 3}f(x)$ does not exist.

$\lim_{x\to 3}g(x)$ does not exist.

$$\lim_{x\to 3}\left[f(x)-g(x)\right]=\lim_{x\to 3}\frac{x-3}{x-3}$$

$$=\lim_{x\to 3}1$$

$$=1$$

5. $f(x)=5x^2-8x$

$f(-2)=5(4)-8(-2)=20+16=36$

$f(1)=5-8=-3$

Slope of secant is $\dfrac{36+3}{-2-1}=-\dfrac{39}{3}$

$=-13.$

6. Slope of a line perpendicular to $y=\dfrac{3}{4}x+5$ is $-\dfrac{4}{5}$.

7. For $f(x)=\dfrac{\sqrt{x^2+100}}{5}$, y-intercept is 2.

8. Through (0, –2), slope –1,

$y=-x-2$ or $x+y+2=0.$

9. **a.** $\lim\limits_{x\to 1} f(x)$ does not exist.

b. $\lim\limits_{x\to 2} f(x)=1$

c. $\lim\limits_{x\to 4^-} f(x)=1$

d. *f* is discontinuous at *x* = 1 and *x* = 2.

10. $P=100\ 000+4000t$

P – population

t – years

a. $t=20$ $\quad P=100\ 000+80\ 000$

$=180\ 000$

Population in 20 years will be 180 000 people.

b. $P(a+h)-P(a)$

$=(100\ 000+4000(a+h))-(100\ 000+4000a)$

$=4000h$

Growth rate:

$\lim\limits_{h\to 0}\dfrac{P(a+h)-Pa}{h}=4000$

Growth rate is 4000 people per year.

11. **a.** Average velocity from $t=2$ to $t=5$:

$\dfrac{s(5)-s(2)}{3}=\dfrac{(40-25)-(16-4)}{3}$

$=\dfrac{15-12}{3}$

$=1$

Average velocity from *t* = 2 to *t* = 5 is 1 km/h.

b. $s(3+h)-s(3)$

$=8(3+h)-(3+h)^2-(24-9)$

$=24+8h-9-6h-h^2-15$

$=2h-h^2$

$v(3)=\lim\limits_{h\to 0}\dfrac{2h-h^2}{h}=2$

Velocity at $t=3$ is 2 km/h.

12. $f(x)=\sqrt{x+11},$

Average rate of change from

$x=5$ to $x=5+h$:

$\dfrac{f(5+h)-f(5)}{h}$

$=\dfrac{\sqrt{16+h}-\sqrt{16}}{h}$

13. $f(x)=\dfrac{x}{x^2-15}$

Slope of the tangent at $x=4$:

$f(4+h)=\dfrac{4+h}{(4+h)^2-15}$

$=\dfrac{4+h}{1+8h+h^2}$

$f(4)=\dfrac{4}{1}$

$f(4+h)-f(4)=\dfrac{4+h}{1+8h+h^2}-4$

$=\dfrac{4+h-4-32h-4h^2}{1+2h+h^2}$

$=-\dfrac{31h-4h^2}{(1+2h+h^2)}$

$\lim\limits_{h\to 0}\dfrac{f(4+h)-f(4)}{h}=\lim\limits_{h\to 0}\dfrac{(-31-4h)}{1+2h+h^2}$

$=-31$

Slope of the tangent at $x=4$ is –31.

14. **a.** $\lim\limits_{x\to 3}\dfrac{4x^2-36}{2x-6}=\lim\limits_{x\to 3}\dfrac{2(x-3)(x+3)}{(x-3)}$

$=12$

b. $\lim\limits_{x\to 2}\dfrac{2x^2-x-6}{3x^2-7x+2}=\lim\limits_{x\to 2}\dfrac{(2x+3)(x-2)}{(x-2)(3x-1)}$

$=\dfrac{7}{5}$

c. $$\lim_{x\to5}\frac{x-5}{\sqrt{x-1}-2}=\lim_{x\to5}\frac{(x-1)-4}{\sqrt{x-1}-2}$$

$$=\lim_{x\to5}\frac{\left(\sqrt{x-1}-2\right)\left(\sqrt{x-1}+2\right)}{\sqrt{x-1}-2}$$

$$=4$$

d. $$\lim_{x\to-1}\frac{x^3+1}{x^4-1}=\lim_{x\to-1}\frac{(x+1)(x^2-x+1)}{(x-1)(x+1)(x^2+1)}$$

$$=\frac{3}{-2(2)}$$

$$=-\frac{3}{4}$$

e. $$\lim_{x\to3}\left(\frac{1}{x-3}-\frac{6}{x^2-9}\right)=\lim_{x\to3}\frac{(x+3)-6}{(x-3)(x+3)}$$

$$=\lim_{x\to3}\frac{1}{x+3}$$

$$=\frac{1}{6}$$

f. $$\lim_{x\to0}\frac{(x+8)^{\frac{1}{3}}-2}{x}=\lim_{x\to0}\frac{(x+8)^{\frac{1}{3}}-2}{(x+8)-8}$$

$$=\lim_{x\to0}\frac{(x+8)^{\frac{1}{3}}-2}{\left((x+8)^{\frac{1}{3}}-2\right)\left((x+8)^{\frac{2}{3}}+2(x+8)^{\frac{1}{3}}+4\right)}$$

$$=\frac{1}{4+4+4}$$

$$=\frac{1}{12}$$

15. $$f(x)=\begin{cases}ax+3, & x>5\\ 8, & x=5\\ x^2+bx+a, & x<5\end{cases}$$

$f(x)$ is continuous.

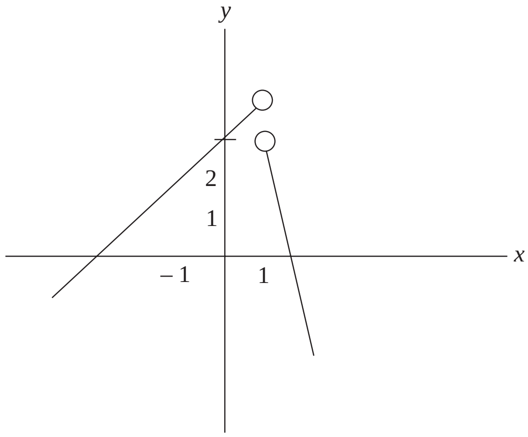
Therefore, $5a+3=8$ $\qquad a=1$

$25+5b+a=8$ $\qquad 5b=-18$

$$b=-\frac{18}{5}$$

16. **a.** $f(0)=3$ **b.** $\lim_{x\to1^+}f(x)=3$

c. $\lim_{x\to1^-}f(x)=4$ **d.** $f(2)=-1$



17. $$f(x)=\begin{cases}\dfrac{x-2}{\sqrt{7x+2}-\sqrt{6x+4}}, & x\ge-\dfrac{2}{7}, x\ne2\\ k, x=2\end{cases}$$

$$\frac{x-2}{\sqrt{7x+2}-\sqrt{6x+4}}\times\frac{\sqrt{7x+2}+\sqrt{6x+4}}{\sqrt{7x+2}+\sqrt{6x+4}}$$

$$=\frac{(x-2)\left(\sqrt{7x+2}+\sqrt{6x+4}\right)}{7x+2-6x-4}$$

$$=\sqrt{7x+2}+\sqrt{6x+4}$$

Now, when $x = 2$,

$k=\sqrt{7(2)+2}+\sqrt{6(2)+4}$

$=4+4$

$k=8.$

Chapter 4 • Derivatives

Review of Prerequisite Skills

1. f.
$$\frac{4p^7\times 6p^9}{12p^{15}}=\frac{24p^{16}}{12p^{15}}$$
$$=2p$$

i.
$$\frac{\left(3a^{-4}\right)\left[2a^3(-b)^3\right]}{12a^5-b^2}=-\frac{6a^{-1}b^3}{12a^5b}$$
$$=-\frac{b}{2a^6}$$

6. d.
$$\frac{(x+y)(x-y)}{5(x-y)}\div\frac{(x+y)^3}{10}$$
$$=\frac{(x+y)}{5}\times\frac{10}{(x+y)^3}$$
$$=\frac{2}{(x+y)^2}\times x\neq y$$

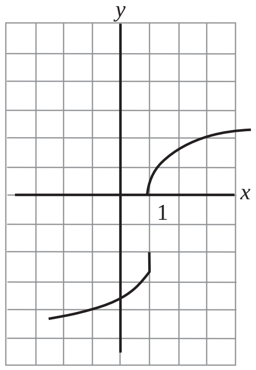
f.
$$\frac{x+1}{x-2}-\frac{x+2}{x+3}=\frac{(x+1)(x+3)-(x+2)(x-2)}{(x-2)(x+3)}$$
$$=\frac{x^2+4x+3-\left(x^2-4\right)}{(x-2)(x+3)}$$
$$=\frac{4x+7}{(x-2)(x+3)}$$

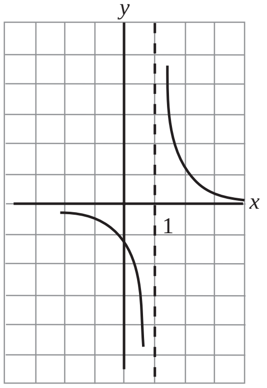
9. c.
$$\frac{2+3\sqrt{2}}{3-4\sqrt{2}}=\frac{2+3\sqrt{2}}{3-4\sqrt{2}}\times\frac{3+4\sqrt{2}}{3+4\sqrt{2}}$$
$$=\frac{6+17\sqrt{2}+24}{9-32}$$
$$=\frac{30+17\sqrt{2}}{-23}$$
$$=-\frac{30+17\sqrt{2}}{23}$$

d.
$$\frac{3\sqrt{2}-4\sqrt{3}}{3\sqrt{2}+4\sqrt{3}}=\frac{3\sqrt{2}-4\sqrt{3}}{3\sqrt{2}+4\sqrt{3}}\times\frac{3\sqrt{2}-4\sqrt{3}}{3\sqrt{2}-4\sqrt{3}}$$
$$=\frac{18-24\sqrt{6}+48}{18-48}$$
$$=\frac{66-24\sqrt{6}}{-30}$$
$$=-\frac{11-4\sqrt{6}}{5}$$

Exercise 4.1

3.





4. b. $f(x)x^2+3x+1;\ \ a=3$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Since $a=3$, $f(3+h)=(3+h)^2+3(3+h)+1$
$$=h^2+9h+19$$
$$f(3)=3^2+3(3)+1$$
$$=19$$

Now $f(3+h)-f(3)=h^2+9h$
$$=h(h+9)$$
$$f'(3)=\lim_{h\to 0}\frac{h(h+9)}{h}$$
$$f'(3)=9$$

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$
$$=\lim_{h\to 0}\frac{(3+h)^2+3(3+h)+1-(9+9+1)}{h}$$
$$=\lim_{h\to 0}\frac{6h+h^2+3h}{h}=\lim_{h\to 0}\frac{9h+h^2}{h}$$
$$=9$$

c. $f(x)=\sqrt{x+1}\ ;\ \ a=0$

$$f'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to0}\frac{\sqrt{h+1}-1}{h}$$

$$=\lim_{h\to0}\frac{\sqrt{h+1}-1}{h}\times\frac{\sqrt{h+1}+1}{\sqrt{h+1}+1}$$

$$=\lim_{h\to0}\frac{h+1-1}{h\left(\sqrt{h+1}+1\right)}$$

$$=\lim_{h\to0}\frac{1}{\sqrt{h+1}+1}$$

$$f'(0)=\frac{1}{2}$$

5. **a.** $f(x)=x^2+3x$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{(x+h)^2+3(x+h)-\left(x^2+3x\right)}{h}$$

$$=\lim_{h\to0}\frac{2hx+h^2+3h}{h}$$

$$=\lim_{h\to0}(2x+3+h)$$

$$f'(x)=2x+3$$

b. $f(x)=\dfrac{3}{x+2}$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{\dfrac{3}{x+h+2}-\dfrac{3}{x+2}}{h}$$

$$=\lim_{h\to0}\frac{3x+6-3x-3h-6}{h(x+h+2)(x+2)}$$

$$=\lim_{h\to0}\frac{-3}{(x+h+2)(x+2)}$$

$$f'(x)=\frac{-3}{(x+2)^2}$$

c.

$$f(x)=\sqrt{3x+2}$$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{\sqrt{3x+3h+2}-\sqrt{3x+2}}{h}\times\frac{\sqrt{3x+3h+2}+\sqrt{3x+2}}{\sqrt{3x+3h+2}+\sqrt{3x+2}}$$

$$=\lim_{h\to0}\frac{3x+3h+2-3x-2}{h\left(\sqrt{3x+3h+2}+\sqrt{3x+2}\right)}$$

$$f'(x)=\frac{3}{2\sqrt{3x+2}}$$

d. $f(x)=\dfrac{1}{x^2}$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h}$$

$$=\lim_{h\to0}\frac{x^2-x^2-2xh-h^2}{h(x+h)^2\left(x^2\right)}$$

$$=\lim_{h\to0}\frac{-2x-h}{(x+h)^2\left(x^2\right)}$$

$$=\frac{-2x}{x^4}$$

$$f'(x)=-\frac{2}{x^3}$$

6. **b.** $y=\dfrac{x+1}{x-1}$

$$\frac{dy}{dx}=\lim_{h\to0}\frac{\dfrac{x+h+1}{x+h-1}-\dfrac{x+1}{x-1}}{h}$$

$$=\lim_{h\to0}\frac{x^2+xh+x-x-h-1-x^2-xh+x-x-h+1}{h(x+h-1)(x-1)}$$

$$=\lim_{h\to0}\frac{-2h}{h(x+h-1)(x-1)}$$

$$\frac{dy}{dx}=-\frac{2}{(x-1)^2}$$

7. $y = 2x^2 - 4x$

Since $y = f(x) = 2x^2 - 4x$

$$f(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = 2(x+h)^2 - 4(x+h)$$

$$= 2x^2 + 4hx + 2h^2 - 4x - 4h$$

$$f(x) = 2x^2 - 4x$$

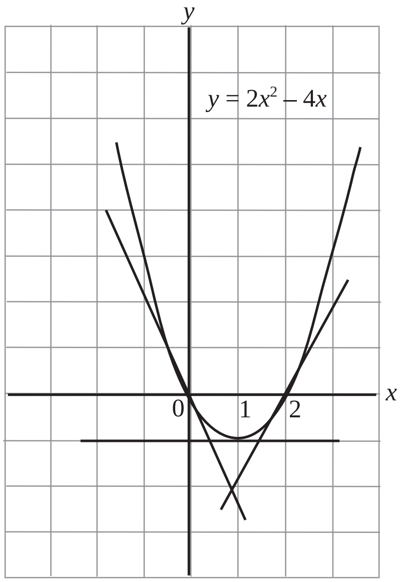
$$f(x+h) - f(x) = 4hx + 2h^2 - 4h$$

$$f'(x) = \lim_{h\to 0}\frac{h(4x + 2h - 4)}{h}$$

$$= \lim_{h\to 0}(4x + 2h - 4)$$

$$f'(x) = 4x - 4$$

Slopes of the tangents at $x = 0$, 1, and 2 are $f'(0) = -4$, $f'(1) = 0$, and $f'(2) = 4$.



8. $s(t) = -t^2 + 8t;$ $t = 0,$ $t = 4,$ $t = 6$

$$v(t) = s'(t) = \lim_{h\to 0}\frac{s(t+h) - s(t)}{h}$$

$$s(t+h) = -(t+h)^2 + 8(t+h)$$

$$= -t^2 - 2ht - h^2 + 8t + 8h$$

$$s(t+h) - s(t) = -2ht - h^2 + 8h$$

$$= h(-2t - h + 8)$$

$$v(t) = \lim_{h\to 0}\frac{h(-2t - h + 8)}{h}$$

$$= \lim_{h\to 0}(-2t - h + 8)$$

$$v(t) = -2t + 8$$

Velocities at $t = 0$, 4, and 6 are $v(0) = 8$, $v(4) = 0$, and $v(6) = -4$.

9. $f(x) = \sqrt{x+1}$, parallel to $x - 6y + 4 = 0$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \times \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$= \lim_{h\to 0}\frac{x+h+1 - (x+1)}{h\left(\sqrt{x+h+1} + \sqrt{x+1}\right)}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{x+h+1} + \sqrt{x+1}}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

The slope of the tangent to $f(x) = \sqrt{x+1}$ is parallel to $x - 6y + 4 = 0$.

$$\therefore f'(x) = \frac{1}{6}$$

$$\frac{1}{2\sqrt{x+1}} = \frac{1}{6}$$

$$\sqrt{x+1} = 3$$

$$x = 8$$

The point of tangency will be $(8, f(8)) = (8, 3)$.

The equation of the line will be $y - 3 = \frac{1}{6}(x - 8)$

or $x - 6y + 10 = 0$.

13. $\frac{1}{x}+\frac{1}{y}=1$ at (2, 2)

$$y=1-\frac{1}{x}=\frac{x-1}{x}$$
$$y=\frac{x}{x-1}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\frac{x+h}{x+h-1}-\frac{x}{x-1}}{h}$$
$$=\lim_{h\to 0}\frac{x^2+xh-x-h-x^2-xh+x}{h(x-1)(x+h-1)}$$
$$=\lim_{h\to 0}\frac{-1}{(x-1)(x+h-1)}$$
$$=\frac{-1}{(x-1)^2}$$

At $x=2$, $f'(x)=-1$.

Slope of the tangent at (2, 2) is –1.

14. $f(x)=x|x|$

For $x<0$, $|x|=-x$ $\therefore f(x)=-x^2$

$x\geq 0$, $|x|=x$ $\therefore f(x)=x^2$

$\therefore f'(x)=-2x,\ x<0$

$f'(x)=2x,\ x\geq 0$

And $f'(x)$ exists for all $x\in R$ and $f'(0)=0$.

15. $f(a)=0,\ f'(a)=6$

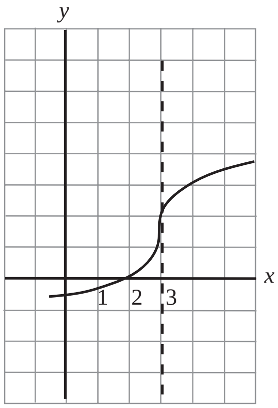
$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=6$$

But $f(a)=0$

$$\therefore \lim_{h\to 0}\frac{f(a+h)}{h}=6$$

and $\lim_{h\to 0}\frac{f(a+h)}{2h}=3.$

16.



$f(x)$ is continuous.

$f(3)=2$

But $f'(3)=\infty$.

(Vertical tangent)

Exercise 4.2

2. **h.** $f(x)=\sqrt[3]{x}$

$=x^{\frac{1}{3}}$

$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$

k. $f(x)=\left(\frac{x}{2}\right)^4$

$=\frac{x^4}{16}$

$f'(x)=\frac{4x^3}{16}$

$=\frac{x^3}{4}$

3. **f.** $h(x)=(2x+3)(x+4)$

$=2x^2+9x+12$

$h'(x)=4x+9$

k.
$$\begin{aligned} s(t) &= \frac{t^5 - 3t^2}{2t} \\ &= \frac{t^4}{2} - \frac{3}{2}t \\ s'(t) &= \frac{4t^3}{2} - \frac{3}{2} \\ &= 2t^3 - \frac{3}{2} \end{aligned}$$

4. e.
$$\begin{aligned} y &= 3x^{\frac{2}{3}} - 6x^{\frac{1}{3}} + x^{-\frac{1}{3}} \\ \frac{dy}{dx} &= \frac{2}{3}\left(3x^{-\frac{1}{3}}\right) - \frac{1}{3}\left(6x^{-\frac{2}{3}}\right) - \frac{1x^{-\frac{4}{3}}}{3} \\ &= 2x^{-\frac{1}{3}} - 2x^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{4}{3}} \end{aligned}$$

i.
$$\begin{aligned} y &= 20x^5 + 3\sqrt[3]{x} + 17 \\ &= 20x^5 + 3x^{\frac{1}{3}} + 17 \\ \frac{dy}{dx} &= 100x^4 + 3 \times \frac{1}{3}x^{-\frac{2}{3}} \\ &= 100x^4 + x^{-\frac{2}{3}} \end{aligned}$$

j.
$$\begin{aligned} y &= \sqrt{x} + 6\sqrt{x^3} + \sqrt{2} \\ &= x^{\frac{1}{2}} + 6x^{\frac{3}{2}} + \sqrt{2} \\ \frac{dy}{dx} &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2} \times 6x^{\frac{1}{2}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + 9x^{\frac{1}{2}} \end{aligned}$$

l.
$$\begin{aligned} y &= \frac{1 + \sqrt{x}}{x} \\ &= \frac{1}{x} + \frac{\sqrt{x}}{x} \\ &= x^{-1} + x^{-\frac{1}{2}} \\ \frac{dy}{dx} &= -x^{-2} - \frac{1}{2}x^{-\frac{3}{2}} \end{aligned}$$

6. b.
$$\begin{aligned} f(x) &= 7 - 6x^{\frac{1}{2}} + 5x^{\frac{2}{3}} \\ f'(x) &= -3x^{-\frac{1}{2}} + \frac{10}{3}x^{-\frac{1}{3}} \\ f'(64) &= -\frac{3}{\sqrt{64}} + \frac{10}{3\sqrt[3]{64}} \\ &= -\frac{3}{8} + \frac{10}{12} \\ &= \frac{11}{24} \end{aligned}$$

7. d.
$$\begin{aligned} y &= \sqrt{16x^3} \\ &= 4x^{\frac{3}{2}} \\ \frac{dy}{dx} &= 6x^{\frac{1}{2}} \end{aligned}$$

At $(4, 32)$,

$$\begin{aligned} \frac{dy}{dx} &= 6(2) \\ \frac{dy}{dx} &= 12. \end{aligned}$$

8. b.
$$\begin{aligned} y &= 2\sqrt{x} + 5 \\ y &= 2x^{\frac{1}{2}} + 5 \\ \frac{dy}{dx} &= x^{-\frac{1}{2}} \end{aligned}$$

At $x = 4$,

$$\frac{dy}{dx} = \frac{1}{3}.$$

d.
$$\begin{aligned} y &= x^{-3}\left(x^{-1} + 1\right) \\ &= x^{-4} + x^{-3} \\ \frac{dy}{dx} &= -4x^{-5} - 3x^{-4} \end{aligned}$$

At $x = 1$,

$$\begin{aligned} \frac{dy}{dx} &= -4 - 3 \\ &= -7. \end{aligned}$$

9. **a.** $y = 2x - \frac{1}{x}$ at $P(0.5, -1)$

$$\frac{dy}{dx} = 2 + \frac{1}{x^2}$$

Slope of the tangent at $x = \frac{1}{2}$ is $2 + 4 = 6$.

Equation $y + 1 = 6\left(x - \frac{1}{2}\right)$

$$6x - y - 4 = 0$$

b. $y = \frac{3}{x^2} - \frac{4}{x^3}$ at $P(-1, 7)$

$$= 3x^{-2} - 4x^{-3}$$

$$\frac{dy}{dx} = -6x^{-3} + 12x^{-4}$$

$$\left.\frac{dy}{dx}\right]_{x=-1} = 6 + 12 = 18$$

$$y - 7 = 18(x + 1)$$

$$18x - y + 25 = 0$$

c. $y = \sqrt{3x^3}$ at $P(3, 9)$

$$\frac{dy}{dx} = \sqrt{3} \times \frac{3}{2} x^{\frac{1}{2}}$$

$$\left.\frac{dy}{dx}\right]_{x=3} = \sqrt{3} \times \frac{3}{2} \times \sqrt{3} = \frac{9}{2}$$

$$y - 9 = \frac{9}{2}(x - 3)$$

$$9x - 2y - 9 = 0$$

d. $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right)$ at $P(1, 2)$

$$= x + x^{-2}$$

$$\frac{dy}{dx} = 1 - 2x^{-3}$$

Slope at $x = 1$ is –1.

$$y - 2 = -1(x - 1)$$

$$x + y - 3 = 0$$

e. $y = \left(\sqrt{x} - 2\right)\left(3\sqrt{x} + 8\right)$ at $\left(2, 2\sqrt{2} - 10\right)$

$$= 3x + 2\sqrt{x} - 16$$

$$\frac{dy}{dx} = 3 + x^{-\frac{1}{2}}$$

At $x = 4$, slope is $3 + \frac{1}{2} = \frac{7}{2}$.

Now, $y = \frac{7}{2}(x - 4)$

or $7x - 2y - 28 = 0$

f. $y = \frac{\sqrt{x} - 2}{\sqrt[3]{x}} = \frac{x^{\frac{1}{2}} - 2}{x^{\frac{1}{3}}} = x^{\frac{1}{6}} - 2x^{-\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{6} x^{-\frac{5}{6}} + \frac{2}{3} x^{-\frac{4}{3}}$$

Slope at $x = 1$ is $\frac{1}{6} + \frac{2}{3} = \frac{5}{6}$.

Now, $y + 1 = \frac{5}{6}(x - 1)$; $5x - 6y - 11 = 0$.

10. A normal to the graph of a function at a point is a line that is perpendicular to the tangent at the given point.

$y = \frac{3}{x^2} - \frac{4}{x^3}$ at $P(-1, 7)$

Slope of the tangent is 18, therefore, the slope of the normal is $-\frac{1}{18}$.

Equation is $y - 7 = -\frac{1}{18}(x + 1)$.

$$x + 18y - 125 = 0$$

11. $y=\frac{3}{\sqrt[3]{x}}=3x^{-\frac{1}{3}}$

Parallel to $x+16y+3-0$

Slope of the line is $-\frac{1}{16}$.

$$\frac{dy}{dx}=-x^{-\frac{4}{3}}$$

$$\therefore x^{-\frac{4}{3}}=\frac{1}{16}$$

$$\frac{1}{x^{\frac{4}{3}}}=\frac{1}{16}$$

$$x^{\frac{4}{3}}=16$$

$$x=(16)^{\frac{3}{4}}=8$$

12. $y=\frac{1}{x}=x^{-1}\ :\ y=x^{3}$

$$\frac{dy}{dx}=-\frac{1}{x^{2}}\quad:\ \frac{dy}{dx}=3x^{2}$$

Now, $-\frac{1}{x^{2}}=3x^{2}$

$$x^{4}=-\frac{1}{3}.$$

No real solution. They never have the same slope.

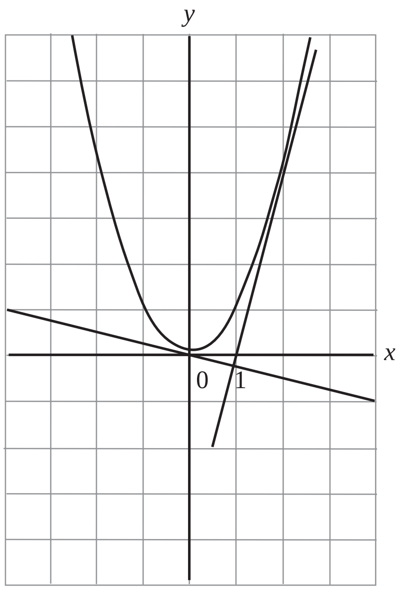
13. $y=x^{2},\ \frac{dy}{dx}=2x$

The slope of the tangent at $A(2,4)$ is 4 and at

$B\left(-\frac{1}{8},\frac{1}{64}\right)$ is $-\frac{1}{4}$.

Since the product of the slopes is –1, the tangents at

$A(2,4)$ and $B\left(-\frac{1}{8},\frac{1}{64}\right)$ will be perpendicular.



14. $y=-x^{2}+3x+4$

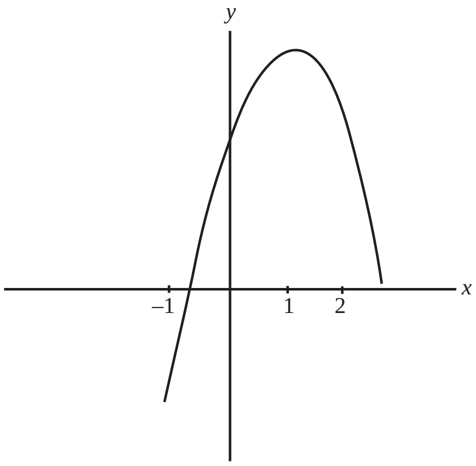
$$\frac{dy}{dx}=-2x+3$$

For $\frac{dy}{dx}=5$,

$$5=-2x+3$$

$$x=-1.$$

The point is $(-1,0)$.



15. $$y = x^3 + 2$$

$$\frac{dy}{dx} = 3x^2, \text{ slope is } 12$$

$$\therefore x^2 = 4$$

$$x = 2 \quad \text{or} \quad x = -2$$

Points are $(2, 10)$ and $(-2, -6)$.

16. $y = \frac{1}{5}x^5 - 10x$, slope is 6

$$\frac{dy}{dx} = x^4 - 10 = 6$$

$$x^4 = 16$$

$$x^2 = 4 \quad \text{or} \quad x^2 = -4$$

$$x = \pm 2 \quad \text{non-real}$$

Tangents with slope 6 are at the points $\left(2, -\frac{68}{5}\right)$ and $\left(-2, \frac{68}{5}\right)$.

17. $y = 2x^2 + 3$

a. Equation of tangent from $A(2, 3)$:
If $x = a$, $y = 2x^2 + 3$.

Let the point of tangency be $P(a, 2a^2 + 3)$.

Now, $\frac{dy}{dx} = 4x$ and $\left.\frac{dy}{dx}\right]_{x=a} = 4a$.

The slope of the tangent is the slope of AP.

$$\therefore \frac{2a^2}{a-2} = 4a.$$

$$2a^2 = 4a^2 - 8a$$

$$2a^2 - 8a = 0$$

$$2a(a-4) = 0$$

$$a = 0 \quad \text{or} \quad a = 4$$

Point (2, 3):

Slope is 0.	Slope is 16.
Equation of tangent is	Equation of tangent is
$y - 3 = 0$.	$y - 3 = 16(x-2)$
	or $16x - y - 29 = 0$.

From the point $B(2, -7)$:

Slope of BP: $\frac{2a^2 + 10}{a-2} = 4a$

$$2a^2 + 10 = 4a^2 - 8a$$

$$2a^2 - 8a - 10 = 0$$

$$a^2 - 4a - 5 = 0$$

$$(a-5)(a+1) = 0$$

$a = 5$	$a = -1$
Slope is $4a = 20$.	Slope is -4.
Equation is	Equation is
$y + 7 = 20(x-2)$	$y + 7 = -4(x-2)$
or $20x - y - 47 = 0$.	or $4x + y - 1 = 0$.

18. $ax - 4y + 21 = 0$ is tangent to $y = \frac{a}{x^2}$ at $x = -2$.

Therefore, the point of tangency is $\left(-2, \frac{a}{4}\right)$.

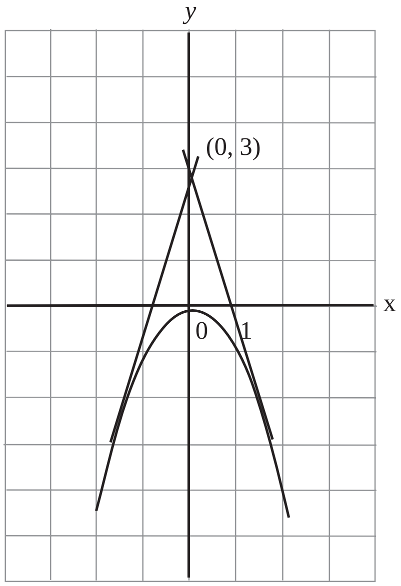
This point lies on the line, therefore,

$$a(-2) - 4\left(\frac{a}{4}\right) + 21 = 0$$

$$-3a + 21 = 0$$

$$a = 7.$$

22.



Let the coordinates of the points of tangency be $A(a, -3a^2)$.

$\frac{dy}{dx} = -6x$, slope of the tangent at A is $-6a$

Slope of PA: $\frac{-3a^2-3}{a} = -6a$

$$-3a^2 - 3 = -6a^2$$
$$3a^2 = 3$$

$a = 1$ or $a = -1$

Coordinates of the points at which the tangents touch the curve are $(1, -3)$ and $(-1, -3)$.

23. $y = x^3 - 6x^2 + 8x$, tangent at $A(3, -3)$

$$\frac{dy}{dx} = 3x^2 - 12x + 8$$

$$\left.\frac{dy}{dx}\right]_{x-3} = 27 - 36 + 8 = -1$$

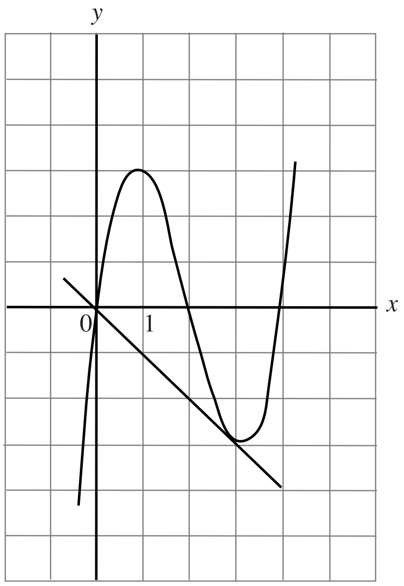
The slope of the tangent at $A(3, -3)$ is –1.

Equation will be

$$y + 3 = -1(x - 3)$$
$$y = -x.$$
$$-x = x^3 - 6x^2 + 8x$$
$$x^3 - 6x^2 + 9x = 0$$
$$x(x^2 - 6x + 9) = 0$$
$$x(x-3)^2 = 0$$

$x = 0$ or $x = 3$

Coordinates are $B(0, 0)$.



24. $\sqrt{x} + \sqrt{y} = 1$

$P(a, b)$ is on the curve, therefore $a \geq 0$, $b \geq 0$.

$$\sqrt{y} = 1 - \sqrt{x}$$
$$y = 1 - 2\sqrt{x} + x$$
$$\frac{dy}{dx} = -\frac{1}{2} \bullet 2x^{-\frac{1}{2}} + 1$$

At $x = a$, slope is $-\frac{1}{\sqrt{a}} + 1 = -\frac{1 + \sqrt{a}}{\sqrt{a}}$.

But $\sqrt{a} + \sqrt{b} = 1$

$$-\sqrt{b} = \sqrt{a} - 1.$$

Therefore, slope is $-\frac{\sqrt{b}}{\sqrt{a}} = -\sqrt{\frac{b}{a}}$.

25. $f(x) = x^n$, $f'(x) = nx^{n-1}$

Slope of the tangent at $x = 1$ is $f'(1) = n$,
The equation of the tangent at
(1, 1) is:

$$y - 1 = n(x - 1)$$
$$nx - y - n + 1 = 0$$

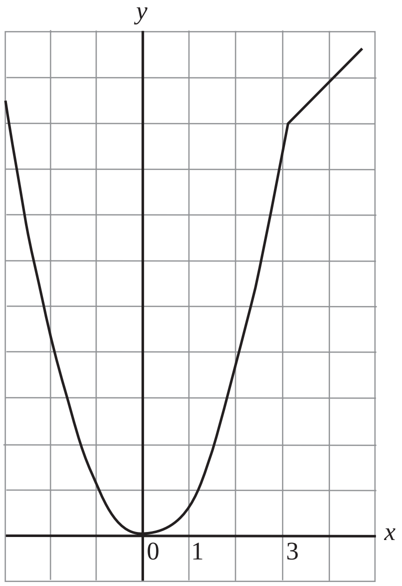
Let $y = 0$, $nz = n - 1$

$$x = \frac{n-1}{n} = 1 - \frac{1}{n}.$$

The x-intercept is $1 - \frac{1}{n}$ as $n \to \infty$, and $\frac{1}{n} \to 0$,

and the x-intercept approaches 1 as $n \to \infty$, the slope of the tangent at (1, 1) increases without bound, and the tangent approaches a vertical line having equation $x - 1 = 0$.

26. **a.**



$$f(x)=\begin{cases} x^2, & x<3 \\ x+6, & x\geq 3\end{cases} \qquad f'(x)=\begin{cases} 2x, & x<3 \\ 1, & x\geq 3\end{cases}$$

$f'(3)$ does not exist.

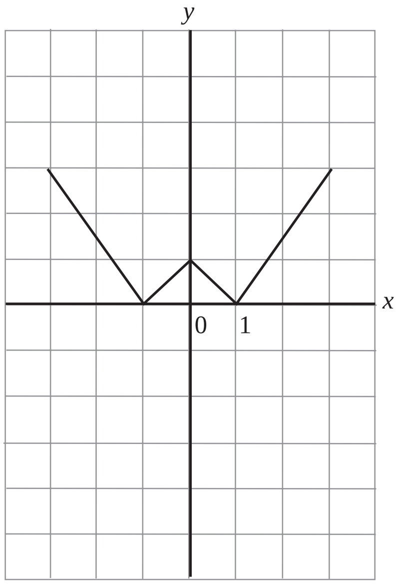
b.

$$f(x)=\begin{cases} 3x^2-6, & x<-\sqrt{2} \text{ or } x>\sqrt{2} \\ 6-3x^2, & -\sqrt{2}<x<\sqrt{2}\end{cases}$$

$$f'(x)=\begin{cases} 6x, & x<-\sqrt{2} \text{ or } x>\sqrt{2} \\ -6x, & -\sqrt{2}\leq x\leq\sqrt{2}\end{cases}$$

$f'(\sqrt{2})$ and $f'(-\sqrt{2})$ do not exist.

c.



$$f(x)=\begin{cases} x-1, x\geq 1 & \text{since} \quad |x-1|=x-1 \\ 1-x, 0\leq x<1 & \text{since} \quad |x-1|=1-x \\ x+1, -1<x<0 & \text{since} \quad |-x-1|=x+1 \\ -x-1, \ x\leq -1 & \text{since} \quad |-x-1|=-x-1\end{cases}$$

$$f'(x)=\begin{cases} 1, & x>1 \\ -1, & 0<x<1 \\ 1, & -1<x<0 \\ -1, & x<-1\end{cases}$$

$f'(0)$, $f'(-1)$, and $f'(1)$ do not exist.

Exercise 4.3

2. **c.** $y=(1-x^2)^4(2x+6)^3$

$$\frac{dy}{dx}=4(1-x^2)^3(-2x)(2x+6)^3+(1-x^2)^4 3(2x+6)^2(2)$$

$$=-8x(1-x^2)^3(2x+6)^3+6(1-x^2)^4(2x+6)$$

4. **e.** $y=x^3(3x+7)^2$

$$\frac{dy}{dx}=3x^2(3x+7)^2+x^3 6(3x+7)$$

At $x=-2$,

$$\frac{dy}{dx}=12(1)^2+(-8)(6)(1)$$

$$=12-48$$

$$=-36.$$

f. $y=(2x+1)^5(3x+2)^4,\ \ x=-1$

$$\frac{dy}{dx}=5(2x+1)^4(2)(3x+2)^4+(2x+1)^5 4(3x+2)^3(3)$$

$$\left.\frac{dy}{dx}\right]_{x=-1}=5(-1)^4(2)(-1)^4+(-1)^5(4)(-1)^3(3)$$

$$=10+12$$

$$=22$$

h. $y=3x(x-4)(x+3),\ x=2$

$$\frac{dy}{dx}=3(x-4)(x+3)+3x(x+3)+3x(x-4)$$

At $x=2$,

$$\frac{dy}{dx}=3(-2)(5)+6(5)+6(-2)$$

$$=-30+30-12$$

$$=-12.$$

5. Tangent to $y=(x^3-5x+2)(3x^2-2x)$ at (1, –2).

$$\frac{dy}{dx}=(3x^2-5)(3x^2-2x)+(x^3-5x+2)(6x-2)$$

$$\left.\frac{dy}{dx}\right]_{x=1}=(-2)(1)+(-2)(4)$$

$$=-2\bullet-8$$

$$=-10$$

Slope of the tangent at (1, –2) is –10.
The equation is $y+2=-10(x-1);\ 10x+y-8=0$.

6. **b.** $y=(x^2+2x+1)(x^2+2x+1)$

$$\frac{dy}{dx}=2(x^2+2x+1)(2x+2)$$

$$(x^2+2x+1)(2x+2)=0$$

$$2(x+1)(x+1)(x+1)=0$$

$$x=-1$$

Point of horizontal tangency is (–1, 0).

7. **b.** $y=x^2(3x^2+4)^2(3-x^3)^4$

$$\frac{dy}{dx}=2x(3x^2+4)^2(3-x^3)^4$$

$$+x^2[2(3x^2+4)(6x)](3-x^3)^4$$

$$+x^2(3x^2+4)^2[4(3-x^3)^3(-3x^2)]$$

8. Determine the point of tangency, and then find the negative reciprocal of the slope of the tangency. Use this information to find the equation of the normal.

$$h(x)=2x(x+1)^3(x^2+2x+1)^2$$

$$h'(x)=2(x+1)^3(x^2+2x+1)^2+2x3(x+1)^2(x^2+2x+1)^2$$

$$+2x(x+1)^3 2(x^2+2x+1)(2x+1)$$

$$h'(-2)=2(-1)^3(1)^2+2(-2)(3)(-1)^2(1)^2+2(-2)(-1)^3(2)(1)(-2)$$

$$=-2-12-16$$

$$=-30$$

9. **a.** $f(x)=g_1(x)g_2(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$

$$f'(x)=g_1^{\ 1}(x)g_2(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$$

$$+g_1(x)g_2^{\ 1}(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$$

$$+g_1(x)g_2(x)g_3^{\ 1}(x)\ \cdots\ g_{n-1}(x)g_n(x)$$

$$+\ \ldots$$

$$+g_1(x)g_2(x)g_3(x)\ \cdots\ g_{n-1}(x)g_n(x)$$

b. $f(x)=(1+x)(1+2x)(1+3x)\ldots(1+nx)$

$$f'(x)=1(1+2x)(1+3x)\ \ldots(1+nx)$$

$$+(1+x)(2)(1+3x)\ \cdots\ (1+nx)$$

$$+(1+x)(1+2x)(3)\ \cdots\ (1+nx)$$

$$+\ \ldots$$

$$+(1+x)(1+2x)(1+3x)\ldots(n)$$

$$\therefore f'(0)=1(1)(1)(1)\ \cdots\ (1)$$

$$+1(2)(1)(1)\ldots\ (1)$$

$$+1(1)(3)(1)\ldots\ (1)$$

$$+\ \ldots$$

$$+(1)(1)(1)\ \ldots\ \ (n)$$

$$=1+2+3+\ \ldots+n$$

$$f'(0)=\frac{n(n+1)}{2}$$

10. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b \qquad (1)$

Horizontal tangent at $(-1, -8)$

$f'(x) = 0$ at $x = -1$

$2a + b = 0$

Since $(2, 19)$ lies on the curve,

$4a + 2b + c = 19. \qquad (2)$

Since $(-1, -8)$ lies on the curve,

$a - b + c = -8. \qquad (3)$

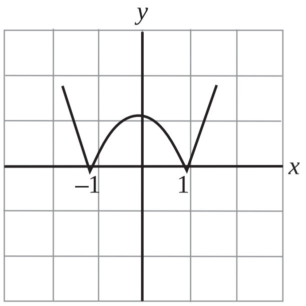
$$\begin{aligned} 4a + 2b + c &= 19 \\ -3a - 3b &= -27 \\ a + b &= 9 \\ 2a + b &= 0 \\ -a &= 9 \end{aligned}$$

$a = -9, \quad b = 18$

$$\begin{aligned} -9 - 18 + c &= -8 \\ c &= 19 \end{aligned}$$

The equation is $y = -9x^2 + 18x + 19$.

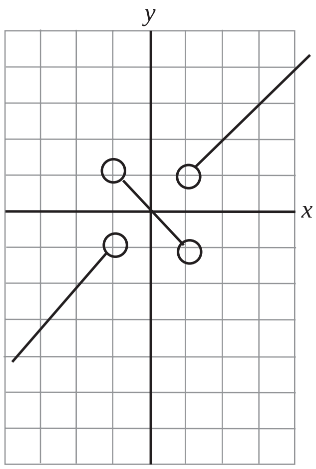
11.



a. $x = 1$ or $x = -1$

b. $f'(x) = 2x, x < -1$ or $x > 1$

$f'(x) = -2x, \ -1 < x < 1.$



c. $f'(-2) = 2(-2) = -4$

$f'(0) = -2(0) = 0$

$f'(3) = 2(3) = 6$

12. $y = \dfrac{16}{x^2} - 1$

$\dfrac{dy}{dx} = -\dfrac{32}{x^3}$

Slope of the line is 4.

$$\begin{aligned} -\frac{32}{x^3} &= 4 \\ 4x^3 &= -32 \\ x^3 &= -8 \\ x &= -2 \\ y &= \frac{16}{4} - 1 \\ &= 3 \end{aligned}$$

Point is at $(-2, 3)$.

Find intersection of line and curve:

$$\begin{aligned} 4x - y + 11 &= 0 \\ y &= 4x + 11. \end{aligned}$$

Substitute,

$$4x + 11 = \frac{16}{x^2} - 1$$

$4x^3 + 11x^2 = 16 - x^2$

or $4x^3 + 12x^2 - 16 = 0.$

Let $x = -2.$

$$\begin{aligned} \text{R.S.} &= 4(-2)^3 + 12(-2)^2 - 16 \\ &= 0 \end{aligned}$$

Since $x = -2$ satisfies the equation, therefore it is a solution.

When $x = -2$, $y = 4(-2) + 11 = 3.$

Intersection point is $(-2, 3)$. Therefore, the line is tangent to the curve.

Exercise 4.4

4. **g.** $y=\frac{x(3x+5)}{(1-x^2)}=\frac{3x^2+5x}{1-x^2}$

$$\frac{dy}{dx}=\frac{(6x+5)(-x^2)-(3x^2+5x)(-2x)}{(1-x^2)^2}$$

$$=\frac{-6x^3-5x^2+6x+5+6x^3+10x^2}{(1-x^2)^2}$$

$$=\frac{5x^2+6x+5}{(1-x^2)^2}$$

5. **c.** $y=\frac{x^2-25}{x^2+25},\quad x=2$

$$\frac{dy}{dx}=\frac{2x(x^2+25)-(x^2-25)(2x)}{(x^2+25)^2}$$

$$\left.\frac{dy}{dx}\right]_{x=2}=\frac{4(29)-(-21)(4)}{(29)^2}$$

$$=\frac{116+84}{29^2}$$

$$=\frac{200}{841}$$

d. $y=\frac{(x+1)(x+2)}{(x-1)(x-2)}, x=4$

$$=\frac{x^2+3x+2}{x^2-3x+2}$$

$$\frac{dy}{dx}=\frac{(2x+3)(x^2-3x+2)-(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2}$$

At $x = 4$:

$$\frac{dy}{dx}=\frac{(11)(6)-(30)(5)}{(9)(4)}$$

$$=-\frac{84}{36}$$

$$=-\frac{7}{3}$$

6. $y=\frac{x^3}{x^2-6}$

$$\frac{dy}{dx}=\frac{3x^2(x^2-6)-x^3(2x)}{(x^2-6)^2}$$

At $(3, 9)$:

$$\frac{dy}{dx}=\frac{3(9)(3)-(27)(6)}{(3)^2}$$

$$=9-18$$

$$=-9$$

The slope of the tangent to the curve at $(3,9)$ is -9.

7. $y=\frac{3x}{x-4}$

$$\frac{dy}{dx}=\frac{3(x-4)-3x}{(x-4)^2}=-\frac{12}{(x-4)^2}$$

Slope of the tangent is $-\frac{12}{25}$.

Therefore, $\frac{12}{(x-4)^2}=\frac{12}{25}$

$\therefore x-4=5$ or $x-4=-5$

$x=9$ or $x=-1$.

Points are $\left(9,\frac{27}{5}\right)$ and $\left(-1,\frac{3}{5}\right)$.

8. $f(x)=\frac{5x+2}{x+2}$

$$f'(x)=\frac{(x+2)(5)-(5x+2)(1)}{(x+2)^2}$$

$$f'(x)=\frac{8}{(x+2)^2}$$

Since $(x+2)^2$ is positive or zero for all $x\in R$, $\frac{8}{(x+2)^2}>0$ for $x\neq-2$. Therefore, tangents to the graph of $f(x)=\frac{5x+2}{x+2}$ do not have a negative slope.

9. **b.**
$$\begin{aligned} y &= \frac{x^2-1}{x^2+x-2} \\ &= \frac{(x-1)(x+1)}{(x+2)(x-1)} \\ &= \frac{x+1}{x+2},\ x \neq 1 \\ \frac{dy}{dx} &= \frac{(x+2)-(x+1)}{(x+2)^2} \\ &= \frac{1}{(x+2)^2} \end{aligned}$$

Curve has horizontal tangents when $\frac{dy}{dx}=0$.

No value of *x* will give a horizontal slope, therefore, there are no such tangents.

10.
$$\begin{aligned} p(t) &= 1000\left(1+\frac{4t}{t^2+50}\right) \\ p'(t) &= 1000\left(\frac{4(t^2+50)-4t(2t)}{(t^2+50)^2}\right) \\ &= \frac{1000(200-4t^2)}{(t^2+50)^2} \\ p'(1) &= \frac{1000(196)}{(51)^2} \doteq 75.36 \\ p'(2) &= \frac{1000(184)}{54^2} \doteq 63.10 \end{aligned}$$

Population is growing at a rate of 75.4 bacteria per hour at $t = 1$ and at 63. One bacteria per hour at $t = 2$.

12. **a.** $s(t)=\frac{10(6-t)}{t+3},\ 0 \le t \le 6\ t=0,\ s(0)=20$

The boat is initially 20 m from the dock.

b.
$$\begin{aligned} v(t) = s'(t) &= 10\left[\frac{(t+3)(-1)-(6-t)(1)}{(t+3)^2}\right] \\ v(t) &= \frac{-90}{(t+3)^2} \end{aligned}$$

At $t=0,\ v(0)=-10$, the boat is moving towards the dock at a speed of 10 m/s. When $s(t)=0$, the boat will be at the dock.

$$\frac{10(6-t)}{t+3}=0,\ t=6.$$

$$v(6)=\frac{-90}{9^2}=-\frac{10}{9}$$

The speed of the boat when it bumps into the dock is $\frac{10}{9}$ m/s.

13.
$$\begin{aligned} f(x) &= \frac{ax+b}{cx+d},\ x \neq -\frac{d}{c} \\ f'(x) &= \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2} \\ f'(x) &= \frac{ad-bc}{(cx+d)^2} \end{aligned}$$

For the tangents to the graph of $y=f(x)$ to have positive slopes, $f'(x)>0$. $(cx+d)^2$ is positive for all $x \in R$. $ad-bc>0$ will ensure each tangent has a positive slope.

Exercise 4.5

4. **b.** If $g(x)=5x-1$ and $f(x)=\sqrt{x}$,

$$\begin{aligned} \text{then } h(x) &= f(g(x)) \\ &= f(5x-1) \\ f(x) &= \sqrt{5x-1}. \end{aligned}$$

e.
$$\begin{aligned} h(x) &= x^4+5x^2+6 \\ &= (x^2+2)(x^2+3) \\ &= (x^2+2)(x^2+2+1) \end{aligned}$$

If $g(x)=x^2+2$ and $f(x)=x(x+1)$,

$$\begin{aligned} \text{then } h(x) &= f(g(x)) \\ &= g(x)(g(x)+1) \\ &= (x^2+2)(x^2+2+1) \\ h(x) &= x^4+5x^2+6. \end{aligned}$$

5. $f(x)=\sqrt{2-x}$ and $f(g(x))=\sqrt{2-x^3}$

$$f(g(x))=\sqrt{2-g(x)}=\sqrt{2-x^3}$$
$$g(x)=x^3$$

6. $g(x)=\sqrt{x},\ f(g(x))=\left(\sqrt{x}+7\right)^2$

$$f(x)=(x+7)^2$$

7. $g(x)=x-3,\ f(g(x))=x^2$

$$\therefore f(x-3)=x^2$$
$$f(x-3)=[(x-3)+3]^2$$
$$\therefore f(x)=(x+3)^2$$

Or, since $g(x)$ is linear and $f(g(x))$ is quadratic, $f(x)$ is a quadratic function.

Let $f(x)=ax^2+bx+c.$

$$\therefore f(g(x))=a(x-3)^2+b(x-3)+c=x^2$$
$$ax^2-bax+ga+bx-3b+c=x^2$$
$$ax^2+x(b-6a)+9a-3b+c=x^2$$

Equating coefficients:

$$a=1$$
$$b-6a=0 \qquad b=6$$
$$9a-3b+c=0 \qquad c=9.$$
$$\therefore f(x)=x^2+6x+9$$
$$f(x)=(x+3)^2$$

8. $f(x)=x^2,\ f(g(x))=x^2+8x+16$

But $f(g(x))=[g(x)]^2$.

$$\therefore [g(x)]^2=x^2+8x+16=(x+4)^2$$
$$g(x)=x+4 \text{ or } g(x)=-x-4$$

9. $f(x)=x+4,\ g(x)=(x-2)^2$

and $f(g(u(x)))=4x^2-8x+8$

$$g(u(x))=(u(x)-2)^2$$

and $f(g(u(x)))=f\left((u(x)-2)^2\right)$

$$=(u(x)-2)^2=4$$
$$=(u(x))^2-4u(x)+8$$

Since $f(g(u(x)))$ is quadratic, $u(x)$ must be linear.

Let $u(x)=ax+b$.

Now,

$$(ax+b)^2-4(ax+b)+8=4x^2-8x+8$$

$$a^2=4,\quad a=2,\ \text{or} \qquad a=-2$$
$$2ab-4a=-8,\ b=0,\ \text{or}\ -4b+8=-8$$
$$b=4.$$
$$\therefore u(x)=2x \text{ or } u(x)=-2x+4$$

10. $f(x)=\dfrac{1}{1-x},\ g(x)=1-x$

a. $g(f(x))=g\left(\dfrac{1}{1-x}\right)$

$$=1-\frac{1}{1-x}$$
$$=\frac{1-x-1}{1-x}$$
$$=-\frac{x}{1-x}=\frac{x}{x-1}$$

b. $f(g(x))=f(1-x)$

$$=\frac{1}{1-(1-x)}$$
$$=\frac{1}{x}$$

11. $f\left(g(x)\right)=g\left(f(x)\right)$

$$3\left(x^2+2x-3\right)+5=\left(3x+5\right)^2+2\left(3x+5\right)-3$$
$$3x^2+6x-9+5=9x^2+30x+25+6x+10-3$$
$$3x^2+6x-4=9x^2+36x+32$$
$$6x^2+30x+36=0$$
$$x^2+5x+6=0$$
$$\left(x+3\right)\left(x+2\right)=0$$
$$x=-3 \text{ or } x=-2$$

12. a. $f(x)=2x-7$

$$f^{-1}(x)=\frac{x+7}{2}$$
$$f\circ f^{-1}=f\left(\frac{x+7}{2}\right)$$
$$=2\left(\frac{x+7}{2}\right)-7$$
$$=x$$

$$f^{-1}\circ f=f^{-1}\left(2x-7\right)$$
$$=\frac{2x-7+7}{2}$$
$$=x$$

b. $f\circ g=f\left(5-2x\right)$

$$=2\left(5-2x\right)-7$$
$$=10-4x-7$$
$$=3-4x$$

$$\left(f\circ g\right)^{-1}=\frac{3-x}{4}$$

Note: $g^{-1}(x)=\dfrac{5-x}{2}$.

$$g^{-1}\times f^{-1}=g^{-1}\left(\frac{x+7}{2}\right)$$
$$=\frac{5-\dfrac{x+7}{2}}{2}$$
$$=\frac{10-x-7}{4}$$
$$=\frac{3-x}{4}$$

Exercise 4.6

3. f. $y=\dfrac{3}{9-x^2}=3\left(9-x^2\right)^{-1}$

$$\frac{dy}{dx}=\frac{6x}{\left(9-x^2\right)^2}$$

i. $y=\left(\dfrac{1+\sqrt{x}}{\sqrt[3]{x^2}}\right)^3=\left(1+\sqrt{x}\right)^3\left(x^2\right)^{-1}$

$$\frac{dy}{dx}=3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2\times\left[\frac{x^{\frac{2}{3}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)-\left(1+x^{\frac{1}{2}}\right)\frac{2}{3}x^{-\frac{1}{3}}}{\left(x^{\frac{2}{3}}\right)^2}\right]$$

$$\frac{dy}{dx}=3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2\left[\frac{\dfrac{x^{\frac{2}{3}}}{2x^{\frac{1}{2}}}-\dfrac{2\left(1+x^{\frac{1}{2}}\right)}{3x^{\frac{1}{3}}}}{x^{\frac{4}{3}}}\right]$$

$$=3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2\left[\frac{3x-4x^{\frac{1}{2}}-4x}{6x^{\frac{5}{6}}x^{\frac{4}{3}}}\right]$$

$$=3\left(\frac{1+\sqrt{x}}{3\sqrt{x^2}}\right)^2\left[\frac{-x-4\sqrt{x}}{6x^{\frac{13}{6}}}\right]$$

$$\frac{-3\left(1+\sqrt{x}\right)^2}{x^{\frac{4}{3}}}\times\frac{x^{\frac{1}{2}}\left(4+\sqrt{x}\right)}{6x^{\frac{13}{6}}}=\frac{-\left(1+\sqrt{x}\right)^2\left(4+\sqrt{x}\right)x^{\frac{3}{6}}}{2x^{\frac{21}{6}}}$$

$$=-\left(\frac{1-\sqrt{x}}{3\sqrt{x^2}}\right)^2\left[\frac{x+4\sqrt{x}}{2x^2 6\sqrt{x}}\right]=-\frac{\left(1+\sqrt{x}\right)^2\left(4+\sqrt{x}\right)}{2x^3}$$

4. **f.** $y=\frac{(2x-1)^2}{(x-2)^3}$

$$\frac{dy}{dx}=\frac{2(x-2)^3(2x-1)(2)-3(2x-1)^2(x-2)^2}{(x-2)^6}$$

$$=\frac{(x-2)^2(2x-1)\left[4(x-2)-3(2x-1)\right]}{(x-2)^6}$$

$$=-\frac{(x-2)^2(2x-1)(2x+5)}{(x-2)^6}$$

k.

$$s=\left(4-3t^3\right)^4(1-2t)^6$$

$$\frac{ds}{dt}=4\left(4-3t^3\right)^3\left(-9t^2\right)(1-2t)^6+6\left(4-3t^3\right)^4(1-2t)^5(-2)$$

$$=12\left(4-3t^3\right)^3(1-2t)^5\left[-3t^2(1-2t)-\left(4-3t^3\right)\right]$$

$$=12\left(4-3t^3\right)^3(1-2t)^5\left(9t^3-3t^2+4\right)$$

$$=12\left(4-3t^3\right)^3(1-2t)^5\left(9t^3-3t^2-4\right)$$

l. $h(x)=\frac{\sqrt{1-x^2}}{1-x}$

$$h'(x)=\frac{\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)(1-x)-\sqrt{1-x^2}(-1)}{(1-x)^2}$$

$$=\left(\frac{-x(1-x)}{\sqrt{1-x^2}}+\frac{1-x^2}{\sqrt{1-x^2}}\right)\frac{1}{(1-x)^2}$$

$$=\frac{1-x}{\sqrt{1-x^2}(1-x)^2}=\frac{1}{(1-x)\sqrt{1-x^2}}$$

6. $y=\left(1+x^3\right)^2$ $\qquad$ $y=2x^6$

$\frac{dy}{dx}=2\left(1+x^3\right)\left(3x^2\right)$ $\qquad$ $\frac{dy}{dx}=12x^5$

For the same slope,

$$6x^2\left(1+x^3\right)=12x^5$$

$$6x^2+6x^5=12x^5$$

$$6x^5-6x^2=0$$

$$6x^2\left(x^3-1\right)=0$$

$$x=0 \quad \text{or} \quad x=1.$$

Curves have the same slope at $x=0$ and $x=1$.

8. $y=\left(x^3-7\right)^5$ at $z=2$

$$\frac{dy}{dx}=5\left(x^3-7\right)^4\left(3x^2\right)$$

$$\left.\frac{dy}{dx}\right]_{x=2}=5(1)^4(12)$$

$$=60$$

Slope of the tangent is 60.

Equation of the tangent at (2, 1) is

$$y-1=60(x-2)$$

$$60x-y-119=0.$$

9. **a.** $y=3u^2-5u+2$

$u=x^2-1,\quad x=2$

$u=3$

$$\frac{dy}{du}=6u-5,\quad \frac{du}{dx}=2x$$

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

$$=(6u-5)(2x)$$

$$=(18-5)(4)$$

$$=13(4)$$

$$=52$$

d. $y=u\left(u^2+3\right)^3,\ u=(x+3)^2,\ x=-2$

$$\frac{dy}{du}=\left(u^2+3\right)^3+6u^2\left(u^2+3\right)^2 \quad \frac{du}{dx}=2(x+3)$$

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\left[4^3+6(4)^2\right]\left[2(1)\right]$$

$$=160\times 2$$

$$=320$$

10. $y=f\left(x^2+3x-5\right),\ x=1,\ f'(-1)=2$

$$\frac{dy}{dx}=f'\left(x^2+3x-5\right)\times(2x+3)$$

$$=f'(1+3-5)\times 5$$

$$=2\times 5$$

$$\frac{dy}{dx}=10$$

11. $y = g(h(x)),\ h(x) = \dfrac{x^2}{x+2}$

$$\frac{dy}{dx} = g'(h(x)) \times h'(x)$$

When $x = 3,\ h(3) = \dfrac{9}{5}$ and $g'\left(\dfrac{9}{5}\right) = -2.$

$$h'(x) = \frac{(x+2)(2x) - x^2(1)}{(x+2)^2}$$

$$h'(x) = \frac{x^2 + 4x}{(x+2)^2}$$

$$h'(5) = \frac{9+12}{25} = \frac{21}{25}$$

At $x = 3,\ \dfrac{dy}{dx} = -2 \times \dfrac{21}{25}$

$$= -\frac{42}{25}.$$

12. $h(x) = f(g(x))$, therefore $R'(x) = f'(g(x)) \times g'(x)$

$f(u) = u^2 - 1,\ g(2) = 3,\ g'(2) = -1$

Now, $h'(2) = f'(g(2)) \times g'(2)$

$$= f'(3) \times g'(2).$$

Since $f(u) = u^2 - 1,\ f'(u) = 2u$, and $f'(3) = 6,$

$\therefore h'(2) = 6(-1)$

$$= -6.$$

13. $h(x) = p(x)q(x)r(x)$

a. $h'(x) = p'(x)q(x)r(x) + p(x) \times q'(x) \times r(x)$
$+ p(x) \times q(x) \times r'(x)$

14. $y = (x^2 + x - 2)^2 + 3$

$$\frac{dy}{dx} = 2(x^2 + x - 2)(2x + 1)$$

At the point $(1, 3)$, $x = 1$ and the slope of the tangent will be $2(1 + 1 - 2)(2 + 1) = 0.$

Equation of the tangent at $(1, 3)$ is $y - 3 = 0$.
Solving this equation with the function, we have

$$(x^2 + x - 2)^2 + 3 = 3$$

$$(x+2)^2(x-1)^2 = 0$$

$$x = -2 \quad \text{or} \quad x = 1$$

Since –2 and 1 are both double roots, the line with equation $y - 3 = 0$ will be a tangent at both $x = 1$ and $x = -2$. Therefore, $y - 3 = 0$ is also a tangent at $(-2, 3)$.

15. $y = \dfrac{x^2(1-x)^3}{(1+x)^3}$

$$= x^2\left[\left(\frac{1-x}{1+x}\right)\right]^3$$

$$\frac{dy}{dx} = 2x\left(\frac{1-x}{1+x}\right)^3 + 3x^2\left(\frac{1-x}{1+x}\right)^2\left[-\frac{(1+x)-(1-x)(1)}{(1+x)^2}\right]$$

$$= 2x\left(\frac{1-x}{1+x}\right)^3 + 3x^2\left(\frac{1-x}{1+x}\right)^2 - \frac{2}{(1+x)^2}$$

$$= 2x\left(\frac{1-x}{1+x}\right)^2\left[\frac{1-x}{1+x} - \frac{3x}{(1+x)^2}\right]$$

$$= 2x\left(\frac{1-x}{1+x}\right)^2\left[\frac{1-x^2-3x}{(1+x)^2}\right]$$

$$= -\frac{2x(x^2+3x-1)(1-x)^2}{(1+x)^4}$$

16. If $y = u^n$, prove $\dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$.

For n = 1, y = u and $\dfrac{dy}{dx} = 1u^{1-1}\dfrac{du}{dy} = \dfrac{du}{dx}$, which is true.

Assume the statement is true for $n = k$, i.e., $y = u^k$, then $\dfrac{dy}{dx} = u^{k-1}\dfrac{du}{dx}$.

For $n = k + 1$ show, $\dfrac{dy}{dx} = (k+1)u^k\dfrac{du}{dx}$.

Now, $y = u^{k+1} = u \times u^k$.

$$\frac{dy}{dx}=\frac{du}{dx}\times u^{k}+u\times ku^{k-1}\frac{du}{dx}$$
$$=\frac{du}{dx}\times u^{k}+k\times u^{k}\times\frac{du}{dx}$$
$$=\frac{du}{dx}\times u^{k}\times(k+1)$$
$$=(k+1)u^{k}\frac{du}{dx}$$

Therefore, if the statement is true for $n = k$, it will be true for $n = k + 1$. Since it is true for $n = 1$, it will be true for $n = 2$, therefore true for all $n \in N$.

17. $f(x)=ax+b,\quad g(x)=cx+d$

$$f(g(x))=f(cx+d)$$
$$=a(cx+d)+b$$
$$=acx+ad+b$$
$$g(f(x))=g(ax+b)$$
$$=c(ax+b)+d$$
$$=acx+bc+d$$

Now, $f(g(x))=g(f(x))$.

$$\therefore acx+ad+b=ccx+bc+d$$
$$ad-d=bc-b$$
$$d(a-1)=b(c-1)$$

If $f(g(x))=g(f(x))$, then $d(a-1)=b(c-1)$.

Review Exercise

4. **f.** $y=(x-1)^{\frac{1}{2}}(x+1)$

$$y'=(x-1)^{\frac{1}{2}}+(x+1)-\frac{1}{2}(x-1)^{-\frac{1}{2}}$$
$$=\sqrt{x-1}+\frac{x+1}{2\sqrt{x-1}}$$
$$=\frac{2x-2+x+1}{2\sqrt{x-1}}$$
$$=\frac{3x-1}{2\sqrt{x-1}}$$

h. $y=\sqrt{(x+3)(x-3)}=(x^{2}-9)^{\frac{1}{2}}$

$$y'=\frac{1}{2}(x^{2}-9)^{-\frac{1}{2}}(2x)$$
$$=\frac{x}{\sqrt{x^{2}-9}}$$

5. **c.** $y=\dfrac{(2x-5)^{4}}{(x+1)^{3}}$

$$y'=\frac{(x+1)^{3}\times 4(2x-5)^{3}-3(2x-5)^{4}(x+1)^{2}}{(x+1)^{6}}$$
$$=\frac{(x+1)^{2}(2x-5)^{3}[4x+4-6x+15]}{(x+1)^{6}}$$
$$y'=\frac{(2x-5)^{3}(19-2x)}{(x+1)^{4}}$$

f. $y=\dfrac{(x^{2}-1)^{3}}{(x^{2}+1)^{3}}=\left(\dfrac{x^{2}-1}{x^{2}+1}\right)^{3}$

$$y'=3\left(\frac{x^{2}-1}{x^{2}+1}\right)^{2}\left[\frac{(x^{2}+1)(2x)-2x(x^{2}-1)}{(x^{2}+1)^{2}}\right]$$
$$=\frac{12x(x^{2}-1)^{2}}{(x^{2}+1)^{4}}$$

i. $y=(1-x^{2})^{3}(6+2x)^{-3}$

$$=\left(\frac{1-x^{2}}{6+2x}\right)^{3}$$
$$y'=3\left(\frac{1-x^{2}}{6+2x}\right)^{2}\left[\frac{(6+2x)(-2x)-(1-x^{2})(2)}{(6+2x)^{2}}\right]$$
$$=\frac{3(1-x^{2})^{2}(-12x-4x^{2}-2+2x^{2})}{(6+2x)^{4}}$$
$$=-\frac{3(1-x^{2})^{2}(2x^{2}+12x+2)}{(6+2x)^{4}}$$
$$=-\frac{3(1-x^{2})^{2}(x^{2}+6x+1)}{8(3-x)^{4}}$$

6. **a.** $g(x) = f(x^2)$

$g'(x) = f'(x^2) \times 2x$

b. $h(x) = 2xf(x)$

$h'(x) = 2f(x) + 2xf'(x)$

7. **b.** $y = \dfrac{u+4}{u-4}, \quad u = \dfrac{\sqrt{x}+x}{10},$

$x = 4$

$u = \dfrac{3}{5}$

$$\frac{dy}{du} = \frac{(u-4)-(u+4)}{(u-4)^2} \qquad \frac{du}{dx} = \frac{1}{10}\left(\frac{1}{2}x^{-\frac{1}{2}}+1\right)$$

$$= -\frac{8}{(u-4)^2} \qquad \left.\frac{du}{dx}\right]_{x=4} = \frac{1}{10}\left(\frac{5}{4}\right) = \frac{1}{8}$$

$$\left.\frac{dy}{du}\right]_{u=\frac{3}{5}} = -\frac{8}{\left(\frac{3}{5}-\frac{20}{5}\right)^2} = -\frac{8(25)}{(-17)^2}$$

$$\left.\frac{dy}{dx}\right]_{x=4} = -\frac{8(25)}{17^2} \times \frac{1}{8} = \frac{25}{289}$$

c. $y = f\left(\sqrt{x^2+9}\right), \quad f'(5) = -2, \quad x = 4$

$$\frac{dy}{dx} = f'\left(\sqrt{x^2+9}\right) \times \frac{1}{2}\left(x^2+9\right)^{-\frac{1}{2}}(2x)$$

$$\frac{dy}{dx} = f'(5) \bullet \frac{1}{2} \bullet \frac{1}{5} \bullet 8$$

$$= -2 \bullet \frac{4}{5}$$

$$= -\frac{8}{5}$$

9. $y = -x^3 + 6x^2$

$y' = -3x^2 + 12x$

$$\begin{aligned} -3x^2 + 12x &= -12 & \qquad -3x^2 + 12x &= -15 \\ x^2 - 4x - 4 &= 0 & x^2 - 4x - 5 &= 0 \\ x &= \frac{4 \pm \sqrt{16+32}}{2} & (x-5)(x+1) &= 0 \\ &= \frac{4 \pm 4\sqrt{3}}{2} & x = 5, \quad x &= -1 \\ x &= 2 \pm 2\sqrt{3} \end{aligned}$$

10. **a. i)** $y = (x^3 - x)^2$

$y' = 2(x^3 - x)(3x^2 - 1)$

Horizontal tangent,

$2x(x^2-1)(3x^2-1) = 0$

$x = 0, \quad x = \pm 1, \quad x = \pm\dfrac{\sqrt{3}}{3}.$

11. **b.** $y = (3x^{-2} - 2x^3)^5$ at $(1,1)$

$y' = 5(3x^{-2} - 2x^3)^4(-6x^{-3} - 6x^2)$

$A + x = 1$

$y' = 5(1)^4(-6-6)$

$= -60$

Equation of the tangent at (1, 1) is

$$y - 1 = -60(x-1)$$

$60x + y - 61 = 0.$

12. $y = 3x^2 - 7x + 5$

$\dfrac{dy}{dx} = 6x - 7$

Slope of $x + 5y - 10 = 0$ is $-\dfrac{1}{5}$.

Since perpendicular, $6x-7=5$

$$\begin{aligned} x&=2 \\ y&=3(4)-14+5 \\ &=3. \end{aligned}$$

Equation of the tangent at (2, 3) is

$$y-3=5(x-2)$$

$$5x-y-7=0.$$

13. $y=8x+b$ is tangent to $y=2x^2$

$$\frac{dy}{dx}=4x$$

Slope of the tangent is 8, therefore $4x=8,\ x=2.$

Point of tangency is (2, 8).

Therefore, $8=16+b,\ b=-8$.

Or $8x+b=2x^2$

$$2x^2-8x-b=0$$

$$x=\frac{8\pm\sqrt{64+8b}}{2(2)}.$$

For tangents, the roots are equal, therefore $64+8b=0,\ \ b=-8$.

Point of tangency is (2, 8), $b=-8$.

15. a. $f(x)=2x^{\frac{5}{3}}-5x^{\frac{2}{3}}$

$$\begin{aligned} f'(x)&=2\times\frac{5}{3}x^{\frac{2}{3}}-5\times\frac{2}{3}x^{-\frac{1}{3}} \\ &=\frac{10}{3}x^{\frac{2}{3}}-\frac{10}{3x^{\frac{1}{3}}} \end{aligned}$$

$$f(x)=0\ \ \therefore x^{\frac{2}{3}}[2x-5]=0$$

$$x=0 \quad \text{or} \quad x=\frac{5}{2}$$

$y=f(x)$ crosses the x-axis at $x=\frac{5}{2}$, and

$$f'(x)=\frac{10}{3}\left(\frac{x-1}{x^{\frac{1}{3}}}\right)$$

$$\begin{aligned} f'\left(\frac{5}{2}\right)&=\frac{10}{3}\times\frac{3}{2}\times\frac{1}{\left(\frac{5}{2}\right)^{\frac{1}{3}}} \\ &=5\times\frac{3\sqrt{2}}{3\sqrt{5}}=5^{\frac{2}{3}}\times2^{\frac{1}{3}} \\ &=(25\times2)^{\frac{1}{3}} \\ &=3\sqrt{50} \end{aligned}$$

b. To find a, let $f(x)=0.$

$$\begin{aligned} \frac{10}{3}x^{\frac{2}{3}}-\frac{10}{3x^{\frac{1}{3}}}&=0 \\ 30x&=30 \\ x&=1 \end{aligned}$$

Therefore $a=1.$

18. $C(x)=\frac{1}{3}x^3+40x+700$

a. $C'(x)=x^2+40$

b. $C'(x)=76$

$$\begin{aligned} \therefore x^2+40&=76 \\ x^2&=36 \\ x&=6 \end{aligned}$$

Production level is 6 gloves/week.

19. $R(x)=750x-\frac{x^2}{6}-\frac{2}{3}x^3$

a. Marginal Revenue

$$R'(x)=750-\frac{x}{3}-2x^2$$

b. $R'(10)=750-\frac{10}{3}-2(100)$

$$=\$546.67$$

20. $D(p) = \frac{20}{\sqrt{p-1}}, \quad p > 1$

$$D'(p) = 20\left(-\frac{1}{2}\right)(p-1)^{-\frac{3}{2}}$$
$$= -\frac{10}{(p-1)^{\frac{3}{2}}}$$
$$D'(5) = -\frac{10}{\sqrt{4}^{3}} = -\frac{10}{8}$$
$$= -\frac{5}{4}$$

Slope of demand curve at (5, 10) is $-\frac{5}{4}$.

Chapter 4 Test

2. *f* is the graph on the right and below the *x*-axis (it's a cubric). f' is the other graph (it is quadratic).

3. $f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{x+h-(x+h)^2-(x-x^2)}{h}$$
$$= \lim_{h\to 0} \frac{x+h-(x^2+2hx+h^2)-x+x^2}{h}$$
$$= \lim_{h\to 0} \frac{h-2hx-h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(1-2x-h)}{h}$$
$$= \lim_{h\to 0}(1-2x-h)$$
$$= 1-2x$$

Therefore, $\frac{d}{dx}(x-x^2) = 1-2x$.

4. **a.** $y = \frac{1}{3}x^3 - 3x^{-5} + 4\pi$

$$\frac{dy}{dx} = x^2 + 15x^{-6}$$

b. $y = 6(2x-9)^5$

$$\frac{dy}{dx} = 30(2x-9)^4(2)$$
$$= 60(2x-9)^4$$

c. $y = \frac{2}{\sqrt{x}} + \frac{x}{\sqrt{3}} + 6\sqrt[3]{x}$

$$= 2x^{-\frac{1}{2}} + \frac{1}{\sqrt{3}}x + 6x^{\frac{1}{3}}$$
$$\frac{dy}{dx} = -x^{-\frac{3}{2}} + \frac{1}{\sqrt{3}} + 2x^{-\frac{2}{3}}$$

d. $y = \left(\frac{x^2+6}{3x+4}\right)^5$

$$\frac{dy}{dx} = 5\left(\frac{x^2+6}{3x+4}\right)^4 \frac{2x(3x+4)-(x^2+6)3}{(3x+4)^2}$$
$$= \frac{5(x^2+6)^4(3x^2+8x-18)}{(3x+4)^6}$$

e. $y = x^2\sqrt[3]{6x^2-7}$

$$\frac{dy}{dx} = 2x(6x^2-7)^{\frac{1}{3}} + x^2\frac{1}{3}(6x^2-7)^{-\frac{2}{3}}(12x)$$
$$= 2x(6x^2-7)^{-\frac{2}{3}}((6x^2-7)+2x^2)$$
$$= 2x(6x^2-7)^{-\frac{2}{3}}(8x^2-7)$$

f. $$y = \frac{4x^5 - 5x^4 + 6x - 2}{x^4}$$
$$= 4x - 5 + 6x^{-3} - 2x^{-4}$$
$$\frac{dy}{dx} = 4 - 18x^{-4} + 8x^{-5}$$
$$= \frac{4x^5 - 18x + 8}{x^5}$$

5. $$y = \left(x^2 + 3x - 2\right)(7 - 3x)$$
$$\frac{dy}{dx} = (2x + 3)(7 - 3x) + \left(x^2 + 3x - 2\right)(-3)$$

At $(1, 8)$,

$$\frac{dy}{dx} = (5)(4) + (2)(-3)$$
$$= 14.$$

The slope of the tangent to $y = \left(x^2 + 3x - 2\right)(7 - 3x)$ at $(1, 8)$ is 14.

6. $$y = 3u^2 + 2u$$
$$\frac{dy}{du} = 6u + 2$$
$$u = \sqrt{x^2 + 5}$$
$$\frac{du}{dy} = \frac{1}{2}\left(x^2 + 5\right)^{-\frac{1}{2}} 2x$$
$$\frac{dy}{dx} = (6u + 2)\left(\frac{x}{\sqrt{x^2 + 5}}\right)$$

At $x = -2, u = 3.$

$$\frac{dy}{dx} = (20)\left(-\frac{2}{3}\right)$$
$$= -\frac{40}{3}$$

7. $$y = \left(3x^{-2} - 2x^3\right)^5$$
$$\frac{dy}{dx} = 5\left(3x^{-2} - 2x^3\right)^4\left(-6x^{-3} - 6x^2\right)$$

At $(1, 1)$,

$$\frac{dy}{dx} = 5(1)^4(-6 - 6)$$
$$= -60.$$

Equation of tangent line at $(1, 1)$ is

$$\frac{y - 1}{x - 1} = -60$$
$$y - 1 = -60x + 60$$
$$60x + y - 61 = 0.$$

8. $$P(t) = \left(t^{\frac{1}{4}} + 3\right)^3$$
$$P'(t) = 3\left(t^{\frac{1}{4}} + 3\right)^2\left(\frac{1}{4}t^{-\frac{3}{4}}\right)$$
$$P'(16) = 3\left(16^{\frac{1}{4}} + 3\right)^2\left(\frac{1}{4} \times 16^{-\frac{3}{4}}\right)$$
$$= 3(2 + 3)^2\left(\frac{1}{4} \times \frac{1}{8}\right)$$
$$= \frac{75}{32}$$

The amount of pollution is increasing at a rate of $\frac{75}{32}$ p.p.m./year.

9. $$y = x^4$$

$$\frac{dy}{dx} = 4x^3$$

$$-\frac{1}{16} = 4x^3$$

Normal line has a slope of 16. Therefore, $\frac{dy}{dx} = -\frac{1}{16}$.

$$x^3 = -\frac{1}{64}$$

$$x = -\frac{1}{4}$$

$$y = -\frac{1}{256}$$

Therefore, $y = x^4$ has a normal line with a slope of 16 at $\left(-\frac{1}{4}, \frac{1}{256}\right)$.

10. $$y = x^3 - x^2 - x + 1$$

$$\frac{dy}{dx} = 3x^2 - 2x - 1$$

For a horizontal tangent line, $\frac{dy}{dx} = 0$.

$$3x^2 - 2x - 1 = 0$$

$$(3x+1)(x-1) = 0$$

$x = -\frac{1}{3}$ or $x = 1$

$y = 1 - 1 - 1 + 1$

$= 0$

$$y = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 1$$

$$= \frac{-1 - 3 + 9 + 27}{27}$$

$$= \frac{32}{27}$$

The required points are $\left(-\frac{1}{3}, \frac{32}{27}\right), (1, 0)$.

11. $$y = x^2 + ax + b$$

$$\frac{dy}{dx} = 2x + a$$

$$y = x^3$$

$$\frac{dy}{dx} = 3x^2$$

Since the parabola and cubic function are tangent at (1, 1), then $2x + a = 3x^2$.

At (1, 1) $2(1) + a = 3(1)^2$

$a = 1.$

Since (1,1) is on the graph of $y = x^2 + x + b$,

$$1 = 1^2 + 1 + b$$

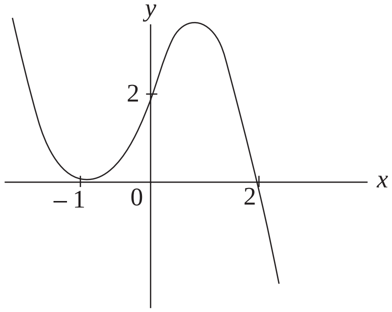
$$b = -1.$$

The required values are 1 and –1 for *a* and *b* respectively.

Cumulative Review Solutions

Chapters 1–4

2. The given function is a polynomial function of degree three. The x-intercepts are –1 and 2. Since –1 is a double root, the graph is tangent to the x-axis at $x = –1$. The y-intercept is 2. Since the coefficient of the x^3 term is negative, the graph goes from the second quadrant to the fourth quadrant.



4. b.

$$\begin{array}{r} 3x^2 - 13x + 50 \\ x+3\overline{)3x^3 - 4x^2 + 11x - 2} \\ \underline{3x^3 + 9x^2} \qquad\qquad\quad \\ -13x^2 + 11x \qquad\; \\ \underline{-13x^2 - 39x} \qquad\; \\ 50x - 2 \\ \underline{50x + 150} \\ -152 \end{array}$$

Thus, $(3x^3 – 4x^2 + 11x – 2) \div (x + 3)$

$= 3x^2 – 13x + 50 – \dfrac{152}{x + 3}$

7. Let $f(x) = x^3 + kx^2 – 4x + 12$. Since $x – 3$ is a factor of $f(x)$, $f(3) = 0$. Thus, $27 + 9k – 12 + 12 = 0$, and $k = –3$.

8. Let $f(x) = x^4 – 2x^3 + 5x^2 – 6x – 8$. To determine whether or not $x – 2$ is a factor of $f(x)$, we evaluate $f(2)$.
$f(2) = 16 – 16 + 20 – 12 – 8 = 0$
Since $f(2) = 0$, $x – 2$ is a factor of $f(x)$

9. We use the Factor Theorem to determine other factors of the given polynomial. We know that for $x – p$ to be a factor, p must be a divisor of 6.
Let $f(x) = x^3 – 2x^2 – 5x + 6$.
Since $f(1) = 1 – 2 – 5 + 6 = 0$, $x – 1$ is a factor of $f(x)$.
Since $f(3) = 27 – 18 – 15 + 6 = 0$, $x – 3$ is a factor of $f(x)$.
Thus, $x^3 – 2x^2 – 5x + 6 = (x + 2), (x – 1)$, and $(x – 3)$.

10. d. Let $f(x) = 5x^3 + 8x^2 + 21x – 10$.

Since $f\left(\dfrac{2}{5}\right) = 0$, $5x – 2$ is a factor.

By long division,
$5x^3 + 8x^2 + 21x – 10$
$= (5x – 2)(x^2 + 2x + 5)$.
The expression $x^2 + 2x + 5$ does not factor in $x \in R$.

11. b.
$$\begin{aligned} x^4 + 5x^2 – 36 &= 0 \\ (x^2 – 4)(x^2 + 9) &= 0 \\ (x – 2)(x + 2)(x^2 + 9) &= 0 \end{aligned}$$
The roots are 2, –2, $3i$, and $–3i$.

d. Let $f(x) = 2x^3 – x^2 – 2x + 1$.

Since $f(1) = f(–1) = f\left(\dfrac{1}{2}\right) = 0$,

$x – 1$, $x + 1$, and $2x – 1$ are factors of $f(x)$.

Thus, $2x^3 – x^2 – 2x + 1 = 0$
$(x – 1)(x + 1)(2x – 1) = 0$.

The roots are 1, –1, and $\dfrac{1}{2}$.

f. Let $f(x) = 3x^3 – 4x^2 + 4x – 1$.

Since $f\left(\dfrac{1}{3}\right) = 0$, $3x – 1$ is a factor of $f(x)$.

By long division or comparing coefficients, the other factor is $x^2 – x + 1$.
Thus, $3x^3 – 4x^2 + 4x – 1 = 0$.
$(3x – 1)(x^2 – x + 1) = 0$

$$x = \frac{1}{3} \text{ or } x = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

The roots are $\dfrac{1}{3}, \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$, and $\dfrac{1}{2} – \dfrac{\sqrt{3}}{2}i$.

13. Let the roots of $x^2 - 9x + 2 = 0$ be r_1 and r_2.

We have $r_1 + r_2 = 9$ and $r_1r_2 = 2$.

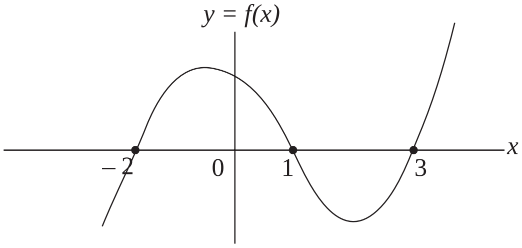
We need to find the quadratic equation whose roots are r_1^2 and r_2^2.

$$\begin{aligned}\text{Since } r_1^2 + r_2^2 &= (r_1 + r_2)^2 - 2r_1r_2 \\ &= 81 - 4 \\ &= 77,\end{aligned}$$

and $r_1^2r_2^2 = (r_1r_2)^2 = 4$, the required equation is $x^2 - 77x + 4 = 0$.

14. **b.** Let $f(x) = (x + 2)(x - 1)(x - 3)$

The graph of $y = f(x)$ is a cubic polynomial that goes from the third quadrant to the first quadrant with x-intercepts –2, 1, and 3.



The solution of the given inequality is the set of values of x for which the graph is above or on the x-axis.

The solution is $-2 \leq x \leq 1$ or $x \geq 3$.

15. **a.** $|x - 2| < 5$

$-5 < x - 2 < 5$

$-3 < x < 7$

$|2x - 3| \leq 5$

b. $-5 \leq 2x - 3 \leq 5$

$-2 \leq 2x \leq 8$

$-1 \leq x \leq 4$

c. $|3x + 1| > 16$

$3x + 1 > 16$ or $3x + 1 < -16$

$3x > 15$ or $3x < -17$

$x > 5$ or $x < -\dfrac{17}{3}$

16. **a.** The average velocity from $t = 1$ to $t = 4$ is

$$\frac{s(4) - s(1)}{4 - 1} = \frac{(32 + 12 + 1) - (2 + 3 + 1)}{3}$$
$$= 13 \text{ m/s.}$$

b. The velocity at any time t is given by

$v(t) = s'(t) = 4t + 3$.

At $t = 3$, $v(3) = 4(3) + 3 = 15$ m/s.

17. $V = \pi r^2h = \pi r^2(r + 3) = 200\pi$

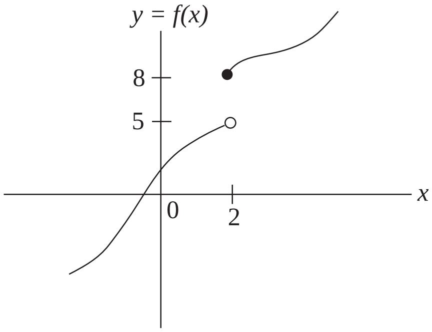
Thus, $r^3 + 3r^2 - 200 = 0$.

Let $f(x) = r^3 + 3r^2 - 200$.

Since $f(5) = 0$, $r - 5$ is a factor of $f(r)$.

By long division or comparing coefficients, $r^3 + 3r^2 - 200 = (r - 5)(r^2 + 8r + 40)$. The equation becomes $(r - 5)(r^2 + 8r + 40) = 0$. The quadratic factor does not have real roots. The radius of the given cylinder is 5 cm.

19.

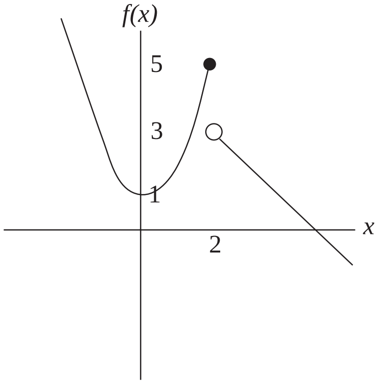


20. Since each of the components of the function $f(x)$ is continuous, the only possible point of discontinuity occurs at $x = 2$.

We have $f(2) = 2(2) + 1 = 5$.

Also, $\lim\limits_{x\to2^-} f(x) = \lim\limits_{x\to2^-} (x^2 + 1) = 5$ and $\lim\limits_{x\to2^+} f(x) = \lim\limits_{x\to2^+} (-x + 5) = 3$.

Since $\lim\limits_{x\to2} f(x)$ does not exist, f is discontinuous at $x = 2$.



22. $\lim\limits_{h\to 0} \dfrac{(4+h)^3 - 64}{h} = \lim\limits_{h\to 0} \dfrac{(4+h)^3 - 4^3}{h}$

The limit is of the form $\lim\limits_{h\to 0} \dfrac{(a+h)^3 - a^3}{h}$ where $a = 4$.

We also know the slope of the tangent line to the graph of $y = f(x)$ at $x = 4$ is defined to be $\lim\limits_{h\to 0} \dfrac{f(4+h) - f(4)}{h}$.

By comparing expressions, we conclude that $f(x) = x^3$.

23. e.

$$\lim_{x\to 2} \frac{x-2}{x^3-8} = \lim_{x\to 2} \frac{x-2}{(x-2)(x^2+2x+4)}$$

$$= \lim_{x\to 2} \frac{1}{x^2+2x+4}$$

$$= \frac{1}{12}$$

f.

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2}$$

$$= \frac{1}{4}$$

24. b.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{x-(x+h)}{(x+h)(x)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+h)(x)}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h)(x)}$$

$$= -\frac{1}{x^2}$$

25. e.

$$\frac{dy}{dx} = \frac{5(4x^2+1)^4(8x)(3x-2^3) - (4x^2+1)^5(3)(3x-2)^2(3)}{(3x-2)^6}$$

$$= \frac{40x(4x^2+1)^4(3x-2) - 9(4x^2+1)^5}{(3x-2)^4}$$

$$= \frac{(4x^2+1)^4(120x^2-80x-36x^2-9)}{(3x-2)^4}$$

$$= \frac{(4x^2+1)^4(84x^2-80x-9)}{(3x-2)^4}$$

f.

$$\frac{dy}{dx} = 5[x^2+(2x+1)^3]^4[2x+3(2x+1)^2(2)]$$

$$= 10[x^2+(2x+1)^3]^4[12x^2+13x+3]$$

27. The slope of the line $6x + 3y - 2 = 0$ *is* -2. We need to find the point on the parabola at which the slope of the tangent line is $\frac{1}{2}$. The slope of the tangent line at any point on the parabola is given by $\dfrac{dy}{dx} = 4x - 4$. To find the point at which the slope is $\frac{1}{2}$, we solve $4x - 4 = \frac{1}{2}$ and get $x = \frac{9}{4}$. The point of contact is $\left(\frac{9}{4}, \frac{-47}{16}\right)$. An equation of the required tangent line is $y + \frac{47}{16} = \frac{1}{2}(x - \frac{9}{4})$ or $8x - 16y - 65 = 0$.

29. To find the point(s) of intersection of the line and the parabola, we solve

$x^2 + 9x + 9 = 3x$

$x^2 + 6x + 9 = 0$

$(x + 3)^2 = 0$

$x = -3.$

Since we have a double root at $x = -3$, the line $y = 3x$ is tangent to the parabola $y = x^2 + 9x + 9$. Hence, the slope of the tangent at the point of intersection is 3.

30. **a.** $p'(t) = 4t + 6$

b. The rate of change of the population in 1990 was $p'(10) = 46$ people per year.

c. We want the value of t when $4t + 6 = 94$

i.e., $4t = 88$

$t = 22.$

The rate of change of population is 94 people per year in the year 2002.

Chapter 5 • Applications of Derivatives

Review of Prerequisite Skills

5. **a.**
$$\begin{aligned} 3(x-2)+2(x-1)-6&=0\\ 3x-6+2x-2-6&=0\\ 5x&=14\\ x&=\frac{14}{5} \end{aligned}$$

e.
$$\begin{aligned} \frac{6}{t}+\frac{t}{2}&=4\\ 12+t^2&=8t\\ t^2-8t+12&=0\\ (t-6)(t-2)&=0\\ \therefore\ t=2 \quad \text{or} \quad t&=6 \end{aligned}$$

f.
$$\begin{aligned} x^3+2x^2-3x&=0\\ x(x^2+2x-3)&=0\\ x(x+3)(x-1)&=0\\ x=0 \quad \text{or} \quad x=-3 \quad \text{or} \quad x&=1 \end{aligned}$$

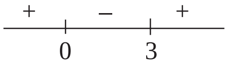
g.
$$\begin{aligned} x^3-8x^2+16x&=0\\ x(x^2-8x+16)&=0\\ x(x-4)^2&=0\\ x=0 \quad \text{or} \quad x&=4 \end{aligned}$$

h.
$$\begin{aligned} 4t^3+12t^2-t-3&=0\\ 4t^2(t+3)-1(t+3)&=0\\ (t+3)(4t^2-1)&=0\\ (t+3)(2t-1)(2t+1)&=0 \end{aligned}$$
$$t=-3 \quad \text{or} \quad t=\frac{1}{2} \quad \text{or} \quad t=-\frac{1}{2}$$

i.
$$\begin{aligned} 4t^4-13t^2+9&=0\\ (4t^2-9)(t^2-1)&=0 \end{aligned}$$
$$t=\pm\frac{9}{4} \quad \text{or} \quad t=\pm 1$$

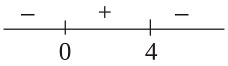
6. **a.**
$$\begin{aligned} 3x-2&>7\\ 3x&>9\\ x&>3 \end{aligned}$$

b. $x(x-3)>0$



$x<0 \quad \text{or} \quad x>3$

c. $-x^2+4x>0$



$x(x-4)<0$

$0<x<4$

Exercise 5.1

2. **d.**
$$\begin{aligned} 3xy^2+y^3&=8\\ 3y^2+3x2y\frac{dy}{dx}+3y^2\frac{dy}{dx}&=0\\ \frac{dy}{dx}(2xy+y^2)&=-y^2\\ \frac{dy}{dx}&=\frac{-y^2}{2xy+y^2} \end{aligned}$$

f.
$$\begin{aligned} 9x^2-16y^2&=-144\\ 18x-32y\frac{dy}{dx}&=0\\ \frac{dy}{dx}&=\frac{9x}{16y} \end{aligned}$$

g.
$$\begin{aligned} -\frac{x^2}{16}+\frac{3}{13}y^2&=1\\ \frac{2x}{16}+\frac{6}{13}y\frac{dy}{dx}&=0\\ 26x+96y\frac{dy}{dx}&=0\\ \frac{dy}{dx}&=-\frac{13x}{48y} \end{aligned}$$

h.
$$\begin{aligned} 3x^2+4xy^3&=9\\ 6x+4y^3+4x3y^2\frac{dy}{dx}&=0\\ 6xy^2\frac{dy}{dx}&=-3x-2y^3\\ \frac{dy}{dx}&=\frac{-3x-2y^3}{6xy^2} \end{aligned}$$

j. $x^3 + y^3 = 6xy$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + \frac{dy}{dx}(6x)$$

$$(3y^2 - 6x)\frac{dy}{dx} = -3x^2 + 6y$$

$$\frac{dy}{dx} = \frac{-3x^2 + 6y}{3y^2 - 6x}$$

$$= \frac{-x^2 + 2y}{y^2 - 2x}$$

k. $x^3y^3 = 144$

$$3x^2y^3 + 3y^2\frac{dy}{dx}x^3 = 0$$

$$\frac{dy}{dx} = -\frac{x^2y^3}{x^3y^2}$$

$$= -\frac{y}{x}$$

m. $xy^3 - x^3y = 2$

$$y^3 + 3y^2\frac{dy}{dx}x - \left[3x^2y + \frac{dy}{dx}x^3\right] = 0$$

$$(3y^2 - x^3)\frac{dy}{dx} = 3x^2y - y^3$$

$$\frac{dy}{dx} = \frac{3x^2y - y^3}{3y^2 - x^3}$$

n. $\sqrt{x} + \sqrt{y} = 5$

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5$$

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}}$$

$$= -\frac{\sqrt{y}}{\sqrt{x}}$$

o. $(x + y)^2 = x^2 + y^2$

$$2(x + y)\left[1 + \frac{dy}{dx}\right] = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}[x + y - y] = x - x - y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

3. a. $x^2 + y^2 = 13$

$$2x + 2y\frac{dy}{dx} = 0$$

At (2, –3),

$$2(2) + 2(-3)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2}{3}.$$

The equation of the tangent at (2, –3) is

$$y = \frac{2}{3}x + b.$$

At (2, –3),

$$-3 = \frac{2}{3}(2) + b$$

$$-9 = 4 + 3b$$

$$-13 = 3b$$

$$-\frac{13}{3} = b.$$

Therefore, the equation of the tangent to $x^2 + y^2 = 13$ is $y = \frac{2}{3} \times -\frac{13}{3}$.

c. $\frac{x^2}{25} - \frac{y^2}{36} = -1$

$$\frac{2x}{25} - \frac{2y}{36}\frac{dy}{dx} = 0$$

$$36x - 25y\frac{dy}{dx} = 0$$

At $(5\sqrt{3}, -12)$,

$$180\sqrt{3} + 300\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3\sqrt{3}}{5}.$$

The equation of the tangent is $y = mx + b$.

At $(5\sqrt{3}, -12)$ and with $m = -\frac{3\sqrt{3}}{5}$,

$$-12 = -\frac{3\sqrt{3}}{5}\left(5\sqrt{3}\right) + b$$

$$-12 = -9 + b$$
$$-3 = b$$

Therefore, the equation of the tangent is

$$y = -\frac{3\sqrt{3}}{5}x - 3.$$

4. $x + y^2 = 1$

The line $x + 2y = 0$ has slope of $-\frac{1}{2}$.

$$1 + 2y\frac{dy}{dx} = 0$$

Since the tangent line is parallel to $x + 2y = 0$,

then $\frac{dy}{dx} = -\frac{1}{2}$.

$$\therefore\ 1 + 2y\left(-\frac{1}{2}\right) = 0$$
$$1 - y = 0$$
$$y = 1$$

Substituting,

$$x + 1 = 1$$
$$x = 0$$

Therefore, the tangent line to the curve $x + y^2 = 1$ is parallel to the line $x + 2y = 0$ at (0, 1).

5. **a.** $5x^2 - 6xy + 5y^2 = 16$

$$10x - \left(6y + \frac{dy}{dx}(6x)\right) + 10y\frac{dy}{dx} = 0 \ (1)$$

At (1, –1),

$$10 - \left(-6 + 6\frac{dy}{dx}\right) - 10\frac{dy}{dx} = 0$$

$$16 - 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 1.$$

b. When the tangent line is horizontal, $\frac{dy}{dx} = 0$.

Substituting,

$$10x - (6y + 0) + 0 = 0.$$

$y = \frac{5}{3}x$ at the point (x_1, y_1) of tangency:

substitute $y_1 = \frac{5}{3}x_1$ into $5x_1^2 - 6x_1y_1 + 5y_1^2 = 16$.

$$5x_1^2 - 6x_1\left(\frac{5}{3}x_1\right) + 5\left(\frac{25}{9}x_1^2\right) = 16$$

$$45x_1^2 - 90x_1^2 + 125x_1^2 = 144$$

$$80x_1^2 = 144$$

$$5x_1^2 = 9$$

$$x_1 = \frac{3}{\sqrt{5}} \quad \text{or} \quad x_1 = -\frac{3}{\sqrt{5}}$$

$$y_1 = \frac{5}{\sqrt{5}} \quad \text{or} \quad y_1 = \sqrt{5}$$

$$y_1 = -\frac{5}{\sqrt{5}} \quad \text{or} \quad y_1 = -\sqrt{5}$$

Therefore, the required points are $\left(\frac{3}{\sqrt{5}}, \sqrt{5}\right)$ and $\left(-\frac{3}{\sqrt{5}}, -\sqrt{5}\right)$.

7. $x^3 + y^3 - 3xy = 17$

$$3x^2 + 3y^2\frac{dy}{dx} - \left[3y + \frac{dy}{dx}(3x)\right] = 0$$

At (2, 3),

$$12 + 27\frac{dy}{dx} - 9 - 6\frac{dy}{dx} = 0$$

$$21\frac{dy}{dx} = -3.$$

The slope of the tangent is $\frac{dy}{dx} = -\frac{1}{7}$.

Therefore, the slope of the normal at (2, 3) is 7.

The equation of the normal at (2, 3) is $\frac{y - 3}{x - 2} = 7$

$y - 3 = 7x - 14$ or $7x - y - 11 = 0$

9. $4x^2y - 3y = x^3$

a. $8xy + \frac{dy}{dx}(4x^2) - 3\frac{dy}{dx} = 3x^2$

$$\frac{dy}{dx}(4x^2 - 3) = 3x^2 - 8xy$$

$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3} \qquad (1)$$

b. $y(4x^2 - 3) = x^3$

$$y = \frac{x^3}{4x^2 - 3}$$

$$\frac{dy}{dx} = \frac{3x^2(4x^2 - 3) - 8x(x^3)}{(4x^2 - 3)^2}$$

$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$

$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2} \qquad (2)$$

We must show that (1) is equivalent to (2).

From (1): $\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$

and substituting, $y = \frac{x^3}{4x^2 - 3}$

$$\frac{dy}{dx} = \frac{3x^2 - 8x\left(\frac{x^3}{4x^2 - 3}\right)}{4x^2 - 3}$$

$$= \frac{3x^2(4x^2 - 3) - 8x^4}{(4x^2 - 3)^2}$$

$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2 - 3)^2}$$

$$= \frac{4x^4 - 9x^2}{(4x^2 - 3)^2} = (2), \text{ as required.}$$

11. $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 10, x \neq\ y \neq\ 0, \frac{dy}{dx} = \frac{y}{x}$

$$\frac{1}{2}\left(\frac{x}{y}\right)^{-\frac{1}{2}} \frac{1y - \frac{dy}{dx}\,x}{y^2} + \frac{1}{2}\left(\frac{y}{x}\right)^{-\frac{1}{2}} \frac{\frac{dy}{dx}\,x - y}{x^2} = 0$$

$$\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \frac{1y - \frac{dy}{dx}\,x}{y^2} + \frac{x^{\frac{1}{2}}}{y^{-\frac{1}{2}}} \frac{\frac{dy}{dx}\,x - y}{x^2} = 0$$

Multiply by x^2y^2:

$$x^{\frac{3}{2}}y^{\frac{1}{2}}\left(y - x\,\frac{dy}{dx}\right) + 2^{\frac{1}{2}}\,y^{\frac{3}{2}}\left(\frac{dy}{dx}\,x - y\right) = 0$$

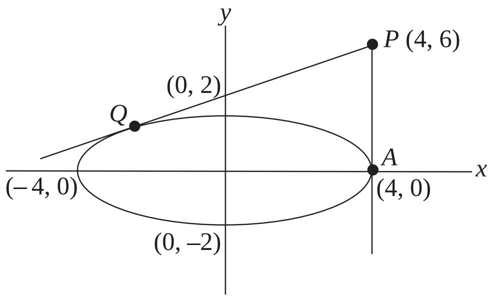
$$x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}}\frac{dy}{dx} + x^{\frac{3}{2}}y^{\frac{3}{2}}\frac{dy}{dx} - x^{\frac{1}{2}}y^{\frac{5}{2}} = 0$$

$$\frac{dy}{dx}\left(x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}}\right) = x^{\frac{1}{2}}y^{\frac{5}{2}} - x^{\frac{3}{2}}y^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}}y^{\frac{3}{2}}\,(y - x)}{x^{\frac{3}{2}}y^{\frac{1}{2}}\,(y - x)}$$

$$\frac{dy}{dx} = \frac{y}{x}, \text{ as required.}$$

12.



Let Q have coordinates

$(q, f(q)) = \left(q, \frac{\sqrt{16 - q^2}}{2}\right), q < 0.$

For $x^2 + 4y^2 = 16$

$$2x + 3y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

At Q, $\frac{dy}{dx} = -\frac{q}{2\sqrt{16 - q^2}}$

The line through P has equation $\frac{y-6}{x-4} = m$.

Since PQ is the slope of the tangent line to $x^2 + 4y^2 = 16$, we conclude:

$m = \frac{dy}{dx}$ at point Q.

$$\therefore \frac{\frac{\sqrt{16-q^2}}{2} - 6}{q-4} = -\frac{q}{2\sqrt{16-q^2}}$$

$$\frac{\sqrt{16-q^2} - 12}{2(q-4)} = -\frac{9}{2\sqrt{16-q^2}}$$

$$16 - q^2 - 12\sqrt{16-q^2} = -q(q-4)$$

$$16 - q^2 - 12\sqrt{16-q^2} = -q^2 + 4q$$

$$4 - q = 3\sqrt{16-q^2}$$

$$\begin{aligned} 16 - 8q + q^2 &= 9(16 - q^2) \\ 16 - 8q + q^2 &= 144 - 9q^2 \\ 10q^2 - 8q - 128 &= 0 \\ 5q^2 - 4q - 64 &= 0 \\ (5q + 16)(q - 4) &= 0 \end{aligned}$$

$q = -\frac{16}{5}$ or $q = 4$ (as expected; see graph)

$f(q) = \frac{6}{5}$ or $f(q) = 0$

$$\frac{dy}{dx} = \frac{\frac{16}{5}}{4\left(\frac{6}{5}\right)} \quad \text{or} \quad f(q) = 0$$

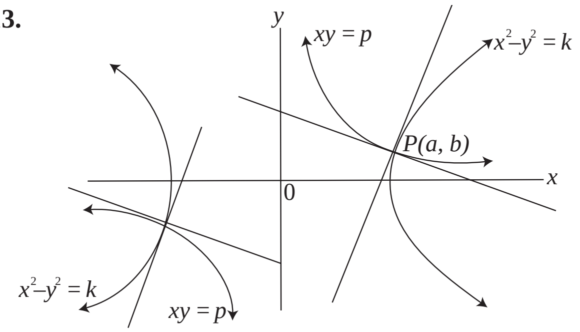
$$= \frac{2}{3}$$

Equation of the tangent at Q is

$\frac{y-6}{x-4} = \frac{2}{3}$ or $2x - 3y + 10 = 0$

or equation of tangent at A is $x = 4$.

13.



Let $P(a, b)$ be the point of intersection where $a \neq 0$ and $b \neq 0$.

For $x^2 - y^2 = k$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

At $P(a, b)$,

$$\frac{dy}{dx} = \frac{a}{b}.$$

For $xy = P$,

$$1 \bullet y + \frac{dy}{dx}x = P$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

At $P(a, b)$,

$$\frac{dy}{dx} = -\frac{b}{a}.$$

At point $P(a, b)$, the slope of the tangent line of $xy = P$ is the negative reciprocal of the slope of the tangent line of $x^2 - y^2 = k$. Therefore, the tangent lines intersect at right angles, and thus, the two curves intersect orthogonally for all values of the constants k and P.

14. $\sqrt{x} + \sqrt{y} = \sqrt{k}$

diff wrt x

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Let $P(a, b)$ be the point of tangency.

$$\therefore\ \frac{dy}{dx} -\frac{\sqrt{b}}{\sqrt{a}}$$

Equation of tangent line l at P is

$$\frac{y-b}{x-a} = -\frac{\sqrt{b}}{\sqrt{a}}.$$

x – intercept is found when $y = 0$.

$$\therefore\ \frac{-b}{x-a} = -\frac{\sqrt{b}}{\sqrt{a}}$$

$$-b\sqrt{a} = -\sqrt{b}\,x + a\sqrt{b}$$

$$x = \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$$

Therefore, the x–intercept is $\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$.

For the y–intercept, let $x = 0$,

$$\frac{y-b}{-a} = -\frac{\sqrt{b}}{\sqrt{a}}.$$

y – intercept is $\frac{a\sqrt{b}}{\sqrt{a}} + b$.

The sum of the intercepts is

$$\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}} + \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{a}}$$

$$= \frac{a^{\frac{3}{2}}b^{\frac{1}{2}} + 2ab + b^{\frac{3}{2}}a^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$

$$= \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}(a + 2\sqrt{a}\sqrt{b} + b)}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$

$$= a + 2\sqrt{a}\sqrt{b} + b$$

$$= \left(a^{\frac{1}{2}} + b^{\frac{1}{2}}\right)^2$$

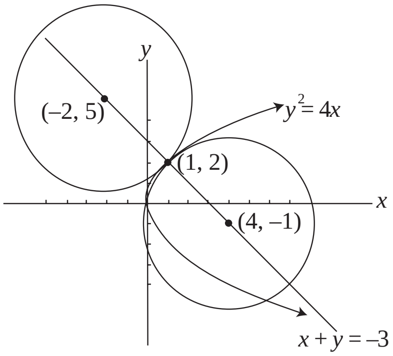
Since $P(a, b)$ is on the curve, then $\sqrt{a} + \sqrt{b} = \sqrt{k}$, or $a^{\frac{1}{2}} + b^{\frac{1}{2}} = k^{\frac{1}{2}}$.

Therefore, the sum of the intercepts

$$= \left(k^{\frac{1}{2}}\right)^2$$

$= k$, as required.

15.



$$y^2 = 4x$$

$$2y\frac{dy}{dx} = 4$$

At (1, 2), $\frac{dy}{dx} = 1$.

Therefore, the slope of the tangent line at (1, 2) is 1 and the equation of the normal is

$$\frac{y-2}{x-1} = -1 \quad \text{or} \quad x + y = 3.$$

The centres of the two circles lie on the straight line $x + y = 3$. Let the coordinates of the centre of each circle be $(p, q) = (p, 3 - p)$. The radius of each circle is $3\sqrt{2}$. Since (1, 2) is on the circumference of the circles,

$$\begin{aligned} (p-1)^2 + (3-p-2)^2 &= r^2 \\ p^2 - 2p + 1 + 1 - 2p + p^2 &= (3\sqrt{2})^2 \\ 2p^2 - 4p + 2 &= 18 \\ p^2 - 2p - 8 &= 0 \\ (p-4)(p+2) &= 0 \end{aligned}$$

$$p = 4 \quad \text{or} \quad p = -2$$

$$\therefore\ q = -1 \quad \text{or} \quad q = 5.$$

Therefore, the centres of the circles are (–2, 5) and (4, –1). The equations of the circles are

$(x + 2)^2 + (y - 5)^2 = 18$

and $(x - 4)^2 + (y + 1)^2 = 18$.

Exercise 5.2

3. **a.** $s(t) = 5t^2 - 3t + 15$

$v(t) = 10t - 3$

$a(t) = 10$

b. $s(t) = 2t^3 + 36t - 10$

$v(t) = 6t^2 + 36$

$a(t) = 12t$

e. $s(t) = \sqrt{t + 1}$

$$v(t) = \frac{1}{2}(t + 1)^{-\frac{1}{2}}$$

$$a(t) = -\frac{1}{4}(t + 1)^{-\frac{3}{2}}$$

f. $s(t) = \dfrac{9t}{t + 3}$

$$v(t) = \frac{9(t + 3) - 9t}{(t + 3)^2}$$

$$= \frac{27}{(t + 3)^2}$$

$a(t) = -54(t + 3)^{-3}$

5. $s = \dfrac{1}{3}t^3 - 2t^2 + 3t$

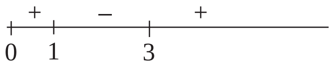
$v = t^2 - 4t + 3$

$a = 2t - 4$

For $v = 0$,

$(t - 3)(t - 1) = 0$

$t = 3$ or $t = 1$.



The direction of the motion of the object changes at $t = 1$ and $t = 3$.

Initial position is $s(0) = 0$.

Solving,

$$0 = \frac{1}{3}t^3 - 2t^2 + 3t$$

$= t^3 - 6t^2 + 9t$

$= t(t^2 - 6t + 9)$

$= t(t - 3)^2$

$\therefore t = 0$ or $t = 3$

$s = 0$ or $s = 0$.

The object returns to its initial position after 3 s.

6. **a.** $s = -\dfrac{1}{3}t^2 + t + 4$

$$v = -\frac{2}{3}t + 1$$

$$v(1) = -\frac{2}{3} + 1$$

$$= \frac{1}{3}$$

$$v(4) = -\frac{2}{3}(4) + 1$$

$$= -\frac{5}{3}$$

For $t = 1$, moving in a positive direction.

For $t = 4$, moving in a negative direction.

b. $s(t) = t(t - 3)^2$

$v(t) = (t - 3)^2 + 2t(t - 3)$

$= (t - 3)(t - 3 + 2t)$

$= (t - 3)(3t - 3)$

$= 3(t - 1)(t - 3)$

$v(1) = 0$

$v(4) = 9$

For $t = 1$, the object is stationary.

$t = 4$, the object is moving in a positive direction.

8. $s(t) = 40t - 5t^2$
$v(t) = 40 - 10t$

a. When $v = 0$, the object stops rising.
$\therefore t = 4$ s

b. Since $s(t)$ represents a quadratic function that opens down because $a = -5 < 0$, a maximum height is attained. It occurs when $v = 0$. Height is a maximum for
$s(4) = 160 - 5(16)$
$= 80$ m.

10. $s(t) = t^{\frac{5}{2}}(7 - t)$

a. $v(t) = \frac{5}{2}t^{\frac{3}{2}}(7 - t) - t^{\frac{5}{2}}$

$$= \frac{35}{2}t^{\frac{3}{2}} - \frac{5}{2}t^{\frac{5}{2}} - t^{\frac{5}{2}}$$

$$= \frac{35}{2}t^{\frac{3}{2}} - \frac{7}{2}t^{\frac{5}{2}}$$

b. $a(t) = \frac{105}{2}t^{\frac{1}{2}} - \frac{35}{4}t^{\frac{3}{2}}$

c. The direction of the motion changes when its velocity changes from a positive to a negative value or visa versa.

$v(t) = \frac{7}{2}t^{\frac{3}{2}}(5 - t) \therefore r(t) = 0$ for $t = 5$

t	$0 \le t < 5$	$t = 5$	$t > 5$
$v(t)$	(+)(+) = +	0	(+)(–) = –

Therefore, the object changes direction at 5 s.

d. $a(t) = 0$ for $\frac{35}{4}t^{\frac{1}{2}}(6 - t) = 0$.

$\therefore t = 0$ or $t = 6$ s.

t	$0 < t < 6$	$t = 6$	$t > 6$
$a(t)$	(+)(+) = +	0	(+)(–) = –

Therefore, the acceleration is positive for
$0 < t < 6$ s.
Note: $t = 0$ yields $a = 0$.

e. At $t = 0$, $s(0) = 0$. Therefore, the object's original position is at 0, the origin.

When $s(t) = 0$,
$t^{\frac{5}{2}}(7 - t) = 0$
$t = 0$ or $t = 7$.
Therefore, the object is back to its original position after 7 s.

12. $s(t) = 6t^2 + 2t$
$v(t) = 12t + 2$
$a(t) = 12$

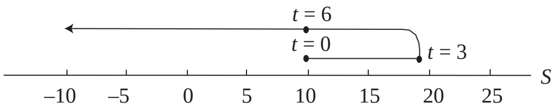
a. $v(8) = 96 + 2 = 98$ m/s
Thus, as the dragster crosses the finish line at $t = 8$ s, the velocity is 98 m/s. Its acceleration is constant throughout the run and equals 12 m/s^2.

b.
$$\begin{aligned} s &= 60 \\ 6t^2 + 2t - 60 &= 0 \\ 2(3t^2 + t - 30) &= 0 \\ 2(3t + 10)(t - 3) &= 0 \end{aligned}$$

$t = \frac{-10}{3}$ or $t = 3$

inadmissible $v(3) = 36 + 2$
$0 \le t \le 8$ $= 38$
Therefore, the dragster was moving at 38 m/s when it was 60 m down the strip.

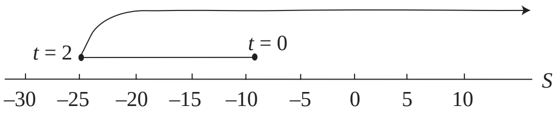
13. a. $s = 10 + 6t - t^2$
$v = 6 - 2t$
$= 2(3 - t)$
$a = -2$
The object moves to the right from its initial position of 10 m from the origin, 0, to the 19 m mark, slowing down at a rate of 2 m/s^2. It stops at the 19 m mark then moves to the left speeding up at 2 m/s^2 as it goes on its journey into the universe. It passes the origin after $(3 + \sqrt{19})$ s.



b. $s = t^3 - 12t - 9$

$$\begin{aligned} v &= 3t^2 - 12 \\ &= 3(t^2 - 4) \\ &= 3(t-2)(t+2) \\ a &= 6t \end{aligned}$$

The object begins at 9 m to the left of the origin, 0, and slows down to a stop after 2 s when it is 25 m to the left of the origin. Then, the object moves to the right speeding up at faster rates as time increases. It passes the origin just before 4 s (approximately 3.7915) and continues to speed up as time goes by on its journey into space.



14. $s(t) = t^5 - 10t^2$

$$\begin{aligned} v(t) &= 5t^4 - 20t \\ a(t) &= 20t^3 - 20 \end{aligned}$$

For $a(t) = 0$,

$$\begin{aligned} 20t^3 - 20 &= 0 \\ 20(t^3 - 1) &= 0 \\ t &= 1. \end{aligned}$$

Therefore, the acceleration will be zero at 1 s.

$$\begin{aligned} s(1) &= 1 - 10 \\ &= -9 \\ &< 0 \\ v(1) &= 5 - 20 \\ &= -15 \\ &< 0 \end{aligned}$$

Since the signs of both s and v are the same at $t = 1$, the object is moving away from the origin at that time.

15. a. $s(t) = kt^2 + (6k^2 - 10k)t + 2k$

$$\begin{aligned} v(t) &= 2kt + (6k^2 - 10k) \\ a(t) &= 2k + 0 \\ &= 2k \end{aligned}$$

Since $k \neq 0$ and $k \in R$, then $a(t) = 2k \neq 0$ and an element of the Real numbers. Therefore, the acceleration is constant.

b. For $v(t) = 0$

$$\begin{aligned} 2kt + 6k^2 - 10k &= 0 \\ 2kt &= 10k - 6k^2 \\ t &= 5 - 3k \\ k &\neq 0 \end{aligned}$$

$$\begin{aligned} s(5-3k) &= k(5-3k)^2 + (6k^2 - 10k)(5-3k) + 2k \\ &= k(25 - 30k + 9k^2) + 30k^2 - 18k^3 - 50k + 30k^2 + 2k \\ &= 25k - 30k^2 + 9k^3 + 30k^2 - 18k^3 - 50k + 30k^2 + 2k \\ &= -9k^3 + 30k^2 - 23k \end{aligned}$$

Therefore, the velocity is 0 at $t = 5 - 3k$, and its position at that time is $-9k^3 + 30k^2 - 23k$.

16. If the ball starts from an initial height of 2 m, then the formulas are $s(t) = 2 + 35t - 5t^2$ and $v(t) = 35 - 10t$. The height is greatest at the instant the upward velocity is 0.

For $v(t) = 0$,

$$\begin{aligned} t &= \frac{35}{10} \\ &= 3.5 \text{ s.} \end{aligned}$$

At $t = 3.5$,

$$\begin{aligned} s(3.5) &= 2 + 35(3.5) - 5(3.5)^2 \\ &= 2 + 122.5 - 61.25 \\ &= 63.25 \text{ m.} \end{aligned}$$

This is much lower than the ceiling of the SkyDome. Thus, a major league pitcher is not likely to hit the ceiling.

17. a. The acceleration is continuous at $t = 0$ if $\lim\limits_{t \to 0} a(t) = a(0)$.

For $t \geq 0$,

$$s(t) = \frac{t^3}{t^2 + 1}$$

$$\begin{aligned} \text{and } v(t) &= \frac{3t^2(t^2+1) - 2t(t^3)}{(t^2+1)^2} \\ &= \frac{t^4 + 3t^2}{(t^2+1)^2} \end{aligned}$$

$$\begin{aligned} \text{and } a(t) &= \frac{(4t^3 + 6t)(t^2+1)^2 - 2(t^2+1)(2t)(t^4 + 3t^2)}{(t^2+1)^2} \\ &= \frac{(4t^3 + 6t)(t^2+1) - 4t(t^4 + 3t^2)}{(t^2+1)^3} \\ &= \frac{4t^5 + 6t^3 + 4t^3 + 6t - 4t^5 - 12t^3}{(t^2+1)^3} \\ &= \frac{-2t^3 + 6t}{(t^2+1)^3} \end{aligned}$$

Therefore, $a(t) = \begin{cases} 0, & t < 0 \\ \dfrac{-2t^3 + 6t}{(t^2 + 1)^3}, & t \geq 0 \end{cases}$

and $v(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t^4 + 3t^2}{(t^2 + 1)^2}, & t \geq 0 \end{cases}$

$$\lim_{t\to 0^-} a(t) = 0, \lim_{t\to 0^+} a(t) = \frac{0}{1}$$
$$= 0.$$

Thus, $\lim_{t\to 0} a(t) = 0$.

Also, $a(0) = \frac{0}{1}$

$= 0$.

Therefore, $\lim_{t\to 0} a(t) = a(0)$.

Thus, the acceleration is continuous at $t = 0$.

b. $\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} \dfrac{t^4 + 3t^2}{t^4 + 2t^2 + 1}$

$$= \lim_{t\to+\infty} \frac{1 + \frac{3}{t^2}}{1 + \frac{2}{t^2} + \frac{1}{t^4}}$$
$$= 1$$

$$\lim_{t\to+\infty} a(t) = \lim_{t\to\infty} \frac{\frac{-2}{t^3} + \frac{6}{t^4}}{1 + \frac{3}{t^2} + \frac{3}{t^4} + \frac{1}{t^6}}$$
$$= \frac{0}{1}$$
$$= 0$$

18. $v = \sqrt{b^2 + 2gs}$

$$v = (b^2 + 2gs)^{\frac{1}{2}}$$
$$\frac{dv}{dt} = \frac{1}{2}\left(b^2 + 2gs\right)^{-\frac{1}{2}} \bullet \left(0 + 2g\frac{ds}{dt}\right)$$
$$a = \frac{1}{2v} \bullet 2gv$$
$$a = g$$

Since g is a constant, a is a constant, as required.

Note: $\dfrac{ds}{dt} = v$

$\dfrac{dv}{dt} = a$

19. $F = \dfrac{m_0 a}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{3}{2}}}$

$$F = m_0 \frac{d\left(\dfrac{v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}\right)}{dt}$$

Using the quotient rule,

$$= \frac{m_0 \dfrac{dv}{dt}\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{1}{2}} - \dfrac{1}{2}\left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}\left(-\dfrac{2v\frac{dv}{dt}}{c^2}\right) \bullet v}{1 - \dfrac{v^2}{c^2}}$$

Since $\dfrac{dv}{dt} = a$,

$$= \frac{m_0\left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}\left[a\left(1 - \dfrac{v^2}{c^2}\right) + \dfrac{v^2a}{c^2}\right]}{1 - \dfrac{v^2}{c^2}}$$

$$= \frac{m_0\left[\dfrac{ac^2 - av^2}{c^2} + \dfrac{v^2a}{c^2}\right]}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$$= \frac{m_0 ac^2}{c^2\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$$= \frac{m_0 a}{\left(1 - \dfrac{v^2}{c^2}\right)^{\frac{3}{2}}}, \text{ as required.}$$

Exercise 5.3

2. $T(x) = \dfrac{200}{1 + x^2}$

a. $\dfrac{dx}{dt} = 2$ m/s

Find $\dfrac{dT(x)}{dt}$ when $x = 5$ m:

$$T(x) = \frac{200}{1 + x^2}$$
$$= 200(1 + x^2)^{-1}$$
$$\frac{dT(x)}{dt} = -200(1 + x^2)^{-2}\,2x\frac{dx}{dt}$$
$$= \frac{-400x}{(1 + x^2)^2} \bullet \frac{dx}{dt}.$$

At a specific time, when $x = 5$,

$$\frac{dT(5)}{dt} = \frac{-400(5)}{(26)^2} \qquad (2)$$

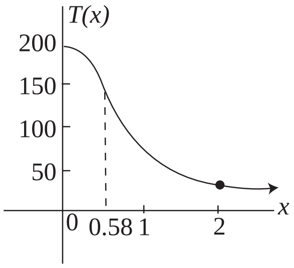
$$= \frac{-4000}{676}$$

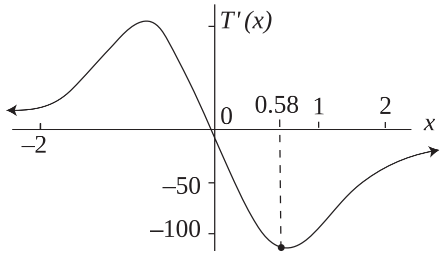
$$= \frac{-1000}{169}$$

$$\frac{dT(5)}{dt} \doteq -5.9.$$

Therefore, the temperature is decreasing at a rate of 5.9°C per s.

b.





c. Solve $T''(x) = 0$.

$$T'(x) = \frac{-400x}{(1 + x^2)^2}$$

$$T''(x) = \frac{-400(1 + x^2)^2 - 2(1 + x^2)(2x)(-400x)}{(1 + x^2)^4}$$

Let $T''(x) = 0$,

$-400(1 + x^2)^2 + 1600x^2(1 + x^2) = 0.$

Divide,

$400(1 + x^2) - (1 + x^2) + 4x^2 = 0$

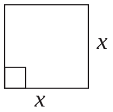
$3x^2 = 1$

$$x^2 = \frac{1}{\sqrt{3}}$$

$$x = \frac{1}{\sqrt{3}}$$

$x > 0$ or $x \doteq 0.58$.

3. Given square



$\frac{dx}{dt} = 5$ cm/s.

Find $\frac{dA}{dt}$ when $x = 10$ cm.

Solution

Let the side of a square be x cm.

$A = x^2$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$

At a specific time, $x = 10$ cm.

$$\frac{dA}{dt} = 2(10)(5)$$

$$= 100$$

Therefore, the area is increasing at 100 cm^2/s when a side is 10 cm.

$P = 4x$

$$\frac{dP}{dt} = 4\frac{dx}{dt}$$

At any time, $\frac{dx}{dt} = 5$.

$$\therefore \frac{dP}{dt} = 20.$$

Therefore, the perimeter is increasing at

20 cm/s.

4. Given cube with sides x cm,

$$\frac{dx}{dt} = 5 \text{ cm/s}.$$

a. Find $\frac{dv}{dt}$ when $x = 5$ cm:

$V = x^3$

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$$

At a specific time, $x = 5$ cm.

$$\frac{dV}{dt} = 3(5)^2(4)$$

$$= 300$$

Therefore, the volume is increasing at 300 cm^3/s.

b. Find $\frac{dS}{dt}$ when $x = 7$ cm.

$$S = 6x^2$$

$$\frac{dS}{dt} = 12x\,\frac{dx}{dt}$$

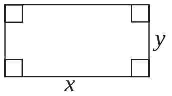
At a specific time, $x = 7$ cm,

$$\frac{dS}{dt} = 12(7)(4)$$

$$= 336.$$

Therefore, the surface area is increasing at a rate of 336 cm^2/s.

5. Given rectangle



$$\frac{dx}{dt} = 2 \text{ cm/s}$$

$$\frac{dy}{dt} = -3 \text{ cm/s}$$

Find $\frac{dA}{dt}$ when $x = 20$ cm and $y = 50$ cm.

Solution

$$A = xy$$

$$\frac{dA}{dt} = \frac{dx}{dt}\,y + \frac{dy}{dt}\,x$$

At a specific time, $x = 20$, $y = 50$,

$$\frac{dA}{dt} = (2)(50) + (-3)(20)$$

$$= 100 - 50$$

$$= 40.$$

Therefore, the area is increasing at a rate of 40 cm^2/s.

6. Given circle with radius r,

$$\frac{dA}{dt} = -5 \text{ m}^2\text{/s}.$$

a. Find $\frac{dr}{dt}$ when $r = 3$ m.

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}$$

When $r = 3$,

$$-5 = 2\pi(3)\,\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-5}{6\pi}.$$

Therefore, the radius is decreasing at a rate of $\frac{5}{6\pi}$ m/s when $r = 3$ m.

b. Find $\frac{dD}{dt}$ when $r = 3$.

$$\frac{dD}{dt} = 2\,\frac{dr}{dt}$$

$$= 2\left(\frac{-5}{6\pi}\right)$$

$$= \frac{-5}{3\pi}$$

Therefore, the diameter is decreasing at a rate of $\frac{5}{3\pi}$ m/s.

7. Given circle with radius r,

$$\frac{dA}{dt} = 6 \text{ km}^2\text{/h}$$

Find $\frac{dr}{dt}$ when $A = 9\pi$ km^2.

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}$$

When $A = 9\pi$,

$$9\pi = \pi r^2$$

$$r^2 = 9$$

$$r = 3$$

$$r > 0.$$

When $r = 3$,

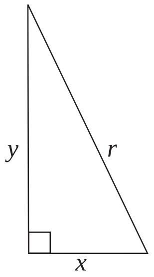
$$\frac{dA}{dt} = 6$$

$$6 = 2\pi(3)\,\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{\pi}.$$

Therefore, the radius is increasing at a rate of $\frac{1}{\pi}$ km/h.

8.



Let x represent the distance from the wall and y the height of the ladder on the wall.

$x^2 + y^2 = r^2$

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2r\dfrac{dr}{dt}$

$x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = r\dfrac{dr}{dt}$

When $r = 5$, $y = 3$,

$x^2 = 25 - 9$

$= 16$

$x = 4$

$x = 4, y = 3, r = 5$

$\dfrac{dx}{dt} = \dfrac{1}{3}, \dfrac{dr}{dt} = 0.$

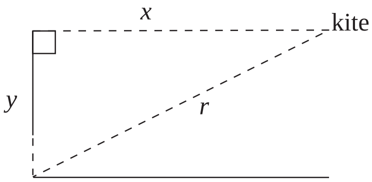
Substituting,

$4\left(\dfrac{1}{3}\right) + 3\left(\dfrac{dy}{dt}\right) = 5(0)$

$\dfrac{dy}{dt} = -\dfrac{4}{9}.$

Therefore, the top of the ladder is sliding down at 4 m/s.

9.



Let the variables represent the distances as shown on the diagram.

$x^2 + y^2 = r^2$

$2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2r\dfrac{dr}{dt}$

$x\dfrac{dx}{dt} + y\dfrac{dy}{dt} = r\dfrac{dr}{dt}$

$x = 30, y = 40$

$r^2 = 30^2 + 40^2$

$r = 50$

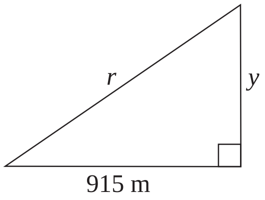
$\dfrac{dr}{dt} = ?, \dfrac{dx}{dt} = 10, \dfrac{dy}{dt} = 0$

$30(10) + 40(0) = 50\left(\dfrac{dr}{dt}\right)$

$\dfrac{dr}{dt} = 8$

Therefore, she must let out the line at a rate of 8 m/min.

10.



Label diagram as shown.

$r^2 = y^2 + 915^2$

$2r\dfrac{dr}{dt} = 2y\dfrac{dy}{dt}$

$r\dfrac{dr}{dt} = y\dfrac{dy}{dt}$

When $y = 1220$, $\dfrac{dy}{dt} = 268$ m/s.

$r = \sqrt{1220^2 + 915^2}$

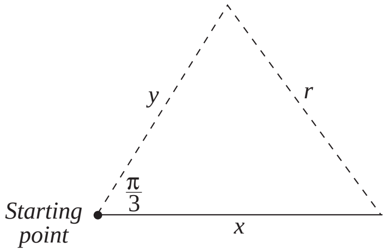
$= 1525$

$\therefore 1525\left(\dfrac{dr}{dt}\right) = 1220 \times 268$

$\dfrac{dr}{dt} = 214$ m/s

Therefore, the camera–to–rocket distance is changing at 214 m/s.

11.



$\dfrac{dx}{dt} = 15$ km/h

$\dfrac{dy}{dt} = 20$ km/h

Find $\dfrac{dr}{dt}$ when $t = 2$ h.

Solution

Let x represent the distance cyclist 1 is from the starting point, $x \geq 0$. Let y represent the distance cyclist 2 is from the starting point, $y \geq 0$ and let r be the distance the cyclists are apart. Using the cosine law,

$r^2 = x^2 + y^2 - 2xy\cos\dfrac{\pi}{3}$

$= x^2 + y^2 - 2xy\left(\dfrac{1}{2}\right)$

$r^2 = x^2 + y^2 - xy$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \left[\frac{dx}{dt}y + \frac{dy}{dt}x\right]$$

At $t = 2$ h, $x = 30$ km, $y = 40$ km

$$\begin{aligned}
\text{and } r^2 &= 30^2 + 40^2 - 2(30)(40)\cos\frac{\pi}{3} \\
&= 2500 - 2(1200)\left(\frac{1}{2}\right) \\
&= 1300 \\
r &= 10\sqrt{13},\ r > 0.
\end{aligned}$$

$$\begin{aligned}
\therefore\ 2(10\sqrt{13})\frac{dr}{dt} &= 2(30)(15) + 2(40)(20) - [15(40) + 20(30)] \\
20\sqrt{13}\frac{dr}{dt} &= 900 + 1600 - [600 - 600] \\
&= 1300 \\
\frac{dr}{dt} &= \frac{130}{2\sqrt{13}} \\
&= \frac{65}{\sqrt{13}} \\
&= \frac{65\sqrt{13}}{13} \\
&= 5\sqrt{13}
\end{aligned}$$

Therefore, the distance between the cyclists is increasing at a rate of $5\sqrt{13}$ km/h after 2 h.

12. Given sphere $v = \frac{4}{3}\pi r^3$

$\frac{dv}{dt} = 8$ cm^3/s.

a. Find $\frac{dr}{dt}$ when $r = 12$ cm.

$$\begin{aligned}
v &= \frac{4}{3}\pi r^3 \\
\frac{dv}{dt} &= 4\pi r^2\frac{dr}{dt}
\end{aligned}$$

At a specific time, when $r = 12$ cm:

$$\begin{aligned}
8 &= 4\pi(12)^2\frac{dr}{dt} \\
8 &= 4\pi(144)\frac{dr}{dt} \\
\frac{1}{72\pi} &= \frac{dr}{dt}.
\end{aligned}$$

Therefore, the radius is increasing at a rate of $\frac{1}{72\pi}$ cm/s.

b. Find $\frac{dr}{dt}$ when $v = 1435$ cm^3.

Solution

$$\begin{aligned}
v &= \frac{4}{3}\pi r^3 \\
\frac{dv}{dt} &= 4\pi r^2\frac{dr}{dt}.
\end{aligned}$$

At a specific time, when $v = 1435$ cm^3:

$$\begin{aligned}
v &= 1435 \\
\frac{4}{3}\pi^3 &= 1435 \\
r^3 &\doteq 342.581015 \\
&\doteq 6.9971486 \\
&= 7 \\
8 &\doteq 4\pi(7)^2\frac{dr}{dt} \\
8 &= 196\pi\frac{dr}{dt} \\
\frac{2}{49\pi} &= \frac{dr}{dt} \\
0.01 &= \frac{dr}{dt}.
\end{aligned}$$

Therefore, the radius is increasing at approximately $\frac{2}{49\pi}$ cm/s (or 0.01 cm/s).

c. Find $\frac{dr}{dt}$ when $t = 33.5$ s.

When $t = 33.5$, $v = 8 \times 33.5$ cm^3:

$$\begin{aligned}
\frac{4}{3}\pi r^3 &= 268 \\
r^3 &\doteq 63.98028712 \\
r &\doteq 3.999589273 \\
&\doteq 4.
\end{aligned}$$

Solution

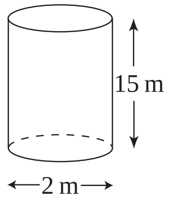
$$\begin{aligned}
v &= \frac{4}{3}\pi r^3 \\
\frac{dv}{dt} &= 4\pi r^2\frac{dr}{dt}
\end{aligned}$$

At $t = 33.5$ s,

$$\begin{aligned}
8 &\doteq 4\pi(4)^2\frac{dr}{dt} \\
8 &= 64\pi\frac{dr}{dt} \\
\frac{1}{8\pi} &= \frac{dr}{dt}.
\end{aligned}$$

Therefore, the radius is increasing at a rate of $\frac{1}{8\pi}$ cm/s (or 0.04 cm/s).

13. Given cylinder



$v = \pi r^2 h$

$$\frac{dv}{dt} = 500 \text{ L/min}$$
$$= 500\ 000 \text{ cm/min}$$

a. Find $\frac{dy}{dt}$.

$v = \pi r^2 h$

Since the diameter is constant at 2 m, the radius is also constant at 1 m = 100 cm.

$$\therefore\ v = 10\ 000\ \pi h$$
$$\frac{dv}{dt} = 10\ 000\pi \frac{dh}{dt}$$
$$500\ 000 = 10\ 000\pi \frac{dh}{dt}$$
$$\frac{50}{\pi} = \frac{dh}{dt}$$

Therefore, the fluid level is rising at a rate of $\frac{50}{\pi}$ cm/min.

b. Find t, the time to fill the cylinder.

$V = \pi r^2 h$

$V = \pi(100)^2(1500)$ cm^3

$V = 15\ 000\ 000\pi$ cm^3

Since $\frac{dv}{dt} = 500\ 000$ cm^3/min,

it takes $\frac{15\ 000\ 000\pi}{500\ 000}$ min,

$= 30\pi$ min to fill

$\doteq 94.25$ min.

Therefore, it will take 94.25 min, or just over 1.5 h to fill the cylindrical tank.

14. There are many possible problems.

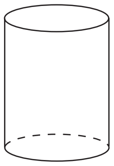
Samples:

i) The diameter of a right-circular cone is expanding at a rate of 4 cm/min. Its height remains constant at 10 cm. Find its radius when the volume is increasing at a rate of 80π cm^3/min.

ii) Water is being poured into a right-circular tank at the rate of 12π m^3/min. Its height is 4 m and its radius is 1 m. At what rate is the water level rising?

iii) The volume of a right-circular cone is expanding because its radius is increasing at 12 cm/min and its height is increasing at 6 cm/min. Find the rate at which its volume is changing when its radius is 20 cm and its height is 40 cm.

15. Given cylinder



$d = 1$ m

$h = 15$ m

$$\frac{dr}{dt} = 0.003 \text{ m/annum}$$
$$\frac{dh}{dt} = 0.4 \text{ m/annum}$$

Find $\frac{dv}{dt}$ at the instant $D = 1$

$v = \pi r^2 h$

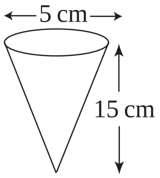
$$\frac{dv}{dt} = \left(2\pi r \frac{dr}{dt}\right)(h) + \left(\frac{dh}{dt}\right)(\pi r^2).$$

At a specific time, when $D = 1$; i.e., $r = 0.5$,

$$\frac{dv}{dt} = 2\pi(0.5)(0.003)(15) + 0.4\pi(0.5)^2$$
$$= 0.045\pi + 0.1\pi$$
$$= 0.145\pi$$

Therefore, the volume of the trunk is increasing at a rate of 0.145π m^3/annum.

16. Given cone



$r = 5$ cm

$h = 15$ cm

$$\frac{dv}{dt} = 2 \text{ cm}^3\text{/min}$$

Find $\frac{dh}{dt}$ when $h = 3$ cm,

$v = \frac{1}{3}\pi r^2 h$.

Using similar triangles, $\frac{r}{h} = \frac{5}{15} = \frac{1}{3}$

$$\therefore r = \frac{h}{3}.$$

Substituting into $v = \frac{1}{3}\pi r^2 h$,

$$v = \frac{1}{3}\pi\left(\frac{h^2}{9}\right)h$$

$$= \frac{1}{27}\pi h^3$$

$$\frac{dv}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

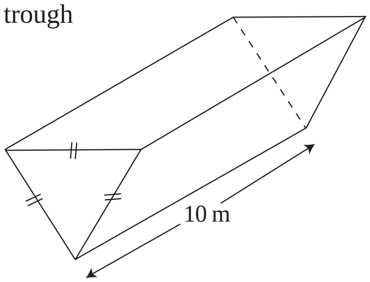
At a specific time, when $h = 3$ cm,

$$-2 = \frac{1}{9}\pi(3)^2 \frac{dh}{dt}$$

$$-\frac{2}{\pi} = \frac{dh}{dt}.$$

Therefore, the water level is being lowered at a rate of $\frac{2}{\pi}$ cm/min when height is 3 cm.

17. Given trough

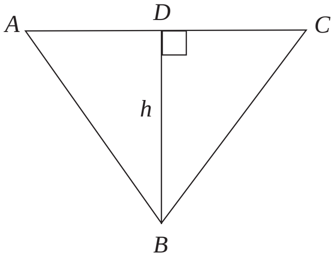


Find a formula for the volume.

v = area of a cross section × length

= area of an equilateral triangle × 10

Let h be the height of any cross section.



Since $\angle C = 60°$, $\angle B = 30°$ and ΔDBC is a special triangle similar to the 1, $\sqrt{3}$, 2 triangle.

Since $DB = h$, then $DC = \frac{h}{\sqrt{3}}$ from similar triangles.

Therefore, $AC = \frac{2h}{\sqrt{3}}$

$$v = \frac{1}{2}AC \times DB \times 10$$

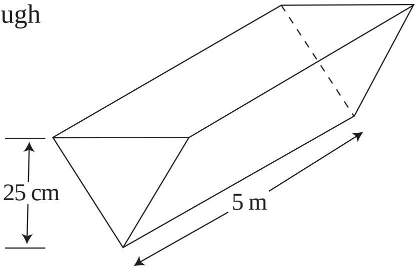
$$= \frac{1}{2} \times \frac{2h}{\sqrt{3}} \times h \times 10$$

$$= \frac{h^2}{\sqrt{3}} \times 10$$

$$= \frac{10h^2}{\sqrt{3}}.$$

Therefore, the volume of the trough of height h is given by $v = \frac{10h^2}{\sqrt{3}}$.

18. Given trough



$$\frac{dv}{dt} = 0.25\,\frac{\text{m}^3}{\text{min}}$$

Find $\frac{dh}{dt}$ when $h = 10$ cm

$= 0.1$ m.

Since the cross section is equilateral, the $v = \frac{h^2}{\sqrt{3}} \times \ell$.

$$v = \frac{h^2}{\sqrt{3}} \times 5.$$

$$\frac{dv}{dt} = \frac{10}{\sqrt{3}}\,h\,\frac{dh}{dt}$$

At a specific, time when $h = 0.1 = \frac{1}{10}$,

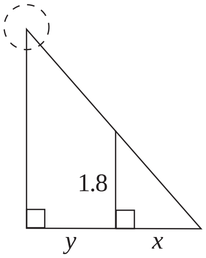
$$0.25 = \frac{10}{\sqrt{3}} \frac{1}{10} \frac{dh}{dt}$$

$$0.25\sqrt{3} = \frac{dh}{dt}$$

$$\frac{\sqrt{3}}{4} = \frac{dh}{dt}$$

Therefore, the water level is rising at a rate of $\frac{\sqrt{3}}{4}$ m/min.

19. Given

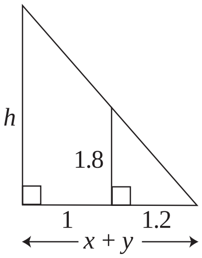


$$\frac{dy}{dt} = 120 \text{ m/min}$$

Find $\frac{dx}{dt}$ when $t = 5$ s.

Solution

Let x represent the length of the shadow. Let y represent the distance the man is from the base of the lamppost. Let h represent the height of the lamppost. At a specific instant, we have



Using similar triangles,

$$\frac{x+y}{h} = \frac{1.2}{1.8}$$

$$\frac{2.2}{h} = \frac{2}{3}$$

$$2h = 6.6$$

$$h = 3.3$$

Therefore, the lamppost is 3.3 m high.

At any time,

$$\frac{x+y}{x} = \frac{3.3}{1.8}$$

$$\frac{x+y}{x} = \frac{11}{6}$$

$$6x + 6y = 11x$$

$$6y = 5x$$

$$6\frac{dy}{dt} = 5\frac{dx}{dt}.$$

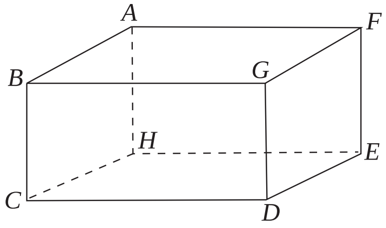
At a specific time, when $t = 5$ seconds $\frac{dy}{dt} = 120$ m/min,

$$6 \times 120 \quad = 5\frac{dx}{dt}$$

$$\frac{dx}{dt} = 144.$$

Therefore, the man's shadow is lengthening at a rate of 144 m/min after 5 s.

20. This question is similar to finding the rate of change of the length of the diagonal of a rectangular prism.



$$20 \text{ m} = \frac{20}{1000} \text{ km}$$

$$= \frac{1}{50} \text{ km}$$

Find $\frac{d(GH)}{dt}$ *at* $t = 10$ s,

$$= \frac{1}{360} \text{ h}.$$

Let BG be the path of the train and CH be the path of the boat:

$$\therefore \frac{d(BG)}{dt} = 60 \text{ km/h and } \frac{d(CH)}{dt} = 20 \text{ km/h}.$$

At $t = \frac{1}{360}$h, $BG = 60 \times \frac{1}{360}$

$$= \frac{1}{6} \text{ km}$$

and $CH = 20 \times \frac{1}{360}$

$$= \frac{1}{18} \text{ km}.$$

Using the Pythagorean Theorem,

$$GH^2 = HD^2 + DG^2$$

and $HD^2 = CD^2 + CH^2$

$$\therefore\ GH^2 = CD^2 + CH^2 + DG^2$$

Since $BG = CD$ and $FE = GD = \frac{1}{50}$, it follows that $GH^2 = BG^2 + CH^2 + \frac{1}{2500}$.

$$2(GH)\frac{d(GH)}{dt} = 2(BG)\frac{d(BG)}{dt} + 2(CH)\frac{d(CH)}{dt}$$

At $t = 10$ s,

$$GH\frac{d(GH)}{dt} = \frac{1}{6}(60) + \frac{1}{18}(20)$$

$$\frac{\sqrt{6331}}{450}\frac{d(GH)}{dt} = \frac{100}{9}$$

$$\frac{d(GH)}{dt} = \frac{45\,000}{9\sqrt{6331}}$$

$$\doteq 62.8.$$

And $GH^2 = \left(\frac{1}{6}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{1}{50}\right)^2$

$$= \frac{1}{36} + \frac{1}{324} + \frac{1}{2500}$$

$$= \frac{911\,664}{29\,160\,000} \div 8$$

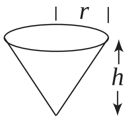
$$GH^2 = \frac{113\,958}{364\,500} \div 18$$

$$= \frac{6331}{202\,500}$$

$$GH = \frac{\sqrt{6331}}{450} = \frac{\sqrt{13 \text{ x } 487}}{450}$$

Therefore, they are separating at a rate of approximately 62.8 km/h.

21. Given cone



$$r = h$$

$$\frac{dv}{dt} = 200 - 20$$

$$= 180 \text{ cm}^3\text{/s}$$

Find $\frac{dh}{dt}$ when $h = 15$ cm.

Solution

$$v = \frac{1}{3}\pi r^2 h \text{ and } r = h$$

$$\therefore\ v = \frac{1}{3}\pi h^3.$$

$$\frac{dv}{dt} = \pi h^2\frac{dh}{dt}$$

At a specific time, $h = 15$ cm.

$$180 = \pi(15)^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{5\pi}$$

Therefore, the height of the water in the funnel is increasing at a rate of $\frac{4}{5\pi}$ cm/s.

Part 2

$$\frac{dv}{dt} = 200 \text{ cm}^3\text{/s}$$

Find $\frac{dh}{dt}$ when $h = 25$ cm.

Solution

$$\frac{dv}{dt} = \pi h^2\frac{dh}{dt}.$$

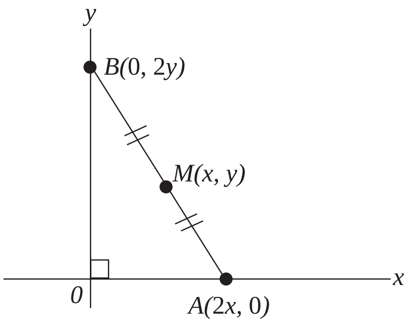
At the time when the funnel is clogged, $h = 25$ cm:

$$200 = \pi(25)^2\frac{dh}{dt}.$$

$$\frac{dh}{dt} = \frac{8}{25\pi}.$$

Therefore, the height is increasing at $\frac{8}{25\pi}$ cm/s.

22.



Let the midpoint of the ladder be (x, y). From similar triangles, it can be shown that the top of the ladder and base of the ladder would have points $B(0, 2y)$ and $A(2x, 0)$ respectively. Since the ladder has length ℓ,

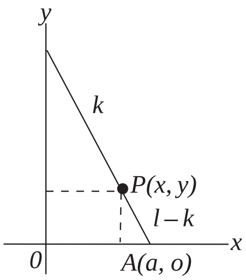
$$(2x)^2 + (2y)^2 = \ell^2$$

$$4x^2 + 4y^2 = \ell^2$$

$$x^2 + y^2 = \frac{\ell^2}{4}$$

$$= \left(\frac{\ell}{2}\right)^2 \text{ is the required equation.}$$

Therefore, the equation of the path followed by the midpoint of the ladder represents a quarter circle with centre (0, 0) and radius $\frac{\ell}{2}$, with $x, y \geq 0$.



Let $P(x, y)$ be a general point on the ladder a distance k from the top of the ladder. Let $A(a, o)$ be the point of contact of the ladder with the ground.

From similar triangles, $\frac{a}{\ell} = \frac{x}{k}$ or $a = \frac{x\ell}{k}$.

Using the Pythagorean Theorem: $y^2 + (a - x)^2 = (\ell - k)^2$,

and substituting $a = \frac{x\ell}{k}$,

$$y^2 + \left(\frac{x\ell}{k} - x\right)^2 = (\ell - k)^2$$

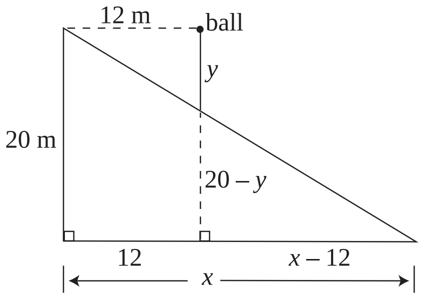
$$y^2 + x^2\left(\frac{\ell - k}{k}\right)^2 = (\ell - k)^2$$

$$\frac{(\ell - k)^2}{k^2}x^2 + y^2 = (\ell - k)^2$$

$$\frac{x^2}{k^2} + \frac{y^2}{(\ell - k)^2} = 1 \text{ is the required equation.}$$

Therefore, the equation is the first quadrant portion of an ellipse.

23.



Let x represent the distance the tip of the ball's shadow is from the base of the lamppost.

Let $\frac{dx}{dt}$ represent the rate at which the shadow is moving along the ground. Let y represent the distance the ball has fallen.

From similar triangles,

$$\frac{20 - y}{x - 12} = \frac{20}{x}$$

$$20x - xy = 20x - 240$$

$$xy = 240$$

$$\frac{dx}{dt}y + \frac{dy}{dt}x = 0.$$

At a specific time,

$$\frac{dx}{dt}(5) + (10)(48) = 0$$

$$\frac{dx}{dt} = -\frac{480}{5}$$

$$= -96.$$

Therefore, the shadow is moving at a rate of 96 m/s.

At any time, t, the height of the ball is $h = 20 - 5t^2$.

When $t = 1$, $h = 20 - 5$,

$= 15$

$\therefore y = 5$.

Also $v = -10t$ and since y increases, $\frac{dy}{dt} = 10$ when $t = 1$.

Section 5.4

Investigation

1. **a.** $f(x) = -x^2 + 6x - 3, 0 \le x \le 5$

$= -(x^2 - 6x + 9 - 9) - 3$

$= -(x - 3)^2 + 6$

maximum of 6 when $x = 3$

b. $f(x) = -x^2 - 2x + 11, -3 \le x \le 4$

$= -(x^2 + 2x + 1 - 1) + 11$

$= -(x + 1)^2 + 12$

maximum of 12 when $x = -1$

c. $f(x) = 4x^2 - 12x + 7, -1 \le x \le 4$

$= 4\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) + 7$

$= 4\left(x - \frac{3}{2}\right)^2 - 2$

minimum value of –2 when $x = \frac{3}{2}$

2. **a.** $f'(x) = -2x + 6 = 0$

$x = 3, c = 3$

b. $f'(x) = -2x - 2$

$x = -1, c = -1$

c. $f'(x) = 8x - 12 = 0$

$x = \frac{3}{2}, c = \frac{3}{2}$

3. The values are the same.

4. **a.** $f(x) = x^3 - 3x^2 - 8x + 10, -2 \le x \le 4$

max at $x \doteq -0.91$, min at $x \doteq 2.91$

b. $f(x) = x^3 - 12x + 5, -3 \le x \le 3$

max at $x \doteq -1.98$, min at $x \doteq 1.98$

c. $f(x) = 3x^3 - 15x^2 + 9x + 23, 0 \le x \le 4$

max at $x \doteq 0.34$, min at $x \doteq 2.98$

d. $f(x) = -2x^3 + 12x + 7, -2 \le x \le 2$

max at $x \doteq 1.41$, min at $x \doteq -1.41$

e. $f(x) = -x^3 - 2x^2 + 15x + 23, -4 \le x \le 3$

max at $x \doteq 1.66$, min at $x \doteq -3.03$

5. **a.** $f'(x) = 3x^2 - 6x - 8 = 0$

$$x = \frac{6 \pm \sqrt{36 + 96}}{6}$$

$$= \frac{6 \pm \sqrt{132}}{6}$$

$x \doteq 2.91$ or $x \doteq -0.91$

b. $f'(x) = 3x^2 - 12 = 0$

$x^2 - 4 = 0$

$x = \pm 2$

c. $f'(x) = 9x^2 - 30x + 9 = 0$

$3x^2 - 10x + 3 = 0$

$(3x - 1)(x - 3) = 0$

$x = \frac{1}{3}$ or $x = 3$

d. $f120(x) = -6x^2 + 12 = 0$

$x^2 - 2 = 0$

$x = \pm\sqrt{2}$

$x = 1.41$ or $x = -1.41$

e. $f'(x) = -3x^2 - 4x + 15 = 0$

$3x^2 + 4x - 15 = 0$

$(3x - 5)(x + 3) = 0$

$x = \frac{5}{3}$ or $x = -3$

6. The values are the same.

7. Set first derivative to zero.

8. **a.** $f(x) = -x^2 + 6x - 3, 4 \le x \le 8$

max at $x = 4$, value 5, min at $x = 8, y = -19$

b. $f(x) = 4x^2 - 12x + 7, 2 \le x \le 6$

max at $x = 6$, value –1, min at $x = 2, y = 79$

c. $f(x) = x^3 - 3x^2 - 9x + 10, -2 \le x \le 6$

max at $x = -2, y = 40$, min at $x = 6, y = -800$

d. $f(x) = x^3 - 12x + 5, 0 \le x \le 5$

max at $x = 5, y = -11$, min at $x = 2, y = 70$

e. $f(x) = x^3 - 5x^2 + 3x + 7, -2 \le x \le 5$

max at $x = 5, y = 20$, min at $x = -2, y = -29$

9. End points of the interval.

Exercise 5.4

3. **a.** $f(x) = x^2 - 4x + 3, 0 \le x \le 3$

$f'(x) = 2x - 4$

Let $2x - 4 = 0$ for max or min

$x = 2$

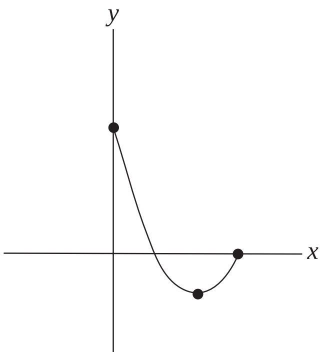
$f(0) = 3$

$f(2) = 4 - 8 + 3 = -1$

$f(3) = 9 - 12 + 3 = 0$

max is 3 at $x = 0$

min is –1 at $x = 2$



c. $f(x) = x^3 - 3x^2, -1 \le x \le 3$

$f'(x) = 3x^2 - 6x$

Let $f'(x) = 0$ for max or min

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0$ or $x = 2$

$f(-1) = -1 - 3$

$= -4$

$f(0) = 0$

$f(2) = 8 - 12$

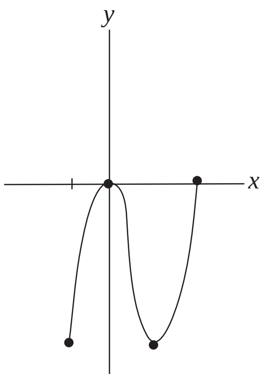
$= -4$

$f(3) = 27 - 27$

$= 0$

min is –4 at $x = -1, 2$

max is 0 at $x = 0, 3$



e. $f(x) = 2x^3 - 3x^2 - 12x + 1, -2 \le x \le 0$

$f'(x) = 6x^2 - 6x - 12$

Let $f'(x) = 0$ for max or min

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

$f(-2) = -16 - 12 + 24 + 1$

$= -3$

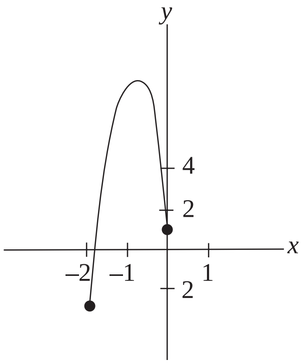
$f(-1) = 8$

$f(0) = 1$

$f(2)$ = not in region

max of 8 at $x = -1$

min of –3 at $x = -2$



4. **b.** $f(x) = 4\sqrt{x} - x, 2 \le x \le 9$

$f'(x) = 2x^{-\frac{1}{2}} - 1$

Let $f'(x) = 0$ for max or min

$\frac{2}{\sqrt{x}} - 1 = 0$

$x = \sqrt{2}$

$x = 4$

$f(2) = 4\sqrt{2} - 2 \doteq 3.6$

$f(4) = 4\sqrt{4} - 4 = 4$

$f(9) = 4\sqrt{9} - 9 = 3$

min value of 3 when $x = 9$

max value of 4 when $x = 4$

c. $f(x) = \frac{1}{x^2 - 2x + 2}, 0 \leq x \leq 2$

$f'(x) = -(x^2 - 2x + 2)^{-2}(2x - 2)$

$= -\frac{2x - 2}{(x^2 - 2x + 2)^2}$

Let $f'(x) = 0$ for max or min.

$-\frac{2x - 2}{(x^2 - 2x + 2)} = 0$

$\therefore 2x - 2 = 0$

$x = 1$

$f(0) = \frac{1}{2}, f(1) = 1, f(2) = \frac{1}{2}$

max value of 1 when $x = 1$

min value of $\frac{1}{2}$ when $x = 0, 2$

e. $f(x) = \frac{4x}{x^2 + 1}, -2 \leq x \leq 4$

$f'(x) = \frac{4(x^2 + 1) - 2x(4x)}{(x^2 + 1)^2}$

$= \frac{-4x^2 + 4}{x^2 + 1}$

Let $f'(x) = 0$ for max or min:

$-4x^2 + 4 = 0$

$x^2 = 1$

$x = \pm 1$

$f(-2) = \frac{-8}{5}$

$f(-1) = \frac{-4}{2}$

$= -2$

$f(1) = \frac{4}{2}$

$= 2$

$f(4) = \frac{16}{17}$

max value of 2 when $x = 1$

min value of –2 when $x = -1$

5. a. $v(t) = \frac{4t^2}{4 + t^3}, t \geq 0$

Interval $1 \leq t \leq 4$

$v(1) = \frac{4}{5}\ v(4)$

$= \frac{16}{17}$

$v'(t) = \frac{(4 + t^3)(8t) - 4t^2(3t^2)}{(4 + t^3)^2} = 0$

$32t + 8t^4 - 12t^4 = 0$

$-4t(t^3 - 8) = 0$

$t = 0, t = 2$

$v(2) = \frac{16}{12} = \frac{4}{3}$

max velocity is $\frac{4}{3}$ m/s

min velocity is $\frac{4}{5}$ m/s

7. a. $E(v) = \frac{1600v}{v^2 + 6400}\ 0 \leq v \leq 100$

$E'(v) = \frac{1600(v^2 + 6400) - 1600v(2v)}{(v^2 + 6400)^2}$

Let $E'(N) = 0$ for max or min

$\therefore 1600v^2 + 6400 \times 1600 - 3200v^2 = 0$

$1600v^2 = 6400 \times 1600$

$v = \pm 80$

$E(0) = 0$

$E(80) = 10$

$E(100) = 9.756$

The legal speed limit that maximizes fuel efficiency is 80 km/h.

8. $C(t) = \frac{0.1t}{(t + 3)^2}, 1 \leq t \leq 6$

$C'(t) = \frac{0(t + 3)^2 - 0.2t(t + 3)}{(t + 3)^4} = 0$

$(t + 3)(0.1t + 0.3 - 0.2t) = 0$

$t = 3$

$C(1) \doteq 0.00625$

$C(3) = 0.0083, C(6) \doteq 0.0074$

The min concentration is at $t = 1$ and the max concentration is at $t = 3$.

9. $P(t) = 2t + \frac{1}{162t + 1}, 0 \le t \le 1$

$$\begin{aligned}
P'(t) &= 2(162t + 1)^{-2}(162) = 0 \\
\frac{162}{(162t + 1)^2} &= 2 \\
81 &= 162^2 + t^2 + 324t + 1 \\
162^2t^2 + 324t - 80 &= 0 \\
81^2t^2 + 81t - 20 &= 0 \\
(81t + 5)(81t - 4) &= 0 \\
t > 0 \therefore t &= \frac{4}{81} \\
&= 0.05
\end{aligned}$$

$P(0) = 1$

$P(0.05) = 0.21$

$P(1) = 2.01$

Pollution is at its lowest level in 0.05 years or approximately 18 days.

10. $r(x) = \frac{1}{400}\left(\frac{4900}{x} + x\right)$

$r'(x) \frac{1}{400}\left(\frac{-4900}{x^2} + 1\right) = 0$

Let $r'(x) = 0$

$$\begin{aligned}
x^2 &= 4900, \\
x &= 70, x > 0 \\
r(30) &= 0.4833 \\
r(70) &= 0.35 \\
r(120) &= 0.402
\end{aligned}$$

A speed of 70 km/h uses fuel at a rate of 0.35 L/km.

Cost of trip is 0.35 × 200 × 0.45 = \$31.50.

11. $C(x) = 3000 + 9x + 0.05x^2, 1 \le x \le 300$

$$\begin{aligned}
\text{Unit cost } u(x) &= \frac{C(x)}{x} \\
&= \frac{3000 + 9x + 0.05x^2}{x} \\
&= \frac{3000}{x} + 9 + 0.05x \\
U'(x) &= \frac{-3000}{x^2} + 0.05
\end{aligned}$$

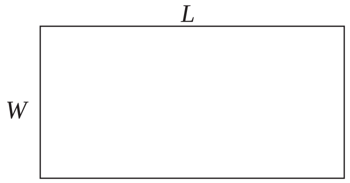
For max or min, let $U'(x) = 0$:

$$\begin{aligned}
0.05x^2 &= 3000 \\
x^2 &= 60\,000 \\
x &\doteq 244.9 \\
U(1) &= 3009.05 \\
U(244) &= 33.4950 \\
U(245) &= 33.4948 \\
U(300) &= 34.
\end{aligned}$$

Production level of 245 units will minimize the unit cost to \$33.49.

Exercise 5.5

1.



Let the length be L cm and the width be W cm.

$$\begin{aligned}
2(L + W) &= 100 \\
L + W &= 50 \\
L &= 50 - W \\
A &= L \bullet W \\
&= (50 - W)(W) \\
A(W) &= -W^2 + 50W \text{ for } 0 \le W \le 50 \\
A'(W) &= -2W + 50
\end{aligned}$$

Let $A'(W) = 0$:

$\therefore -2W + 50 = 0$

$W = 25$

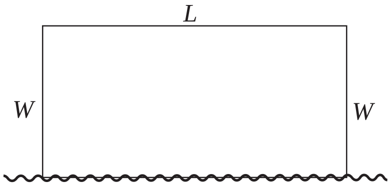
$A(0) = 0$

$$\begin{aligned}
A(25) &= 25 \times 25 \\
&= 625
\end{aligned}$$

$A(50) = 0.$

The largest area is 625 cm^2 and occurs when $W = 25$ cm and $L = 25$ cm.

3.



Let the length of L m and the width W m.

$$\begin{aligned}
2W + L &= 600 \\
L &= 600 - 2W \\
A &= L \bullet W \\
&= W(600 - 2W) \\
A(W) &= -2w^2 + 600W, 0 \le W \le 300 \\
A'(W) &= -4w + 600
\end{aligned}$$

For max or min, let $\frac{dA}{dW} = 0$:

$\therefore W = 50$

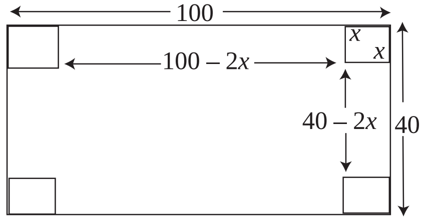
$A(0) = 0$

$$\begin{aligned}
A(150) &= -2(150)^2 + 600 \times 150 \\
&= 45\,000
\end{aligned}$$

$A(300) = 0.$

The largest area of 45 000 m^2 occurs when $W = 150$ m and $L = 300$ m.

4. Let dimensions of cut be x cm by x cm. Therefore, the height is x cm.



Length of the box is $100 - 2x$.
Width of the box is $40 - 2x$.
$V = (100 - 2x)(40 - 2x)(x)$ for domain $0 \leq x \leq 20$
Using Algorithm for Extreme Value,

$$\frac{dV}{dx} = (100 - 2x)(40 - 4x) + (40x - 2x^2)(-2)$$
$$= 4000 - 480x + 8x^2 - 80x + 4x^2$$
$$= 12x^2 - 560x + 4000$$

Set $\frac{dV}{dx} = 0$

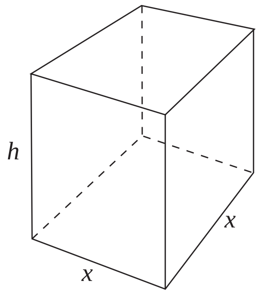
$3x^2 - 140x + 1000 = 0$

$$x = \frac{140 \pm \sqrt{7600}}{6}$$

$$x = \frac{140 \pm 128.8}{6}$$

$x = 8.8$ or $x = 37.9$
Reject $x = 37.9$ since $0 \leq x \leq 20$
When $x = 0$, $V = 0$
$x = 8.8$, $V = 28\ 850$ cm^2
$x = 20$, $V = 0$.
Therefore, the box has a height of 8.8 cm, a length of $100 - 2 \times 8.8 = 82.4$ cm, and a width of $40 - 3 \times 8.8 = 22.4$ cm.

5.



Let the base be x by x and the height be h
$x^2h = 1000$

$$\therefore h = \frac{1000}{x^2} \qquad (1)$$

Surface area $= 2x^2 + 4xh$
$$A = 2x^2 + 4xh \qquad (2)$$
$$= 2x^2 + 4x\left(\frac{1000}{x^2}\right)$$
$$= 2x^2 + \frac{4000}{x} \text{ for domain } 0 \leq x \leq 10\sqrt{2}$$

Using the max min Algorithm,

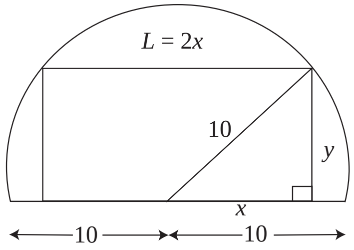
$$\frac{dA}{dx} = 4x - \frac{4000}{x^2} = 0$$

$x \neq 0$, $4x^3 = 4000$
$x^3 = 1000$
$x = 10$
$\therefore A = 200 + 400 = 600$ cm^2
Step 2: At $x \to 0$, $A \to \infty$

Step 3: At $x = 10\sqrt{10}$, $A = 2000 + \frac{4000}{10\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$
$$= 2000 + 40\sqrt{10}$$

Minimum area is 600 cm^2 when the base of the box is 10 cm by 10 cm and height is 10 cm.

6.



Let the length be $2x$ and the height be y. We know $x^2 + y^2 = 100$.
$\therefore y = \pm\sqrt{100 - x^2}$
Omit negative area $= 2xy$
$= 2x\sqrt{100 - x^2}$ for domain
$0 \leq x \leq 10$
Using the max min Algorithm,

$$\frac{dA}{dx} = 2\sqrt{100 - x^2} + 2y \bullet \frac{1}{2}(100 - x^2)^{-\frac{1}{2}}(-2x).$$

Let $\frac{dA}{dx} = 0$.

$$\therefore 2\sqrt{100-x^2} - \frac{2x^2}{\sqrt{100-x^2}} = 0$$

$$\therefore 2(100-x^2) - 2x^2 = 0$$

$$\therefore 100 = 2x^2$$

$$x^2 = 50$$

$x = 5\sqrt{2}$, $x > 0$. Thus, $y = 5\sqrt{2}$, $L = 10\sqrt{2}$

Part 2: If $x = 0$, $A = 0$

Part 3: If $x = 10$, $A = 0$

The largest area occurs when $x = 5\sqrt{2}$ and the area is

$10\sqrt{2}\sqrt{100 - 50}$

$= 10\sqrt{2}\sqrt{50}$

= 100 square units.

7. **a.** Let the radius be r cm and the height be h cm.

Then $\pi r^2 h = 1000$

$$h = \frac{1000}{\pi r^2}$$

Surface Area: $A = 2\pi r^2 + 2\pi rh$

$$= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{4},\ 0 \le r \le \infty$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

For max or min, let $\frac{dA}{dr} = 0$.

$$4\pi r - \frac{2000}{r^2} = 0$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \doteq 5.42$$

When $r = 0$, $A \to \infty$

$r = 5.42$ $A \doteq 660.8$

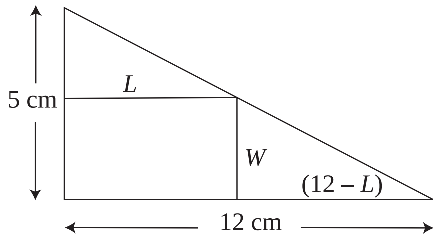
$r \to \infty$, $A \to \infty$

The minimum surface area is approximately 661 cm^3 when $r = 5.42$.

b. $r = 5.42$, $h = \frac{1000}{\pi(5.42)^2} \doteq 10.84$

$$\frac{h}{d} = \frac{10.84}{2 \times 5.42} = \frac{1}{1}$$

8. **a.**



Let the rectangle have length L cm on the 12 cm leg and width W cm on the 5 cm leg.

$A = LW$ (1)

By similar triangles, $\frac{12 - L}{12} = \frac{W}{5}$

$$\therefore 60 - 5L = 12W$$

$L = \frac{60 - 12W}{5}$ (2)

$A = \frac{(60 - 12W)W}{5}$ for domain $0 \le W \le 5$

Using the max min Algorithm,

$\frac{dA}{dW} = \frac{1}{5}[60 - 24W] = 0$, $W = \frac{60}{24} = 2.5$ cm.

When $W = 2.5$ cm, $A = \frac{(60 - 30) \times 2.5}{5} = 15$ cm^2.

Step 2: If $W = 0$, $A = 0$

Step 3: If $W = 5$, $A = 0$

The largest possible area is 15 cm^2 and occurs when $W = 2.5$ cm and $L = 6$ cm.

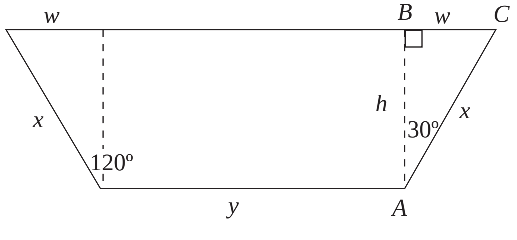
9. **a.** Let the base be y cm, each side x cm and the height h cm.

$2x + y = 60$

$y = 60 - 2x$

$$A = yh + 2 \times \frac{1}{2}(wh)$$

$$= yh + wh$$



From ΔABC

$$\frac{h}{x} = \cos 30°$$

$$h = x \cos 30°$$

$$= \frac{\sqrt{3}}{2}x$$

$$\frac{w}{x} = \sin 30°$$

$$w = x \sin 30°$$

$$= \frac{1}{2}x$$

Therefore, $A = (60 - 2x)\left(\frac{\sqrt{3}}{2}x\right) + \frac{x}{2} \times \frac{\sqrt{3}}{2}x$

$$A(x) = 30\sqrt{3}x - \sqrt{3}x^2 + \frac{\sqrt{3}}{4}x^2, 0 \le x \le 30$$

Apply the Algorithm for Extreme Values,

$$A'(x) = 30\sqrt{3} - 2\sqrt{3}x + \frac{\sqrt{3}}{2}x$$

Now, set $A'(x) = 0$

$$30\sqrt{3} - 2\sqrt{3}x + \frac{\sqrt{3}}{2}x = 0.$$

Divide by $\sqrt{3}$:

$$30 - 2x + \frac{x}{2} = 0$$

$$x = 20.$$

To find the largest area, substitute $x = 0$, 20, and 30.

$$A(0) = 0$$

$$A(20) = 30\sqrt{3}(20) - \sqrt{3}(20)^2 + \frac{\sqrt{3}}{4}(20)^2$$

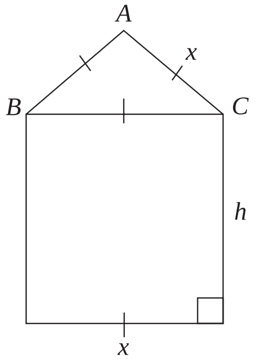
$$= 520$$

$$A(30) = 30\sqrt{3}(30) - \sqrt{3}(30)^2 + \frac{\sqrt{3}}{4}(30)^2$$

$$\doteq 390$$

The maximum area is 520 cm^2 when the base is 20 cm and each side is 20 cm.

10. a.



$$4x + 2h = 6$$

$$2x + h = 3 \text{ or } h = 3 - 2x$$

$$\text{Area} = xh + \frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x$$

$$= x(3 - 2x) + \frac{\sqrt{3}x^2}{4}$$

$$A(x) = 3x - 2x^2 + \frac{\sqrt{3}}{4}x^2$$

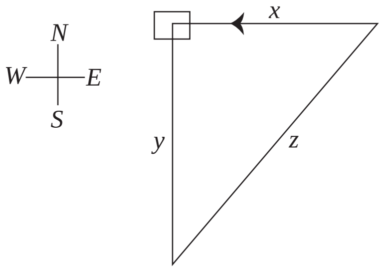
$$A'(x) = 3 - 4x + \frac{\sqrt{3}}{2}x, 0 \le x \le 1.5$$

For max or min, let $A'(x) = 0$, $x \doteq 1.04$.

$A(0) = 0$, $A(1.04) \doteq 1.43$, $A(1.5) \doteq 1.42$

The maximum area is approximately 1.43 cm^2 and occurs when $x = 0.96$ cm and $h = 1.09$ cm.

11.



Let z represent the distance between the two trains.

After t hours, $y = 60t$, $x = 45(1 - t)$

$$z^2 = 3600t^2 + 45^2(1 - t)^2, 0 \le t \le 1$$

$$2z\frac{dz}{dt} = 7200t - 4050(1 - t)$$

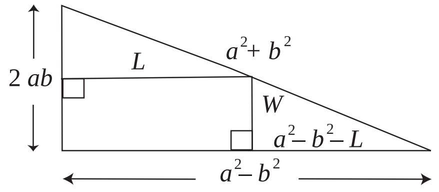
$$\frac{dz}{dt} = \frac{7200t - 4050(1 - t)}{2\sqrt{3600t^2 + 45^2(1 - t)^2}}$$

For max or min, let $\frac{dz}{dt} = 0$.

$\therefore\ 7200t - 4050(1 - t) = 0$
$t = 0.36$
When $t = 0$, $z^2 = 45^2$, $z = 45$
$t = 0.36$, $z^2 = 3600(0.36)^2 + 45^2(1 - 0.36)^2$
$z^2 = 129$
$z = 36$
$t = 1$, $z^2 = \sqrt{3600} = 60$

The closest distance between the trains is 36 km and occurs at 0.36 h after the first train left the station.

12.



$$\frac{a^2 - b^2 - L}{a^2 - b^2} = \frac{W}{2ab}$$

$$W = \frac{2ab}{a^2 - b^2}(a^2 - b^2 - L)$$

$$A = LW = \frac{2ab}{a^2 - b^2}[a^2L - b^2L - L^2]$$

Let $\frac{dA}{dL} = a^2 - b^2 - 2L = 0$,

$$L = \frac{a^2 - b^2}{2}$$

$$\text{and } W = \frac{2ab}{a^2 - b^2}\left[a^2 - b^2 - \frac{a^2 - b^2}{2}\right]$$
$$= ab.$$

The hypothesis is proven.

13. Let the height be h and the radius r.

Then, $\pi r^2 h = k$, $h = \frac{k}{\pi r^2}$.

Let M represent the amount of material,

$M = 2\pi r^2 + 2\pi rh$

$$= 2\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2k}{r},\ 0 \le r \le \infty$$

Using the max min Algorithm,

$$\frac{dM}{dr} = 4\pi r - \frac{2k}{r^2}$$

Let $\frac{dM}{dr} = 0$, $r^3 = \frac{k}{2\pi}$, $r \neq 0$ or $r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$.

When $r \to 0$, $M \to \infty$
$r \to \infty$, $M \to \infty$

$$r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$d = 2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$h = \frac{k}{\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}}} = \frac{k}{\pi} \bullet \frac{(2\pi)^{\frac{2}{3}}}{k^{\frac{2}{3}}} = \frac{k^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} \bullet 2^{\frac{2}{3}}$$

Min amount of material is $M = 2\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}} + 2k\left(\frac{2\pi}{k}\right)^{\frac{1}{3}}$.

$$\text{Ratio } \frac{h}{d} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \bullet 2^{\frac{2}{3}}}{2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \bullet 2^{\frac{2}{3}}}{2^{\frac{2}{3}}\left(\frac{k}{\pi}\right)^{\frac{1}{3}}} = \frac{1}{1}$$

14.

Cut the wire at P and label diagram as shown. Let AP form the circle and PB the square.

Then, $2\pi r = x$

$$r = \frac{x}{2\pi}$$

and the length of each side of the square is $\frac{100 - x}{4}$.

$$\text{Area of circle } = \pi\left(\frac{x}{2\pi}\right)^2$$
$$= \frac{x^2}{4\pi}$$

$$\text{Area of square } = \left(\frac{100 - x}{4}\right)^2$$

The total area is

$$A(x) = \frac{x^2}{4\pi} + \left(\frac{100 - x}{4}\right)^2,\text{ where } 0 \le x \le 100.$$

$$A'(x) = \frac{2x}{4\pi} + 2\left(\frac{100 - x}{4}\right)\left(-\frac{1}{4}\right)$$

$$= \frac{x}{2\pi} - \frac{100 - x}{8}$$

For max or min, let $A'(x) = 0$.

$$\frac{x}{2\pi} - \frac{100 - x}{8} = 0$$

$$x = \frac{100\pi}{r} + \pi \doteq 44$$

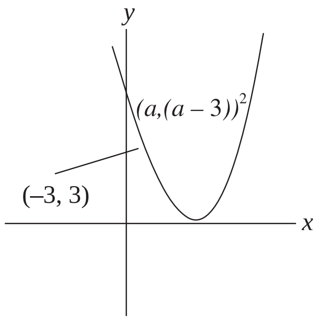
$$A(0) = 625$$

$$A(44) = \frac{44^2}{4\pi} + \left(\frac{100 - 44}{4}\right)^2 \doteq 350$$

$$A(100) = \frac{100^2}{4\pi} \doteq 796$$

The minimum area is 350 cm^2 when a piece of wire of approximately 44 cm is bent into a circle. The maximum area is 796 cm^2 and occurs when all of the wire is used to form a circle.

15.



Any point on the curve can be represented by $(a, (a - 3)^2)$.

The distance from (–3, 3) to a point on the curve is

$d = \sqrt{(a + 3)^2 + ((a - 3)^2 - 3)^2}$.

To minimize the distance, we consider the function

$d(a) = (a + 3)^2 + (a^2 - 6a + 6)^2$.

In minimizing $d(a)$, we minimize d since $d > 1$ always.

For critical points, set $d'(a) = 0$.

$d'(a) = 2(a + 3) + 2(a^2 - 6a + 6)(2a - 6)$

If $d'(a) = 0$,

$$a + 3 + (a^2 - 6a + 6)(2a - 6) = 0$$

$$2a^3 - 18a^2 + 49a - 33 = 0$$

$$(a - 1)(2a^2 - 16a + 33) = 0$$

$$a = 1 \text{ or } a = \frac{16 \pm \sqrt{-8}}{4}$$

There is only one critical value, $a = 1$.

To determine whether $a = 1$ gives a minimal value, we use the second derivative test:

$$d'(a) = 6a^2 - 36a + 49$$

$$d''(1) = 6 - 36 + 49$$

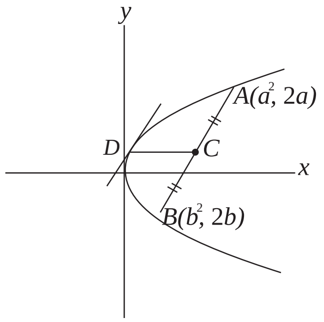
$$\geq 0.$$

Then, $d(1) = 4^2 + 1^2$

$= 17$.

The minimal distance is $d = \sqrt{17}$, and the point on the curve giving this result is (1, 4).

16.



Let the point A have coordinates $(a^2, 2a)$. (Note that the x-coordinate of any point on the curve is positive, but that the y-coordinate can be positive or negative. By letting the x-coordinate be a^2, we eliminate this concern.) Similarly, let B have coordinates $(b^2, 2b)$. The slope of AB is

$$\frac{2a - 2b}{a^2 - b^2} = \frac{2}{a + b}.$$

Using the mid-point property, C has coordinates

$$\left(\frac{a^2 + b^2}{2}, a + b\right).$$

Since CD is parallel to the x-axis, the y-coordinate of D is also $a + b$. The slope of the tangent at D is given by $\frac{dy}{dx}$ for the expression $y^2 = 4x$.

Differentiating,

$$2y\frac{dy}{dx} = 4$$

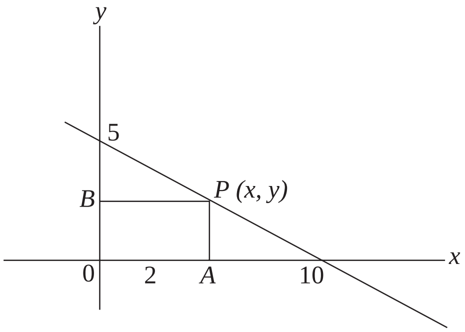
$$\frac{dy}{dx} = \frac{2}{y}$$

And since at point D, $y = a + b$,

$$\frac{dy}{dx} = \frac{2}{a + b}.$$

But this is the same as the slope of AB. Then, the tangent at D is parallel to the chord AB.

17.



Let the point $P(x, y)$ be on the line $x + 2y - 10 = 0$.

Area of $\Delta APB = xy$

$x + 2y = 10$ or $x = 10 - 2y$

$A(y) = (10 - 2y)y$

$= 10y - 2y^2, 0 \leq y \leq 5$

$A'(y) = 10.4y$

For max or min, let $A'(y) = 0$ or $10 - 4y = 0$, $y = 2.5$,

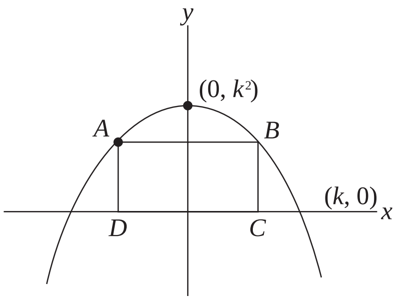
$A(0) = 0$

$A(2.5) = (10 - 5)(2.5) = 12.5$

$A(5) = 0$.

The largest area is 12.5 units squared and occurs when P is at the point (5, 2.5).

18.



A is $(-x, y)$ and $B(x, y)$

Area $= 2xy$ where $y = k^2 - x^2$

$A(x) = 2x(k^2 - x^2)$

$= 2k^2x - 2x^3, -k \leq x \leq k$

$A'(x) = 2k^2 - 6x^2$

For max or min, let $A'(x) = 0$,

$6x^2 = 2k^2$

$$x = \pm \frac{k}{\sqrt{3}}$$

When $x = \pm \dfrac{k}{\sqrt{3}}$, $y = k^2 - \left(\dfrac{k}{\sqrt{3}}\right)^2 = \dfrac{2}{3}k^2$

Max area is $A = \dfrac{2k}{\sqrt{3}} \times \dfrac{2}{3}k^2 = \dfrac{4k^3}{3\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{4k^3}{9}$ square units.

Exercise 5.6

1. **a.** $C(625) = 75(\sqrt{625} - 10)$

$= 1125$

Average cost is $\dfrac{1125}{625} = \$1.80$.

b. $C(x) = 75(\sqrt{x} - 10)$

$= 75\sqrt{x} - 750$

$$C'(x) = \frac{75}{2\sqrt{x}}$$

$$C'(1225) = \frac{75}{2\sqrt{1225}} = \$1.07$$

c. For a marginal cost of \$0.50/L,

$$\frac{75}{2\sqrt{x}} = 0.5$$

$75 = \sqrt{x}$

$x = 5625$

The amount of product is 5625 L.

3. $L(t) = \dfrac{6t}{t^2 + 2t + 1}$

a. $L'(t) = \dfrac{6(t^2 + 2t + 1) - 6t(2t + 2)}{(t^2 + 2t + 1)^2}$

$= \dfrac{-6t^2 + 6}{(t^2 + 2t + 1)^2}$

Let $L'(t) = 0$, then $-6t^2 + 6 = 0$,

$t^2 = 1$

$t = \pm 1$.

b. $L(1) = \dfrac{6}{1 + 2 + 1} = \dfrac{6}{4} = 1.5$

4. $C = 4000 + \dfrac{h}{15} + \dfrac{15\,000\,000}{h}$, $1000 \leq h \leq 20\,000$

$$\frac{dC}{dh} = \frac{1}{15} - \frac{15\,000\,000}{h^2}$$

Set $\dfrac{dC}{dh} = 0$, therefore, $\dfrac{1}{15} = \dfrac{15\,000\,000}{h^2} = 0$,

$h^2 = 225\,000\,000$

$h = 15\,000$, $h > 0$.

Using the max min Algorithm, $1000 \leq h \leq 20\,000$.

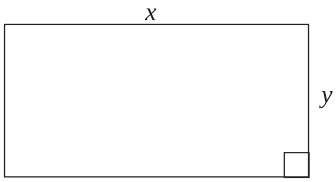
$$\text{When } h = 1000,\ C = 4000 + \frac{1000}{15} + \frac{15\,000\,000}{1000},$$
$$\doteq 19\,067.$$

$$\text{When } h = 15\,000,\ C = 4000 + \frac{15\,000}{15} + \frac{15\,000\,000}{15\,000},$$
$$= 6000.$$

When $h = 20\,000$, $C \doteq 6083$.

The minimum operating cost of \$6000/h occurs when the plane is flying at 15 000 m.

5.



Label diagram as shown and let the side of length x cost \$6/m and the side of length y be \$9/m.

Therefore, $(2x)(6) + (2y)(9) = 9000$

$$2x + 3y = 1500.$$

Area $A = xy$

But $y = \dfrac{1500 - 2x}{3}$.

$$\therefore\ A(x) = x\left(\frac{1500 - 2x}{3}\right)$$
$$= 500x - \frac{2}{3}x^2 \text{ for domain } 0 \le x \le 500$$

$$A'(x) = 500 - \frac{4}{3}x$$

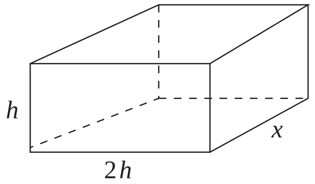
Let $A'(x) = 0$, $x = 375$.

Using max min Algorithm, $0 \le x \le 500$,

$$A(0) = 0,\ A(375) = 500(375) - \frac{2}{3}(375)^2$$
$$= 93\,750$$
$$A(500) = 0.$$

The largest area is 93 750 m^2 when the width is 250 m by 375 m.

6.



Label diagram as shown.

We know that $(x)(2h)(h) = 20\,000$

$$\text{or } h^2x = 10\,000$$
$$x = \frac{10\,000}{h^2}$$

Cost $C = 40(2hx) + 2xh(200) + 100(2)(2h^2 + xh)$

$$= 80xh + 400xh + 400h^2 + 200xh$$
$$= 680xh + 400h^2$$

Since $x = \dfrac{10\,000}{h^2}$,

$$C(h) = 680h\left(\frac{10\,000}{h^2}\right) + 400h^2,\ 0 \le h \le 100$$

$$C(h) = \frac{6\,800\,000}{h} + 400h^2$$

$$C'(h) = -\frac{6\,800\,000}{h^2} + 800h.$$

Let $C'(h) = 0$,

$$800h^3 = 6\,800\,000$$
$$h^3 = 8500$$
$$h \doteq 20.4.$$

Apply max min Algorithm,

As $h \to 0$ $C(0) \to \infty$

$$C(20.4) = \frac{6\,800\,000}{20.4} + 400(20.4)^2$$
$$= 499\,800$$
$$C(100) = 4\,063\,000.$$

Therefore, the minimum cost is about \$500 000.

7. Let the height of the cylinder be h cm, the radius r cm. Let the cost for the walls be $\$k$ and for the top $\$2k$.

$$V = 1000 = \pi r^2 h \text{ or } h = \frac{1000}{\pi r^2}$$

The cost $C = (2\pi r^2)(2k) + (2\pi rh)k$

$$\text{or } C = 4\pi kr^2 + 2\pi kr\left(\frac{1000}{\pi r^2}\right)$$

$$C(r) = 4\pi kr^2 + \frac{2000k}{r},\ r \geq\ 0$$

$$C'(r) = 8\pi kr - \frac{2000k}{r^2}$$

Let $C'(r) = 0$, then $8\pi kr = \dfrac{2000k}{r^2}$

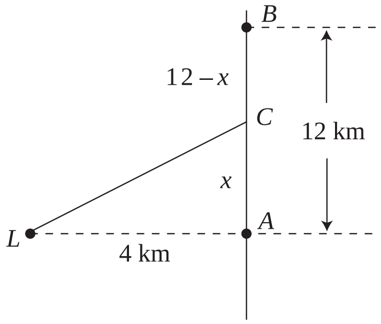
$$\text{or } r^3 = \frac{2000}{8\pi}$$

$$r \doteq 4.3$$

$$h = \frac{1000}{\pi(4.3)^2} = 17.2.$$

Since $r \geq\ 0$, minimum cost occurs when $r = 4.3$ cm and $h = 17.2$ cm.

8.



Let the distance AC be x km. Therefore, $CB = 12 - x$

$CL = \sqrt{16 + x^2}$.

Cost $C = 6000\sqrt{16 + x^2} + 2000(12 - x),\ 0 \leq\ x \leq\ 12$

$$\frac{dC}{dx} = \frac{1}{2} \times 6000 \times (16 + x^2)^{-\frac{1}{2}}(2x) + 2000(-1)$$

$$= \frac{6000x}{\sqrt{16 + x^2}} - 2000$$

Set $\dfrac{dC}{dx} = 0$

$$\frac{6000x}{\sqrt{16 + x^2}} = 2000$$

$$3x = \sqrt{16 + x^2}$$

$$9x^2 = 16 + x^2$$

$$x^2 = 2$$

$$x \doteq 1.4,\ x \geq\ 0$$

Using max min Algorithm:

when $x = 0$, $\quad C = 6000 \times 4 + 2000(12) = \$48\ 000$

$x = 1.4$, $C = 6000 \times \sqrt{16 + (1.4)^2} + 2000(12 - 1.4)$

$\doteq \$46\ 627$

$x = 12$, $C = 6000 \times \sqrt{16 + 12^2} \doteq 75\ 895.$

The minimum cost occurs when point C is 1.4 km from point A or about 10.6 km south of the power plant.

9. Let the number of fare changes be x. Now, ticket price is $\$20 + \$0.5x$. The number of passengers is $10\ 000 - 200x$.

The revenue $R(x) = (10\ 000 - 200x)(20 + 0.5x)$,

$R'(x) = -200(20 + 0.5x) + 0.5(1000 - 200x)$

$= -4000 - 100x + 5000 - 100x.$

Let $R'(x) = 0$:

$200x = 1000$

$x = 5.$

The new fare is $\$20 + \$0.5(5) = \$22.50$ and the maximum revenue is $\$202\ 500$.

10. Cost $C - \left(\dfrac{v^3}{2} + 216\right) \times t$

where $vt = 500$ or $t = \dfrac{500}{v}$.

$$C(v) = \left(\frac{v^3}{2} + 216\right)\left(\frac{500}{v}\right)$$

$$= 250v^2 + \frac{108\ 000}{v},\text{ where, } v \geq\ 0.$$

$$C'(v) = 500v - \frac{108\ 000}{v^2}$$

Let $C'(v) = 0$, then $500v = \dfrac{108\ 000}{v^2}$

$$v^3 = \frac{108\ 000}{500}$$

$$v^3 = 216$$

$$v = 6.$$

The most economical speed is 6 nautical miles/h.

11. Let the number of increases be n.

New speed = $110 + n$.

Fuel consumption = $(8 - 0.1n)$ km/L.

For a 450 km trip:

$$\text{fuel consumption} = \left(\frac{450}{8 - 0.1n}\right)\text{L},$$

$$\text{fuel cost} = \left(\frac{450}{8 - 0.1n}\right)0.68$$

$$\text{Time for Trip} = \frac{D}{v} = \left(\frac{450}{110} + n\right)\text{h}$$

Cost = Cost of driver + fixed cost + fuel

$$C(n) = 35\left(\frac{450}{110 + n}\right) + 15.50\left(\frac{450}{110 + n}\right) + \left(\frac{450}{8 - 0.1n}\right)0.68$$

$$C'(n) = \frac{-15\,750}{(110 + n)^2} - \frac{6975}{(110 + n)^2} + \frac{30.6}{(8 - 0.1n)^2}$$

Let $C'(n) = 0$:

$$\frac{30.6}{(8 - 0.1n)^2} = \frac{22\,725}{(110 + n)^2}$$

$$\frac{(110 + n)^2}{(8 - 0.1n)^2} = \frac{22\,725}{30.6}$$

$$\frac{110 + n}{8 - 0.1n} = \pm\sqrt{742.6} = \pm 27.3$$

$110 + n = 27.3(8 - 0.1n)$

$n \doteq 29$

or $110 + n = -27.3(8 - 0.1n)$

$n \doteq 190.$

For $r \doteq 29$, new speed = 139 km/h

$n \doteq 190$, new speed = 300 km/h, which is not possible.

The speed is 139 km/h.

12. a. Let the number of \$0.50 increases be n.

New price = $10 + 0.5n$.

Number sold = $200 - 7n$.

Revenue $R(n) = (10 + 0.5n)(200 - 7n)$

$= 2000 + 30n - 3.5n^2$

Profit $P(n) = R(n) - C(n)$

$= 2000 + 30n - 3.5n^2 - 6(200 - 7n)$

$= 800 + 72n - 3.5n^2$

$P'(n) = 72 - 7n$

Let $P'(n) = 0$,

$72 - 7n = 0$, $n \doteq 10$.

Price per cake = 10 + 5 = \$15

Number sold = 200 – 70 = 130

b. Since 200 – 165 = 35, it takes 5 price increases to reduce sales to 165 cakes.

New price is 10 + 0.5 × 5 = \$12.50.

c. If you increase the price, the number sold will decrease. Profit in situations like this will increase for several price increases and then it will decrease because too many customers stop buying.

13. $P(x) = R(x) - C(x)$

Marginal Revenue = $R'(x)$.

Marginal Cost = $C'(x)$.

Now $P'(x) = R'(x) - C'(x)$.

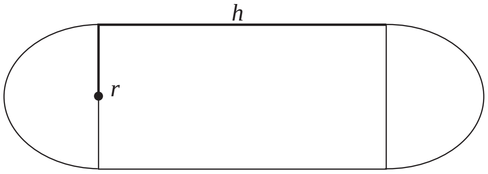
The critical point occurs when $P'(x) = 0$.

If $R'(x) = C'(x)$ then $P'(x) = R'(x) - R'(x)$

$= 0.$

Therefore, the profit function has a critical point when the marginal revenue equals the marginal cost.

14.



Label diagram as shown. Let cost of cylinder be \$k/m^3.

$V = 200$

$= \pi r^2 h + \frac{4}{3}\pi r^3$

Note: Surface Area = Total cost C

Cost $C = (2\pi rh)k + (4\pi r^2)2k$

But, $200 = \pi r^2 h + \frac{4}{8}\pi r^3$ or $600 = 3\pi r^2 h + 4\pi r^2$

Therefore, $h = \frac{600 - 4\pi r^3}{3\pi r^2}$.

$$C(r) = 2k\pi r\left(\frac{600 - 4\pi r^3}{3\pi r^2}\right)1 + 8k\pi r^2$$

$$= 2k\left(\frac{600 - 4\pi r^3}{3r}\right) + 8k\pi r^2$$

Since $h \le 16$, $r \le \left(\frac{600}{4\pi}\right)^{\frac{1}{3}}$ or $0 \le r \le 3.6$

$$C(r) = \frac{400k}{r} - \frac{8k\pi r^2}{3} + 3k\pi r^2$$

$$= \frac{400k}{r} + \frac{16k\pi r^2}{3}$$

$$C'(r) = -\frac{400k}{r^2} + \frac{32k\pi r}{3}$$

Let $C'(r) = 0$

$$\frac{400k}{r^2} = \frac{32k\pi r}{3}$$

$$\frac{50}{r^2} = \frac{4\pi r}{3}$$

$$4\pi r^3 = 150$$

$$r^3 = \frac{150}{4\pi}$$

$$r = 2.29$$

$$h \doteq 8.97 \text{ m}$$

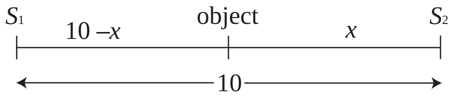
Note: $C(0) \rightarrow \infty$

$C(2.3) \doteq 262.5k$

$C(3.6) \doteq 330.6k$

The minimum cost occurs when $r = 230$ cm and h is about 900 cm.

15.



Note: $S_2 = 35_1$

Let x be the distance from S_2 to the object.

$I = \frac{ks}{x^2}$, where S is the strength of the source and x is the distance to the source.

$$I_1 = \frac{ks}{(10 - x)^2}$$

$$I_2 = \frac{k(35)}{x^2},\ 0 < x < 10$$

$$I = \frac{ks}{(10 - x)^2} + \frac{3ks}{x^2}$$

$$\frac{dI}{dx} = \frac{-2ks}{(10 - x)^3} - \frac{6ks}{x^3}$$

Let $\frac{dI}{dx} = 0$. Therefore, $\frac{2ks}{(10 - x)^3} = \frac{6ks}{x^3}$:

$$x^3 = 3(10 - x)^3$$

$$x = \sqrt[3]{3}(10 - x)$$

$$x \doteq 1.44(10 - x)$$

$$2.4x = 14.4$$

$$x \doteq 5.9.$$

Minimum illumination occurs when $x = 5.9$ m.

16. $v(r) = Ar^2(r_0 - r),\ 0 \le r \le r$

$$v(r) = Ar_0r^2 - Ar^3$$

$$v'(r) = 2Ar_0r - 3Ar^2$$

Let $v'(r) = 0$:

$$2Ar_0r - 3Ar^2 = 0$$

$$2r_0r - 3r^2 = 0$$

$$r(2r_0 - 3r) = 0$$

$$r = 0 \text{ or } r = \frac{2r_0}{3}.$$

$$v(0) = 0$$

$$v\left(\frac{2r_0}{3}\right) = A\left(\frac{4}{9}r_0^{\ 2}\right)\left(r_0 - \frac{2r_0}{3}\right)$$

$$= \frac{4}{27}r_0A$$

$$A(r_0) = 0$$

The maximum velocity of air occurs when radius is $\frac{2r_0}{3}$.

Review Exercise

1. **d.** $x^2y^{-3} + 3 = y$

$$2xy^{-3} - 3x^2y^{-4}\frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2xy^{-3}}{1 + 3x^2y^{-4}}$$

$$= \frac{\frac{2x}{y^3}}{1 + \frac{3x^2}{y^4}}$$

$$= \frac{\frac{2x}{y^3}}{\frac{y^4 + 3x^2}{y^4}}$$

$$= \frac{2xy}{3x^2y^4}$$

2. **b.** $(x^2 + y^2)^2 = 4x^2y$

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 8xy + 4x^2\frac{dy}{dx}$$

At (1, 1),

$$2(1 + 1)\left(2 + 2\frac{dy}{dx}\right) = 8 \times 1 \times 1 + 4(1)^2\frac{dy}{dx}$$

$$8 + 8\frac{dy}{dx} = 8 + 4\frac{dy}{dx}$$

$$\frac{dy}{dx} = 0.$$

3. $x^{-2}y^{6} + 2y^{-2} - 6 = 0$

$$-2x^{-3}y^{6} + 6x^{-2}y^{5}\frac{dy}{dx} - 4y^{-3}\frac{dy}{dx} = 0$$

At (0.5, 1):

$$-2(0.5)^{-3}(1)^{6} + 6(0.5)^{-2}(1)^{5}\frac{dy}{dx} - 4(1)^{-3}\frac{dy}{dx} = 0$$

$$-16 + 24\frac{dy}{dx} - 4\frac{dy}{dx} = 0$$

$$20\frac{dy}{dx} = 16$$

$$\frac{dy}{dx} = \frac{4}{5}.$$

At (0.5, –1):

$$-2(0.5)^{-3}(-1)^{6} + 6(0.5)^{-2}(-1)^{5}\frac{dy}{dx} - 4(-1)^{-3}\frac{dy}{dx} = 0$$

$$-16 - 24\frac{dy}{dx} + 4\frac{dy}{dx} = 0$$

$$20\frac{dy}{dx} = -16$$

$$\frac{dy}{dx} = \frac{-4}{5}.$$

6. $3x^{2} - y^{2} = 7$

$$6x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{6x}{2y}$$

$$= \frac{3x}{y}$$

$$\frac{d^{2}y}{dx^{2}} = \frac{3y - 3x\frac{dy}{dx}}{y^{2}}$$

$$= \frac{3y - 3x\left(\frac{3x}{y}\right)}{y^{2}}$$

$$= \frac{3y^{2} - 9x^{2}}{y^{2}}$$

But, $3y^{2} - 9x^{2} = -3(3x^{2} - y^{2})$

$= -3 \times 7$

$= -21.$

Therefore, $y'' = \dfrac{-21}{y^{2}}.$

7. $s(t)= t^{2} + (2t - 3)^{\frac{1}{2}}$

$$V = S'(t) = 2t + \frac{1}{2}(2t - 3)^{-\frac{1}{2}}(2)$$

$$= 2t + (2t - 3)^{-\frac{1}{2}}$$

$$a = S''(t) = 2 - \frac{1}{2}(2t - 3)^{-\frac{3}{2}}(2)$$

$$= 2 - (2t - 3)^{-\frac{3}{2}}$$

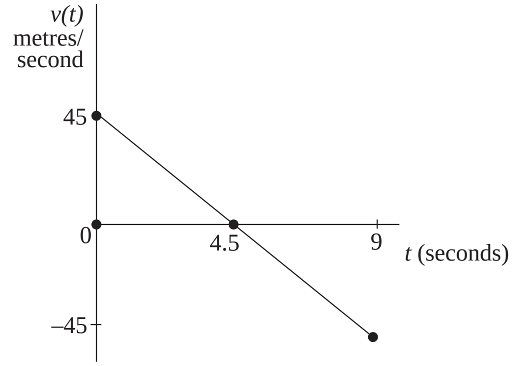
9. $s(t) = 45t - 5t^{2}$

$v(t) = 45 - 10t$

For $v(t) = 0$, $t = 4.5$.

t	$0 \leq t < 4.5$	4.5	$t > 4.5$
v(t)	+	0	–

Therefore, the upward velocity is positive for $0 \leq t < 4.5$ s, zero for $t = 4.5$ s, negative for $t > 4.5$ s.



10. a. $f(x) = 2x^{3} - 9x^{2}$

$f'(x) = 6x^{2} - 18x$

For max min, $f'(x) = 0$:

$6x(x - 3) = 0$

$x = 0$ or $x = 3$.

x	**$f(x) = 2x^{3} - 9x^{2}$**	
–2	–52	min
0	0	max
3	–27	
4	–16	

The minimum value is –52.
The maximum value is 0.

c. $f(x) = 2x + \dfrac{18}{x}$

$f'(x) = 2 - 18x^{-2}$

For max min, $f'(x) = 0$:

$$\frac{18}{x^2} = 2$$
$$x^2 = 9$$
$$x = \pm 3.$$

x	$f(x) = 2x + \dfrac{18}{x}$
1	20
3	12
5	$10 + \dfrac{18}{5} = 13.6$

The minimum value is 12.
The maximum value is 20.

11. $s(t) = 62 - 16t + t^2$

$v(t) = -16 + 2t$

a. $s(0) = 62$

Therefore, the front of the car was 62 m from the stop sign.

b. When $v = 0$, $t = 8$.

$$\begin{aligned}\therefore\ s(8) &= 62 - 16(8) + (8)^2\\ &= 62 - 128 + 64\\ &= -2\end{aligned}$$

Yes, the car goes 2 m beyond the stop sign before stopping.

c. Stop signs are located two or more metres from an intersection. Since the car only went 2 m beyond the stop sign, it is unlikely the car would hit another vehicle travelling perpendicular.

12. $y^3 - 3xy - 5 = 0$

$$3y^2\frac{dy}{dx} - 3y - \frac{dy}{dx}(3x) = 0$$

At $(2, -1)$:

$$3\frac{dy}{dx} + 3 - 6\frac{dy}{dx} = 0$$

$$1 = \frac{dy}{dx}.$$

Equation of tangent at $(2, -1)$ is

$$\frac{y+1}{x-2} = 1$$

$y = x - 3.$

Therefore, the equation of the tangent at $(2, -1)$ to $y^3 - 3xy - 5 = 0$ is $y = x - 3$.

13. $s(t) = 1 + 2t - \dfrac{8}{t^2 + 1}$

$v(t) = 2 + 8(t^2 + 1)^{-2}(2t) = 2 + \dfrac{16t}{(t^2 + 1)^2}$

$$\begin{aligned}a(t) &= 16(t^2 + 1)^{-2} + 16t(-2)(t^2 + 1)^{-3}\,2t\\ &= 16(t^2 + 1)^{-2} - 64t^2(t^2 + 1)^{-3}\\ &= 16(t^2 + 1)^{-3}[t^2 + 1 - 4t^2]\end{aligned}$$

For max min velocities, $a(t) = 0$:

$3t^2 = 1$

$t = \pm\dfrac{1}{\sqrt{6}}.$

t	$v(t) = 2 + \dfrac{16t}{(t^2 + 1)^2}$
0	2 min
$\dfrac{1}{\sqrt{3}}$	$2 + \dfrac{\frac{16}{\sqrt{3}}}{\left(\frac{1}{3} + 1\right)^2} = 2 + \dfrac{\frac{16\sqrt{3}}{3}}{\frac{16}{9}} = 2 + 3\sqrt{3}$ max
2	$2 + \dfrac{32}{25} = 3.28$

The minimum value is 2.
The maximum value is $2 + 3\sqrt{3}$.

14. $u(x) = 625x^{-1} + 15 + 0.01x$

$u'(x) = -625x^{-2} + 0.01$

For a minimum, $u'(x) = 0$

$$x^2 = 62\,500$$
$$x = 250$$

x	$u(x) = \dfrac{625}{x} + 0.01x$
1	625.01
250	2.5 + 2.5 = 5 min
500	$\dfrac{625}{500} + 5 = 6.25$

Therefore, 250 items should be manufactured to ensure unit waste is minimized.

15. **iii)** $C(x) = \sqrt{x} + 5000$

a. $C(400) = 20 + 5000$

$= \$5020$

b. $C(400) = \dfrac{5020}{400}$

$= \$12.55$

c. $C'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}$

$= \dfrac{1}{2\sqrt{x}}$

$C'(400) = \dfrac{1}{40}$

$= \$0.025$

$\doteq \$0.03$

$C'(401) = \dfrac{1}{2\sqrt{401}}$

$= \$0.025$

$\doteq \$0.03$

The cost to produce the 401st item is \$0.03.

iv) $C(x) = 100x^{-\frac{1}{2}} + 5x + 700$

a. $C(400) = \dfrac{100}{20} + 2000 + 700$

$= \$2705$

b. $C(400) = \dfrac{2750}{400}$

$= \$6.875$

$= \$6.88$

c. $C'(x) = -50x^{-\frac{3}{2}} + 5$

$C'(400) = \dfrac{-50}{(20)^3} + 5$

$= 5.00625$

$= \$5.01$

$C'(401) = \$5.01$

The cost to produce the 401st item is \$5.01.

16. $C(x) = 0.004x^2 + 40x + 16\,000$

Average cost of producing x items is

$C(x) = \dfrac{C(x)}{x}$

$C(x) = 0.004x + 40 + \dfrac{16\,000}{x}$

To find the minimum average cost, we solve

$$\begin{aligned} C'(x) &= 0 \\ 0.004 - \frac{16\,000}{x^2} &= 0 \\ 4x^2 - 16\,000\,000 &= 0 \\ x^2 &= 4\,000\,000 \\ x &= 2000,\ x > 0. \end{aligned}$$

From the graph, it can be seen that $x = 2000$ is a minimum. Therefore, a production level of 2000 items minimizes the average cost.

17. **b.** $s(t) = -t^3 + 4t^2 - 10$

$s(0) = -10$

Therefore, its starting position is at -10.

$s(3) = -27 + 36 - 10$

$= -1$

$v(t) = -3t^2 + 8t$

$v(3) = -27 + 24$

$= -3$

Since $s(3)$ and $v(3)$ are both negative, the object is moving away from the origin and towards its starting position.

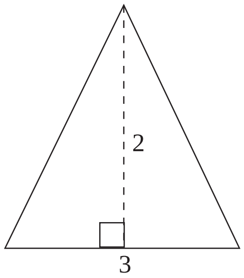
18. Given cone $v = \dfrac{1}{3}\pi r^2 h$

$\dfrac{dv}{dt} = 9\dfrac{m^3}{h}$

Slopes of sides $= \dfrac{2}{3}$

$\dfrac{rise}{run} = \dfrac{2}{3}$

$\therefore \dfrac{h}{r} = \dfrac{2}{3}$



a. Find $\frac{dh}{dt}$ when $r = 6$ m.

Solution

$$v = \frac{1}{3}\pi r^2 h$$

Using $\frac{h}{r} = \frac{2}{3}$

$$r = \frac{3}{2h}$$

$$\therefore\ r = \frac{1}{3}\pi\left(\frac{9h^2}{4}\right)h$$

$$v = \frac{3}{4}\pi h^3$$

$$\frac{dv}{dt} = \frac{9}{4}\pi h^2\ \frac{dh}{dt}$$

At a specific time, $r = 6$ m:

$$\frac{h}{6} = \frac{2}{3}$$

$$h = 4 \text{ m.}$$

$$9 = \frac{9}{4}\pi(4)^2\ \frac{dh}{dt}$$

$$9 = 36\pi\ \frac{dh}{dt}$$

$$\frac{1}{4\pi} = \frac{dh}{dt}$$

Therefore, the altitude is increasing at a rate of $\frac{1}{4\pi}$ m/h when $r = 6$ m.

b. Find $\frac{dr}{dt}$ when $h = 10$ m.

Solution

$$v = \frac{1}{3}\pi r^2 h$$

Using $\frac{h}{r} = \frac{2}{3}$,

$$h = \frac{2}{3}r$$

$$v = \frac{1}{3}\pi r^2\left(\frac{2}{3}r\right)$$

$$v = \frac{2}{9}\pi r^3$$

$$\frac{dv}{dt} = \frac{2}{3}\pi r^2\ \frac{dr}{dt}$$

At a specific time, $h = 10$ m,

$$\frac{10}{r} = \frac{2}{3}$$

$$r = 15 \text{ m}$$

$$9 = \frac{2}{3}\pi(15)^2\ \frac{dr}{dt}$$

$$9 = 150\pi\ \frac{dr}{dt}$$

$$\frac{3}{50\pi} = \frac{dr}{dt}.$$

Therefore, the radius is increasing at a rate of $\frac{3}{50\pi}$ m/h when $h = 10$ m.

19. Given $\frac{dv}{dt} = 1$ cm^3/s

Surface area = circular with $h = 0.5$ cm.

Volume is a cylinder.

Find $\frac{dA}{dt}$.

Solution

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\ \frac{dr}{dt}$$

$$v = \pi r^2 h$$

But $h = 0.5$ cm,

$$v = \frac{1}{2}\pi r^2$$

$$\frac{dv}{dt} = \pi r\ \frac{dr}{dt}$$

At a specific time,

$$1 = \pi r\ \frac{dr}{dt}$$

$$\frac{1}{\pi r} = \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\left(\frac{1}{\pi r}\right)$$

$$= 2.$$

Therefore, the top surface area is increasing at a rate of 2 cm^2/s.

20. Given cube $s = 6x^2$

$$v = x^3$$

$$\frac{ds}{dt} = 8 \text{ cm}^2\text{/s},$$

find $\frac{dv}{dt}$ when $s = 60$ cm^2.

Solution

$$v = x^2$$

$$\frac{dv}{dt} = 3x^2 \frac{dx}{dt}$$

At a specific time, $s = 60$ cm^2,

$$\therefore 6x^2 = 60$$

$$x^2 = 10.$$

Also, $s = 6x^2$.

$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

At a specific time, $x = \sqrt{10}$,

$$8 = 12\sqrt{10} \frac{dx}{dt}$$

$$\frac{2}{3\sqrt{10}} = \frac{dx}{dt}$$

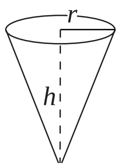
$$\frac{dv}{dt} = 3(10)\left(\frac{2}{3\sqrt{10}}\right)$$

$$= \frac{20}{\sqrt{10}}$$

$$= 2\sqrt{10}.$$

Therefore, the volume is increasing at a rate of $2\sqrt{10}$ cm^3/s.

21. Given cone



$$v = \frac{1}{3}\pi r^2 h$$

$$\frac{dv}{dt} = -10 \text{ cm}^3\text{/min}$$

$$\frac{dh}{dt} = -2 \text{ cm/min}$$

Find $\frac{h}{r}$ when $h = 8$ cm.

Solution

Let $\frac{h}{r} = k$, k is a constant $k \in R$

$$v = \frac{1}{3}\pi r^2 h \text{ and } r = \frac{h}{k}$$

$$\therefore v = \frac{1}{3}\pi \frac{h^2}{k^2} h$$

$$v = \frac{1}{3k^2} \pi h^3$$

$$\frac{dv}{dt} = \frac{1}{k^2} \pi h^2 \frac{dh}{dt}.$$

At a specific time, $h = 8$ cm,

$$-10 = \frac{1}{k^2} \pi(8)^2(-2)$$

$$-10 = \frac{-128\pi}{k^2}$$

$$k^2 = \frac{64\pi}{5}$$

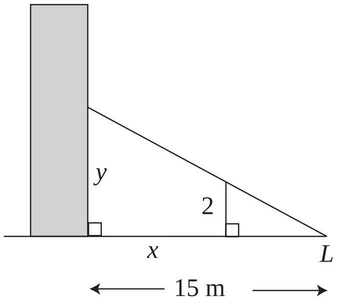
$$k = \frac{8\sqrt{\pi}}{\sqrt{5}}$$

$$\therefore \frac{h}{r} = 8\sqrt{\frac{\pi}{5}}$$

$$= \frac{8\sqrt{5\pi}}{5}.$$

Therefore, the ratio of the height to the radius is $8\sqrt{5\pi} : 5$.

22. Given



Since the angle formed from the light to the top of the man's head decreases as he walks towards the building, the length of his shadow on the building is decreasing.

Solution

Let x represent the distance he is from the wall. Therefore, $\frac{dx}{dt} = -2$, since he is walking towards the building. Let y be the length of his shadow on the building. Therefore, $\frac{dy}{dt}$ represents the rate of change of the length of his shadow.

Using similar triangles,

$$\frac{y}{15} = \frac{2}{15 - x}$$

$$15y - xy = 30$$

$$15y - 30 = xy$$

$$15\frac{dy}{dt} = \frac{dx}{dt}y + \frac{dy}{dt}x$$

At a specific time, $15 - x = 4$,

$\therefore x = 11$.

And using $\frac{y}{15} = \frac{2}{15 - x}$,

$$15\left(\frac{dy}{dt}\right) = (-2)(7.5) + \left(\frac{dy}{dt}\right)(11)$$

$$4\frac{dy}{dt} = -15$$

$$\frac{dy}{dt} = \frac{-15}{4}$$

Therefore, the length of his shadow is decreasing at a rate of 3.75 m/s.

23. $s = 27t^3 + \frac{16}{t} + 10, t > 0$

a. $v = 81t^2 - \frac{16}{t^2}$

$$81t^2 - \frac{16}{t^2} = 0$$

$$81t^4 = 16$$

$$t^4 = \frac{16}{81}$$

$$t = \pm\frac{2}{3}$$

$$t > 0$$

Therefore, $t = \frac{2}{3}$.

b.

t	$0 < t < \frac{2}{3}$	$t = \frac{2}{3}$	$t > \frac{2}{3}$
$\frac{ds}{dt}$	–	0	+

A minimum velocity occurs at $t = \frac{2}{3}$ or 0.67.

c. $a = \frac{dv}{dt} = 162t + \frac{32}{t^3}$

At $t = \frac{2}{3}$, $a = 162 \times \frac{2}{3} + \frac{32}{\frac{8}{27}}$

$= 216$

Since $a > 0$, the particle is accelerating.

24. Let the base be x cm by x cm and the height h cm.
Therefore, $x^2h = 10\ 000$.

$$A = x^2 + 4xh$$

But $h = \frac{10\ 000}{x^2}$,

$$A(x) = x^2 + 4x\left(\frac{10\ 000}{x^2}\right)$$

$$= x^2 + \frac{400\ 000}{x}, \text{ for } x \geq 5$$

$$A'(x) = 2x - \frac{400\ 000}{x^2},$$

Let $A'(x) = 0$, then $2x = \frac{400\ 000}{x^2}$

$$x^3 = 200\ 000$$

$$x = 27.14.$$

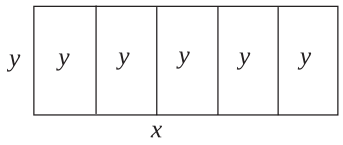
Using the max min Algorithm,

$A(5) = 25 + 80\ 000 = 80\ 025$

$A(27.14) \doteq 15\ 475$

The dimensions of a box of minimum area is 27.14 cm for the base and height 13.57 cm.

25. Let the length be x and the width y.



$$P = 2x + 6y \text{ and } xy = 12\ 000 \text{ or } y = \frac{12\ 000}{x}$$

$$P(x) = 2x + 6 \times \frac{12\ 000}{x}$$

$$P(x) = 2x + \frac{72\ 000}{x}, 10 \leq x \leq 1200\ (5 \times 240)$$

$$A'(x) = 2 - \frac{72\ 000}{x^2}$$

Let $A'(x) = 0$,

$$2x^2 = 72\ 000$$

$$x^2 = 36\ 000$$

$$x \doteq 190.$$

Using max min Algorithm,

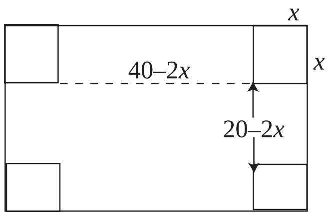
$A(10) = 20 + 7200 = 7220\text{ m}^2$

$A(190) \doteq 759\text{ m}^2$

$A(1200) = 1\ 440\ 060$

The dimensions for the minimum amount of fencing is a length of 190 m by a width of approximately 63 m.

26.



Let the width be w and the length $2w$.

Then, $2w^2 = 800$

$w^2 = 400$

$w = 20, w > 0.$

Let the corner cuts be x cm by x cm. The dimensions of the box are shown. The volume is

$V(x) = x(40 - 2x)(20 - 2x)$

$= 4x^3 - 120x^2 - 800x, 0 \leq \ x \leq \ 10$

$V'(x) = 12x^2 - 240x - 800$

Let $V'(x) = 0$:

$12x^2 - 240x - 800 = 0$

$3x^2 - 60x - 200 = 0$

$$x = \frac{60 \pm \sqrt{3600 - 2400}}{6}$$

$x \doteq 15.8$ or $x = 4.2$, but $x \leq \ 10$.

Using max min Algorithm,

$V(0) = 0$

$V(4.2) = 1540 \text{ cm}^2$

$V(10) = 0.$

Therefore, the base is

$40 - 2 \times 4.2 = 31.6$

by $20 - 2 \times 4.2 = 11.6$.

The dimensions are 31.6 dm^2, by 11.6 dm, by 4.2 dm.

27. Let the radius be r cm and the height h cm.

$V = \pi r^2 h = 500$

$A = 2\pi r^2 + 2\pi rh$

Since $h = \frac{500}{\pi r^2}$, $6 \leq \ h \leq \ 15$

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{1000}{r} \text{ for } 2 \leq \ r \leq \ 5$$

$$A'(r) = 4\pi r - \frac{1000}{r^2}.$$

Let $A'(r) = 0$, then $4\pi r^3 = 1000$,

$$r^3 = \frac{1000}{4\pi}$$

$r \doteq 4.3.$

Using max min Algorithm,

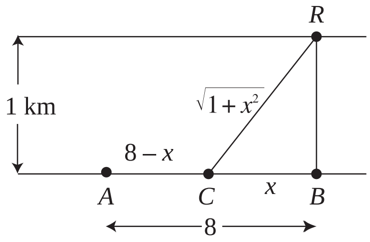
$A(2) \doteq 550$

$A(4.3) \doteq 349$

$A(5) \doteq 357$

For a minimum amount of material, the can should be constructed with a radius of 4.3 cm and a height of 8.6 cm.

28.



Let x be the distance CB, and $8 - x$ the distance AC. Let the cost on land be \$$k$ and under water \$$1.6k$.

The cost $C(x)k(8 - x) + 1.6k\sqrt{1 + x^2}, 0 \leq \ x \leq \ 8.$

$$C'(x) = -k + 1.6k \times \frac{1}{2}(1 + x^2)^{-\frac{1}{2}}(2x)$$

$$= -k + \frac{1.6kx}{\sqrt{1 + x^2}}$$

Let $C'(x) = 0$,

$$-k + \frac{1.6kx}{\sqrt{1 + k^2}} = 0$$

$$\frac{1.6x}{\sqrt{1 + k^2}} = 1$$

$1.6x = \sqrt{1 + x^2}$

$2.56x^2 = 1 + x^2$

$1.56x^2 = 1$

$x^2 \doteq 0.64$

$x = 0.8, x > 0.$

Using max min Algorithm,

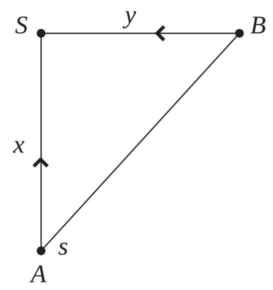
$A(0) = 9.6k$

$A(0.8) = k(8 - 0.8) + 1.6k\sqrt{1 + (0.8)^2} = 9.25k$

$A(8) = 12.9k$

The best way to cross the river is to run the pipe 8 – 0.8 or 7.2 km along the river shore and then cross diagonally to the refinery.

29.



Let y represent the distance the westbound train is from the station and x the distance of the northbound train from the station S. Let t represent time after 10:00.

Then $x = 100t$, $y = (120 - 120t)$

Let the distance AB be z.

$$z = \sqrt{(100t)^2 + (120 - 120t)^2},\ 0 \leq t \leq 1$$

$$\frac{dz}{dt} = \frac{1}{2}\left[(100t)^2 + (120 - 120t)^2\right]^{-\frac{1}{2}}$$

$$\left[2 \times 100 \times 100t - 2 \times 120 \times (120(1 - t)\right]$$

Let $\frac{dz}{dt} = 0$, that is

$$\frac{2 \times 100 \times 100t - 2 \times 120 \times 120(1 - t)}{2\sqrt{(100t)^2 + (120 - 120t)^2}} = 0$$

$$\begin{aligned} \text{or } 20\ 000t &= 28\ 800(1 - t) \\ 48\ 800t &= 288\ 000 \\ t &= \frac{288}{488} \doteq 0.59 \text{ h or } 35.4 \text{ min.} \end{aligned}$$

When $t = 0$, $z = 120$.

$t = 0.59$

$$\begin{aligned} z &= \sqrt{(100 \times 0.59)^2 + (120 - 120 \times 0.59)^2} \\ &= 76.8 \text{ km} \end{aligned}$$

$t = 1$, $z = 100$

The closest distance between trains is 76.8 km and occurs at 10:35.

30. Let the number of price increases be n.

New selling price $= 100 + 2n$.

Number sold $= 120 - n$.

$$\begin{aligned} \text{Profit} &= \text{Revenue} - \text{Cost} \\ P(n) &= (100 + 2n)(120 - n) - 70(120 - n),\ 0 \leq n \leq 120 \\ &= 3600 + 210n - 2n^2 \\ P'(n) &= 210 - 4n \end{aligned}$$

Let $P'(n) = 0$

$$\begin{aligned} 210 - 4n &= 0 \\ n &= 52.5. \end{aligned}$$

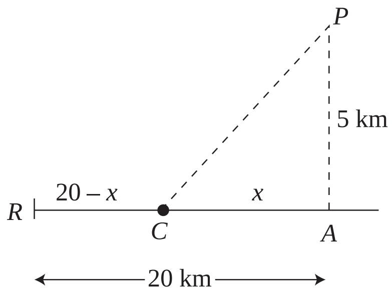
Therefore, $n = 52$ or 53.

Using max min Algorithm,

$$\begin{aligned} P(0) &= 3600 \\ P(52) &= 9112 \\ P(53) &= 9112 \\ P(120) &= 0 \end{aligned}$$

The maximum profit occurs when the CD players are sold at \$204 for 68 and at \$206 for 67 CD players.

31.



Let x represent the distance AC.

Then, $RC = 20 - x$ and 4.

$PC = \sqrt{25 + x^2}$

The cost:

$$C(x) = 100\ 000\sqrt{25 + x^2} + 75\ 000(20 - x),\ 0 \leq x \leq 20$$

$$C'(x) = 100\ 000 \times \frac{1}{2}\left(25 + x^2\right)^{-\frac{1}{2}}(2x) - 75\ 000.$$

Let $C'(x) = 0$,

$$\frac{100\ 000x}{\sqrt{25 + x^2}} - 75\ 000 = 0$$

$$\begin{aligned} 4x &= 3\sqrt{25 + x^2} \\ 16x^2 &= 9(25 + x^2) \\ 7x^2 &= 225 \\ x^2 &\doteq 32 \\ x &\doteq 5.7. \end{aligned}$$

Using max min Algorithm,

$$\begin{aligned} A(0) &= 100\ 000\sqrt{25} + 75\ 000\ (20) = 2\ 000\ 000 \\ A(5.7) &= 100\ 000\sqrt{25 + 5.7^2} + 75\ 000(20 - 5.7) \\ &= 1\ 830\ 721.60 \\ A(20) &= 2\ 061\ 552.81. \end{aligned}$$

The minimum cost is \$1 830 722 and occurs when the pipeline meets the shore at a point C, 5.7 km from point A, directly across from P.

Chapter 5 Test

1. $x^2 + 4xy - y^2 = 8$

$$2x + 4y + 4x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$x + 2y + \frac{dy}{dx}(2x - y) = 0$$

$$\frac{dy}{dx} = \frac{-x - 2y}{2x - y}$$

$$\frac{dy}{dx} = \frac{x + 2y}{y - 2x}$$

2. $3x^2 + 4y^2 = 7$

$$6x + 8y\frac{dy}{dx} = 0$$

At $P(-1, 1)$,

$$-6 + 8\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3}{4}.$$

Equation of tangent line at $P(-1, 1)$ is

$$\frac{y - 1}{x + 1} = \frac{3}{4}$$

$$3x + 3 = 4y - 4$$

$$3x - 4y + 7 = 0.$$

3. **a.** Average velocity from $t = 1$ to $t = 6$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(6) - s(1)}{6 - 1}$$

$$= \frac{(216 - 324 + 144 + 5) - (1 - 9 + 24 + 5)}{5}$$

$$= 4 \text{ m/s}.$$

The average velocity from $t = 1$ to $t = 6$ is 4 m/s.

b. Object is at rest when $v = 0$:

$$0 = 3t - 18t + 24$$
$$= 3(t^2 - 6t + 8)$$
$$= 3(t - 4)(t - 2)$$
$$t = 2 \text{ or } t = 4.$$

Therefore, the object is at rest at 2 s and 4 s.

c. $v(t) = 3t^2 - 18t + 24$

$a(t) = 6t - 18$

$a(5) = 30 - 18$

$= 12$

Therefore, the acceleration after 5 s is 12 m/s^2.

d. $s(3) = 27 - 81 + 72 + 5$

$= 23$

$v(3) = 27 - 54 + 24$

$= -3$

Since the signs of $s(3)$ and $v(3)$ are different, the object is moving towards the origin.

4. Given: circle $\frac{dr}{dt} = 2$ m/s,

find $\frac{dA}{dt}$ when $r = 60$.

Solution

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

At a specific time, $r = 60$,

$$\frac{dA}{dt} = 2\pi(60)(2)$$
$$= 240\pi.$$

Therefore, the area is increasing at a rate of 240π m^2/s.

5. Given: sphere $\frac{dr}{dt} = 2$ m/min,

find $\frac{dv}{dt}$ when $r = 8$ m.

Solution

$$v = \frac{4}{3}\pi r^3$$

$$\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}.$$

At a specific time, $r = 8$ m:

$$\frac{dv}{dt} = 4\pi(64)(2)$$
$$= 512\pi.$$

a. Therefore the volume is increasing at a rate of 512π m^3/min.

b. The radius is increasing, therefore the volume is also increasing. Answers may vary.

6. Given: cube $V = x^3$, $S = 6x^2$, $\frac{dV}{dt} = 2$ cm^3/min,

find $\frac{dS}{dt}$ when $x = 5$.

Solution

$$S = 6x^2$$

$$\frac{dS}{dt} = 12x\,\frac{dx}{dt}$$

At a specific time, $x = 5$:

$$\frac{dS}{dt} = 12(5)\left(\frac{2}{75}\right)$$

$$= \frac{8}{5}$$

$$V = x^3$$

$$\frac{dV}{dt} = 3x^2\,\frac{dx}{dt}$$

When $x = 5$, $\frac{dV}{dt} = 2$

$$2 = 3(5)^2\,\frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{2}{75}$$

$$S = 6x^2$$

$$\frac{dS}{dt} = 12x\,\frac{dx}{dt}$$

When $x = 5$, $\frac{dx}{dt} = \frac{2}{75}$.

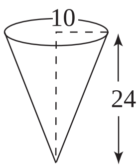
Therefore, $\frac{dS}{dt} = 12(5)\left(\frac{2}{75}\right)$

$$= \frac{8}{5}$$

$$= 1.6.$$

Therefore, the surface area of the cube is increasing at a rate of 1.6 cm^2/min.

7.



$$\frac{dv}{dt} = 20 \text{ m}^3\text{/min}$$

Find $\frac{dh}{dt}$ when $h = 16$.

Solution

$$v = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{h} = \frac{10}{24} = \frac{5}{12}$$

$$v = \frac{5}{12}h.$$

Substituting into v,

$$v = \frac{1}{3}\pi\left(\frac{25}{144}h^2\right)h$$

$$v = \frac{25}{432}\,\pi h^3.$$

$$\frac{dv}{dt} = \frac{25}{144}\,\pi h^2\,\frac{dh}{dt}$$

At a specific time, $h = 16$:

$$20 = \frac{25}{144}\,\pi(16)^2\,\frac{dh}{dt}$$

$$20 = \frac{6400\pi}{144}\,\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{20\pi}.$$

Therefore, the depth of the water is increasing at a rate of $\frac{9}{20\pi}$ m/min.

8. $f(x) = \frac{x^2 - 1}{x + 2}$

$$f'(x) = \frac{2x(x+2)–(x^2-1)(1)}{(x+2)^2}$$

$$= \frac{x^2 + 4x + 1}{(x+2)^2}$$

For max min, $f'(x) = 0$:

$$x^2 + 4x + 1 = 0$$

$$x = \frac{-4 \pm\sqrt{16-4}}{2}$$

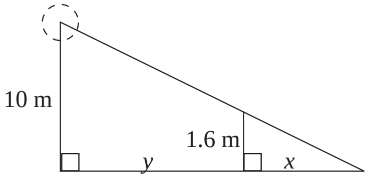
$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm\sqrt{3}.$$

x	$f(x) = \frac{x^2 - 2}{x + 2}$
-1	0
$-2 + \sqrt{3}$	$\frac{4 - 4\sqrt{3} + 3 - 1}{\sqrt{3}} = \frac{6 - 4\sqrt{3}}{\sqrt{3}} = 2\sqrt{3} - 4$ $\doteq -0.536$ min
3	$\frac{8}{5} = 1.6$ max $\sqrt{3}$

Therefore, the minimum value is $(2\sqrt{3} - 4)$ and the maximum value is 1.6.

9.



Find $\frac{dx}{dt}$ when $y = 8$ m.

Let x represent the distance the tip of her shadow is from the point directly beneath the spotlight.

Let y represent the distance she is from the point directly beneath the spotlight.

$\frac{dy}{dx} = 6$ m/s

Solution

$$\frac{x}{x - y} = \frac{10}{1.6} = 6.25$$

$$x = 6.25x - 6.25y$$

$$6.25y = 5.25x$$

$$6.25\frac{dy}{dt} = 5.25\frac{dx}{dt}$$

At a specific time, $y = 8$ m:

$$6.25(6) = 5.25\frac{dx}{dt}$$

$$7.1 \doteq \frac{dx}{dt} \quad \text{or} \quad = 21\frac{dx}{dt}.$$

At a specific time,

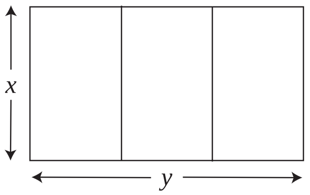
$$25(6) = 2\frac{dx}{dt}$$

$$\frac{150}{21} = \frac{dx}{dt}$$

$$\frac{50}{7} = \frac{dx}{dt}.$$

Therefore, her shadow's head is moving away at a rate of 7.1 m/s.

10.



Let x represent the width of the field in m, $x > 0$.

Let y represent the length of the field in m.

$4x + 2y = 2000$ (1)

$A = xy$ (2)

From (1): $y = 1000 - 2x$. Restriction $0 < x < 500$

Substitute into (2):

$$A(x) = x(1000 - 2x)$$

$$= 1000x - 2x^2$$

$$A'(x) = 1000 - 4x.$$

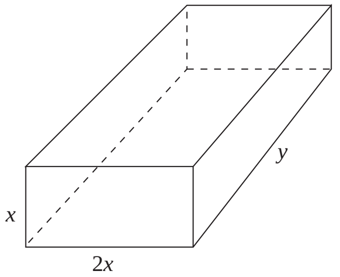
For a max min, $A'(x) = 0$, $x = 250$

x	$A(x) = x(1000 - 2x)$
0	$\lim_{x \to 0^+} A(x) = 0$
250	$A(250) = 125\,000$ max
1000	$\lim_{x \to 1000} A(x) = 0$

$\therefore$ $x = 250$ and $y = 500$.

Therefore, each paddock is 250 m in width and $\frac{500}{3}$ m in length.

11.



Let x represent the height.
Let $2x$ represent the width.
Let y represent the length.
Volume $10\ 000 = 2x^2y$
Cost:

$$\begin{aligned}C &= 0.02(2x)y + 2(0.05)(2x^2) + 2(0.05)(xy) + 0.1(2xy)\\ &= 0.04xy + 0.2x^2 + 0.1xy + 0.2\,xy\\ &= 0.34xy + 0.2x^2\end{aligned}$$

But $y = \dfrac{10\ 000}{2x^2} = \dfrac{5000}{x^2}$.

$$\begin{aligned}\text{Therefore, } C(x) &= 0.34x\left(\frac{5000}{x^2}\right) + 0.2x^2\\ &= \frac{1700}{x} + 0.2x^2,\ x \geq 0\\ C'(x) &= \frac{-1700}{x^2} + 0.4x.\end{aligned}$$

Let $C'(x) = 0$:

$$\begin{aligned}\frac{-1700}{x^2} + 0.4x &= 0\\ 0.4x^3 &= 1700\\ x^3 &= 4250\\ x &\doteq 16.2.\end{aligned}$$

Using max min Algorithm,
$C(0)\rightarrow\infty$

$$C(16.2) = \frac{1700}{16.2} + 0.2(16.2)^2 = 157.4.$$

Minimum when $x = 16.2$, $2x = 32.4$ and $y = 19.0$.
The required dimensions are 162 m, 324 m by 190 m.

Chapter 6 • Exponential Functions

Review of Prerequisite Skills

3. d. $\dfrac{2^{-1}+2^{-2}}{3^{-1}}$

$$= \frac{\frac{1}{2}+\frac{1}{4}}{\frac{1}{3}}$$

$$= \frac{12}{12}\left[\frac{\frac{1}{2}+\frac{1}{4}}{\frac{1}{3}}\right]$$

$$= \frac{6+3}{4}$$

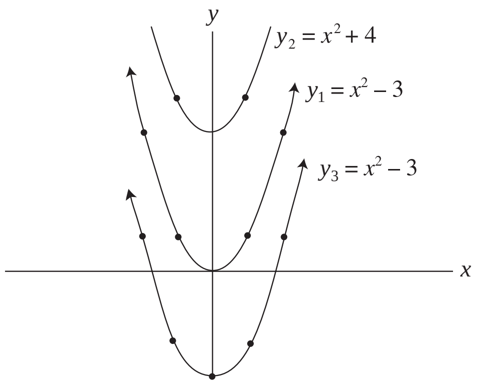
$$= \frac{9}{4}$$

5. b. (i) y_1 transforms to y_2 by a vertical shift upwards of four units.

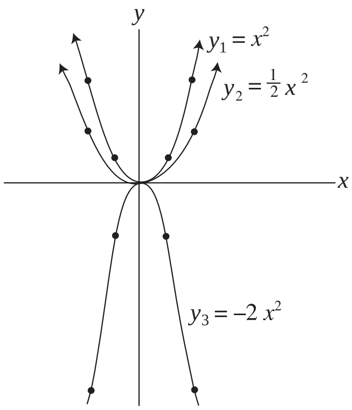
(ii) y_1 transforms to y_3 by a vertical shift downwards of three units. These are called translations.

c. The graph of $y = x^2 - 2$, shifted vertically upwards four units, becomes the graph of $y = x^2 + 2$.

d. When a positive or negative constant is added to a function, it results in a vertical shift of the graph of the function. For a positive constant, the shift is upwards that many units and for a negative constant, the shift is downward that many units.



6. a.



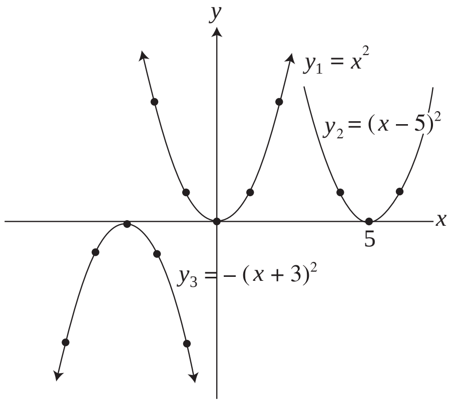
b. (i) The graph of y_1 is vertically compressed by one-half to form the graph of y_2.

(ii) The graph of y_1 is stretched vertically by two and it is reflected in the x-axis.

c. The graph of $y = 3x^2 + 25$ is the vertical stretch by three of the graph of $y = x^2$ and is shifted upwards 25 units.

d. For the function $y = c \bullet f(x)$ where c is a constant, the function is a transformation of $y = f(x)$.
If $c < 0$, the graph of the function is reflected in the x-axis.
If $0 < c < 1$, the graph of the function is compressed by a factor of c.
If $c > 1$, the graph of the function is stretched by a factor of c.

7. a.



b. (i) The graph of y_1 is shifted to the right five units to the graph of y_2.

(ii) The graph of y_1 is shifted to the left three units and reflected in the x-axis.

c. The graph of $y = (x + 6)^2 - 7$ is the lateral shift of six units to the left and seven units vertically downwards.

d. When a positive or negative constant is added to the independent variable in a function, there is a lateral shift or translation. If the number is positive, it causes a shift to the left; if the number is negative it causes a shift to the right. In order to keep the same *y*-value, if the number is negative, the *x*-value must be increased to compensate, or decreased if the number is positive.

Exercise 6.1

1. **a.** $(7^3)^2 \div 7^4$
$= 7^6 \div 7^4$
$= 7^{6-4}$
$= 7^2$
$= 49$

b. $(0.4)^5 \div (0.4)^3$
$= (0.4)^2$
$= 1.6$

c. $(\sqrt{3})^5 \times (\sqrt{3})^3$
$= (\sqrt{3})^8$
$= (3^{\frac{1}{2}})^8$
$= 3^4$
$= 81$

d. $25^{\frac{3}{2}}$
$= 5^3$
$= 125$

e. $(-8)^{\frac{2}{3}}$
$= \left(\sqrt[3]{-8}\right)^2$
$= (-2)^2$
$= 4$

f. $(-2)^3 \times (-2)^3$
$= [(-2)(-2)]^3$
$= 4^3$
$= 64$

g. $4^{-2} - 8^{-1}$
$= \frac{1}{4^2} - \frac{1}{8}$
$= \frac{1}{16} - \frac{2}{16}$
$= -\frac{1}{16}$

i. $(0.3)^3 \div (0.3)^5$
$= (0.3)^{-2}$
$= \left(\frac{3}{10}\right)^{-2}$
$= \left(\frac{10}{3}\right)^2$
$= \frac{100}{9}$

k. $(3^2)^3 \div 3^{-2}$
$= 3^6 \div 3^{-2}$
$= 3^{6+2}$
$= 3^8$
$= 6561$

o. $(6^3)^4 \div 12^6$
$= \frac{6^{12}}{12^6}$
$= \frac{6^{12}}{6^6 \times 2^6}$
$= \frac{6^6}{2^6}$
$= 3^6$
$= 729$

2. **g.** $\frac{5x^3y^{-4}}{2x^{-2}y^2}$
$= \frac{5x^{3+2}}{2y^{2+4}}$
$= \frac{5x^5}{2y^6}$

h. $\frac{\pi x^2 y}{4xy^3}$
$= \frac{\pi x^{2-1}}{4y^{3-1}}$
$= \frac{\pi x y^{-2}}{4}$
$= \frac{\pi x}{4y^2}$

k. $(a^2b^{-1})^{-3}$
$= \frac{1}{(a^2b^{-1})^3}$ or $= a^{-6}b^3$
$= \frac{1}{a^6b^{-3}}$ or $= \frac{b^3}{a^6}$
$= \frac{b^3}{a^6}$

l. $(ab)^4\left(\frac{a^{-2}}{b^{-2}}\right)^2$

$= a^4b^4\left(\frac{a^{-4}}{b^{-4}}\right)$

$= a^0b^8$

$= b^8$

3. **b.** $\left(a^{\frac{1}{4}}b^{-\frac{1}{3}}\right)^{-2}$

$= a^{-\frac{1}{2}}b^{\frac{2}{3}}$

$= \frac{b^{\frac{2}{3}}}{a^{\frac{1}{2}}}$

$= \frac{\sqrt[3]{b^2}}{\sqrt{a}}$

d. $\frac{\left(4x^2y^{\frac{1}{3}}\right)^{\frac{1}{2}}}{\left(8xy^{\frac{1}{4}}\right)^{\frac{1}{3}}}$

$= \frac{2xy^{\frac{1}{6}}}{2x^{\frac{1}{3}}y^{\frac{1}{12}}}$

$= x^{1-\frac{1}{3}}y^{\frac{1}{6}-\frac{1}{12}}$

$= x^{\frac{2}{3}}y^{\frac{1}{2}}$

e. $\frac{(4a^{-2})(2a^3b^2)}{12a^4b^3}$

$= \frac{8a^1b^2}{12a^4b^3}$

$= \frac{2a^{-3}b^{-1}}{3}$

$= \frac{2}{3a^3b}$

f. $\frac{(5x^{-2}y^0)^3}{(25x^2y)^{\frac{1}{2}}}$

$= \frac{5^3x^{-6}y^0}{5xy^{\frac{1}{2}}}$

$= \frac{5^2x^{-7}}{y^{\frac{1}{2}}}$

$= \frac{25}{x^7\sqrt{y}}$

4. **d.** $\sqrt{2}a^{\frac{1}{2}} \times \sqrt{32}a^{\frac{3}{4}}$

$= \sqrt{64}a^{\frac{1}{2}+\frac{2}{4}}$

$= 8a^{\frac{5}{4}}$ or $= 8\sqrt[4]{a^5}$

h. $\sqrt[3]{5^2} \div \sqrt[4]{5^5}$

$= 5^{\frac{2}{3}} \div 5^{\frac{5}{4}}$

$= 5^{\frac{2}{3}-\frac{5}{4}}$

$= 5^{\frac{8}{12}-\frac{15}{12}}$

$= 5^{-\frac{7}{12}}$

i. $(\sqrt[3]{t})^2 \times \sqrt{t^5}$

$= t^{\frac{2}{3}} \times t^{\frac{5}{2}}$

$= t^{\frac{4}{6}} \times t^{\frac{15}{6}}$

$= t^{\frac{4+15}{6}}$

$= t^{\frac{19}{6}}$

5. **a.** $\frac{3^{-1}+3^{-2}}{3^{-3}}$

$= \left[\frac{3^{-1}+3^{-2}}{3^{-3}}\right] \times \frac{3^3}{3^3}$ or $= \frac{\frac{1}{3}+\frac{1}{3^2}}{\frac{1}{3^3}}$

$= \frac{3^2+3}{3^0}$ or $= \frac{\frac{1}{3}+\frac{1}{9}}{\frac{1}{27}}$

$= 9 + 3$

$= 12$

$= \left[\frac{\frac{1}{3}+\frac{1}{9}}{\frac{1}{27}}\right] \times \frac{27}{27}$

$= \frac{9+3}{1}$

$= 12$

c. $\dfrac{(p^2q + pq^3)^3}{p^3q^4}$

$= \dfrac{\left[pq(p + q^2)\right]^3}{p^3q^4}$

$= \dfrac{p^3q^3(p + q^2)^3}{p^3q^4}$

$= \dfrac{(p + q^2)^3}{q}$

d. $\dfrac{x^{-2} - x^{-3}}{2x}$

$= \dfrac{x^{-3}(x - 1)}{2x}$

$= \dfrac{x - 1}{2x^{3+1}}$

$= \dfrac{x - 1}{2x^4}$

e. $\dfrac{3t - 2t^{-1}}{t^3}$

$= \left[\dfrac{3t - 2t^{-1}}{t^3}\right] \times \dfrac{t}{t}$

$= \dfrac{3t^2 - 2}{t^4}$

f. $\dfrac{3p^2 - p^{-3}}{p^4} \times \dfrac{p^3}{p^3} = \dfrac{3p^5 - 1}{p^7}$

6. **a.** $\dfrac{x^{\frac{3}{2}} - x^{\frac{1}{2}} - x^{-1}}{x^{-\frac{1}{2}}}$

$= x^{\frac{1}{2}}\left[x^{\frac{3}{2}} - x^{\frac{1}{2}} - x^{-1}\right]$

$= x^2 - x - x^{-\frac{1}{2}}$

b. $\dfrac{4 - \sqrt{x}}{x^{\frac{3}{2}}}$

$= \dfrac{(4 - x^{\frac{1}{2}})}{(x^{\frac{3}{2}})} \times \dfrac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

$= \dfrac{4x^{\frac{1}{2}} - x}{x^{\frac{4}{2}}}$

$= \dfrac{4\sqrt{x} - x}{x^2}$

c. $\dfrac{x - 9}{x^{\frac{1}{2}} - 3}$

$= \dfrac{(\sqrt{x} - 3)(\sqrt{x} + 3)}{(\sqrt{x} - 3)}$

$= \sqrt{x} + 3$

d. $\dfrac{x - 1}{\sqrt{x} - x}$

$= \dfrac{(\sqrt{x} + 1)(\sqrt{x} - 1)}{\sqrt{x}(1 - \sqrt{x})}$

$= \dfrac{(\sqrt{x} + 1)(\sqrt{x} - 1)}{-\sqrt{x}(\sqrt{x} - 1)}$

$= -\dfrac{\sqrt{x} + 1}{\sqrt{x}}$

7. $64^{\frac{1}{6}} = 8^{\frac{1}{3}}$

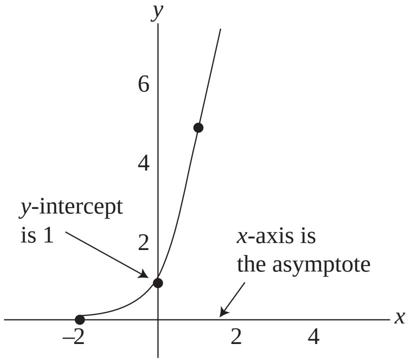
64 cannot be expressed as a power of 8, that is 8^2. But $(8^2)^{\frac{1}{6}}$ is the power of a power and we then keep the base and multiply the exponents to get $8^{2 \times \frac{1}{6}}$ or $8^{\frac{1}{3}}$.

Section 6.2

Investigation:

1. **e.** No, it only approaches the x-axis, even for very large negative values for x.

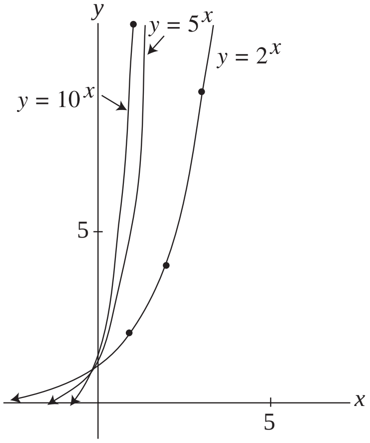
2.



Domain: $x \in R$

Range: $y > 0, y \in R$

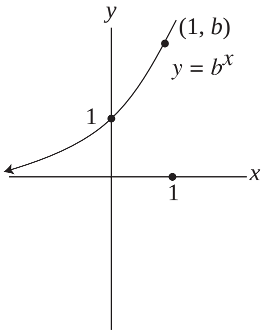
3.



a. The curves all have the same y-intercept of 1, they have the same domain, $x \in R$, and the same range, $y > 0, y \in R$. As well, the curves show functions that are increasing.

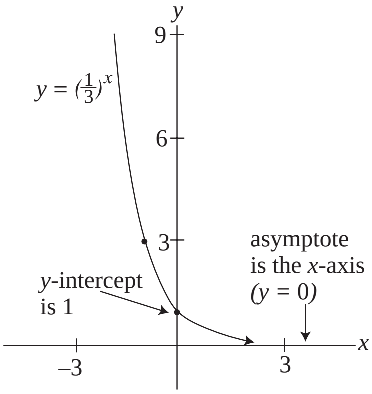
b. The curve of $y = 7^x$ will lie between the curves of $y = 5^x$ and $y = 10^x$, having the point (0, 1) in common with them.

4.



The graph of $y = b^x$ increases from left to right. It has a y-intercept of 0 and no x-intercept, that is it has a horizontal asymptote of $y = 0$. The domain is the set of real numbers and the range is all values of y, greater than zero, i.e., only positive values. The graph appears in the first and second quadrants only.

5.

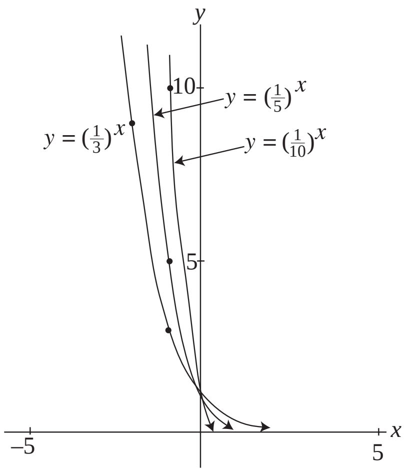


Domain: $x \in R$

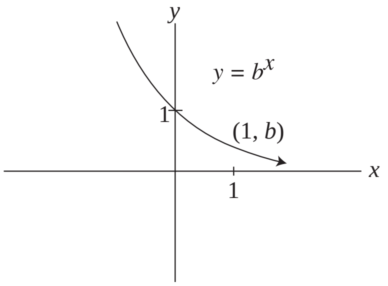
Range: $y > 0, y \in R$

6. **a.** These curves have the same y-intercept of 1, and the same asymptote, $y = 0$. Also, all curves are descending from the second quadrant to first quadrant.

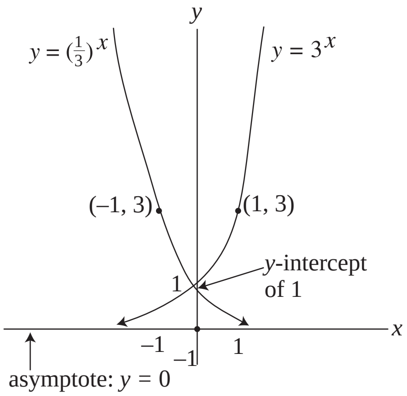
b. The graph of $y = \left(\frac{1}{7}\right)^x$ will be between $y = \left(\frac{1}{5}\right)^x$ and $y = \left(\frac{1}{10}\right)^x$.



7. The curve $y = b^x$ where $0 < b < 1$ is a decreasing curve with y-intercept of 1, an asymptote of $y = 0$, and moves from second quadrant to first quadrant only. It has a domain of $x \in R$, and a range of $y > 0$, $y \in$ R.



8.



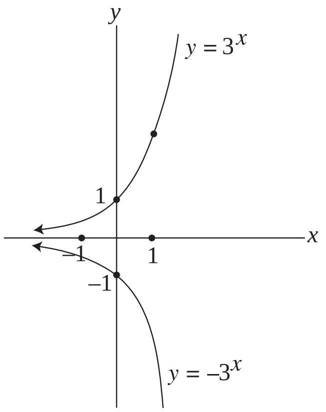
a. If $y = 3^x$ is reflected in the y-axis, its image is

$$y = \left(\frac{1}{3}\right)^x.$$

b. The curves $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$ share the same y-intercept of 1, and the asymptote of $y = 0$. Also, both curves exist only above the x-axis, changing at the same rate.

c. The curves differ in that $y = 3^x$ is increasing, whereas $y = \left(\frac{1}{3}\right)^x$ is decreasing.

9.



a. The reflection of $y = 3^x$ in the x-axis gives the graph of $y = –3^x$ as its image.

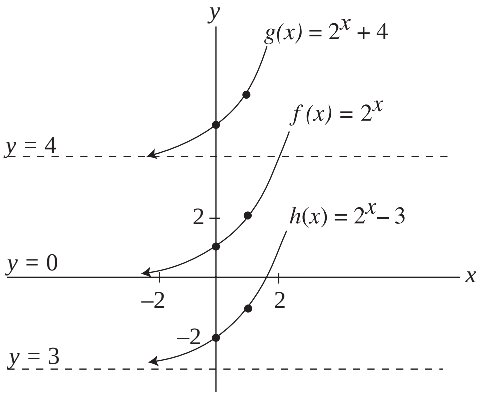
b. The graphs have the same asymptote and the same shape.

c. The curves have different y-intercepts; the graph of $y = 3^x$ has a y-intercept of 1; the graph of $y = \left(\frac{1}{3}\right)^x$ has a y-intercept of –1. The graph of $y = 3^x$ exists in the first and second quadrants only. The graph of $y = \left(\frac{1}{3}\right)^x$ exists in the third and fourth quadrants only.

Section 6.3

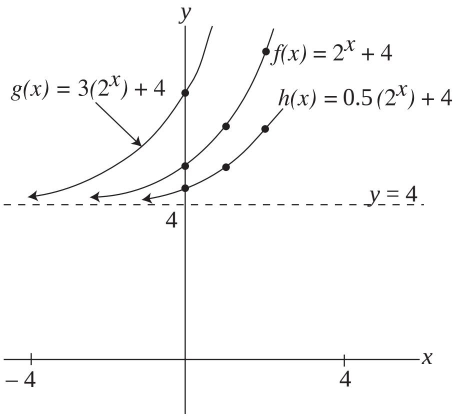
Investigation:

1. **a.**



c. The horizontal asymptote also moves that many units. If c is positive, it moves up c units. If c is negative, it moves down c units. The asymptote for $y = ab^x + c$ is $y = c$.

2. **a.**

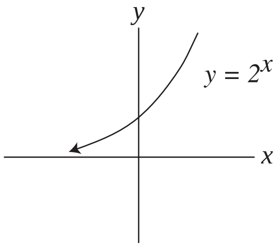


b. $f(x)$ is transformed to $g(x)$ by a dilation of 3. $f(x)$ is transformed to $h(x)$ by a dilation of 0.5. If the function is multiplied by a positive number that is greater than 1, it results in a "stretch," if it is between 0 and 1, it results in a "compression."

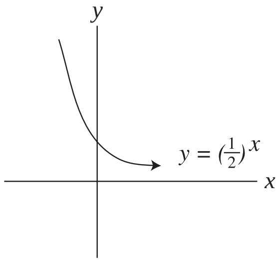
c. The asymptote remains the same, $y = c$; in this case $y = 4$. Since $ab^x > 0$ for any value of $b > 0$, then the function is always greater than 4.

Exercise 6.3

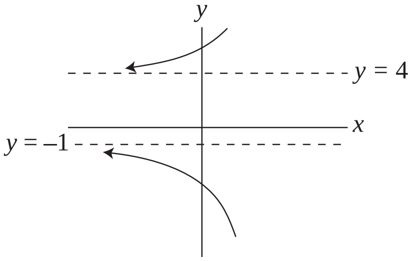
4. For $y = ab^x + c$, $b > 0$, note that the independent variable, x, is in the exponent. So, the graph will be always increasing if $b > 0$ and decreasing if $0 < b < 1$. For example if $y = 2^x$, the graph is



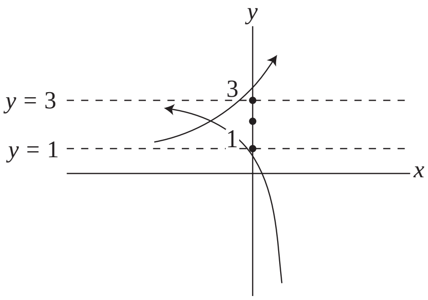
Whereas if $y = \left(\frac{1}{2}\right)^x$, the graph is



Since for $b > 0$, $y = ab^x + c$ will either be greater than c (if $a > 0$) or less than c (if $a < 0$), making $y = c$ the equation of the asymptote. For example, for $y = 3(2^x) + 4$ and $y = -3(2^x) - 1$:



The y-intercept is found by $x = 0$, so the y-intercept for $y = ab^x + c$ is $y = ab^0 + c = a + c$. For example, for $y = 2.3^x + 1$, the y-intercept is $2 + 1$ or 3, and for $y = -2(4)^x + 3$, the y-intercept is $-2(4^0) + 3 = -2 + 3 = 1$.



Exercise 6.4

3. **a.** $P = 5000(1.07)^t$

b. **(i)** To find the population in 3 years, we let $t = 3$:

$$P = 5000(1.07)^3 \doteq 6125.$$

The population will be approximately 6125 in 3 years.

(ii) To find the population in 15 years, $t = 15$:

$P = 5000(1.07)^{15} \doteq 13\,795.$

The population will be approximately 13 795 in 15 years.

c. For the population to double, it must reach 2(5000) or 10 000.

Let $P = 10\,000$

$10\,000 = 5000(1.07)^t$

$2 = 1.07^t$

By trial and error, we find

$1.07^{10} \doteq 1.97$

$1.07^{10.5} \doteq 2.03$

so $t \doteq 10.25$.

The population will double in $10\frac{1}{4}$ years.

5. *Solution 1*

Depreciation of 15% is equivalent to a value of 85%. The value of the car is given by

$V_1 = V_0(0.85)^t$ where t is the time in years. Since the value now is \$4200 and five years have passed, then

$4200 = V_0(0.85)^5$

or $V_0 = \dfrac{4200}{0.85^5}$

$= 9466.$

The car was originally worth approximately \$9500.

Solution 2

A depreciation of 15% is equivalent to a value of (1 – 0.15) or 0.85. If we consider today's value of \$4200 and the time of original value to be five years ago, then

$V = 4200(0.85)^{-5}$

$= 9466.$

The car was worth approximately \$9500, five years ago.

6. A decline in value of 8.3% is equivalent to a value of 100% – 8.3% = 91.7%.

$\therefore$ the value of the dollar is $V = 1(0.917)^t$, where t is the time in years.

$V_5 = 1(0.917)^5$

$\doteq 0.65$

Five years later the Canadian dollar has a purchasing value of \$0.65.

7. For a normal pancreas, the secretion rate is 4% per minute. So, the amount of dye remaining is a rate of 100% – 4% = 96%. The amount of dye left is $A = 0.50\ (0.96)^t$, where t is the time in minutes. After 20 minutes, the amount of dye remaining is

$A = 0.5(0.96)^{20}$

or $A \doteq 0.22$ g.

8. Let the present population be P_0, and doubling time be five days. The population function can be expressed as $P = P_0(2)^{\frac{t}{5}}$, where t is the time in days.

a. For a population 16 times as large,

$$P = 16P_0$$
$$16P_0 = P_0(2)^{\frac{t}{5}}$$
$$16 = 2^{\frac{t}{5}}$$
$$2^4 = 2^{\frac{t}{5}}$$
$$\frac{t}{5} = 4$$
$$t = 20$$

In 20 years, the population will be 16 times larger.

b. For a population $\frac{1}{2}$ of its present size,

$$P = \frac{1}{2}P_0$$
$$\frac{1}{2}P_0 = P_0(2)^{\frac{t}{5}}$$
$$\frac{1}{2} = 2^{\frac{t}{5}}$$
$$2^{-1} = 2^{\frac{t}{5}}$$
$$\frac{t}{5} = -1$$
$$t = -5$$

Five years ago, the population was $\frac{1}{2}$ of its present size.

c. For a population $\frac{1}{4}$ of its present size,

$$P = \frac{1}{4}P_0$$

$$\frac{1}{4}P_0 = P_0(2)^{\frac{t}{5}}$$

$$\frac{1}{4} = 2^{\frac{t}{5}}$$

$$2^{-2} = 2^{\frac{t}{5}}$$

$$\frac{t}{5} = -2$$

$$t = -10.$$

Ten years ago, the population was $\frac{1}{4}$ of its present size.

d. For a population $\frac{1}{32}$ of its present size,

$$\text{let } P = \frac{1}{32}P_0$$

$$\frac{1}{32}P_0 = P_0(2)^{\frac{t}{5}}$$

$$\frac{1}{32} = 2^{\frac{t}{5}}$$

$$2^{-5} = 2^{\frac{t}{5}}$$

$$-5 = \frac{t}{5}$$

$$t = -25.$$

Twenty-five years ago, the population was $\frac{1}{32}$ of its present size.

9. **a.** Due to inflation, cost can be expressed as $C = C_0(1.02)^t$, where t is the time in years and C_0 is the cost today.

10. If an element decays at a rate of 12% per hour, it leaves only 88% per hour. The amount that remains can be given as $A = 100(0.88)^t$, where t is the time in hours.

a. $A = 100(0.88)^t$, where t is the time in hours.

$$A_{10} = 100(0.88)^{10}$$
$$\doteq 27.85$$

There is approximately 28 g left after 10 h.

b. $A_{30} = 100(0.88)^{30}$

$$\doteq 2.16$$

There is approximately 2 g left after 30 h.

c. $40 = 100(0.88)^t$

$$0.4 = 0.88^t$$
$$t \doteq 7.17$$

After about 7 hours there is 40 grams left.

11. Since the sodium has a half-life, the base for the half-life is $\frac{1}{2}$. After t hours, the amount of radioactive sodium is given by $A = 160\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where t is the time in hours, and h is the half-life in hours.

a. $20 = 160\left(\frac{1}{2}\right)^{\frac{45}{h}}$

$$0.125 = 0.5^{\frac{45}{h}}$$

$$\frac{45}{h} = 3$$

$$h = \frac{45}{3}$$

$$= 15$$

The half-life of Na^{24} is 15 h.

b. $A = 160\left(\frac{1}{2}\right)^{\frac{t}{15}}$

c. $100 = A_0\left(\frac{1}{2}\right)^{\frac{12}{15}}$

$$100 = A_0(0.5)^{0.8}$$

$$A_0 = 100(0.5)^{-0.8}$$

$$= 174$$

The assistant must make 174 mg.

d. $A = 20(0.5)^{\frac{12}{15}}$

$\doteq 11.5$

After another 12 hours, the 20 mg will be reduced to only 11.5 mg of Na^{24}.

12. The number of bacteria can be expressed as $N = N_0(1.15)^t$, where t is the time in hours.

a. For the colony to double in size, $N = 2N_0$.

$\therefore 2N_0 = N_0(1.15)^t$

$2 = 1.15^t$

$t = 5$

In five years, the colony will double in size.

b. In 10 hours,

$1.3 \times 10^3 = N_0(1.15)^{10}$

$$N_0 = \frac{1.3 \times 10^3}{1.15^{10}}$$

$\doteq 321.$

There were approximately 320 bacteria initially.

14. Assuming that the population of a city grows consistently, the population can be expressed as $P = P_0(1 + r)^t$, or for this city $P = 125\ 000(1 + r)^t$, where r is the rate of growth and t is the time years from 1930. If the population was 500 000 in 1998,

$500\ 000 = 125\ 000(1 + r)^{1998-1930}$

$4 = (1 + r)^{68}$

$1 + r = \sqrt[68]{4}$

$1 + r = 1.02$

$r = 0.02059591.$

So, the population grows at 2% per year.

a. The population in 2020 is

$P = 125\ 000(1.02)^{2020-1930}$

$= 125\ 000(1.02)^{90}$

$\doteq 783\ 000.$

In 2020, the population will be approximately 783 000.

b. For one million population,

$1\ 000\ 000 = 125\ 000(1.02)^t$

$8 = 1.02^t$

$t \doteq 105$

The population will reach one million in 105 years from 1930, or in the year 2035.

15. Assuming a constant inflation rate of 3% per year, the cost of a season's ticket is $C = 900(1.03)^6$ $\doteq 1074.65$. The father should put aside about \$1075. Alternately, the father should invest the \$900 in a secure account, which over six years should earn enough interest to compensate for the cost of living.

16. For virus A:

$P_A = P_0(3)^{\frac{t}{8}}$, where t is time in hours

$P_A = 1000(3)^{\frac{24}{8}}$

$= 1000.3^3$

$= 27\ 000$

For virus B:

$P_B = P_0(2)^{\frac{t}{4.8}}$

$P_B = 1000(2)^{\frac{24}{4.8}}$

$= 1000.2^5$

$= 32\ 000$

The virus B culture has more after 24 h.

17. Answers may vary.

Exercise 6.5

1. **d.** $y = 996.987(1.143)^x$, where x is the number of time intervals in hours

For 10 000,

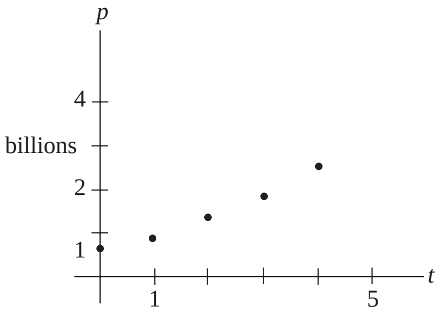
$10\ 000 = 996.987(1.143)^x$

$10.030 \doteq 1.143^x$

$x \doteq 17.25$

There will be 10 000 bacteria in 17 h and 15 min.

2. **a.**



b. In the year 2050, the time interval will be

$$\frac{2050 - 1750}{50} = 6$$

$$\begin{aligned} y &= 0.66(1.462)^6 \\ &\doteq 6.445. \end{aligned}$$

The world population could reach 6.45 billion in the year 2050 if the growth pattern continues.

c. For a population of seven billion people,

$$\begin{aligned} \frac{7}{1.06} &= 0.66(1.462)^x \\ &= 1.462^x \\ x &\doteq 6.22. \end{aligned}$$

But the time interval is 50 years, so in 6.22(50) or 311 years from 1860. The population is expected to reach seven billion in the year 2161.

3. a. Using the $\boxed{\text{ExpReg}}$ function on the graphing calculator, we find the curve of best fit is $y = 283.843(1.032)^x$, where y is CO_2 concentration in parts/million, and x is the time interval of 20 years.

b. For 1930, 3.5 intervals of 20 years have passed.

$$\begin{aligned} \therefore\ y &= 283.843(1.032)^{3.5} \\ &\doteq 317 \end{aligned}$$

For 1990, 6.5 time intervals have passed.

$$\begin{aligned} \therefore\ y &= 283.843(1.032)^{6.5} \\ &\doteq 348 \end{aligned}$$

The estimated concentration of carbon dioxide was 317 parts per million in 1930 and 348 parts per million in 1990.

c. Let $y = 390$

$$\begin{aligned} 390 &= 283.843(1.032)^x \\ 1.374 &\doteq 1.032^x \\ x &\doteq 10.1 \end{aligned}$$

If the trend continues, concentration will reach 390 parts per million in 1860 + 10.1(20), or in 2062.

4. a. The curve of best fit is $y = 9.277(2.539)^x$, where y is the amount of stored nuclear waste in million curies and t is the number of time intervals of five years.

b. In 1983, 13 years have passed, or $\frac{13}{5} = 2.6$ time intervals.

$$\begin{aligned} y &= 9.277(2.539)^{2.6} \\ &\doteq 105 \end{aligned}$$

The amount of waste stored in 1983 is about 105 million curies.

c. Let $y = 800$.

$$\begin{aligned} 800 &= 9.277(2.539)^x \\ 86.2 &\doteq 2.539^x \\ x &\doteq 4.78 \end{aligned}$$

But x is in five-year intervals. So, in 4.78×5, or about 24 years (in 1970 + 24 or 1994), the amount of waste would reach 800 million curies. However, this contradicts the evidence that the amount of waste is only 600 million in 1995. Therefore, by extrapolation of the scatter plot, we see that waste will reach 800 million curies in 1997.

6. In order to predict the mathematical model that best suits the data, we can graph the data to get a sense of the curve of best fit, or we can take first, second, or third differences to investigate a polynomial function. If we use a graphing calculator, we may use the regression functions and then test our data to see if it is appropriate to our situation.

Review Exercise

1. a. $(3^{-2} + 2^{-3})^{-1}$

$$\begin{aligned} &= \left(\frac{1}{9} + \frac{1}{8}\right)^{-1} \\ &= \left(\frac{8}{72} + \frac{9}{72}\right)^{-1} \\ &= \left(\frac{17}{72}\right)^{-1} \\ &= \frac{72}{17} \end{aligned}$$

b. $\dfrac{3^{-3}}{3^{-1} - 3^{-2}}$

$$= \frac{3^{-3}}{3^{-1} - 3^{-2}} \times \frac{3^3}{3^3} \qquad \text{or} \qquad \frac{\frac{1}{27}}{\frac{1}{3} - \frac{1}{9}}$$

$$= \frac{3^0}{3^2 - 3} \qquad\qquad = \frac{1}{9 - 3}$$

$$= \frac{1}{9 - 3} \qquad\qquad = \frac{1}{6}$$

$$= \frac{1}{6}$$

c. $\dfrac{3^{-8}}{3^{-6} \times 3^{-5}}$

$$= \frac{3^{-8}}{3^{-11}} \qquad \text{or} \qquad = 3^{-8+6+5}$$

$$= 3^3 \qquad\qquad = 3^3$$

$$= 27 \qquad\qquad = 27$$

2. **b.** $\left(\dfrac{54}{250}\right)^{\frac{2}{3}}$

$$= \left(\frac{27}{125}\right)^{\frac{2}{3}}$$

$$= \left[\frac{3^3}{5^3}\right]^{\frac{2}{3}}$$

$$= \frac{3^2}{5^2}$$

$$= \frac{9}{25}$$

c. $\sqrt[4]{\dfrac{1}{16}}$

$$= \frac{1}{2}$$

3. **b.** $\sqrt[3]{\dfrac{x^{\frac{1}{3}}\sqrt{x}}{\sqrt[3]{x^2}}}$

$$= \left(\frac{x^{\frac{1}{3}} \bullet x^{\frac{1}{2}}}{x^{\frac{2}{3}}}\right)^{\frac{1}{3}}$$

$$= \left(\frac{x^{\frac{5}{6}}}{x^{\frac{4}{6}}}\right)^{\frac{1}{3}}$$

$$= (x^{\frac{1}{6}})^{\frac{1}{3}}$$

$$= x^{\frac{1}{18}}$$

d. $(16^{p+q})(8^{p-q})$

$$= (2 \times 8)^{p+q}(8)^{p-q} \qquad \text{or} \qquad = (2^4)^{p+q}(2^3)^{p-q}$$

$$= 2^{p+q} \times 8^{p+q} \times (8)^{p-q} \qquad\qquad = 2^{4p+4q+3p-3q}$$

$$= 2^{p+q} \times 8^{2p} \qquad\qquad = 2^{7p+q}$$

$$= 2^{p+q} \times (2^3)^{2p}$$

$$= 2^{p+q+6p}$$

$$= 2^{7p+q}$$

4. **a.** $1 + 8x^{-1} + 15x^{-2}$

$$= (1 + 5x^{-1})(1 + 3x^{-1}) \qquad \text{or} \qquad = \left(1 + \frac{5}{x}\right)\left(1 + \frac{3}{x}\right)$$

b. $x^{\frac{1}{2}} - x^{\frac{5}{2}}$

$$= x^{\frac{1}{2}}(1 - x^{\frac{4}{2}}) \qquad \text{or} \qquad = \sqrt{x}(1 - x^2)$$

$$= x^{\frac{1}{2}}(1 - x^2)$$

c. $x^{-1} + x^{-2} - 12x^{-3}$

$$= x^{-1}(1 + x^{-1} - 12x^{-2}) \qquad \text{or} \qquad \frac{1}{x}\left(1 + \frac{4}{x}\right)\left(1 - \frac{3}{x}\right)$$

$$= x^{-1}(1 + 4x^{-1})(1 - 3x^{-1})$$

d. $x^{\frac{3}{2}} - 25x^{-\frac{1}{2}}$

$$= x^{-\frac{1}{2}}(x^2 - 25) \qquad \text{or} \qquad \frac{1}{\sqrt{x}}(x - 5)(x + 5)$$

$$= x^{-\frac{1}{2}}(x - 5)(x + 5)$$

8. The experiment can be modelled with the exponential function $N = N_0(b)^t$, where N is the number of cells, and t is the time lapsed in hours.

Solution 1

$$\begin{aligned} \text{For 2 hours: } 1600 &= 50b^2 \\ b^2 &= 32 \\ \text{For 6 hours: } N &= 50b^6 \\ &= 50(b^2)^3 \\ &= 50(32)^3 \\ &= 1\ 638\ 400 \end{aligned}$$

Solution 2

$$\begin{aligned} \text{For 2 hours: } 1600 &= 50b^2 \\ b^2 &= 32 \\ \text{Since } b > 0,\ b &= \sqrt{32} \\ \therefore\ N &= 50\left(\sqrt{32}\right)^t \\ \text{For 6 hours, } N &= 50\left(\sqrt{32}\right)^6 \\ &= 1\ 638\ 400 \end{aligned}$$

After six hours, there will be 1 638 400 bacteria cells.

9. The radioactive decay can be modelled by

$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where A is the amount in mg, t is the time in days, h is the half-life in days.

$$\therefore\ 5 = 40\left(\frac{1}{2}\right)^{\frac{24}{h}}$$

$$\frac{1}{8} = \left(\frac{1}{2}\right)^{\frac{24}{h}}$$

$$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{\frac{24}{h}}$$

$$3 = \frac{24}{h}$$

$$h = 8$$

The half-life of iodine 131 is eight days.

10. a. Using a graphing calculator's exponential regression function $\boxed{\text{ExpReg}}$, we find the curve of best fit is $y = 29\ 040.595(1.0108)^x$.

b. Using this model, in the year 2010,

$$\begin{aligned} x &= 2010 - 1994 \\ &= 16 \\ y &= 29\ 040.595(1.0108)^{16} \\ &\doteq 34\ 486.5. \end{aligned}$$

So, the population of Canada in 2010 will be 34 487 000.

c. Let $y = 35\ 000$ thousands.

$$\begin{aligned} \therefore\ 35\ 000 &= 29\ 040.595(1.0108)^x \\ 1.0108^x &\doteq 1.20521 \\ x &\doteq 17.376 \end{aligned}$$

Canada's population may reach 35 million in 1994 + 17, or 2011.

11. a. (i) Average rate of change between 1750 and 1800 is

$$\frac{203 - 163}{1800 - 1750} = \frac{40}{50}$$

$$= 0.8$$

Population changed at 0.8 million per year.

(ii) Average rate of change between 1950 and 1998 is

$$\frac{729 - 547}{1998} - 1950 = \frac{182}{48}$$

$$\doteq 3.79$$

Population changed at 3.79 million per year.

(iii) The population rate in Europe increased five-fold from the end of the eighteenth century to the end of the twentieth century.

b. (i) Average rate of change between 1800 and 1850 is

$$\frac{26 - 7}{1850 - 1800} = \frac{19}{50}$$

$$= 0.38$$

Population increased at a rate of 0.38 million per year.

(ii) Average rate of change between 1950 and 1998 is

$$\frac{305-172}{1998-1950}=\frac{133}{48}$$

$$=2.77$$

The population increased at a rate of 277 million per year.

(iii) The population rate in North America increased seven-fold from the mid-1800’s to the end of the twentieth century.

(c) North America experienced huge population surges due to immigration. As well, North America was still an agrarian society, whereas Europe was more industrial and agrarian societies tend to have a higher birth rate. European birth rates fell due to housing squeeze.

Chapter 6 Test

1. a. $\left(4^{\frac{1}{2}}\right)^3$

$= 4^{\frac{3}{2}}$ or $= (2)^3$

$= 2^3$ $= 8$

$= 8$

b. $\left[5^{\frac{1}{3}} \div 5^{\frac{1}{6}}\right]^{12}$

$= \left[5^{\frac{2}{6}-\frac{1}{6}}\right]^{12}$

$= \left[5^{\frac{1}{6}}\right]^{12}$

$= 5^2$

$= 25$

c. $4^{-1} + 2^{-3} - 5^0$

$= \frac{1}{4} + \frac{1}{8} - 1$

$= \frac{2}{8} + \frac{1}{8} - \frac{8}{8}$

$= -\frac{5}{8}$

d. $(\sqrt{2})^3 \times (\sqrt{2})^5$

$= (\sqrt{2})^8$

$= (2^{\frac{1}{2}})^8$

$= 2^4$

$= 16$

e. $\dfrac{2^{-1} + 2^{-2}}{2^{-3}}$

$= \dfrac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{8}}$

$= \dfrac{4 + 2}{1}$

$= 6$

f. $(-5)^{-3} \times (5)^2$

$= (-5)^{-3} \times (-5)^2$

$= (-5)^{-1}$

$= -\frac{1}{5}$

2. a. $\dfrac{a^4 \cdot a^{-3}}{a^{-2}}$

$= \dfrac{a^1}{a^{-2}}$

$= a^3$

b. $(3x^2y)^2$

$= 9x^4y^2$

c. $(x^4y^{-2})^2 \cdot (x^2y^3)^{-1}$

$= (x^8y^{-4})(x^{-2}y^{-3})$

$= x^6y^{-7}$

d. $(x^{a+b})(x^{a-b})$

$= x^{2a}$

e. $\dfrac{x^{p^2-q^2}}{x^{p+q}}$

$= x^{p^2-q^2-p-q}$

f. $\dfrac{\sqrt{x} \bullet \sqrt[3]{x}}{x^{-1}}$

$= \left(\dfrac{x^{\frac{1}{2}} \bullet x^{\frac{1}{3}}}{x^{-1}}\right)^{\frac{1}{2}}$

$= \left(\dfrac{x^{\frac{5}{6}}}{x^{-1}}\right)^{\frac{1}{2}}$

$= \left(x^{\frac{11}{6}}\right)^{\frac{1}{2}}$

$= x^{\frac{11}{12}}$

3. $\dfrac{x - 16}{x^{\frac{1}{2}} - 4}$

$= \dfrac{(x^{\frac{1}{2}} - 4)(x^{\frac{1}{2}} + 4)}{x^{\frac{1}{2}} - 4}$

$= x^{\frac{1}{2}} + 4$ or $\sqrt{x} + 4$

4. For $f(x) = b^x$, the sign of $f(x)$ will be positive if $b > 0$. If b is such that $0 < b < 1$, then the function will always decrease, but if $b > 1$, then the function will always increase. If $b = 1$, the function $f(x) = 1$ is a horizontal line.

5. $y = 2\left(\dfrac{1}{3}\right)^x - 5$

a. **(i)** The equation of the asymptote is $y = -5$.

(ii) Let $x = 0$

$y = 2\left(\dfrac{1}{3}\right)^0 - 5$

$= 2 - 5$

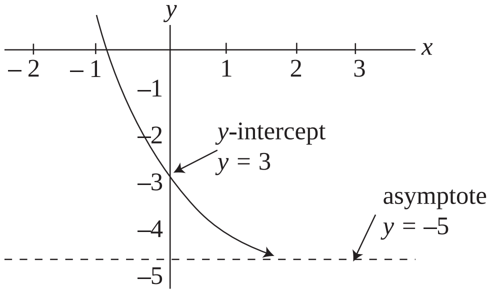
$= -3$

The y-intercept is -3.

(iii) The function is always decreasing.

(iv) Domain is x, $x \in R$. Range is $y > -5$, $y \in R$.

b.



6. The value of the dresser is given by $V = 3500(1.07)^t$, where t is the number of years since 1985.

In 2001, $V = 3500(1.07)^{2001-1985}$

$= 3500(1.07)^{16}$

$\doteq 10\ 332.57.$

The dresser is worth approximately \$10 330 in 2001.

7. If the population is decreasing by 8% per year, then the base of the exponential function is $1 - 0.08$ or 0.92. The population is given by $P = 4500(0.92)^t$, where t is the number of years since 1998.

For 2004, $P = 4500(0.92)^{2004-1998}$

$= 4500(0.92)^6$

$= 2728.6.$

The population estimate is 2729 for 2004.

8. The amount of polonium is given by $A = A_0\left(\dfrac{1}{2}\right)^{\frac{t}{h}}$ where A is the amount, t is the number of minutes passed, and h is the half-life in minutes.

So, $\dfrac{1}{16} = 1\left(\dfrac{1}{2}\right)^{\frac{14}{h}}$

$\left(\dfrac{1}{2}\right)^4 = \left(\dfrac{1}{2}\right)^{\frac{14}{h}}$

$4 = \dfrac{14}{h}$

$h = \dfrac{14}{4}$

$= 3.5.$

The half-life of the sample is 3.5 minutes.

9. **a.** The curve of best fit (using the $\boxed{\text{ExpReg}}$ function) is given as

$y = 0.6599(1.4619)^x$

or $y = 0.660(1.462)^x$ to three decimal places.

b. Since the time intervals are in 50-year divisions,

the time interval for 2300 is

$\frac{2300 - 1750}{50}$ or 11.

$y = 0.66(1.462)^{11}$

$\doteq 43.05$

The population estimate for 2300 is 43 billion.

c. Assuming that our estimate is correct, the population density for 2300 is

$\frac{20 \times 10^6 \text{ hectares}}{43 \times 10^9 \text{ people}}$ or 0.465×10^{-3}.

Each person will have 0.465×10^{-3} hectares or $0.465 \times 10^{-3} \times 10^4$ m² which is 4.65 m².

d. No, since there are so many other facts that determine population and may alter the pure exponential function. Answers may vary.

10. **a.** $f(x) = 2^x + 3$

b. It appears that the equation of the asymptote is $y = 3$. $\therefore$ $c = 3$. For the given y-intercept of 4, when $x = 0$, substituting gives $4 = b^0 + 3$.
Looking at the point (1, 5) and substituting, we find $5 = b^1 + 3$.

$\therefore$ $b = 2$.

Chapter 7 • The Logarithmic Function and Logarithms

Review of Prerequisite Skills

4. The increase in population is given by

$$f(x) = 2400(1.06)^x$$
$$f(20) = 2400(1.06)^{20}$$
$$\doteq 7697.$$

The population in 20 years is about 7700.

5. The function representing the increase in bacteria population is $f(t) = 2000(2^{\frac{t}{4}})$. Determine t when $f(t) = 512\ 000$:

$$512\ 000 = 2000(2^{\frac{t}{4}})$$
$$256 = 2^{\frac{t}{4}}$$
$$\frac{t}{4} = 8$$
$$t = 32.$$

The bacteria population will be 512 000 in 32 years.

6. **a.** The function is $A(t) = 5\left(\frac{1}{2}\right)^{\frac{t}{1620}}$.

When $t = 200$, $A(t) = 5\left(\frac{1}{2}\right)^{\frac{200}{1620}}$

$$= 4.59.$$

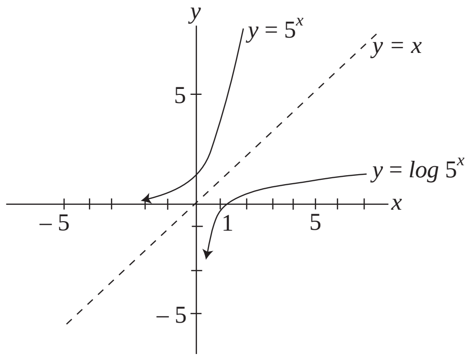
They will have 4.59 g in 200 years.

b. Determine t when $A(t) = 4$.

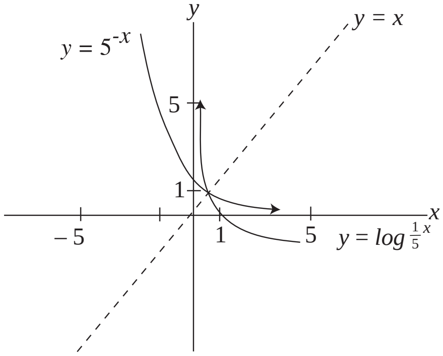
$$4 = 5\left(\frac{1}{2}\right)^{\frac{t}{1620}}$$
$$\left(\frac{1}{2}\right)^{\frac{t}{1620}} = 0.8$$
$$\frac{t}{1620}\log 0.5 = \log 0.8$$
$$t = 1620\,\frac{\log 0.8}{\log 0.5}$$
$$= 521.52$$

Exercise 7.1

4.



5.



6. **a.** Let $\log_2 8 = x$.

Then $2^x = 8$, by definition.

But $2^3 = 8$.

$\therefore x = 3$

So, $\log_2 8 = 3$.

c. Let $\log_3 81 = x$.

$3^x = 81$

$\therefore x = 4$

So, $\log_3 81 = 4$.

e. Let $\log_2\left(\frac{1}{8}\right) = x$.

$$2^x = \frac{1}{8}$$
$$2^x = 2^{-3}$$
$$x = -3$$

So, $\log_2\left(\frac{1}{8}\right) = -3$.

g. Let $\log_5\sqrt{5} = x$.

$$\therefore\ 5^x = \sqrt{5}$$
$$5^x = 5^{\frac{1}{2}}$$
$$x = \frac{1}{2}$$

So, $\log_5\sqrt{5} = \frac{1}{2}$.

i. Let $\log_2\sqrt[4]{32} = x$.

$$\therefore\ 2^x = \sqrt[4]{32}$$
$$2^x = (32)^{\frac{1}{4}}$$
$$2^x = (2^5)^{\frac{1}{4}}$$
$$2^x = 2^{\frac{5}{4}}$$
$$\therefore\ x = \frac{5}{4}$$

So, $\log_2\sqrt[4]{32} = \frac{5}{4}$.

7. **a.** $\log_6 36 - \log_5 25$

$= 2 - 2$

$= 0$

b. $\log_9\left(\frac{1}{3}\right) + \log_3\left(\frac{1}{9}\right)$

$$= \log_9(3^{-1}) + \log_3(9^{-2})$$
$$= \log_9\left[(9)^{\frac{1}{2}}\right]^{-1} + \log_3[(3^2)]^{-2}$$
$$= \log_9 9^{-\frac{1}{2}} + \log_3(3^{-4})$$
$$= -\frac{1}{2} + (-4)$$
$$= -4\frac{1}{2} \quad \text{or} \quad -\frac{9}{2} \quad \text{or} \quad -4.5$$

c. $\log_6\sqrt{36} - \log_{25}5$

$$= \log_6(36)^{\frac{1}{2}} - \log_{25}(25^{\frac{1}{2}})$$
$$= \log_6 6 - \log_{25}(25^{\frac{1}{2}})$$
$$= 1 - \frac{1}{2}$$
$$= \frac{1}{2}$$

d. $\log_3\sqrt[4]{27}$

$$= \log_3(27)^{\frac{1}{4}}$$
$$= \log_3(3^3)^{\frac{1}{4}}$$
$$= \log_3 3^{\frac{3}{4}}$$
$$= \frac{3}{4}$$

e. $\log_3(9 \times \sqrt[5]{9})$

$$= \log_3(3^2 \times 9^{\frac{1}{5}})$$
$$= \log_3(3^2 \times 3^{\frac{2}{6}})$$
$$= \log_3 3^{\frac{12}{5}}$$
$$= \frac{12}{5} \text{ by definition of logarithms}$$

f. $\log_2 16^{\frac{1}{3}}$

$$= \log_2 (2^4)^{\frac{1}{3}}$$
$$= \log_2 2^{\frac{4}{3}}$$
$$= \frac{4}{3}$$

8. **b.** $\log_4 x = 2$

$x = 4^2$

$x = 16$

d. $\log_4\left(\frac{1}{64}\right) = x$

$$4^x = \frac{1}{64}$$
$$4^x = 4^{-3}$$
$$x = -3$$

f. $\log_{\frac{1}{4}} x = -2$

$$x = \left(\frac{1}{4}\right)^{-2}$$
$$= 4^2$$
$$= 16$$

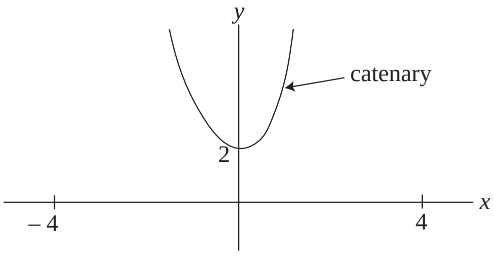
9. To find the value of a logarithm, you can use the $\boxed{\text{LOG}}$ button on a calculator for those which have base 10. The log can be rounded to the number of decimals appropriate to the problem.
If, however, the number can be expressed as a power of the base of the logarithm, then the exponent of the power *is* the logarithm. Since 16 can be written as 2^4, $\log_2 16 = \log_2 2^4$, = 4. By definition $\log_b b^x = x$. If the number cannot be so expressed, we can use the calculator to find $\log_a b$ by finding

$$\frac{\log_{10} b}{\log_{10} a} \bullet \text{ i.e., } \log_2 16 = \frac{\log_{10} 16}{\log_{10} 2}$$

$$= 4 \text{ by calculator.}$$

10.

x	y
±4	$81\frac{4}{81}$
±3	$27\frac{1}{27}$
±2	$9\frac{1}{9}$
±1	$3\frac{1}{3}$
0	2



11. Integer values for y exist where x is a power of 10 with an integer exponent. If $y > -20$, the smallest number is 10^{-19}. If $x \le 1000$, the largest number is $1000 = 10^3$. There are integers from –19 to 3 that satisfy the condition, so there are $3 - (-19) + 1 = 23$ integer values of y possible.

Section 7.2

Investigation:

2. $\log_b 1 = 0$

3. $\log_b b = 1$

4. $\log_b b^x = x$

5. $b^{\log_b x} = x$

Exercise 7.2

6. **a.** $y = \log_3 x$

b. $y = \log_3 (9x)$

$= \log_3 9 + \log_3 x$

$= 2 + \log_3 x$

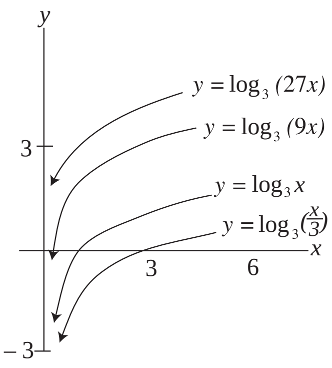
c. $y = \log_3 (27x)$

$= \log_3 27 + \log_3 x$

$= 3 + \log_3 x$

d. $y = \log_3 \left(\frac{x}{3}\right)$

$= \log_3 x - 1$



7. **a.** $y = \log_3 x$

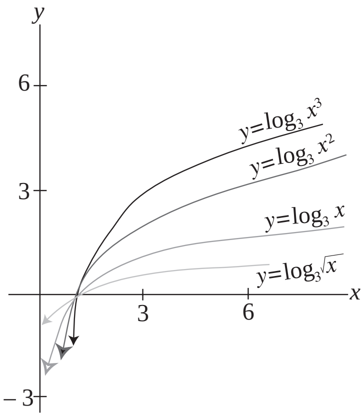
b. $y = \log_3 x^2$
$= 2\log_3 x$

c. $y = \log_3 x^3$
$= 3\log_3 x$

d. $y = \log_3\sqrt{x}$

$= \log_3 x^{\frac{1}{2}}$

$= \frac{1}{2}\log 3$



8. **a.** $\log_3 135 - \log_3 5$

$= \log_3 \frac{135}{5}$

$= \log_3 27$

$= 3$

b. $\log_2 40 + \log_2\left(\frac{4}{5}\right)$

$= \log_2\left(40 \times \frac{4}{5}\right)$

$= \log_2 32$

$= 5$

c. $\log_8 640 - \log_8 10$

$= \log_8 \frac{640}{10}$

$= \log_8 64$

$= 2$

d. $\log_5(2.5) + \log_5 10$

$= \log_5(2.5 \times 10)$

$= \log_5(25)$

$= 2$

e. $\log_2 224 - \log_2 7$

$= \log_2 \frac{224}{7}$

$= \log_2 32$

$= 5$

f. $\log_3 36 + \log_3\left(\frac{3}{4}\right)$

$= \log_3\left(36 \times \frac{3}{4}\right)$

$= \log_3 27$

$= 3$

9. **a.** $\log_3 3 + \log_5 1$

$= 1 + 0$

$= 1$

b. $\log_3 18 + \log_3\left(\frac{3}{2}\right)$

$= \log_3\left(18 \times \frac{3}{2}\right)$

$= \log_3 27$

$= 3$

c. $\log_4 16 - \log_4 1$

$= 2 - 0$ or $\log_4 16 - \log_4 1$

$= 2$ $\log_4(16 \times 1)$

$= 2$

d. $\log_5 5^3$

$= 3$

e. $\log_2 40 - \log_2\left(\frac{5}{2}\right)$

$= \log_2\left(40 \div \frac{5}{2}\right)$

$= \log_2\left(40 \times \frac{2}{5}\right)$

$= \log_2 16$

$= 4$

f. $\log_4 4^4 + \log_3 3^3$

$= 4 + 3$

$= 7$

g. $\log_2 14 + \log_2\left(\frac{4}{7}\right)$

$$= \log_2\left(14 \times \frac{4}{7}\right)$$

$$= \log_2 8$$

$$= 3$$

h. $\log_5 200 - \log_5 8$

$$= \log_5\left(\frac{200}{8}\right)$$

$$= \log_5 25$$

$$= 2$$

10. a. $\log_a \sqrt[3]{x^2y^4}$

$$= \log_a (x^2y^4)^{\frac{1}{3}}$$

$$= \log_a\left(x^{\frac{2}{3}} \bullet y^{\frac{4}{3}}\right)$$

$$= \log_a x^{\frac{2}{3}} + \log_a y^{\frac{4}{3}}$$

$$= \frac{2}{3}\log_a x + \frac{4}{3}\log_a y$$

b. $\log_a \sqrt{\frac{x^3y^2}{w}}$

$$= \log_a (x^3y^2w^{-1})^{\frac{1}{2}}$$

$$= \frac{1}{2}\log_a (x^3y^2w^{-1})$$

$$= \frac{1}{2}[\log_a x^3 + \log_a y^2 + \log_a w^{-1}]$$

$$= \frac{1}{2}[3\log_a x + 2\log_a y - \log_a w]$$

c. $\log_a \frac{x^3y^4}{\sqrt{x^{\frac{1}{4}}y^{\frac{2}{3}}}}$

$$= \log_a\left[\frac{x^3y^4}{x^{\frac{1}{4}}y^{\frac{2}{3}})^{\frac{1}{2}}}\right]$$

$$= \log_a\left(\frac{x^3y^4}{x^{\frac{1}{8}}y^{\frac{1}{3}}}\right)$$

$$= \log_a\left(x^{\frac{23}{8}}y^{\frac{11}{3}}\right)$$

$$= \log_a x^{\frac{23}{8}} + \log_a y^{\frac{11}{3}}$$

$$= \frac{23}{8}\log_a x + \frac{11}{3}\log_a y$$

d. $\log_a\left(\frac{x^5}{y^3}\right)^{\frac{1}{4}}$

$$= \frac{1}{4}\log_a (x^5y^{-3})$$

$$= \frac{1}{4}[\log_a x^5 + \log_a y^{-3}]$$

$$= \frac{1}{4}[5\log_a x - 3\log_a y]$$

11. a.

$$10^{2x} = 495$$

$$\log 10^{2x} = \log 495$$

$$2x \log 10 = \log 495$$

$$2x = 495$$

$$x = \log \frac{495}{2}$$

$$\doteq 1.347$$

b.

$$10^{3x} = 0.473$$

$$\log 10^{3x} = \log 0.473$$

$$3x\log 10 = \log 0.473$$

$$x = \frac{\log 0.473}{3}$$

$$\doteq -0.1084$$

c.

$$10^{-x} = 31.46$$

$$\log 10^{-x} = 31.46$$

$$-x\log 10 = 31.46$$

$$-x = 31.46$$

$$x = -31.46$$

d.

$$7^x = 35.72$$

$$x\log 7 = 35.72$$

$$x = \frac{\log 35.72}{\log 7}$$

$$\doteq 1.8376$$

e.

$$(0.6)^{4x} = 0.734$$

$$\log(0.6)^{4x} = \log 0.734$$

$$4x\log 0.6 = \log 0.734$$

$$x = \frac{\log 0.734}{4\log 0.6}$$

$$\doteq 0.1513$$

f.
$$\begin{aligned}(3.482)^{-x} &= 0.0764\\ \log 3.482^{-x} &= \log 0.0764\\ -x\log 3.482 &= \log 0.0764\\ x &= \log\frac{0.0764}{-\log 3.482}\\ &\doteq 2.0614\end{aligned}$$

12. b.
$$\begin{aligned}7^{x+9} &= 56\\ \log 7^{x+9} &= \log 56\\ (x+9)\log 7 &= \log 56\\ x+9 &= \frac{\log 56}{\log 7}\\ x &= \frac{\log 56}{\log 7} - 9\\ &\doteq -6.93\end{aligned}$$

c.
$$\begin{aligned}5^{3x+4} &= 25\\ 5^{3x+4} &= 5^2\\ 3x+4 &= 2\\ 3x &= -2\\ x &= -\frac{2}{3}\end{aligned}$$

d.
$$\begin{aligned}10^{2x+1} &= 95\\ \log 10^{2x+1} &= \log 95\\ (2x+1)\log 10 &= \log 95\\ 2x+1 &= \log 95\\ 2x &= \log 95 - 1\\ x &\doteq 0.4889\end{aligned}$$

e.
$$\begin{aligned}6^{x+5} &= 71.4\\ (x+5)\log 6 &= \log 71.4\\ x+5 &= \frac{\log 71.4}{\log 6}\\ x &\doteq -2.6163\end{aligned}$$

f.
$$\begin{aligned}3^{5-2x} &= 875\\ (5-2x)\log 3 &= \log 875\\ 5-2x &= \frac{\log 875}{\log 3}\\ 5-2x &\doteq 6.1662\\ -2x &\doteq 1.662\\ x &\doteq -0.5841\end{aligned}$$

13. a.
$$\begin{aligned}2\times 3^x &= 7\times 5^x\\ \frac{3^x}{5^x} &= \frac{7}{2}\\ \left(\frac{3}{5}\right)^x &= 3.5\\ x\log(0.6) &= \log 3.5\\ x &= \frac{\log 3.5}{\log 0.6}\\ &\doteq -2.452\end{aligned}$$

b. $12^x = 4\times 8^{2x}$

Solution 1

$$\begin{aligned}x\log 12 &= \log 4 + 2x\log 8\\ x\log 12 - 2x\log 8 &= \log 4\\ x(\log 12 - 2\log 8) &= \log 4\\ x\left(\log\frac{12}{8^2}\right) &= \log 4\\ x\left(\log\frac{3}{16}\right) &= \log 4\\ x &= \frac{\log 4}{\log\frac{3}{16}} \text{ or } x \doteq -0.828\end{aligned}$$

Solution 2

$$\begin{aligned}12^x &= 2^2\times(2^3)^{2x}\\ 12^x &= 2^{6x+2}\\ x\log 12 &= (6x+2)\log 2\\ x(\log 12 - 6\log 2) &= 2\log 2\\ x(\log\frac{12}{2^6}) &= 2\log 2\\ x &= \frac{2\log 2}{\log\frac{2}{16}} \text{ or } x = -0.828\end{aligned}$$

c. $4.6\times 1.06^{2x+3} = 5\times 3^x$
$$\begin{aligned}\log 4.6 + (2x+3)\log 1.06 &= \log 5 + x\log 3\\ \log 4.6 + 2x\log 1.06 + 3\log 1.06 &= \log 5 + x\log 3\\ x(2\log 1.06 - \log 3) &= \log 5 - \log 4.6 - 3\log 1.06\\ x\left(\log\frac{1.06^2}{3}\right) &= \log\left(\frac{5}{4.6\times 1.06^3}\right)\\ x &\doteq \frac{\log(0.9126)}{\log(0.3745)}\\ x &\doteq 0.093\end{aligned}$$

d. $2.67 \times 7.38^x = 9.36^{5x-2}$

Solution 1

$$\begin{aligned}
2.67 \times 7.38x &= 9.365\,x - 2 \\
\log 2.67 + x\log 7.38 &= (5x - 2)\log 9.36 \\
\log 2.67 + x\log 7.38 &= 5x\log 9.36 - 2\log 9.36 \\
x(\log 7.38 - 5\log 9.36) &= -\log 2.67 - 2\log 9.36 \\
x\left(\log \frac{7.38}{9.36^5}\right) &= -\log(2.67 \times 9.36^2) \\
x(\log .000103) &\doteq -\log(233.9) \\
x &\doteq 0.59
\end{aligned}$$

Solution 2

$$\begin{aligned}
2.67 \times 7.38^x &= 9.36^{5x-2} \\
2.67 &= \frac{9.36^{5x-2}}{7.38^x} \\
\log 2.67 &= \log 9.36^{5x-2} - \log 7.38^x \\
\log 2.67 &= (5x - 2)\log 9.36 - x\log 7.38 \\
\log 2.67 &= 5x\log 9.36 - 2\log 9.36 - x\log 7.38^x \\
\log 2.67 + 2\log 9.36 &= x(5\log 9.36 - \log 7.38) \\
\log(2.67 \times 9.36^2) &= x\left(\log \frac{9.36^5}{7.38}\right) \\
\log(233.913) &\doteq x(\log 9734.707) \\
x &\doteq \frac{\log 233.918}{\log 9734.707} \\
x &\doteq 0.59
\end{aligned}$$

e. $12 \times 6^{2x-1} = 11^{x+3}$

Solution 1

$$\begin{aligned}
\log 12 + (2x - 1)\log 6 &= (x + 3)\log 11 \\
\log 12 + 2x\log 6 - \log 6 &= x\log 11 + 3\log 11 \\
x(2\log 6 - \log 11) &= 3\log 11 - \log 12 + \log 6 \\
x\left(\log \frac{62}{11}\right) &= \log\left(\frac{11^3 \times 6}{12}\right) \\
x(\log 3.273) &\doteq \log(6665.5) \\
x &\doteq 5.5
\end{aligned}$$

Solution 2

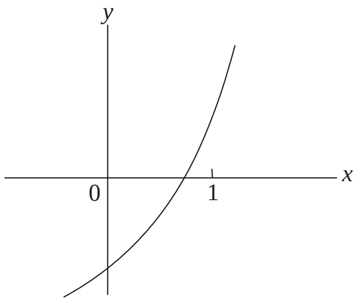
$$\begin{aligned}
12 \times 6^{2x-1} &= 11^{x+3} \\
12 \times \frac{6^{2x}}{6} &= 11^x \bullet 11^3 \\
2 \bullet 6^{2x} &= 1331 \bullet 11^x \\
\frac{6^{2x}}{11^x} &= 665.5 \\
2x\log 6 - x\log 11 &= \log 665.5 \\
x\left(\log \frac{6^2}{11}\right) &= \log 665.5 \\
x &\doteq 5.5
\end{aligned}$$

f.

$$\begin{aligned}
7 \times 0.43^{2x} &= 9 \times 6^{-x} \\
6^x \times 0.43^{2x} &= \frac{9}{7} \\
x\log 6 + 2x\log 0.43 &\doteq \log 1.2857 \\
x(\log 6 + 2\log 0.43) &\doteq \log 1.2857 \\
x(\log(6 \times 0.43^2)) &\doteq \log 1.2857 \\
x &\doteq 2.42
\end{aligned}$$

g. $5^x + 3^{2x} = 92$
Since we cannot take logarithms of a sum,
let $y = 5^x + 3^{2x} - 92$.
Graphing on a graphing calculator,
when $y = 0$, $x \doteq 1.93$.

h. $4 \times 5^x - 3(0.4)^{2x} = 11$
Let $y = 4 \times 5^x - 3x(0.4)^{2x} - 11$.
Graphing on a graphing calculator to find the value of x for $y = 0$, we find $x \doteq 0.64$.



14. a. $\frac{1}{3}\log_a x + \frac{1}{4}\log_a y - \frac{2}{5}\log_a w$

$$= \log_a \sqrt[3]{x} + \log_a \sqrt[4]{y} - \log_a w^{\frac{2}{5}}$$

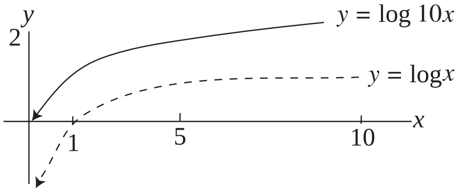
$$= \log_a\left[\frac{\sqrt[3]{x}\,\sqrt[4]{y}}{\sqrt[5]{w^2}}\right]$$

b. $(4\log_5 x - 2\log_5 y) \div 3\log_5 w$

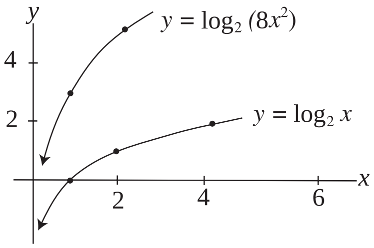
$= (\log_5 x^4 - \log_5 y^2) \div \log_5 w^3$

$= \log_5\left(\frac{x^4}{y^2}\right) \div \log_5 w^3$

15. a. Since $\log(10x)$ can be written as $\log 10 + \log x$ or $1 + \log x$, the transformation is a dilation horizontally and a vertical translation of one upwards.



b. Since $y = \log_2(8x^2)$ can be written as $y = \log_2 8 + \log_2 x^2$ or $y = 3 + 2\log_2 x$, the transformation is a stretch vertically by a factor of two and a vertical translation of three upwards.



c. Since $y = \log_3(27x^3)$ can be written as $y = \log_3 27 + \log_3 x^3 = 3 + 3\log_3 x$, the transformation is a vertical stretch of three times the original and an upwards vertical translation of three units.

16. a. $\log_3\left(27 \bullet \sqrt[3]{81}\right) + \log_5\left(125 \bullet \sqrt[4]{5}\right)$

$= \log_3 27 + \log_3 \sqrt[3]{81} + \log_5 125 + \log_5 \sqrt[4]{5}$

$= 3 + \frac{1}{3}\log_3 81 + 3 + \frac{1}{4}\log_5 5$

$= 3 + \frac{1}{3}(4) + 3 + \left(\frac{1}{4}\right)(1)$

$= 6 + \frac{4}{3} + \frac{1}{4}$

$= 7\,\frac{7}{12}$

b. $\log_4\left(2 \bullet \sqrt{32}\right) + \log_{27} \sqrt{3}$

$= \log_4 2 + \log_4 \sqrt{32} + \log_{27} \sqrt{3}$

$= \log_4 4^{\frac{1}{2}} + \log_4 32^{\frac{1}{2}} + \log_{27} 3^{\frac{1}{2}}$

$= \frac{1}{2} + \log_4 2^{\frac{5}{2}} + \log_{27} (27^{\frac{1}{3}})^{\frac{1}{2}}$

$= \frac{1}{2} + \log_4(4^{\frac{1}{2}})^{\frac{5}{2}} + \log_{27}(27^{\frac{1}{4}})$

$= \frac{1}{2} + \frac{5}{4} + \frac{1}{6}$

$= \frac{23}{12}$

17. a. Given $y = 3 \log x$.

(i) If x is multiplied by 2,
then $y = 3 \log 2x$
$= 3[\log 2 + \log x]$
$= 3 \log 2 + 3 \log x$.
So, the value of y increases by $3 \log 2$, or about 0.9.

(ii) If x is divided by 2,
then $y = 3 \log \frac{x}{2}$
$= 3[\log x - \log 2]$
$= 3 \log x - 3 \log 2$.
So, the value of y decreases by $3 \log 2$, or 0.9.

b. Given $y = 5 \log x$.

(i) If x is replaced by $4x$, then
$y = 5 \log 4x$
$= 5[\log 4 + \log x]$
$= 5 \log 4 + 5 \log x$,
so that y is increased by $5 \log 4$, or about 3.01.

(ii) If x is replaced by $\frac{x}{5}$, then
$y = 5(\log \frac{x}{5})$
$= 5[\log x - \log 5]$
$= 5 \log x - 5 \log 5$,
so that y is decreased by $5 \log 5$, or about 3.5.

Exercise 7.3

1. c. $2 \log_5 x = \log_5 36$
$\log_5 x^2 = \log_5 36$
$\therefore x^2 = 36$
$x = \pm 6$

The logarithm of a negative number is not defined. The root $x = –6$ is inadmissible.
By inspection $x = 6$ is admissible.
$\therefore\ x = 6$

d. $2 \log x = 4 \log 7$
$\log x^2 = \log 7^4$
$\therefore\ x^2 = 7^4 \text{ or } (7^2)^2$
$x = \pm 49$
But the logarithm of a negative number is not defined. The root $x = –49$ is inadmissible.
If $x = 49$, L.S. $\doteq$ 1.690, R.S. $\doteq$ 1.690
$\therefore\ x = 49$

2. **c.** $2^x – 1 = 4$
$2x = 5$
$\log 2^x = \log 5$
$x \log 2 = \log 5$
$$x = \frac{\log 5}{\log 2}$$
$\doteq 2.32$

d. $7 = 12 – 4^x$
$4^x = 5$
$\log 4^x = \log 5$
$x \log 4 = \log 5$
$$x = \frac{\log 5}{\log 4}$$
$\doteq 1.16$

3. **a.** $\log x = 2 \log 3 + 3 \log 2$
$\log x = \log 3^2 + \log 2^3$
$\log x = \log 3^2 \times 2^3$
$\log x = \log 72$
$x = 72$

c. $\log x^2 = 3 \log 4 – 2 \log 2$
$\log x^2 = \log 4^3 – \log 2^2$
$$\log x^2 = \log \frac{4^3}{2^2}$$
$x^2 = 16$
$x = \pm 4$
Both answers verify, so there are two roots, ± 4.

d. $\log\sqrt{x} = \log 1 – 2 \log 3$
$$\log \sqrt{x} = \log \frac{1}{3^2}$$
$$\sqrt{x} = \frac{1}{9}$$
$$x = \frac{1}{81}$$

e. $\log x^{\frac{1}{2}} – \log x^{\frac{1}{3}} = \log 2$
$$\log\left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}\right) = \log 2$$
$\log(x^{\frac{1}{6}}) = \log 2$
$x^{\frac{1}{6}} = 2$
$x = 2^6$
$x = 64$

f. $\log_4(x + 2) + \log_4(x – 3) = \log_4 9$
$\log_4(x + 2)(x – 3) = \log_4 9$
$(x + 2)(x – 3) = 9$
$x^2 – x – 6 = 9$
$x^2 – x – 15 = 0$
$$x = \frac{1 \pm \sqrt{1 – 4(1)(–15)}}{2(1)}$$
$$x = \frac{1 \pm \sqrt{61}}{2}$$

4. **a.** $\log_6(x + 1) + \log_6(x + 2) = 1$
$\log_6(x + 1)(x + 2) = 1$
In exponential form,
$(x + 1)(x + 2) = 6^1$
$x^2 + 3x + 2 = 6$
$x^2 + 3x – 4 = 0$
$(x + 4)(x – 1) = 0$
$x = –4$ or $x = 1$.
But $x = –4$, then $(x + 1) < 0$ and its logarithm is undefined.
$\therefore\ x = 1$

b. $\log_7(x + 2) + \log_7(x – 4)= 1$
$\log_7(x + 2)(x – 4) = 1$
In exponential form,
$(x + 2)(x – 4) = 7^1$
$x^2 – 2x – 8 = 7$
$x^2 – 2x – 15 = 0$
$(x – 5)(x + 3) = 0$
$x = 5$ or $x = –3$.
But if $x = –3$, $(x + 2) < 0$ and its logarithm is not defined.
$\therefore\ x = 5$

c.
$$\begin{aligned}\log_2(x+2) &= 3-\log_2 x\\ \log_2(x+2)+\log_2 x &= 3\\ \log_2 x(x+2) &= 3\end{aligned}$$

In exponential form,

$$\begin{aligned}x(x+2) &= 2^3\\ x^2+2x-8 &= 0\\ (x+4)(x-2) &= 0\\ x &= -4 \text{ or } x = 2.\end{aligned}$$

But $\log_2 x$ is not defined for $x = -4$.

$\therefore\ x = 2$

d.
$$\begin{aligned}\log_4 x+\log_4(x+6) &= 2\\ \log_4 x(x+6) &= 2\end{aligned}$$

In exponential form,

$$\begin{aligned}x(x+6) &= 4^2\\ x^2+6x-16 &= 0\\ (x+8)(x-2) &= 0\\ x &= -8 \text{ or } x = 2.\end{aligned}$$

But $\log_4 x$ is not defined for $x = -8$.

$\therefore\ x = 2$

e.
$$\log_5(2x+2)-\log_5(x-1) = \log_5(x+1)$$
$$\log_5\left(\frac{2x+2}{x-1}\right) = \log_5(x+1),\ x \neq 1$$
$$\begin{aligned}\frac{2x+2}{x-1} &= x+1\\ 2x+2 &= (x+1)(x-1)\\ 2x+2 &= x^2-1\\ x^2-2x-3 &= 0\\ (x-3)(x+1) &= 0\\ x &= 3 \text{ or } x = -1\end{aligned}$$

But $\log_5(x+1)$ is not defined for $x = -1$.

$\therefore\ x = 3$

5. There are no solutions, since logarithms are only defined for a base greater than one and for a number greater than zero, i.e., $\log_6 x$ is defined for $b > 1$ and $x > 0$.

6. If a car depreciates 15% per year, it is worth 85% of its value each year, and so its value can be written as: $V = V_0(0.85)^t$, where t is the time in years.

For half its value, $0.5V_0 = V_0(0.85)^t$
$$\begin{aligned}0.5 &= 0.85^t\\ \log 0.5 &= t\log 0.85\\ t &= \frac{\log 0.5}{\log 0.85}\\ &\doteq 4.265\end{aligned}$$

It will depreciate to half of its value in $4\frac{1}{4}$ years or 4 years, 3 months.

7. The amount of radioactive carbon can be modelled by $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{5760}}$, where t is the age in years. For dating of the animal skeleton:
$$\begin{aligned}\frac{3}{4}A_0 &= A_0\left(\frac{1}{2}\right)^{\frac{t}{5760}}\\ 0.75 &= (0.5)^{\frac{t}{5760}}\\ \log 0.75 &= \frac{t}{5760}\log 0.5\\ t &= \left[\frac{\log 0.75}{\log 0.5}\right]5760\\ &\doteq 2390.6\end{aligned}$$

The animal skeleton is approximately 2400 years old.

8. Let the time required before replacement be t years.

The amount of Co^{60} is $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{5.24}}$
$$\begin{aligned}0.45A_0 &= A_0(0.5)^{\frac{t}{5.24}}\\ 0.45 &= (0.5)^{\frac{t}{5.24}}\\ \log 0.45 &= \frac{t}{5.24}\log 0.5\\ t &= \left[\frac{\log 0.45}{\log 0.5}\right]\times 5.24\\ &\doteq 6.04\end{aligned}$$

The cobalt should be replaced every six years.

9. The amount of carbon^{14} can be modelled as
$$\begin{aligned}C &= \text{Co}(0.5)^{\frac{t}{5760}}\\ 4.2\times 10^{10} &= 5.0\times 10^{10}(0.5)^{\frac{t}{5760}}\\ 0.84 &= 0.5^{\frac{t}{5760}}\\ \log 0.84 &= \frac{t}{5760}\log 0.5\\ t &= \left[\frac{\log 0.84}{\log 0.5}\right]\times 5760\\ t &\doteq 1449\end{aligned}$$

The relic is only about 1450 years old, so it cannot be authentic.

10. $\log_2(\log_3 a) = 2$

In exponential form,

$\log_3 a = 2^2$

$\log_3 a = 4.$

In exponential form,

$a = 3^4$

$= 81.$

11. $\log_{2n}(1944) = \log_n(486\sqrt{2})$

Let $\log_n(486\sqrt{2}) = x$.

In exponential form,

$n^x = 486\sqrt{2}.$ (1)

Also, $\log_{2n}(1944) = x$.

In exponential form,

$(2n)^x = 1944.$ (2)

Dividing equation (2) by (1), we have

$$\frac{(2n)^x}{n^x} = \frac{1944}{486\sqrt{2}}$$

$$\left(\frac{2n}{n}\right)^x = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$2^x = 2^{\frac{3}{2}}$$

$$x = \frac{3}{2} \text{ or } 1.5.$$

Substituting into equation (1):

$$\begin{aligned} n^{1.5} &= 486\sqrt{2} \\ (n^{1.5})^4 &= (486\sqrt{2})^4 \\ n^6 &= 472\ 392^2 \\ &\doteq 2.23 \times 10^{11} \end{aligned}$$

Exercise 7.4

2. If an earthquake has a magnitude of five on the Richter scale, then $\log\left[\frac{I_1}{I_0}\right] = 5$ or $I_1 = 10^5 I_0$.

If the second earthquake has a magnitude of six, then $\log\left[\frac{I_2}{I_0}\right] = 6$ or $I_2 = 10^6 I_0$.

Comparing the two, $\frac{I_2}{I_1} = \frac{10^6 I_0}{10^5 I_0}$

$= 10.$

So, the second quake is 10 times as intense as the first.

3. If the sound is 1 000 000 or 10^6 times as loud as one you can just hear, $\frac{I}{I_0} = 10^6$.

The loudness of the sound is given by

$L = 10\log\left[\frac{I}{I_0}\right].$

The loudness of the sound:

$= 10\log[10^6]$

$= 10(6)$

$= 60.$

The loudness is 60 decibels.

4. The definition of pH is given by

$$\begin{aligned} \text{pH} &= -\log[H^+] \\ \text{For this liquid, pH} &= -\log[8.7 \times 10^{-6}] \\ &= -(\log 8.7 + \log 10^{-6}) \\ &= -(\log 8.7 - 6) \\ &= 6 - \log 8.7 \\ &\doteq 6 - 0.9395. \\ \text{The pH is then} &\doteq 5.06. \end{aligned}$$

5. For the earthquake of magnitude two, $I_2 = 10^2 I_0$.

For the earthquake of magnitude four, $I_4 = 10^4 I_0$.

Comparing the intensities,

$$\frac{I_4}{I_2} = \frac{10^4 I_0}{10^2 I_0}$$

$$= 10^2.$$

So, the larger earthquake is 100 times as intense as the smaller.

6. For the earthquake measuring 4, $I_4 = 10^4 I_0$.

For the earthquake in China measuring 8.6, $I_{8.6} = 10^{8.6} I_0$.

$$\begin{aligned} \text{Comparing them, } \frac{I_{8.6}}{I_4} &= \frac{10^{8.6} I_0}{10^4 I_0} \\ &= 10^{8.6-4} \\ &= 10^{4.6} \\ \text{or } &\doteq 39\ 811. \end{aligned}$$

The earthquake in China was almost 40 000 times as intense as the lesser one.

7. **a.** For the earthquake in Pakistan, $I_P = 10^{6.8} I_0$.

For the earthquake in California, $I_C = 10^{6.1} I_0$.

Comparing $\frac{I_P}{I_C} = \frac{10^{6.8} I_0}{10^{6.1} I_0} = 10^{6.8-6.1} \doteq 5.01.$

The quake in Pakistan was five times as intense as that in California.

8. For the earthquake in Chile, $I_C = 10^{8.3}I_0$.

For the earthquake in Taiwan, $I_T = 10^{7.6}I_0$.

Comparing the two quakes, $\frac{I_T}{I_C} = \frac{10^{8.3}I_0}{10^{7.6}I_0}$

$= 10^{0.7}$

$\doteq 5.$

The earthquake in Chile was five times more intense.

9. The loudness of a sound is given by $L = 10 \log\left(\frac{I}{I_o}\right)$.

For her defective muffler,

$$120 = 10 \log\left(\frac{I_d}{I_0}\right)$$

$$12 = \log\left(\frac{I_d}{I_0}\right).$$

Solving for $\frac{I_d}{I_0} = 10^{12}$

$I_d = 10^{12}I_0$.

For the new muffler,

$$75 = 10 \log\left(\frac{I_n}{I_0}\right)$$

$$7.5 = \log\left[\frac{I_n}{I_0}\right] \quad \text{or} \quad \log\frac{I_n}{I_0} = 10^{7.5}.$$

So, $I_n = 10^{7.5}I_0$.

Comparing the sounds,

$$\frac{I_d}{I_n} = \frac{10^{12}I_0}{10^{7.5}I_0}$$

$$= 10^{12-7.5}$$

$$\doteq 31\ 623.$$

So, the sound with a defective muffler is almost 32 000 times as loud as the sound with a new muffler.

10. The loudness level is given by $L = 10 \log\left[\frac{I}{I_0}\right]$.

For a baby with colic,

$$75 = 10 \log\left[\frac{I_c}{I_0}\right] \quad \text{or} \quad \log\left[\frac{I_c}{I_o}\right] = 7.5.$$

$\therefore I_c = 10^{7.5}I_0$

For a sleeping baby,

$$35 = 10 \log\left[\frac{I_s}{I_0}\right] \quad \text{or} \quad \log\left[\frac{I_s}{I_o}\right] = 3.5.$$

$\therefore I_s = 10^{3.5}I_0$

Comparing the noise level, $\frac{I_c}{I_s} = \frac{10^{7.5}I_0}{10^{3.5}I_0}$

$= 10^4.$

The noise level with a baby with colic is 10 000 times as loud as when the baby is asleep.

11. For a space shuttle, $I_s = 10^{18}I_0$.

For a jet engine, $I_j = 10^{14}I_0$.

Comparing, $\frac{I_s}{I_j} = \frac{10^{18}I_0}{10^{14}I_0}$

$= 10^4$ or 10 000.

A space shuttle launch is 10 000 times as loud as a jet engine.

12. For open windows, $I_1 = 10^{7.9}I_0$.

For closed windows, $I_2 = 10^{6.8}I_0$.

Comparing $\frac{I_1}{I_2} = \frac{10^{7.9}I_0}{10^{6.8}I_0}$

$= 10^{1.1}$ or 12.6.

Closing the windows reduces the noise by a factor of about 13.

13. The pH level is defined by pH = $-\log[H^+]$.

For milk, $6.50 = -\log[H^+]$ or $\log[H^+] = -6.5$

$[H^+] = 10^{-6.5}$

$= 3.2 \times 10^{-7}$

14. The hydrogen ion concentration of milk of magnesia is 3.2×10^{-7} mol/L.

The pH level is defined by pH = $-\log[H^+]$.

For milk of magnesia, $10.50 = -\log[H^+]$

$\log[H^+] = -10.5$

$[H^+] = 10^{-10.5}$

$\doteq 3.2 \times 10^{-11}.$

Milk of magnesia has an ion concentration of 3.2×10^{-1} mol/L.

Exercise 7.5

1. **b.** $\log_7 124 = \dfrac{\log 124}{\log 7}$

$\doteq 2.477$

c. $\log_6 3.24 = \dfrac{\log 3.24}{\log 6}$

$\doteq 0.656$

2. **a.** To prove $\dfrac{1}{\log_5 a} + \dfrac{1}{\log_3 a} = \dfrac{1}{\log_{15} a}$

$$\begin{aligned}
\text{L.S.} &= \frac{1}{\log_5 a} + \frac{1}{\log_3 a}\\
&= log_a 5 + log_a 3\\
&= log_a (5 \times 3)\\
&= log_a 15\\
&= \frac{1}{\log_{15} a}\\
&= \text{R.S.}
\end{aligned}$$

$\therefore \dfrac{1}{\log_5 a} + \dfrac{1}{\log_5 a} = \dfrac{1}{\log_{15} a}$

b. To prove $\dfrac{1}{\log_8 a} - \dfrac{1}{\log_2 a} = \dfrac{1}{\log_4 a}$

$$\begin{aligned}
\text{L.S.} &= \frac{1}{\log_8 a} - \frac{1}{\log_2 a}\\
&= \log_a 8 - \log_a 2\\
&= \log_a\left(\frac{8}{2}\right)\\
&= \log_a 4\\
&= \frac{1}{\log_4 a}\\
&= \text{R.S.}
\end{aligned}$$

$\therefore \dfrac{1}{\log_8 a} - \dfrac{1}{\log_2 a} = \dfrac{1}{\log_4 a}$

c. To prove $\dfrac{2}{\log_6 a} = \dfrac{1}{\log_{36} a}$

$$\begin{aligned}
\text{L.S.} &= \frac{2}{\log_6 a}\\
&= 2(\log_a 6)\\
&= \log_a 6^2\\
&= \log_a 36\\
&= \frac{1}{\log_{36} a}\\
&= \text{R.S.}
\end{aligned}$$

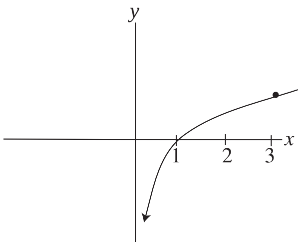
$\therefore \dfrac{2}{\log_6 a} = \dfrac{1}{\log_{36} a}$

d. To prove $\dfrac{2}{\log_8 a} - \dfrac{4}{\log_2 a} = \dfrac{1}{\log_4 a}$

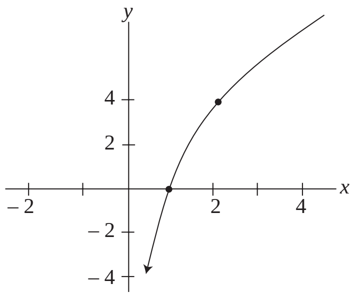
$$\begin{aligned}
\text{L.S.} &= \frac{2}{\log_8 a} - \frac{4}{\log_2 a}\\
&= 2\log_a 8 - 4\log_a 2\\
&= \log_a 8^2 - \log_a 2^4\\
&= \log_a\left(\frac{8^2}{2^4}\right)\\
&= \log_a 4\\
&= \frac{1}{\log_4 a}\\
&= \text{R.S.}
\end{aligned}$$

$\therefore \dfrac{2}{\log_8 a} - \dfrac{4}{\log_2 a} = \dfrac{1}{\log_4 a}$

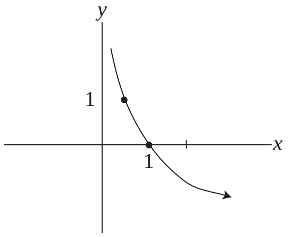
3. **a.** $y = \log_3 x$



b. $y = \log_{0.5} x$

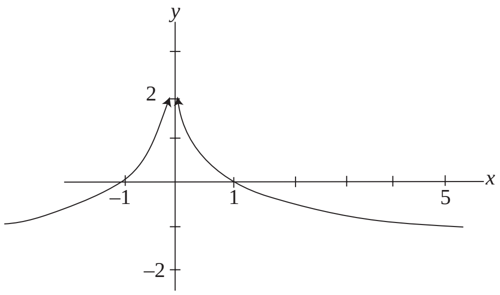


c. $y = 4\log_2 x$



d. $y = \log_{0.2} x^2$

Since x^2 is always positive, we can include negative values for x as well.



Solution 1

Given $a > 1$, $b > 1$,
show $(\log_a b)(\log_b a) = 1$.
Proof:

$$\log_a b = \frac{1}{\log_b a}$$

$$\therefore \text{ L.S.} = (\log_a b)(\log_b a)$$

$$= \frac{1}{\log_b a} \times \log_b a$$

$$= 1$$

$$= \text{R.S.}$$

Solution 2

$$(\log_a b)(\log_b a)$$

$$= (\log_a b) \times \frac{1}{(\log_a b)}$$

$$= 1$$

$$= \text{R.S.}$$

6. Noting that the L.S. has $(a + b)$, we find an expression for it in terms of $a^2 + b^2$.

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$\therefore\ a^2 + b^2 = (a + b)^2 - 2ab$$

But, we are given that $a^2 + b^2 = 23ab$.

$$\therefore\ (a + b)^2 - 2ab = 23ab$$

$$(a + b)^2 = 25ab \quad \text{or} \quad \frac{(a + b)^2}{25} = ab$$

That is $\left[\frac{(a + b)}{5}\right]^2 = ab$.

Taking logarithms of both sides, we have

$$\log\left[\frac{(a + b)}{5}\right]^2 = \log ab$$

$$2\log\left(\frac{a + b}{5}\right) = \log a + \log b$$

$$\therefore\ \log\left(\frac{a + b}{5}\right) = \frac{1}{2}(\log a + \log b).$$

7. $\log_a \frac{1}{x} = \log_{\frac{1}{a}} x$

Let $\log_a \frac{1}{x} = b$.

Then, in exponential form

$$a^b = \frac{1}{x}$$

$$a^b = x^{-1} \quad \text{or} \quad x = a^{-b}.$$

Taking logarithms of both sides, we have

$$\log_{\frac{1}{a}} x = \log_{\frac{1}{a}} a^{-b}$$

$$= -b \log \frac{1}{a} a$$

$$= -b \log_{\frac{1}{a}}\left(\frac{1}{a}\right)^{-1}$$

$$= -b(-1)$$

$$= b$$

$$= \log_a\left(\frac{1}{x}\right).$$

8. *Solution 1*

$$\log_a b = p^3$$

In exponential form, $a^{p^3} = b$ (1)

$$\log_b a = \frac{4}{p^2}$$

In exponential form $b^{\frac{4}{p^2}} = a$ (2)

Substituting for b from equation (1):

$$\left(\frac{p^3}{a}\right)^{\frac{4}{p^2}} = a$$

$$a^{4p} = a^1$$

$$\therefore\ 4p = 1$$

$$p = \frac{1}{4}$$

Solution 2

$$\log_a b = p^3$$

$$\therefore\ \log_b a = \frac{1}{p^3}$$

But, $\log_b a = \frac{4}{p^2}$:

$$\therefore\ \frac{4}{p^2} = \frac{1}{p^3}$$

$$4p = 1$$

$$p = \frac{1}{4}.$$

Solution 3

Since $\log_a b = p^3$,

$$\begin{aligned}\frac{1}{\log_a b} &= \frac{1}{p^3}\\ \log_b a &= \frac{1}{p^3}\\ \frac{4}{p^2} &= \frac{1}{p^3}\\ 4p &= 1\\ p &= \frac{1}{4}.\end{aligned}$$

9. Noting that $a^3 - b^3$ is given, but $a - b$ is required, we find $(a - b)$ in terms of $a^3 - b^3$.

Since $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$,

$$(a - b)^3 = (a^3 - b^3) - 3a^2b + 3ab^2.$$

But, it is given that $a^3 - b^3 = 3a^2b + 4ab^2$,

$$\begin{aligned}\therefore (a - b)^3 &= (3a^2b + 5ab^2) - 3a^2b + 3ab^2\\ (a - b)^3 &= 8ab^2\\ \frac{(a - b)^3}{8} &= ab^2\\ \left(\frac{a - b}{2}\right)^3 &= ab^2.\end{aligned}$$

Taking the logarithms of both sides,

$$\begin{aligned}\log\left(\frac{a - b}{2}\right)^3 &= \log(ab^2)\\ 3\log\left(\frac{a - b}{2}\right) &= \log a + 2\log b\\ \log\left(\frac{a - b}{2}\right) &= \frac{1}{3}(\log a + 2\log b).\end{aligned}$$

Review Exercise

2. **c.** $\log_5 \sqrt[3]{25} - \log_3\sqrt{27}$

$$\begin{aligned}&= \log_5 25^{\frac{1}{3}} - \log_3 27^{\frac{1}{3}}\\ &= \log_5(5^2)^{\frac{1}{3}} - \log_3(3^3)^{\frac{1}{3}}\\ &= \log_5 5^{\frac{2}{3}} - \log_3 3\\ &= \frac{2}{3} - 1 \quad \text{or} \quad -\frac{1}{3}\end{aligned}$$

d. $7^{\log_7 5}$

Let $\log_7 5 = x$.

$$\begin{aligned}7^x &= 5\\ \therefore 7^{\log_7 5} &= 7^{\log_7 7x}\\ &= 7^x\\ &= 5\end{aligned}$$

3. **b.** $\log(x + 3) + \log x = 1$

$\log x(x + 3) = 1$

In exponential form:

$$\begin{aligned}x(x + 3) &= 10^1\\ x^2 + 3x - 10 &= 0\\ (x + 5)(x - 2) &= 0\end{aligned}$$

$x + 5 = 0 \quad \text{or} \quad x - 2 = 0$

$x = -5 \quad \text{or} \quad x = 2$

But $\log x$ is not defined for $x = -5$,

$\therefore\ x = 2.$

c. $\log_5(x + 2) - \log_5(x - 1) = 2\log_5 3$

$$\begin{aligned}\log_5\left[\frac{x + 2}{x - 1}\right] &= \log_5 3^2,\ x \neq\ 1\\ \frac{x + 2}{x - 1} &= 9\\ x + 2 &= 9x - 9\\ -8x &= -11\\ x &= \frac{11}{8}\end{aligned}$$

d. $\dfrac{\log(35 - x^3)}{\log(5 - x)} = 3$

$$\begin{aligned}\log(35 - x^3) &= 3\log(5 - x)\\ \log(35 - x^3) &= \log(5 - x)^3\\ 35 - x^3 &= (5 - x)^3\\ 35 - x^3 &= 125 - 3(5)^2x + 3(5)x^2 - x^3\\ 35 - x^3 &= 125 - 75x + 15x^2 - x^3\\ 15x^2 - 75x + 90 &= 0\\ x^2 - 5x + 6 &= 0\\ (x - 3)(x - 2) &= 0\end{aligned}$$

$x - 3 = 0 \quad \text{or} \quad x - 2 = 0$

$x = 3 \quad \text{or} \quad x = 2$

4. For an earthquake of 7.2, the intensity is $I_J = 10^{7.2}I_0$.

For an earthquake of 6.9, the intensity is $I_A = 10^{6.9}I_0$.

$$\text{Comparing, } \frac{I_J}{I_A} = \frac{10^{7.2}I_0}{10^{6.9}I_0}$$
$$= 10^{7.2-6.9}$$
$$= 10^{0.3}$$
$$\doteq 1.99.$$

The earthquake in Kobe was twice as intense as that in Armenia.

5. Loudness of sound is given by $L = 10\log\left(\frac{I}{I_0}\right)$, where L is in decibels and I is the intensity.

$$\text{Morning noise is } 50 = 10\log\left(\frac{I_M}{I_0}\right)$$
$$5 = \log\left(\frac{I_m}{I_0}\right) \quad \text{or} \quad \frac{I_m}{I_0} = 10^5$$
$$I_m = 10^5 I_0.$$
$$\text{Similarly for noon noise: } 100 = 10\log\left(\frac{I_N}{I_0}\right)$$
$$10 = \log\left(\frac{I_n}{I_0}\right)$$
$$I_n = 10^{10}I_0.$$
$$\text{Comparing, } \frac{I_n}{I_m} = \frac{10^{10}I_0}{10^5 I_0}$$
$$= 10^5.$$

The noise at noon in the cafeteria is 10^5 or 100 000 times as loud as in the morning.

6. pH is defined as pH = $-\log[H^+]$.

For this liquid, $5.62 = -\log[H^+]$:

$$\log[H^+] = -5.62$$
$$[H^+] = 10^{-5.62}$$
$$\doteq 2.3988 \times 10^{-6}.$$

The hydrogen ion concentration is approximately 2.4×10^{-6} moles/L.

8. **a.** $\log_{19}264$

$$= \frac{\log 264}{\log 19}$$
$$\doteq 1.894$$

b. $\log_5 34.62$

$$= \frac{\log 34.62}{\log 5}$$
$$\doteq 2.202$$

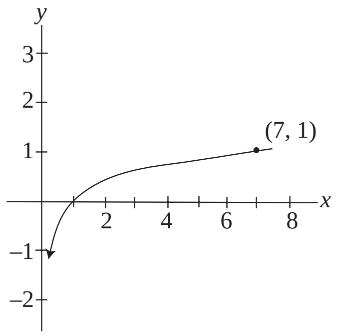
9. $$\frac{2}{\log_9 a} - \frac{1}{\log_3 a} = \frac{3}{\log_3 a}$$

$$\text{L.S.} = \frac{2}{\log_9 a} - \frac{1}{\log_3 a}$$
$$= 2\log_a 9 - \log_a 3$$
$$= \log_a 9^2 - \log_a 3$$
$$= \log_a \frac{81}{3}$$
$$= \log_a 27$$
$$\text{R.S.} = \frac{3}{\log_3 a}$$
$$= 3\log_a 3$$
$$= \log_a 3^3$$
$$= \log_a 27$$

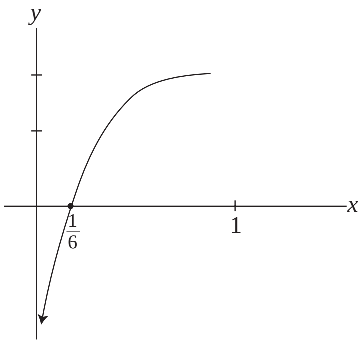
L.S. = R.S.

$$\therefore \frac{2}{\log_a a} - \frac{1}{\log_3 a} = \frac{3}{\log_3 a}$$

10. **a.** $y = \log_7 x$



b. $y = 2\log_6(6x)$



Chapter 7 Test

1. **a.** $\log_3 27 = 3$

$(3^3 = 27)$

b. $\log_5 125 = 3$

$(5^3 = 125)$

c. $\log_2 \frac{1}{16} = -4$

$\left(2^{-4} = \frac{1}{16}\right)$

d. $\log_5 \sqrt[4]{25}$

$= \log_5 25^{\frac{1}{4}}$

$= \frac{1}{4} \log_5 25$

$= \frac{1}{4} \times 2$

$= \frac{1}{2}$

e. $\log_2 8 + \log_3 9$

$= 3 + 2$

$= 5$

f. $\log_3 9^{\frac{1}{3}}$

$= \frac{1}{3} \log_3 9$

$= \frac{1}{3} \times 2$

$= \frac{2}{3}$

2. **a.** $\log_2 \frac{8}{5} + \log_2 10$

$= \log_2 \frac{8}{5} \times 10$

$= \log_2 16$

$= 4\ (2^4 = 16)$

b. $\log_6 108 - \log_6 3$

$= \log_6 \frac{108}{3}$

$= \log_6 36$

$= 2\ (6^2 = 36)$

3. Vertical stretch by a factor of two, translated two units up.

4. **a.** $2 \log x = 3 \log 4$

$\log x = \frac{3}{2} \log 4$

$\log x = \log 4^{\frac{3}{2}}$

$x = 4^{\frac{3}{2}}$

$x = 8$

b. $\log x + \log 3 = \log 12$

$\log 3x = \log 12$

$3x = 12$

$x = 4$

c. $\log_2(x + 2) + \log_2 x = 3$

$\log_2 x(x + 2) = 3$

$x(x + 2) = 2^3$

$x^2 + 2x - 8 = 0$

$(x + 4)(x - 2) = 0$

$x = -4$ or $x = 2$

But $x > 0$, therefore $x = -4$ is inadmissible.

Verify $x = 2$.

L.S. $= \log_2 4 + \log_2 2$

$= \log_2 8$

$= 3$

$=$ R.S.

Therefore, $x = 2$.

d. $\log_2(x - 2) + \log_2(x + 1) = 2$

$\log_2(x - 2)(x + 1) = 2$

$x^2 - x - 2 = 4$

$x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0$

$x = 3$ or $x = -2$

If $x = 3$,

L.S. $= \log_2 1 + \log_2 4$

$= \log_2 4$

$= 2$

$=$ R.S.

If $x = -2$,

L.S. $= \log_2 0 + \log_2(-1)$, which is not possible.

Therefore, $x = -2$ is inadmissible, and the answer is $x = 3$.

5. $\log_3(-9) = x$

or $3^x = -9$

There is no real value for x such that a power of 3 is a negative number.

6. The formula for half-life is given by $A(t) - A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$.

$A_0 = 20$

$A(t) = 15$

$t = 7$

$$15 = 20\left(\frac{1}{2}\right)^{\frac{7}{h}}$$

$$\frac{15}{20} = \left(\frac{1}{2}\right)^{\frac{7}{h}}$$

Take the logarithm of both sides by

$15 - \log 20 = \frac{7}{h} \log 0.5$:

$h(\log 15 - \log 20) = 7 \log 0.5$

$$h = \frac{7 \log 0.5}{\log 15 - \log 20}$$

$= 16.87$.

The half-life is 16.87 h.

7. For the earthquake in Tokyo, $I_T = 10^{8.3}I_0$.

For the earthquake in Guatemala, $I_G = 10^{7.5}I_0$.

Comparing the two earthquakes, $\frac{I_T}{I_G} = \frac{10^{8.3}I_0}{10^{7.5}I_0}$

$= 10^{0.8}$

$\doteq 6.3$.

The earthquake in Tokyo was six times more intense than the earthquake in Guatemala.

8. The loudness of sound is given by $L = 10 \log\left(\frac{I}{I_0}\right)$.

For the subway platform, $60 = 10 \log\left(\frac{I_s}{I_0}\right)$

$$\frac{I_s}{I_0} = 10^6$$

$$I_s = 10^6 I_0$$

For the subway train, $90 = 10 \log \left(\frac{I_r}{I_0}\right)$

$$\log\left(\frac{I_T}{I_0}\right) = 9$$

$$\frac{I_T}{I_0} = 10^9$$

Then, $I_T = 10^9 I_0$.

Comparing the sounds, $\frac{I_T}{I_S} = \frac{10^9 I_0}{10^6 I_0}$

$= 10^3$

$= 1000$.

The noise level is 1000 times more intense when the train arrives.

9. The pH level is defined by pH = $-\log[H^+]$.

For the liquid, $8.31 = -\log[H^+]$

$\log[H^+] = -8.31$

$H^+ = 10^{-8.31}$

$= 4.90 \times 10^{-9}$.

The hydrogen ion concentration is 4.9×10^{-9} moles/L.

10. Prove $\frac{3}{\log_2 a} = \frac{1}{\log_8 a}$.

L.S. $= \frac{3}{\log_2 a}$

$= 3\log_a 2$

R.S. $= \frac{1}{\log_8 a}$

$= \log_a 8$

$= \log_a 2^3$

$= 3\log_a 2$

Therefore, $\frac{3}{\log_2 a} = \frac{1}{\log_8 a}$.

11. $\log_a b = \frac{1}{x}$ and $\log_b \sqrt{a} = 3x^2$, $x = \frac{1}{6}$

$\log_a b = \frac{1}{x}$

$a^{\frac{1}{x}} = b$

$a = b^x$

$\log_b \sqrt{a} = 3x^2$

$b^{3x^2} = a^{\frac{1}{2}}$

$a = b^{6x^2}$

Therefore, $b^{6x^2} = b^x$ and $6x^2 = x$.

But $x \neq 0$, therefore, $6x = 1$,

$x = \frac{1}{6}$, as required.

Cumulative Review Solutions

Chapters 5–7

1. **c.** $x^2 + 16y^2 = 5x + 4y$

We differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(x^2 + 16y^2) = \frac{d}{dx}(5x + 4y)$$

$$2x + 32y\frac{dy}{dx} = 5 + 4\frac{dy}{dx}$$

$$(32y - 4)\frac{dy}{dx} = 5 - 2x$$

$$\frac{dy}{dx} = \frac{5 - 2x}{32y - 4}.$$

d. $2x^2 - xy + 2y = 5$

We differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(2x^2 - xy + 2y) = \frac{d}{dx}(5)$$

$$4x - \left(y + x\frac{dy}{dx}\right) + 2\frac{dy}{dx} = 0$$

$$(2 - x)\frac{dy}{dx} = y - 4x$$

$$\frac{dy}{dx} = \frac{y - 4x}{2 - x}.$$

f. $(2x + 3y)^2 = 10$

We differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(2x + 3y)^2 = \frac{d}{dx}(10)$$

$$2(2x + 3y)\left(2 + 3\frac{dy}{dx}\right) = 0$$

$$4(2x + 3y) + 6(2x + 3y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2}{3}.$$

2. **c.** $xy^2 + x^2y = 2$ at (1, 1)

The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$. We differentiate both sides of the equation with respect to x:

$$(1)y^2 + x\left(2y\frac{dy}{dx}\right) + (2x)y + x^2\frac{dy}{dx} = 0.$$

$$\text{At } (1, 1),\ 1 + 2\frac{dy}{dx} + 2 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1.$$

An equation of the tangent line at (1, 1) is
$y - 1 = -(x - 1)$ or $x + y - 2 = 0$

d. $y^2 = \frac{3x^2 + 9}{7x^2 - 4}$ at (1, 2)

The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$. We differentiate both sides of the equation with respect to x:

$$2y\frac{dy}{dx} = \frac{(6x)(7x^2 - 4) - (3x^2 + 9)(14x)}{(7x^2 - 4)^2}.$$

$$\text{At } (1, 2),\ 4\frac{dy}{dx} = \frac{(6)(3) - (12)(14)}{3^2}$$

$$\frac{dy}{dx} = -\frac{25}{6}.$$

An equation of the tangent line at (1, 2) is

$y - 2 = -\frac{25}{6}(x - 1)$ or $25x + 6y - 37 = 0$

3. **d.** $f(x) = x^4 - \frac{1}{x^4}$

$$f'(x) = 4x^3 + \frac{4}{x^5}$$

$$f''(x) = 12x^2 - \frac{20}{x^6}$$

4. **b.** $y = (x^2 + 4)(1 - 3x^3)$

$$\begin{aligned}\frac{dy}{dx} &= (2x)(1 - 3x^3) + (x^2 + 4)(-9x^2)\\ &= 2x - 6x^4 - 9x^4 - 36x^2\\ &= 2x - 15x^4 - 36x^2\end{aligned}$$

$$\frac{d^2y}{dx^2} = 2 - 60x^3 - 72x$$

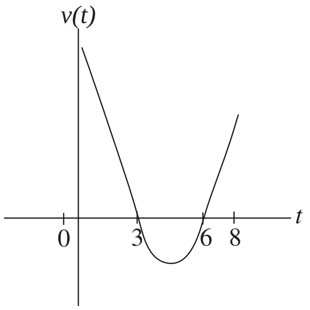
5. $s(t) = 3t^3 - 40.5t^2 + 162t$ for $0 \le t \le 8$

a. The position of the object at any time t in the interval is $s(t) = 3t^3 - 40.5t^2 + 162t$. The velocity of the object at any time t in the interval is $v(t) = s'(t) = 9t^2 - 81t + 162$. The acceleration of the object at any time t in the interval is $a(t) = v'(t) = 18t - 81$.

b. The object is stationary when $v(t) = 0$, i.e., when

$$\begin{aligned}9t^2 - 81t + 162 &= 0\\ t^2 - 9t + 18 &= 0\\ (t - 3)(t - 6) &= 0\\ t &= 3, 6.\end{aligned}$$

The object is stationary at $t = 3$ and at $t = 6$. The object is advancing (moving to the right) when $v(t) > 0$, i.e., when $9t^2 - 81t + 162 > 0$
$t^2 - 9t + 18 > 0$.



From the graph, we conclude that the object is advancing in the intervals $0 \le t < 3$ or $6 < t \le 8$. The object is retreating (moving to the left) when $v(t) < 0$. From the graph, we conclude that the object is retreating in the interval $3 < t < 6$.

c. The velocity is not changing when $a(t) = 0$.

Solving $18t - 81 = 0$ gives $t = \frac{9}{2}$.

d. The velocity of the object is decreasing when $a(t) < 0$.

Solving $18t - 81 < 0$ gives $t < \frac{9}{2}$.

e. The velocity of the object is increasing when $a(t) > 0$.
Solving $18t - 81 > 0$ gives $t > \frac{9}{2}$.

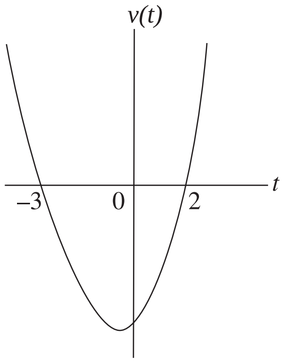
6. $x(t) = 2t^3 + 3t^2 - 36t + 40$

a. The velocity of the particle at time t is $v(t) = x'(t) = 6t^2 + 6t - 36$.

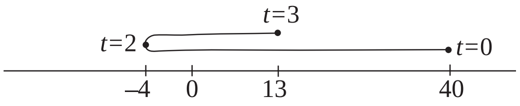
b. The acceleration of the particle at time t is $a(t) = x''(t) = 12t + 6$.

c. The particle is stationary when

$$\begin{aligned}v(t) &= 0\\ 6t^2 + 6t - 36 &= 0\\ (t + 3)(t - 2) &= 0\\ t &= 2, -3 \text{ is inadmissible.}\end{aligned}$$



Since $v(t) < 0$ for $0 \le t < 2$, the particle is moving to the left during this interval. The particle is stationary at $t = 2$. Since $v(t) > 0$ for $t > 2$, the particle is moving to the right for $2 < t \le 3$. The positions of the particle at times 0, 2, and 3 are 40, –4, and 13, respectively.



The total distance travelled by the particle during the first three seconds is 44 + 17 = 61.

7. **a. (ii)** $C(900) = \sqrt{900} + 8000$

$= \$8030$

b. (ii) The average cost per item is $\frac{8030}{900} = \$8.92$.

c. (ii) The marginal cost is $C'(x) = \frac{1}{2\sqrt{x}}$.

Thus, $C'(900) = \frac{1}{60} = \$0.017$.

The cost of producing the 901st item is $C(901) - (900) = 8030.017 - 8030$
$= \$0.017$.

8. $C(x) = 3x^2 + x + 48$

a. The average cost of producing x units is given by

$$\overline{C}(x) = \frac{C(x)}{x}$$

$$= \frac{3x^2 + x + 48}{x}$$

$$= 3x + 1 + \frac{48}{x}.$$

Hence, $\overline{C}(3) = \$26$, $\overline{C}(4) = \$25$, $\overline{C}(5) = \$25.60$, and $\overline{C}(6) = \$27$.

b. We first graph $y = 3x + 1 + \frac{48}{x}$, using the window $x_{min} = 1$, $x_{max} = 10$, $y_{min} = 20$, and $y_{max} = 30$.

One way of estimating the minimum value is to use the trace function. We can use the $\boxed{\text{ZOOM}}$ box to get a more accurate estimate of the minimum value. A second method to estimate the minimum value of $\overline{C}(x)$ is to use the $\boxed{\text{CALCULATE}}$ mode and press the minimum function. Enter a left bound (3.5), a right bound (4.5), and an estimate (4, or a value close to 4). The minimum will then be displayed. In this case, the minimum value displayed will be 25.

9. **a.** We differentiate both sides of the equation with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \text{ and } \frac{dy}{dt} = -\frac{x\frac{dx}{dt}}{y}.$$

To determine the value of $\frac{dy}{dt}$, we need the value of x, y, and $\frac{dx}{dt}$.

When $x = 3$, $y = \pm 3\sqrt{3}$.

There are two possible values for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = -\frac{12}{3\sqrt{3}} = -\frac{4}{\sqrt{3}} \text{ or } \frac{dy}{dt} = -\frac{12}{-3\sqrt{3}} = \frac{4}{\sqrt{3}}.$$

11. The volume of the spherical piece of ice at any time t is given by $V = \frac{4}{3}\pi r^3$. To find the rates of change of the volume and radius, we differentiate the equation with respect to t:

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

We are given that $\frac{dV}{dt} = -5$. When $r = 4$, we have $-5 = 64\pi\frac{dr}{dt}$ and $\frac{dr}{dt} = -\frac{5}{64\pi} \approx -0.025$ cm/min.

The surface area of a sphere is $A = 4\pi r^2$.

Thus, $\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$.

When $r = 4$, $\frac{dA}{dt} = 8\pi(4)\left(-\frac{5}{64\pi}\right) = -\frac{5}{2}$ cm^2/min.

Since the ice is melting, both the radius and the surface area are decreasing.

12. The volume of sand in the pile at any time t is

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3$, since $h = r$.

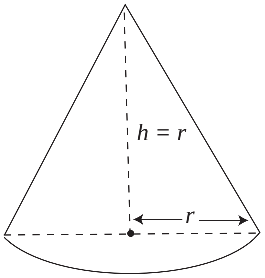
To find the rates of change of the volume and the height, we differentiate the equation with respect to t:

$$\frac{dV}{dt} = \pi h^2\frac{dh}{dt} \text{ and } \frac{dh}{dt} = \frac{dV}{dt}/\pi h^2.$$

To find the value of h when $V = 1050$, we solve:

$$1050 = \frac{1}{3}\pi h^3$$

$$h = \sqrt[3]{\frac{3150}{\pi}} \approx 10$$



When the volume of sand in the pile is 1050 m^3, the height of the pile is increasing at the rate:

$$\frac{dh}{dt} = \frac{10}{100\pi} = \frac{1}{10\pi} \text{ m/h.}$$

14. **e.** $\dfrac{e^3e^{-2x}}{e^{-x}}$

$= e^{3-2x+x}$

$= e^{3-x}$

f. $(e^{4x})^3$

$= e^{12x}$

15. **b.**
$$\begin{aligned} 3x^2 - 3 &= 81^x \\ 3^{x^2-3} &= 3^{4x} \\ x^2 - 3 &= 4x \\ x^2 - 4x - 3 &= 0 \end{aligned}$$

$$x = 4 \pm \frac{\sqrt{16 + 12}}{2}$$

$$= 2 \pm \sqrt{7}$$

d.
$$\begin{aligned} 2^{2x} - 12(2^x) + 32 &= 0 \\ (2^x)^2 - 12(2^x) + 32 &= 0 \\ (2^x - 4)(2^x - 8) &= 0 \\ x^x = 4 \text{ or } x^x &= 8 \\ x = 2 \text{ or } x &= 3 \end{aligned}$$

e. $e^x = 1$

$x = \ln 1$

$= 0$

f.
$$\begin{aligned} e^{2x} + e^x - 2 &= 0 \\ (e^x)^2 + e^x - 2 &= 0 \\ (e^x + 2)(e^x - 1) &= 0 \end{aligned}$$

$e^x = -2$, inadmissible since $e^x > 0$ for all x

or $e^x = 1$

$x = 0$

16. $N(t) = \dfrac{80\ 000}{1 + 10e^{-0.2t}}$

a. The number of subscribers after six months will be:

$N(6) = \dfrac{80\ 000}{1 + 10e^{-1.2}} \doteq 19\ 940.$

b. Eventually, the number of subscribers will be:

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{80\ 000}{1 + \dfrac{10}{e^{0.2t}}}$$

$$= \frac{80\ 000}{1 + 0}$$

$$= 80\ 000.$$

18. **a.** $C(t) = C_0(1 + 0.05)^t$, where C_0 is the present cost of goods and t is in years from now.

b. $C(10) = 39.95(1.05)^{10} = 65.07$

Ten years from now, a mechanical inspection of your car will cost $65.07.

c. $C(10) = 40.64 = C_0(1.05)^{10}$

$$C_0 = \frac{40.64}{(1.05)^{10}} = 24.95$$

The price of an oil change today is $24.95.

19. **a.** $V(t) = 30\ 000(1 - 0.25)^t$, t in years from purchase date.

b. The value of the car two years after the purchase date is $V(2) = 30\ 000(0.75)^2 = \$16\ 875$.

c. $V(t) = 30\ 000(0.75)^t = 3000$

$$(0.75)^t = \frac{1}{10}$$

$$t\ln(0.75) = -\ln 10$$

$$t = \frac{-\ln 10}{\ln(0.75)} \doteq 8$$

In approximately eight years, the car will be worth $3000.

22. **f.** $\log_4\left(\dfrac{1}{8}\right) = \log_4(2^{-3})$

$= \log_4(4^{-\frac{3}{2}})$

$= -\dfrac{3}{2}$

k. $10^{-10\log 3}$

$= \dfrac{1}{10^{10\log 3}}$

$= \dfrac{1}{10^{\log 3^{10}}}$

$= \dfrac{1}{3^{10}}$

l. $a^{8\log_a\sqrt{a}}$

$= a^{\log_a a^4}$

$= a^4$

24. **c.** $2\log 3 - \dfrac{1}{2}\log(x^2 + 1) = \log 9 - \log\sqrt{x^2 + 1}$

$$= \log\left(\frac{9}{\sqrt{x^2 + 1}}\right)$$

d. $\log x - 4\log(x-5) + \frac{2}{3}\log\sqrt{x+1}$

$= \log x - \log(x-5)^4 + \log\,[(x+1)^{\frac{1}{2}}]^{\frac{2}{3}}$

$= \log\left(\frac{x(x+1)^{\frac{1}{3}}}{(x-5)^4}\right)$

26. c. $x - 3\log_3 243 = 4\log_2\sqrt{512}$

$x - 3(5) = \frac{1}{2} \bullet 4\log_2 2^9$

$x - 15 = 2.9$

$x = 33$

e. $2\log_3(4x+1) = 4$

$\log_3(4x+1) = 2$

$4x + 1 = 3^2$

$4x = 8$

$x = 2$

f. $\log_{12} x - \log_{12}(x-2) + 1 = 2$

$\log_{12}\left(\frac{x}{x-2}\right) = 1$

$\frac{x}{x-2} = 12^1$

$x = 12x - 24$

$11x = 24$

$x = \frac{24}{11}$

j. $(\log x)^2 + 3\log x - 10 = 0$

$(\log x + 5)(\log x - 2) = 0$

$\log x = -5$ or $\log x = 2$

$x = 10^{-5}$ or $x = 10^2$

27. $SL = 10\log(I\ 10^{12})$

a. $SL = 10\log(2.51 \times 10^{-5} \times 10^{12})$

$= 10\log(2.51 \times 10^7)$

$= 10[\log 2.51 + \log 10^7)$

$= 10[0.3997 + 7)$

$\doteq 74$ dB

b. $SL = 10\log(6.31 \times 10^{-4} \times 10^{12})$

$= 10[\log 6.31 + 8]$

$= 10[0.80 + 8]$

$= 88$ dB

The sound level in the room is bearable to the human ear.

c. $50 = 10\log(I \times 10^{12})$

$5 = \log(I \times 10^{12})$

$I \times 10^{12} = 10^5$

$I = 10^{-7}$

$= 1.0 \times 10^{-7}$

d. $110 = 10\log(I \times 10^{12})$

$11 = \log(I \times 10^{12})$

$I \times 10^{12} = 10^{11}$

$I = 10^{-1} = 1.0 \times 10^{-1}$

Chapter 8 • Derivatives of Exponential and Logarithmic Functions

Review of Prerequisite Skills

4. **a.** $\log_2 32$

Since $32 = 2^5$, $\log_2 32 = 5$.

b. $\log_{10} 0.0001$

Since $0.0001 = 10^{-4}$, $\log_{10} 0.0001 = -4$.

c. $\log_{10} 20 + \log_{10} 5$

$= \log_{10}(20 \times 5)$

$= \log_{10} 100$

$= \log_{10} 10^2$

$= 2$

d. $\log_2 20 - \log_2 5$

$$= \log_2\left(\frac{20}{5}\right)$$

$= \log_2 4$

$= 2$

e. $3^{2\log_3 5}$

$= (3^{\log_3 5})^2$

$= 5^2$

$= 25$

f. $\log_3(5^3 9^{-3} 25^{-\frac{3}{2}})$

$= \log_3 5^3 + \log_3 9^{-3} + \log_3 25^{-\frac{3}{2}}$

$= \log_3 5^3 + \log_3 3^{-6} + \log_3 5^{-3}$

$= 3\log_3 5 - 6 - 3\log_3 5$

$= -6$

5. **a.** $\log_2 80$, $b = e$

$$= \frac{\log_e 80}{\log_e 2}$$

$$= \frac{\ln 80}{\ln 2}$$

$\doteq 6.322$

b. $3\log_5 22 - 2\log_5 15$, $b = 10$

$$= 3\left(\frac{\log_{10} 22}{\log_{10} 5}\right) - 2\left(\frac{\log_{10} 15}{\log_{10} 5}\right)$$

$$= \frac{3\log_{10} 22 - 2\log_{10} 15}{\log_{10} 5}$$

$\doteq 2.397$

Exercise 8.1

4. **c.** $f(x)\ \dfrac{e^{-x^3}}{x}$

$$f'(x) = \frac{-3x^2e^{-x^3}(x) - e^{-x^3}}{x^2}$$

d. $s = \dfrac{e^{3t^2}}{t^2}$

$$\frac{ds}{dt} = 6te^{3t^2}(t^2) - 2t(e^{3t^2})$$

$$= \frac{2e^{3t^2}[3t^2 - 1]}{t^3}$$

h. $g(t) = \dfrac{e^{2t}}{1 + e^{2t}}$

$$g'(t) = \frac{2e^{2t}(1 + e^{2t}) - 2e^{2t}(e^{2t})}{(1 + e^{2t})^2}$$

$$= \frac{2e^{2t}}{(1 + e^{2t})^2}$$

5. **a.** $f'(x) = \dfrac{1}{3}(3e^{3x} - 3e^{-3x})$

$= e^{3x} - e^{-3x}$

$f'(1) = e^3 - e^{-3}$

c. $h'(z) = 2z(1 + e^{-z}) + z^2(-e^{-z})$

$h'(-1) = 2(-1)(1 + e) + (-1)^2(-e^1)$

$= -2 - 2e - e$

$= -2 - 3e$

7. $y = e^x$

Slope of the tangent is $\dfrac{dy}{dx} = e^x$.

Slope of the given line is –3.

Slope of the perpendicular line is $\dfrac{1}{3}$.

Therefore, $e^x = \dfrac{1}{3}$:

$x \ln e = \ln 1 - \ln 3$

$x = -\ln 3$

$\doteq -1.099$.

The point where the tangent meets the curve has $x = -\ln 3$ and $y = 3^{-\ln 3}$

$$= \frac{1}{3}.$$

The equation of the tangent is

$$y - \frac{1}{3} = \frac{1}{3}(x + \ln 3) \quad \text{or} \quad y = 0.3x + 0.6995.$$

8. The slope of the tangent line at any point is given by

$$\frac{dy}{dx} = (1)(e^{-x}) + x(-e^{-x})$$

$$= e^{-x}(1 - x).$$

At the point $(1, e^{-1})$, the slope is $e^{-1}(0) = 0$. The equation of the tangent line at the point A is

$$y - e^{-1} = 0(x - 1) \quad \text{or} \quad y = \frac{1}{e}.$$

9. The slope of the tangent line at any point on the curve is $\frac{dy}{dx} = 2xe^{-x} + x^2(-e^{-x})$

$$= (2x - x^2)(e^{-x})$$

$$= \frac{2x - x^2}{e^x}.$$

Horizontal lines have slope equal to 0.

We solve $\frac{dy}{dx} = 0$

$$\frac{x(2 - x)}{e^x} = 0.$$

Since $e^x > 0$ for all x, the solutions are $x = 0$ and $x = 2$. The points on the curve at which the tangents are horizontal are $(0, 0)$ and $\left(2, \frac{4}{e^2}\right)$.

10. If $y = \frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$,

then $y' = \frac{5}{2}\left(\frac{1}{5}e^{\frac{x}{5}} - \frac{1}{5}e^{-\frac{x}{5}}\right)$,

and $y'' = \frac{5}{2}\left(\frac{1}{25}e^{\frac{x}{5}} + \frac{1}{25}e^{-\frac{x}{5}}\right)$

$$= \frac{1}{25}\left[\frac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})\right]$$

$$= \frac{1}{25}y.$$

11. b. $\frac{d^ny}{dx^n} = (-1)^n(3^n)e^{-3x}$

12. In this question, y is an implicitly defined function of x.

a.

$$\frac{dy}{dx} - \frac{de^{xy}}{dx} = 0$$

$$\frac{dy}{dx} - e^{xy}\left((1)y + x\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} - ye^{xy} - xe^{xy}\frac{dy}{dx} = 0$$

At the point $(0, 1)$, we get

$$\frac{dy}{dx} - 1 - 0 = 0 \quad \text{and} \quad \frac{dy}{dx} = 1.$$

The equation of the tangent line at $A(0, 1)$ is $y - 1 = x$ or $y = x + 1$.

b. $\frac{d}{dx}(x^2e^y) = 0$

$$2xe^y + x^2e^y\frac{dy}{dx} = 0$$

At the point $(1, 0)$, we get

$$2 + \frac{dy}{dx} = 0$$

and $\frac{dy}{dx} = -2$.

The equation of the tangent line at $B(1, 0)$ *is* $y = -2(x - 1)$ or $2x + y - 2 = 0$.

c. It is difficult to determine y as an explicit function of x.

13. a. When $t = 0$, $N = 1000[30 + e^0] = 31\,000$.

b. $\frac{dN}{dt} = 1000\left[0 - \frac{1}{30}e^{\frac{-t}{30}}\right] = -\frac{100}{3}e^{-\frac{t}{30}}$

c. When $t = 20h$, $\frac{dN}{dt} = -\frac{100}{3}e^{-\frac{2}{3}} \doteq -17$ bacteria/h.

d. Since $e^{-\frac{t}{30}} > 0$ for all t, there is no solution to $\frac{dN}{dt} = 0$.

Hence, the maximum number of bacteria in the culture occurs at an endpoint of the interval of domain.

When $t = 50$, $N = 1000[30 + e^{-\frac{5}{3}}] \doteq 30\ 189$.

The largest number of bacteria in the culture is 31 000 at time $t = 0$.

14. **a.** $v = \dfrac{ds}{dt} = 160\left(\dfrac{1}{4} - \dfrac{1}{4}\, e^{-\frac{t}{4}}\right)$

$= 40(1 - e^{\frac{t}{4}})$

b. $a = \dfrac{dv}{dt} = 40\left(\dfrac{1}{4}\, e^{-\frac{t}{4}}\right) = 10e^{-\frac{t}{4}}$

From **a.**, $v = 40(1 - e^{-\frac{t}{4}})$,

which gives $e^{-\frac{t}{4}} = 1 - \dfrac{v}{40}$.

Thus, $a = 10\left(1 - \dfrac{v}{40}\right) = 10 - \dfrac{1}{4}v$.

c. $v_T = \lim\limits_{t\to\infty} v$

$$v_T = \lim_{t\to\infty} 40(1 - e^{-\frac{t}{4}})$$

$$= 40 \lim_{t\to\infty}\left(1 - \frac{1}{e^{\frac{t}{4}}}\right)$$

$$= 40(1), \text{ since } \lim_{t\to\infty} \frac{1}{e^{\frac{t}{4}}} = 0$$

The terminal velocity of the skydiver is 40 m/s.

d. Ninety-five per cent of the terminal velocity is

$\dfrac{95}{100}(40) = 38$ m/s.

To determine when this velocity occurs, we solve

$$40(1 - e^{-\frac{t}{4}}) = 38$$

$$1 - e^{-\frac{t}{4}} = \frac{38}{40}$$

$$e^{-\frac{t}{4}} = \frac{1}{20}$$

$$e^{\frac{t}{4}} = 20$$

$$\text{and } \frac{t}{4} = \ln 20,$$

which gives $t = 4$

$\ln 20 \doteq 12$ s.

The skydiver's velocity is 38 m/s, 12 s after jumping. The distance she has fallen at this time is

$$S = 160(\ln 20 - 1 + e^{-\ln 20})$$

$$= 160\left(\ln 20 - 1 + \frac{1}{20}\right)$$

$$\doteq 327.3 \text{ m.}$$

15. **a.** The given limit can be rewritten as

$$\lim_{h\to 0} \frac{e^h - 1}{h} = \lim_{h\to 0} \frac{e^{0+h} - e^0}{h}$$

This expression is the limit definition of the derivative at $x = 0$ for $f(x) = e^x$.

$$\left[f'(0) = \lim_{h\to 0} \frac{e^{0+h} - e^0}{h}\right]$$

Since $f'(x) = \dfrac{de^x}{dx} = e^x$, the value of the given limit is $e^0 = 1$.

b. Again, $\lim\limits_{h\to 0} \dfrac{e^{2+h} - e^2}{h}$ is the derivative of e^x at $x = 2$.

Thus, $\lim\limits_{h\to 0} \dfrac{e^{2+h} - e^2}{h} = e^2$.

16. For $y = Ae^{mt}$, $\dfrac{dy}{dt} = Ame^{mt}$ and $\dfrac{d^2y}{dt^2} = Am^2e^{mt}$.

Substituting in the differential equation gives

$$Am^2e^{mt} + Ame^{mt} - 6Ae^{mt} = 0$$

$$Ae^{mt}(m^2 + m - 6) = 0.$$

Since $Ae^{mt} \neq 0$, $m^2 + m - 6 = 0$

$$(m + 3)(m - 2) = 0$$

$$m = -3 \quad \text{or} \quad m = 2.$$

17. **a.** $D_x \sinh x = \cosh x$

$$D_x \sinh x = D_x\left[\frac{1}{2}(e^x - e^{-x})\right]$$

$$= \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

b. $D_x \cosh x = \sinh x$

$$D_x \cosh x = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

c. $D_x \tanh x = \dfrac{1}{(\cosh x)^{2g}}$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Since $\tanh x = \dfrac{\sinh x}{\cosh x}$,

$$D_x \tanh x = \frac{(D_x \sinh x)(\cosh x) - (\sinh x)(D_x \cosh x)}{(\cosh x)^2}$$

$$= \frac{\frac{1}{2}\left(e^x + e^{-x}\right)\left(\frac{1}{2}\right)\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right)\left(\frac{1}{2}\right)\left(e^x - e^{-x}\right)}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}(4)}{(\cosh x)^2}$$

$$= \frac{1}{(\cosh x)^2}$$

Exercise 8.2

2. Since $e = \lim_{h\to 0} (1 + h)^{\frac{1}{h}}$, let $h = \frac{1}{n}$. Therefore,

$$e = \lim_{\frac{1}{n}\to 0} \left(1 + \frac{1}{n}\right)^n.$$

But as $\frac{1}{n} \to 0$, $n \to \infty$.

Therefore, $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

$$\begin{aligned} \text{If } n = 100,\ e &\doteq \left(1 + \frac{1}{100}\right)^{100} \\ &= 1.01^{100} \\ &\doteq 2.70481. \end{aligned}$$

Try $n = 100\,000$, etc.

4. **f.** $g(z) = 1\text{n}\,(e^{-z} + ze^{-z})$

$$\begin{aligned} g'(z) &= \frac{1}{e^{-z} + ze^{-z}}\left[-e^{-z} + (e^{-z} - ze^{-z})\right] \\ &= \frac{-ze^{-z}}{e^{-z} + ze^{-z}} \end{aligned}$$

h.

$$\begin{aligned} h(u) &= e^{\sqrt{u}} \ln u^{\frac{1}{2}} \\ &= e^{\sqrt{u}}\left(\frac{1}{2} \ln u\right) \\ h'(u) &= e^{\sqrt{u}}\left(\frac{1}{2\sqrt{u}}\right)\left(\frac{1}{2}\ln u\right) + \frac{1}{2}\left(\frac{1}{u}\right)e^{\sqrt{u}} \\ &= \frac{1}{2}\, e^{\sqrt{u}}\left(\frac{1}{2}e^{1\sqrt{u}} \ln u + \frac{1}{u}\right) \end{aligned}$$

i. $f(x) = \ln\left(\frac{x^2 + 1}{x - 1}\right)$

$$\begin{aligned} f'(x) &= \frac{\frac{1}{x^2 + 1}}{x - 1}\left[\frac{2x(x - 1) - (x^2 + 1)}{(x - 1)^2}\right] \\ &= \frac{x - 1}{x^2 + 1}\left[\frac{x^2 - 2x - 1}{(x - 1)^2}\right] \\ &= \frac{x^2 - 2x - 1}{(x - 1)(x^2 + 1)} \end{aligned}$$

5. **a.**

$$\begin{aligned} g(x) &= e^{2x-1} \ln (2x - 1) \\ g'(x) &= e^{2x-1}(2) \ln (2x - 1) + \left(\frac{1}{2x - 1}\right)(2)e^{2x-1} \\ g'(1) &= e^{2}(2) \ln (1) + 1(2)\, e^{1} \\ &= 2e \end{aligned}$$

b.

$$\begin{aligned} f(t) &= 1\text{n}\left(\frac{t - 1}{3t + 5}\right) \\ f'(t) &= \left(\frac{3t + t}{t - 1}\right)\left[\frac{3t + 5 - 3(t - 1)}{(3t + 5)^2}\right] \\ f'(5) &= \frac{20}{4}\left[\frac{20 - 12}{20^2}\right] \\ &= \frac{8}{4 \times 20} \\ &= \frac{1}{10} \\ &= 0.1 \end{aligned}$$

6. **a.**

$$\begin{aligned} f(x) &= \ln (x^2 + 1) \\ f'(x) &= \left(\frac{1}{1 + x^2}\right)(2x) \\ &= \frac{2x}{1 + x^2} \end{aligned}$$

Since $1 + x^2 > 0$ for all x, $f'(x) = 0$ when $2x = 0$, i.e., when $x = 0$.

b.

$$\begin{aligned} f(x) &= (1\text{n}\, x + 2x)^{\frac{1}{3}} \\ f'(x) &= \frac{1}{3}(\ln x + 2x)^{-\frac{2}{3}}\left(\frac{1}{x} + 2\right) \\ &= \frac{\frac{1}{x} + 2}{3(\ln x + 2x)^{\frac{2}{3}}} \end{aligned}$$

$f'(x) = 0$ if $\frac{1}{x} + 2 = 0$ and $(\ln x + 2x)^{\frac{2}{3}} \neq 0$.

$\frac{1}{x} + 2 = 0$ when $x = -\frac{1}{2}$.

Since $f(x)$ is defined only for $x > 0$, there is no solution to $f'(x) = 0$.

c.

$$\begin{aligned} f(x) &= (x^2 + 1)^{-1} \ln(x^2 + 1) \\ f'(x) &= -(x^2 + 1)^{-2}\, (2x)\ln(x^2 + 1) + (x^2 + 1)^{-1}\left(\frac{2x}{x^2 + 1}\right) \\ &= \frac{2x(1 - \ln(x^2 + 1))}{(x^2 + 1)^2} \end{aligned}$$

Since $(x^2 + 1)^2 \geq 1$ for all x, $f'(x) = 0$,
when $2x(1 - \ln(x^2 + 1)) = 0$.
Hence, the solution is

$$x = 0 \quad \text{or} \quad \begin{aligned} \ln (x^2 + 1) &= 1 \\ x^2 + 1 &= e \\ x &= \pm \sqrt{e - 1}. \end{aligned}$$

7. **a.** $f(x) = \dfrac{\ln \sqrt[3]{x}}{x}$

$$= \frac{\frac{1}{3}\ln x}{x}$$

$$f'(x) = \frac{1}{3}\left[\frac{\frac{1}{x} \bullet x - \ln x}{x^2}\right]$$

At the point (1, 0), the slope of the tangent line is

$$f'(1) = \frac{1}{3}\left[\frac{1-0}{1}\right]$$

$$= \frac{1}{3}.$$

The equation of the tangent line is $y = \frac{1}{3}\left(x - 1\right)$

or $x - 3y - 1 = 0$.

b. Use the $\boxed{y =}$ button to define $f(x)$ and set the window so $-1 \le x \le 4$ and $-2 \le y \le 0.5$.

Select $\boxed{2^{\text{ND}}}$ $\boxed{\text{DRAW}}$ and pick menu item five to draw the tangent at the point (1, 0).

c. The calculator answer is $y = 0.31286x - 0.31286$.

This can be improved using the $\boxed{\text{ZOOM}}$ feature.

8. The line defined by $3x - 6y - 1 = 0$ has slope $\frac{1}{2}$.

For $y = \ln x - 1$, the slope at any point is $\frac{dy}{dx} = \frac{1}{2}$.

Therefore, at the point of tangency $\frac{1}{x} = \frac{1}{2}$,

or $x = 2$ and $y = \ln 2 - 1$.

The equation of the tangent is

$$y - (\ln 2 - 1) = \frac{1}{2}\left(x - 2\right)$$

or $x - 2y + (2\ln 2 - 4) = 0$

9. **a.** For a horizontal tangent line, the slope equals 0.

We solve:

$$f'(x) = 2(x \ln x)\left(\ln x + x \bullet \frac{1}{x}\right) = 0$$

$x = 0$ or $\ln x = 0$ or $\ln x = -1$

No ln in the domain $x = 1$ $x = e^{-1} = \frac{1}{e}$

The points on the graph of $f(x)$ at which there are horizontal tangents are $\left(\frac{1}{e}, \frac{1}{e^2}\right)$ and (1, 0).

b. Graph the function and use the $\boxed{\text{TRACE}}$ and $\boxed{\text{CALC}}$ $\frac{dy}{dx}$ features to determine the points where $\frac{dy}{dx} = 0$.

c. The solution in **a.** is more precise and efficient.

11. $v(t) = 90 - 30\ln(3t + 1)$

a. At $t = 0$, $v(0) = 90 - 30\ln(1) = 90$ km/h.

b. $a = v'(t) = \dfrac{-30}{3t+1} \bullet 3 = \dfrac{-90}{3t+1}$

c. At $t = 2$, $a = -\frac{90}{7} \doteq -12.8$ km/h/s.

d. The car is at rest when $v = 0$.

We solve:

$$v(t) = 90 - 30\ln(3t + 1) = 0$$

$$\ln(3t + 1) = 3$$

$$3t + 1 = e^3$$

$$t = \frac{e^3 - 1}{3} = 6.36 \text{ s.}$$

12. **a.** $\text{pH} = -\log_{10}(6.3 \times 10^{-5})$

$= -[\log_{10}6.3 + \log_{10}10^{-5}]$

$\doteq -[0.7993405 - 5]$

$\doteq 4.20066$

The pH value for tomatoes is approximately 4.20066.

b. $H(t) = 30 - 5t - 25(e^{-\frac{t}{5}} - 1)$

$$\text{pH} = -\frac{\ln(30 - 5t - 25(e^{-\frac{t}{5}} - 1))}{\ln 10}$$

$$= -\frac{1}{\ln 10}\ln(55 - 5t - 25e^{-\frac{t}{5}})$$

$$\frac{d}{dt}\text{pH} = -\frac{1}{\ln 10} \bullet \frac{-5 + 5e^{-\frac{t}{5}}}{55 - 5t - 25e^{-\frac{t}{5}}}$$

$$= -\frac{1}{\ln 10} \bullet \frac{-1 + e^{-\frac{t}{5}}}{11 - t - 5e^{-\frac{t}{5}}}$$

When $t = 10$ s, $\dfrac{d}{dt}\text{pH} = -\dfrac{1}{\ln 10} \bullet \dfrac{-1 + e^{-2}}{1 - 5e^{-2}}$

$$= \frac{1}{\ln 10} \bullet \frac{e^2 - 1}{e^2 - 5}$$

$$\doteq 1.16.$$

13. $\dfrac{d^2F}{dS^2} = F - 18ke^{-2S}$

$F = k(e^{-s} - 6e^{-2S})$

$\dfrac{dF}{dS} = k(-e^{-s} + 12e^{-2S})$

$\dfrac{d^2F}{dS^2} = k(e^{-s} - 24e^{-2S})$

$= k(e^{-s} - 6e^{-2S} - 18e^{-2S})$

$= k(e^{-S} - 6e^{-2S}) - 18ke^{-2S}$

$= F - 18ke^{-2S}$

14. a. We assume y is an implicitly defined function of x, and differentiate implicitly with respect to x.

$$(1)(e^{4}) + x(e^{y})\frac{dy}{dx} + \frac{dy}{dx}\ln x + y\left(\frac{1}{x}\right) = 0$$

At the point (1, ln 2) the derivative equation simplifies to

$$(1)(e^{\ln 2}) + (1)(e^{\ln 2})\frac{dy}{dx} + \frac{dy}{dx}\ln(1) + \ln 2(1) = 0$$

$$2 + 2\frac{dy}{dx} + 0 + \ln 2 = 0$$

$$\frac{dy}{dx} = \frac{-2 - \ln 2}{2}$$

The slope of the tangent to the curve at (1, ln 2) is $-\dfrac{2 + \ln 2}{2}$

b. $\ln\sqrt{xy} = 0$

$$\frac{1}{2}\ln(xy) = 0$$

$$\ln(xy) = 0$$

$$xy = e^{0} = 1$$

$$y = \frac{1}{x}$$

$$\frac{dy}{dx} = -\frac{1}{x^{2}}$$

The slope of the tangent to the curve at $\left(\frac{1}{3}, 3\right)$ is –9.

15. By definition, $\dfrac{d}{dx}\ln x = \lim\limits_{h\to 0}\dfrac{\ln(2+h) - \ln 2}{h}$

$$= \frac{1}{x}.$$

The derivative of ln x at $x = 2$ is

$$\lim_{h\to 0}\frac{\ln(2+h) - \ln 2}{h} = \frac{1}{2}.$$

16. a.

$$\left(1+\frac{1}{n}\right)^{n} = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^{2} + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^{3} + \ldots$$

$$= 1 + 1 + \frac{1}{2!}(1)\left(1-\frac{1}{n}\right) + \frac{1}{3!}(1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{1}{2!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \ldots$$

Thus, $e = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \ldots$

b. $S_3 = 1 + 1 + \dfrac{1}{2!} = 2.5$

$$S_4 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} = 2.\dot{6}$$

$$S_5 = S_4 + \frac{1}{4!} = 2\frac{2}{3} + \frac{1}{24} = 2.708\dot{3}$$

$$S_7 = S_6 + \frac{1}{6!} = 2.71\dot{6} + \frac{1}{720} = 2.7180\dot{5}$$

17. a. $y = \ln|x| = \begin{cases}\ln x, \text{ if } x > 0\\ \ln(-x), \text{ if } x < 0\end{cases}$

$$\frac{dy}{dx} = \begin{cases}\dfrac{1}{x}, \text{ if } x > 0\\ \dfrac{1}{-x} \bullet -1, \text{ if } x < 0\end{cases}$$

$$= \begin{cases}\dfrac{1}{x}, \text{ if } x > 0\\ \dfrac{1}{x}, \text{ if } x < 0\end{cases}$$

Thus, $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$ for all $x \neq 0$.

b. $\dfrac{dy}{dx} = \dfrac{1}{2x+1} \bullet 2 = \dfrac{2}{2x+1}$

c. $\dfrac{dy}{dx} = 2x\ln|x| + x^{2}\left(\dfrac{1}{x}\right)$

$$= 2x\ln|x| + x$$

Exercise 8.3

2. e. $f(x) = \dfrac{3^{\frac{x}{2}}}{x^{2}}$

$$f'(x) = \frac{\frac{1}{2}\ln 3(3^{\frac{x}{2}})(x^{2}) - 2x(3^{\frac{x}{2}})}{x^{4}}$$

$$= \frac{x\ln 3(3^{\frac{x}{2}}) - 4(3^{\frac{x}{2}})}{2x^{4}}$$

$$= \frac{3^{\frac{x}{2}}[x\ln 3 - 4]}{2x^{4}}$$

f. $\dfrac{\log_5(3x^{2})}{\sqrt{x+1}}$

$$f'(x) = \frac{\frac{1}{\ln 5(3x^{2})}(6x)(\sqrt{x+1}) - \frac{1}{2}(x+1)^{-\frac{1}{2}}(\log_5 3x^{2})}{x+1}$$

3. **a.** $f(t) = \log_2\left(\frac{t+1}{2t+7}\right)$

$$f(t) = \log_2(t+1) - \log_2(2t+1)$$

$$f'(t) = \frac{1}{(t+1)\ln 2} - \frac{2}{(2t+1)\ln 2}$$

$$f'(3) = \frac{1}{4\ln 2} - \frac{2}{13\ln 2}$$

$$= \frac{13}{52\ln 2} - \frac{8}{52\ln 2}$$

$$= \frac{5}{52\ln 2}$$

b. $h(t) = \log_3[\log_2(t)]$

$$h'(t) = \frac{1}{\ln 3(\log_2 t)} \times \frac{1}{\ln 2(t)}$$

$$h'(8) = \frac{1}{\ln 3 \log_2 8} \times \frac{1}{8(\ln 2)}$$

$$= \frac{1}{3\ln 3} \times \frac{1}{8\ln 2}$$

$$= \frac{1}{24(\ln 3)(\ln 2)}$$

5. **a.** $\frac{dy}{dx} = \log_{10}(x^2 - 3x)(\ln 10(10^{2x-9})2) + 10^{2x-9}\left[\frac{1(2x-3)}{\ln 10\,(x^2 - 3x)}\right]$

At $x = 5$, $\frac{dy}{dx} = 2\log_{10}10[\ln 10(10)) + 10\left[\frac{7}{\ln 10}\right]$

$$= 20\ln 10 + \frac{7}{\ln 10}$$

$$\doteq 49.1.$$

When $x = 5$, $y = 10(\log_{10}10)$

$= 10.$

Equation is $y - 10 = 49.1(x - 5)$ or $y = 49.1x - 237.5$.

b. When using a graphing calculator it is necessary to use the $\boxed{\text{ZOOM}}$ function to get the x-coordinate close to 5.

6. $y = 20 \times 10\left(\frac{t-5}{10}\right)$

To find the point where the curve crosses the y-axis, set $t = 0$.

Thus, $y = 20(10^{-\frac{1}{2}})$

$$= \frac{20}{\sqrt{10}}$$

$$= 2\sqrt{10}.$$

The point of tangency is $(0, 2\sqrt{10})$.

The slope of the tangent is given by

$$\frac{dy}{dx} = 20\left(10^{\frac{t-5}{10}}\right)(\ln 10)\left(\frac{1}{10}\right).$$

At $(0, 2\sqrt{10})$ the slope of the tangent is $\frac{2\ln 10}{\sqrt{10}}$.

The equation of the tangent is $y - 2\sqrt{10} = \frac{2\ln 10}{10}(x - 0)$

or $2\ln 10x - \sqrt{10}y + 20 = 0$

7. **a.** For $f(x) = \log_2(\log_2(x))$ to be defined, $\log_2 x > 0$.

For $\log_2 x > 0$, $x > 2^0 = 1$.

Thus, the domain of $f(x)$ *is* $x > 1$.

b. The x-intercept occurs when $f(x) = 0$.

Thus, $\log_2(\log_2 x) = 0$

$\log_2 x = 1$

$x = 2.$

The slope of the tangent is given by

$$f'(x) = \frac{1}{(\log_2 x)(\ln 2)} \bullet \frac{1}{x\ln 2}.$$

At $x = 2$, the slope is

$$f'(2) = \frac{1}{(1)(\ln 2)} \bullet \frac{1}{2\ln 2}$$

$$= \frac{1}{2(\ln 2)^2}.$$

c. It is difficult to directly graph logarithm functions having a base other than 10 or e.

8. $s = 40 + 3t + 0.01t^2 + \ln t$

a. $$v = \frac{ds}{dt} = 3 + 0.02t + \frac{1}{t}$$

When $t = 20$, $v = 3 + 0.4 + 0.05$
$= 3.45$ cm/min.

b. $$a = \frac{dv}{dt} = 0.02 - \frac{1}{t^2}$$

When $a = 0.01$,

$$0.02 - \frac{1}{t^2} = 0.01$$

$$\frac{1}{t^2} = 0.01$$

$$t^2 = 100$$

$$t = 10.$$

After 10 minutes, the acceleration is 10 cm/min/min.

9. $P = 0.5(10^9)\, e^{0.20015t}$

a. $$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{0.20015t}$$

In 1968, $t = 1$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{0.20015}$

$\doteq 0.12225 \times 10^9$ dollars/annum.

In 1978, $t = 11$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{11 \times 0.20015}$

$\doteq 0.90467 \times 10^9$ dollars/annum.

In 1978, the rate of increase of debt payments was \$904,670,000/annum compared to \$122,250,000/annum in 1968.

b. In 1988, $t = 21$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{21 \times 0.20015}$

$\doteq 6.69469 \times 10^9$ dollars/annum.

In 1998, $t = 31$ and $\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{31 \times 0.20015}$

$\doteq 49.5417 \times 10^9$ dollars/annum.

Note the continuing increase in the rates of increase of the debt payments.

10. a. For an earthquake of intensity I,

$$R = \log_{10}\left(\frac{I}{I_0}\right).$$

For an earthquake of intensity $10I$,

$$R = \log_{10}\left(\frac{10I}{I_0}\right).$$

$$= \log_{10}\left(\frac{I}{I_0}\right) + \log_{10}10$$

$$= \log_{10}\left(\frac{I}{I_0}\right) + 1.$$

The Richter magnitude of an earthquake of intensity $10I$ *is* 1 greater than that of intensity I.

b. $R = \log_{10} I - \log_{10} I_0$

$$\frac{dR}{dt} = \frac{1}{I \ln 10} \bullet \frac{dI}{dt} - 0$$

We are given that $\frac{dI}{dt} = 100$ and $I = 35$.

Thus, $\frac{dR}{dt} = \frac{1}{35 \ln 10} \bullet 100 \doteq 1.241$ units/s.

When the intensity of an earthquake is 35 and increasing at the rate of 100 units/s, the Richter magnitude is increasing at the rate of 1.24 units/s.

11. b. Rewrite $y = 7^x$ *as* $y = e^{x\ln 7}$ and graph using $\boxed{y =}$.

c. The factor ln 7 is a power used to transform $y = e^x$ to $y = (e^x)^{\ln 7}$.

12. b. Rewrite $y = \log_5 x$ *as* $y = \frac{\ln x}{\ln 5} = \frac{1}{\ln 5}\ln x$, and graph using $\boxed{y =}$.

c. Since $\frac{1}{\ln 5} < 1$, multiplying ln x by $\frac{1}{\ln 5}$ causes the graph of $y = \ln x$ to be compressed vertically.

Exercise 8.4

1. a. $f(x) = e^{-x} - e^{-3x}$ on $0 \le \times \le 10$

$f'(x) = -e^{-x} + 3e^{-3x}$

Let $f'(x) = 0$, therefore $e^{-x} + 3e^{-3x} = 0$.

Let $e^{-x} = w$, when $-w + 3w^3 = 0$.

$w(-1 + 3w^2) = 0$.

Therefore, $w = 0$ or $w^2 = \frac{1}{3}$

$w = \pm\frac{1}{\sqrt{3}}$.

But $w \ge 0$, $w = +\frac{1}{\sqrt{3}}$.

When $w = \frac{1}{\sqrt{3}}$, $e^{-x} = \frac{1}{\sqrt{3}}$,

$$-x\ln e = \ln 1 - \ln\sqrt{3}$$

$$x = \frac{\ln\sqrt{3} - \ln 1}{1}$$

$$= \ln\sqrt{3}$$

$$\doteq 0.55.$$

$f(0) = e^0 - e^0$

$= 0$

$f(0.55) \doteq -4.61$

$f(100) = e^{-100} - e^{-300} \doteq 3.7$

Absolute maximum is about 3.7 and absolute minimum is about –4.61.

b. $g(t) = \frac{e^t}{1 + \ln t}$ on $1 \le t \le 12$

$$g'(t) = \frac{e^t(1 + \ln t) - \frac{1}{t}(e^t)}{(1 + \ln t)^2}$$

Let $g'(t) = 0$:

$e^t(1 + \ln t) - \frac{1}{t}(e^t) = 0$

Since $e^t \ne 0$, $1 + \ln t - \frac{1}{t} = 0$,

$$\ln t = \frac{1}{t} - 1.$$

Therefore, $t = 1$ by inspection.

$g(1) = \frac{e^1}{1 + \ln 1} = 2.7$

$g(12) = \frac{e^{12}}{1 + \ln 12} \doteq 46\ 702$

The maximum value is about 46 202 and the minimum value is 2.7.

c. $m(x) = (x + 2)e^{-2x}$ on $-4 \le x \le 4$

$m'(x) = e^{-2x} + (-2)(x + 2)e^{-2x}$

Let $m'(x) = 0$.

$e^{-2x} \ne 0$, therefore, $1 + (-2)(x + 2) = 0$

$$x = \frac{-3}{2}$$

$$= -1.5.$$

$m(-4) = -2e^8 \doteq -5961$

$m(-1.5) = 0.5e^3 \doteq 10$

$m(4) = 6e^{-8} \doteq 0.0002$

The maximum value is about 10 and the minimum value is about –5961.

d. $s(t) = \ln\left(\frac{t^2 + 1}{t^2 - 1}\right) + 6\ln t$ on $1.1 \le t \le 10$

$= \ln(t^2 + 1) - \ln(t^2 - 1) + 6\ln t$

$s'(t) = \frac{2t}{t^2 + 1} - \frac{2t}{t^2 - 1} + \frac{6}{t}$

Let $s'(t) = 0$,

$\frac{-4t + 6(t^2 - 1)(t^2 + 1)}{t(t^2 + 1)(t^2 - 1)} = 0$ or $-4t + 6(t^4 - 1) = 0$

$$3t^4 - 2t - 3 = 0$$

$t \doteq 1.2$ (using a calculator).

$s(1.1) = \ln\left(\frac{1.1^2 + 1}{1.1^2 - 1}\right) + 6\ln(1.1) \doteq 2.9$

$s(1.2) = \ln\left(\frac{1.2^2 + 1}{1.2^2 - 1}\right) + 6\ln(1.2) \doteq 2.8$

$s(10) = \ln\left(\frac{10^2 + 1}{10^2 - 1}\right) + 6\ln(10) \doteq 13.84$

The maximum value is about 13.84 and the minimum is about 2.8.

4. **a.** $P(x) = 10^6[1 + (x - 1)e^{-0.001x}],\ 0 \le x \le 2000$

Using the Algorithm for Extreme Values, we have

$P(0) = 10^6[1 - 1] = 0$

$P(2000) = 10^6[1 + 1999e^{-2}] \doteq 271.5 \times 10^6.$

Now,

$P'(x) = 10^6\left[(1)e^{-0.001x} + (x - 1)(-0.001)e^{-0.001x}\right]$

$= 10^6e^{-0.001x}(1 - 0.001x + 0.001)$

Since $e^{-0.001x} > -$ for all x,

$P'(x) = 0$ when $1.001 - 0.001x = 0$

$x = \dfrac{1.001}{0.001} = 1001.$

$P(1001) = 10^6[1 + 1000e^{-1.001}] \doteq 368.5 \times 10^6$

The maximum monthly profit will be 368.5 × 106 dollars when 1001 items are produced and sold.

b. The domain for $P(x)$ becomes $0 \le x \le 500$.

$P(500) = 10^6[1 + 499e^{-0.5}] = 303.7 \times 10^6$

Since there are no critical values in the domain, the maximum occurs at an endpoint. The maximum monthly profit when 500 items are produced and sold is 303.7×10^6 dollars.

5. $R(x) = 40x^2e^{-0.4x} + 30,\ 0 \le x \le 8$

We use the Algorithm for Extreme Values:

$R'(x) = 80xe^{-0.4x} + 40x^2(-0.4)e^{-0.4x}$

$= 40xe^{-0.4x}(2 - 0.4x)$

Since $e^{-0.4x} > 0$ for all x, $R'(x) = 0$ when

$x = 0$ or $2 - 0.4x = 0$

$x = 5.$

$R(0) = 30$

$R(5) \doteq 165.3$

$R(8) \doteq 134.4$

The maximum revenue of 165.3 thousand dollars is achieved when 500 units are produced and sold.

6. The speed of the signal is $S(x) = kv(x)$

$= kx^2 \ln\left(\dfrac{1}{x}\right)$

$= kx^2(\ln 1 - \ln x)$

$= -kx^2 \ln x.$

Since $x = \dfrac{r}{R}$, we have $\dfrac{R}{10} \le r \le \dfrac{9R}{10}$

$\dfrac{1}{10} \le \dfrac{r}{R} \le \dfrac{9}{10}$

$\dfrac{1}{10} \le x \le \dfrac{9}{10}.$

Now, $S'(x) = -k\left(2x \ln x + x^2\left(\dfrac{1}{x}\right)\right)$

$= -kx(2 \ln x + 1).$

$S'(x) = 0$ when $\ln x = -\dfrac{1}{2}$ $\left(\text{since } x \ge \dfrac{1}{10}, x \ne 0\right)$

$x = e^{-\frac{1}{2}}$

$\doteq 0.6065.$

$S(0.1) = 0.023k$

$S(e^{-\frac{1}{2}}) = 0.184k$

$S(0.9) = 0.08k$

The maximum speed of the signal is $0.184k$ units when $x \doteq 0.61$.

7. $C(h) = 1 + h(\ln h)^2,\ 0.2 \le h \le 1$

$C'(h) = (\ln h)^2 + 2h \ln h \bullet \dfrac{1}{h}$

$= (\ln h)^2 + 2 \ln h$

$= \ln h\,(\ln h + 2)$

$C'(h) = 0$ when $\ln h = 0$ or $\ln h = -2$

$h = 1$ or $h = e^{-2} \doteq 0.135.$

Since the domain under consideration is $0.2 \le h \le 0.75$, neither of the critical values is admissible.

$C(0.2) \doteq 1.52$

$C(0.75) \doteq 1.06$

The student's intensity of concentration level is lowest at the 45 minute mark of the study session.

8. $P(t) = 100(e^{-t} - e^{-4t}),\ 0 \le t \le 3$

$P'(t) = 100(-e^{-t} + 4e^{-4t})$

$= 100e^{-t}(-1 + 4e^{-3t})$

Since $e^{-t} > 0$ for all t, $P'(t) = 0$ when

$4e^{-3t} = 1$

$e^{-3t} = \dfrac{1}{4}$

$-3t = \ln(0.25)$

$t = \dfrac{-\ln(0.25)}{3}$

$= 0.462.$

$P(0) = 0$

$P(0.462) \doteq 47.2$

$P(3) \doteq 4.98$

$P'(t) = 100(4e^{-4t})$

$= \dfrac{400}{e^{4t}} > 0$ for all t

Since there are no critical values in the given interval, the maximum value will occur at an endpoint.
$P(0) = 0$
$P(3) \doteq 4.98$
The highest percentage of people spreading the rumour is 4.98% and occurs at the 3 h point.

9. $C = 0.015 \times 10^9 e^{0.07533t}$

a. $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{0.20015t}$

In 1968, $t = 1$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{0.20015}$

$\doteq 0.1225 \times 10^9$ dollars/year

In 1978, $t = 11$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{11x0.20015}$

$\doteq 0.90467 \times 10^9$ dollars/year

In 1978 the rate of increase of debt payments was $904 670 000/year compared to $122 250 000/year in 1968.

b. In 1987, $t = 20$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{20x0.20015}$

$\doteq 5.48033 \times 10^9$ dollars/year

In 1989, $t = 22$ and $\frac{dp}{dt} = 0.5(10^9)(0.20015)e^{22x0.20015}$

$\doteq 8.17814 \times 10^9$ dollars

c. In 1989, $P = 0.5(10^9)(e^{20x0.20015})$

$\doteq 27.3811 \times 10^9$ dollars

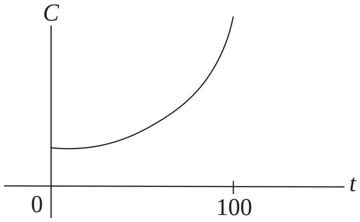
In 1989, $P = 0.5(10^9(e^{22x0.20015})$

$\doteq 40.8601 \times 10^9$ dollars

Year	Amount Paid	Rate of Change	%
1987	$\$27.3811 \times 10^9$	$\$5.48033 \times 10^9$/year	20.02
1989	$\$40.8601 \times 10^9$	$8.17814/year	20.02

9. $C = 0.015 \times 10^9 e^{0.07533t}$, $0 \leq t \leq 100$

a.



b. $\frac{dC}{dt} = 0.015 \times 10^9 \times 0.07533 e^{0.07533t}$

In 1947, $t = 80$ and the growth rate was

$\frac{dC}{dt} = 0.46805 \times 10^9$ dollars/year.

In 1967, $t = 100$ and the growth rate was

$\frac{dC}{dt} = 2.1115 \times 10^9$ dollars/year.

The ratio of growth rates of 1967 to that of 1947 is

$\frac{2.1115 \times 10^9}{0.46805 \times 10^9} = \frac{4.511}{1}$.

The growth rate of capital investment grew from 468 million dollars per year in 1947 to 2.112 billion dollars per year in 1967.

c. In 1967, the growth rate of investment as a percentage of the amount invested is

$\frac{2.1115 \times 10^9}{28.0305 \times 10^9} \times 100 = 7.5\%$.

d. In 1977, $t = 110$

$C = 59.537 \times 10^9$ dollars

$\frac{dC}{dt} = 4.4849 \times 10^9$ dollars/year.

e. Statistics Canada data shows the actual amount of U.S. investment in 1977 was 62.5×10^9 dollars. The error in the model is 3.5%.

f. In 2007, $t = 140$.
The expected investment and growth rates are
$C = 570.490 \times 10^9$ dollars and $\frac{dC}{dt} = 42.975 \times 10^9$ dollars/year.

10. $C(t) = \frac{k}{b-a}(e^{at} - e^{-bt}),\ b > a > 0,\ k > 0,\ t \geq 0$

$$C^1(t) = \frac{k}{b-a}(-ae^{-at} + be^{-bt})$$

$$C^1(t) = 0 \text{ when } be^{-bt} - ae^{-at} = 0$$

$$be^{-bt} = ae^{-at}$$

$$\frac{b}{a} = \frac{e^{bt}}{e^{at}} = e^{(b-a)t}$$

$$(b-a)t = \ln\left(\frac{b}{a}\right)$$

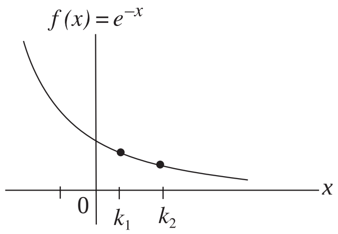
$$t = \frac{\ln(b) - \ln(a)}{b-a}$$

Since $\frac{b}{a} > 1$, $\ln\left(\frac{b}{a}\right) > 0$ and hence the value of t is a positive number. If $t = 0$, $C(0) = \frac{k}{b-a}(1-1) = 0$

Also, $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty}\left[\frac{k}{b-a}\left(\frac{1}{e^{at}} - \frac{1}{e^{bt}}\right)\right]$

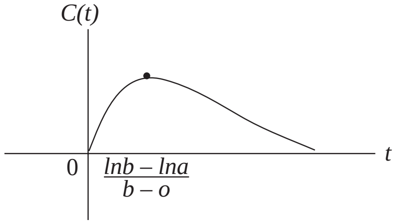
$$= \frac{k}{b-a}(0-0) = 0$$

Since $f(x) = e^{-x}$ is a decreasing function throughout its domain, if $k_1 < k_2$ then $e^{-k_1} > e^{-k_2}$



Since $a < b$, $at < bt$ where a, b, t are all positive.
Thus, $e^{-at} > e^{-bt}$ for all $t > 0$.
Hence, $C(t) > 0$ for all $t > 0$.
Since there is only one critical value, the largest concentration of the drug in the blood occurs at

$$t = \frac{\ln b - \ln a}{b-a}$$



11. a. The growth function is $N = 2^{\frac{t}{5}}$.
The number killed is given by $K = e^{\frac{t}{3}}$.
After 60 minutes, $N = 2^{12}$.
Let T be the number of minutes after 60 minutes.
The population of the colony at any time, T, after the first 60 minutes is
$P = N - k$

$$= 2\,\frac{60+T}{5} - e^{\frac{T}{3}},\ T \geq 0$$

$$\frac{dP}{dt} = 2^{\frac{60+T}{5}}\left(\frac{1}{5}\right)\ln 2 - \frac{1}{3}e^{\frac{T}{3}}$$

$$= 2^{12+\frac{T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$

$$= 2^{12} \bullet 2^{\frac{T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$

$$\frac{dP}{dt} = 0 \text{ when } 2^{12}\,\frac{\ln 2}{5}\,2^{\frac{T}{5}} = \frac{1}{3}e^{\frac{T}{3}}$$

$$\text{or } 3\,\frac{\ln 2}{5} \bullet 2^{12}\,2^{\frac{T}{5}} = e^{\frac{T}{3}}.$$

We take the natural logarithm of both sides:

$$\ln\left(3.2^{12}\,\frac{\ln 2}{5}\right) + \frac{T}{5}\ln 2 = \frac{T}{3}$$

$$7.4404 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$

$$T = \frac{7.4404}{0.1947} = 38.2 \text{ min.}$$

At $T = 0$, $P = 2^{12} = 4096$.
At $T = 38.2$, $P = 478\,158$.
For $T > 38.2$, $\frac{dP}{dt}$ is always negative.
The maximum number of bacteria in the colony occurs 38.2 min after the drug was introduced. At this time the population numbers 478 158.

b. $P = 0$ when $2^{\frac{60+T}{5}} = e^{\frac{T}{3}}$

$$\frac{60+T}{5}\ln 2 = \frac{T}{3}$$

$$12\ln 2 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$

$$T = 42.72$$

The colony will be obliterated 42.72 minutes after the drug was introduced.

12. Let t be the number of minutes assigned to study for the first exam and $30 - t$ minutes assigned to study for the second exam. The measure of study effectiveness for the two exams is given by

$$E(t) = E_1(t) + E_2(30 - t),\ 0 \leq t \leq 30$$

$$= 0.5(10 + te^{-\frac{t}{10}}) + 0.6\left(9 + (30 - t)e^{-\frac{30-t}{20}}\right)$$

$$E'(t) = 0.5\left(e^{-\frac{t}{10}} - \frac{1}{10}te^{-\frac{t}{10}}\right) + 0.6\left(-e^{-\frac{30-t}{5}} + \frac{1}{20}(30 - t)e^{-\frac{30-t}{20}}\right)$$

$$= 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{30-t}{5}}(-20 + 30 - t)$$

$$= (0.05e^{-\frac{t}{10}} + 0.03e^{-\frac{30-t}{5}})(10 - t)$$

$E'(t) = 0$ when $10 - t = 0$

$t = 10$ (The first factor is always a positive number.)

$E(0) = 5 + 5.4 + 18e^{-\frac{3}{2}} = 14.42$

$E(10) = 16.65$

$E(30) = 11.15$

For maximum study effectiveness, 10 h of study should be assigned to the first exam and 20 h of study for the second exam.

13. The solution starts in a similar way to that of **12.** The effectiveness function is

$$E(t) = 0.5(10 + te^{-\frac{t}{10}}) + 0.6\left(9 + (25 - t)e^{-\frac{25-t}{20}}\right).$$

The derivative simplifies to

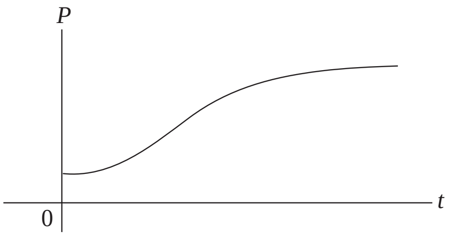
$$E'(t) = 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{25-t}{20}}(5 - t).$$

This expression is very difficult to solve analytically. By calculation on a graphing calculator, we can determine the maximum effectiveness occurs when $t = 8.16$ hours.

14. $P = \dfrac{aL}{a + (L - a)e^{-kLt}}$

a. We are given $a = 100$, $L = 10\ 000$, $k = 0.0001$.

$$P = \frac{10^6}{100 + 9900e^{-t}} = \frac{10^4}{1 + 99e^{-t}} = 10^4(1 + 99e^{-t})^{-1}$$



b. We need to determine when the derivative of the growth rate $\left(\frac{dP}{dt}\right)$ is zero, i.e., when $\frac{d^2P}{dt^2} = 0$.

$$\frac{dP}{dt} = \frac{-10^4(-99e^{-t})}{(1 + 99e^{-t})^2} = \frac{990\ 000e^{-t}}{(1 + 99e^{-t})^2}$$

$$\frac{d^2P}{dt^2} = \frac{-990\ 000e^{-t}(1 + 99e^{-t})^2 - 990\ 000e^{-t}(2)(1 + 99e^{-t})(-99e^{-t})}{(1 + 99e^{-t})^4}$$

$$= \frac{-990\ 000e^{-t}(1 + 99e^{-t}) + 198(990\ 000)e^{-2t}}{(1 + 99e^{-t})^3}$$

$$\frac{d^2P}{dt^2} = 0 \text{ when } 990\ 000e^{-t}(-1 - 99e^{-t} + 198e^{-t}) = 0$$

$$99e^{-t} = 1$$

$$e^t = 99$$

$$t = \ln 99$$

$$\doteq 4.6$$

After 4.6 days, the rate of change of the growth rate is zero. At this time the population numbers 5012.

c. When $t = 3$, $\frac{dP}{dt} = \frac{990\ 000e^{-3}}{(1 + 99e^{-3})^2} \doteq 1402$ cells/day.

When $t = 8$, $\frac{dP}{dt} = \frac{990\ 000e^{-8}}{(1 + 99e^{-8})^2} \doteq 311$ cells/day.

The rate of growth is slowing down as the colony is getting closer to its limiting value.

Exercise 8.5

3. **a.** $y = f(x) = x^x$

$\ln y = x \ln x$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = x^x(\ln x + 1)$$

$$f'(e) = e^e(\ln e + 1) = 2e^e$$

b. $s = e^t + t^e$

$$\frac{ds}{dt} = e^t + et^{e-1}$$

When $t = 2$, $\frac{ds}{dt} = e^2 + e \bullet 2^{e-1}$

c. $y = \dfrac{(x-3)^2\sqrt[3]{x+1}}{(x-4)^5}$

Let $y = f(x)$.

$$\ln y = 2\ln(x-3) + \frac{1}{3}\ln(x+1) - 5\ln(x-4)$$

$$\frac{1}{y}\bullet\frac{dy}{dx} = \frac{2}{x-3} + \frac{1}{3\,(x+1)} - \frac{5}{x-4}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x-3} + \frac{1}{3\,(x+1)} - \frac{5}{x-4}\right)$$

$$f'(7) = f(7)\left(\frac{2}{4} + \frac{1}{24} - \frac{5}{3}\right)$$

$$= \frac{32}{243}\left(-\frac{27}{24}\right) = -\frac{4}{27}$$

4. $y = x(x^2)$

The point of contact is (2, 16). The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$. We take the natural logarithm of both sides and differentiate implicitly with respect to x.

$$y = x^{(x^2)}$$

$$\ln y = x^2 \ln x$$

$$\frac{1}{y}\bullet\frac{dy}{dx} = 2x\ln x + x$$

At the point (2, 16), $\frac{dy}{dx} = 16(4\ln 2 + 2)$.

The equation of the tangent line at (2, 16) is
$y - 16 = 32(2\ln 2 + 1)(x - 2)$.

5. $y = \dfrac{1}{(x+1)(x+2)(x+3)}$

We take the natural logarithm of both sides and differentiate implicitly with respect to x to find $\frac{dy}{dx}$, the slope of the tangent line.

$$\text{In } y = \ln(1) - \ln(x+1) - \ln(x+2) - \ln(x+3)$$

$$\frac{1}{y}\bullet\frac{dy}{dx} = -\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}$$

The point of contact is $\left(0, \frac{1}{6}\right)$.

At $\left(0, \frac{1}{6}\right)$, $\frac{dy}{dx} = \frac{1}{6}\left(-1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}\left(-\frac{11}{6}\right) = -\frac{11}{36}$.

6. $y = x^{\frac{1}{x}}$, $x > 0$

We take the natural logarithm of both sides and differentiate implicitly with respect to x to find $\frac{dy}{dx}$, the slope of the tangent.

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

$$\frac{1}{y}\bullet\frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2}$$

We want the values of x so that $\frac{dy}{dx} = 0$.

$$\frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2} = 0$$

Since $x^{\frac{1}{x}} \neq 0$ and $x^2 > 0$, we have $1 - \ln x = 0$

$\ln x = 1$

$x = e$.

The slope of the tangent is 0 at $(e, e^{\frac{1}{e}})$.

7. We want to determine the points on the given curve at which the tangent lines have slope 6. The slope of the tangent at any point on the curve is given by $\frac{dy}{dx} = 2x + \frac{4}{x}$.

To find the required points, we solve:

$$2x + \frac{4}{x} = 6$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

$$x = 1 \text{ or } x = 2.$$

The tangents to the given curve at (1, 1) and (2, 4 + 4 ln 2) have slope 6.

8. We first must find the equation of the tangent at A(4, 16). We need $\frac{dy}{dx}$ for $y = x^{\sqrt{x}}$.

$$\text{In } y = \sqrt{x}\ln x$$

$$\frac{1}{y}\bullet\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}\ln x + \sqrt{x}\bullet\frac{1}{x}$$

$$= \frac{\ln x + 2}{2\sqrt{x}}$$

At (4, 16), $\frac{dy}{dx} = 16\frac{(\ln 4 + 2)}{4} = 4\ln 4 + 8$.

The equation of the tangent is
$y - 16 = (4\ln 4 + 8)(x - 4)$.
The y-intercept is $-16(\ln 4 + 1)$.

The x-intercept is $\frac{-4}{\ln 4 + 2} + 4 = \frac{4 \ln 4 + 4}{\ln 4 + 2}$.

The area of ΔOBC is $\frac{1}{2}\left(\frac{4 \ln 4 + 4}{\ln 4 + 2}\right)(16)(\ln 4 + 1)$,

which equals $\frac{32(\ln 4 + 1)^2}{\ln 4 + 2}$.

9. $s(t) = t^{\frac{1}{t}}$, $t > 0$

a. $\ln (s(t)) = \frac{1}{t} \ln t$

Differentiate with respect to t:

$$\frac{1}{s(t)} \bullet s'(t) = \frac{\frac{1}{t} \bullet t - \ln t}{t^2}$$

$$= \frac{1 - \ln t}{t^2}.$$

Thus, $v(t) = s(t) \bullet \left(\frac{1 - \ln t}{t^2}\right) = t^{\frac{1}{t}}\left(\frac{1 - \ln t}{t^2}\right)$.

Now, $a(t) = v'(t) = s'(t)\left(\frac{1 - \ln t}{t^2}\right) + s(t)\left(\frac{-\frac{1}{t} \bullet t^2 - (1 - \ln t)(2t)}{t^4}\right)$.

Substituting for $s(t)$ and $s'(t) = v(t)$ gives

$$a(t) = t^{\frac{1}{t}}\left(\frac{1 - \ln t}{t^2}\right)^2 + t^{\frac{1}{t}}\left(\frac{2t \ln t - 3t}{t^4}\right)$$

$$= \frac{t^{\frac{1}{t}}}{t^4}[1 - 2 \ln t + (\ln t)^2 + 2t \ln t - 3t]$$

b. Since $t^{\frac{1}{t}}$ and t^2 are always positive, the velocity is zero when $1 - \ln t = 0$ or when $t = e$.

$$a(e) = \frac{e^{\frac{1}{e}}}{e^4}[1 - 2 + 1 + 2e - 3e]$$

$$= -\frac{e^{\frac{1}{e}}}{e^3}$$

$$= -e^{\frac{1}{e} - 3}$$

Review Exercise

2. **b.** $y = \frac{x \ln x}{e^x}$

$$\ln y = \ln x + \ln (\ln x) - \ln e^x$$

$$= \ln x + \ln (\ln x) - e^x$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{1}{x} + \frac{1}{\ln x}\left(\frac{1}{x}\right) - 1$$

$$= \frac{x \ln x}{e^x}\left[\frac{1}{x} + \frac{1}{x \ln x} - 1\right]$$

d. $y = \frac{(x + 2)(x - 4)^5}{(2x^3 - 1)^2}$

$$\ln y = \ln (x + 2) + 5 \ln (x - 4) - 2 \ln (2x^3 - 1)$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{1}{x + 2} + \frac{5}{x - 4} - \frac{12x^2}{2x^3 - 1}$$

$$\frac{dy}{dx} = \frac{(x + 2)(x - 4)^5}{(2x^3 - 1)^2}\left[\frac{1}{x + 2} + \frac{5}{x - 4} - \frac{12x^2}{2x^3 - 1}\right]$$

f. $y = \left(\sqrt{x^2 + 3}\right)^{e^x}$

$$\ln y = e^x \ln (x^2 + 3)^{\frac{1}{2}}$$

$$= \frac{e^x}{2} \ln (x^2 + 3)$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = \frac{e^x}{2} \ln (x^2 + 3) + \frac{e^x}{2} \bullet \frac{2x}{x^2 + 3}$$

$$\frac{dy}{dx} = \left(\sqrt{x^2 + 3}\right)^{e^x}\left[\frac{e^x}{2} \ln (x^2 + 3) + \frac{e^x}{2} \bullet \frac{2x}{x^2 + 3}\right]$$

g. $y = \left(\frac{30}{x}\right)^{2x}$

$$\ln y = 2x[\ln 30 - \ln x]$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = 2[\ln 30 - \ln x] + 2x\left(\frac{-1}{x}\right)$$

$$= \left(\frac{30}{x}\right)^{2x}[2 \ln 30 - 2 \ln x - 2]$$

h. $e^{xy} = \ln (x + y)$

$$e^{xy}\left[x \frac{dy}{dx} + y\right] = \frac{1}{x + y}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left[xe^{xy} - \frac{1}{x + y}\right] = \frac{1}{x + y} + ye^{xy}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x + y} + ye^{xy}}{xe^{xy} - \frac{1}{x + y}} = \frac{1 + (x + y)\, ye^{xy}}{x\,(x + y)e^{xy} - 1}$$

3. **b.** $f(x) = [\ln(3x^2 - 6x)]^4$

$$f'(x) = 4[\ln(3x^2 - 6x)]^3 \bullet \frac{6x-6}{3x^2-6}$$

Let $f'(x) = 0$, therefore, $\ln(3x^2 - 6x) = 0$

$$3x^2 - 6x = 1$$

$$3x^2 - 6x - 1 = 0$$

$$x = \frac{6 \pm \sqrt{48}}{6}$$

$$= \frac{6 \pm 4\sqrt{3}}{6}$$

$$= \frac{3 + 2\sqrt{3}}{3}$$

or

$$\frac{6x-6}{3x^2-6} = 0$$

$$6x - 6 = 0$$

$$x = 1.$$

But $3x^2 - 6x > 0$ or $3x(x - 2) > 0$.

Therefore, $x > 2$ or $x < 0$.

Only solution is $x = \frac{3+2\sqrt{3}}{3}$ and $\frac{3+2\sqrt{3}}{3}$.

6. $y = \frac{\ln x^2}{2x}$

$$= \frac{2 \ln x}{x}$$

$$\frac{dy}{dx} = \frac{2\left(\frac{1}{x}\right)x - 2\ln x}{x^2}$$

$$= \frac{2 - 2\ln x}{x^2}$$

At $x = 4$, $\frac{dy}{dx} = \frac{2 - 2\ln 4}{16}$

$$= \frac{1 - \ln 4}{8}.$$

At $x = 4$, $\frac{y = 2\ln 4}{4}$

$$= \frac{\ln 4}{2}.$$

The equation of the tangent is

$$y - \frac{\ln 4}{2} = \frac{1 - \ln 4}{8}(x - 4).$$

$8y - 4\ln 4 = (1 - \ln 4)x - 4 + 4\ln 4$

$(1 - \ln 4)x - 8y + 8\ln 4 - 4 = 0$

7. $y = \frac{e^{2x} - 1}{e^{2x} + 1}$

$$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x}+1) - (e^{2x}-1)(2e^{2x})}{(e^{2x}+1)^2}$$

$$= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}}{(e^{2x}+1)^2}$$

Now, $1 - y^2 = \frac{1 - e^{4x} - 2e^{2x} + 1}{(e^{2x}+1)^2}$

$$= \frac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}}{(3^{2x}+1)^2} = \frac{dy}{dx}$$

8. $y = e^{kx}$

a. $y' - 7y = 0$

$ke^{kx} - 7e^{kx} = 0$

$e^{kx}(k - 7) = 0$

$k = 7$ since $e^{kx} \neq 0$

b. $y'' - 16y = 0$

$k^2e^{kx} - 16e^{kx} = 0$

$e^{kx}(k^2 - 16) = 0$

$k = \pm\ 4$, since $e^{kx} \neq 0$

c. $y'' - y'' - 12y' = 0$

$k^3e^{kx} - k^2e^{kx} - 12ke^{kx} = 0$

$ke^{kx}(k^2 - k - 12) = 0$

$ke^{kx}(k + 3)(k - 4) = 0$

$k = -3$ or $k = 0$ or $k = 4$, since $e^{kx} \neq 0$

9. The slope of the required tangent line is 3.
The slope at any point on the curve is given by

$$\frac{dy}{dx} = 1 + e^{-x}.$$

To find the point(s) on the curve where the tangent has slope 3, we solve:

$1 + e^{-x} = 3$

$e^{-x} = 2$

$-x = \ln 2$

$x = -\ln 2.$

The point of contact of the tangent is $(-\ln 2, -\ln 2 - 2)$.

The equation of the tangent line is

$y + \ln 2 + 2 = 3(x + \ln 2)$ or $3x - y + 2\ln 2 - 2 = 0.$

10. The slope of the tangent line to the given curve at any point is

$$\frac{dy}{dx} = \frac{e^x(1 + \ln x) - e^x\left(\frac{1}{x}\right)}{(1 + \ln x)^2}$$

At the point (1, e), the slope of the tangent is $\frac{e–e}{1} = 0$.

Since the tangent line is parallel to the x-axis, the normal line is perpendicular to the x-axis. The line through (1, e) perpendicular to the x-axis has equation $x = 1$.

11. **a.** $\frac{dN}{dt} = 2000\left[e^{-\frac{t}{20}} - \frac{1}{20}te^{-\frac{t}{20}}\right]$

$= 2000e^{-\frac{t}{20}}\left[1 - \frac{t}{20}\right]$

Since $e^{-\frac{t}{20}} > 0$ for all t, $\frac{dN}{dt} = 0$,

when $1 - \frac{t}{20} = 0$

$t = 20.$

The growth rate of bacteria is zero bacteria per day on day 20.

b. When $t = 10$, $N = 2000[30 + 10e^{-\frac{1}{2}}]$

$\doteq 72\ 131$

$m = \sqrt[3]{72\ 131 + 1000}$

$\doteq 41.81.$

On day 10, there will be 42 newly infected mice.

12. $g(t) = \frac{\ln(t^3)}{2t}$

$= \frac{3 \ln t}{2}, t > 1$

$g'(t) = \frac{\frac{3}{t} \bullet 2t - (3 \ln t)(2)}{4t^2}$

$= \frac{6 - 6 \ln t}{4t^2}$

Since $t > 1$, $g'(t) – 0$ when $6 - 6 \ln t = 0$

$\ln t = 1$

$t = e.$

Now, $\lim_{t\to1^+} g(t) = \lim_{t\to1^+}\left(\frac{3 \ln (t)}{2t}\right) = 0$

$g(e) = \frac{3}{23}$

$\doteq 0.552$

For $t > e$, $\ln t > 1$ and $g'(t) < 0$

Thus, the maximum measure of effectiveness of this medicine is 0.552 and occurs at $t = 2.718$ h after the medicine was given.

13. $m(t) = t \ln (t) + 1$ for $0 < t \leq 4$

$m'(t) = \ln (t) + 1$

$m'(t) = 0$ when $\ln (t) + 1 = 0$

$t = e^{-1}$

For $0 < t < e^{-1}$, $m'(t) < 0$.

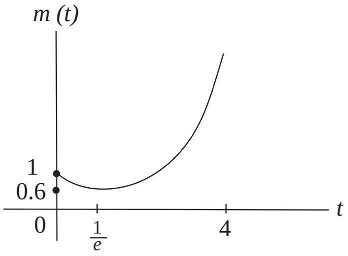
Thus, $m(t)$ is decreasing over this interval.

$\lim_{t\to0^+} (t \ln t + 1) = 1$ (by investigating the graph of $m(t)$)

$m(e^{-1}) \doteq .632$

$m(4) \doteq 6.545$

During the first four years, a child's ability to memorize is lowest at 0.368 years of age and highest at four years.



14. **a.** $c_1(t) = te^{-t}$; $c_1(0) = 0$

$c_1'(t) = e^{-t} - te^{-t}$

$= e^{-t}(1 - t)$

Since $e^{-t} > 0$ for all t, $c_1'(t) = 0$ when $t = 1$.

Since $c_1'(t) > 0$ for $0 \leq t < 1$, and $c_1'(t) < 0$ for all $t > 1$, $c_1(t)$ has a maximum value of $\frac{1}{e} \doteq 0.368$ at $t = 1$ h.

$c_2(t) = t^2e^{-t}$; $c_2(0) = 0$

$c_2'(t) = 2te^{-t} - t^2e^{-t}$

$= te^{-t}(2 - t)$

$c_2'(t) = 0$ when $t = 0$ or $t = 2$.

Since $c_2'(t) > 0$ for $0 < t < 2$ and $c_2'(t) < 0$ for all $t > 2$, $c_2(t)$ has a maximum value of $\frac{4}{e^2} \doteq 0.541$ at $t = 2$ h. The larger concentration occurs for drug c_2.

b. $c_1(0.5) = 0.303$
$c_2(0.5) = 0.152$
In the first half-hour, the concentration of c_1 increases from 0 to 0.303, and that of c_2 increases from 0 to 0.152. Thus, c_1 has the larger concentration over this interval.

15. $T(x) = 10\left(1 + \frac{1}{x}\right)(0.9)^{-x}$

a. $T'(x) = 10\left(-\frac{1}{x^2}\right)(0.9)^{-x} + 10(1 + \frac{1}{x})(0.9)^{-1}(-1)(\ln(0.9))$

$$= 10(0.9)^{-x}\left[-\frac{1}{x^2} - \ln(0.9)\left(1 + \frac{1}{x}\right)\right]$$

b. Since $(0.9)^{-x} > 0$ for all x, $T'(x) = 0$ when

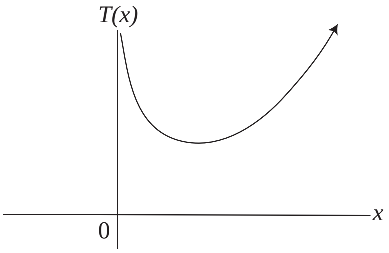
$$-\frac{1}{x^2} - \ln(0.9) - \frac{\ln(0.9)}{x} = 0.$$

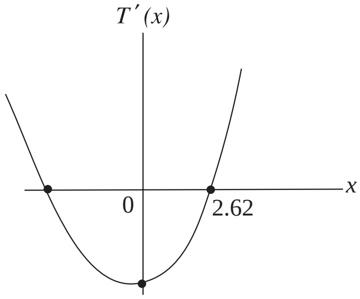
To find an approximate solution, we use $l(0.9) \doteq -0.1$. The quadratic equation becomes

$$-\frac{1}{x^2} + 0.1 + \frac{0.1}{x} = 0$$

$$\begin{aligned} 0.1x^2 + 0.1x - 1 &= 0,\ x \neq 0 \\ x^2 + x - 10 &= 0 \\ x &= \frac{-1 \pm \sqrt{1 + 40}}{2} \\ &= 2.7, \text{ since } x \geq 0. \end{aligned}$$

Note: Using $\ln(0.9) \doteq -0.105$ yields $x \doteq 2.62$. Since $T'(x) < 0$ for $0 < x < 2.62$, and $T'(x) > 0$ for $x > 2.62$, $T(x)$ has a minimum value at $x = 2.62$.





16. $v(x) = Kx^2 \ln\left(\frac{1}{x}\right)$

a. $v(x) = 2x^2 \ln\left(\frac{1}{x}\right) = -2x^2 \ln x$

$$\begin{aligned} v\left(\frac{1}{2}\right) &= 2\left(\frac{1}{4}\right)(\ln 2) \\ &= \frac{\ln 2}{2} \\ &= 0.347 \end{aligned}$$

b. $v'(x) = -4x \ln x - 2x^2\left(\frac{1}{x}\right)$

$$\begin{aligned} &= -4x \ln x - 2x \\ v'\left(\frac{1}{2}\right) &= 2 \ln 2 - 1 \\ &\doteq 1.386 \end{aligned}$$

17. $C(t) = K(e^{-2t} - e^{-5t})$

a. $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} K\left(\frac{1}{e^{2t}} - \frac{1}{e^{5t}}\right)$

$$\begin{aligned} &= K(0 - 0) \\ &= 0 \end{aligned}$$

b. $C'(t) = K(-2e^{-2t} + 5e^{-5t})$

$$C'(t) = 0 \text{ when } -\frac{2}{e^{2t}} + \frac{5}{e^{5t}} = 0$$

$$\frac{5}{e^{5t}} = \frac{2}{e^{2t}}$$

$$\frac{5}{2} = e^{5t-2t} = e^{3t}$$

$$3t = \ln\left(\frac{5}{2}\right)$$

$$t = \frac{\ln\left(\frac{5}{2}\right)}{3} \doteq 0.305$$

The rate is zero at $t = 0.305$ days or 7.32 h.

Chapter 8 Test

1. $y = e - 2x^2$

a. $\frac{dy}{dx} = -4xe^{-2x^2}$

b. $y = \ln(x^2 - 6)$

$$\frac{dy}{dx} = \frac{1}{x^2 - 6} \bullet 2x = \frac{2x}{x^2 - 6}$$

c. $y = 3^{x^2 + 3x}$

$$\frac{dy}{dx} = 3x^2 + 3x \bullet \ln 3 \bullet (2x + 3)$$

d. $y = \frac{e^{3x} + e^{-3x}}{2}$

$$\frac{dy}{dx} = \frac{1}{2}[3e^{3x} - 3e^{-3x}]$$

$$= \frac{3}{2}[e^{3x} - e^{-3x}]$$

e. $y = (4x^3 - x)\log_{10}(2x - 1)$

$$\frac{dy}{dx} = (12x^2 - 1)\log_{10}(2x - 1) + (4x^3 - x) \bullet \frac{1}{(2x - 1)\ln 10} \bullet 2$$

f. $y = \frac{\ln(x + 4)}{x^3}$

$$\frac{dy}{dx} = \frac{\frac{1}{x + 4} \bullet x^3 - \ln(x + 4) \bullet 3x^2}{x^6}$$

$$= \frac{\frac{x}{x + 4} - \ln(x + 4)}{x^4}$$

2. $f(t) = \ln(3t^2 + t)$

$$f'(t) = \frac{1}{3t^2 + t} \bullet (6t + 1)$$

Thus, $f'(2) = \frac{13}{14}$.

3. $y = x^{\ln x}$, $x > 0$

To find $\frac{dy}{dx}$, we take the natural logarithm of both sides and differentiate implicitly with respect to x.

$$y = x^{\ln x}$$

$$\ln y = \ln x \ln x = (\ln x)^2$$

$$\frac{1}{y} \bullet \frac{dy}{dx} = 2 \ln x \bullet \frac{1}{x}$$

The point of contact is (e, e).

At this point, $\frac{1}{e} \bullet \frac{dy}{dx} = 2 \bullet \frac{1}{e}$

$$\frac{dy}{dx} = 2.$$

The slope of the tangent at (e, e) is 2.

4. $x^2y + x\ln x = 3y$, $x > 0$

We differentiate implicitly with respect to x.

$$2xy + x^2\frac{dy}{dx} + \ln x + x \bullet \frac{1}{x} = 3\frac{dy}{dx}$$

$$2xy + \ln x + 1 = \frac{dy}{dx}(3 - x^2)$$

$$\frac{dy}{dx} = \frac{2xy + \ln x + 1}{3 - x^2}$$

Alternate Solution

y can be expressed explicitly as a function of x.

$$y = \frac{x \ln x}{3 - x^2}$$

$$\frac{dy}{dx} = \frac{(\ln x + 1)(3 - x^2) - x \ln x\,(-2x)}{(3 - x^2)^2}$$

$$= \frac{x^2 \ln x + 3 \ln x - x^2 + 3}{(3 - x^2)^2}$$

5. Since $e^{xy} = x$, $xy = \ln x$.

$$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x} \bullet x - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

At $x = 1$, $\frac{dy}{dx} = \frac{1 - \ln 1}{1} = 1$.

Alternate Solution

$e^{xy} = x$

We differentiate implicitly with respect to x

$$e^{xy}\left(y + x\frac{dy}{dx}\right) = 1$$

$$x\frac{dy}{dx} = \frac{1}{e^{xy}} - y$$

When $x = 1$, $y = 0$.

Thus, $\frac{dy}{dx} = \frac{1}{e^0} - 0 = 1$

6. $y'' + 3y' + 2y = 0$

$y = e^{Ax}$, $y' = Ae^{Ax}$, $y'' = A^2e^{Ax}$

The differential equation is

$$A^2e^{Ax} + 3Ae^{Ax} + 2e^{Ax} = 0$$
$$e^{Ax}(A^2 + 3A + 2) = 0$$
$$(A + 1)(A + 2) = 0,\ e^{Ax} \neq 0$$
$$A = -1 \text{ or } A = -2.$$

7. The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 3^x \ln 3 + \ln x + 1$$

At $A(1, 3)$, the slope of the tangent is $3 \ln 3 + 1$.

The slope of the normal line is $-\frac{1}{3 \ln 3 + 1}$.

The equation of the normal line is

$$y - 3 = -\frac{1}{3 \ln 3 + 1}(x - 1).$$

8. $v(t) = 10e^{-kt}$

a. $a(t) = v'(t) = -10ke^{-kt}$
$= -k(10e^{-kt})$
$= -kv(t)$

Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of k.

b. At time $t = 0$, $v = 10$ cm/s.

c. When $v = 5$, we have $10e^{-kt} = 5$

$$e^{-kt} = \frac{1}{2}$$
$$-kt = \ln\left(\frac{1}{2}\right) = -\ln 2$$
$$t = \frac{\ln 2}{k}.$$

After $\frac{\ln 2}{k}$ s have elapsed, the velocity of the particle is 5 cm/s. The acceleration of the particle is $-5k$ at this time.

9. We have Profit = Revenue – Cost:

$P(p) = 4000[e^{0.01(p-100)} + 1] - 50p,\ 100 \leq p \leq 250$

We apply the Algorithm for Extreme Values:

$P'(p) = 4000[e^{0.01(p-100)}(0.01)] - 50.$

For critical values, we solve $P'(p) = 0$

$$40e^{0.01(p-100)} - 50 = 0$$
$$e^{0.01(p-100)} = \frac{5}{4}$$
$$0.01(p - 100) = \ln(1.25)$$
$$p - 100 = 100 \ln(1.25)$$
$$p = 100 \ln(1.25) + 100$$
$$\doteq 122.3.$$

Since the number of jackets produced must be an integer, we evaluate P for $p = 100, 122, 123$, and 250.

$P(100) = 3000$

$P(122) = 2884.81$

$P(123) = 2884.40$

$P(250) = 9426.76$

The maximum profit of \$9426.76 occurs when 250 jackets are produced and sold. The price per jacket is given by Revenue ÷ number of jackets. Thus, selling price per jacket is

$$\frac{R(250)}{250} = \frac{21\ 926.76}{250}$$
$$= \$87.71.$$

Chapter 9 • Curve Sketching

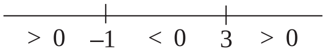
Review of Prerequisite Skills

2. **c.** $t^2 - 2t < 3$

$t^2 - 2t - 3 < 0$

$(t - 3)(t + 1) < 3$

Consider $t = 3$ and $t = –1$.

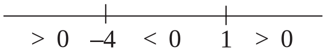


The solution is $–1 < t < 3$.

d. $x^2 + 3x - 4 > 0$

$(x + 4)(x - 1) > 0$

Consider $t = –4$ and $t = 1$.



The solution is $x < –4$ or $x > 1$.

4. **b.** $\lim\limits_{x\to 2} \dfrac{x^2 + 3x - 10}{x - 2}$

$= \lim\limits_{x\to 2} \dfrac{(x + 5)(x - 2)}{x - 2}$

$= \lim\limits_{x\to 2} (x + 5)$

$= 7$

Exercise 9.1

1. **c.** $f(x) = (2x - 1)^2(x^2 - 9)$

$f'(x) = 2(2x - 1)(2)(x^2 - 9) + 2x(2x - 1)^2$

Let $f'(x) = 0$:

$2(2x - 1)(2(x^2 - 9) + x(2x - 1)) = 0$

$2(2x - 1)(4x^2 - x - 18) = 0$

$2(2x - 1)(4x - 9)(x + 2) = 0$

$x = \dfrac{1}{2}$ or $x = \dfrac{9}{4}$ or $x = –2$.

The points are $\left(\dfrac{1}{2}, 0\right)$, (2.24, –48.2), and (–2, –125).

3. **a.** $y = x^7 - 430x^6 - 150x^3$

$\dfrac{dy}{dx} = 7x^6 - 2580x^5 - 450x^2$

If $x = 10$, $\dfrac{dy}{dx} < 0$.

If $x = 1000$, $\dfrac{dy}{dx} > 0$.

The curve rises upward in quadrant one.

c. $y = x\ln x - x^4$

$\dfrac{dy}{dx} = x\left(\dfrac{1}{x}\right) + \ln x - 4x^3$

$= 1 + \ln x - 4x^3$

If $x = 10$, $\dfrac{dy}{dx} < 0$.

If $x = 1000$, $\dfrac{dy}{dx} < 0$.

The curve is decreasing downward into quadrant four.

5. **b.** $f(x) = x^5 - 5x^4 + 100$

$f'(x) = 5x^4 - 20x^3$

Let $f'(x) = 0$:

$5x^4 - 20x^3 = 0$

$5x^3(x - 4) = 0$

$x = 0$ or $x = 4$.

x	$x < 0$	0	$0 < x < 4$	4	$x > 4$
$f'(x)$	+	0	–	0	+
Graph	Increasing		Decreasing		Increasing

d. $f(x) = \dfrac{x - 1}{x^2 + 3}$

$f'(x) = \dfrac{x^2 + 3 - 2x(x - 1)}{(x^2 + 3)^2}$

Let $f'(x) = 0$, therefore, $-x^2 + 2x + 3 = 0$.

Or $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = –1$

x	$x < –1$	–1	$–1 < x < 3$	3	$x > 3$
$f'(x)$	–	0	+	0	–
Graph	Decreasing		Increasing		Decreasing

e. $f(x) = x\ln(x)$

$$f'(x) = \ln x + \frac{1}{x}(x)$$

$$= \ln x + 1$$

Let $f'(x) = 0$:

$\ln x + 1 = 0$

$\ln x = -1$

$$x = e^{-1} = \frac{1}{e} = 0.37.$$

x	$x \leq 0$	$0 < x < 0.37$	0.37	$x > 0.37$
$f'(x)$	No values	–	0	–
Graph		Decreasing		Increasing

6. $f'(x) = (x - 1)(x + 2)(x + 3)$

Let $f'(x) = 0$:

then $(x - 1)(x + 2)(x + 3) = 0$

$x = 1$ or $x = -2$ or $x = -3$.

x	$x < -3$	–3	$-3 < x < -3$	–2	$-2 < x < 1$	1	$x > 1$
$f'(x)$	–	0	+	0	–	0	+
Graph	Decreasing		Increasing		Decreasing		Increasing

7. $g'(x) = (3x - 2)\ln(2x^2 - 3x + 2)$

Let $g'(x) = 0$:

then $(3x - 2)\ln(2x^2 - 3x + 2) = 0$

$3x - 2 = 0$ or $\ln(2x^2 - 3x + 2) = 0$

$$x = \frac{2}{3} \text{ or } 2x^2 - 3x + 2 = e^0$$

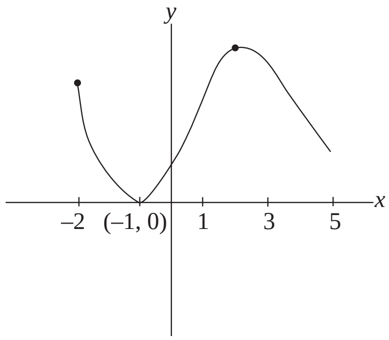
$$2x^2 - 3x + 2 = 1$$

$$2x^2 - 3x + 1 = 0$$

$$(2x - 1)(x - 1) = 0$$

$$x = \frac{1}{2} \text{ or } x = 1.$$

x	$x < \frac{1}{2}$	$\frac{1}{2} < x < \frac{2}{3}$	$\frac{2}{3} < x < 1$	$x > 1$
$f'(x)$	–	+	–	+
Graph	Decreasing	Increasing	Decreasing	Increasing



9. $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

Since $f(x)$ increases to $(-3, 18)$ and then decreases, $f'(3) = 0$.

Therefore, $27 - 6a + b = 0$ or $6a - b = 27$. (1)

Since $f(x)$ decreases to the point $(1, -14)$ and then increases, $f'(1) = 0$.

Therefore, $3 + 2a + b = 0$ or $2a + b = -3$. (2)

Add (1) to (2): $8a = 24$ and $a = 3$.

When $a = 3$, $b = 6 + b = -3$ or $b = -9$.

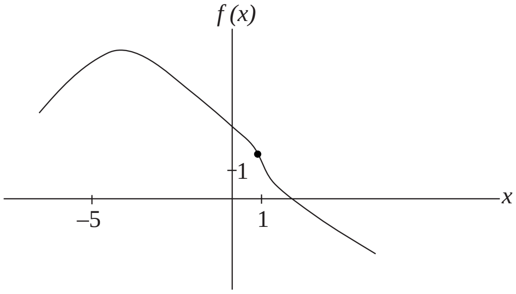
Since $(1, -14)$ is on the curve and $a = 3$, $b = -9$,

then $-14 = 1 + 3 - 9 + c$

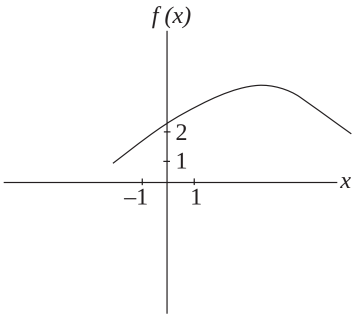
$c = -9$.

The function is $f(x) = x^3 + 3x^2 - 9x - 9$.

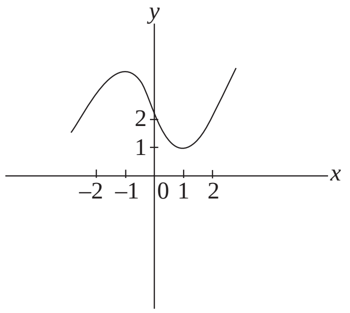
10.



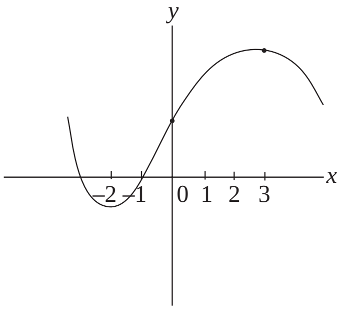
11. a.



b.



c.



12. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

Let $f'(x) = 0$, then $x = \frac{-b}{2a}$.

If $x < \frac{-b}{2a}$, $f'(x) < 0$, therefore the function is decreasing.

If $x > \frac{-b}{2a}$, $f'(x) > 0$, therefore the function is increasing.

13. Let $y = f(x)$ and $u = g(x)$.

Let x_1 and x_2 be any two values in the interval $a \le x \le b$ so that $x_1 < x_2$.

Since $x_1 < x_2$, both functions are increasing:

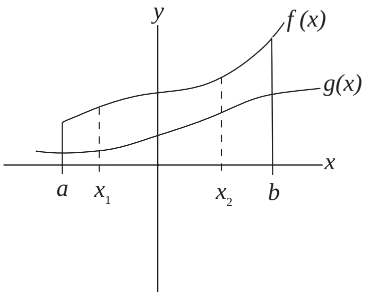
$f(x_2) > f(x_1)$ (1)

$g(x_2) > g(x_1)$ (2)

$yu = f(x) \bullet g(x)$.

(1) × (2) results in $f(x_2) \bullet g(x_2) > f(x_1)g(x_1)$.

The function yu or $f(x) \bullet g(x)$ is strictly increasing.



14. Let x_1, x_2 be in the interval $a \le x \le b$, such that $x_1 < x_2$.

Therefore, $f(x_2) > f(x_1)$, and $g(x_2) > g(x_1)$.

In this case, $f(x_1)$, $f(x_2)$, $g(x_1)$, and $g(x_2) < 0$.

Multiplying an inequality by a negative will reverse its sign.

Therefore, $f(x_2) \bullet g(x_2) < f(x_1) \bullet g(x_1)$.

But L.S. > 0 and R.S. > 0.

Therefore, the function fg is strictly increasing.

Exercise 9.2

3. **b.** $f(x) = \frac{2x}{x^2 + 9}$

$$f'(x) = \frac{2(x^2 + 9) - 2x(2x)}{(x^2 + 9)^2} = \frac{18 - 4x^2}{(x^2 + 9)^2}$$

Let $f'(x) = 0$:

therefore, $18 - 2x^2 = 0$

$x^2 = 9$

$x = \pm 3$.

x	$x < -3$	-3	$-3 < x < 3$	3	$x > 3$
$f'(x)$	–	0	+	0	–
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing

Local minimum at (–3, –0.3) and local maximum at (3, 0.3).

c. $y = xe^{-4x}$

$$\frac{dy}{dx} = e^{-4x} - 4xe^{-4x}$$

Let $\frac{dy}{dx} = 0$, $e^{-4x}(1 - 4x) = 0$:

$e^{-4x} \neq 0$ or $(1 - 4x) = 0$

$x = \frac{1}{4}$.

x	$x < \frac{1}{4}$	$\frac{1}{4}$	$x > \frac{1}{4}$
$f'(x)$	+	0	–
Graph	Increasing	Local Max	Decreasing

At $x = \frac{1}{4}$, $y = \frac{1}{4}e^{-1} = \frac{1}{4e}$.

Local maximum occurs at $\left(\frac{1}{4}, \frac{1}{4e}\right)$.

d. $y = \ln(x^2 - 3x + 4)$

$$\frac{dy}{dx} = \frac{2x - 3}{x^2 - 3x + 4}$$

Let $\frac{dy}{dx} = 0$, therefore, $2x - 3 = 0$

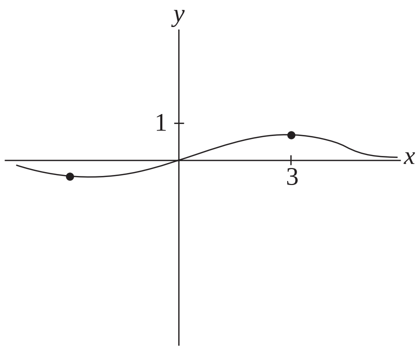
$$x = \frac{3}{2} = 1.5.$$

x	$x < 1.5$	1.5	$x > 1.5$
$f'(x)$	–	0	+
Graph	Decreasing	Local Min	Increasing

Local minimum at (1.5, ln 1.75).

4. **b.** $f(x) = \frac{2x}{x^2 + 9}$

The x-intercept is 0 and the y-intercept is 0.



c. $y = xe^{-4x}$

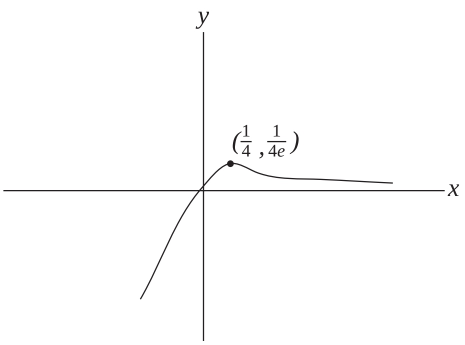
x-intercept, let $y = 0$,

$$0 = xe^{-4x}$$

Therefore, $x = 0$.

y-intercept, let $x = 0$,

$$y = 0.$$



d. $y = \ln(x^2 - 3x + 4)$

x-intercept, let $y = 0$,

$\ln x^2 - 3x + 4 = 0$

$x^2 - 3x + 4 = 1$

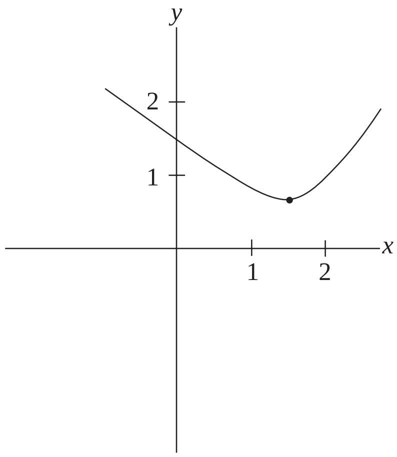
$x^2 - 3x + 3 = 0.$

No solution, since $0 = 9 - 12 < 0$

y-intercept, let $x = 0$,

$y = \ln 4$

$= 1.39.$



5. **b.** $s = -t^2e^{-3t}$

$$\frac{ds}{dt} = -2te^{-3t} + 3e^{-3t}(t^2)$$

Let $\frac{ds}{dt} = 0.$

$$te^{-3t}[-2 + 3t] = 0$$

$$t = 0 \text{ or } t = \frac{2}{3}.$$

t	$t < 0$	$t = 0$	$0 < t < \frac{2}{3}$	$t = \frac{2}{3}$	$t > \frac{2}{3}$
$\frac{ds}{dt}$	+	0	–	0	+
Graph	Increasing	Local Max	Decreasing	Local Min	Increasing

Critical points are (0, 0) and $\left(\frac{2}{3}, -0.06\right)$.

Tangent is parallel to t-axis.

c. $y = (x-5)^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}\left(x-5\right)^{\frac{-2}{3}}$$

$$= \frac{1}{3(x-5)^{\frac{2}{3}}}$$

$$\frac{dy}{dx} \neq 0$$

The critical point is at (5, 0), but is neither a maximum or minimum. The tangent is not parallel to x-axis.

f. $y = x^2 - 12x^{\frac{1}{3}}$

$$\frac{dy}{dx} = 2x - \frac{1}{3}\left(12x^{\frac{-2}{3}}\right)$$

$$= 2x - \frac{4}{x^{\frac{2}{3}}}$$

Let $\frac{dy}{dx} = 0$. Then, $2x = \frac{4}{x^{\frac{2}{3}}}$:

$$2x^{\frac{5}{3}} = 4$$

$$x^{\frac{5}{3}} = 2$$

$$x = 2^{\frac{3}{5}} = \sqrt[5]{2^3}$$

$$x \doteq 1.5.$$

Critical points are at $x = 0$ and $x = 1.5$.

x	$x < 0$	$x = 0$	$0 < x < 1.5$	$x = 1.5$	$x > 1.5$
$\frac{dy}{dx}$	–	undefined	–	0	+
Graph	Decreasing	Vertical Tangent	Decreasing	Local Min	Increasing

Critical points are at (0, 0) and (1.5, –11.5).
Local minimum is at (1.5, –11.5).
Tangent is parallel to y-axis at (0, 0).
Tangent is parallel to x-axis at (1.5, –11.5).

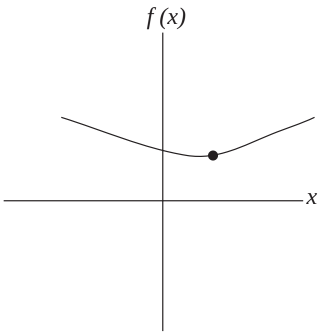
7. **e.** $f(x) = \sqrt{x^2 - 2x + 2}$

$$f'(x) = \frac{2x-2}{2\sqrt{x^2-2x+2}}$$

Let $f'(x) = 0$, then $x = 1$.
Also, $x^2 - 2x + 2 \geq 0$ for all x.

x	$x < 1$	$x = 1$	$x > 1$
$f'(x)$	–	0	+
Graph	Decreasing	Local Min	Increasing

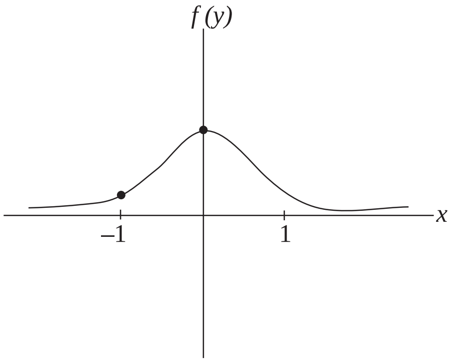
Local minimum is at (1, 1).



g. $f(x) = e^{-x^2}$
$f'(x) = -2xe^{-x^2}$
Let $f'(x) = 0$, then $x = 0$.

x	$x < 0$	0	$x > 0$
$f'(x)$	+	0	–
Graph	Increasing	Local Max	Decreasing

When $x = 0$, $e^0 = 1$.
Local maximum point is at (0, 1).



h. $f(x) = x^2 \ln x$

$$f'(x) = 2x \ln x + x^2\left(\frac{1}{x}\right)$$

$$= 2x \ln x + x$$

Let $f'(x) = 0$:

$2x \ln x + x = 0$

$x(2 \ln x + 1) = 0$

$x = 0$ or $\ln x = -\frac{1}{2}$.

But, $x > 0$, then $x \neq 0$,

$x = e^{-\frac{1}{2}} = 0.61$.

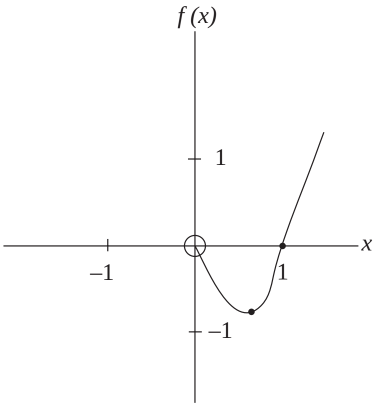
x	$0 < x < 0.61$	0.61	$x > 0.61$
$f'(x)$	–	0	+
Graph	Decreasing	Local Min	Increasing

Local minimum is at $x = 0.61$ and $f(0.61)$:

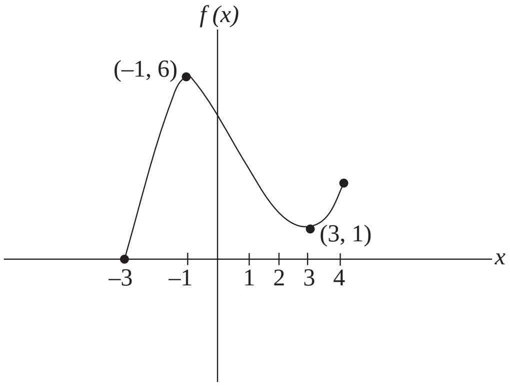
$= 2(0.61) \ln 0.61 + 0.61$

$= -0.64$.

Critical point is (0.61, –0.64).



9.



10. $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

Since a relative maximum occurs at $x = 3$, then $2ax + b = 0$ at $x = 3$.

Or, $6a + b = 0$.

Also, at (0, 1), $1 = 0 + 0 + c$ or $c = 1$.

Therefore, $y = ax^2 + bx + 1$.

Since (3, 12) lies on the curve,

$12 = 0a + 3b + 1$

$9a + 3b = 11$

$6a + b = 0$.

Since $b = -6a$,

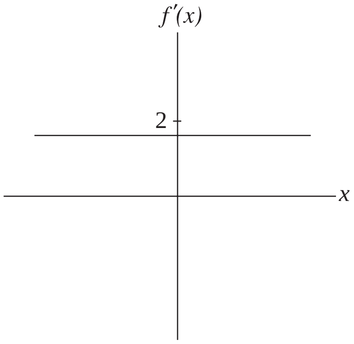
then $9a - 18a = 11$

$$\text{or } a = \frac{-11}{9}$$

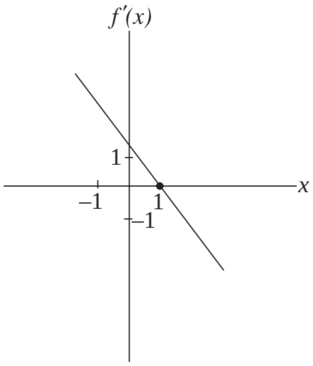
$$b = \frac{22}{3}.$$

The equation is $y = \frac{-11}{9}x^2 + \frac{22}{3}x + 1$.

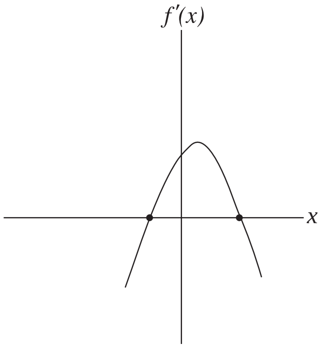
11. a.



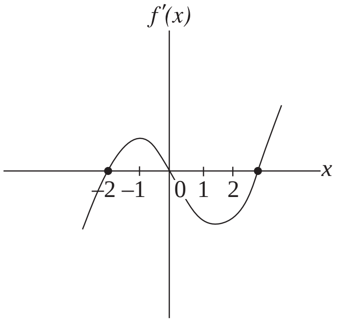
b.



c.



d.



12. $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$

a. $f'(x) = 12x^3 + 3ax^2 + 2bx + c$

At $x = 0$, $f'(0) = 0$, then $f'(0) = 0 + 0 + 0 + c$

or $c = 0$.

At $x = –2$, $f'(–2) = 0$,

$–96 + 12a – 4b = 0$. (1)

Since (0, –9) lies on the curve,

$–9 = 0 + 0 + 0 + 0 + d$

or $d = –9$.

Since (–2, –73) lies on the curve,

$–73 = 48 – 8a + 4b + 0 – 9$

$–8a + 4b = –112$

or $2a – b = 28$ (2)

Also, from (1): $3a – b = 24$

$2a – b = –28$

$a = –4$

$b = –36$.

The function is $f(x) = 3x^4 – 4x^3 – 36x^2 – 9$.

b. $f'(x) = 12x^3 – 12x^2 – 72x$

Let $f'(x) = 0$:

$x^3 – x^2 – 6x = 0$

$x(x – 3)(x + 2) = 0$.

Third point occurs at $x = 3$,

$f(3) = –198$.

x	$x < –2$	–2	$–2 < x < 0$	0	$0 < x < 3$	3	$x > 3$
$f'(x)$	–	0	+	0	–	0	+
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

Local minimum is at (–2, –73) and (3, –198).
Local maximum is at (0, –9).

13. a. $y = 4 – 3x^2 – x^4$

$$\frac{dy}{dx} = –6x – 4x^3$$

Let $\frac{dy}{dx} = 0$:

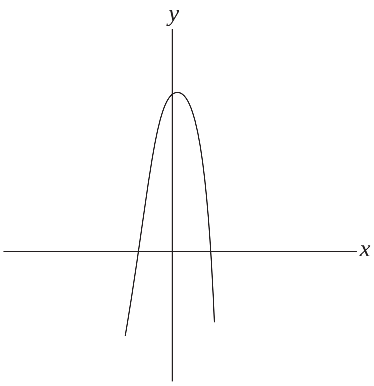
$–6x – 4x^3 = 0$

$–2x(2x^2 + 3) = 0$

$x = 0$ or $x^2 = \frac{–3}{2}$; inadmissible.

x	$x < 0$	0	$x > 0$
$\frac{dy}{dx}$	+	0	–
Graph	Increasing	Local Max	Decreasing

Local maximum is at (0, 4).



b. $y = 3x^5 – 5x^3 –30x$

$$\frac{dy}{dx} = 15x^4 – 15x^2 – 30$$

Let $\frac{dy}{dx} = 0$:

$15x^4 – 15x^2 – 30 = 0$

$x^4 – x^2 – 2 = 0$

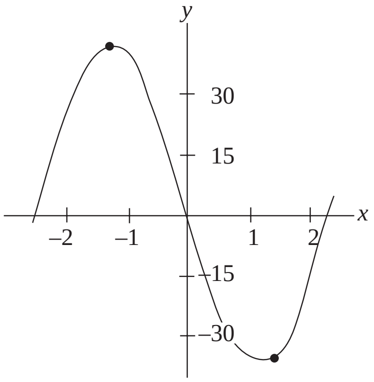
$(x^2 – 2)(x^2 + 1) = 0$

$x^2 = 2$ or $x^2 = –1$

$x = \pm\sqrt{2}$; inadmissible.

At $x = 100$, $\frac{dy}{dx} > 0$.

Therefore, function is increasing into quadrant one, local minimum is at (1.41, –39.6) and local maximum is at (–1.41, 39.6).



14. $h(x) = \frac{f(x)}{g(x)}$

Since $f(x)$ has a local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$.
Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

If $x < c$, $f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.
If $x > c$, $f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.
Since for $x < c$, $h'(x) > 0$ and for $x > c$, $h'(x) < 0$.
Therefore, $h(x)$ has a local maximum at $x = c$.

Exercise 9.3

2. $f(x) = \frac{g(x)}{h(x)}$

Conditions for a vertical asymptote:
$h(x) = 0$ must have at least one solution s, and $\lim_{x \to x_1} f(x) = \infty$.

Conditions for a horizontal asymptote:
$\lim_{x \to \infty} f(x) = k$, where $k \in R$,

or $\lim_{x \to -\infty} f(x) = k$ where $k \in R$.

Condition for an oblique asymptote is that the highest power of $g(x)$ must be one more than the highest power of $k(x)$.

6. **a.** $y = \frac{x - 3}{x + 5}$

$$\lim_{x \to -5^+} \frac{x-3}{x+5} = -\infty, \lim_{x \to -5^-} \frac{x-3}{x+5} = +\infty \qquad (1)$$

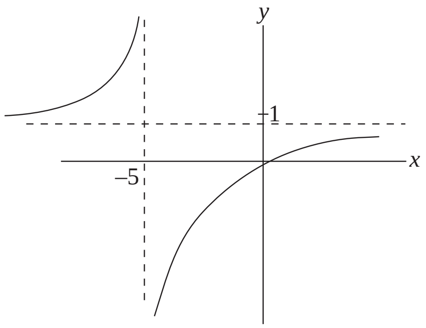
Vertical asymptote at $x = -5$.

$$\lim_{x \to \infty} \frac{x-3}{x+5} = 1, \lim_{x \to -\infty} \frac{x-3}{x+5} = 1 \qquad (2)$$

Horizontal asymptote at $y = 1$.

$$\frac{dy}{dx} = \frac{x + 5 - x + 3}{(x+5)^2} = \frac{8}{(x+5)^2} \qquad (3)$$

Since $\frac{dy}{dx} \neq 0$, there are no maximum or minimum points.



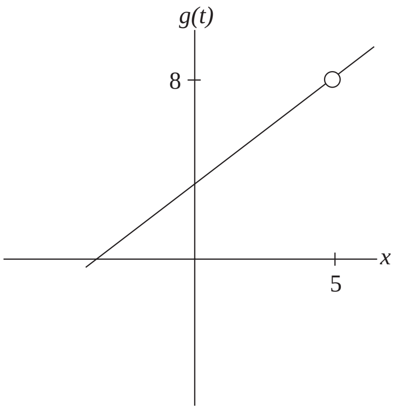
c. $g(t) = \frac{t^2 - 2t - 15}{t - 5}$

Discontinuity at $t = 5$.

$$\lim_{t \to 5^-} \frac{(t-5)(t+3)}{t-5} = \lim_{t \to 5^-} (t + 2) = 8$$

$$\lim_{t \to 5^+} (t + 3) = 8$$

No asymptote at $x = 5$. The curve is of the form $t + 3$.



d. $p(x) = \frac{15}{6 - 2e^x}$

Discontinuity when $6 - 2e^x = 0$

$$e^x = 3$$

$$x = \ln 3 \doteq 1.1.$$

$\lim\limits_{x \to 1.1^-} \frac{15}{6 - 2e^x} = +\infty$, $\lim\limits_{x \to 1.1^+} \frac{15}{6 - 2e^x} = +\infty$ (1)

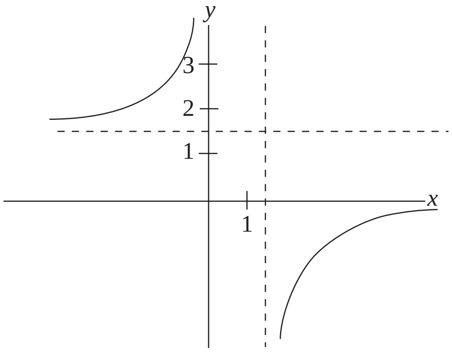
Vertical asymptote at $x \doteq 1.1$.

Horizontal asymptote: $\lim\limits_{x \to \infty} \frac{15}{6 - 2e^x} = 0$ from below, (2)

$\lim\limits_{x \to \infty} \frac{15}{6 - 2e^x} = \frac{15}{6}$ from above.

$p'(x) = \frac{-15(-2e^x)}{(6 - 2e^x)^2}$ (3)

True if $e^x = 0$, which is not possible. No maximum or minimum points.



e. $y = \frac{(2 + x)(3 - 2x)}{x^2 - 3x}$

Discontinuity at $x = 0$ and $x = 3$ (1)

$\lim\limits_{x \to 0^+} \frac{(2 + x)(3 - 2x)}{x^2 - 3x} = +\infty$

$\lim\limits_{x \to 0^-} \frac{(2 + x)(3 - 2x)}{x^2 - 3x} = -\infty$

$\lim\limits_{x \to 3^+} \frac{(2 + x)(3 - 2x)}{x^2 - 3x} = +\infty$

$\lim\limits_{x \to 3^-} \frac{(2 + x)(3 - 2x)}{x^2 - 3x} = -\infty$

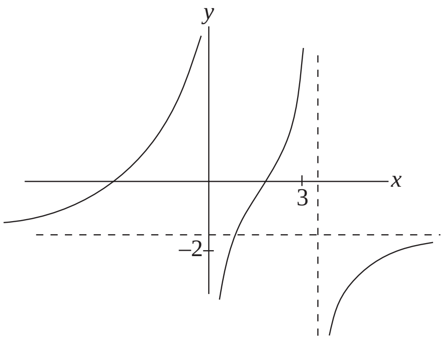
Vertical asymptotes at $x = 0$ and $x = 3$.

Horizontal asymptote. (2)

$$\lim_{x \to \infty} \frac{(2 + x)(3 - 2x)}{x^2 - 3x} = \lim_{x \to \infty} \frac{\frac{6}{x^2} - \frac{1}{x} - 2}{1 - \frac{3}{x}} = -2$$

$$\lim_{x \to -\infty} \frac{\frac{6}{x^2} - \frac{1}{x} - 2}{1 - \frac{3}{x}} = -2$$

Horizontal asymptote at $y = -2$.



f. $P = \frac{10}{n^2 + 4}$

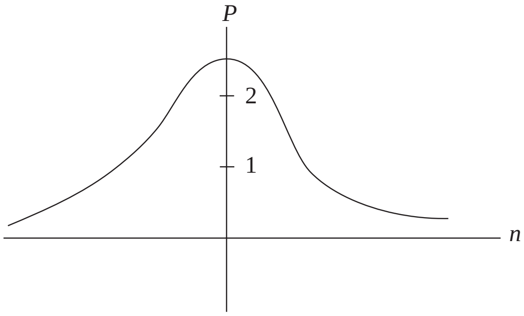
No discontinuity

$\lim\limits_{n \to \infty} p = 0$, $\lim\limits_{n \to \infty} p = 0$

$$\frac{dp}{dn} = \frac{-10(2n)}{(n^2 + 4)^2}$$

$\frac{dp}{dn} = 0$, then $n = 0$

Maximum point is at (0, 2.5).



7. **b.** $f(x) = \dfrac{2x^2 + 9x + 2}{2x + 3}$

$$\begin{array}{rl} & \quad\quad\quad\;\; x + 3 \\ 2x + 3 & \big)\overline{2x^2 + 9x + 2} \\ & \;\underline{2x^2 + 3x} \\ & \quad\quad\quad 6x + 2 \\ & \quad\quad\quad \underline{6x + 9} \\ & \quad\quad\quad\quad\; -7 \end{array}$$

$$f(x) = \frac{2x^2 + 9x + 2}{2x + 3}$$

$$= x + 3 - \frac{7}{2x + 3}$$

Oblique asymptote is at $y = x + 3$.

d. $f(x) = \dfrac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$

$$\begin{array}{rl} & \quad\quad\quad\quad\quad\quad\; x + 3 \\ x^2 - 4x + 3 & \big)\overline{x^3 - x^2 - 9x + 15} \\ & \;\underline{x^3 - 4x^2 + 3x} \\ & \quad\quad 3x^2 - 12x + 15 \\ & \quad\quad \underline{3x^2 - 12x + 9} \\ & \quad\quad\quad\quad\quad\quad\quad 6 \end{array}$$

$$f(x) = x + 3 + \frac{6}{x^2 - 4x + 3}$$

Oblique asymptote is at $y = x + 3$.

8. **b.** Oblique asymptote is at $y = x + 3$.

Consider $x > \dfrac{-3}{2}$ and $x < \dfrac{-3}{2}$.

Consider $x = 0$.

$f(0) = \dfrac{2}{3}$ and for the oblique asymptote $y = 3$.

Therefore, the oblique asymptote is above the

curve for $x > \dfrac{-3}{2}$.

The curve approaches the asymptote from below.
Consider $x = -2$.

$$f(-2) = \frac{8 - 18 + 2}{-1}$$

$$= 8$$

For the oblique asymptote, $y = 1$.
Therefore, the curve is above the oblique
asymptote and approaches the asymptote
from above.

9. **a.** $f(x) = \dfrac{3 - x}{2x + 5}$

Discontinuity is at $x = -2.5$.

$$\lim_{x \to -2.5^-} \frac{3 - x}{2x + 5} = -\infty$$

$$\lim_{x \to -2.5^+} \frac{3 - x}{2x + 5} = +\infty$$

Vertical asymptote is at $x = -2.5$.

Horizontal asymptote:

$$\lim_{x \to \infty} \frac{3 - x}{2x + 5} = -\frac{1}{2}, \lim_{x \to -\infty} \frac{3 - x}{2x + 5} = -\frac{1}{2}.$$

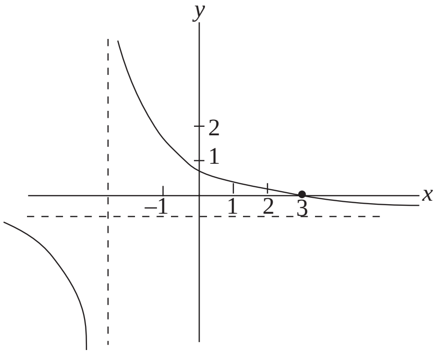
Horizontal asymptote is at $y = -\dfrac{1}{2}$.

$$f'(x) = \frac{-(2x + 5) - 2(3 - x)}{(2x + 5)^2} = \frac{-11}{(2x + 5)^2}$$

Since $f'(x) \neq 0$, there are no maximum or minimum points.

y-intercept, let $x = 0$, $y = \dfrac{3}{5} = 0.6$

x-intercept, let $y = 0$, $\dfrac{3 - x}{2x + 5} = 0$, $x = 3$



d. $s(t) = t + \dfrac{1}{t}$

Discontinuity is at $t = 0$.

$$\lim_{t \to 0^+} \left(t + \frac{1}{t}\right) = +\infty$$

$$\lim_{t \to 0^-} \left(t + \frac{1}{t}\right) = -\infty$$

Oblique asymptote is at $s(t) = t$.

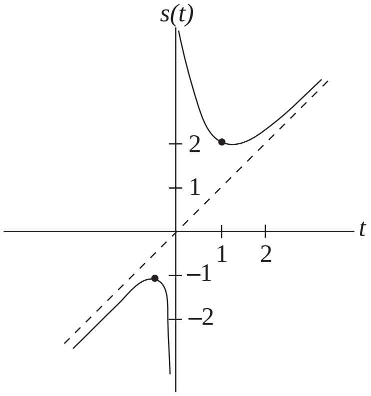
$$s'(t) = 1 - \frac{1}{t^2}$$

Let $s'(t) = 0$, $t^2 = 1$

$t = \pm 1$.

t	$t < –1$	$t = –1$	$–1 < t < 0$	$0 < t < 1$	$t = 1$	$t > 1$
$s'(t)$	+	0	–	–	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

Local maximum is at (–1, –2) and local minimum is at (1, 2).



e. $g(x) = \dfrac{2x^2 + 5x + 2}{x + 3}$

Discontinuity is at $x = –3$.

$$\frac{2x^2 + 5x + 2}{x + 3} = 2x – 1 + \frac{5}{x + 3}$$

Oblique asymptote is at $y = 2x – 1$.

$\lim\limits_{x \to -3^+} g(x) = +\infty$, $\lim\limits_{x \to -3^-} g(x) = -\infty$

$$g'(x) = \frac{(4x + 5)(x + 3) – (2x^2 + 5x + 2)}{(x + 3)^2}$$

$$= \frac{2x^2 + 12x + 13}{(x + 3)^2}$$

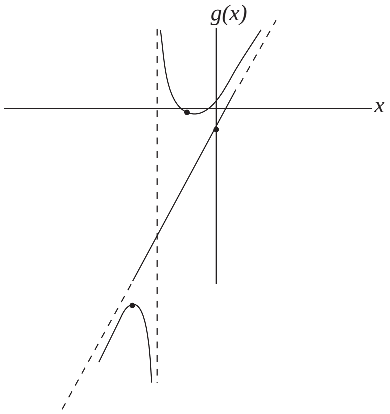
Let $g'(x) = 0$, therefore, $2x^2 + 12x + 13 = 0$:

$$x = \frac{–12 \pm \sqrt{144 – 104}}{4}$$

$x = –1.4$ or $x = –4.6$.

x	$x < –4.6$	$–4.6$	$–4.6 < x < –3$	$–3$	$–3 < x < –1.4$	$x = 1.4$	$x > –1.4$
$g'(x)$	+	0	–	Undefined	–	0	+
Graph	Increasing	Local Max	Decreasing	Vertical Asymptote	Decreasing	Local Min	Increasing

Local maximum is at (–4.6, –10.9) and local minimum is at (–1.4, –0.7).



f. $s(t) = \dfrac{t^2 + 4t – 21}{t – 3}$, $t \geq –7$

$$= \frac{(t + 7)(t – 3)}{(t – 3)}$$

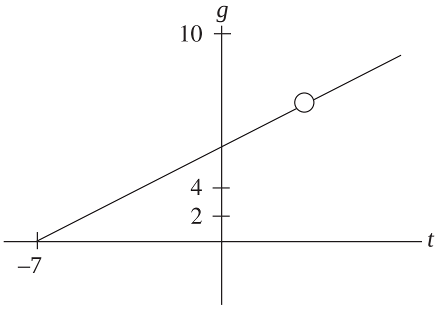
Discontinuity is at $t = 3$.

$$\lim_{x \to 3^+} \frac{(t + 7)(t – 3)}{t – 3)} = \lim_{x \to 3^+} (t + 7)$$

$$= 10$$

$$\lim_{x \to 3^-} (t + 7) = 10$$

There is no vertical asymptote.
The function is the straight line $s = t + 7$, $t \geq –7$.



11. $f(x) = \dfrac{ax + 5}{3 – bx}$

Vertical asymptote is at $x = –4$.
Therefore, $3 – bx = 0$ at $x = – 5$.
That is, $3 – b(–5) = 0$

$$b = \frac{3}{5}.$$

Horizontal asymptote is at $y = –3$.

$$\lim_{x \to \infty} \left(\frac{ax + 5}{3 – bx} \right) = –3$$

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right)=\lim_{x\to\infty}\left(\frac{a+\frac{5}{x}}{\frac{3}{x}-b}\right)=\frac{-a}{b}$$

But $-\frac{a}{b}=-3$ or $a=3b$.

But $b=\frac{3}{5}$, then $a=\frac{9}{5}$.

12. **a.** $$\lim_{x\to\infty}\frac{x^2+1}{x+1}=\lim_{x\to\infty}\frac{x+\frac{1}{x}}{1+\frac{1}{x}}$$

$$=\infty$$

$$\lim_{x\to\infty}\frac{x^2+2x+1}{x+1}=\lim_{x\to\infty}\frac{(x+1)(x+1)}{(x+1)}$$

$$=\lim_{x\to\infty}(x+1)$$

$$=\infty$$

b. $$\lim_{x\to\infty}\left[\frac{x^2+1}{x+1}-\frac{x^2+2x+1}{x+1}\right]$$

$$=\lim_{x\to\infty}\frac{x^2+1-x^2-2x-1}{x+1}$$

$$=\lim_{x\to\infty}\frac{-2x}{x+1}$$

$$=\lim_{x\to\infty}\frac{-2}{1+\frac{1}{x}}=-2$$

13. $f(x)=\dfrac{2x^2-2x}{x^2-9}$

Discontinuity is at $x^2-9=0$ or $x=\pm3$.

$\lim_{x\to3^+}f(x)=+\infty$

$\lim_{x\to3^-}f(x)=-\infty$

$\lim_{x\to-3^+}f(x)=-\infty$

$\lim_{x\to-3^-}f(x)=+\infty$

Vertical asymptotes are at $x=3$ and $x=-3$.

Horizontal asymptote:

$\lim_{x\to\infty}f(x)=2$ (from below)

$\lim_{x\to\infty}f(x)=2$ (from above)

Horizontal asymptote is at $y=2$.

$$f'(x)=\frac{(4x-2)(x^2-9)-2x(2x^2-2x)}{(x^2-9)^2}$$

$$=\frac{4x^3-2x^2-36x+18-4x^3+4x^2}{(x^2-9)^2}$$

$$=\frac{2x^2-36x+18}{(x^2-9)^2}$$

Let $f'(x)=0$, $2x^2-36x+18=0$
or $x^2-18x+9=0$.

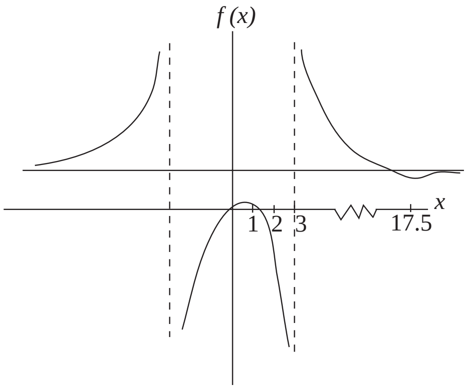
$$x=\frac{18\pm\sqrt{18^2-36}}{2}$$

$x=0.51$ or $x=17.5$

$y=0.057$ or $y=1.83$.

x	$-3<x<0.51$	0.51	$0.51<x<3$	$3<x<17.5$	17.5	$x>17.5$
$f'(x)$	+	0	–	–	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

Local maximum is at (0.51, 0.057) and local minimum is at (17.5, 1.83).



14. $y=\dfrac{x^2+3x+7}{x+2}$

$$\begin{array}{r} x+1 \\ x+2\;\overline{)\,x^2+3x+7} \\ \underline{x^2+2x} \\ x+7 \\ \underline{x+2} \\ 5 \end{array}$$

$$y=\frac{x^2+3x+7}{x+2}=x+1+\frac{5}{x+2}$$

Oblique asymptote is at $y=x+1$.

Exercise 9.4

2. **a.** $y = x^3 - 6x^2 - 15x + 10$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For critical values, we solve $\frac{dy}{dx} = 0$:

$$\begin{aligned} 3x^2 - 12x - 15 &= 0 \\ x^2 - 4x - 5 &= 0 \\ (x-5)(x+1) &= 0 \end{aligned}$$

$x = 5$ or $x = -1$

The critical points are (5, –105) and (–1, 20).

Now, $\frac{d^2y}{dx^2} = 6x - 12$.

At $x = 5$, $\frac{d^2y}{dx^2} = 18 > 0$. There is a local minimum at this point.

At $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$. There is a local maximum at this point.

The local minimum is (5, –105) and the local maximum is (–1, 20)

b. $y = \frac{25}{x^2 + 48}$

$\frac{dy}{dx} = -\frac{50x}{(x^2 + 48)^2}$

For critical values, solve $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.

Since $x^2 + 48 > 0$ for all x, the only critical point is $\left(0, \frac{25}{48}\right)$.

$$\frac{d^2y}{dx^2} = -50(x^2 + 48)^{-2} + 100x(x^2 + 48)^{-3}(2x)$$

$$= -\frac{50}{(x^2 + 48)^2} + \frac{200x^2}{(x^2 + 48)^3}$$

At $x = 0$, $\frac{d^2y}{dx^2} = -\frac{50}{48^2} < 0$. The point $\left(0, \frac{25}{48}\right)$ is a local maximum.

c. $s = t + t^{-1}$

$\frac{ds}{dt} = 1 - \frac{1}{t^2}, t \neq 0$

For critical values, we solve $\frac{ds}{dt} = 0$:

$$\begin{aligned} 1 - \frac{1}{t^2} &= 0 \\ t^2 &= 1 \\ t &= \pm 1. \end{aligned}$$

The critical points are (–1, –2) and (1, 2).

$\frac{d^2s}{dt^2} = \frac{2}{t^3}$

At $t = -1$, $\frac{d^2s}{dt} = -2 < 0$. The point (–1, –2) is a local maximum. At $t = 1$, $\frac{d^2s}{dt^2} = 2 > 0$. The point (1, 2) is a local minimum.

d. $y = (x - 3)^3 + 8$

$\frac{dy}{dx} = 3(x - 3)^2$

$x = 3$ is a critical value.

The critical point is (3, 8).

$\frac{d^2y}{dx^2} = 6(x - 3)$

At $x = 3$, $\frac{d^2y}{dx^2} = 0$.

The point (3, 8) is neither a relative (local) maximum or minimum.

3. **a.** For possible point(s) of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$6x - 8 = 0$$

$$x = \frac{4}{3}.$$

Interval	$x < \frac{4}{3}$	$x = \frac{4}{3}$	$x > \frac{4}{3}$
$f''(x)$	< 0	$= 0$	> 0
Graph of of $f(x)$	Concave Down	Point of Inflection	Concave Up

The point $\left(\frac{4}{3}, -14\frac{20}{27}\right)$ is a point of inflection.

b. For possible point(s) of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$\frac{200x^2 - 50x^2 - 2400}{(x^2 + 48)^3} = 0$$

$150x^2 = 2400.$

Since $x^2 + 48 > 0$:

$x = \pm 4.$

Interval	$x < -4$	$x = -4$	$-4 < x < 4$	$x = 4$	$x > 4$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$\left(-4, \frac{25}{64}\right)$ and $\left(4, \frac{25}{64}\right)$ are points of inflection.

c. $\frac{d^2s}{dt^2} = \frac{3}{t^2}$

Interval	$t < 0$	$t = 0$	$t > 0$
$f''(t)$	< 0	Undefined	> 0
Graph of $f(t)$	Concave Down	Undefined	Concave Up

The graph does not have any points of inflection.

d. For possible points of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$6(x - 3) = 0$$

$$x = 3.$$

Interval	$x < 3$	$x = 3$	$x > 3$
$f''(x)$	< 0	$= 0$	> 0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

(3, 8) is a point of inflection.

4. **a.** $f(x) = 2x^3 - 10x + 3$ at $x = 2$

$f'(x) = 6x^2 - 10$

$f''(x) = 12x$

$f''(2) = 24 > 0$

The curve lies above the tangent at (2, –1).

b. $g(x) = x^2 - \frac{1}{x}$ at $x = -1$

$g'(x) = 2x + \frac{1}{x^2}$

$g''(x) = 2 - \frac{2}{x^3}$

$g''(-1) = 2 + 2 = 4 > 0$

The curve lies above the tangent line at (–1, 2).

c. $s = e^t \ln t$ at $t = 1$

$$\frac{ds}{dt} = e^t \ln t + \frac{e^t}{t}$$

$$\frac{d^2s}{dt^2} = e^t \ln t + \frac{e^t}{t} + \frac{e^t}{t} - \frac{e^t}{t^2}$$

At $t = 1$, $\frac{d^2s}{dt^2} = 0 + e + e - e = e > 0.$

The curve is above the tangent line at (1, 0).

d. $p = \frac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

$$p = w(w^2 + 1)^{-\frac{1}{2}}$$

$$\frac{dp}{dw} = (w^2 + 1)^{-\frac{1}{2}} + w\left(-\frac{1}{2}\right)(w^2 + 1)^{-\frac{3}{2}}(2w)$$

$$= (w^2 + 1)^{-\frac{1}{2}} - w^2(w^2 + 1)^{-\frac{3}{2}}$$

$$\frac{d^2p}{dw^2} = -\frac{1}{2}(w^2+1)^{-\frac{3}{2}}(2w) - 2w(w^2+1)^{-\frac{3}{2}} +$$
$$w^2\left(\frac{3}{2}\right)(w^2+1)^{-\frac{5}{2}}\left(2w\right)$$

At $w = 3$, $\frac{d^2p}{dw^2} = -\frac{3}{10\sqrt{10}} - \frac{6}{10\sqrt{10}} + \frac{81}{100\sqrt{10}}$

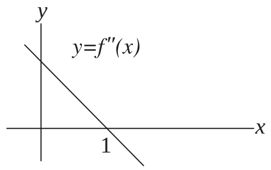
$$= -\frac{9}{100\sqrt{10}} < 0.$$

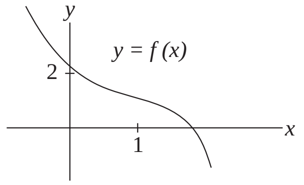
The curve is below the tangent line at $\left(3, \frac{3}{\sqrt{10}}\right)$.

5. **(i) a.** $f''(x) > 0$ for $x < 1$
Thus, the graph of $f(x)$ is concave up on $x < 1$.
$f''(x) \leq 0$ for $x > 1$. The graph of $f(x)$ is concave down on $x > 1$.

(i) b. There is a point of inflection at $x = 1$.

(i) c.

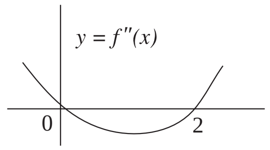


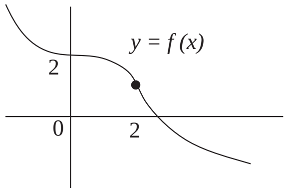


(ii) a. $f''(x) > 0$ for $x < 0$ or $x > 2$
The graph of $f(x)$ is concave up on $x < 0$ or $x > 2$.
The graph of $f(x)$ is concave down on $0 < x < 2$.

(ii) b. There are points of inflection at $x = 0$ and $x = 2$.

(ii) c.





6. For any function $y = f(x)$, find the critical points, i.e., the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist. Evaluate $f''(x)$ for each critical value. If the value of the second derivative at a critical point is positive, the point is a local minimum. If the value of the second derivative at a critical point is negative, the point is a local maximum.

7. Step 4: Use the first derivative test or the second derivative test to determine the type of critical points that may be present.

8. **a.** $f(x) = x^4 + 4x^3$
(i) $f'(x) = 4x^3 + 12x^2$
$f''(x) = 12x^2 + 24x$
For possible points of inflection, solve $f''(x) = 0$:
$12x^2 + 24x = 0$
$12x(x + 2) = 0$
$x = 0$ or $x = -2$.

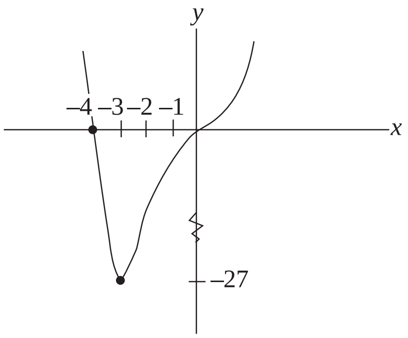
Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

The points of inflection are (–2, –16) and (0, 0).

(ii) If $x = 0$, $y = 0$.
For critical points, we solve $f'(x) = 0$:
$4x^3 + 12x^2 = 0$
$4x^2(x + 3) = 0$
$x = 0$ or $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$f'(x)$	< 0	0	> 0	$= 0$	> 0
Graph of $f(x)$	Decreasing	Local Min	Increasing		Increasing

If $y = 0$, $x^4 + 4x^3 = 0$
$x^3(x + 4) = 0$
$x = 0$ or $x = -4$.
The x-intercepts are 0 and –4.



b. $y = x - \ln x$

(i) $\frac{dy}{dx} = 1 - \frac{1}{x}$

$\frac{d^2y}{dx^2} = \frac{1}{x^2}$

Since $x > 0$, $\frac{d^2y}{dx^2} > 0$ for all x. The graph of $y = f(x)$ is concave up throughout the domain.

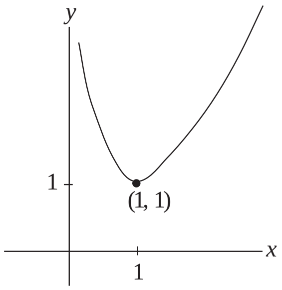
(ii) There are no x– or y-intercepts ($x > \ln x$ for all $x > 0$).

For critical points, we solve $\frac{dy}{dx} = 0$:

$1 - \frac{1}{x} = 0$

$x = 1.$

Interval	$0 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	< 0	$= 0$	> 0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing



c. $y = e^x + e^{-x}$

(i) $\frac{dy}{dx} = e^x - e^{-x}$

$\frac{d^2y}{dx^2} = e^x + e^{-x} > 0$, since $e^x > 0$ and $e^{-x} > 0$ for all x.

The graph of $y = f(x)$ is always concave up.

(ii) For critical points, we solve $\frac{dy}{dx} = 0$:

$e^x - e^{-x} = 0$

$e^x = \frac{1}{e^x}$

$(e^x)^2 = 1$

$e^x = 1$, since $e^x > 0$

$x = 0.$

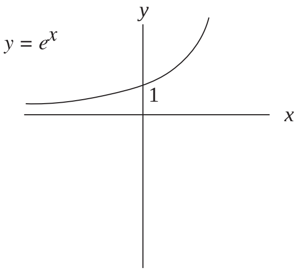
There are no x-intercepts ($e^x + e^{-x} > 0$ for all x).

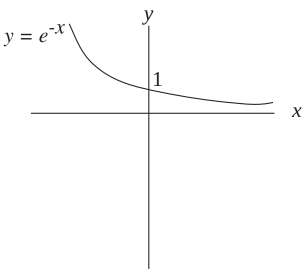
The y-intercept is $1 + 1 = 2$.

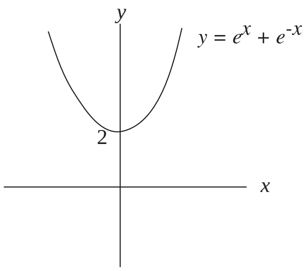
Interval	$x < 0$	$x = 0$	$x > 0$
$\frac{dy}{dx}$	< 0	$= 0$	> 0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing

$\lim_{x\to-\infty} (e^x + e^{-x}) = \infty$

$\lim_{x\to\infty} (e^x + e^{-x}) = \infty$







d. $g(w) = \frac{4w^2 - 3}{w^3}$

$= \frac{4}{w} - \frac{3}{w^3}, w \neq 0$

(i) $g'(w) = -\frac{4}{w^2} + \frac{9}{w^4}$

$= \frac{9 - 4w^2}{w^4}$

$g''(w) = \frac{8}{w^3} - \frac{36}{w^5}$

$= \frac{8w^2 - 36}{w^5}$

For possible points of inflection, we solve $g''(w) = 0$:

$8w^2 - 36 = 0$, since $w^5 \neq 0$

$w^2 = \frac{9}{2}$

$w = \pm \frac{3}{\sqrt{2}}.$

Interval	$w < -\frac{3}{\sqrt{2}}$	$w = -\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}} < w < 0$	$< w < 0 < \frac{3}{\sqrt{2}}$	$w = \frac{3}{\sqrt{2}}$	$w > \frac{3}{\sqrt{2}}$
$g'(w)$	< 0	$= 0$	> 0	< 0	0	> 0
Graph of $g(w)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

The points of inflection are $\left(-\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$

and $\left(\frac{3}{\sqrt{2}}, \frac{8\sqrt{2}}{9}\right)$.

(ii) There is no y-intercept.

The x-intercept is $\pm \frac{3}{\sqrt{2}}$.

For critical values, we solve $g'(w) = 0$:

$9 - 4w^2 = 0$ since $w^4 \neq 0$

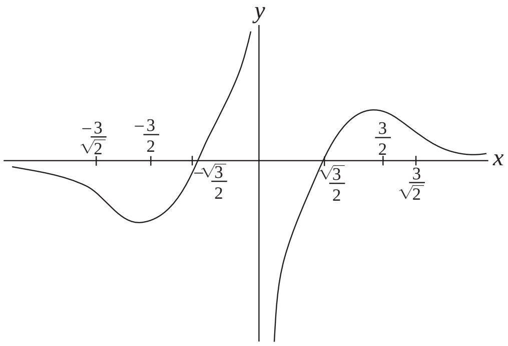
$$w = \pm \frac{3}{2}.$$

Interval	$w < -\frac{3}{2}$	$w = -\frac{3}{2}$	$-\frac{3}{2} < w < 0$	$0 < w < \frac{3}{2}$	$w = \frac{3}{2}$	$w > \frac{3}{2}$
$g'(w)$	< 0	$= 0$	> 0	> 0	$= 0$	< 0
Graph $g(wP)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing

$$\lim_{w \to 0^-} \frac{4w^2 - 3}{w^3} = \infty, \lim_{w \to 0^+} \frac{4w^2 - 3}{w^3} = -\infty$$

$$\lim_{w \to -\infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0, \lim_{w \to \infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0$$

Thus, $y = 0$ is a horizontal asymptote and $x = 0$ is a vertical asymptote.



9. The graph is increasing when $x < 2$ and when $2 < x < 5$.

The graph is decreasing when $x > 5$.

The graph has a local maximum at $x = 5$.

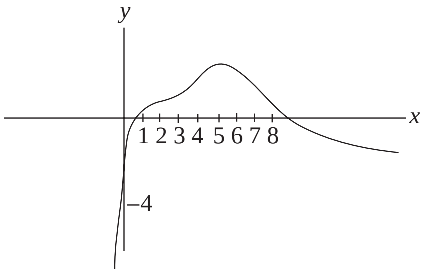
The graph has a horizontal tangent line at $x = 2$.

The graph is concave down when $x < 2$ and when $4 < x < 7$.

The graph is concave up when $2 < x < 4$ and when $x > 7$.

The graph has points of inflection at $x = 2$, $x = 4$, and $x = 7$.

The y-intercept of the graph is -4.



10. $f(x) = ax^3 + bx^2 + c$

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

Since (2, 11) is a relative extremum, $f(2) = 12a + 4b = 0$.

Since (1, 5) is an inflection point, $f''(1) = 6a + 2b = 0$.

Since the points are on the graph,

$a + b + c = 5$ and

$\underline{8a + 4b + c = 11}$

$7a + 3b = 6$

$9a + 3b = 0$

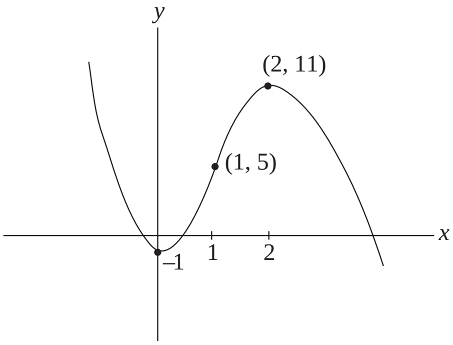
$2a = -6$

$a = -3$

$b = 9$

$c = -1.$

Thus, $f(x) = -3x^3 + 9x^2 - 1$.



11. $f(x) = (x+1)^{\frac{1}{2}} + bx^{-1}$

$$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}} - bx^{-2}$$

$$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}} + 2bx^{-3}$$

Since the graph of $y = f(x)$ has a point of inflection at $x = 3$:

$$-\frac{1}{4}(4)^{-\frac{3}{2}} + \frac{2b}{27} = 0$$

$$-\frac{1}{32} + \frac{2b}{27} = 0$$

$$b = \frac{27}{64}.$$

12. $f(x) = ax^4 + bx^3$

$$f'(x) = 4ax^3 + 3bx^2$$

$$f''(x) = 12ax^2 + 6bx$$

For possible points of inflection, we solve $f''(x) = 0$:

$$12ax^2 + 6bx = 0$$

$$6x(2ax + b) = 0$$

$$x = 0 \text{ or } x = -\frac{b}{2a}.$$

The graph of $y = f''(x)$ is a parabola with x-intercepts 0 and $-\frac{b}{2a}$.

We know the values of $f''(x)$ have opposite signs when passing through a root. Thus, at $x = 0$ and at $x = -\frac{b}{2a}$, the concavity changes as the graph goes through these points. Thus, $f(x)$ has points of inflection at $x = 0$ and $x = -\frac{b}{2a}$.

To find the x-intercepts, we solve $f(x) = 0$

$$x^3(ax + b) = 0$$

$$x = 0 \text{ or } x = -\frac{b}{a}.$$

The point midway between the x-intercepts has x-coordinate $-\frac{b}{2a}$.

The points of inflection are (0, 0) and $\left(-\frac{b}{2a}, -\frac{b^4}{16a^3}\right)$.

13. a. $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4} = x - 2 + \frac{8x - 8}{x^2 - 4}$ (by division of polynomials). The graph has discontinuities at $x = \pm 2$.

$$\left.\begin{aligned} \lim_{x\to -2^-}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= -\infty \\ \lim_{x\to -2^+}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= \infty \end{aligned}\right\} = -2 \text{ is a vertical asymptote.}$$

$$\left.\begin{aligned} \lim_{x\to 2^-}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= -\infty \\ \lim_{x\to 2^+}\left(x - 2 + \frac{8x-8}{x^2-4}\right) &= \infty \end{aligned}\right\} = 2 \text{ is a vertical asymptote.}$$

When $x = 0$, $y = 0$.

Also, $y = \frac{x(x^2 - 2x + 4)}{x^2 - 4} = \frac{x[(x-1)^2 + 3]}{x^2 - 4}$.

Since $(x-1)^2 + 3 > 0$, the only x-intercept is $x = 0$.

Since $\lim_{x\to\infty} \frac{8x - 8}{x^2 - 4} = 0$, the curve approaches the value $x - 2$ as $x \to \infty$. This suggests that the line $y = x - 2$ is an oblique asymptote. It is verified by the limit $\lim_{x\to\infty} [x - 2 - f(x)] = 0$. Similarly, the curve approaches $y = x - 2$ as $x \to -\infty$.

$$\frac{dy}{dx} = 1 + \frac{8(x^2 - 4) - 8(x-1)(2x)}{(x^2 - 4)^2}$$

$$= 1 - \frac{8(x^2 - 2x + 4)}{(x^2 - 4)^2}$$

We solve $\frac{dy}{dx} = 0$ to find critical values:

$$8x^2 - 16x + 32 = x^4 - 8x^2 + 16$$

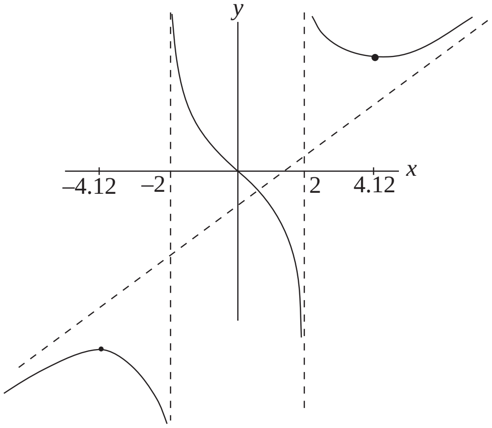
$$x^4 - 16x^2 - 16 = 0$$

$$x^2 = 8 + 4\sqrt{5} \quad (8 - 4\sqrt{5} \text{ is inadmissible})$$

$$x \doteq \pm 4.12.$$

Interval	$x <$ -4.12	$x =$ -4.12	-4.12 $< x < 2$	$-2< x$ < 2	$2 < x$ < 4.12	$x =$ 4.12	$x >$ 4.12
$\frac{dy}{dx}$	> 0	$= 0$	< 0	< 0	< 0	0	> 0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Decreasing Min	Local	Increasing

$\lim_{x\to\infty} y = \infty$ and $\lim_{x\to-\infty} y = -\infty$



Exercise 9.5

1. **a.** $y = x^3 - 9x^2 + 15x + 30$

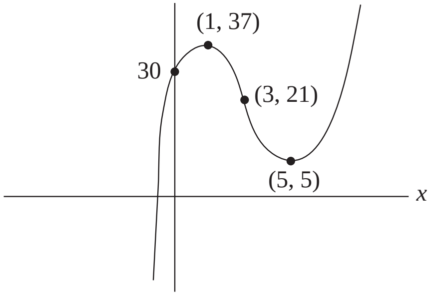
We know the general shape of a cubic polynomial with leading coefficient positive. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 3x^2 - 18x + 15$$

Set $\frac{dy}{dx} = 0$ to find the critical values:

$$3x^2 - 18x + 15 = 0$$
$$x^2 - 6x + 5 = 0$$
$$(x - 1)(x - 5) = 0$$
$$x = 1 \text{ or } x = 5.$$

The local extrema are (1, 37) and (5, 5).



b. $f(x) = 4x^3 + 18x^2 + 3$

The graph is that of a cubic polynomial with leading coefficient negative. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 12x^2 + 36x$$

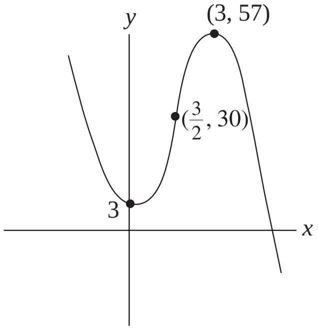
To find the critical values, we solve $\frac{dy}{dx} = 0$:

$$-12x(x - 3) = 0$$
$$x = 0 \text{ or } x = 3.$$

The local extrema are (0, 3) and (3, 57).

$$\frac{d^2y}{dx^2} = -24x + 36$$

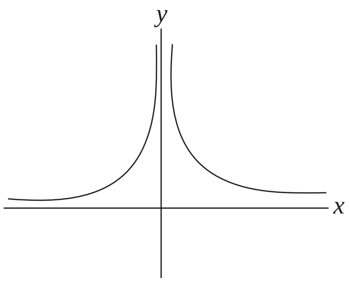
The point of inflection is $\left(\frac{3}{2}, 30\right)$.



c. $y = 3 + \frac{1}{(x + 2)^2}$

We observe that $y = 3 + \frac{1}{(x + 2)^2}$ is just a translation of $y = \frac{1}{x^2}$.

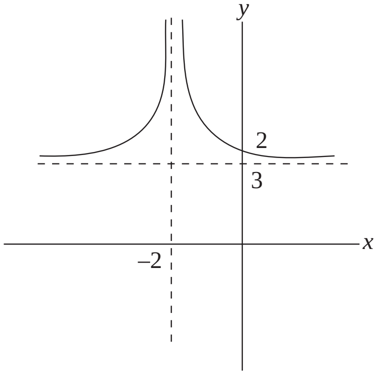
The graph of $y = \frac{1}{x^2}$ is



The reference point (0, 0) for $y = \frac{1}{x^2}$ becomes the point (–2, 3) for $y = 3 + \frac{1}{(x + 2)^2}$. The vertical asymptote is $x = -2$, and the horizontal asymptote is $y = 3$.

$\frac{dy}{dx} = -\frac{2}{(x+2)^3}$, hence there are no critical points.

$\frac{d^2y}{dx^2} = \frac{6}{(x+2)^4} > 0$, hence the graph is always concave up.



d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

We know the general shape of a fourth degree polynomial with leading coefficient positive. The local extrema will help refine the graph.

$f'(x) = 4x^3 - 12x^2 - 16x + 48$

For critical values, we solve $f'(x) = 0$

$$x^3 - 3x^2 - 4x + 12 = 0.$$

Since $f'(2) = 0$, $x - 2$ is a factor of $f'(x)$.

The equation factors are $(x-2)(x-3)(x+2) = 0$.

The critical values are $x = -2, 2, 3$.

$f''(x) = 12x^2 - 24x - 16$

Since $f''(-2) = 80 > 0$, $(-2, -80)$ is a local minimum.

Since $f''(2) = -16 < 0$, $(2, 48)$ is a local maximum.

Since $f''(3) = 20 > 0$, $(3, 45)$ is a local minimum.

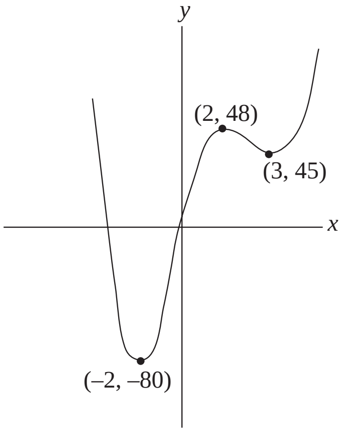
The graph has x-intercepts 0 and –3.2.

The points of inflection can be found by solving $f''(x) = 0$:

$3x^2 - 6x - 4 = 0$

$x = \frac{6 \pm \sqrt{84}}{6}$

$x \doteq -\frac{1}{2}$ or $\frac{5}{2}$.



e. $y = \frac{2x}{x^2 - 25}$

There are discontinuities at $x = -5$ and $x = 5$.

$$\lim_{x \to -5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty \quad \text{and} \quad \lim_{\to -5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$$

$$\lim_{x \to 5^-} \left(\frac{2x}{x^2 - 25}\right) = -\infty \quad \text{and} \quad \lim_{\to 5^+} \left(\frac{2x}{x^2 - 25}\right) = \infty$$

$x = -5$ and $x = 5$ are vertical asymptotes.

$$\frac{dy}{dx} = \frac{2(x^2 - 25) - 2x(2x)}{(x^2 - 25)^2} = -\frac{2x^2 + 50}{(x^2 - 25)^2} < 0 \text{ for}$$

all x in the domain. The graph is decreasing throughout the domain.

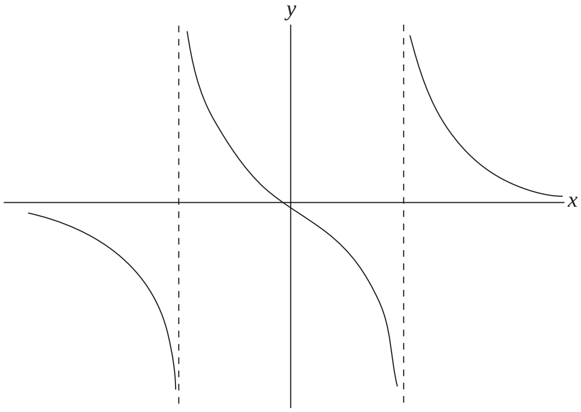
$$\left.\begin{aligned} \lim_{x \to \infty} \left(\frac{2x}{x^2 - 25}\right) &= \lim_{x \to \infty} \left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) \\ &= 0 \\ \lim_{x \to -\infty} \left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) &= 0 \end{aligned}\right\} y = 0 \text{ is a horizontal asymptote.}$$

$$\frac{d^2y}{dx^2} = -\frac{4x(x^2 - 25)^2 - (2x^2 + 50)(2)(x^2 - 25)(2x)}{(x^2 - 25)^4}$$

$$= \frac{4x^3 + 300x}{(x^2 - 25)^3} = \frac{4x(x^2 + 75)}{(x^2 - 25)^3}$$

There is a possible point of inflection at $x = 0$.

Interval	$x < -5$	$-5 < x < 0$	$x = 0$	$0 < x < 5$	$x > 5$
$\frac{d^2y}{dx^2}$	< 0	> 0	$= 0$	< 0	> 0
Graph of y	Concave Down	Concave Up	Point of Inflection	Concave Down	Concave Up



f. $y = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

The graph of $y = f(x)$ is always above the x-axis. The y-intercept is $\frac{1}{\sqrt{2\pi}} \doteq 0.4$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} (-x)$$

$\frac{dy}{dx} = 0$ when $x = 0$. Thus, (0, 0.4) is a critical point.

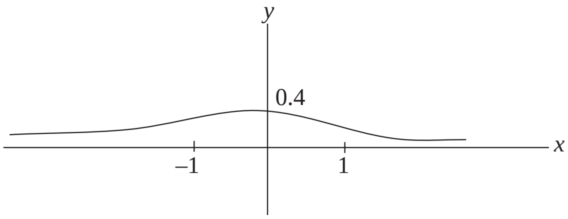
$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{x^2}{2}} (-x)(-x) + e^{-\frac{x^2}{2}} (-1) \right)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^2 - 1)$$

When $x = 0$, $\frac{d^2y}{dx^2} < 0$. Thus, (0, 0.4) is a local maximum. Possible points of inflection occur when $x^2 - 1 = 0$ or $x = -1$ and $x = 1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	> 1
$\frac{d^2y}{dx^2}$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$$\lim_{x \to \infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) = 0 \text{ and } \lim_{x \to \infty} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) = 0$$



g. $y = \frac{6x^2 - 2}{x^3}$

$= \frac{6}{x} - \frac{2}{x^3}$

There is a discontinuity at $x = 0$.

$$\lim_{x \to 0^-} \frac{6x^2 - 2}{x^3} = \infty \text{ and } \lim_{x \to 0^+} \frac{6x^2 - 2}{x^3} = -\infty$$

The y-axis is a vertical asymptote. There is no y-intercept. The x-intercept is $\pm \frac{1}{\sqrt{3}}$.

$$\frac{dy}{dx} = -\frac{6}{x^2} + \frac{6}{x^4} = \frac{-6x^2 + 6}{x^4}$$

$\frac{dy}{dx} = 0$ when $6x^2 = 6$

$x = \pm 1$

Interval	$x < -1$	$x = -1$	$-1 < x < 0$	$0 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	> 0	$= 0$	< 0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing

There is a local minimum at (–1, –4) and a local maximum at (1, 4).

$$\frac{d^2y}{dx^2} = \frac{12}{x^3} - \frac{24}{x^5} = \frac{12x^2 - 24}{x^5}$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$ ($x^5 \neq 0$):

$12x^2 = 24$

$x = \pm \sqrt{2}$.

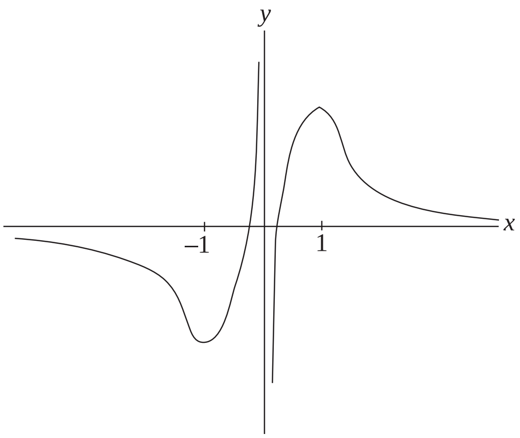
Interval	$x < -\sqrt{2}$	$x = -\sqrt{2}$	$-\sqrt{2} < x < 0$	$0 < x < \sqrt{2}$	$x = \sqrt{2}$	$x > \sqrt{2}$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0	< 0	$= 0$	> 0
Graph of $y = f(x)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

There are points of inflection at $\left(-\sqrt{2}, -\frac{5}{\sqrt{2}}\right)$ and $\left(\sqrt{2}, \frac{5}{\sqrt{2}}\right)$.

$$\lim_{x\to\infty} \frac{6x^2 - 2}{x^3} = \lim_{x\to\infty} \frac{\frac{6}{x} - \frac{2}{x^3}}{1} = 0$$

$$\lim_{x = -\infty} \frac{\frac{6}{x} - \frac{6}{x^3}}{x} = 0$$

The x-axis is a horizontal asymptote.



h. $s = \frac{50}{1 + 5e^{-0.01t}}, t \geq 0$

When $t = 0$, $s = \frac{50}{6}$.

$$\frac{ds}{dt} = 50(-1)(1 + 5e^{-0.01t})^{-2}\,(5e^{-0.01t})(-0.01)$$

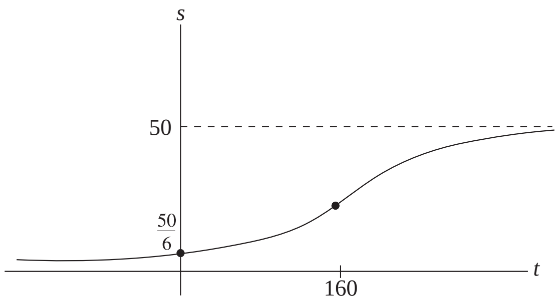
$$= \frac{2.5e^{-0.01t}}{(1 + 5e^{-0.01t})^2}$$

Since $\frac{ds}{dt} > 0$ for all t, s is always increasing.

$$\lim_{t\to\infty} \left(\frac{50}{1 + 5e^{-0.01t}}\right) = 50$$

$$\lim_{t\to-\infty} \left(\frac{50}{1 + 5e^{-0.01t}}\right) = 0$$

Thus, $s = 50$ is a horizontal asymptote for large values of t, and $s = 0$ is a horizontal asymptote for large negative values of t. It can be shown that there is a point of inflection at $t \doteq 160$.



i. $y = \frac{x + 3}{x^2 - 4}$

There are discontinuities at $x = -2$ and at $x = 2$.

$$\lim_{x\to-2^-} \left(\frac{x + 3}{x^2 - 4}\right) = \infty \text{ and } \lim_{x\to-2^+} \left(\frac{x + 3}{x^2 - 4}\right) = -\infty$$

$$\lim_{x\to2^-} \left(\frac{x + 3}{x^2 - 4}\right) = -\infty \text{ and } \lim_{x\to2^-} \left(\frac{x + 3}{x^2 - 4}\right) = \infty$$

There are vertical asymptotes at $x = -2$ and $x = 2$.

When $x = 0$, $y = -\frac{3}{4}$. The x-intercept is –3.

$$\frac{dy}{dx} = \frac{(1)(x^2 - 4) - (x + 3)(2x)}{(x^2 - 4)^2}$$

$$= \frac{-x^2 - 6x - 4}{(x^2 - 4)^2}$$

For critical values, we solve $\frac{dy}{dx} = 0$:

$x^2 + 6x + 4 = 0$

$$x = -\frac{6 \pm \sqrt{36 - 16}}{2}$$

$$= -3 \pm \sqrt{5}$$

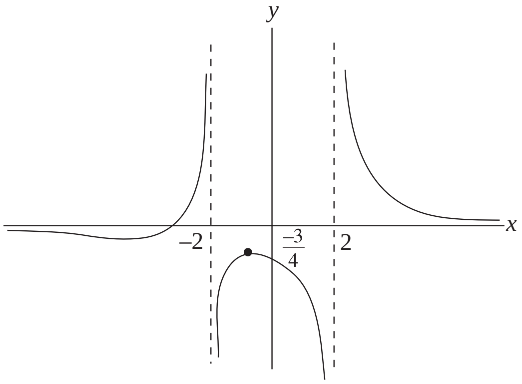
$$\doteq -5.2 \text{ or } -0.8.$$

Interval	$x < -5.2$	$x = -5.2$	$-5.2 < x < -2$	$-2 < x < -0.8$	$x = -0.8$	$-0.8 < x < 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	> 0	$= 0$	< 0	< 0
Graph of y	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

$$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(\frac{\frac{1}{x}+\frac{3}{x^2}}{1-\frac{4}{x^2}}\right) = 0$$

$$\lim_{x\to-\infty}\left(\frac{\frac{1}{x}+\frac{3}{x^2}}{1-\frac{4}{x^2}}\right) = 0$$

The x-axis is a horizontal asymptote.



j. $y = \dfrac{x^2 - 3x + 6}{x - 1}$

$= x - 2 + \dfrac{4}{x-1}$

$$\begin{array}{r|l} & \;\; x - 2 \\ \hline x - 1 & x^2 - 3x + 6 \\ & \underline{x^2 - x} \\ & \quad -2x + 6 \\ & \quad \underline{-2x + 2} \\ & \qquad\quad 4 \end{array}$$

There is a discontinuity at $x = 1$.

$$\lim_{x\to1^-}\left(\frac{x^2-3x+6}{x-1}\right) = -\infty$$

$$\lim_{x\to1^+}\left(\frac{x^2-3x+6}{x-1}\right) = \infty$$

Thus, $x = 1$ is a vertical asymptote.
The y-intercept is –6.
There are no x-intercepts ($x^2 - 3x + 6 > 0$ for all x in the domain).

$$\frac{dy}{dx} = 1 - \frac{4}{(x-1)^2}$$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$$\begin{aligned} 1 - \frac{4}{(x-1)^2} &= 0 \\ (x-1)^2 &= 4 \\ x - 1 &= \pm 2 \\ x &= -1 \text{ or } x = 3. \end{aligned}$$

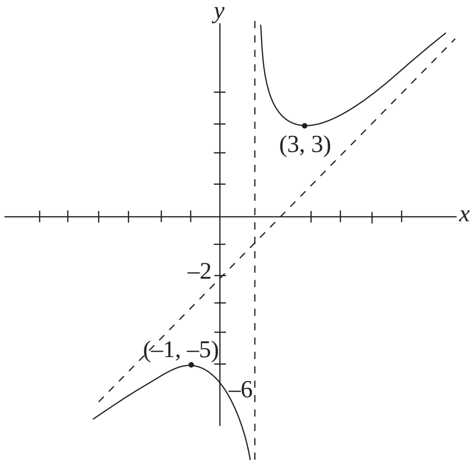
Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$1 < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	> 0	$= 0$	< 0	< 0	$= 0$	> 0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

$$\frac{d^2y}{dx^2} = \frac{8}{(x-1)^3}$$

For $x < 1$, $\dfrac{d^2y}{dx^2} < 0$ and y is always concave down.

For $x > 1$, $\dfrac{d^2y}{dx^2} > 0$ and y is always concave up.

The line $y = x - 2$ is an oblique asymptote.



k. $c = te^{-t} + 5$
When $t = 0$, $c = 5$.

$$\frac{dc}{dt} = e^{-t} - te^{-t} = e^{-t}(1 - t)$$

Since $e^{-t} - te^{-t} = e^{-t}(1 - t)$
Since $e^{-t} > 0$, the only value for which

$\dfrac{dc}{dt} = 0$ is $t = 1$.

Interval	$t < 1$	$t = 1$	$t > 1$
$\frac{dc}{dt}$	> 0	$= 0$	< 0
Graph of c	Increasing	Local Max	Decreasing

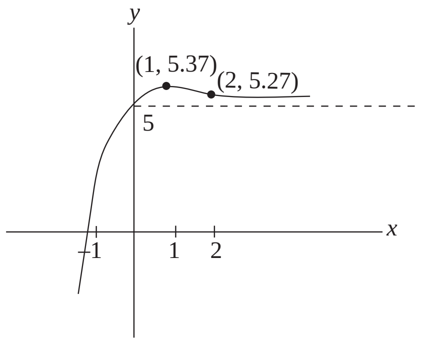
$$\lim_{x\to\infty} (te^{-t} + t) = 5$$

$$\lim_{x\to-\infty} (te^{-t} + t) = -\infty$$

$$\frac{d^2c}{dt^2} = -e^{-t} - e^{-t} + te^{-t} = e^{-t}(t - 2)$$

$$\frac{d^2c}{dt^2} = 0 \text{ when } t = 2$$

Interval	$t < 2$	$t = 2$	$t > 0$
$\frac{d^2c}{dt^2}$	< 0	$= 0$	> 0
Graph of c	Concave Down	Point of Inflection	Concave Up



1. $y = x(\ln x)^3, x > 0$

$$\frac{dy}{dx} = (\ln x)^3 + x(3)(\ln x)^2\left(\frac{1}{x}\right) = (\ln x)^2(\ln x + 3)$$

$$\frac{dy}{dx} = 0 \text{ when } \ln x = 0 \text{ or } \ln x = -3$$

$$x = 1 \text{ or } x = e^{-3} \doteq 0.05$$

Interval	$0 < x < 0.05$	$x = 0.05$	$0.05 < x < 1$	$x = 1$	$x > 1$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	0	> 0
Graph of y	Decreasing	Local Min	Increasing	Stationary Point	Increasing

There is no y-intercept. The x-intercept is 1.

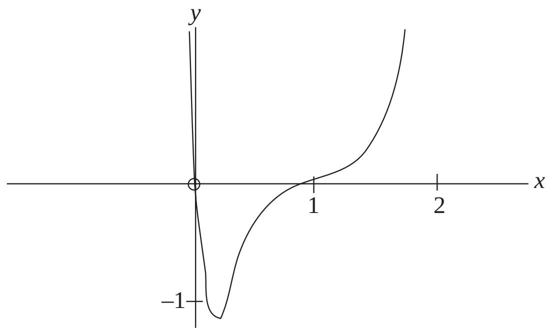
$$\frac{d^2y}{dx^2} = 3(\ln x)^2\left(\frac{1}{x}\right) + 6(\ln x)\left(\frac{1}{x}\right) = 3\,\frac{\ln x}{x}(\ln x + 2)$$

$$\frac{d^2y}{dx^2} = 0 \text{ when } \ln x = 0 \text{ or } \ln x = -2$$

$$x = 1 \quad \text{or} \quad x = e^{-2} \doteq 0.14$$

Interval	$0 < x < 0.14$	$x = 0.14$	$0.14 < x < 1$	$x = 1$	$x > 1$
$\frac{d^2y}{dx^2}$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$$\lim_{x\to\infty} [x(\ln x)^3] = \infty$$



2. $y = ax^3 + bx^2 + cx + d$

Since (0, 0) is on the curve $d = 0$:

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

At $x = 2$, $\frac{dy}{dx} = 0$.

Thus, $12a + 4b + c = 0$.

Since (2, 4) is on the curve, $8a + 4b + 2c = 4$ or $4a + 2b + c = 2$.

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

Since (0, 0) is a point of inflection, $\frac{d^2y}{dx^2} = 0$ when $x = 0$.

Thus, $2b = 0$

$b = 0.$

Solving for a and c:

$$\begin{aligned} 12a + c &= 0 \\ 4a + c &= 2 \\ 8a &= -2 \\ a &= -\frac{1}{4} \\ c &= 3. \end{aligned}$$

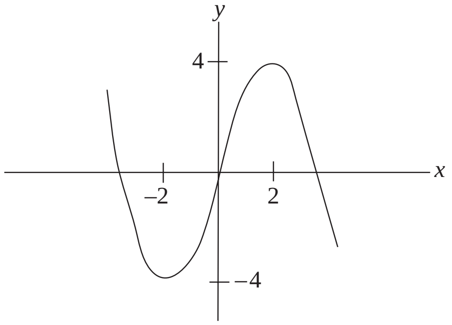
The cubic polynomial is $y = -\frac{1}{4}x^3 + 3x$.

The y-intercept is 0. The x-intercepts are found by setting $y = 0$:

$$-\frac{1}{4}x(x^2 - 12) = 0$$

$x = 0$, or $x = \pm 2\sqrt{3}$.

Let $y = f(x)$. Since $f(-x) = \frac{1}{4}x^3 - 3x = -f(x)$, $f(x)$ is an odd function. The graph of $y = f(x)$ is symmetric when reflected in the origin.



3. $g(x) = \frac{8e^x}{e^{2x}} + 4$

There are no discontinuities. The graph is always above the x-axis. The y-intercept is $\frac{8}{5}$.

$$g'(x) = \frac{8e^x(e^{2x} + 4) - 8e^x(e^{2x})2)}{(e^{2x} + 4)^2}$$

$$= \frac{8e^x(4 - e^{2x})}{(e^{2x} + 4)^2}$$

The only critical values occur when $4 - e^{2x} = 0$

$$\begin{aligned} e^{2x} &= 4 \\ 2x &= \ln 4 \\ x &= \ln\frac{4}{2} \\ &= \ln 2. \end{aligned}$$

For $x < \ln 2$, $g'(x) > 0$

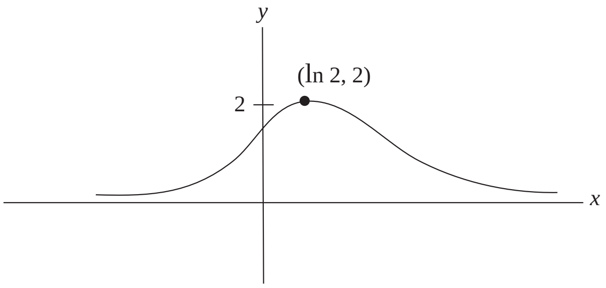
For $x > \ln 2$, $g'(x) < 0$

Thus, (ln 2, 2) is a local maximum point.

$$\left.\begin{aligned} &\lim_{x\to\infty}\left(\frac{8e^x}{e^{2x} + 4}\right) = \lim_{x\to\infty}\left(\frac{8}{e^x + \frac{4}{e^x}}\right) = 0 \\ &\lim_{x\to-\infty}\left(\frac{8e^x}{e^{2x} + 4}\right) = \frac{0}{0 + 4} = 0 \end{aligned}\right\}$$

Hence, the x-axis is a horizontal asymptote.

It is very cumbersome to evaluate $g''(x)$. Since there is a horizontal tangent line at the local maximum (ln 2, 2) and the x-axis is a horizontal asymptote, it is reasonable to conclude that there are two points of inflection. (It can be shown to be true.)



4. $y = e^x + \frac{1}{x}$

There is a discontinuity at $x = 0$.

$\lim_{x\to0^-}\left(e^x + \frac{1}{x}\right) = -\infty$ and $\lim_{x\to0^+}\left(\frac{e^x + 1}{x}\right) = \infty$

Thus, the y-axis is a vertical asymptote.

$$\frac{dy}{dx} = e^x - \frac{1}{x^2}$$

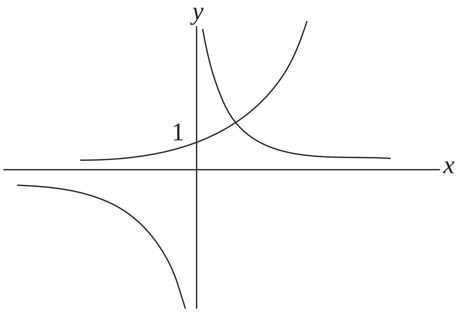
To find the critical values, we solve $\frac{dy}{dx} = 0$:

$$e^x - \frac{1}{x^2} = 0$$

This equation does not have a simple analytic solution. Solving $\frac{d^2y}{dx^2} = 0$ is even more cumbersome.

We use a different approach to sketch $y = e^x + \frac{1}{x}$.

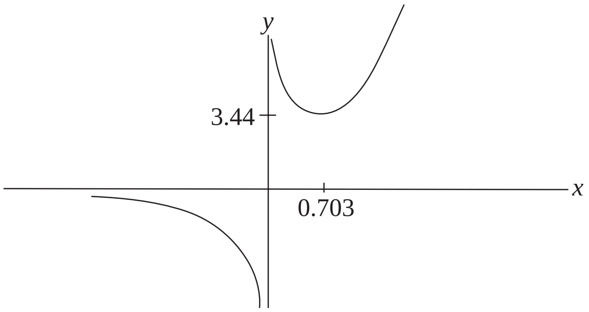
We use the method of adding functions. The given function is the sum of $y_1 = e^x$ and $y_2 = \frac{1}{x}$.



For $x > 0$, the sum of the two functions is always positive. The resulting graph will be in the first quadrant. The graph of $y_2 = \frac{1}{x}$ dominates for values near 0, and the graph of $y_1 = e^x$ dominates for large values of x. It appears that this branch of the graph will have a relative minimum value. (A calculator solution of $\frac{dy}{dx} = 0$ verifies a relative minimum at $x \doteq 0.703$.)

For $x < 0$, the graph of $y_2 = \frac{1}{x}$ dominates the sum.

There are no points of inflection.



5. $f(x) = \frac{k - x}{k^2 + x^2}$

There are no discontinuities.

The y-intercept is $\frac{1}{k}$ and the x-intercept is k.

$$f'(x) = \frac{(-1)(k^2 + x^2) - (k - x)(2x)}{(k^2 + x^2)^2}$$

$$= \frac{x^2 - 2kx - k^2}{(k^2 + x^2)^2}$$

For critical points, we solve $f'(x) = 0$:

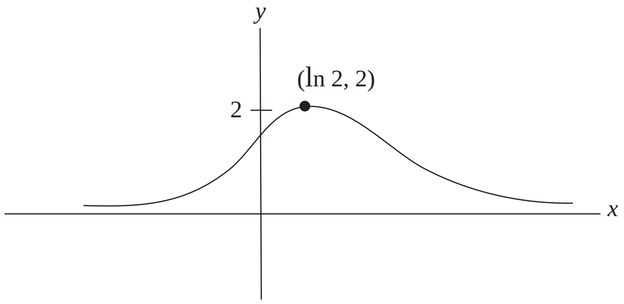
$$x^2 - 2kx - k^2 = 0$$
$$x^2 - 2kx + k^2 = 2k^2$$
$$(x - k)^2 = 2k^2$$
$$x - k = \pm\sqrt{2}\,k$$
$$x = (1 + \sqrt{2})k \text{ or } x = (1 - \sqrt{2})k.$$

Interval	$x <$ $-0.41k$	$x \doteq$ $0.41k$	$-0.41k <$ $x < 2.41k$	$x \doteq$ $2.41k$	$x >$ $2.41k$
$f'(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing

$$\lim_{x\to\infty}\left(\frac{k - x}{k^2 + x^2}\right) = \lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

$$\lim_{x\to-\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

Hence, the x-axis is a horizontal asymptote.



6. $g(x) = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$

There are no discontinuities.

$$g'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x+3)^{\frac{2}{3}} + x^{\frac{1}{3}}\left(\frac{2}{3}\right)(x+3)^{-\frac{1}{3}}(1)$$

$$= \frac{x+3+2x}{3x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}} = \frac{3(x+1)}{3x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}}$$

$$= \frac{x+1}{x^{\frac{2}{3}}(x+3)^{\frac{1}{3}}}$$

$g'(x) = 0$ when $x = -1$.

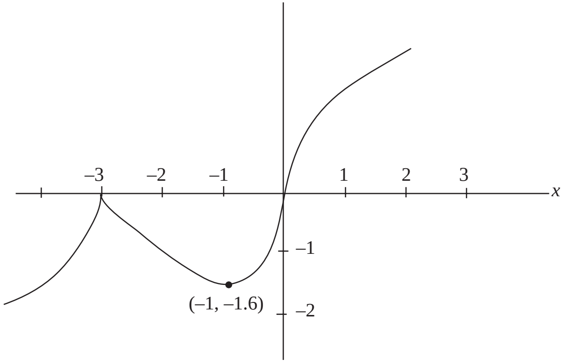
$g'(x)$ doesn't exist when $x = 0$ or $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x > 0$
$g'(x)$	> 0	Does Not Exist	< 0	$= 0$	> 0	Does Not Exist	> 0
Graph of $g(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing		Increasing

There is a local maximum at (–3, 0) and a local minimum at (–1, –1.6). The second derivative is algebraically complicated to find. It can be verified that

$$g''(x) = \frac{-2}{x^{\frac{5}{3}}(x+3)^{\frac{2}{3}}}.$$

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$g''(x)$	> 0	Does Not Exist	> 0	Does Not Exist	< 0
Graph $g(x)$	Concave Up	Cusp	Concave Up	Point of Inflection	Concave Down



7. a. $$f(x) = \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x^2}}}, \text{ since } x > 0$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}}$$

$$= 1$$

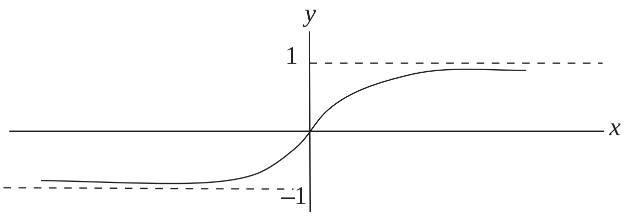
$y = 1$ is a horizontal asymptote to the right hand branch of the graph.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{-x\sqrt{1+\frac{1}{x^2}}}, \text{ since } |x| = -x \text{ for } x < 0$$

$$= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}}$$

$$= -1$$

$y = -1$ is a horizontal asymptote to the left hand branch of the graph.



b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

$$= \frac{\left(\sqrt{t^2 + 4t} - \sqrt{t^2 + t}\right)\left(\sqrt{t^2 + 4t} + \sqrt{t^2 + t}\right)}{}$$

$$= \frac{3t}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$$

$$= \frac{3t}{|t|\sqrt{1 + \frac{4}{t}} + |t|\sqrt{1 + \frac{1}{t}}}$$

$$\lim_{t \to \infty} g(t) = \frac{3}{1 + 1} = \frac{3}{2}, \text{ since } |t| = t \text{ for } t > 0$$

$$\lim_{t \to -\infty} g(t) = \frac{3}{-1 - 1} = -\frac{3}{2}, \text{ since } |t| = -t \text{ for } t < 0$$

$y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes.

8. $y = ax^3 + bx^2 + cx + d$

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

$$\frac{d^2y}{dx^2} = 6ax + 2b = 6a\left(x + \frac{b}{3a}\right)$$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$:

$x = -\frac{b}{3a}$.

The sign of $\frac{d^2y}{dx^2}$ changes as x goes from values less than $\frac{-b}{3a}$ to values greater than $\frac{-b}{3a}$. Thus, there is a point of inflection at $x = \frac{-b}{3a}$.

At $x = \frac{b}{3a}$, $\frac{dy}{dx} = 3a\left(\frac{-b}{3a}\right)^2 + 2b\left(\frac{-b}{3a}\right) + c = c - \frac{b^2}{3a}$.

Review Exercise

1. **a.**
$$y = e^{nx}$$
$$\frac{dy}{dx} = ne^{nx}$$
$$\frac{d^2y}{dx^2} = n^2e^{nx}$$

b. $f(x) = \ln(x + 4)^{\frac{1}{2}}$

$$= \frac{1}{2}\ln(x + 4)$$

$$f'(x) = \frac{1}{2} \bullet \frac{1}{x + 4} = \frac{1}{2(x + 4)}$$

$$f''(x) = -\frac{1}{2} \bullet \frac{1}{(x + 4)^2} = -\frac{1}{2(x + 4)^2}$$

c. $s = \frac{e^t - 1}{e^t + 1}$

$$\frac{ds}{dt} = \frac{e^t(e^t + 1) - (e^t - 1)(e^t)}{(e^t + 1)^2}$$

$$= \frac{2e^t}{(e^t + 1)^2}$$

$$\frac{d^2s}{dt^2} = \frac{2e^t(e^t + 1)^2 - 2e^t(2)(e^t + 1)(e^t)}{(e^t + 1)^4}$$

$$= \frac{2e^{2t} + 2e^t - 4e^{2t}}{(e^t + 1)^3}$$

$$= \frac{2e^t(1 - e^t)}{(e^t + 1)^3}$$

d. $g(t) = \ln\left(t + \sqrt{1 + t^2}\right)$

$$g'(t) = \frac{1}{t + \sqrt{1 + t^2}} \bullet \left(1 + \frac{1}{2}\left(1 + t^2\right)^{-\frac{1}{2}}(2t)\right)$$

$$= \frac{1 + \frac{t}{\sqrt{1 + t^2}}}{t + \sqrt{1 + t^2}}$$

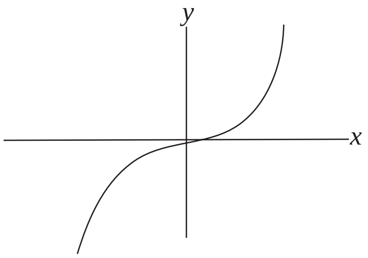
$$= \frac{\frac{\sqrt{1 + t^2} + t}{\sqrt{1 + t^2}}}{t + \sqrt{1 + t^2}}$$

$$= \frac{1}{\sqrt{1 + t^2}}$$

$$g''(t) = -\frac{1}{2}(1 + t^2)^{-\frac{3}{2}}(2t)$$

$$= \frac{-t}{(1 + t^2)^{\frac{3}{2}}}$$

3. No. A counter example is sufficient to justify the conclusion. The function $f(x) = x^3$ is always increasing yet the graph is concave down for $x < 0$ and concave up for $x > 0$.



4. **a.** $f(x) = -2x^3 + 9x^2 + 20$

$f'(x) = -6x^2 + 18x$

For critical values, we solve:

$f'(x) = 0$

$-6x(x - 3) = 0$

$x = 0$ or $x = 3$.

$f''(x) = -12x + 18$

Since $f''(0) = 18 > 0$, (0, 20) is a local minimum point. The tangent to the graph of $f(x)$ is horizontal at (0, 20). Since $f''(3) = -18 < 0$, (3, 47) is a local maximum point. The tangent to the graph of $f(x)$ is horizontal at (3, 47).

b. $g(t) \; \frac{e^{-2t}}{t^2}$

$g(t) = e^{-2t}t^{-2}, t \neq 0$

$g'(t) = -2e^{-2t}t^{-2} + e^{-2t}(-2t^{-3})$

$$= -\frac{2e^{-2t}(t + 1)}{t^3}$$

Since $e^{-2t} > 0$ for all t, and $g(t)$ has a discontinuity at $t = 0$, the only critical value is $t = -1$.

Interval	$t < -1$	$t = -1$	$-1 < t < 0$	$t > 0$
$g'(t)$	< 0	$= 0$	> 0	< 0
Graph of $g(t)$	Decreasing	Local Min	Increasing	Decreasing

There is a local minimum at $(-1, e^2)$. The tangent line at $(-1, e^2)$ is parallel to the x-axis.

c. $h(x) = \frac{x - 3}{x^2 + 7}$

$$h'(x) = \frac{(1)(x^2 + 7) - (x - 3)(2x)}{(x^2 + 7)^2}$$

$$= \frac{7 + 6x - x^2}{(x^2 + 7)^2}$$

$$= \frac{(7 - x)(1 + x)}{(x^2 + 7)^2}$$

Since $x^2 + 7 > 0$ for all x, the only critical values occur when $h'(x) = 0$. The critical values are $x = 7$ and $x = -1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 7$	$x = 7$	$x > 7$
$h'(x)$	< 0	$= 0$	> 0	$= 0$	< 0
Graph of $h(t)$	Decreasing	Local Min	Increasing	Local Max	Decreasing

There is a local minimum at $\left(-1, -\frac{1}{2}\right)$ and a local maximum at $\left(7, \frac{1}{14}\right)$. At both points, the tangents are parallel to the x-axis.

d. $k(x) = \ln (x^3 - 3x^2 - 9x)$

The domain of $k(x)$ is the set of all x such that $x^3 - 3x^2 - 9x > 0$.

Let $g(x) = x^2 - 3x^2 - 9x$.

The x-intercepts of the graph of $g(x)$ are found by solving $g(x) = 0$:

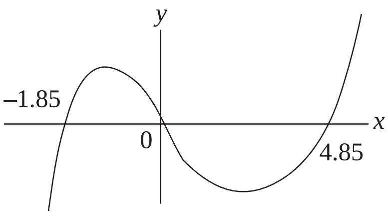
$x(x^2 - 3x - 9) = 0$

$x = 0$ or $x = \frac{3 \pm \sqrt{9 + 36}}{2}$

$= \frac{3 \pm 3\sqrt{5}}{2}$

$= 4.85$ or -1.85.

The graph of $y = g(x)$ is



Thus, the domain of $k(x)$ is $-1.85 < x < 0$ or $x > 4.85$.

$$k'(x) = \frac{3x^2 - 6x - 9}{x^3 - 3x^2 - 9x}.$$

Since the denominator $x^3 - 3x^2 - 9x > 0$, the only critical values of $k(x)$ result from

$$\begin{aligned} 3x^2 - 6x - 9 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0 \end{aligned}$$

$x = -1$ or $x = 3$ (this value is not in the domain).

Interval	$-1.85 < x < -1$	$x = -1$	$-1 < x < 0$	$x > 4.85$
$k'(x)$	> 0	$= 0$	< 0	> 0
Graph of $k(x)$	Increasing	Local Max	Decreasing	Increasing

Thus, $(-1, \ln 5)$ is a local maximum. The tangent line is parallel to the x-axis at $(-1, \ln 5)$.

6. **a.** $y = \frac{2x}{x - 3}$

There is a discontinuity at $x = 3$.

$$\lim_{x\to 3^-}\left(\frac{2x}{x-3}\right) = -\infty \text{ and } \lim_{x\to 3^+}\left(\frac{2x}{x-3}\right) = \infty$$

Therefore, $x = 3$ is a vertical asymptote.

b. $g(x) = \frac{x - 5}{x + 5}$

There is a discontinuity at $x = -5$.

$$\lim_{x\to -5^-}\left(\frac{x-5}{x+5}\right) = \infty \text{ and } \lim_{x\to -5^+}\left(\frac{x-5}{x+5}\right) = -\infty$$

Therefore, $x = -5$ is a vertical asymptote.

c. $s = \frac{s}{2e^x - 8}$

There is a discontinuity when $2e^x - 8 = 0$ or $x = \ln 4$.

$$\lim_{x\to \ln 4^-}\left(\frac{5}{2e^x - 8}\right) = -\infty \text{ and } \lim_{x\to \ln 4^+}\left(\frac{5}{2e^x - 8}\right) = \infty$$

Therefore, $x = \ln 4$ is a vertical asymptote.

d. $$\begin{aligned} f(x) &= \frac{x^2 - 2x - 15}{x + 3} \\ &= \frac{(x+3)(x-5)}{x+3} \\ &= x - 5,\ x \neq -3 \end{aligned}$$

There is a discontinuity at $x = -3$.

$$\lim_{x\to -3^+} f(x) = -8 \text{ and } \lim_{x\to -3^-} f(x) = -8$$

There is a hole in the graph of $y = f(x)$ at $(-3, -8)$.

7. **a.** $$\begin{aligned} f(w) &= \frac{\ln w^2}{w} \\ &= (2\ln|w|)(w^{-1}) \end{aligned}$$

$$\begin{aligned} f'(w) &= \left(\frac{2}{w}\right)(w^{-1}) + (2\ln|w|)(-w^{-2}) \\ &= 2w^{-2} - 2w^{-2}\ln|w| \end{aligned}$$

$$\begin{aligned} f''(w) &= -4w^{-3} + 4w^{-3}\ln|w| - 2w^{-2}\left(\frac{1}{w}\right) \\ &= -6w^{-3} + 4w^{-3}\ln|w| \\ &= \frac{4\ln|w| - 6}{w^3} \end{aligned}$$

For possible points of inflection, we solve $f''(w) = 0$.

Note: $w^3 \neq 0$.

$4\ln|w| = 6$

$w = \pm e^{\frac{3}{2}}$

Interval	$w < -e^{\frac{3}{2}}$	$w = -e^{\frac{3}{2}}$	$-e^{\frac{3}{2}} < w < 0$	$0 < w < e^{\frac{3}{2}}$	$w = e^{\frac{3}{2}}$	$w > e^{\frac{3}{2}}$
$f''(w)$	< 0	$= 0$	> 0	< 0	$= 0$	> 0
Graph $f(w)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

The points of inflection are $\left(-e^{\frac{3}{2}}, -\frac{3}{e^{\frac{3}{2}}}\right)$ and $\left(e^{\frac{3}{2}}, \frac{3}{e^{\frac{3}{2}}}\right)$.

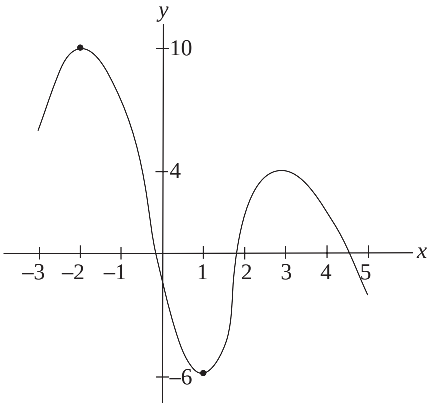
b. $$\begin{aligned} g(t) &= te^t \\ g'(t) &= e^t + te^t \\ g''(t) &= e^t + e^t + te^t = e^t(t + 2) \end{aligned}$$

Since $e^t > 0$, $g''(t) = 0$ when $t = -2$.

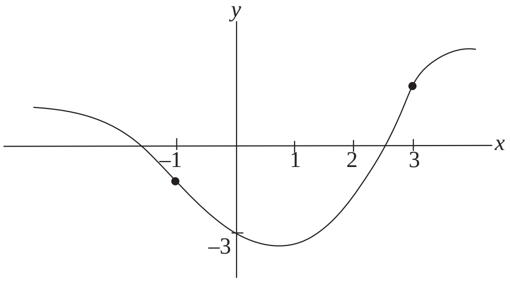
Interval	$t < -2$	$t = -2$	$t > -2$
$g''(t)$	< 0	$= 0$	> 0
Graph of $g(t)$	Concave Down	Point of Inflection	Concave Up

There is a point of inflection at $\left(-2, -\frac{2}{e^2}\right)$

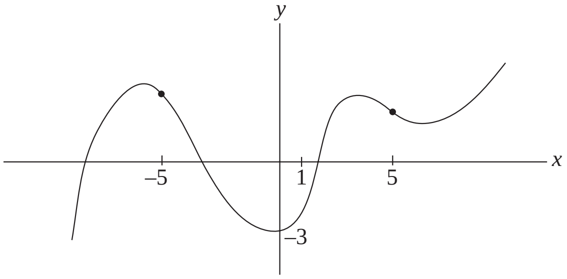
8.



9. **c.** **(i)**



(ii)



10. **a.** $g(x) = \dfrac{ax + b}{(x-1)(x-4)}$

$= \dfrac{ax + b}{x^2 - 5x + 4}$

$$g'(x) = \frac{a(x^2 - 5x + 4) - (ax + b)(2x - 5)}{(x^2 - 5x + 4)^2}$$

Since the tangent at (2, –1) has slope 0, $g'(2) = 0$.

Hence, $\dfrac{-2a + 2a + b}{4} = 0$ and $b = 0$.

Since (2, –1) is on the graph of $g(x)$:

$$-1 = \frac{2a + b}{-2}$$

$2a + 0 = 2$

$a = 1.$

Therefore $g(x) = \dfrac{x}{(x-1)(x-4)}$.

b. There are discontinuities at $x = 1$ and at $x = 4$.

$\lim_{x \to 1^-} g(x) = \infty$ and $\lim_{x \to 1^+} g(x) = -\infty$

$\lim_{x \to 4^-} g(x) = -\infty$ and $\lim_{x \to 4^+} g(x) = \infty$

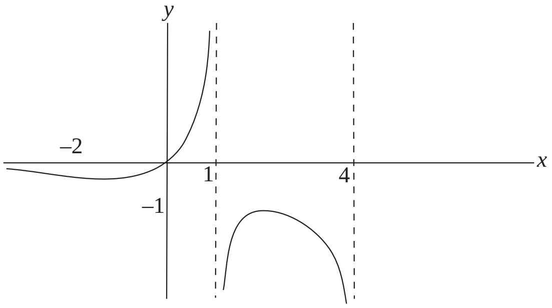
$x = 1$ and $x = 4$ are vertical asymptotes.
The y-intercept is 0.

$$g'(x) = \frac{4 - x^2}{(x^2 - 5x + 4)^2}$$

$g'(x) = 0$ when $x = \pm 2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 1$	$1 < x < 2$	$x = 2$	$2 < x < 4$	$x > 4$
$g'(x)$	< 0	0	> 0	> 0	0	< 0	< 0
Graph of $g(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local minimum at $\left(-2, -\dfrac{1}{9}\right)$ and a local maximum at (2, –1).



11. a. $f(x) = \dfrac{2x^2 - 7x + 5}{2x - 1}$

$f(x) = x - 3 + \dfrac{2}{2x - 1}$

The equation of the oblique asymptote is $y = x - 3$.

$$\begin{array}{r l} & x - 3 \\ 2x - 1 & \overline{\big)\,2x^2 - 7x + 5} \\ & \underline{2x^2 - x} \\ & \quad -6x + 5 \\ & \quad \underline{-6x + 3} \\ & \qquad\quad 2 \end{array}$$

$$\lim_{x\to\infty} [y - f(x)] = \lim_{x\to\infty} \left[x - 3 - \left(x - 3 + \frac{2}{2x - 1}\right)\right]$$

$$= \lim_{x\to\infty} \left[-\frac{2}{2x - 1}\right] = 0$$

b. $f(x) = \dfrac{4x^3 - x^2 - 15x - 50}{x^2 - 3x}$

$f(x) = 4x + 11 + \dfrac{18x - 50}{x^2 - 3x}$

$$\begin{array}{r l} & 4x + 11 \\ x^2 - 3x & \overline{\big)\,4x^3 - x^2 - 15x - 50} \\ & \underline{4x^3 - 12x^2} \\ & \qquad 11x^2 - 15x \\ & \qquad \underline{11x^2 - 33x} \\ & \qquad\qquad 18x - 50 \end{array}$$

$\lim_{x\to\infty} [y - f(x)]$

$$= \lim_{x\to\infty} \left[4x + 11 - \left(4x + 11 + \frac{18x - 50}{x^2 - 3x}\right)\right]$$

$$= \lim_{x\to\infty} \left[\frac{\frac{18}{x} - \frac{50}{x^2}}{1 - \frac{3}{x}}\right]$$

$= 0$

12. a. $y = x^4 - 8x^2 + 7$

This is a fourth degree polynomial and is continuous for all x. The y-intercept is 7.

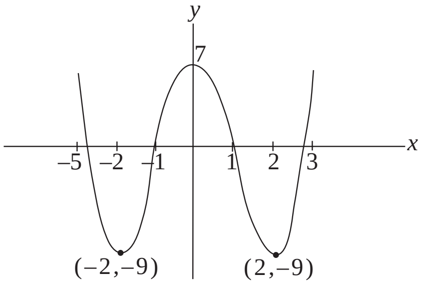
$$\frac{dy}{dx} = 4x^3 - 16x$$

$$= 4x(x - 2)(x + 2)$$

The critical values are $x = 0, -2$, and 2.

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

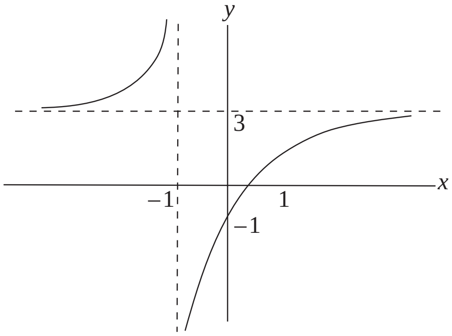
There are local minima at $(-2, -9)$ and at $(2, -9)$, and a local maximum at $(0, 7)$.



b. $f(x) = \dfrac{3x - 1}{x + 1}$

$= 3 - \dfrac{4}{x + 1}$

From experience, we know the graph of $y = -\dfrac{1}{x}$ is



The graph of the given function is just a transformation of the graph of $y = -\dfrac{1}{x}$. The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = 3$. The y-intercept is -1 and there is an x-intercept at $\dfrac{1}{3}$.

c. $g(x) = \frac{x^2 + 1}{4x^2 - 9}$

$$= \frac{x^2 + 1}{(2x - 3)(2x + 3)}$$

The function is discontinuous at $x = -\frac{3}{2}$ and at $x = \frac{3}{2}$.

$$\lim_{x \to -\frac{3}{2}^-} g(x) = \infty$$

$$\lim_{x \to -\frac{3}{2}^+} g(x) = -\infty$$

$$\lim_{x \to \frac{3}{2}^-} g(x) = -\infty$$

$$\lim_{x \to -\frac{3}{2}^+} g(x) = \infty$$

Hence, $x = -\frac{3}{2}$ and $x = \frac{3}{2}$ are vertical asymptotes.

The y-intercept is $-\frac{1}{9}$.

$$g'(x) = \frac{2x(4x^2 - 9) - (x^2 + 1)(8x)}{(4x^2 - 9)^2} = \frac{-26x}{(4x^2 - 9)^2}$$

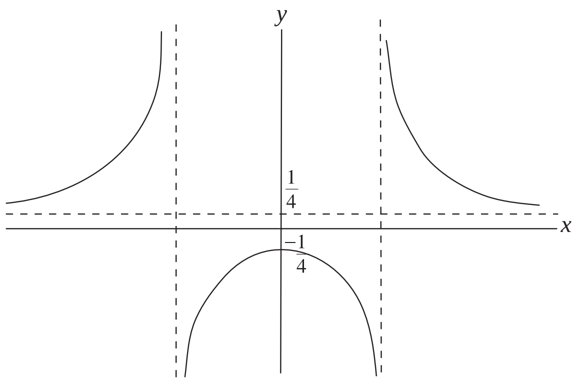
$g'(x) = 0$ when $x = 0$.

Interval	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$g'(x)$	> 0	> 0	$= 0$	< 0	< 0
Graph $g(x)$	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local maximum at $\left(0, -\frac{1}{9}\right)$.

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{4 - \frac{9}{x^2}} = \frac{1}{4} \text{ and } \lim_{x \to -\infty} g(x) = \frac{1}{4}$$

Hence, $y = \frac{1}{4}$ is a horizontal asymptote.



d. $y = 3x^2 \ln x, x > 0$

$$\frac{dy}{dx} = 6x \ln x + 3x^2 \left(\frac{1}{x}\right) = 3x(2 \ln x + 1)$$

Since $x > 0$, the only critical value is when $2 \ln x + 1 = 0$

$$\ln x = -\frac{1}{2}$$

$$x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}.$$

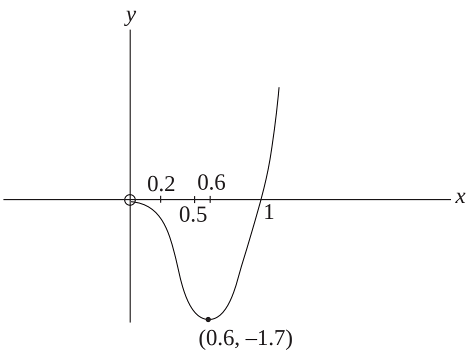
Interval	$0 < x < \frac{1}{\sqrt{e}}$	$x = \frac{1}{\sqrt{e}}$	$x > \frac{1}{\sqrt{e}}$
$\frac{dy}{dx}$	< 0	$= 0$	> 0
Graph of y	Decreasing	Local Min	Increasing

$$\frac{d^2y}{dx^2} = 6 \ln x + 6x\left(\frac{1}{x}\right) + 3 = 6 \ln x + 9$$

$$\frac{d^2y}{dx^2} = 0 \text{ when } \ln x = -\frac{3}{2}$$

$$x = e^{-\frac{3}{2}}$$

Interval	$0 < x < e^{-\frac{3}{2}}$	$x = e^{-\frac{3}{2}}$	$x > e^{-\frac{3}{2}}$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0
Graph of y	Concave Down	Point of Inflection	Concave Up



e. $h(x) = \dfrac{x}{x^2 - 4x + 4}$

$$= \frac{x}{(x-2)^2} = x(x-2)^{-2}$$

There is a discontinuity at $x = 2$

$\lim\limits_{x\to 2^-} h(x) = \infty = \lim\limits_{x\to 2^+} h(x)$

Thus, $x = 2$ is a vertical asymptote. The y-intercept is 0.

$h'(x) = (x-2)^{-2} + x(-2)(x-2)^{-3}(1)$

$$= \frac{x - 2 - 2x}{(x-2)^3}$$

$$= \frac{-2 - x}{(x-2)^3}$$

$h'(x) = 0$ when $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 2$	$x > 2$
$h'(x)$	< 0	$= 0$	> 0	< 0
Graph of $h(x)$	Decreasing	Local Min	Increasing	Decreasing

There is a local minimum at $(-2, -\frac{1}{8})$.

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} = 0$$

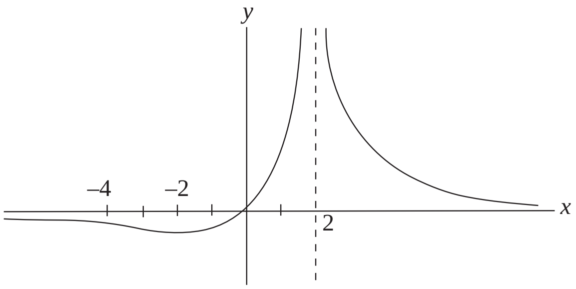
Similarly, $\lim\limits_{x\to -\infty} h(x) = 0$

The x-axis is a horizontal asymptote.

$$\begin{aligned} h''(x) &= -2(x-2)^{-3} - 2(x-2)^{-3} + 6x(x-2)^{-4} \\ &= -4(x-2)^{-3} + 6x(x-2)^{-4} \\ &= \frac{2x + 8}{(x-2)^4} \end{aligned}$$

$h''(x) = 0$ when $x = -4$

The second derivative changes signs on opposite sides of $x = -4$. Hence, $\left(-4, -\frac{1}{9}\right)$ is a point of inflection.



f. $f(t) = \dfrac{t^2 - 3t + 2}{t - 3}$

$$= t + \frac{2}{t-3}$$

Thus, $f(t) = t$ is an oblique asymptote. There is a discontinuity at $t = 3$.

$\lim\limits_{t\to 3^-} f(t) = -\infty$ and $\lim\limits_{t\to 3^+} f(t) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{2}{3}$.

The x-intercepts are $t = 1$ and $t = 2$.

$f'(t) = 1 - \dfrac{2}{(t-3)^2}$

$$f'(t) = 0 \text{ when } 1 - \frac{2}{(t-3)^2} = 0$$

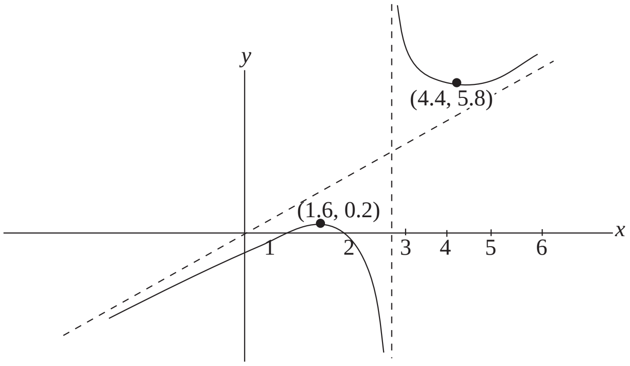
$$(t-3)^2 = 2$$

$$t - 3 = \pm\sqrt{2}$$

$$t = 3 \pm \sqrt{2}.$$

Interval	$t < 3 - \sqrt{2}$	$t = 3 - \sqrt{2}$	$3 - \sqrt{2} < t < 3$	$3 < t < 3 + \sqrt{2}$	$t = 3 + \sqrt{2}$	$t > 3 + \sqrt{2}$
$f'(t)$	> 0	$= 0$	< 0	< 0	$= 0$	> 0
Graph of $f(t)$	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

(1.6, 0.2) is a local maximum and (4.4, 5.8) is a local minimum.



g. $s = te^{-3t} + 10$

At $t = 0$, $s = 10$.

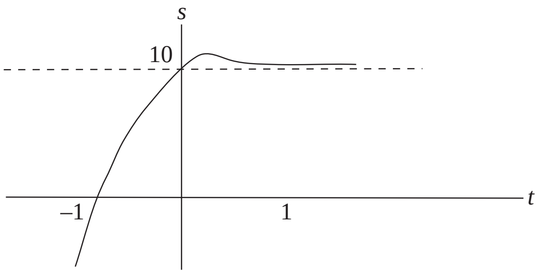
$$\frac{ds}{dt} = e^{-3t} + te^{-3t}(-3) = e^{-3t}(1 - 3t)$$

Since $e^{-3t} > 0$, $\frac{ds}{dt} = 0$ when $t = \frac{1}{3}$.

Interval	$t < \frac{1}{3}$	$t = \frac{1}{3}$	$t > \frac{1}{3}$
$\frac{ds}{dt}$	> 0	0	< 0
Graph of s	Increasing	Total Maximum	Decreasing

$\left(\frac{1}{3}, 10 + \frac{1}{3e}\right)$ is a local maximum point.

Since s is always decreasing for $t > \frac{1}{3}$, and te^{-3t} is positive for $t > \frac{1}{3}$, the graph will always be above the line $s = 10$, but it is approaching the line $s = 10$ as $t \to \infty$. Thus, $s = 10$, it is a horizontal asymptote. Since s is continuous for all t, has a local maximum at $\left(\frac{1}{3}, 10 + \frac{1}{3}e\right)$, and has $s = 10$ as a horizontal asymptote, we conclude that there is an inflection point at a value of $t > \frac{1}{3}$. (It can be shown that there is an inflection point at $t = \frac{2}{3}$.)



h. $P = \frac{100}{1 + 50e^{-0.2t}}$

When $t = 0$, $P = \frac{100}{51} \doteq 1.99$

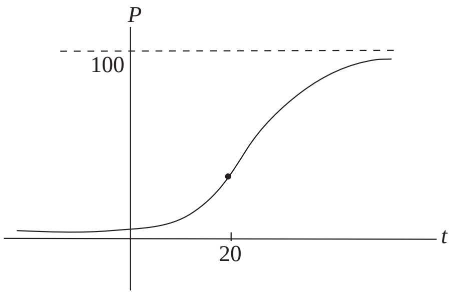
$$\frac{dP}{dt} = -100(1 + 50e^{-0.2t})^{-2}(50e^{-0.2t})(-0.2)$$

$$= \frac{1000e^{-0.2t}}{(1 + 50e^{-0.2t})^2}$$

Since $\frac{dP}{dt} > 0$ for all t, the graph is always increasing.

$$\lim_{t\to\infty}\left(\frac{100}{1 + 50e^{-0.2t}}\right) = 100 \text{ and } \lim_{t\to-\infty}\left(\frac{100}{1 + 50e^{-0.2t}}\right) = 0$$

Thus, $P = 100$ is a horizontal asymptote for large positive values of t, and $P = 0$ (the horizontal axis) is a horizontal asymptote for large negative values of t. It can be shown that there is a point of inflection at $t \doteq 20$.



13. $$P = 10^4te^{-0.2t} + 100,\ t \ge 0$$

a. $$\frac{dP}{dt} = 10^4\left[e^{-0.2t} + te^{-0.2t}(-0.2)\right]$$
$$= 10^4e^{-0.2t}\left[1 - 0.2t\right]$$

$$\frac{dP}{dt} = 0 \text{ when } t = \frac{1}{0.2} = 5.$$

Since $\frac{dP}{dt} > 0$ for $0 \le t < 5$ and $\frac{dP}{dt} < 0$ for $t > 5$,

the maximum population of the colony is $P = 10^4(5)e^{-1} \doteq 18\,994$ and it occurs on the fifth day after the creation of the colony.

b. The growth rate of the colony is the function

$\frac{dP}{dt}$. The rate of change of the growth rate is

$$\frac{d^2P}{dt^2} = 10^4\left[e^{-0.2t}(-0.2)(1 - 0.2t) + e^{-0.2t}(-0.2)\right]$$

$$= 10^4e^{-0.2t}[0.04t - 0.4].$$

To determine when $\frac{d^2P}{dt^2}$ starts to increase, we need

$\frac{d^3P}{dt^3}$.

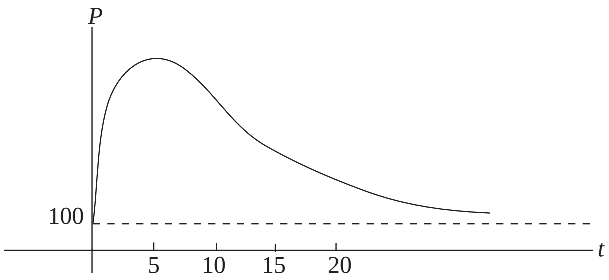
$$\frac{d^3P}{dt^3} = 10^4\left[e^{-0.2t}(-0.2)(0.04t - 0.4) + e^{-0.2t}(0.04)\right]$$

$$= 10^4e^{-0.2t}\left[0.12 - 0.008t\right]$$

$$= 80e^{-0.2t}\ (15 - t)$$

Since $\frac{d^3P}{dt^3} > 0$ for $0 \le t < 15$ and $\frac{d^3P}{dt^3} < 0$ for

$t > 15$, $\frac{d^2P}{dt^2}$ is increasing from the moment the colony is formed and continues for the first 15 days.



14. $y = \ln\left[\frac{x^2 + 1}{x^2 - 1}\right], \frac{x^2 + 1}{x^2 - 1} > 0$

Since $x^2 + 1 > 0$ for all x, for y to be defined, $x^2 - 1 > 0$. The domain is $x < -1$ or $x > 1$.
y can be written as $y = \ln(x^2 + 1) - \ln(x^2 - 1)$.

Thus, $$\frac{dy}{dx} = \frac{2x}{x^2 + 1} - \frac{2x}{x^2 - 1}$$

$$= \frac{-4x}{x^4 - 1} = -4x(x^4 - 1)^{-1}$$

$$\frac{d^2y}{dx^2} = -4(x^4 - 1)^{-1} - 4x(-1)(x^4 - 1)^{-2}(4x^3)$$

$$= \frac{-4x^4 + 4 + 16x^4}{(x^4 - 1)^2} = \frac{4 + 12x^4}{(x^4 - 1)^2}.$$

Since $x \ne \pm 1$, $\frac{d^2y}{dx^2}$ is positive for all x in the domain.

15. **a.** $$f(x) = \frac{2x + 4}{x^2 - k^2}$$

$$f'(x) = \frac{2(x^2 - k^2) - (2x + 4)(2x)}{(x^2 - k^2)^2}$$

$$= -\frac{2x^2 + 8x + 2k^2}{(x^2 - k^2)^2}$$

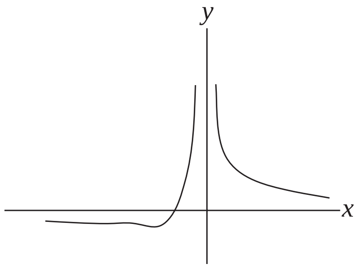
For critical values, $f'(x) = 0$ and $x \ne \pm k$:
$x^2 + 4x + k^2 = 0$

$$x = \frac{-4 \pm \sqrt{16 - 4k^2}}{2}.$$

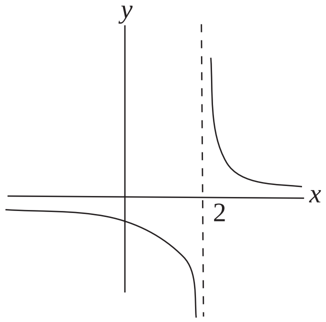
For real roots, $16 - 4k^2 \ge 0$
$$-2 \le k \le 2.$$
The conditions for critical points to exist are $-2 \le k \le 2$ and $x \ne \pm k$.

b. There are three different graphs that result for values of k chosen.

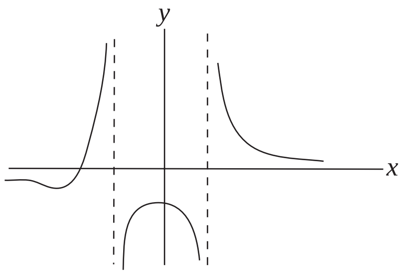
(i) $k = 0$



(ii) $k = 2$



(iii) For all other values of k, the graph will be similar to that of 1(*i*) in Exercise 9.5.



Chapter 9 Test

1. **a.** $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$

b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$

c. $(-9, 1)$, $(-6, -2)$, $(0, 1)$, $(8, -2)$

d. $x = -3$, $x = 4$

e. $f''(x) > 0$

f. $-3 < x < 0$ or $4 < x < 8$

g. $(-8, 0)$, $(10, -3)$

2. **a.** $g(x) = 2x^4 - 8x^3 - x^2 + 6x$
$g'(x) = 8x^3 - 24x^2 - 2x + 6$
To find the critical points, we solve $g'(x) = 0$:
$8x^3 - 24x^2 - 2x + 6 = 0$
$4x^3 - 12x^2 - x + 3 = 0$
Since $g'(3) = 0$, $(x - 3)$ is a factor.
$(x - 3)(4x^2 - 1) = 0$
$x = 3$ or $x = -\frac{1}{2}$ or $x = \frac{1}{2}$.

Note: We could also group to get
$4x^2(x - 3) - (x - 3) = 0$.

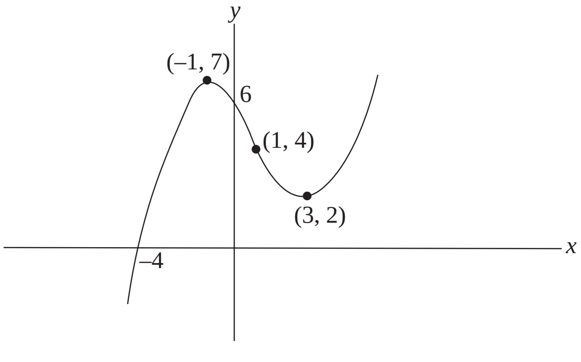
b. $g''(x) = 24x^2 - 48x - 2$

Since $g''\left(-\frac{1}{2}\right) = 28 > 0$, $\left(-\frac{1}{2}, -\frac{17}{8}\right)$ is a local maximum.

Since $g''\left(\frac{1}{2}\right) = -20 < 0$, $\left(\frac{1}{2}, \frac{15}{8}\right)$ is a local maximum.

Since $g''(3) = 70 > 0$, $(3, -45)$ is a local minimum.

3.



4. $g(x) = \dfrac{x^2 + 7x + 10}{(x - 3)(x + 2)}$

The function $g(x)$ is not defined at $x = -2$ or $x = 3$. At $x = -2$, the value of the numerator is 0. Thus, there is a discontinuity at $x = -2$, but $x = -2$ is not a vertical asymptote.

At $x = 3$, the value of the numerator is 40. $x = 3$ is a vertical asymptote.

$$g(x) = \frac{(x + 2)(x + 5)}{(x - 3)(x + 2)} = \frac{x + 5}{x - 3},\ x \neq\ -2$$

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} \left(\frac{x + 5}{x - 3}\right)$$
$$= -\frac{3}{5}$$

$$\lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} \left(\frac{x + 5}{x - 3}\right)$$
$$= -\frac{3}{5}$$

There is a hole in the graph of $g(x)$ at $\left(-2, -\dfrac{3}{5}\right)$.

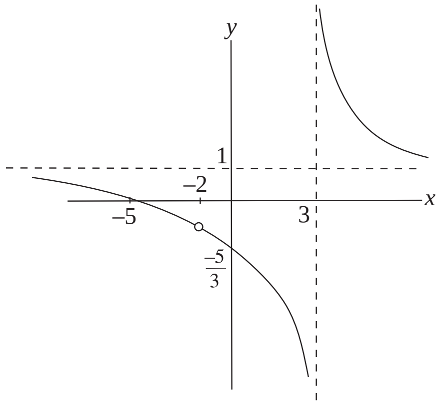
$$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} \left(\frac{x + 5}{x - 3}\right)$$
$$= -\infty$$

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} \left(\frac{x + 5}{x - 3}\right)$$
$$= \infty$$

There is a vertical asymptote at $x = 3$.

Also, $\lim\limits_{x \to \infty} g(x) = \lim\limits_{x \to -\infty} g(x) = 1$.

Thus, $y = 1$ is a horizontal asymptote.



5. $g(x) = e^{2x}(x^2 - 2)$

$g'(x) = e^{2x}(2)(x^2 - 2) + e^{2x}(2x)$

$= 2e^{2x}(x^2 + x - 2)$

To find the critical points, we solve $g'(x) = 0$:

$2e^{2x}(x^2 + x - 2) = 0$

$(x + 2)(x - 1) = 0$, since $e^{2x} > 0$ for all x

$x = -2$ or $x = 1$.

Interval	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$g'(x)$	> 0	0	< 0	0	> 0
Graph of $g(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing

The function $g(x)$ has a local maximum at $\left(-2, \dfrac{2}{e^4}\right)$ and a local minimum at $(1, -e^2)$.

6. $f(x) = \dfrac{2x + 10}{x^2 - 9}$

$= \dfrac{2x + 10}{(x - 3)(x + 3)}$

There are discontinuities at $x = -3$ and at $x = 3$.

$$\left.\begin{aligned}\lim_{x \to -3^-} f(x) &= \infty \\ \lim_{x \to -3^+} f(x) &= -\infty\end{aligned}\right\}\ x = -3 \text{ is a vertical asymptote.}$$

$$\left.\begin{aligned}\lim_{x \to 3^-} f(x) &= -\infty \\ \lim_{x \to 3^+} f(x) &= \infty\end{aligned}\right\}\ x = 3 \text{ is a vertical asymptote.}$$

The y-intercept is $-\dfrac{10}{9}$ and $x = -5$ is an x-intercept.

$$f'(x) = \frac{2(x^2 - 9) - (2x + 10)(2x)}{(x^2 - 9)^2}$$
$$= \frac{-2x^2 - 20x - 18}{(x^2 - 9)^2}$$

For critical values, we solve $f'(x) = 0$:

$x^2 + 10x + 9 = 0$

$(x + 1)(x + 9) = 0$

$x = -1$ or $x = -9$.

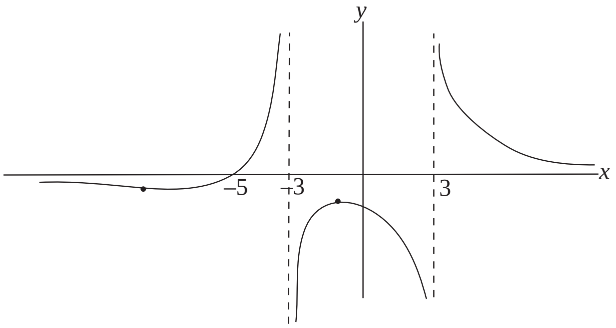
Interval	$x < -9$	$x = -9$	$-9 < x < -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 3$	$x > 3$
$f'(x)$	< 0	0	> 0	> 0	0	< 0	< 0
Graph $f(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and (–1, –1) is a local maximum.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}} = 0 \text{ and}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(\frac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}}\right) = 0$$

$y = 0$ is a horizontal asymptote.



7. $y = x^2 + \ln(kx)$
$= x^2 + \ln k + \ln x$

$$\frac{dy}{dx} = 2x + \frac{1}{x}$$

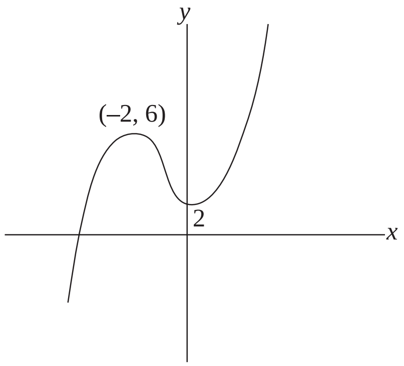
$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}$$

The second derivative is independent of k. There is not enough information to determine k.

8. $f(x) = x^3 + bx^2 + c$

a. $f'(x) = 3x^2 + 2bx$
Since $f'(-2) = 0$, $12 - 4b = 0$
$b = 3$.
Also, $f(-2) = 6$.
Thus, $-8 + 12 + c = 6$
$c = 2$.

b. $f'(x) = 3x^2 + 6x$
$= 3x(x + 2)$
The critical points are (–2, 6) and (0, 2).
$f''(x) = 6x + 6$
Since $f''(-2) = -6 < 0$, (–2, 6) is a local maximum.
Since $f'(0) = 6 > 0$, (0, 2) is a local minimum.



9. $y = x^{\frac{2}{3}}(x - 5)$
$= x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}}$$

$$= \frac{5}{3}x^{-\frac{1}{3}}(x - 2)$$

$$= \frac{5(x-2)}{3x^{\frac{1}{3}}}$$

The critical values are $x = 2$ when $\frac{dy}{dx} = 0$,

and $x = 0$ when $\frac{dy}{dx}$ does not exist.

Interval	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	> 0	Does Not Exist	< 0	= 0	> 0
Graph of $y = f(x)$	Increasing	Local Max	Decreasing	Local Min	Increasing

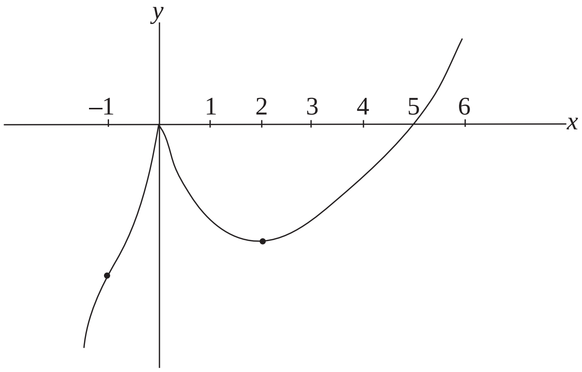
$$\frac{d^2y}{dx^2} = \frac{10}{9}x^{-\frac{1}{3}} + \frac{10}{9}x^{-\frac{4}{3}}$$

$$= \frac{10}{9}\left(\frac{x+1}{x^{\frac{4}{3}}}\right)$$

There are possible points of inflection at $x = -1$ and $x = 0$.

Interval	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x > 0$
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0	Does Not Exist	> 0
Graph of $y = f(x)$	Concave Down	Point of Inflection	Concave Up	Cusp	Concave Up

The y-intercept is 0. There are x-intercepts at 0 and 5.



10. $y = x^2 e^{kx} + p$

$$\frac{dy}{dx} = 2xe^{kx} + x^2(ke^{kx})$$

$$= xe^{kx}(2 + kx)$$

a. When $x = \frac{2}{3}$, $\frac{dy}{dx} = 0$.

Thus, $0 = \frac{2}{3}e^{\frac{2}{3}k}\left(2 + \frac{2}{3}k\right)$.

Since $e^{\frac{2}{3}k} > 0$, $2 + \frac{2}{3}k = 0$

$k = -3$.

b. The parameter p represents a vertical translation of the graph of $y = x^2e^{-3x}$.

Cumulative Review Solutions

Chapters 3–9

2. **c.** $\lim\limits_{x\to 2} \dfrac{x^3-8}{x-2}$

$= \lim\limits_{x\to 2} \dfrac{(x-2)(x^2+2x+4)}{x-2}$

$= \lim\limits_{x\to 2} x^2 + 2x + 4$

$= 12$

f. $\lim\limits_{x\to 0} \dfrac{\left(\sqrt{2+x}-\sqrt{2}\right)}{\sqrt{2}x} \times \dfrac{\left(\sqrt{2+x}+\sqrt{2}\right)}{\left(\sqrt{2+x}+\sqrt{2}\right)}$

$= \lim\limits_{x\to 0} \dfrac{2+x-2}{x\sqrt{2}\left(\sqrt{2+x}+\sqrt{2}\right)}$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{2}\left(\sqrt{2+x}+\sqrt{2}\right)}$

$= \dfrac{1}{4}$

4. **b.** $y = \dfrac{2-x}{x^2}$

$\dfrac{dy}{dx} = \lim\limits_{h\to 0} \left[\dfrac{2-(x+h)}{(x+h)^2} - \dfrac{2-x}{x^2}\right]$

$= \lim\limits_{h\to 0} \dfrac{2x^2 - x^2(x+h) - (2-x)(x^2+2xh+h^2)}{hx^2(x+h)^2}$

$= \lim\limits_{h\to 0} \dfrac{2x - x^3 - x^2h - 2x^2 + x^3 - 4xh + 2x^2h - 2h^2 + xh^2}{h(x^2)(x+h)^2}$

$= \lim\limits_{h\to 0} \dfrac{h[-x^2 - 4x + 2x^2 - 2h + xh]}{h(x^2)(x+h)^2}$

$= \dfrac{x^2-4x}{x^4}$

$= \dfrac{x-4}{x^3}$

5. **d.** $s = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$

$\dfrac{ds}{dt} = \dfrac{[e^t + e^{-t}][e^t + e^{-t}] - (e^t - e^{-t})(e^t - e^{-t})}{(e^t + e^{-t})^2}$

$= \dfrac{e^{2t} + 2 + e^{-2t} - (e^{2t} - 2 + e^{-2t})}{(e^t + e^{-t})^2}$

$= \dfrac{4}{(e^t + e^{-t})^2}$

f. $s = (1\text{n }t + e^t)t$

$\dfrac{ds}{dt} = \left[\dfrac{1}{t} + e^t\right]t + \left[\ln t + e^t\right]$

$= 1 + te^t + \ln t + e^t$

6. **c.** $w = \sqrt{3x + \dfrac{1}{x}}$

$\dfrac{dw}{dx} = \dfrac{1}{2}\left(3x + \dfrac{1}{x}\right)^{-\frac{1}{2}}\left(3 - \dfrac{1}{x^2}\right)$

$= \dfrac{1}{2}\left(\dfrac{x}{3x^2+1}\right)^{\frac{1}{2}}\left(\dfrac{3x^2-1}{x^2}\right)$

$= \dfrac{1}{2}\dfrac{(3x^2-1)^{\frac{1}{2}}}{x^{\frac{3}{2}}}$

h. $1\text{n }(x^2y) = 2y$

$\dfrac{1}{x^2y}\left[x^2\dfrac{dy}{dx} + 2xy\right] = 2\dfrac{dy}{dx}$

$\dfrac{1}{y}\dfrac{dy}{dx} + \dfrac{2}{x} = 2\dfrac{dy}{dx}$

$\dfrac{dy}{dx}\left(2 - \dfrac{1}{y}\right) = \dfrac{2}{x}$

$\dfrac{dy}{dx} = \dfrac{2}{x}\left(\dfrac{y}{2y-1}\right) = \dfrac{2y}{2xy-x}$

8. **c.** $s = 3\sqrt{\frac{2 + 3t}{2 - 3t}}$

$$s = 3\left(\frac{2 + 3t}{2 - 3t}\right)^{\frac{1}{2}}$$

$$\frac{ds}{dt} = 3\left[\frac{1}{2}\left(\frac{2 + 3t}{2 - 3t}\right)^{-\frac{1}{2}}\right]\left[\frac{3(2 - 3t)-(-3)(2 + 3t)}{(2 - 3t)^2}\right]$$

$$= \frac{3}{2}\left[\left(\frac{2 - 3t}{2 + 3t}\right)^{\frac{1}{2}}\right]\left[\frac{12 + 12t}{(2 - 3t)^2}\right]$$

$$= \frac{18}{(2 + 3t)^{\frac{1}{2}}(2 - 3t)^{\frac{3}{2}}}$$

f. $x^3 + 3x^2y + y^3 = c^3$

$$3x^2 + 6xy + 3x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3x^2 + 3y^2) = -3x^2 - 6xy$$

$$\frac{dy}{dx} = \frac{-3(x^2 + 2xy)}{3(x^2 + y^2)} = \frac{-(x^2 + 2xy)}{x^2 + y^2}$$

9. $\frac{ds}{dt} = e^{t^2} + te^{t^2}(2t)$

At $x = \pi$, $\frac{ds}{dt} = e^{\pi^2} + \pi e^{\pi^2}(2\pi)$

$$= e^{\pi^2}(1 + 2\pi^2).$$

At $t = \pi$, $s = \pi e^{\pi^2}$ or the point is $(\pi, \pi e^{\pi^2})$.

The equation is

$$y - \pi e^{\pi^2} = e^{\pi^2}(1 + 2\pi^2)(x - \pi).$$

10. $y = e^{kx}$, $y' = ke^{kx}$, $y'' = k^2e^{kx}$, $y''' = k^3e^{kx}$

a. $y'' - 3y' + 2y = 0$

$k^2e^{kx} - 3ke^{kx} + 2e^{kx} = 0$

Since $e^{kx} \neq 0$,

$k^2 - 3k + 2 = 0$

$(k - 2)(k - 1) = 0$

$k = 2$ or $k = 1$.

b. $y''' - y'' - 4y' + 4y = 0$

$e^{kx}[k^3 - k^2 - 4k + 4] = 0$ or $k^3 - k^2 - 4k + 4 = 0$

$k^2(k - 1) + 4(k - 1) = 0$

$(k - 1)(k - 2)(k + 2) = 0$

$k = 1, 2$ or -2

12. $y^2 = e^{2x} + 2y - e$

When $y = 2$, therefore $4 = e^{2x} + 4 - e$

$e^{2x} = e$

$2x = 1$

$x = \frac{1}{2}$

$y^2 = e^{2x} + 2y - e$

$2y\frac{dy}{dx} = 2e^{2x} + 2\frac{dy}{dx}$

At $x = \frac{1}{2}$, $y = 2$:

$4\frac{dy}{dx} = 2e + 2\frac{dy}{dx}$

$2\frac{dy}{dx} = 2e$

$\frac{dy}{dx} = e.$

13. $x^2 - xy + 3y^2 = 132$

Using implicit differentiation:

$$2x - y - x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

The slope of $x - y = 2$ is 1, therefore, $\frac{dy}{dx} = 1.$

Substituting,

$2x - y - x + 6y = 0$ or $x + 5y = 0$

Substitute $x = -5y$ into $x^2 - xy + 3y^2 = 132$:

$25y^2 + 5y^2 + 3y^2 = 132$

$33y^2 = 132$

$y^2 = 4$

$y = \pm 2$

$y = 2$ or $y = -2$

$x = -10$ or $x = 10.$

The equations are $y - 2 = x + 10$ or $y = x + 12$,

and $y + 2 = x - 10$ or $y = x - 12$.

14. ***Note:*** the point (3, 2) is not on the curve $y = x^2 - 7$.

Let any point on the curve be $(a, a^2 - 7)$:

$\frac{dy}{dx} = 2x$ or at $x = a$ $\frac{dy}{dx} = 2a.$

Equation of the tangent is

$y - (a^2 - 7) = 2a(x - a).$

Since (3, 2) lies on the line, therefore,

$2 - a^2 + 7 = 2a(3 - a)$

$a^2 - 6a + 5 = 0$

$(a - 5)(a - 1) = 0$

$a = 5$ or $a = 1$

If $a = 5$, the equation of the tangent is
$y - 18 = 10(x - 5)$ or $y = 10x - 32$.
If $a = 1$, the equation of the tangent is
$y + 6 = 2(x - 1)$ or $y = 2x - 8$.

15. Slope of $3x + 9y = 8$ is $-\frac{1}{3}$.

The slope of the tangent is 3:

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

Therefore, $1 + \frac{1}{x} = 3$

$$\frac{1}{x} = 2$$

$$x = 2.$$

When $x = 2$, $y = 2 + \ln 2$, and $\frac{dy}{dx} = \frac{3}{2} = m$.

The equation of the tangents is
$y -(2 + \ln 2) = \frac{3}{2}(x - 2)$ or $6x - 2y -(2 \ln 2 + 2) = 0$

17. **b.** Average velocity in the fourth second is

$$\frac{s(4) - s(3)}{4 - 3} = \frac{-4 -(4)}{1} = 0.$$

18. **a.** Surface area is $A = 4\pi r^2$.

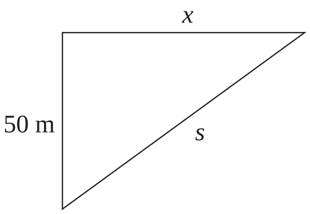
Determine $\frac{dA}{dt}$:

$$\frac{dA}{dt} = 8\pi r\,\frac{dr}{dt}.$$

When $\frac{dr}{dt} = 2$ and $r = 7$,

$$\frac{dA}{dt} = 8\pi(2)(7) = 112 \text{ mm}^2\text{/s}.$$

19.



Let x represent the horizontal distance and s the length of the string.
$s^2 = x^2 + 50^2$

Determine $\frac{ds}{dt}$ when $s = 100$.

Differentiate with respect to t:

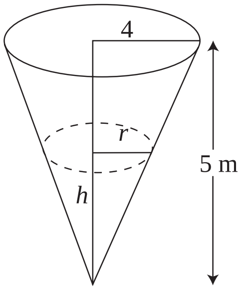
$$2s\,\frac{ds}{dt} = 2x\,\frac{dx}{dt}$$

$\frac{dx}{dt} = 3$ and when $s = 100$, $x^2 = 100^2 - 50^2$
$x = 50\sqrt{3}$.

Therefore, $100\,\frac{ds}{dt} = 50\sqrt{3}(3)$

$$\frac{dA}{dt} = \frac{3\sqrt{2}}{2} \text{ m/s}.$$

20.



Let r represent the radius of the water and h the height of the water. The volume of water is

$$V = \frac{1}{3}\pi r^2 h.$$

b. $\frac{r}{h} = \frac{4}{5}$

$$h = \frac{5}{4}r$$

$$V = \frac{1}{3}\pi r^2(\frac{5}{4}r)$$

$$= \frac{5\pi r^3}{12}$$

c. Determine $\frac{dh}{dt}$ when $r = 3$ m or 300 cm:

$$v = \frac{5\pi r^3}{12}$$

$$\frac{dv}{dt} = \frac{5\pi}{12}(3r^2)\frac{dv}{dt}$$

$$= \frac{5\pi}{4}\, r^2\, \frac{dr}{dt}.$$

When $v = 10\ 000$, $r = 300$:

$$10\ 000 = \frac{5\pi}{12}(300)^2\, \frac{dr}{dt}$$

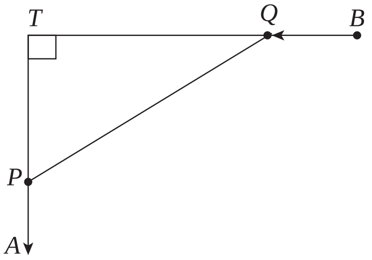
$$\frac{dr}{dt} = \frac{4 \times 10\ 000}{5 \times \pi \times 90\ 000} = \frac{4}{45\pi}.$$

But $h = \frac{5}{4}r$ or $\frac{dh}{dt} = \frac{5}{4}\,\frac{dr}{dt}$

$$= \frac{5}{4} \times \frac{4}{45\pi}$$

$$= \frac{1}{9\pi} \text{ cm/min.}$$

22.



Car B is travelling west and Car A is travelling south.

$BT = 2 \times 100$

$= 200$ km

$TP = 80\ t$

$QT = 200 - 100t$

Let $PQ = s$, therefore,

$s^2 = (80t)^2 + (200 - 100t)^2$

Differentiate with respect to t:

$$2s\,\frac{ds}{dt} = 160(80t) + 2(200 - 100t)(-100).$$

To determine when the two cars were closest, let $\frac{ds}{dt} = 0$.

$$160(80t) - 200(200 - 100t) = 0$$

$$160(80t) = 200(200 - 100t) = 0$$

$$160(80t) + 200(100t) = 200 \times 200$$

$$32\ 800t = 40\ 000$$

$$t \doteq 1.22$$

The time is 1 h 13 min or at 14:13.

23. b. $w = 4 - \frac{100}{2^2 + 25}$

$$\frac{dw}{dz} = -200z(z^2 + 25)^{-2}$$

$$= \frac{200z}{(z^2 + 25)^2}$$

For extreme points, $\frac{dw}{dz} = 0$.

Therefore, $z = 0$ and the point is $(0, 0)$.

$$\frac{d^2w}{dz^2} = \frac{200(z^2 + 25)^2 - 2(z^2 + 25)(2x)200z}{(z^2 + 25)^2}$$

For a point of inflection, let $\frac{d^2w}{dz^2} = 0$:

$$200(z^2 + 25)^2 - 800z^2(z^2 + 25) = 0$$

$$(z^2 + 25)^2 - 4z^2(z^2 + 25) = 0 \text{ or } z^2 + 25 - 4z^2$$

$$= 0$$

$$3z^2 = 25$$

$$z^2 = \frac{25}{3} \text{ or } z = \pm\frac{5}{\sqrt{3}}$$

$$w = 4 - \frac{100}{\frac{25}{3} + 25}$$

$$= 4 - \frac{300}{100}$$

$$= 1.$$

The points of inflection are $\left(\frac{5}{\sqrt{3}}, 1\right)$ and $\left(\frac{-5}{\sqrt{3}, 1}\right)$.

d. $y = x^3e^{-2x}$

$$\frac{dy}{dx} = 3x^2e^{-2x} + (-2)(e^{-2x})x^3 = 3x^2e^{-2x} - 2x^3e^{-2x}$$

For extreme values, let $\frac{dy}{dx} = 0$:

$$3x^2e^{-2x} - 2x^3e^{-2x} = 0$$

$$3x^2 - 2x^3 = 0$$

$$x^2(3 - 2x) = 0$$

$$x = 0 \text{ or } x = \frac{3}{2}$$

$$y = 0 \text{ or } y = \frac{27}{8}\, e^{-3}.$$

Local extreme points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{27}{8}e^{-3}\right)$.

For points of inflection:

$$\frac{d^2y}{dx^2} = 6xe^{-2x} - 6x^2e^{-2x} - 6x^2e^{-2x} + 4x^3e^{-2x}.$$

For point of inflection, let $\frac{d^2y}{dx^2} = 0$:

$$\begin{aligned} 6x - 6x^2 - 6x^2 + 4x^3 = 0 \text{ or } 4x^3 - 12x^2 + 6x &= 0 \\ 2x^3 - 6x^2 + 3x &= 0 \\ x(2x^2 - 6x + 3) &= 0 \\ x = 0 \text{ or } 2x^2 - 6x + 3 &= 0 \\ \text{or } x = \frac{3 + \sqrt{3}}{2} \text{ or } x = \frac{3 - \sqrt{3}}{2}. \end{aligned}$$

The points of inflection are (0, 0),

$$\left(\frac{3 + \sqrt{3}}{2}, \frac{\left(3 + \sqrt{3}\right)^3}{8} e^{-(\sqrt{3}+\sqrt{3})}\right),$$

and $\left(\frac{3 - \sqrt{3}}{2}, \frac{(3 - \sqrt{3})^3}{8} e^{\sqrt{3}-3}\right)$.

f. $n = 10pe^{-p} + 2$

$$\frac{dn}{dp} = 10\, e^{-p} - 10\, pe^{-p}$$

$$\frac{d^2n}{dp^2} = -10e^{-p} - 10e^{-p} + 10\, pe^{-p}$$

For extreme points, let $\frac{dn}{dp} = 0$:

$$\begin{aligned} 10e^{-p} - 10pe^{-p} &= 0 \\ p &= 1 \\ n &= 10e^{-1} + 2. \end{aligned}$$

The extreme point is $(1, 10e^{-1} + 2)$

For points of inflection, let $\frac{d^2n}{dp^2} = 0$:

$$\begin{aligned} -10e^{-p} - 10e^{-p} + 10pe^{-p} &= 0 \\ -20 + 10p &= 0 \\ p &= 2 \\ n &= 20e^{-2} + 2. \end{aligned}$$

The point of inflection is $(2, 20e^{-2} + 2)$.

24. a. $y = \frac{8}{x^2 - 9}$

Discontinuity is at $x = \pm 3$.

$$\lim_{x \to -3^-} \frac{8}{x^2 - 9} = +\infty$$

$$\lim_{x \to -3^+} \frac{3}{x^2 - 9} = -\infty$$

$$\lim_{x \to -3^-} \frac{8}{x^2 - 9} = -\infty$$

$$\lim_{x \to -3^+} \frac{8}{x^2 - 9} = +\infty$$

Vertical asymptotes at $x = 3$ and $x = -3$:

$$\lim_{x \to \infty} \frac{8}{x^2 - 9} = 0 \text{ and } \lim_{x \to \infty} \frac{8}{x^2 - 9} = 0.$$

Horizontal asymptote at $y = 0$:

$$\frac{dy}{dx} = -8(x^2 - 9)^{-2}(2x) = \frac{16x}{(x^2 - 9)^2}.$$

Let $\frac{dy}{dx} = 0$, $\frac{-16x}{(x^2 - 9^2)} = 0$. Therefore, $x = 0$.

The local maximum is at $\left(0, -\frac{8}{9}\right)$.

b. $y = \frac{4x^3}{x^2 - 1}$

Discontinuous at $x = \pm 1$.

$$\lim_{x \to -1^-} \left(\frac{4x^3}{x^2 - 1}\right) = -\infty$$

$$\lim_{x \to -1^+} \left(\frac{4x^3}{x^2 - 1}\right) = +\infty$$

$$\lim_{x \to -1^-} \frac{4x^3}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{4x^3}{x^2 - 1} = +\infty$$

Vertical asymptote is at $x = 1$ and $x = -1$.

$$y = \frac{4x^3}{x^2 - 1} = 4x + \frac{4x}{x^2 - 1}$$

Oblique asymptote is at $y = 4x$.

$$\frac{dy}{dx} = \frac{12x^2(x^2 - 1) - 2x(4x^3)}{(x^2 - 1)^2}$$

For extreme values, let $\frac{dy}{dx} = 0$.

$$\begin{aligned} 12x^2(x^2 - 1) - 8x^4 &= 0 \\ 4x^2(x^2 - 3) &= 0 \\ x = 0 \text{ or } x &= \pm\sqrt{3} \end{aligned}$$

Critical points are $(0, 0)$, $(\sqrt{3}, 6\sqrt{3})$, $(-\sqrt{3}, -6\sqrt{3})$.

25. a. $p = \frac{10n^2}{n^2 + 25}$

There are no discontinuities.

The curve passes through point (0, 0).

Determine extreme values and points of inflection:

$$\begin{aligned} \frac{dp}{dn} &= \frac{20n(n^2 + 25) - 2n(10n^2)}{(n^2 + 25)^2} \\ &= \frac{500n}{(n^2 + 25)^2} \end{aligned}$$

$$\begin{aligned} \frac{d^2p}{dn^2} &= \frac{500(n^2 + 25)^2 - 2(n^2 + 25)(2n)(500n)}{(n^2 + 25)^4} \\ &= \frac{500(25 - 3n^2)}{(n^2 + 25)^3}. \end{aligned}$$

Let $\frac{dp}{dn} = 0$, therefore, $500n = 0$ or $n = 0$ and $p = 0$.

Let $\frac{d^2p}{dn^2} = 0$

$$\begin{aligned} 3n^2 &= 25 \\ n^2 &= \frac{25}{3} \\ n &= \pm\frac{5}{\sqrt{3}} \\ &\doteq \pm 2.9 \\ p &= 2.5 \end{aligned}$$

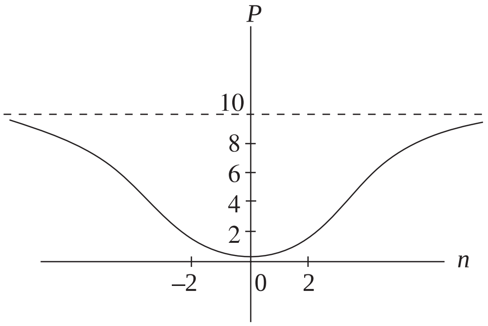
Points of inflection are at (2.9, 2.5) and (–2.9, 2.5).

When $n = 0$, $\frac{d^2p}{dn^2} > 0$, therefore a minimum point occurs at (0, 0).

Horizontal asymptote: $\lim\limits_{n\to\infty} \frac{10}{1 + \frac{25}{n^2}} = 10$

and $\lim\limits_{n\to\infty} \frac{10}{1 + \frac{25}{n^2}} = 10$.

Horizontal asymptote at $y = 10$.



b. $y = x \ln(3x)$

Note: $x > 0$ for y to be defined.

There is no y-intercept.

Determine extreme values and points of inflection:

$$\begin{aligned} \frac{dy}{dx} &= \ln 3x + x\left(\frac{3}{3x}\right) \\ &= \ln 3x + 1 \end{aligned}$$

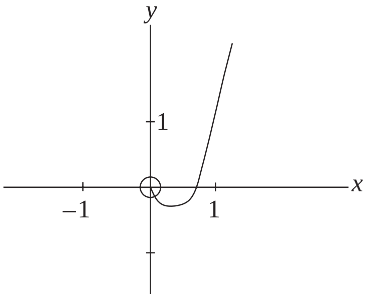
$$\frac{d^2y}{dx^2} = \frac{3}{3x} = \frac{1}{x}.$$

Let $\frac{dy}{dx} = 0$, $\ln 3x = -1$

$$\begin{aligned} 3x &= e^{-1} \\ x &= \frac{1}{3e} \text{ and } y = -\frac{1}{3e}. \end{aligned}$$

$\frac{d^2y}{dx^2} \neq 0$, no point of inflection

When $x = \frac{1}{3e}$, $\frac{d^2y}{dx^2} > 0$, therefore a minimum point occurs at $\left(\frac{1}{3e}, -\frac{1}{3e}\right)$ or (0.12, –0.12).



c. $y = \frac{3x}{x^2 - 4}$

Discontinuity is at $x^2 - 4 = 0$ or $x = \pm 2$.

$$\lim_{x \to -2^+} \frac{3x}{x^2 - 4} = +\infty$$

$$\lim_{x \to -2^-} \frac{3x}{x^2 - 4} = -\infty$$

$$\lim_{x \to -2^+} \frac{3x}{x^2 - 4} = +\infty$$

$$\lim_{x \to -2^-} \frac{3x}{x^2 - 4} = -\infty$$

Vertical asymptotes are at $x = 2$ and $x = -2$.

Horizontal asymptote:

$$\lim_{x \to \infty} \frac{3x}{x^2 - 4} = 0$$

$$\lim_{x \to -\infty} \frac{3x}{x^2 - 4} = 0.$$

Horizontal asymptote at $y = 0$.
Determine extreme values and points of inflection:

$$\frac{dy}{dx} = \frac{3(x^2 - 4) - 3x(2x)}{(x^2 - 4)^2}$$

$$= \frac{-3x^2 - 12}{(x^2 - 4)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-6x(x^2 - 4)^2 - 2(x^2 - 4)(2x)(-3x^2 - 12)}{(x^2 - 4)^4}$$

$$= \frac{6x(x^4 + 8x^2 - 48)}{(x^2 - 4)^4}$$

$$= \frac{6x(x^2 + 12)(x^2 - 4)}{(x^2 - 4)^4}$$

$$= \frac{6x(x^2 + 4)}{(x^2 - 4)^3}$$

Let $\frac{dy}{dx} = 0$ or $-3x^2 - 12 = 0$

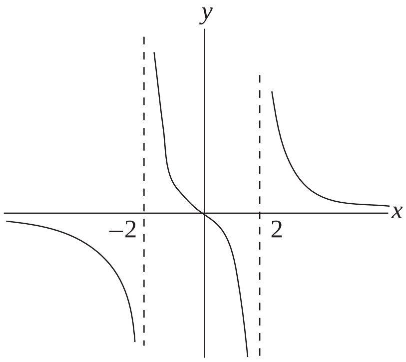
$$x^2 = -4.$$

There are no real values for x. There are no extreme points.
For points of inflection:

$\frac{d^2y}{dx^2} = 0$ or $6x(x^2 + 4) = 0$

$$x = 0.$$

Point of inflection is $(0, 0)$.



d. $y = 10^{-\frac{x^2}{4}}$
y-intercept, let $x = 0$, $y = 1$.
Determine extreme values and points of inflection:

$$\frac{dy}{dx} = -\frac{x}{2} \ln 10 \left(10^{\frac{-x^2}{4}}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-\ln 10}{2}\left(10^{\frac{-x^2}{4}}\right) + \left(\frac{-x}{2}\right) \ (\ln 10)$$

$$= \left(\frac{-x}{2} \ \ln 10 \left(10^{\frac{-x^2}{4}}\right)\right)$$

$$= -\frac{\ln 10}{2}\left(10^{\frac{-x^2}{4}}\right) + \frac{x^2}{4}(\ln 10)^2\left(10^{\frac{-x^2}{4}}\right).$$

Let $\frac{dy}{dx} = 0$, therefore $\frac{-x}{2} \ln 10 \left(10^{\frac{-x^2}{4}}\right) = 0$ or $x = 0$.

Let $\frac{d^2y}{dx^2} = 0$, that is:

$$\frac{\ln 10}{2}\left(10^{\frac{-x^2}{4}}\right) \left[-1 + \frac{x^2}{4} \ln 10\right] = 0$$

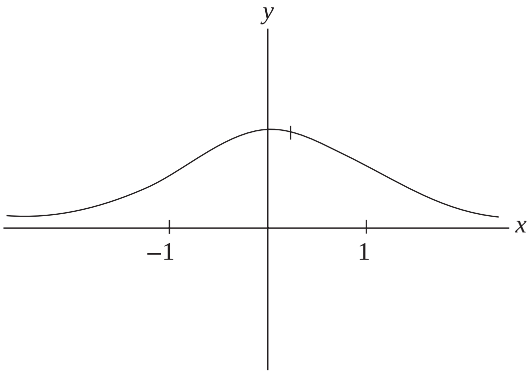
$$\frac{x^2}{4}(\ln 10) = 1$$

$$x^2 = \frac{4}{\ln 10}$$

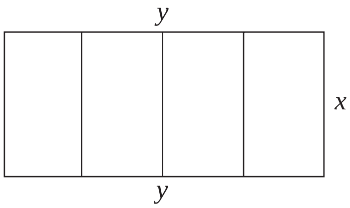
$$x = \pm 1.3$$
$$y = 0.38.$$

Points of inflection occur at $(1.3, 0.38)$ and $(-1.3, 0.38)$.

At $x = 0$, $\frac{d^2y}{dx^2} < 0$, therefore a maximum point occurs at $(0, 1)$.



26.



Let the length be y and the width be x in metres.

$$\begin{aligned} 5x + 2y &= 750 \\ A &= xy \\ \text{But, } 2y &= 750 - 5x \\ y &= \frac{750 - 5x}{2},\ 0 \le x \le 150 \\ A(x) &= x\left(\frac{750 - 5x}{2}\right) \\ &= 375x - \frac{5x^2}{2}. \\ A'(x) &= 375 - 5x \end{aligned}$$

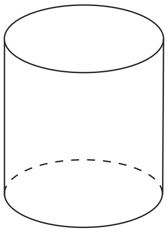
Let $A'(x) = 0$, $x = 187.5$.

Using the max min algorithm,

$$\begin{aligned} A(0) &= 0 \\ A(187.5) &= 14\,062.5 \text{ m}^2 \\ A(150) &= 0. \end{aligned}$$

The maximum area of the four pens is 14 062.5 m^2.

27.



Let the radius be r and the height be h in cm.

Minimize the surface area:

500 mL = 500 cm^3

$$\begin{aligned} V &= \pi r^2 h, \\ 500 &= \pi r^2 h \\ h &= \frac{500}{\pi r^2} \\ A &= 2\pi r^2 + 2\pi r h \\ A &= 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right) \\ A(r) &= 2\pi r^2 + \frac{1000}{r},\ 1 \le r \le 15 \\ A'(r) &= 4\pi r - \frac{1000}{r^2} \end{aligned}$$

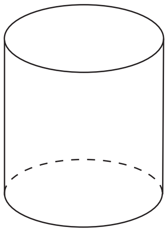
Let $A'(r) = 0$, $4\pi r^3 = 1000$:

$$\begin{aligned} r^3 &= \frac{250}{\pi} \\ r &\doteq 4.3. \end{aligned}$$

$A(1) = 1006.3$, $A(4.3) = 348.7$, $A(15) = 1480.4$

The minimum amount of material is used when the radius is 4.3 cm and the height is 8.6 cm.

28.



Let the radius be r and the height h.

Minimize the cost:

$C = 2\pi r^2(0.005) + 2\pi rh(0.0025)$

$V = \pi r^2 h = 4000$

$$h = \frac{4000}{\pi r^2}$$

$$C(r) = 2\pi r^2(0.005) + 2\pi r\left(\frac{4000}{\pi r^2}\right)(0.0025)$$

$$= 0.01\pi r^2 + \frac{20}{r},\ 1 \le r \le 36$$

$$C'(r) = 0.02\pi r - \frac{20}{r^2}.$$

For a maximum or minimum value, let $C'(r) = 0$.

$$0.02\pi r - \frac{20}{r^2} = 0$$

$$r^3 = \frac{20}{0.02\pi}$$

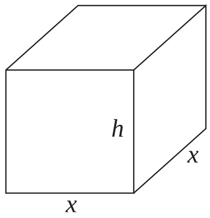
$$r \doteq 6.8$$

Using the max min algorithm:

$C(1) = 20.03$, $C(6.8) = 4.39$, $C(36) = 41.27$.

The dimensions for the cheapest box are a radius of 6.8 cm and a height of 27.5 cm.

29. **a.**



$h + 2x = 140$

$h = 140 - 2x$

b. $V = x^2h$

$= x^2(140 - 2x)$

$= 140x^2 - 2x^3,\ 0 \le x \le 70$

$V'(x) = 280x - 6x^2$

For a maximum or minimum value, let $V'(x) = 0$:

$280x - 6x^2 = 0$

$x(280 - 6x) = 0$

$$x = 0 \text{ or } x = \frac{280}{6}$$

$$\doteq 46.7.$$

Using the max min algorithm:

$V(0) = 0$

$V(46.6) = 101\,629.5\text{ cm}^3$

$V(70) = 0$.

The box of maximum volume has dimensions of 46.7 cm by 46.7 cm by 46.6 cm.

30. $R(x) = x(50 - x^2) = 50x - x^3,\ 0 \le x \le \sqrt{50}$

$R'(x) = 50 - 3x^2$

For a maximum value, let $R'(x) = 0$:

$3x^2 = 50$

$x \doteq 4.1$.

Using the max min algorithm:

$R(0) = 0$, $R(4.1) \doteq 136$, $R(\sqrt{50}) = 0$.

The maximum revenue is \$136 when the price is about \$4.10.

31. $$p = \frac{4000}{1 + 3e^{-0.1373t}}$$

a. For the maximum population, determine:

$$\lim_{t\to\infty} \frac{4000}{1 + 3e^{-0.1373t}} = 4000.$$

The maximum population expected is 4000.

b. $$\frac{dp}{dt} = -4000(1 + 3e^{-0.1373t})^{-2}(3\,(-0.1373)e^{-0.1373t})$$

$$= \frac{1647.6e^{-0.1373t}}{(1 + 3e^{-0.1373t})^2}$$

$$\frac{d^2p}{dt^2} = \frac{1647.6(-0.1373)(e^{-0.1373t})(1 + 3e^{-0.1373t^2}) - 2(1 + 3e^{-0.1373t})(-0.4119e^{-0.1373t})(1647.6e^{-0.1373t})}{(1 + 3e^{-0.1373t})^4}$$

Let $\frac{d^2p}{dt^2} = 0$. To find when the rate of change of the growth rate started to decrease:

Let $1647.6e^{-0.1373t}(1 + 3e^{-0.1373t})[-0.1373 - 0.4119e^{-0.1373t} + 0.8238e^{-0.1373t}] = 0$

or $0.4119e^{-0.1373t} = 0.1373$

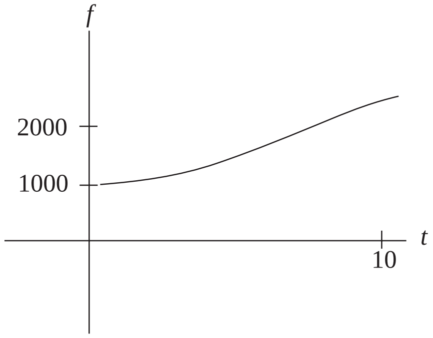
$e^{-0.1373t} = 0.3333$

$$t = \frac{\ln 0.3333}{-0.1373t}$$

$$\doteq 8$$

The rate of change of the growth rate started to decrease after eight years.

c.



d. Data must be collected for six more years.

32. $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

Relative maximum at (1, –7), therefore $f'(1) = 0$:

$3a + 2b + c = 0 \qquad (1)$

Point of inflection at (2, –11), therefore $f''(2) = 0$:

$12a + 2b = 0$

$6a + b = 0 \qquad (2)$

Since (1, –7) is on the curve, then

$a + b + c + d = –7 \qquad (3).$

Since (2, –11) is on the curve, then,

$8a + 4b + 2c + d = –1 \qquad (4)$

$(4) – (3)$: $7a + 3b + c = –4 \qquad (5)$

$(5) – (1)$: $4a + b = –4$

$6a + b = –0$

$–2a = –4$

$a = 2$

$b = –12.$

Substitute in (1): $6 – 24 + c = 0$

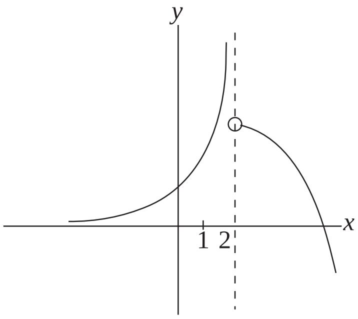
$c = 18.$

Substitute in (3): $2 – 12 + 18 + d = – 7$

$d = –15.$

The function is $f(x) = 2x^3 – 12x^2 + 18x – 15.$

33.



34. a. $f(x) = 1 + (x + 3)^2, –2 \le x \le 6$

$f'(x) = 2(x + 3)$

For critical values, we solve $f'(x) = 0$:

$2(x + 3) = 0$

$x = –3$, not in the domain.

$f(–2) = 1 + 1 = 2$

$f(6) = 1 + 81 = 82$

The minimum value is 2 and the maximum value is 82.

$f(x) = 1 + (x – 3)^2, –2 \le x \le 6$

$f'(x) = 2(x – 3)$

For critical values we solve $f'(x) = 0$

$2(x – 3) = 0$

$x = 3.$

$f(–2) = 1 + 1 = 2$

$f(3) = 1 + 0 = 1$

$f(6) = 1 + 81 = 82$

The minimum value is 1 and the maximum value is 82.

b. $f(x) = x + \frac{1}{\sqrt{x}}, 1 \le x \le 9$

$$f'(x) = 1 – \frac{1}{2}x^{-\frac{3}{2}}$$

$$= \frac{2x^{\frac{3}{2}} – 1}{2x^{\frac{3}{2}}}$$

For critical values, we solve $f'(x) = 0$ or $f'(x)$ does not exist:

$f'(x) = 0$ when $2x^{\frac{3}{2}} – 1 = 0$

$$x = \left(\frac{1}{2}\right)^{\frac{3}{2}}.$$

Since $x \neq 0$, there are no values for which $f'(x)$ does not exist. The critical value is not in the domain of f.

$f(1) = 1 + \frac{1}{1} = 2$

$f(9) = 9 + \frac{1}{3} = 9^{\frac{1}{3}}$

The minimum value of f is 2 and the maximum value is $9^{\frac{1}{3}}$.

c. $f(x) = \dfrac{e^x}{1 + e^x}, 0 \le x \le 4$

$$f'(x) = \frac{e^x(1 + e^x) - e^x(e^x)}{(1 + e^x)^2}$$

$$= \frac{e^x}{(1 + e^x)^2}$$

Since $e^x > 0$ for all x, there are no critical values.

$$f(0) = \frac{1}{1 + 1}$$

$$= \frac{1}{2}$$

$$f(4) = \frac{e^4}{1 + e^4} \doteq .982$$

The minimum value of f is $\frac{1}{2}$ and the maximum value is .982

d. $f(x) = x + \ln(x), 1 \le x \le 5$

$$f'(x) = 1 + \frac{1}{x}$$

$$= \frac{x + 1}{x}$$

Since $1 \le x \le 5$, there are no critical values.

$f(1) = 1 + 0 = 1$

$f(5) = 5 + \ln 5 \doteq 6.609$

The minimum value of f is 1 and the maximum value is 5 + ln5.

35. Let the number of \$30 price reductions be n.
The resulting number of tourists will be $80 + n$ where $0 \le n \le 70$. The price per tourist will be $5000 - 30n$ dollars. The revenue to the travel agency will be $(5000 - 30n)(80 + n)$ dollars. The cost to the agency will be $250\,000 + 300(80 + n)$ dollars.

Profit = Revenue – Cost

$P(n) = (5000 - 30n)(80 + n) - 250\,000 - 300(80 + n)$, $0 \le n \le 70$

$$P'(n) = -30(80 + n) + (5000 - 30n)(1) - 300$$

$$= 2300 - 60n$$

$P'(n) = 0$ when $n = 38^{\frac{1}{3}}$

Since n must be an integer, we now evaluate $P(n)$ for $n = 0, 38, 39$, and 70. (Since $P(n)$ is a quadratic function whose graph opens downward with vertex at $38^{\frac{1}{3}}$, we know $P(38) > P(39)$.)

$P(0) = 126\,000$

$P(38) = (3860)(118) - 250\,000 - 300(118) = 170\,080$

$P(39) = (3830)(119) - 250\,000 - 300(119) = 170\,070$

$P(70) = (2900)(150) - 250\,000 - 300(150) = 140\,000$

The price per person should be lowered by \$1140 (38 decrements of \$30) to realize a maximum profit of \$170 080.

36. $x^2 + xy + y^2 = 19$

To find the coordinates of the points of contact of the tangents, substitute $y = 2$ in the equation of the given curve.

$$x^2 + 2x - 15 = 0$$

$$(x + 5)(x - 3) = 0$$

$$x = -5 \text{ or } x = 3$$

The points on the curve are $(-5, 2)$ and $(3, 2)$.

The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$.

To find $\frac{dy}{dx}$, we differentiate implicitly:

$$2x + (1)y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0.$$

At $(3, 2)$, $6 + 2 + 3\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\frac{8}{7}.$$

The equation of the tangent line to the curve at $(3, 2)$ is

$y - 2 = -\frac{8}{7}(x - 3)$ or $8x + 7y - 38 = 0$.

At $(-5, 2)$, $-10 + 2 - 5\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -8.$$

The equation of the tangent line to the curve at $(-5, 2)$ is $y - 2 = -8(x + 5)$ or $8x + y + 38 = 0$.

37. $y = \dfrac{4}{x^2 - 4}$

There are discontinuities at $x = -2$ and at $x = 2$.

$$\left.\begin{aligned}\lim_{x\to -2^-}\left(\frac{4}{x^2 - 4}\right) &= \infty \\ \lim_{x\to -2^+}\left(\frac{4}{x^2 - 4}\right) &= -\infty\end{aligned}\right\}\ x = -2 \text{ is a vertical asymptote.}$$

$$\left.\begin{aligned}\lim_{x\to 2^-}\left(\frac{4}{x^2 - 4}\right) &= -\infty \\ \lim_{x\to 2^+}\left(\frac{4}{x^2 - 4}\right) &= \infty\end{aligned}\right\}\ x = 2 \text{ is a vertical asymptote.}$$

$$\lim_{x\to\infty}\left(\frac{4}{x^2 - 4}\right) = 0 = \lim_{x\to -\infty}\left(\frac{4}{x^2 - 4}\right)$$

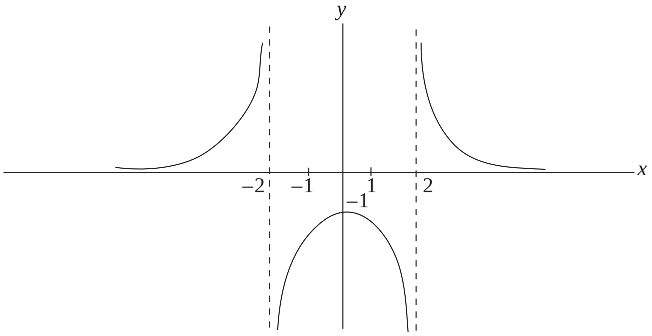
Thus, $y = 0$ is a horizontal asymptote.

$$\frac{dy}{dx} = 4(-1)(x^2 - 4)^{-2}(2x) = \frac{-8x}{(x^2 - 4)^2}$$

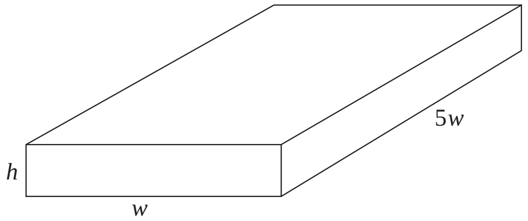
The only critical value is $x = 0$ (since $x = \pm 2$).

Interval	$x < -2$	$-2 < x < 0$	0	$0 < x < 2$	$x > 0$
$\frac{dy}{dx}$	> 0	> 0	$= 0$	< 0	< 0
Graph of y	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local maximum at (0, –1).



38.



We are given that $\frac{dh}{dt} = -2$ m/week and $\frac{dw}{dt} = -3$ m/week.

The volume of the portion of the iceberg above the water is $v = 5w^2h$.

We differentiate the volume expression with respect to t:

$$\frac{dv}{dt} = 10\,w\,\frac{dw}{dt}\,h + 5w^2\,\frac{dh}{dt}.$$

$$= -30wh - 10w^2$$

When $h = 60$ and $w = 300$,

$$\frac{dv}{dt} = -30(300)(60) - 10(300)^2$$

$$= -1\,440\,000.$$

The volume of the portion of the iceberg above water is decreasing at the rate of 1 440 000 m^3/week.

39. $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

Since the points (–2, 3) and (1, 0) are on the curve, we have

$-8a + 4b - 2c + d = 3$ (1)

and $a + b + c + d = 0$. (2)

Since $x = -2$ and $x = 1$ are critical values and $f(x)$ is a polynomial function, we have $f'(-2) = 0 = f'$. (1)

Thus, $12a - 4b + c = 0$ (3)

and $3a + 2b + c = 0$ (4)

Solving the system of equations yields:

from (1) + (2): $-9a + 3b - 3c = 3$

$-3a + b - c = 1$ (5)

(5) + (4): $3b = 1$

$$b = \frac{1}{3}.$$

(3) – (4): $9a - 6b = 0$

$9a = 2$

$$a = \frac{2}{9}$$

From (4): $c = -3a - 2b = -\frac{2}{3} - \frac{2}{3} = -\frac{4}{3}$

From (2): $d = -(a + b + c) = -\left(\frac{2}{9} + \frac{1}{3} - \frac{4}{3}\right) = \frac{7}{9}$

Thus, $(a, b, c, d) = \left(\frac{2}{9}, \frac{1}{3}, -\frac{4}{3}, \frac{7}{9}\right)$.

40. a. $y = x^3 + 2x^2 + 5x + 2,\ x = -1$

$$\frac{dy}{dx} = 3x^2 + 4x + 5$$

The slope of the tangent line at (–1, –2) is 4.

Thus, the slope of the normal line at (–1, –2) is $-\frac{1}{4}$.

The equation of the normal is $y + 2 = -\frac{1}{4}(x + 1)$ or $x + 4y + 9 = 0$.

b. $y = x^{\frac{1}{2}} + x^{\frac{1}{2}}$, at (4, 2.5)

$$\frac{dy}{dx} + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{x - 1}{2x^{\frac{3}{2}}}$$

The slope of the tangent line at (4, 2.5) is $\frac{3}{16}$.

Thus, the slope of the normal line at (4, 2.5) is $-\frac{16}{3}$.

The equation of the normal line is

$$y - \frac{5}{2} = -\frac{16}{3}(x - 4) \text{ or } 32x + 6y - 143 = 0.$$

Appendix A

Exercise

4. **b.** $\cos\theta = -\frac{2}{3} = \frac{x}{r}$ and θ is an angle in the third quadrant.

Since $x^2 + y^2 = r^2$, $4 + y^2 = 9$

$$y = -\sqrt{5}.$$

Hence, $\sin\theta = -\frac{\sqrt{5}}{3}$ and $\tan\theta = \frac{\sqrt{5}}{2}$.

c. $\tan\theta = -2 = \frac{y}{x}$ and θ is an angle in the fourth quadrant.

Since $x^2 + y^2 = r^2$, $1 + 4 = r^2$

$$r = \sqrt{5}.$$

Hence, $\sin\theta = -\frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$.

7. **a.** $\tan x + \cot x = \sec x \csc x$

$$\begin{aligned}\text{L.S.} &= \tan x + \cot x\\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x}\\ &= \frac{1}{\cos x + \sin x}\end{aligned}$$

$$\begin{aligned}\text{R.S.} &= \sec x \csc x\\ &= \frac{1}{\cos x} \bullet \frac{1}{\sin x}\\ &= \frac{1}{\cos x \sin x}\end{aligned}$$

Therefore, $\tan x + \cot x = \sec s \csc x$.

b. $\frac{\sin x}{1 - \sin x}\ \tan x + \sec x$

$$\begin{aligned}\text{L.S.} &= \frac{\sin x}{1 - \sin^2 x}\\ &= \sin \frac{x}{\text{cox}^2 x}\end{aligned}$$

$$\begin{aligned}\text{R.S.} &= \tan x \sec x\\ &= \frac{\sin x}{\cos x} \bullet \frac{1}{\cos x}\\ &= \frac{\sin x}{\cos^2 x}\end{aligned}$$

Therefore, $\frac{\sin x}{1 - \sin^2 x} = \tan x \sec x$.

c. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

$$\begin{aligned}\text{L.S.} &= \sin^4 x - \cos^4 x\\ &= (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)\\ &= (1)(1 - \cos^2 x - \cos^2 x)\\ &= 1 - 2\cos^2 x\end{aligned}$$

Therefore, $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$.

d. $\frac{1}{1 + \sin x} = \sec^2 x - \frac{\tan x}{\cos x}$

$$\begin{aligned}\text{R.S.} &= \sec^2 x - \frac{\tan x}{\cos x}\\ &= \frac{1}{\cos^2 x} - \frac{\frac{\sin x}{\cos x}}{\cos x}\\ &= \frac{1 - \sin x}{\cos^2 x}\\ &= \frac{1 - \sin x}{1 - \sin^2 x}\\ &= \frac{1 - \sin x}{(1 - \sin x)(1 + \sin x)}\\ &= \frac{1}{1 + \sin x}\end{aligned}$$

Therefore, $\frac{1}{1 + \sin x} = \sec^2 x - \frac{\tan x}{\cos x}$.

8. **a.** $6 \sin x - 3 = 1 - 2 \sin x$

$$\begin{aligned}8 \sin x &= 4\\ \sin x &= \frac{1}{2}\\ x &= \frac{\pi}{6}, \frac{5\pi}{6}.\end{aligned}$$

b.

$$\begin{aligned}\cos^2 x - \cos x &= 0\\ \cos x (\cos x - 1) &= 0\\ \cos x = 0 \text{ or } \cos x &= 1\end{aligned}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = 0, 2\pi$$

c. $2 \sin x \cos x = 0$

$\sin 2x = 0$ where $0 \le 2x \le 4\pi$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

d. $2 \sin^2 x - \sin x - 1 = 0$

$(2 \sin x + 1)(\sin x - 1) = 0$

$\sin x = -\frac{1}{2}$ or $\sin x = 1$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$ or $x = \frac{\pi}{2}$

e. $\sin x + \sqrt{3} \cos x = 0$

$\sin x = -\sqrt{3} \cos x$

$\tan x = -\sqrt{3}$

$x = \frac{2\pi}{3}, \frac{5\pi}{3}$

f. $2 \sin^2 x - 3 \cos x = 0$

$2(1 - \cos^2 x) - 3 \cos x = 0$

$2 - 3 \cos x - 2 \cos^2 x = 0$

$(2 + \cos x)(1 - 2 \cos x) = 0$

$\cos x = -2$ or $\cos x = \frac{1}{2}$

no solutions or $x = \frac{\pi}{3}, \frac{5\pi}{3}$

Exercise A1

3. **a.** $\sin(W + T) = \sin W \cos T + \cos W \sin T$

$$= \frac{3}{5} \bullet \frac{12}{13} + \frac{4}{5} \bullet \frac{5}{13}$$

$$= \frac{36 + 20}{65}$$

$$= \frac{56}{65}$$

b. $\cos(W - T) < \sin(W + T)$

$\cos(W - T) = \cos W \cos T + \sin W \sin T$

$$= \frac{4}{5} \bullet \frac{12}{13} + \frac{3}{5} \bullet \frac{5}{13}$$

$$= \frac{48 + 15}{65}$$

$$= \frac{63}{65}$$

4. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$\sin(A - B) = \sin(A + (-B))$

$= \sin A \cos(-B) + \cos A \sin(-B)$

$= \sin A \cos B - \cos A \sin B$

5. **a.** $\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = \cos(A + A)$

$= \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$

b. $\cos 2A = 2 \cos^2 A - 1$

$\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

c. $\cos 2A = \cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

7. **a.** $\cos 75° = \cos(45° + 30°)$

$= \cos 45° \cos 30° - \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}} \bullet \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \bullet \frac{1}{2}$$

$$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

c. $\cos 105° = \cos(60° + 45°)$

$= \cos 60° \cos 45° - \sin 60° \sin 45°$

$$= \frac{1}{2} \bullet \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \bullet \frac{1}{\sqrt{2}}$$

$$= \frac{1 - \sqrt{3}}{2\sqrt{2}}$$

8. **a.** $\sin\left(\frac{\pi}{3} + x\right) = \sin \frac{\pi}{3} \cos x + \cos \frac{\pi}{3} \sin x$

$$= \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x$$

b. $\cos\left(x + \frac{3\pi}{4}\right) = \cos x \cos \frac{3\pi}{4} - \sin x \sin \frac{3\pi}{4}$

$$= -\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x$$

9. Since $\cos A = \frac{x}{r} = \frac{1}{3}$ and $0 < A < \frac{\pi}{2}$, we have

$1^2 + 1^2 + x^2 = 0$, so $x = \sqrt{8} = 2\sqrt{2}$.

Since $\sin B = \frac{1}{4} = \frac{y}{r}$ and $\frac{\pi}{2} < B < \pi$, we have

$x^2 + 1^2 = 16$, so $x = -\sqrt{15}$.

a. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \frac{1}{3} \bullet \frac{-\sqrt{15}}{4} - \frac{2\sqrt{2}}{3} \bullet \frac{1}{4}$$

$$= \frac{-\sqrt{15} - 2\sqrt{2}}{12}$$

b. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{2\sqrt{2}}{3} \bullet -\frac{\sqrt{15}}{4} + \frac{1}{3} \bullet \frac{1}{4}$$

$$= \frac{-2\sqrt{30} + 1}{12}$$

c. $\cos 2A = 2\cos^2 A - 1$

$$= 2\left(\frac{1}{3}\right)^2 - 1$$

$$= -\frac{7}{9}$$

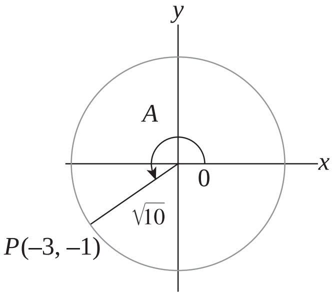
d. $\sin 2B = 2\sin B \cos B$

$$= 2\left(\frac{1}{4}\right)\left(\frac{-\sqrt{15}}{4}\right)$$

$$= \frac{-\sqrt{15}}{8}$$

10. Since $\tan A = \frac{y}{x} = \frac{1}{3}$ and $\pi < A < \frac{3\pi}{2}$, we have

$1^2 + 1^2 + 3^2 = r^2$, so $r = \sqrt{10}$, $x = -3$, and $y = -1$.



a. $\sin 2A = 2\sin A \cos A$

$$= 2\left(-\frac{1}{\sqrt{10}}\right)\left(-\frac{3}{10}\right)$$

$$= \frac{3}{5}$$

b. $\cos 2A = 2\cos^2 A - 1$

$$= 2\left(-\frac{3}{\sqrt{10}}\right)^2 - 1$$

$$= \frac{4}{5}$$

Since $2\pi < 2A < 3\pi$ and both $\sin 2A$ and $\cos 2A$ are positive, angle $2A$ must be in the first quadrant.

11. **a.** $\cos^4 A - \sin^4 A = \cos 2A$

L.S. $= \cos^4 A - \sin^4 A$

$= (\cos^2 A - \sin^2 A)(\cos^2 A + \sin^2 A)$

$= \cos^2 A - \sin^2 A$

$= \cos 2A$

= R.S.

The identity is true.

b. $1 + \sin 2\alpha = (\sin \alpha + \cos \alpha)^2$

R.S. $= (\sin \alpha + \cos \alpha)^2$

$= \sin^2 \alpha + 2\sin \alpha \cos \alpha + \cos^2 \alpha$

$= 1 + \sin 2\alpha$

= L.S.

The identity is true.

c. $\sin(A + B) \bullet \sin(A - B) = \sin^2 A - \sin^2 B$

L.S. $= \sin(A + B) \bullet \sin(A - B)$

$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B$

$= \sin^2 A - \sin^2 B$

= R.S.

The identity is true.

d. $\dfrac{\cos W - \sin 2W}{\cos 2W + \sin W - 1} = \cot W$

$$\text{L.S.} = \frac{\cos W - \sin 2W}{\cos 2W + \sin W - 1}$$

$$= \frac{\cos W - 2\sin W \cos W}{1 - 2\sin^2 W + \sin W - 1}$$

$$= \frac{\cos W(1 - 2\sin W)}{\sin W(1 - 2\sin W)}$$

$$= \cot W$$

$$= \text{R.S.}$$

The identity is true.

e. $\dfrac{\sin^2 \theta}{1 - \cos 2\theta} = 2\csc 2\theta - \tan\theta$

$$\text{L.S.} = \frac{\sin 2\theta}{1 - \cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{1 - (1 - 2\sin^2\theta)}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$\text{R.S.} = 2\csc 2\theta - \tan\theta$$

$$= \frac{2}{\sin 2\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{2}{2\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1 - \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

The identity is true.

f. $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1 + \cos\theta}$

$$\text{R.S.} = \frac{\sin\theta}{1 + \cos\theta}$$

$$= \frac{\sin\left(2 \bullet \dfrac{\theta}{2}\right)}{1 + \cos\left(2 \bullet \dfrac{\theta}{2}\right)}$$

$$= \frac{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{1 + 2\cos^2\dfrac{\theta}{2} - 1}$$

$$= \frac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}$$

$$= \text{L.S.}$$

The identity is true.

12. $f(x) = \sin 3x \csc x - \cos 3x \sec x$

Using the identities proven in question 12,

$f(x) = \sin 3x \csc x - \cos 3x \sec x$

$$= (3\sin x - 4\sin^3 x) \bullet \frac{1}{\sin x} - (4\cos^3 x - 3\cos x) \bullet \frac{1}{\cos x}$$

$$= 3 - 4\sin^2 x - 4\cos^2 x + 3$$

$$= 6 - 4(\sin^2 x + \cos^2 x)$$

$$= 2.$$

13. a. $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} = \tan\dfrac{\theta}{2}$

$$\text{L.S.} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}$$

$$= \frac{1 + 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} - (1 - 2\sin^2\theta\,\dfrac{\theta}{2})}{1 + 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} + 2\cos^2\dfrac{\theta}{2} - 1}$$

$$= \frac{2\sin\dfrac{\theta}{2}(\cos\dfrac{\theta}{2} + \sin\dfrac{\theta}{2})}{2\cos\dfrac{\theta}{2}(\sin\dfrac{\theta}{2} + \cos\dfrac{\theta}{2})}$$

$$= \tan\frac{\theta}{2}$$

$$= \text{R.S.}$$

The identity is true.

b. $\dfrac{\cos 2W}{1 + \sin 2W} = \dfrac{\cot W - 1}{\cot W + 1}$

$$\text{R.S.} = \frac{\cot W - 1}{\cot W + 1}$$

$$= \frac{\dfrac{\cos W}{\sin W} - 1}{\dfrac{\cos W}{\sin W} + 1}$$

$$= \frac{\cos W - \sin W}{\cos W + \sin W} \bullet \frac{\cos W + \sin W}{\cos W + \sin W}$$

$$= \frac{\cos^2 W - \sin^2 W}{\cos^2 W + 2\sin W \cos W + \sin^2 W}$$

$$= \frac{\cos 2W}{1 + \sin 2W}$$

$$= \text{L.S.}$$

The identity is true.

c. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin 3\theta = \sin(2\theta + \theta)$$

$$= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$$

$$= 2\sin \theta \cos^2\theta + (1 - 2\sin^2\theta)\sin \theta$$

$$= 2\sin \theta(1 - \sin^2\theta) + \sin \theta - 2\sin^3\theta$$

$$= 3\sin \theta - 4\sin^3 \theta$$

The identity is true.

d. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\cos 3\theta = \cos(2\theta + \theta)$$

$$= \cos 2\theta \cos \theta - \sin 2\theta \cos \theta$$

$$= (2\cos^2\theta - 1)\cos \theta - 2\sin^2\theta \cos \theta$$

$$= 2\cos^3\theta - \cos \theta - 2(1 - \cos^2\theta)\cos \theta$$

$$= 4\cos^3\theta - 3\cos \theta$$

The identity is true.

14. $\sin \beta + \cos \beta = \sin \beta \cos \beta$

Thus, $(\sin \beta + \cos \beta)^2 = \left(\dfrac{1}{2} \sin 2\beta\right)^2$

$$\sin^2\beta + 2\sin \beta \cos \beta + \cos^2\beta = \frac{1}{4} \sin^2 2\beta$$

$$1 + \sin 2\beta = \frac{1}{4} \sin^2 2\beta$$

$$\sin^2 2\beta - 4\sin 2\beta - 4 = 0,\ 0 \le 2\beta \le 4\pi$$

$$\sin 2\beta = \frac{4 \pm 4\sqrt{2}}{2}$$

$$= 2 + 2\sqrt{2},\ 2 - 2\sqrt{2} \text{ (inadmissible)}$$

$$\sin 2\ \beta = -\ 0.8284271247.$$

Thus, possible values for 2β are 236°, 304°, 596°, and 664°. Possible values for β are 118°, 152°, 298°, and 332°. Upon verification, the solutions are 152° and 298°.

15. $b^2 \sin 2C + c^2\sin 2B = 2bc \sin A$

In ΔABD, $|BD| = c \cos B$

and $|AD| = c \sin B.$

In ΔADC, $|DC| = b \cos C$

and $|AD| = b \sin C.$

The area of ΔABC is

$$A = \frac{1}{2}\ |BC|\ |AD|$$

$$= \frac{1}{2}\ [C \cos B + b \cos C]\,|AD|$$

$$= \frac{1}{2}\ c \cos B \bullet |AD| + \frac{1}{2}\ b \cos C \bullet |AD|$$

$$= \frac{1}{2}\ c \cos B \bullet c \sin B + \frac{1}{2}\ b \cos C \bullet b \sin C$$

$$= \frac{1}{2}\ C^2 \sin B \cos B + \frac{1}{2}\ b^2 \sin C \cos C$$

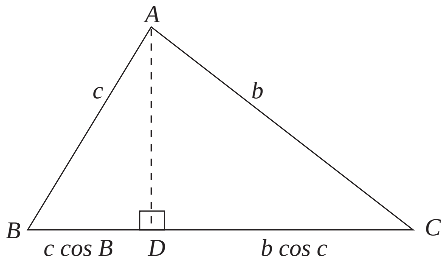
$$= \frac{1}{4}\ C^2 \sin 2B + \frac{1}{4}\ b^2 \sin 2C$$

But, the area of ΔABC is also

$$A = \frac{1}{2}\ bc \sin A$$

Thus, $\dfrac{1}{4}\ b^2 \sin 2C + \dfrac{1}{4}\ C^2 \sin 2B = \dfrac{1}{2}\ bc \sin A$

and $b^2 \sin 2C + C^2 \sin 2B = 2bc \sin A.$



16. **a.** $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)}$$

$$= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

$$= \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}}{1 + \dfrac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

c. $\tan 2A = \tan(A + A)$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$= \frac{2 \tan A}{1 - \tan^2 A}$$

Or, $\tan 2A = \dfrac{\sin 2A}{\cos 2A}$

$$= \frac{2 \sin A \cos A}{\cos^2 A - \sin^2 A}$$

$$= \frac{2 \tan A}{1 - \tan^2 A}.$$

17. $\sin(A + B) = \sin A \cos B + \cos A \sin B$ (1)

$\sin(A - B) = \sin A \cos B - \cos A \sin B$ (2)

Let $A + B = x$ and $A - B = y$.

Thus, $A = \dfrac{x + y}{2}$ and $B = \dfrac{x - y}{2}$.

a. Adding (1) and (2) gives

$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right).$$

b. Subtracting (2) from (1) gives

$$\sin x - \sin y = 2 \cos\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right).$$

18. In any ΔABC:

$A + B + C = \pi$

$C = \pi - (A + B)$

$$\sin\frac{C}{2} = \sin\left(\frac{\pi}{2} - \frac{A + B}{2}\right)$$

$$= \cos\frac{A + B}{2}. \qquad (1)$$

In $\dfrac{2\sin\dfrac{C}{2}\sin\left(\dfrac{A - B}{2}\right)}{\sin B}$

Substitute (1) above to obtain:

$$\frac{2\cos\left(\dfrac{A + B}{2}\right)\sin\left(\dfrac{A - B}{2}\right)}{\sin B}.$$

Using the identity in **17.b.**, this becomes

$$\frac{\sin A - \sin B}{\sin B}$$

$$= \frac{\sin A}{\sin B} - 1.$$

But $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ or $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$.

Then $2\sin\dfrac{c}{2}\sin\left(\dfrac{A - B}{2}\right)$

$$= \frac{a}{b} - 1$$

$$= \frac{a - b}{b}.$$

Exercise A2

1. **l.** $y = 2x^3 \sin x - 3x \cos x$

$$\frac{dy}{dx} = 6x^2 \sin x + 2x^3 \cos x - 3 \cos x + 3x \sin x$$

$$= (6x^2 + 3x) \sin x + (2x^3 - 3) \cos x$$

n. $y = \dfrac{\cos 2x}{x}$

$$\frac{dy}{dx} = \frac{(-2 \sin 2x)(x) - (\cos 2x)(1)}{x^2}$$

$$= \frac{-2x \sin 2x - \cos 2x}{x^2}$$

$y = \cos(\sin 3x)$

o. $y = \cos(\sin 2x)$

$$\frac{dy}{dx} = (-\sin(\sin 2x))(2 \cos 2x)$$

$$= -2 \cos 2x \sin(\sin 2x)$$

q. $y = \tan^2(x^3)$

$$\frac{dy}{dx} = 2 \tan(x^3)(\sec^2 x^3)(3x^2)$$

$$= 6x^2 \sec^2 x^3 \tan x^3$$

r. $y = e^x(\cos x + \sin x)$

$$\frac{dy}{dx} = e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)$$

$$= 2e^x \cos x$$

2. **b.** $f(x) = \tan x,\ x = \frac{\pi}{4}$

The point of contact is $\left(\frac{\pi}{2}, 1\right)$.

The slope of the tangent at any point is $f'(x) = \sec^2 x$.

At $\left(\frac{\pi}{4}, 1\right)$ the slope of the tangent line is $\sec^2 \frac{\pi}{4} = 2$.

An equation of the tangent line is $y - 1 = 2\left(x - \frac{\pi}{4}\right)$.

d. $f(x) = \sin 2x, \cos x,\ x = \frac{\pi}{2}$

The point of contact is $\left(\frac{\pi}{2}, 0\right)$. The slope of the tangent line at any point is $f'(x) = 2 \cos 2x - \sin x$.

At $\left(\frac{\pi}{2}, 0\right)$, the slope of the tangent line is

$2 \cos\pi - \sin \frac{\pi}{2} = -3$.

An equation of the tangent line is $y = -3\left(x - \frac{\pi}{2}\right)$.

e. $f(x) = \cos\left(2x + \frac{\pi}{3}\right),\ x = \frac{\pi}{4}$

The point of contact is $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$. The slope of the tangent line at any point is $f'(x) = -2\sin\left(2x + \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$, the slope of the tangent line is

$-2\sin\left(\frac{5\pi}{6}\right) = -1$.

An equation of the tangent line is

$$y + \frac{\sqrt{3}}{2} = -\left(x - \frac{\pi}{4}\right).$$

3. **b.** $(\cos(x + y))\left(1 + \frac{dy}{dx}\right) = 0$

$$\frac{dy}{dx} = -1$$

c. $2 \sec^2 2x = (-\sin 3y)\left(3\frac{dy}{dx}\right)$

$$\frac{dy}{dx} = -\frac{2 \sec^2 2x}{3 \sin 3y}$$

d. $\frac{dy}{dx} = (-\sin(xy))\left(y + x\frac{dy}{dx}\right)$

$$\frac{dy}{dx}(1 + x \sin(xy)) = -y \sin(xy)$$

$$\frac{dy}{dx} = -\frac{y \sin(xy)}{1 + x \sin(xy)}$$

e. $\sin y + x \cos y \frac{dy}{dx} + (\sin(x + y))\left(1 + \frac{dy}{dx}\right) = 0$

$$\frac{dy}{dx}(x \cos y + \sin(x + y)) = -\sin y - \sin(x + y)$$

$$\frac{dy}{dx} = -\frac{\sin y + \sin(x + y)}{x \cos y + \sin(x + y)}$$

4. $\frac{d}{dx} \cos x = -\sin x$

Consider $f(x) = \cos x$.

$$\text{Thus, } f'(x) = \lim_{h\to0} \frac{\cos(x + h) - \cos x}{h}$$

$$= \lim_{h\to0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \cos x \lim_{h\to0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to0} \frac{\sin h}{h}$$

$$= \cos x \bullet 0 - \sin x \bullet 1$$

$$= -\sin x$$

5. $$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x}$$
$$= \frac{d}{dx}(\sin x)^{-1}$$
$$= -(\sin x)^{-2}\cos x$$
$$= -\frac{\cos x}{\sin^2 x}$$
$$= -\csc x \cot x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x}$$
$$= \frac{d}{dx}(\cos x)^{-1}$$
$$= -(\cos x)^{-2}(-\sin x)$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \sec x \tan x$$

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x}$$
$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{\sin^2 x}$$
$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$
$$= -\frac{1}{\sin^2 x} = -\csc^2 x$$

6. **a.** If x is in degrees, $\lim\limits_{x\to 0}\frac{\sin x}{x} = \frac{\pi}{180}$ and $\lim\limits_{x\to 0}\frac{\cos x - 1}{x} = 0$.

b. If x is in degrees,

$$\frac{d}{dx}\sin x = \frac{\pi}{180}\cos x \text{ and } \frac{d}{dx}\cos x = -\frac{\pi}{180}\sin x.$$

Exercise A3

3. **a.** $y = \cos x + \sin x, \le x \le 2\pi$

We use the Algorithm for Extreme Values.

$f'(x) = -\sin x + \cos x$

Solving $f'(x) = 0$ yields:

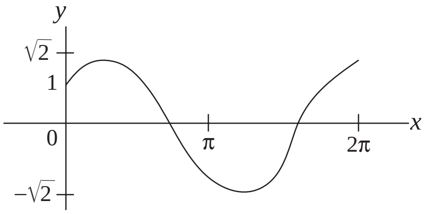
$-\sin x + \cos x = 0$

$\sin x = \cos x$

$\tan x = 1$

$x = \frac{\pi}{4}, \frac{5\pi}{4}.$

x	0	$\frac{\pi}{4}$	$\frac{5\pi}{4}$	2π
$f(x)$	1	$\sqrt{2}$	$-\sqrt{2}$	1



The maximum value is $\sqrt{2}$ when $x = \frac{\pi}{4}$ and the minimum value is $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.

b. $y = x + 2\cos x, -\pi \le x \le \pi$

We use the Algorithm for Extreme Values.

$f'(x) = 1 - 2\sin x$

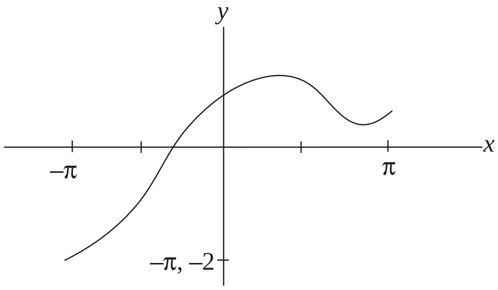
Solving $f'(x) = 0$ yields:

$1 - 2\sin x = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}.$

x	$-\pi$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	π
$f(x)$	–5.14	2.26	0.89	1.14
$f(x)$	$-\pi-2$	$\frac{\pi}{6}+\sqrt{3}$	$\frac{5\pi}{6}-\sqrt{3}$	$\pi - 2$



The maximum value is $\frac{\pi}{6} + \sqrt{3} \doteq 2.26$, when $x = \frac{\pi}{6}$ and the minimum value is $-\pi - 2 \doteq -5.14$ when $x = -\pi$.

4. The velocity of the object at any time t is $v = \frac{ds}{dt}$.

Thus, $v = 8(\cos(10\pi t))(10\pi)$

$= 80\pi \cos(10\pi t)$.

The acceleration at any time t is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Hence, $a = 80\pi\,(-\sin(10\pi t))(10\pi) = -800\pi^2 \sin(10\pi t)$.

Now, $\frac{d^2s}{dt^2} + 100\pi^2 s = -800\pi^2 \sin(10\pi t) + 100\pi^2(8\sin(10\pi t)) = 0$.

5. Since $s = 5\cos\left(2t + \frac{\pi}{4}\right)$, $v = \frac{ds}{dt} = 5\left(-\sin\left(2t + \frac{\pi}{4}\right)\right)$

$$= -10\sin\left(2t + \frac{\pi}{4}\right),$$

$$\text{and } a = \frac{dv}{dt} = -10\left(\cos\left(2t + \frac{\pi}{4}\right)\right)$$

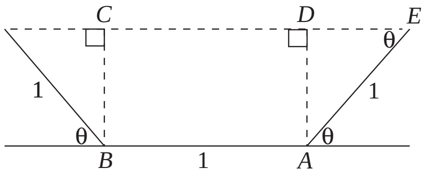
$$= -20\cos\left(2t + \frac{\pi}{4}\right).$$

The maximum values of the displacement, velocity, and acceleration are 5, 10, and 20, respectively.

6. Let $A(\theta)$ be the cross-sectional area when the bending angle is θ radians. We restrict θ to the interval $0 \le \theta \le \frac{\pi}{2}$, because bending past the vertical will reduce the area. Since the channel is symmetrical,

$A(\theta) = \text{Area(rectangle } ABCD + 2 \times \Delta\text{ADE)}$

$= |AB|\,|AD| + |DE|\,|AD|$



From ΔADE, $|AD| = \sin\theta$ and $|DE| = \cos\theta$.

Thus, $A(\theta) = (1)(\sin\theta). + (\cos\theta)(\sin\theta),\ 0 \le \theta \le \frac{\pi}{4}$.

To find the maximum value of A(θ), we apply the Algorithm for Extreme Values:

$A'(\theta) = \cos\theta - \sin^2\theta + \cos^2\theta$

Solving $A'(\theta) = 0$ yields:

$\cos^2\theta - \sin^2\theta + \cos\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$.

Since $0 \le \theta \le \frac{\pi}{2}$, we discard the second case and $\theta = \frac{\pi}{3}$.

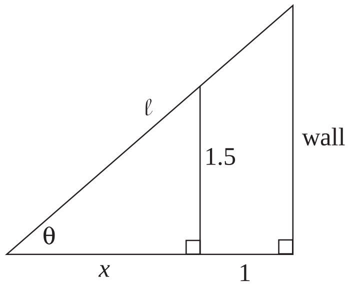
θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$A(\theta)$	0	$\frac{3\sqrt{3}}{4}$	1

Since $\frac{3\sqrt{3}}{4} > 1$, $A(\theta)$ attains its maximum when $\theta = \frac{\pi}{3}$. Thus, a bending angle of $\frac{\pi}{3}$ radians will maximize the cross-sectional area of the channel.

7. Let ℓ be the length of the ladder, θ be the angle between the foot of the ladder and the ground, and x be the distance of the foot of the ladder from the fence, as shown.

Thus, $\frac{x+1}{\ell} = \cos\theta$ and $\frac{1.5}{x} = \tan\theta$

$x + 1 = \ell\cos\theta$ where $x = \frac{1.5}{\tan\theta}$.



Replacing x, $\frac{1.5}{\tan\theta} + 1 = \ell\cos\theta$

$$\ell = \frac{1.5}{\sin\theta} + \frac{1}{\cos\theta},\ 0 < \theta < \frac{\pi}{2}$$

$$\frac{d\ell}{d\theta} = -\frac{1.5\cos\theta}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{-1.5\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta}.$$

Solving $\frac{d\ell}{d\theta} = 0$ yields:

$\sin^3\theta - 1.5\cos^3\theta = 0$

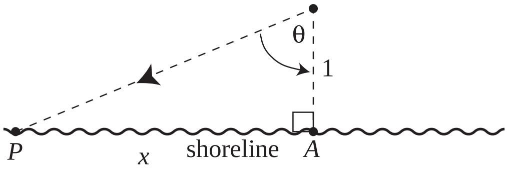
$\tan^3\theta = 1.5$

$\tan\theta = \sqrt[3]{1.5}$

$\theta \doteq 0.46365.$

The length of the ladder corresponding to this value of θ is $\ell \doteq 4.5$ m. As $\theta \to 0^+$ and $\frac{\pi^-}{2}$, ℓ increases without bound. Therefore, the shortest ladder that goes over the fence and reaches the wall has a length of 4.5 m.

8. Let *P*, the point on the shoreline where the light beam hits, be *x* km from *A* at any time *t*, and θ be the angle between the light beam and the line from the lighthouse perpendicular to the shore.



The relationship between *x* and θ is

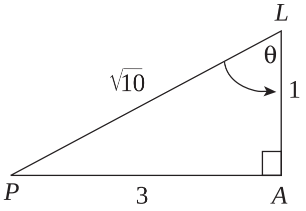
$\tan \theta = \frac{x}{1} = x.$

We differentiate implicitly with respect to *t*:

$$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}.$$

We know $\frac{d\theta}{dt} = \frac{1}{6}(2\pi) = \frac{\pi}{3}$ radians/min,

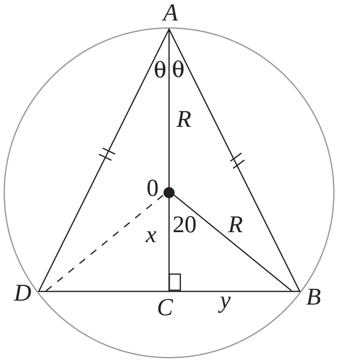
$$\text{so } \frac{dx}{dt} = \frac{\pi}{3}\sec^2\theta.$$



When $x = 3$, $\sec \theta = \sqrt{10}$ and $\frac{dx}{dt} = \frac{10\pi}{3}$.

When the runner is illuminated by the beam of light, the spot is moving along the shore at $\frac{10\pi}{3}$ km/min.

9. Let 0 be the centre of the circle with line segments drawn and labelled, as shown.



In ΔOCB, $\angle COB = 2\theta$.

Thus, $\frac{y}{R} = \sin 2\theta$ and $\frac{x}{R} = \cos 2\theta$,

so $y = R \sin 2\theta$ and $x = R \cos 2\theta$.

The area *A* of ΔABD is

$$A = \frac{1}{2}|DB|\,|AC|$$
$$= y(R + x)$$
$$= R \sin 2\theta\,(R + R \cos 2\theta)$$
$$= R^2(\sin 2\theta + \sin 2\theta \cos 2\theta), \text{ where } 0 < 2\theta < \pi$$

$$\frac{dA}{d\theta} = R^2\,(2\cos 2\theta + 2\cos 2\theta \cos 2\theta + \sin 2\theta(-2 \sin 2\theta)).$$

We solve $\frac{dA}{d\theta} = 0$:

$$2\cos^2 2\theta - 2\sin^2 2\theta + 2\cos 2\theta = 0$$
$$2\cos^2 2\theta + \cos 2\theta - 1 = 0$$
$$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$

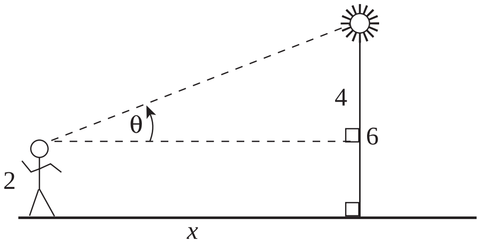
$\cos 2\theta = \frac{1}{2}$ or $\cos 2\theta = -1$

$2\theta = \frac{\pi}{3}$ or $10 = \pi$,

$2\theta = \pi$ (not in domain).

As $2\theta \to 0$, $A \to 0$ and as $2\theta \to \pi$, $A \to 0$. The maximum area of the triangle is $\frac{3\sqrt{3}}{4}R^2$ when $2\theta = \frac{\pi}{3}$, i.e., $\theta = \frac{\pi}{6}$.

10. Let the distance the man is from the street light at any time be x m, and the angle of elevation of the man's line of sight to the light be θ radians, as shown.



The relationship between x and θ is $\tan\theta = \frac{4}{x}$.
We differentiate implicitly with respect to t:

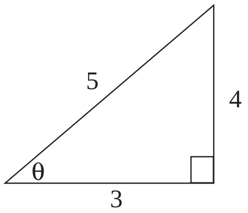
$$\sec^2\theta \bullet \frac{d\theta}{dt} = -\frac{4}{x^2} \bullet \frac{dx}{dt}.$$

Since the man is approaching the light, x is decreasing, so $\frac{dx}{dt} = -2$.

Thus, $\frac{d\theta}{dt} = \frac{8}{\sec^2\theta \bullet x^2} = \frac{8\cos^2\theta}{x^2}$.

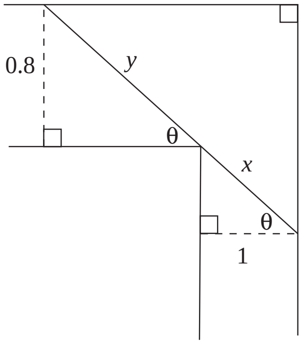
When $x = 3$, $\cos\theta = \frac{3}{5}$

and $\frac{d\theta}{dt} = \frac{8 \bullet \frac{9}{25}}{9} = 0.32$.



When the man is 3 m from the street light, the angle of elevation of his line of sight to the light is increasing at the rate of 0.32 radians/s.

11. The longest pole that can fit around the corner is determined by the minimum value of $x + y$. Thus, we need to find the minimum value of $\ell = x + y$.



From the diagram, $\frac{0.8}{y} = \sin\theta$ and $\frac{1}{x} = \cos\theta$.

Thus, $\ell = \frac{1}{\cos\theta} + \frac{0.8}{\sin\theta}$, $0 \le \theta \le \frac{\pi}{2}$:

$$\frac{d\ell}{d\theta} = \frac{1\sin\theta}{\cos^2\theta} - \frac{0.8\cos\theta}{\sin^2\theta}$$

$$= \frac{0.8\sin^3\theta - \cos^3\theta}{\cos^2\theta\sin^2\theta}.$$

Solving $\frac{d\ell}{d\theta} = 0$ yields:

$$0.8\sin^3\theta - \cos^3\theta = 0$$
$$\tan^3\theta = 1.25$$
$$\tan\theta = \sqrt[3]{1.25}$$
$$\tan\theta \doteq 1.077$$
$$\theta \doteq 0.822.$$

Now, $\ell = \frac{0.8}{\cos(0.822)} + \frac{1}{\sin(0.822)} \doteq 2.5$.

When θ = 0, the longest possible pole would have a length of 0.8 m. When $\theta = \frac{\pi}{2}$, the longest possible pole would have a length of 1 m. Therefore, the longest pole that can be carried horizontally around the corner is one of length 2.5 m.

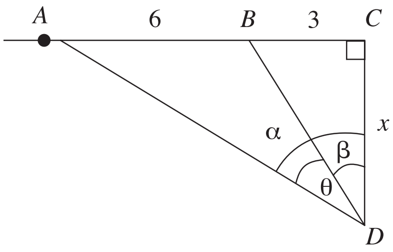
12. We want to find the value of x that maximizes θ.

Let $\angle ADC = \alpha$ and $\angle BDC = \beta$.

Thus, $\theta = \alpha - \beta$:

$$\tan\theta = \tan(\alpha - \beta)$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$



From the diagram, $\tan\alpha = \frac{9}{x}$ and $\tan\beta = \frac{3}{x}$.

$$\text{Hence, } \tan\theta = \frac{\frac{9}{x} - \frac{3}{x}}{1 + \frac{27}{x^2}}$$

$$= \frac{9x - 3x}{x^2 + 27}$$

$$= \frac{6x}{x^2 + 27}.$$

We differentiate implicitly with respect to x:

$$\sec^2\theta \frac{d\theta}{dx} = \frac{6(x^2 + 27) - 6x(2x)}{(x^2 + 27)^2}$$

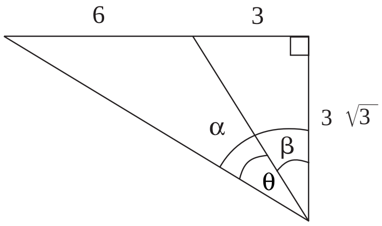
$$\frac{d\theta}{dx} = \frac{162 - 6x^2}{\sec^2\theta(x^2 + 27)^2}$$

Solving $\frac{d\theta}{dx} = 0$ yields:

$$162 - 6x^2 = 0$$

$$x^2 = 27$$

$$x = 3\sqrt{3}.$$



Thus, $\theta = \alpha - \beta$:

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\tan\beta = \frac{1}{\sqrt{3}}$$

$$\beta = \frac{\pi}{6}$$

$$\tan\alpha = \frac{9}{3\sqrt{3}} = \sqrt{3}$$

$$\alpha = \frac{\pi}{3}.$$

As $x \to 0$, both α and β approach $\frac{\pi}{2}$ and $\theta \to 0$.

As $x \to \infty$, both α and β approach 0 and $\theta \to 0$.
Thus, the maximum viewing angle for Paul Kariya is $\frac{\pi}{6}$ or 30°.

Appendix B

Exercise B1

2. **a.** The general antiderivative of $f(x) = 12x^2 - 24x + 1$

is $F(x) = 12\left(\frac{1}{3}x^3\right) - 24\left(\frac{1}{2}x^2\right) + x + C$

$= 4x^3 - 12x^2 + x + C.$

Since $F(1) = -2$, we have:

$F(1) = 4 - 12 + 1 + C = -2.$

Thus, $-7 + C = -2$

$C = 5.$

The specific antiderivative is $F(x) = 4x^3 - 12x^2 + x + 5$.

b. The general antiderivative of $f(x) = 3\sqrt{x} - \sin x$ is

$F(x) = 3\left(\frac{2}{3}x^{\frac{3}{2}}\right) + \cos x + C$

$= 2x^{\frac{3}{2}} + \cos x + C.$

Since $F(0) = 0$, we have:

$F(0) = 0 + 1 + C = 0.$

Thus, $C = -1$ and the specific antiderivative is

$F(x) = 2x^{\frac{3}{2}} + \cos x - 1.$

d. The general antiderivative of $f(x) = e^{3x} - \frac{1}{2x}$ is

$F(x) = \frac{1}{3}e^{3x} - \frac{1}{2}\ln x + C.$

Since $F(1) = e^3$, we have:

$F(1) = \frac{1}{3}e^3 - \frac{1}{2}\ln 1 + C$

$= \frac{1}{3}e^3 - 0 + C = e^3.$

Thus, $C = \frac{2}{3}e^3$ and the specific antiderivative is

$F(x) = \frac{1}{3}e^{3x} - \frac{1}{2}\ln x + \frac{2}{3}e^3.$

e. The general antiderivative of $f(x) = \frac{x^2}{\sqrt{x^3 + 1}}$ is

$F(x) = (2)\left(\frac{1}{3}\right)(x^3 + 1)^{\frac{1}{2}} + C.$

$= \frac{2}{3}(x^3 + 1)^{\frac{1}{2}} + C$

Since $F(0) = 4$, we have:

$F(0) = \frac{2}{3}(0 + 1)^{\frac{1}{2}} + C$

$= \frac{2}{3} + C = 4.$

Thus, $C = \frac{10}{3}$ and the specific antiderivative is

$F(x) = \frac{2}{3}\sqrt{x^3 + 1} + \frac{10}{3}.$

f. The general antiderivative of $f(x) = \cos x \sin^4 x$ is

$F(x) = \frac{1}{5}(\sin x)^5 + C.$

Since $F(0) = -1$, we have:

$F(0) = \frac{1}{5}(\sin 0)^5 + C$

$= C = -1.$

The specific antiderivative is $F(x) = \sin^5 x - 1$.

3. We wish to determine a function $P(t)$ that gives the population at any time t. We are given that the rate of change of the population is $3 + 4t^{\frac{1}{3}}$.

Using $P'(t) = 3 + 4t^{\frac{1}{3}}$, we can determine $P(t)$, the general antiderivative:

$P(t) = 3t + 4\left(\frac{3}{4}t^{\frac{4}{3}}\right) + C$

$= 3t + 3t^{\frac{4}{3}} + C.$

In order to determine the specific population function, we use the fact that the current population is 10 000, i.e., $P(0) = 10\,000$:

$P(0) = 10\,000$

$0 + 0 + C = 10\,000$

$C = 10\,000.$

Thus, the population at any time t is given by

$P(t) = 3t + 3t^{\frac{4}{3}} + 10\,000$. Six months from now the population will be

$P(6) = 3(6) + 3(6)^{\frac{4}{3}} + 10\,000$

$= 10\,081.$

4. We first need to determine a function $V(t)$ that gives the volume of water in the tank at any time t. Since the water is leaking from the tank at the rate of $\frac{t}{50}$ L/min, we have $V'(t) = \frac{t}{50}$.

Thus, $V(t) = -\frac{1}{50}\left(\frac{1}{2}t^2\right) + C$

$= -\frac{1}{100}t^2 + C.$

Since $V = 400$ L at time $t = 0$, $C = 400$.

The volume of water in the tank at any time t is

$V(t) = -\frac{1}{100}t^2 + 400.$

To determine when the tank will be empty, we solve $V(t) = 0$:

$$-\frac{1}{100}t^2 + 400 = 0$$

$$t^2 = 40\,000$$

$$t = 200,\ t \geq 0.$$

The tank will be empty 200 min or 3 h 20 min from the time at which there were 400 L of water in it.

5. Let the measure of the inner radius of a water pipe at any time t be $r(t)$. We are given that $r'(t) = -0.02e^{-0.002t}$ cm/year.

a. Thus, $r(t) = -0.02\left(\frac{1}{-0.002}e^{-0.002t}\right) + C$

$$= 10\,e^{-0.002t} + C.$$

Since $r = 1$ when $t = 0$:

$$1 = 10\,e^{\circ} + C$$

$$= 10 + C$$

$$C = -9.$$

The inner radius of a pipe at any time t is $r(t) = 10e^{-0.002t} - 9$.

b. When $t = 3$ years,

$$r(3) = 10\,e^{(-0.002)(3)} - 9$$

$$= 0.94$$

After three years, the inner radius of a pipe will be 0.94 cm.

c. The pipe will be completely blocked when $r = 0$. To determine when this occurs, we solve $r(t) = 0$:

$$10\,e^{-0.002t} - 9 = 0$$

$$e^{-0.002t} = \frac{9}{10}$$

$$-0.002t = \ln(0.9), \text{ by definition}$$

$$t = \frac{\ln(0.9)}{-0.002}$$

$$= 52.68.$$

The pipe will be completely blocked in approximately 52.7 years from the time when its inner radius is 1 cm.

6. Let the height of the tree at any time t be $h(t)$. We are given that:

$$h'(t) = \frac{20}{t + 30} \text{ m/year.}$$

Thus, $h(t) = 20\ln(t + 30) + C$.

Since $h = 3$ when $t = 0$:

$$3 = 20\ln(30) + C$$

$$C = 3 - 20\ln(30).$$

Ten years later, the height of the tree will be

$$h(10) = 20\ln(40) + 3 - 20\ln(30)$$

$$= 20\ln\left(\frac{40}{30}\right) + 3$$

$$\doteq 8.75 \text{ m.}$$

Exercise B2

1. **b.** We are given $v(t) = s'(t) = 3e^t - \frac{1}{t + 1}$.

Thus, $s(t) = 3e^t - \ln(t + 1) + C$.

Since $s(0) = 2$:

$$3e^{\circ} - \ln(1) + C = 2$$

$$3 - 0 + C = 2$$

$$C = -1.$$

Thus, $s(t) = 3e^t - \ln(t + 1) - 1$.

c. We are given $v(t) = s'(t) = 2[1 - (t + 1)^{-2}]$.

Hence, $s(t) = 2[t - (-(t + 1)^{-1})] + C$

$$= 2\left[t + \frac{1}{t + 1}\right] + C.$$

Since $s(0) = 0$:

$$s\left[0 + \frac{1}{1}\right] + C = 0$$

$$2 + C = 0$$

$$C = -2$$

and $s(t) = 2\left[t + \frac{1}{t + 1}\right] - 2$.

d. We are given $v(t) = s'(t) = 3\cos(\pi t)$

Thus, $s(t) = 3\left(\frac{1}{\pi}\sin(\pi t)\right) + C$

$$= \frac{3}{\pi}\sin(\pi t) + C.$$

Since $s(0) = -1$:

$$\frac{3}{\pi}\sin(0) + C = -1$$

$$C = -1$$

and $s(t) = \frac{3}{\pi}\sin(\pi t) - 1$.

2. **a.** We are given $a(t) = v'(t) = -2$.

Thus, $v(t) = -2t + C_1$.

Since $v(0) = 10$:

$0 + C_1 = 10$.

Hence, $v(t) = s'(t) = -2t + 10$

and $s(t) = -t^2 + 10t + C_2$.

Since $s(0) = 0$, $0 + 0 + C_2 = 0$.

The velocity and position functions are $v(t) = -2t + 10$ and $s(t) = -t^2 + 10t$.

b. We are given that $a(t) = v'(t) = (3t + 1)^{\frac{1}{2}}$.

$$\text{Thus, } v(t) = \frac{2}{3}(3t + 1)^{\frac{3}{2}}\left(\frac{1}{3}\right) + C_1$$

$$= \frac{2}{9}(3t + 1)^{\frac{3}{2}} + C_1.$$

Since $v(0) = 0$:

$$\frac{2}{9}(1)^{\frac{3}{2}} + C_1 = 0$$

$$\frac{2}{9} + C_1 = 0$$

$$C_1 = -\frac{2}{9}.$$

$$\text{Hence, } v(t) = s'(t) = \frac{2}{9}(3t + 1)^{\frac{3}{2}} - \frac{2}{9}$$

$$\text{and } s(t) = \frac{2}{9}\left(\frac{2}{5}(3t + 1)^{\frac{5}{2}}\left(\frac{1}{3}\right)\right) - \frac{2}{9}\,t + C_2$$

$$= \frac{4}{135}(3t + 1)^{\frac{5}{2}} - \frac{2}{9}t + C_2.$$

Since $s(0) = 0$:

$$\frac{4}{135}(1)^{\frac{5}{2}} - 0 + C_2 = 0$$

$$C_2 = -\frac{4}{135}$$

$$\text{and } s(t) = \frac{4}{135}(3t + 1)^{\frac{5}{2}} - \frac{2}{9}t - \frac{4}{135}.$$

Alternate Solution

$$a(t) = v'(t) = 15(t + 1)^{\frac{1}{2}}$$

$$v(t) = 15\left[\frac{2}{3}(t + 1)^{\frac{3}{2}}\right] + C_1$$

$$= 10(t + 1)^{\frac{3}{2}} + C_1$$

Since $v(0) = 0$:

$$10(1)^{\frac{3}{2}} + C_1 = 0$$

$$C_1 = -10$$

$$v(t) = s'(t) = 10(t + 1)^{\frac{3}{2}} - 10.$$

$$s(t) = 10\left[\frac{2}{5}(t + 1)^{\frac{5}{2}}\right] - 10t + C_2$$

$$= 4(t + 1)^{\frac{5}{2}} - 10t + C_2.$$

Since $s(0) = 0$:

$$4(1)^{\frac{5}{2}} - 0 + C_2 = 0$$

$$C_2 = -4$$

and $s(t) = 4(t + 1)^{\frac{5}{2}} - 10t - 4$.

c. We are given that $a(t) = v'(t) = \cos(t) + \sin(t)$.

Thus, $v(t) = \sin(t) - \cos(t) + C_1$.

Since $v(0) = 3$:

$$\sin(0) - \cos(0) + C_1 = 3$$

$$0 - 1 + C_1 = 3$$

$$C_1 = 4.$$

Hence, $v(t) = s'(t) = \sin(t) - \cos(t) + 4$

and $s(t) = -\cos(t) - \sin(t) + 4t + C_2$.

Since $s(0) = 0$:

$$-\cos(0) - \sin(0) + 4(0) + C_2 = 0$$

$$-1 + C_2 = 0$$

$$C_2 = 1$$

and $s(t) = -\cos(t) - \sin(t) + 4t + 1$.

d. We are given that $a(t) = v'(t) = 4(1 + 2t)^{-2}$.

$$\text{Thus, } v(t) = 4\left[-\frac{1}{2}(1 + 2t)^{-1}\right] + C_1$$

$$= -2(1 + 2t)^{-1} + C_1.$$

Since $v(0) = 0$, $-2(1)^{-1} + C_1 = 0$

$$C_1 = 2.$$

$$\text{Hence, } v(t) = s'(t) = -\frac{2}{1 + 2t} + 2$$

$$\text{and } s(t) = -2\left[\frac{1}{2}\ln(1 + 2t)\right] + 2t + C_2$$

$$= -\ln(1 + 2t) + 2t + C_2.$$

Since $s(0) = 8$:

$$-\ln(1) + 0 + C_2 = 8$$

$$C_2 = 8.$$

Thus, $s(t) = -\ln(1 + 2t) + 2t + 8$.

3. **a.** Let the position of the stone above ground at any time t be $s(t)$. Since the only acceleration is due to the force of gravity, we know:

$a(t) = v'(t) = -9.81$.

Thus, $v(t) = -9.81t + C_1$.

Since the stone is dropped, we know the initial velocity is 0 m/s:

$$v(0) = 0$$

$$-9.81(0) + C_1 = 0$$

$$C_1 = 0.$$

Hence, the velocity of the stone at any time t is $v(t) = –9.81t$. The position of the stone at any time t is the antiderivate of $v(t)$.

So, $s(t) = –9.81\left(\frac{t^2}{2}\right) + C_2$.

Since the stone is dropped from a height of 450 m, we know:

$$s(0) = 450$$
$$0 + C_2 = 450.$$

The position of the stone at any time t is $s(t) = –4.905t^2 + 450$.

b. To determine when the stone reaches the ground, we solve $s(t) = 0$.

$$–4.905t^2 + 450 = 0$$
$$t^2 = 91.74$$
$$t = \pm 9.58.$$

It takes approximately 9.58 s for the stone to reach the ground.

c. The approximate velocity of the stone when it strikes the ground is

$$v(9.58) = –9.81(9.58)$$
$$= –94 \text{ m/s}.$$

4. **a.** From **3. a.**, $v(t) = –9.81t + C_1$.

The initial velocity is –10 m/s:

$$v(0) = –10$$
$$0 + C_1 = –10$$
$$C_1 = –10.$$

Hence, the velocity of the stone at any time t is $v(t) = –9.81t – 10$. The position of the stone at any time t is the antiderivative of $v(t)$.

Thus, $s(t) = –9.81\left(\frac{t^2}{2}\right) – 10t + C_2$.

Since $s(0) = 450$:

$$0 + 0 + C_2 = 450$$

and $s(t) = –4.905t^2 – 10t + 450$.

b. To determine when the stone reaches the ground, we solve:

$$s(t) = 0$$
$$–4.905t^2 – 10t + 450 = 0$$
$$4.905t^2 + 10t – 450 = 0$$
$$t = –10 \pm \frac{\sqrt{100 – 4(4.905)(–450)}}{9.81}$$
$$\doteq 8.6, –10.7.$$

It takes approximately 8.6 s for the stone to reach the ground.

c. The approximate velocity of the stone when it strikes the ground is

$$v(8.6) = –9.81(8.6) – 10$$
$$= –94.4 \text{ m/s}.$$

5. **a.** From **3. a.**, $v(t) = –9.81t + C_1$.

The initial velocity is 10 m/s:

$$v(0) = 10$$
$$0 + C_1 = 10$$
$$C_1 = 10.$$

The velocity of the stone at any time t is $v(t) = –9.81t + 10$. The position of the stone at any time t is the antiderivative of $v(t)$.

Thus, $s(t) = –4.905t^2 + 10t + C_2$

Since $s(0) = 450$:

$$0 + 0 + C_2 = 450$$
$$C_2 = 450$$

and $s(t) = –4.905t^2 + 10t + 450$.

b. To determine when the stone reaches the ground, we solve:

$$s(t) = 0$$
$$–4.905t^2 + 10t + 450 = 0$$
$$t = –10 \pm \frac{\sqrt{100 – 4(–4.905)(450)}}{9.81}$$
$$\doteq 10.7, 8.6.$$

It takes approximately 10.7 s for the stone to reach the ground.

c. The approximate velocity of the stone when it strikes the ground is

$$v(10.7) = –9.81(10.7) + 10$$
$$\doteq –95 \text{ m/s}.$$

6. Let the constant acceleration of the airplane be a m/s^2. The velocity of the airplane at any time t is $v(t) = at + C_1$.

Since the airplane starts from rest:

$$v(0) = 0$$
$$0 + C_1 = 0$$
$$C_1 = 0.$$

The velocity of the airplane at any time t is $v(t) = at$.

The position of the plane at any time t is the antiderivative of $v(t)$.

Thus, $s(t) = a\frac{t^2}{2} + C_2$.

We know that $s(0) = 0$.

Thus, $0 + C_2 = 0$

$C_2 = 0$.

The position of the airplane at any time t is

$s(t) = a\frac{t^2}{2}$.

Let the elapsed time from start to liftoff be T.

Thus, $v(T) = aT = 28$

and $s(T) = a\frac{T^2}{2} = 300$.

Solving for T yields $T = \frac{600}{28} = \frac{150}{7}$ s.

The constant acceleration of the airplane is

$$a = \frac{28}{\frac{150}{7}} \doteq 1.3 \text{ m/s}^2.$$

7. First, change 80 km/h to $\frac{80\,000}{3600} = 22.2$ m/s and

100 km/h $= \frac{100\,000}{3600}$

$= 27.2$ m/s.

Let acceleration be a.

Therefore, $v(t) = at + C$.

When $t = 0$, $v = 22.2$, therefore, $22.2 = 0 + C$

or $C = 22.2$

$v(t) = at + 22.2$.

When $t = 5$, $v = 27.2$, therefore, $27.2 = 5a + 22.2$

$a = 1.12$.

The acceleration is 1.1 m/s^2.

8. We are given that the acceleration of the car is $a(t) = -10$ m/s^2.

Thus, the velocity of the car during the braking period is $v(t) = -10t + C_1$.

The distance that the car travels during the braking interval is $s(t) = -5t^2 + C_1t + C_2$.

Since $s(0) = 0$:

$0 + 0 + C_2 = 0$

$C_2 = 0$.

Let the time it takes to stop after applying the brakes be T. We know that $v(T) = 0$.

Thus, $-10T + C_1 = 0$

$C_1 = 10T$.

Since the braking distance is 50 m, $s(T) = 50$.

Thus, $-5T^2 + C_1T = 50$

Since $C_1 = 10T$, $-5T^2 + 10T^2 = 50$

$5T^2 = 50$

$T^2 = 10$

$T = \sqrt{10}$.

The velocity of the car when the brakes were first applied is $v(0) = C_1 = 10\sqrt{10}$

$\doteq 32$ m/s.

9. Let the position the stone is above ground at any time be $s(t)$. The acceleration of the stone due to gravity is $a(t) = -9.81$.

Thus, $v(t) = -9.81t + C_1$.

Since the stone is dropped from rest,

$v(0) = 0$

$0 + C_1 = 0$

$C_1 = 0$.

Thus, $v(t) = -9.81t$.

Now, $s(t) = -4.905t^2 + C_2$.

The initial position of the stone is $s(0) = C_2$.

Thus, the height of the building is C_2.

Let the time it takes for the stone to reach the ground be T s.

We are given $v(T) = -50$ m/s.

Thus, $-9.81T = -50$

$T = \frac{50}{9.81}$.

We also know that $s(T) = 0$.

Hence, $-4.905T^2 + C_2 = 0$

$C_2 = 4.905\left(\frac{50}{9.81}\right)^2$

$\doteq 127$.

The height of the building is approximately 127 m.

Exercise B3

1. **a.** Let P represent the population of the bacteria culture after t hours. We are given that

$\frac{dP}{dt} = kP$, where $k > 0$.

The population at any time t is given by $P(t) = Ce^{kt}$.

We know $P(0) = 200$, so $200 = Ce^{k(0)} = C$.

The population function is $P(t) = 200e^{kt}$.

We also know that $P\left(\frac{1}{2}\right) = 600$,

$$\begin{aligned} \text{so } 600 &= 200e^{\frac{k}{2}} \\ e^{\frac{k}{2}} &= 3 \\ \frac{k}{2} &= \ln(3) \\ k &= 2\ln(3) \\ &\doteq 2.2. \end{aligned}$$

Hence, the population at any time t is given by $P(t) = 200e^{2.2t}$.

b. After 20 minutes, the population is

$$\begin{aligned} P\left(\frac{1}{3}\right) &= 200e^{(2.2)\left(\frac{1}{3}\right)} \\ &\doteq 416. \end{aligned}$$

c. We solve:

$$\begin{aligned} P(t) &= 10\,000 \\ 200e^{2.2t} &= 10\,000 \\ e^{2.2t} &= 50 \\ 2.2t &= \ln(50) \\ t &= \frac{\ln(50)}{2.2} \\ &\doteq 1.8. \end{aligned}$$

The population will be 10 000 after 1.8 h.

2. **a.** Let P represent the population at any time t after 1980.

We are given that $\frac{dP}{dt} = kP$, where $k > 0$.

The population at any time t is $P(t) = Ce^{kt}$.

We know that $P(0) = 150\,000$,

so $150\,000 = Ce^{k(0)} = C$.

The population function is $P(t) = 150\,000e^{kt}$.

We also know that $P(20) = 250\,000$,

$$\begin{aligned} \text{so } 250\,000 &= 150\,000e^{20k} \\ e^{20k} &= \frac{5}{3} \\ 20k &= \ln\left(\frac{5}{3}\right) \\ k &= \frac{\ln\left(\frac{5}{3}\right)}{20} \\ &\doteq 0.026. \end{aligned}$$

Hence, the population at any time t after 1980 is $P(t) = 150\,000e^{0.026t}$.

b. In 2010, $t = 30$

$$\begin{aligned} P(30) &= 150\,000e^{(0.026)(30)} \\ &\doteq 327\,221. \end{aligned}$$

3. **a.** Let P be the amount of Polonium–210 present at any time t.

We are given that $\frac{dP}{dt} = kP$, where $k < 0$.

The half-life of Polonium–210 present at any time is $P(t) = Ce^{kt}$.

We know that $P(0) = 200$, so $200 = Ce^{k(0)} = C$.

The half-life of Polonium–210 is 140 days, $P(140) = 100$.

Thus, $100 = 200e^{140k}$

$$\begin{aligned} e^{140k} &= \frac{1}{2} \\ 140k &= \ln(0.5) \\ k &= \frac{\ln(0.5)}{140} \\ &\doteq -0.005. \end{aligned}$$

Hence, the mass of Polonium–210 remaining after t days is given by $P(t) = 200e^{-0.005t}$.

b. The amount of Polonium–210 remaining after 50 days is

$$\begin{aligned} P(50) &= 200e^{-0.005(50)} \\ &\doteq 156 \text{ mg}. \end{aligned}$$

c. We want to determine the number of days it takes for the mass of Polonium–210 remaining to be 5 mg.

We solve: $5 = 200e^{-0.005t}$

$$e^{-0.005t} = \frac{5}{200}$$

$$-0.005t = 1\text{n}(0.025)$$

$$t = \frac{\ln(0.025)}{-0.005}$$

$$\cong 738.$$

It takes approximately 738 days for 200 mg of Polonium–210 to decay to 5 mg.

4. Let P be the population of Central America at any time t.

We are told that $\frac{dP}{dt} = 0.035y$.

Thus, the population at any time t is $P(t) = Ce^{0.035t}$.

Let the population be P_0 at a given starting point in time.

Hence, we know $P(0) = P_0$, so $P_0 = Ce^{0.035(0)} = C$ and the population function becomes $P(t) = P_0e^{0.035t}$.

We want to find the value of t so that $P(t) = 2P_0$ doubles the initial population.

Thus, $2P_0 = P_0e^{0.035t}$

$$e^{0.035t} = 2$$

$$0.035t = \ln(2)$$

$$t = \frac{\ln(2)}{0.035}$$

$$\doteq 19.8.$$

It takes approximately 20 years for the population of Central America to double.

5. Since the town has a limiting population of 16 000, the population at any time t is represented by the Logistic Model. The differential equation satisfied by the population P is

$$\frac{dP}{dt} = kP(16\,000 - P).$$

The solution to this differential equation is

$$P(t) = \frac{16\,000}{1 + Ce^{16\,000kt}}.$$

Using 1950 as our starting point in time, $P(0) = 10\,000$.

Thus, $10\,000 = \frac{16\,000}{1 + Ce^{16\,000k(0)}}$

$$1 + C = 1.6$$

$$C = 0.6.$$

The population equation becomes

$$P(t) = \frac{16\,000}{1 + 0.6e^{16\,000kt}}$$

We also know that $P(20) = 12\,000$.

Hence, $12\,000 = \frac{16\,000}{1 + 0.6e^{16\,000k(20)}}$

$$1 + 0.6e^{320\,000k} = \frac{4}{3}$$

$$e^{320\,000k} = \frac{\frac{1}{3}}{0.6}$$

$$320\,000k = \ln\left(\frac{1}{1.8}\right)$$

$$k = \frac{\ln\left(\frac{1}{1.8}\right)}{320\,000}$$

$$\doteq -1.8368 \times 10^{-6}.$$

The population equation is $P(t) = \frac{16\,000}{1 + 0.6e^{-0.029389t}}$

In 2005, the population of the town will be

$$P(55) = \frac{16\,000}{1 + 0.6e^{-0.029389(55)}}$$

$$\doteq 14\,296.$$

6. Since Easter Island has a carrying capacity of 25 000 rabbits, the rabbit population at any time t is given by the Logistic Model. The population equation is

$$P(t) = \frac{25\,000}{1 + Ce^{25\,000kt}}.$$

Using 1995 as the starting time, $P(0) = 20\,000$.

Thus, $20\,000 = \frac{25\,000}{1 + Ce^{25\,000k(0)}}$

$$1 + C = 1.25$$

$$C = 0.25.$$

The population equation becomes

$$P(t) = \frac{25\,000}{1 + 0.25e^{25\,000kt}}.$$

We also know that $P(3) = 22\,000$.

Hence, $22\,000 = \frac{25\,000}{1 + 0.25e^{75\,000k}}$

$$1 + 0.25e^{75\,000k} = \frac{25}{22}$$

$$e^{75\,000k} = \frac{\frac{3}{22}}{0.25} = \frac{6}{11}$$

$$7500k = \ln\left(\frac{6}{11}\right)$$

$$25\ 000k = \frac{1}{3}\ln\left(\frac{6}{11}\right).$$

Note: The population equation has the value of 25 000k as part of the exponent of e. As such, we can use this value which does not require us to find the value of k. The rabbit population of Easter Island at

any time t is $P(t) = \dfrac{25\ 000}{1 + 0.25e^{\frac{1}{3}\ln\left(\frac{6}{11}\right)t}}$.

7. Using the given information, the differential equation that describes the temperature of the potato at any time t is

$$\frac{dT}{dt} = k(T - 20).$$

The general solution is $T(t) = 20 + Ce^{kt}$.

Since the initial temperature is 80°C, we know $T(0) = 80$.

Thus, $80 = 20 + Ce^{k(0)}$

$$C = 60.$$

The temperature function becomes $T(t) = 20 + 60e^{kt}$.

We also know that $T(15) = 40$.

Hence, $40 = 20 + 60e^{15k}$

$$e^{15k} = \frac{1}{3}$$

$$15k = \ln\left(\frac{1}{3}\right)$$

$$k = \frac{1}{15}\ln\left(\frac{1}{3}\right).$$

The temperature function is $T(t) = 20 + 60e^{\frac{1}{15}\ln\left(\frac{1}{3}\right)t}$

To find how long the restaurant server has to get the potato to a customer's table, we solve $T(t) = 50$:

$$50 = 20 + 60e^{\frac{1}{15}\ln\left(\frac{1}{3}\right)t}$$

$$e^{\frac{1}{15}\ln\left(\frac{1}{3}\right)t} = \frac{1}{2}$$

$$\frac{1}{15}\ln\left(\frac{1}{3}\right)t = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{15\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)} \doteq 9.46.$$

The server has about 9.5 min.

8. The temperature of the coil at any time t is $T(t) = 27 + Ce^{kt}$.

We know that $T(0) = 684$.

Thus, $684 = 27 + Ce^{k(0)}$

$$C = 657.$$

The temperature function becomes $T(t) = 27 + 657e^{kt}$.

We also know that $T(4) = 246$.

Hence, $246 = 27 + 657e^{4k}$

$$e^{4k} = \frac{1}{3}$$

$$4k = \ln\left(\frac{1}{3}\right)$$

$$k = \frac{1}{4}\ln\left(\frac{1}{3}\right).$$

The temperature function is $T(t) = 27 + 657e^{\frac{1}{4}\ln\left(\frac{1}{3}\right)t}$.

To find out how long it will take the coil to cool to a temperature of 100°C, we solve:

$$T(t) = 100$$

$$100 = 27 + 657e^{\frac{1}{4}\ln\left(\frac{1}{3}\right)t}$$

$$e^{\frac{1}{4}\ln\left(\frac{1}{3}\right)t} = \frac{1}{9}$$

$$\frac{1}{4}\ln\left(\frac{1}{3}\right)t = \ln\left(\frac{1}{9}\right)$$

$$t = \frac{4\ln\left(\frac{1}{9}\right)}{\ln\left(\frac{1}{3}\right)} = 8.$$

It will take 8 min for the coil to cool from 684°C to 100°C.

Answers

CHAPTER 1 POLYNOMIAL FUNCTIONS

Review of Prerequisite Skills

1. a. $(P + r)^2$ **b.** $(4n + 1)^2$ **c.** $(3u + 5)^2$ **d.** $(v + 3)(v + 1)$
e. $(2w + 1)(w + 1)$ **f.** $(3k + 1)(k + 2)$ **g.** $(7y + 1)(y + 2)$
h. $(5x - 1)(x - 3)$ **i.** $(3v - 5)(v - 2)$

2. a. $(5x - y)(5x + y)$ **b.** $(m - p)(m + p)$ **c.** $(1 - 4r)(1 + 4r)$
d. $(7m - 8)(7m + 8)$ **e.** $(pr - 10x)(pr + 10x)$
f. $3(1 - 4y)(1 + 4y)$ **g.** $(x + n + 3)(x + n - 3)$
h. $(7u + x - y)(7u - x + y)$ **i.** $(x^2 + 4)(x + 2)(x - 2)$

3. a. $(k + p)(x - y)$ **b.** $(f + g)(x - y)$ **c.** $(h + 1)(h^2 + 1)$
d. $(x - d)(1 + x - d)$ **e.** $(2y + z - 1)(2y + z + 1)$
f. $(x - z - y)(x - z + y)$

4. a. $2(2x + 3)(x - 1)$ **b.** $4(7s - 5t)(s + t)$
c. $(y + r - n)(y - r + n)$ **d.** $8(1 + 5m)(1 - 2m)$
e. $(3x - 2)(2x - 3)$ **f.** $(y + 1)(y^2 - 5)$ **g.** $10(3y + 4)(2y - 3)$
h. $2(5x^2 + 19x + 10)$ **i.** $3(3x - 4)(3x + 4)$

5. a. $(12x + 4y - 5u)(12x - 16y + 5u)$ **b.** $g(1 - x)(1 + x)$
c. $(y - 1)(y^4 + y^2 + 1)$ **d.** $(n^2 + w^2)^2$
e. $(-x + 14y - z)(7x - 2y + 7z)$ **f.** $(u + 1)(4u + 3)(2u - 1)$
g. $(p - 1 + y + z)(p - 1 - y - z)$ **h.** $(3y^2 + 2)^2$
i. $(ax + m)(bx + n)$ **j.** $\left(x + \frac{1}{x}\right)^2$

Exercise 1.2

1. $f(x) = x^2 - 5x + 4$
2. $f(x) = 3x - 4$
3. $f(x) = 2x^2 + 5x - 3$
4. $f(x) = 2x^2 - 7x - 4$
5. $f(x) = 2x^3 - 5x^2 - 21x + 36$
6. $f(x) = x^3 - 15x - 20$
7. $f(x) = x^3 - x^2 - 14x + 24$
8. $f(x) = 2x^3 + x^2 - 13x + 6$
9. $f(x) = x^4 - 10x^3 + 35x^2 - 52x + 24$
10. $f(x) = 2^{x-1}$
11. a. $V = -0.0374t^3 + 0.1522t^2 + 0.1729t$
b. maximum volume of 0.8863 L at 3.2 s
12. a. $f(t) = t^3 - 27t^2 + 3t + 403t$ **b.** 1999 **c.** 57 000

Exercise 1.3

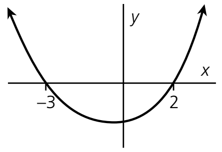
1. a. $17 = 5(3) + 2$ **b.** $42 = 7(6) + 0$ **c.** $73 = 12(6) + 1$
d. $90 = 6(15) + 0$ **e.** $103 = 10(10) + 3$ **f.** $75 = 15(5) + 0$
2. a. The remainder is not zero. **b.** The remainder is zero.
c. possible solution from Question 1: **1.d.** 15; **1.f.** 5
d. 15 **f.** 5
3. The dividend equals the product of the divisor and the quotient added to the remainder of the division.
4. a. $x - 2$ **b.** $x^2 + 3x - 2$ **c.** 5 **d.** $x^3 + x^2 - 8x + 9$
5. $f(x) = 3x^2 - 8x^2 + 8x + 26$
6. $f(x) = x^4 + x^2$
7. a. $x^3 - 3x^2 + x + 2 = (x + 2)(x^2 - 5x + 11) - 20$
b. $x^3 + 4x^2 - 3x - 2 = (x - 1)(x^2 + 5x + 2)$
c. $2x^3 - 4x^2 - 3x + 5 = (x - 3)(2x^2 + 2x + 3) + 14$
d. $3x^3 + x^2 - x - 6 = (x + 1)(3x^2 - 2x + 1) - 7$
e. $3x^3 - 4 = (x - 4)(3x + 12) + 44$
f. $x^3 - 2x - 4 = (x - 2)(x^2 + 2x + 2)$
g. $4x^3 + 6x^2 - 6x - 9 = (2x + 3)(2x^2 - 3)$
h. $3x^3 - 11x^2 + 21x - 7 = (3x - 2)(x^2 - 3x + 5) + 3$
i. $(3x - 2)(2x^2 - 1) + 7$
j. $3x^3 + 7x^2 + 5x + 1 = (3x + 1)(x^2 + 2x + 1)$
8. a. No. **b.** Yes. **c.** No. **d.** No. **e.** Yes. **f.** Yes. **g.** Yes. **h.** No.
i. No. **j.** Yes. The degree of the remainder is less.
9. a. $x^3 + 3x^2 + 14x + 53, R = 220$
b. $2x^3 - 2x^2 - x + 1$ **c.** $4x^2 - 8x + 16$
d. $x^4 + x^3 + x^2 + x + 1$
10. $x + 6, x - 1$
11. $x^2 - x - 1$ with $R = -5$
12. $x^2 - x$
13. $x^2 + 3x + 2$
14. $r(x) = 0$
15. 0, 1
16. a. $r = 0$ **b.** 1, 2, 3, 4; 1, 2, 3, 4, 5, 6; 1, 2, 3, ..., $n - 1$
17. a. $x^3 + 4x^2 - 5x - 9 = (x - 2)(x\mathrm{v} + 6x + 7) + 5$
$x^2 + 6x + 7 = (x + 1)(x + 5) + 2$ **b.** Yes.
c. $r = r_1 + (x - 2)r_2$ or $r_2x + (r_1 - 2r_2)$

Exercise 1.4

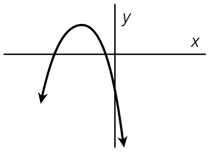
1. Find $f(1)$.
2. a. -10 **b.** -13 **c.** $-\frac{47}{8}$ or 5.875 **d.** $\frac{-171}{8}$ or -21.375
3. a. 12 **b.** 3 **c.** -25 **d.** 1 **e.** 17 **f.** -16
4. a. -2 **b.** 58 **c.** 13 **d.** 0 **e.** 11 **f.** -5 **g.** 1 **h.** 3
5. a. 4 **b.** 2 **c.** 5
6. $m = 2, g = 1$
7. $m = -\frac{4}{9}, g = \frac{13}{9}$
8. $24x + 73$
9. $42x - 39$
10. a. 4 **b.** 3 **c.** 2 **d.** -1 **e.** 9
11. $f(x) + 2$
12. a. $(x^2 + x + 3)(x^2 - x + 3)$ **b.** $(3y^2 + 2y + 2)(3y^2 - 2y + 2)$
c. $(x^2 + 2x + 5)(x^2 - 2x + 5)$
d. $(2x^2 + 2x + 3)(2x^2 - 2x + 3)$

Review Exercise

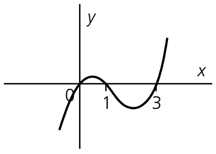
1. a.



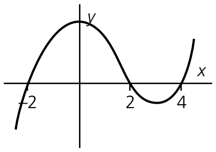
b.



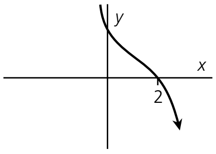
c.



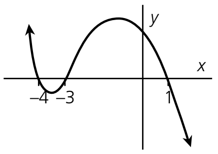
d.



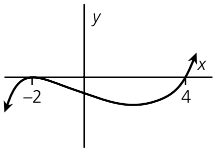
e.



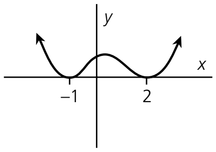
f.



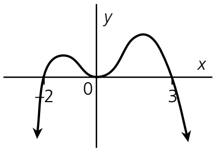
g.



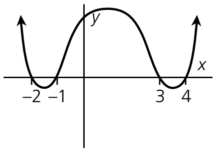
h.



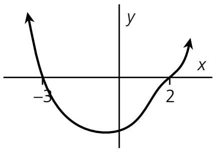
i.



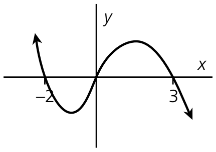
j.



k.



l.



2. a. $f(x) = x^3 - 5x^2 + 10x - 11$
b. $f(x) = 2x^3 - 3x^2 + 12x + 4$
c. $f(x) = x^4 - 14x^2 + 5x - 1$
d. not enough information given
e. not enough information given

3. a. $x^3 - 2x^2 + 3x - 1 = (x - 3)(x^2 + x + 6) + 17$
b. $2x^3 + 5x + 4 = (x + 2)(2x^2 - 4x + 13) - 22$
c. $4x^3 + 8x^2 - x + 1 = (2x + 1)(2x^2 + 3x - 2) + 3$
d. $x^4 - 4x^3 + 3x^2 - 3 = (x^2 + x - 2)(x^2 - 5x + 10) - 20x + 17$

4. a. 3 **b.** 1 **c.** −33 **d.** −1 **e.** $\frac{22}{9}$

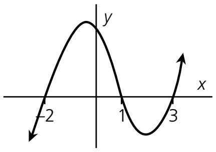
5. a. $x^3 + 2x^2 - x + 2 = (x - 1)(x + 1)(x + 2)$
b. $x^3 - 3x^2 - x + 3 = (x - 3)(x - 1)(x + 1)$
c. $6x^3 + 31x^2 + 25x - 12 = (2x + 3)(3x - 1)(x + 4)$

6. a. $k = \frac{1}{2}$ **b.** $r = 2$, $g = 5$

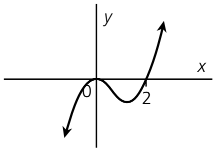
Chapter 1 Test

1. a. $2(3x - 56)(3x + 56)$ **b.** $(pm + 1)(m^2 + 1)$
c. $2(3x - 2)(2x - 3)$ **d.** $(x + y - 3)(x - y + 3)$

2. a.



b.



3. a. $q(x) = x^2 - 7x + 20$ $r(x) = -44$
b. $q(x) = x^2 + 3x + 3$ $r(x) = 11$

4. Yes.

5. −40

6. $k = 3$

7. a. Yes. **b.** $f(x) = 2x^3 - 3x^2 + 5x - 8$

8. $c = \frac{-14}{3}$, $d = -\frac{5}{3}$

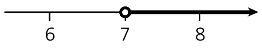
9. $(x - 3)(x + 3)$

CHAPTER 2 POLYNOMIAL EQUATIONS AND INEQUALITIES

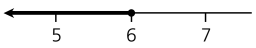
Review of Prerequisite Skills

1. a. −3 **b.** no solution **c.** $\frac{11}{4}$ or 2.75 **d.** 1

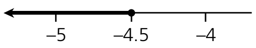
2. a. $x > 7$



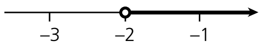
b. $x \le 6$



c. $x \le -4.5$



d. $x > -2$



3. a. 0 **b.** 15 **c.** 10 **d.** 0

4. a. −2 **b.** 13 **c.** −52 **d.** $\frac{53}{8}$

5. a. $(x - 6)(x - 8)$ **b.** $(y - 2)(y - 1)$ **c.** $(3x - 7)(x - 1)$
d. $3(x - 5)(x + 5)$ **e.** $(3x - 1)(2x + 3)$ **f.** $x(x + 8)(x - 7)$
g. $4x(x + 5)$ **h.** $3x(x - 2)(x + 2)$ **i.** $2(3x + 2)(x - 3)$

6. a. 0, 4 **b.** 3, −2 **c.** −3, −2 **d.** −6, −3 **e.** 5, −3 **f.** −1, $\frac{4}{7}$
g. 1, $\frac{7}{3}$ **h.** −3, 0, 3 **i.** $\frac{1}{3}$, 4

7. a. 1.5, −5.5 **b.** 2.3, −0.6 **c.** $\frac{-1 \pm i\sqrt{35}}{6}$ **d.** 5.7, −0.7
e. 3, −0.5 **f.** 1.5, −0.7 **g.** $\frac{3 \pm i\sqrt{31}}{4}$ **h.** −6, 1 **i.** 8.3, 0.7

Exercise 2.1

1. 0

2. a. $(x - 5)$ **b.** Divide.

3. $(x + 1)$, $(x - 2)$, $(x + 3)$

4. a. Yes. **b.** No. **c.** Yes. **d.** No. **e.** No. **f.** Yes.

5. b. $x - 3$ **c.** $x^2 + x + 1$

6. b. $x + 2$ **c.** $x^2 - 4x + 3$

7. a. $(x - 1)(x^2 + x - 3)$ **b.** $(x + 2)(x - 1)(x + 1)$
c. $(y - 1)(y^2 + 20y + 1)$ **d.** $(x + 1)(x^2 + x + 4)$
e. $(y - 2)(y^2 + y + 1)$ **f.** $(x - 4)(x^2 - 5x + 2)$
g. $(x + 2)(x - 3)(x^2 - 7x + 2)$ **h.** $(x + 2)(x - 8)(x^2 + 1)$

8. 2.5

9. 1.5

10. a. $(x - 3)(x^2 + 3x + 9)$ **b.** $(y + 2)(y^2 - 2y + 4)$
c. $(5u - 4r)(25u^2 + 20ur + 16r^2)$
d. $2(10w + y)(100w - 10wy + y^2)$
e. $(x + y - uz)(x^2 + 2xy + y^2 + xuz + yuz + u^2z^2)$
f. $(5)(u - 4x - 2y)(u^2 + 4ux + 2uy + 16x^2 + 16xy + 4y^2)$

12. b. $x^3 + x^2y + xy^2 + y^3$ **c.** $(x - 3)(x^3 + 3x^2 + 9x + 27)$

13. b. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$
c. $(x - 2)(x^4 - 2x^3 + 4x^2 - 8x + 16)$

14. b. $x^{n-1} + x^{n-2}y + x^{n-3}y^2 + ... + y^{n-1}$

17. If n is odd.

18. $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

19. No.

Exercise 2.2

1. a. $\pm\frac{1}{2}, \pm\frac{5}{2}, \pm 1, \pm 5$ **b.** $-\frac{1}{3}, \frac{2}{3}$ **c.** $\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}$
d. $\pm 1, \pm 2, \pm 4, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$ **e.** $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}$ **f.** $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$

2. $5(2x - 3)(x - 2)$

3. $-2(x - 3)(4x + 3)(x + 2)$

4. a. $(2x-1)(x^2+x+1)$ **b.** $(x+2)(x-1)(5x-2)$
c. $(x-2)(2x-1)(3x-1)$ **d.** $(x+3)(2x-5)(3x-1)$
e. $(x+2)(x-2)(5x^2+x-2)$ **f.** $(3x+1)(2x-1)(3x-2)$
g. $(x-2)(3x-2)(x^2+x+1)$ **h.** $(x-4)(4x-3)(x^2+1)$
5. a. $(x+2)(px^2-(p+q)\times 7q)$ **b.** $(x-1)(ax-2)(bx+1)$

Exercise 2.3

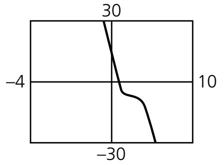
1. the factors of 8
2. $(x-1)(x+2)(x-4)=0$
3. a. $f(x)=kx(x-2)(x+3)$ **b.** $f(x)=2x(x-2)(x+3)$
4. a. $f(x)=k(x-1)(x+1)(x+2)$
b. $f(x)=-\frac{1}{2}(x-1)(x+1)(x+2)$
5. a. $f(x)=k(x+2)(x+1)(x-1)(x-3)$
b. $f(x)=\frac{1}{2}(x+2)(x+1)(x-1)(x-3)$
6. $(x-1)(x-2)(5x-3)=0$
7. 2
8. a. $-4, 5$ **b.** $-1 \pm 3i$ **c.** $0, 2, -5$ **d.** $0, 2, -2$ **e.** $-1, 0, 1$
f. $\pm i, \pm 1$ **g.** $-1, 0, 4$ **h.** $\frac{3}{2}, \frac{-3 \pm 3i\sqrt{3}}{4}$ **i.** $-2, 3, 3$
j. $2, 3, 4$ **k.** $-1, -1, 2$ **l.** $3, 3, -4$ **m.** $5, 1 \pm \sqrt{3}$
n. $2, \frac{1 \pm \sqrt{33}}{2}$
9. a. $1, \frac{-7 \pm \sqrt{17}}{4}$ **b.** $-4, -1, \frac{1}{4}$ **c.** $-2, 3\frac{3}{5}$ **d.** $0, \frac{1}{2}, \pm 2$
e. $\pm 3, \pm 2$ **f.** $\pm i, \pm 7$ **g.** $-2, \frac{-1 \pm \sqrt{3}i}{2}$ **h.** $-2, \frac{-7 \pm \sqrt{13}}{2}$
10. a. $\pm 1, \pm i, \pm \sqrt{3}, \pm i\sqrt{3}$ **b.** $2, -1, -1 \pm 3i, \frac{1 \pm 2\sqrt{3}}{2}$
c. $-2, -1, 2, 3$ **d.** $-\frac{1}{3}, -\frac{1}{4}, 3, 4$ **e.** $\frac{-1 \pm \sqrt{34}}{3}, \frac{-1 \pm \sqrt{2}}{3}$
f. $-8, 2, -3 \pm i\sqrt{21}$
11. 5 cm
12. a. $-7.140, 0.140$ **b.** $-2.714, 1.483, 3.231$ **c.** $1, -0.732, 2732$
d. $-2.278, -1.215, 1.215, 2.278$
13. 3 cm, 4 cm, 5 cm
14. 6.64 m
15. 3.1 s

Exercise 2.4

1. a. $-5, 11$ **b.** $\frac{5}{2}, \frac{9}{2}$ **c.** $\frac{7}{3}, \frac{-8}{3}$
2. a. $x^2-3x+7=0$ **b.** $x^2+6x+4=0$ **c.** $25x^2-5x-2=0$
d. $12x^2+13x+3=0$ **e.** $3x^2+33x-2=0$
3. a. $x^2-10x+21=0$ **b.** $x^2-3x-40=0$
c. $3x^2-10x+3=0$ **d.** $8x^2-10x+3=0$
e. $125x^2+85x-12=0$ **f.** $x^2-4x+5=0$
4. -6
5. $6, k=21$
6. $x^2-4x-13=0$
7. $2x^2-37x+137=0$
8. $x^2+7x+9=0$
9. $16x^2-97x+4=0$
10. $x^2+10x+5=0$
11. $4x^2-40x+1=0$
12. $8x^2+40x+1=0$
13. $x_1+x_2+x_3=\frac{-b}{a}, x_1x_2+x_1x_3+x_2x_3=\frac{c}{a}, x_1x_2x_3=-\frac{d}{a}$
14. $2x^3-13x^2+22x-8=0$
15. $x^3-10x^2+31x-32=0$
16. $x_1+x_2+x_3+x_4=-\frac{b}{a}$,
$x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4=\frac{c}{a}$,
$x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4=-\frac{d}{a}, x_1x_2x_3x_4=\frac{e}{a}$

Exercise 2.5

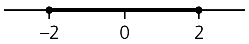
1. a. $f(x)>0$ for $x<-3, 0<x<4$ $f(x)<0$ for $-3<x<0$, $x>4$ **b.** $f(x)>0$ for $-2<x<1, x>4$ $f(x)<0$ for $x<-2, 1<x<4$ **c.** $f(x)>0$ for $x<-3, 0<x<2$ $f(x)<0$ for $-3<x<0$
2. a. $0<x<2$ **b.** $-3 \leq x \leq 1$ **c.** $2 \leq x \leq 5$
d. $x<-3$ or $x>0.5$ **e.** $x=2$ **f.** $x \leq -3, 0 \leq x \leq 3$
g. $x<-1, 1<x<5$ **h.** $x \leq -2, 0.5 \leq x \leq 1$
i. $-3.1 \leq x \leq -.2$ or $x \geq 3.3$ **j.** R
3. a.



b. $t<59.15$ °C **c.** $t>270.50$ °C
4. between 1.96 and 4.16 s
5. $3.27<w<3.30$ in cm

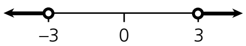
Exercise 2.6

1. a. 10 **b.** 19 **c.** 4 **d.** 6
2. a.



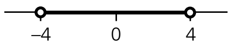
$-2 \leq x \leq 2$

b.



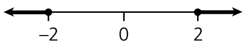
$x>3$ or $x<-3$

c.



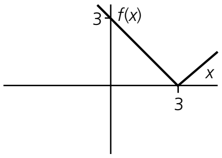
$-4<x<4$

d.

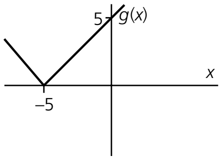


$x \geq 2$ or $x \leq -2$

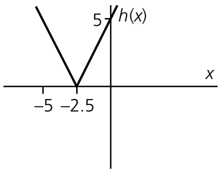
3. a.



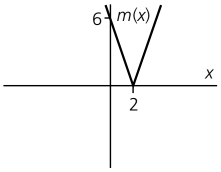
b.



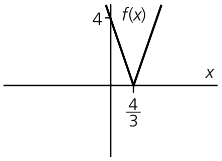
c.



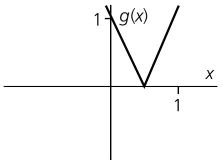
d.



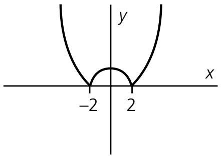
e.



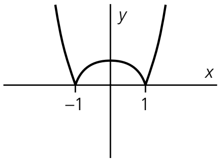
f.



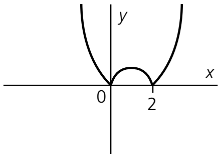
4. a.



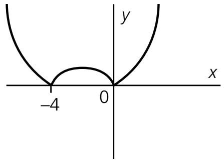
b.



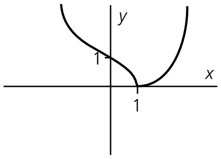
c.



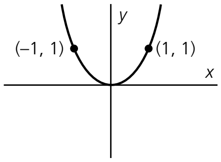
d.



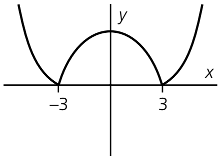
e.



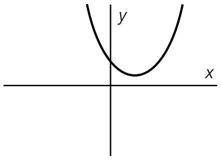
f.



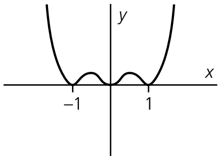
6. a.



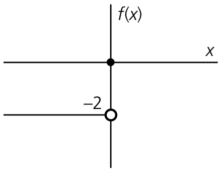
b.



c.



7. a. $x = 4, -3$ **b.** $x = \frac{4}{3}, \frac{-8}{3}$ **c.** $-6 \le x \le 12$
d. $x \ge 1$ or $x \le -9$ **e.** $-\frac{1}{2} < x < \frac{7}{2}$ **f.** no solution
8. a. -1 **b.** 0.8 **c.** $4, \frac{4}{3}$ **d.** $x > \frac{1}{2}$ **e.** $x < \frac{2}{5}$
f. $x \le -1$ or $x \ge \frac{5}{3}$ **g.** $-2, 4$ **h.** 0
9. none
10.



Review Exercise

1. a. $(x + 3)$ **b.** $(3x - 2)$
2. a. $y = a(x - 4)(x - 1)(x + 2)$ **b.** $y = -(x - 4)(x - 1)(x + 2)$
3. a. No. **b.** Yes.
4. $(x - 5)(x^2 - x + 1)$
5. a. $\frac{3}{4}$ **b.** $\frac{35}{33}$
6. a. $(x - 1)(x^2 - x + 1)$
b. $(x - 1)(x - 2)(x - 3)$ **c.** $(2x - 3y)(4x^2 + 6xy + 9y^2)$
d. $3(x + 2x - pr)(x^2 + 4xw + 4w^2 + prx + 2wpr + p^2r^2)$
8. a. $(2x + 3)(x^2 + x + 1)$ **b.** $(x - 1)(3x + 5)(3x - 1)$
9. a. Yes **b.** No
10. a. $(3x - 1)(x^2 - x + 1)$ **b.** $(2x - 5)(x^2 + 3x + 1)$
c. $(5x - 1)(3x - 1)(2x - 1)$
11. a. -2.5 **b.** $0, -5, 5$ **c.** $-2, 1 \pm i\sqrt{3}$ **d.** $1, 3, -3$
e. $-4, 4, \pm 2i$ **f.** $1, \frac{3 \pm \sqrt{21}}{2}$ **g.** $2, \frac{1 \pm i\sqrt{3}}{2}$
h. $-1, 3, \frac{-3 \pm 3i\sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}$ **i.** $-1 \pm \sqrt{5}, -1 \pm i\sqrt{2}$
12. a. $x \doteq \pm 1.414$ **b.** $x \doteq -10.196, 0.196$
c. $x \doteq -1.377, -0.274, 2.651$ **d.** $x \doteq -1.197$
e. $x \doteq \pm 2.857, \pm 1.356$ **f.** $x \doteq -5.67$
13. $x_2 = 3$ and $k = -1$
14. $x^2 + 5x + 2 = 0$
15. a. $x_1 + x_2 = \frac{1}{2}, x_1x_2 = 2$ **b.** $15x^2 - x - 2 = 0$
c. $x^2 - 6x + 13 = 0$ **d.** $x_2 = -\frac{2}{3}, k = -1$
e. $x^2 + x - 4 = 0$ **f.** $4x^2 - x - 2 = 0$
16. a. $-4 < x < 2$ **b.** $x \le -2$ or $x \ge 1$ **c.** $x \le 0$
d. $-1 < x < 1$ or $x > 2$ **e.** $x = 0$ **f.** R
g. $-2.8 < x < -.72$ or $.72 < x < 2.8$
h. $-1.44 < x < 1$ or $x > 1.38$
17. a. $\frac{-10}{3}, 4$ **b.** $-4 < x < 2$ **c.** $x \le -1$ or $x \ge 4$
18. 5 cm

Chapter 2 Test

1. No.
2. a. $(x - 1)(x^2 + 4x + 2)$ **b.** $(x + 1)(2x - 3)(x - 3)$
c. $(x + 1)(x - 1)^3$
3. $(3x - 2)(x^2 + 2x + 2)$
4. a. $3, \frac{-3 \pm 3i\sqrt{3}}{2}$ **b.** $1, \frac{3 \pm i\sqrt{3}}{2}$ **c.** $0, \frac{1}{2}, 3$ **d.** $\pm 2, \pm 1$
5. $x^2 - 8x + 20 = 0$
6. Yes.
7. a. $-2 < x < 3$ or $x < -2$ **b.** $-2 \le x \le 0$ or $x \ge 2$
c. $x < -7$ or $x > 2$
8. a. 3 zeros, positive, cubic (3rd)
b. 2 zeros, positive, quartic (4th)
c. 3 zeros, negative, cubic (3rd)
9. a. 173.9 cm **b.** 6.52 kg

CHAPTER 3 INTRODUCTION TO CALCULUS

Review of Prerequisite Skills

1. a. -3 **b.** -2 **c.** 12 **d.** -1 **e.** $\frac{-2}{3}$ **f.** $\frac{-2}{3}$ **g.** 4 **h.** -4 **i.** $\frac{-5}{6}$
j. -1 **k.** $-\frac{41}{10}$ **l.** -1
2. a. $y = 4x - 2$ **b.** $y = -2x + 5$
c. $y + 5 = 0$ **d.** $2x - 3y + 12 = 0$ **e.** $6x - 5y + 36 = 0$
f. $x + y - 2 = 0$ **g.** $6x - y + 2 = 0$ **h.** $4x - y = 0$
i. $7x - y - 27 = 0$ **j.** $3x + y - 6 = 0$ **k.** $x + 3 = 0$
l. $y - 5 = 0$
3. a. $\frac{-5}{52}$ **b.** $\frac{-3}{13}$ **c.** 0 **d.** $\frac{5}{52}$
4. a. 6 **b.** $\sqrt{3}$ **c.** 9
5. a. $\frac{-1}{2}$ **b.** -1 **c.** 5 **d.** 1 **e.** 10^6
6. a. $\frac{5\sqrt{2}}{2}$ **b.** $\frac{6\sqrt{3} + \sqrt{6}}{3}$ **c.** $\frac{6 + 4\sqrt{3}}{3}$ **d.** $\frac{3 - \sqrt{3}}{6}$
e. $\frac{-5\sqrt{7} - 20}{9}$ **f.** $-6 - 4\sqrt{3}$ **g.** $\frac{-15 + 10\sqrt{3}}{2}$
h. $\frac{-6\sqrt{6} - 15\sqrt{2}}{13}$ **i.** $\frac{20 + 2\sqrt{5}}{19}$
7. a. $\frac{2}{5\sqrt{2}}$ **b.** $\frac{3}{6\sqrt{3} + \sqrt{6}}$ **c.** $\frac{-9}{5\sqrt{7} + 4}$ **d.** $\frac{-13}{6\sqrt{6} + 15\sqrt{2}}$
e. $\frac{-1}{\sqrt{3} + \sqrt{7}}$ **f.** $\frac{1}{2\sqrt{3} - 7}$
8. a. $(x - 2)(x + 2)$ **b.** $x(x - 1)(x + 1)$ **c.** $(x + 3)(x - 2)$
d. $(2x - 3)(x - 2)$ **e.** $x(x + 1)(x + 1)$ **f.** $(x + 2)(x^2 - 2x + 4)$
g. $(3x - 4)(9x^2 + 12x + 16)$ **h.** $(x - 2)(x^2 + 3)$
i. $(x - 1)(x + 2)(2x - 3)$
9. a. $x \in R$ **b.** $x \in R$ **c.** $x \ge -5, x \in R$ **d.** $x \in R$
e. $x \ne 1, x \in R$ **f.** $x \in R$ **g.** $x \ge 9, x \in R$ **h.** $x \ne 0, x \in R$
i. $x \ne 5, x \in R$ **j.** $x \ne 4, -1, -5, x \in R$ **k.** $x \ne 3, \frac{1}{2}, x \in R$
l. $x \ne -2, 1, -5, x \in R$

Exercise 3.1

1. a. 3 **b.** $\frac{-5}{3}$ **c.** $\frac{-1}{3}$
2. a. $\frac{-1}{3}$ **b.** $\frac{-7}{13}$

3. a. $x - y = 0$ **b.** $y = 8x + 6$ **c.** $3x - 5y - 15 = 0$
d. $x - 5 = 0$

4. a. $4 + h$ **b.** $75 + 15h + h^2$ **c.** $108 + 54h + 12h^2 + h^3$
d. $\frac{-1}{1 + h}$ **e.** $6 + 3h$ **f.** $12 + 6h + h^2$ **g.** $\frac{-3}{4(4 + h)}$ **h.** $\frac{1}{4 + 2h}$

5. a. $\frac{1}{\sqrt{16 + h} + 4}$ **b.** $\frac{h + 5}{\sqrt{h^2 + 5h + 4} + 2}$ **c.** $\frac{1}{\sqrt{5 + h} + \sqrt{5}}$

6. a. $6 + 3h$ **b.** $3 + 3h + h^2$ **c.** $\frac{1}{\sqrt{9 + h} + 3}$

7. a. $P(2, 8)$

Q	Slope of *PQ*
(3, 27)	19
(2.5, 15.625)	15.25
(2.1, 9.261)	12.61
(2.01, 8.120601)	12.0601
(1, 1)	7
(1.5, 3.375)	9.25
(1.9, 6.859)	11.41
(1.99, 7.880599)	11.9401

b. 12 **c.** $12 + 6h + h^2$ **d.** 12

8. a. -12 **b.** 5 **c.** 12

9. a. $\frac{1}{2}$ **b.** $\frac{1}{4}$ **c.** $\frac{5}{6}$

10. a. -2 **b.** $\frac{-1}{2}$ **c.** $\frac{-1}{25}$

11. a. 1 **b.** -1 **c.** 9 **d.** $\frac{1}{4}$ **e.** $\frac{-1}{10}$ **f.** $\frac{-3}{4}$ **g.** $\frac{-1}{6}$ **h.** $\frac{-1}{16}$

16. $\frac{-5}{4}$

17. 1600 papers/month

18. (2, 4)

19. $\left(-2, \frac{28}{3}\right), \left(-1, \frac{26}{3}\right), \left(1, \frac{-26}{3}\right), \left(2, \frac{-28}{3}\right)$

Exercise 3.2

1. 0 s and 4 s

2. a. slope of the secant between two points $(2, s(2))$ and $(9, s(9))$
b. slope of the tangent at $(6, s(6))$

3. slope of the tangent to $y = \sqrt{x}$ at (4, 2)

4. a. between A and B **b.** greater

7. a. 5 m/s, 25 m/s, 75 m/s **b.** 55 m/s **c.** 20 m/s

8. a. i) 72 km/h **ii)** 64.8 km/h **iii)** 64.08 km/h **c.** 64 km/h

9. a. 15 terms **b.** 16 terms/h

10. a. $-\frac{1}{3}$ mg/h

11. $\frac{1}{50}$ s/m

12. $-\frac{12}{5}$°C/km

13. 2 s, 0 m/s

14. a. \$4800 **b.** \$80/ball **c.** $0 < x < 8$

15. a. 6 **b.** -1 **c.** $\frac{1}{10}$

18. 200π m²/m

Exercise 3.3

1. a. $\frac{8}{11}$ **b.** π

4. a. -5 **b.** 10 **c.** 100 **d.** -8 **e.** 4 **f.** 8

5. 1

6. a. 0 **b.** 2 **c.** -1 **d.** 2

7. a. 2 **b.** 1 **c.** does not exist

8. a. 8 **b.** 2 **c.** 2

9. 5

10. a. 0 **b.** 0 **c.** 5 **d.** $\frac{-1}{2}$ **e.** $\frac{1}{5}$ **f.** does not exist

11. a. does not exist **b.** 2 **c.** 2 **d.** does not exist

13. $m = -3, b = 1$

14. $a = 3, b = 2, c = 0$

15. b. 6, 4 **c.** 2000 **d.** $2\frac{1}{2}$ years after the spill, or $8\frac{1}{2}$ years in total.

Exercise 3.4

4. a. 1 **b.** 1 **c.** $\frac{100}{9}$ **d.** $5\pi^3$ **e.** 2 **f.** $\sqrt{3}$

5. a. 2 **b.** $\sqrt{2}$

7. a. 4 **b.** 4 **c.** 7 **d.** 1 **e.** $\frac{-7}{3}$ **f.** 27 **g.** 0 **h.** $\frac{7}{2}$ **i.** $\frac{1}{2}$ **j.** $-\frac{1}{4}$ **k.** $\frac{1}{4}$
l. $-\frac{1}{\sqrt{7}}$ **m.** $\frac{-1}{2}$ **n.** $\frac{3}{4}$ **o.** 1

8. a. $\frac{1}{12}$ **b.** -27 **c.** $\frac{1}{6}$ **d.** $\frac{1}{2}$ **e.** $\frac{1}{12}$ **f.** $\frac{1}{12}$

9. a. 0 **b.** 0 **c.** 4 **d.** -1 **e.** 0 **f.** $\frac{2}{3}$ **g.** -16 **h.** $\frac{1}{4}$ **i.** $\frac{-3}{2}$ **j.** $\frac{1}{2}$
k. $2x$ **l.** $\frac{1}{32}$

10. a. does not exist **b.** does not exist **c.** does not exist **d.** 0

11. b. $V = 0.08213T + 22.4334$ **c.** $T = \frac{V - 22.4334}{0.08213}$

13. a. 27 **b.** -1 **c.** 1

14. a. 0 **b.** 0

15. a. 0 **b.** 0 **c.** $\frac{1}{2}$

16. 2

17. No.

18. $b = 2$

19. $m = 6, b = 9$

Exercise 3.5

4. a. 3 **b.** 0 **c.** 0 **d.** ± 3 **e.** $-3, 2$ **f.** 3

5. a. $x \in R$ **b.** $x \in R$ **c.** $x < 0, 0 < x < 5, x > 5, x \in R$
d. $x \geq -2, x \in R$ **e.** $x \in R$ **f.** $x \in R$

7. continuous everywhere

8. No.

9. 0, 100, 200, and 500

10. Yes.

11. discontinuous at $x = 2$

12. $k = 16$

13. $a = -1, b = 6$

14. a. 1, -1, does not exist **b.** discontinuous at $x = 1$

Review Exercise

1. a. -3 **b.** 7 **c.** $2x - y - 5 = 0$

2. a. $-\frac{1}{3}$ **b.** $\frac{1}{2}$ **c.** $\frac{-1}{27}$ **d.** $\frac{-5}{4}$

3. a. 2 **b.** 2

4. a. -5 m/s; -15 m/s **b.** -40 m/s **c.** -60 m/s

5. a. 0.0601 g **b.** 6.01 g/min **c.** 6 g/min

6. a. 7×10^5 tonnes **b.** 1.8×10^5 tonnes/year
c. 1.5×10^5 tonnes/year **d.** 7.5 years

7. a. 10 **b.** 7, 0 **c.** $t = 3, t = 4$

9. a. $x = -1, x = 1$ **b.** do not exist

10. not continuous at $x = 3$

11. a. $x = 1, x = -2$ **b.** $\lim_{x \to 1} f(x) = \frac{2}{3}, \lim_{x \to -2} f(x)$ does not exist

12. a. does not exist **b.** 0 **c.** $\frac{37}{7}$, does not exist

13. $\frac{1}{3}$

x	$f(x)$
1.9	0.34483
1.99	0.33445
1.999	0.33344
2.001	0.33322
2.01	0.33223
2.1	0.32258

b. $\frac{1}{2}$

x	$f(x)$
0.9	0.52632
0.99	0.50251
0.999	0.50025
1.001	0.49975
1.01	0.49751
1.1	0.47619

14.

x	$f(x)$
–0.1	0.29112
–0.01	0.28892
–0.001	0.2887
0.001	0.28865
0.01	0.28843
0.1	0.28631

15. a.

x	$f(x)$
2.1	0.24846
2.01	0.24984
2.001	0.24998
2.0001	0.25

c. $\frac{1}{4}$

16. a. 10, slope of the tangent to $y = x^2$ at $x = 5$
b. $\frac{1}{4}$, slope of the tangent to $y = \sqrt{x}$ at $x = 4$
c. $\frac{-1}{16}$, slope of the tangent to $y = \frac{1}{x}$ at $x = 4$
d. $\frac{1}{147}$, slope of the tangent to $y = \sqrt[3]{x}$ at $x = 343$

17. a. $\frac{-3}{2}$ **b.** $5a^2 - 3a + 7$ **c.** does not exist **d.** 1 **e.** -12 **f.** 4
g. $\frac{1}{3}$ **h.** $10a$ **i.** $\frac{-3}{7}$ **j.** $\frac{3}{5}$ **k.** 1 **l.** -1 **m.** $\frac{3}{2}$ **n.** $\frac{\sqrt{3}-2}{2}$
o. $\frac{1}{\sqrt{5}}$ **p.** -3 **q.** 0 **r.** 16 **s.** 48 **t.** $-\frac{1}{4}$ **u.** 2

Chapter 3 Test

5. -13
6. $\frac{-4}{3}$
7. 2
8. $x + y + 2 = 0$
9. a. does not exist **b.** 1 **c.** 1 **d.** 1, 2
10. a. 1.8×10^5 **b.** 4000 people/year
11. a. 1 km/h **b.** 2 km/h
12. $\frac{\sqrt{16+h}-4}{h}$
13. -31
14. a. 12 **b.** $\frac{7}{5}$ **c.** 4 **d.** $\frac{-3}{4}$ **e.** $\frac{1}{6}$ **f.** $\frac{1}{12}$
15. $a = 1, b = \frac{-18}{5}$
17. $k = 8$

CHAPTER 4 DERIVATIVES

Review of Prerequisite Skills

1. a. 5^{11} **b.** a^8 **c.** 4^{18} **d.** $-8a^6$ **e.** $6m^{13}$ **f.** $2p$ **g.** $\frac{1}{a^2b^7}$ **h.** $48e^{18}$
i. $-\frac{b}{2a^6}$
2. a. $x^{\frac{7}{6}}$ **b.** $4x^4$ **c.** $a^{\frac{2}{3}}$
3. a. $\frac{-3}{2}$ **b.** 2 **c.** $\frac{-3}{5}$ **d.** 1
4. a. $x + 2y - 5 = 0$ **b.** $3x - 2y + 16 = 0$ **c.** $4x + 3y - 7 = 0$
5. a. $2x^2 - 5xy - 3y^2$ **b.** $x^3 - 5x^2 + 10x - 8$
c. $12x^2 + 36x - 21$ **d.** $-13x + 42y$ **e.** $29x^2 - 2xy + 10y^2$
f. $-13x^3 - 12x^2y + 4xy^2$
6. a. $\frac{15x}{2}, x \neq -2, 0$ **b.** $\frac{y-5}{4y^2(y+2)}, y \neq 5$ **c.** $\frac{8}{9}, h \neq -k$
d. $\frac{2}{(x+y)^2}, x \neq y$ **e.** $\frac{11x^2 - 8x + 7}{2x(x-1)}$ **f.** $\frac{4x+7}{(x-2)(x+3)}$
7. a. $2a(5a - 3)$ **b.** $(2k - 3)(2k + 3)$ **c.** $(x + 4)(x - 8)$
d. $(y - 14)(y + 3)$ **e.** $(3a - 7)(a + 1)$ **f.** $(6x + 5)(x + 2)$
g. $(x - 1)(x + 1)(x^2 + 1)$ **h.** $(x - y)(x^2 + xy + y^2)$
i. $(r - 1)(r + 1)(r - 2)(r + 2)$
8. a. $(a - b)(a^2 + ab + b^2)$
b. $(a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$
c. $(a - b)(a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6)$
d. $(a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + ab^{n-2} + b^{n-1})$
9. a. $\frac{3\sqrt{2}}{2}$ **b.** $\frac{4\sqrt{3}-\sqrt{6}}{3}$ **c.** $-\frac{30+17\sqrt{2}}{23}$ **d.** $-\frac{11-4\sqrt{6}}{5}$

Exercise 4.1

1. a. $x \in R, x \neq -2$ **b.** $x \in R, x \neq 2$ **c.** $x \in R$ **d.** $x \in R, x \neq 1$
e. $x \in R$ **f.** $x > 2, x \in R$
4. a. 2 **b.** 9 **c.** $\frac{1}{2}$
5. a. $2x + 3$ **b.** $\frac{-3}{(x+2)^2}$ **c.** $\frac{3}{2\sqrt{3x+2}}$ **d.** $\frac{-2}{x^3}$
6. a. -7 **b.** $\frac{-2}{(x-1)^2}$ **c.** $6x$
7. $-4, 0, 4$
8. 8 m/s, 0 m/s, -4 m/s
9. $x - 6y + 10 = 0$
10. a. 0 **b.** 1 **c.** m **d.** $2ax + b$
12. a and e, b and f, c and d
13. -1
14. $f'(0) = 0$
15. 3
16. $f(x) = (x - 3)^{\frac{1}{3}}$, answers will vary

Exercise 4.2

2. a. 4 **b.** 0 **c.** $4x + 1$ **d.** $\frac{1}{2\sqrt{x}}$ **e.** $12x^2$ **f.** $3x^2 - 2x$ **g.** $-2x + 5$
h. $\frac{1}{3\sqrt[3]{x^2}}$ **i.** x^3 **j.** $18x$ **k.** $\frac{x^3}{4}$ **l.** $-3x^{-4}$
3. a. $\frac{dy}{dx} = 2x - 3$ **b.** $f'(x) = 6x^2 + 10x - 4$
c. $v'(t) = 18t^2 - 20t^4$ **d.** $s'(t) = \frac{-2}{t^3}, t > 0$ **e.** $f'(x) = 6x^5$
f. $h'(x) = 4x + 11$ **g.** $\frac{ds}{dt} = 4t^3 - 6t^2$ **h.** $g'(x) = 20x^4$
i. $\frac{dy}{dx} = x^4 + x^2 - x$ **j.** $g'(x) = 40x^7$ **k.** $s'(t) = 2t^3 - \frac{3}{2}$
l. $g'(x) = 7f'(x)$ **m.** $h'(x) = \frac{21}{x^8}$ **n.** $\frac{dy}{dx} = m$
4. a. $2x^{\frac{-4}{5}}$ **b.** $5x^{\frac{2}{3}}$ **c.** $-9x^{\frac{-5}{2}}$ **d.** $8x^7 + 8x^{-9}$
e. $2x^{\frac{-1}{3}} - 2x^{\frac{-2}{3}} - \frac{1}{3}x^{\frac{-4}{3}}$ **f.** $2x^{\frac{-3}{2}} + 6x^{-2}$ **g.** $-18x^{-4} - 4x^{-3}$
h. $-18x^{-3} + \frac{3}{2}x^{\frac{-1}{2}}$ **i.** $100x^4 + x^{\frac{-2}{3}}$ **j.** $\frac{1}{2}x^{\frac{-1}{2}} + 9x^{\frac{1}{2}}$

k. $1.5x^{0.5} + 3x^{-1.25}$ **l.** $-x^{-2} - \frac{1}{2}x^{\frac{-3}{2}}$

5. a. $-4t + 7$ **b.** $5 - t^2$ **c.** $2t - 6$

6. a. $\frac{191}{4}$ **b.** $\frac{11}{24}$

7. a. 12 **b.** 5 **c.** $-\frac{1}{2}$ **d.** 12

8. a. 9 **b.** $\frac{1}{2}$ **c.** 4 **d.** -7

9. a. $6x - y - 4 = 0$ **b.** $18x - y + 25 = 0$ **c.** $9x - 2y - 9 = 0$
d. $x + y - 3 = 0$ **e.** $7x - 2y - 28 = 0$ **f.** $5x - 6y - 11 = 0$

10. $x + 18y - 125 = 0$

11. 8 or -8

12. No

14. $(-1, 0)$

15. $(2, 10), (-2, -6)$

17. a. $y - 3 = 0, 16x - y - 29 = 0;$
b. $20x - y - 47 = 0, 4x + y - 1 = 0$

18. 7

19. a. 50 km **b.** 0.12 km/m

20. 0.29 min, 1.71 min

21. -20 m/s

22. $(1, -3), (-1, -3)$

23. $(0, 0)$

25. $1 - \frac{1}{n}$, approaches 1

26. a. $f'(x) = \begin{cases} 2x, x < 3 \\ 1, x > 3 \end{cases}$

$f'(x)$ does not exist at $(3, 9)$.

b. $f'(x) = \begin{cases} 6x, x < -\sqrt{2} \\ -6x, -\sqrt{2} < x < \sqrt{2} \\ 6x, x > \sqrt{2} \end{cases}$

$f'(x)$ does not exist at $(-\sqrt{2}, 0)$, $(\sqrt{2}, 0)$.

c. $f'(x) = \begin{cases} -1, x < -1 \\ 1, -1 < x < 0 \\ -1, 0 < x < 1 \\ 1, x > 0 \end{cases}$

$f'(x)$ does not exist at $(-1, 0)$, $(0, 1)$, $(1, 0)$.

Exercise 4.3

1. a. $2x - 4$ **b.** $6x^2 - 2x$ **c.** $12x - 17$ **d.** $-8x + 26$
e. $45x^8 - 80x^7 + 2x - 2$ **f.** $-8t^3 + 2t$

2. a. $15(5x + 1)^2(x - 4) + (5x + 1)^3$
b. $6x(3 + x^3)^5 + 15x^2(3x^2 + 4)(3 + x^3)^4$
c. $-8x(1 - x^2)^3(2x + 6)^3 + 6(1 - x^2)^4(2x + 6)^2$

4. a. 9 **b.** -4 **c.** -9 **d.** 6 **e.** -36 **f.** 22 **g.** 671 **h.** -12

5. $10x + y - 8 = 0$

6. a. $(14, -450)$ **b.** $(-1, 0)$

7. a. $3(x + 1)^2(x + 4)(x - 3)^2 + (x + 1)^3(x - 3)^2 +$
$2(x + 1)^3(x + 4)(x - 3)$
b. $2x(3x^2 + 4)^2(3 - x^3)^4 + 12x^3(3x^2 + 4)(3 - x^3)^4 -$
$12x^4(3x^2 + 4)^2(3 - x^3)^3$

8. -30

9. a. $f'(x) = g'(x)g_2(x) \ldots g_{11}(x) + g_1(x)\, g'_2(x)\, g'_3(x) \ldots g_{11}(x)$
$+ g_1(x)\, g_2(x)\, g'_3(x) \ldots g_{11}(x) + \ldots$
$+ g_1(x)\, g_2(x) \ldots g_{x-1}(x)\, g'_{11}(x)$ **b.** $\frac{n(n + 1)}{2}$

10. $f(x) = 3x^2 + 6x - 5$

11. a. ± 1 **b.** $f'(x) = 2x, x < -1$ or $x > 1;$
$f'(x) = -2x, -1 < x < 7$ **c.** $-4, 0, 6$

Exercise 4.4

2. $f'(x) = 1, g'(x) = 2x^{\frac{-1}{3}}, h'(x) = \frac{-1}{2x^6}, \frac{dy}{dx} = 8x, \frac{ds}{dt} = 1$

4. a. $\frac{1}{(x + 1)^2}$ **b.** $\frac{x^2 + 2x}{(x + 1)^2}$ **c.** $\frac{13}{(t + 5)^2}$ **d.** $\frac{7}{(x + 3)^2}$ **e.** $\frac{2x^4 - 3x^2}{(2x^2 - 1)^2}$
f. $\frac{-2x}{(x^2 + 3)^2}$ **g.** $\frac{5x^2 + 6x + 5}{(1 - x^2)^2}$ **h.** $\frac{x^2 + 4x - 3}{(x^2 + 3)^2}$ **i.** $\frac{x^2 + 6x + 1}{(3x^2 + x)^2}$

5. a. $\frac{13}{4}$ **b.** $\frac{7}{25}$ **c.** $\frac{200}{841}$ **d.** $\frac{-7}{3}$

6. -9

7. $\left(9, \frac{27}{5}\right), (-1, \frac{3}{5})$

9. a. $(0, 0), (8, 32)$ **b.** $(1, 0)$

10. $p'(1) \cong 75.36, p'(2) \cong 63.10$

11. $4x - 6y - 5 = 0$

12. a. 20 m **b.** -1.1 m/s

13. $ad - bc > 0$

Exercise 4.5

1. a. 0 **b.** 0 **c.** -1 **d.** $\sqrt{15}$ **e.** $\sqrt{x^2 - 1}$ **f.** $x - 1$

2. a. $f(g(x)) = x, x \geq 0; g(f(x)) = |x|, x \in R; f \circ g \neq g \circ f$
b. $f(g(x)) = \frac{1}{x^2 + 1}, x \in R; g(f(x)) = \frac{1}{x^2} + 1, x \neq 0; f \circ g \neq g \circ f$
c. $f(g(x)) = \frac{1}{\sqrt{x + 2}}, x > -2; g(f(x)) = \sqrt{\frac{1 + 2x}{x}}, x < -\frac{1}{2}$ or
$x > 0; f \circ g \neq g \circ f$

3. a. $3\sqrt{x} + 1$ **b.** $\frac{1}{\sqrt{x + 1}}$ **c.** $(3x + 1)^3$
d. $\sqrt{x^3}$ **e.** $\frac{1}{\sqrt{x} + 1}$ **f.** $3x^3 + 1$ **g.** $\frac{1}{3\sqrt{x + 2}}$ **h.** $3x\sqrt{x} + 1$
i. $\frac{1}{(\sqrt{x} + 1)^3}$

4. a. $f(x) = x^4, g(x) = 2x^2 - 1$ **b.** $f(x) = \sqrt{x}, g(x) = 5x - 1$
c. $f(x) = \frac{1}{x}, g(x) = x - 4$ **d.** $f(x) = x^{\frac{5}{2}}, g(x) = 2 - 3x$
e. $f(x) = x(x + 1), g(x) = x^2 + 2$
f. $f(x) = x^2 - 9x, g(x) = x + 1$

5. $g(x) = x^3$

6. $f(x) = (x + 7)^2$

7. $f(x) = (x + 3)^2$

8. $g(x) = x + 4$ or $g(x) = -x - 4$

9. $u(x) = 2x$ or $u(x) = -2x + 4$

10. a. $\frac{x}{x - 1}$ **b.** $\frac{1}{x}$ **11.** $-2, -3$ **12. a.** x

Exercise 4.6

2. a. $8(2x + 3)^2$ **b.** $-6(5 - x)^5$ **c.** $6x(x^2 - 4)^2$ **d.** $-15x^2(7 - x^3)^4$
e. $4(4x + 3)(2x^2 + 3x - 5)^3$ **f.** $5(5x - x^2)^4(5 - 2x)$
g. $-6x(\pi^2 - x^2)^2$ **h.** $4(-1 + 2x - 3x^2)(1 - x + x^2 - x^3)^3$
i. $-12(2 - x)^3[(2 - x)^4 + 16]^2$ **j.** $\frac{2}{\sqrt{4x + 1}}$ **k.** $\frac{5}{2\sqrt{5x + 7}}$
l. $\frac{x}{\sqrt{x^2 - 3}}$ **m.** $\frac{-10x}{(x^2 - 16)^6}$ **n.** $\frac{-x}{\sqrt{x^2 + 4^3}}$ **o.** $\frac{-1}{2\sqrt{x}(\sqrt{x} + 1)^2}$
p. $\frac{2(1 + u^{\frac{1}{3}})^5}{\sqrt[3]{u^2}}$ **q.** $3\sqrt{2x - 5}$ **r.** $\frac{2(x + 2)^2(x - 1)}{x^2}$

3. a. $\frac{-6}{x^3}$ **b.** $\frac{6}{x^4}$ **c.** $\frac{-1}{(x + 1)^2}$ **d.** $\frac{-2x}{(x^2 - 4)^2}$ **e.** $\frac{-8}{x^3}$ **f.** $\frac{6x}{(9 - x^2)^2}$
g. $\frac{-10x - 1}{(5x^2 + x)^2}$ **h.** $\frac{-4(2x + 1)}{(x^2 + x + 1)^5}$ **i.** $\frac{(1 + \sqrt{x})^2(\sqrt{x} - 2)}{x^3}$

4. a. $3(3x + 5)(x + 4)^2(x - 3)^5$ **b.** $\frac{(3x + 1)(x + 3)}{(1 - x^2)^2}$
c. $4(2x - 1)^3(2 - 3x)^3(7 - 12x)$
d. $\frac{-2(x^2 - 3x - 1)}{(x^2 + 1)^2}$ **e.** $3x^2(3x - 5)(4x - 5)$

f. $\frac{-(2x-1)(2x+5)}{(x-2)^4}$ **g.** $4x^3(1-4x^2)^2(1-10x^2)$

h. $\frac{48x(x^2-3)^3}{(x^2+3)^5}$ **i.** $6x(2x^3+3x+3)(x^2+3)^2(x^3+3)$

j. $\frac{1}{(1+x^2)^{\frac{3}{2}}}$ **k.** $12(4-3t^3)^3(1-2t)^5(9t^3-3t^2-4)$

l. $\frac{1}{(1-x)\sqrt{1-x^2}}$

5. a. $\frac{91}{36}$ **b.** $\frac{7}{48\pi}$

6. $x = 0, x = 1$

7. $\frac{-1}{4}$

8. $60x - y - 119 = 0$

9. a. 52 **b.** 78 **c.** 54 **d.** 320 **e.** $\frac{9728}{27}$ **f.** $\frac{-1}{8}$ **g.** -48608

10. 10

11. $\frac{-42}{25}$

12. -6

13. a. $h'(x) = p'(x)g(x)r(x) + p(x)q'(x)r(x) + p(x)q(x)r'(x)$
b. -344

15. $\frac{-2x(x^2+3x-1)(1-x)^2}{(1+x)^4}$

17. $(a-1)d = (c-1)b$

Technology Extension

b. i) 6 **ii)** 3 **iii)** 32 **iv)** 6 **v)** $\frac{-3}{4}$ **vi)** -4 **vii)** 6 **viii)** -1

Review Exercise

2. a. $4x - 5$ **b.** $\frac{1}{2\sqrt{x-6}}$ **c.** $\frac{4}{(4-x)^2}$

3. a. $2x - 5$ **b.** $-3x^2$ **c.** $\frac{3}{4}x^{\frac{-1}{4}}$ **d.** $20x^{-5}$ **e.** $\frac{-28}{3x^5}$ **f.** $\frac{-1}{(x-3)^2}$

g. $\frac{-2x}{(x^2+5)^2}$ **h.** $\frac{12x}{(3-x^2)^3}$ **i.** $\frac{-1}{2\sqrt{2-x}}$ **j.** $\frac{7x+2}{\sqrt{7x^2+4x+1}}$

k. $60x^3(5x^4+\pi)^2$ **l.** $\frac{-4}{x^5} - \frac{6x^2}{5(x^3-4)^{\frac{7}{5}}}$

4. a. $\frac{2x^3+2}{x^3}$ **b.** $\frac{\sqrt{x}}{2}(7x^2-3)$ **c.** $\frac{-4}{x^2\sqrt{x}}$ **d.** $\frac{-3x+2}{x^3}$ **e.** $\frac{-5}{(3x-5)^2}$

f. $\frac{3x-1}{2\sqrt{x-1}}$ **g.** $\frac{-1}{3\sqrt{x}\sqrt[3]{(\sqrt{x}+2)^5}}$ **h.** $\frac{x}{\sqrt{x^2-9}}$ **i.** $1, x \neq -4$

j. $2x + 6$

5. a. $20x^3(x-1)(2x-6)^5$ **b.** $\frac{2x^2+1}{\sqrt{x^2+1}}$ **c.** $\frac{(2x-5)^3(2x+23)}{(x+1)^4}$

d. $\frac{x^2+15}{3(x^2+5)^{\frac{4}{3}}}$ **e.** $\frac{318(10x-1)^5}{(3x+5)^7}$ **f.** $\frac{12x(x^2-1)^2}{(x^2+1)^4}$ **g.** $\frac{-1}{(x^2-1)^{\frac{3}{2}}}$

h. $(x-2)^2(x^2+9)^3(11x^2-16x+27)$

i. $-6(1-x^2)^2(6+2x)^{-4}(3x^2+6x-1)$

j. $\frac{(3x^2-2)(15x^2-62)}{\sqrt{x^2-5}}$

6. a. $g'(x) = f(x^2) \cdot 2x$ **b.** $h'(x) = 2f(x) + 2xf'(x)$

7. a. $\frac{92}{9}$ **b.** $\frac{25}{289}$ **c.** $\frac{-8}{5}$

8. $\frac{-2}{3}$

9. $2 \pm 2\sqrt{3}, 5, -1$

10. a. i. $\pm 2, 0$ **ii.** $0, \pm 1, \pm\frac{\sqrt{3}}{3}$

11. a. $160x - y + 16 = 0$ **b.** $60x + y - 61 = 0$

12. $5x - y - 7 = 0$

13. $(2, 8), b = -8$

14. c. $(0, 0), \left(3\sqrt{2}, \frac{9\sqrt{2}}{2}\right), \left(-3\sqrt{2}, \frac{-9\sqrt{2}}{2}\right)$ **d.** -14

15. a. $\sqrt[3]{50} \doteq 3.68$ **b.** 1

16. a. 9, 19 **b.** 1.7 words/min, 2.3 words/min

17. a. $\frac{30t}{(\sqrt{9+t^2})^3}$ **b.** Yes. The limit of $N(t)$ as $t \to 0$ is 0.

18. a. $x^2 + 40$ **b.** 6 gloves/week

19. a. $750 - \frac{x}{3} - 2x^2$ **b.** $546.67

20. $\frac{-5}{4}$

Chapter 4 Test

3. $f'(x) = 1 - 2x$

4. a. $x^2 + 15x^{-6}$ **b.** $60(2x-9)^4$ **c.** $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{x^3}} - \frac{2}{\sqrt[3]{x^2}}$

d. $\frac{5(x^2+6)^4(3x^2+8x-18)}{(3x+4)^6}$ **e.** $\frac{16x^3-14x}{\sqrt[3]{6x^2-7^2}}$ **f.** $\frac{4x^5-18x+8}{x^5}$

5. 14

6. $\frac{-40}{3}$

7. $60x + y - 61 = 0$

8. $\frac{75}{32}$ p.p.m./year

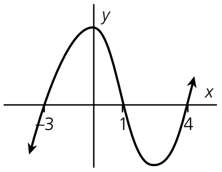
9. $\left(-\frac{1}{4}, \frac{1}{256}\right)$

10. $(1, 0), \left(-\frac{1}{3}, \frac{32}{27}\right)$

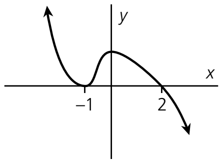
11. $a = 1, b = -1$

Cumulative Review Chapters 1–4

1.



2.



3. $y = 2x^3 - 3x^2 - 24$

4. a. $x^2 - x - 2$ **b.** $3x^2 - 13x + 50, R: -153$
c. $x^2 + x - 5, R: -5x + 4$

5. 27

6. 6

7. -3

8. $(x - 2)$ is a factor.

9. $(x - 3)$ and $(x - 1)$

10. a. $(x-4)(x+2)(x+5)$ **b.** $(x-2)(x+2)(x+5)$
c. $(2x+1)(x-2)(x+2)$ **d.** $(5x-2)(x^2+2x+5)$

11. a. $1, -2, -2$ **b.** $2, -2, 3i, -3i$ **c.** $1, -2, -3$ **d.** $1, -1, \frac{1}{2}$
e. $1, 1, -3$ **f.** $\frac{1}{3}, \frac{1 \pm 3i}{2}$

12. $-4, \frac{5}{2}$

13. $x^2 - 77x + 4 = 0$

14. a. $-2 < x < 3$ **b.** $-2 \leq x \leq 1, x \geq 3$

15. a. $-3 < x < 7$
b. $-1 \leq x \leq 4$ **c.** $x > 5$ or $x < -\frac{17}{3}$

16. a. 13 m/s **b.** 15 m/s

17. 5

18. a. 3 **b.** 1 **c.** 3 **d.** 2 **e.** No.

19. Answers will vary.

20. at $x = 2$

21. 2

22. x^3

23. a. $-\frac{1}{5}$ **b.** does not exist **c.** $-\frac{1}{9}$ **d.** 2 **e.** $\frac{1}{12}$ **f.** $\frac{1}{4}$
24. a. $6x + 1$ **b.** $-x^{-2}$
25. a. $-8x + 5$ **b.** $3x^2(2x^3 + 1)^{-\frac{1}{2}}$ **c.** $6(x + 3)^{-2}$
d. $4x(x^2 + 3)(4x^5 + 5x + 1) + (20x^4 + 5)(x^2 + 3)^2$
e. $\frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$
f. $5[x^2 + (2x + 1)^3]^4[2x + 6(2x + 1)^2]$
26. $4x + 36 = 10$
27. $8x - 16y -65 = 0$
28. $\left[\frac{18}{(x^3 + 2)^2} + \frac{4}{(x^3 + 2)} + 5\right] \cdot \frac{-3x^2}{(x^3 + 2)^2}$
29. 3
30. a. $4t + 6$ **b.** 46 people/year **c.** 2002

CHAPTER 5 APPLICATIONS OF DERIVATIVES

Review of Prerequisite Skills

5. a. $\frac{14}{5}$ **b.** -13 **c.** 3, 1 **d.** $-\frac{1}{2}$, 3 **e.** 2, 6 **f.** -3, 0, 1 **g.** 0, 4
h. $-\frac{1}{2}, \frac{1}{2}, 3$ **i.** $\pm\frac{9}{2}, \pm 1$
6. a. $x > 3$ **b.** $x < 0$ or $x > 3$ **c.** $0 < x < 4$
7. a. 25 cm² **b.** 48 cm² **c.** 49π cm² **d.** 36π cm²
8. a. $S = 56\pi$ cm², $V = 48\pi$ cm³ **b.** $h = 6$ cm, $S = 80\pi$ cm²
c. $r = 6$ cm, $S = 144\pi$ cm² **d.** $h = 7$ cm, $V = 175\pi$ cm³
9. a. $V = 972\pi$ cm³, $S = 324\pi$ cm²
b. $V = 36\pi$ cm³, $S = 36\pi$ cm² **c.** $r = 3$, $S = 36\pi$ cm²
d. $r = 5\sqrt{10}$ cm, $V = \frac{5000\sqrt{10}}{3}\pi$ cm³
10. a. 16π cm³ **b.** 9 cm **c.** $\frac{9}{2}$ cm
11. a. $S = 54$ cm², $V = 27$ cm³ **b.** $S = 30$ cm², $V = 5\sqrt{5}$ cm³
c. $S = 72$ cm², $V = 24\sqrt{3}$ cm³ **d.** $S = 24k^2$ cm², $V = 8k^3$ cm³

Exercise 5.1

2. a. $-\frac{x}{y}$ **b.** $\frac{x}{y}$ **c.** $\frac{x^2}{5y}$ **d.** $-\frac{y}{2xy + y^2}, y \neq 0$ **e.** $\frac{3x^2}{20y^3}$ **f.** $\frac{9x}{16y}$
g. $-\frac{13x}{48y}$ **h.** $-\frac{3x + 2y^3}{6xy^2}$ **i.** $-\frac{2x}{2y + 5}$ **j.** $\frac{2y - x^2}{y^2 - 2x}$ **k.** $-\frac{y}{x}, y \neq 0$
l. $\frac{1}{1 + 5y^4}$ **m.** $\frac{3x^2y - y^3}{3y^2 - x^3}$ **n.** $-\frac{\sqrt{y}}{\sqrt{x}}$ **o.** $-\frac{y}{x}$
3. a. $2x - 3y - 13 = 0$ **b.** $2x - 3y + 25 = 0$
c. $3\sqrt{3}x + 5y + 15 = 0$ **d.** $11x - 10y + 81 = 0$
4. (0, 1)
5. a. 1 **b.** $\left(\frac{3\sqrt{5}}{5}, \sqrt{5}\right), \left(-\frac{3\sqrt{5}}{5}, -\sqrt{5}\right)$
6. -10
7. $7x - y - 11 = 0$
8. $x - 2y - 3 = 0$
9. a. $\frac{3x^2 - 8xy}{4x^2 - 3}$
10. a. 1 **b.** 1 **c.** 1 **d.** 2
12. $x - 4 = 0, 2x - 3y + 10 = 0$
15. $x^2 + y^2 - 8x + 2y - 1 = 0, x^2 + y^2 + 4x - 10y + 11 = 0$

Exercise 5.2

2. a. $90x^8 + 90x^4$ **b.** $-\frac{1}{4}x^{-\frac{3}{2}}$ **c.** 2
3. a. $v(t) = 10t - 3, a(t) = 10$ **b.** $v(t) = 6t^2 + 3b, a(t) = 12t$
c. $v(t) = 1 - 6t^{-2}, a(t) = 12t^{-3}$ **d.** $v(t) = 2t - 6, a(t) = 2$
e. $v(t) = \frac{1}{2}(t + 1)^{-\frac{1}{2}}, a(t) = -\frac{1}{4}(t + 1)^{-\frac{3}{2}}$
f. $v(t) = \frac{27}{(t + 3)^2}, a(t) = -\frac{54}{(t + 3)^3}$
4. a. i) $t = 3$ **ii)** $1 < t < 3$ **iii)** $3 < t < 5$ **b. i)** $t = 3, t = 7$
ii) $1 < t < 3, 7 < t < 9$ **iii)** $3 < t < 7$
5. $v(t) = t^2 - 4t + 3$, $a(t) = 2t - 4$, direction changes at $t = 3$ and $t = 1$ returns to original position at $t = 3$
6. a. positive at $t = 1$, negative at $t = 4$
b. neither at $t = 1$, positive at $t = 4$
c. negative at $t = 1$, positive at $t = 4$
7. a. $2t - 6$ **b.** 3 s
8. a. $t = 4$ **b.** 80 m
9. a. 3 m/s **b.** 2 m/s²
10. a. $v(t) = \frac{35}{2}t^{\frac{3}{2}} - \frac{7}{2}t^{\frac{5}{2}}, a(t) = \frac{105}{4}t^{\frac{1}{2}} - \frac{35}{4}t^{\frac{3}{2}}$ **b.** $t = 5$ **c.** $t = 5$
d. $0 < t < 3$ **e.** $t = 7$
11. a. 25 m/s **b.** 31.25 m **c.** $t = 5$, 25 m/s
12. a. Velocity is 98 m/s, acceleration is 12 m/s². **b.** 38 m/s
13. a. $v(t) = 6 - 2t, a(t) = -2$, 19 m **b.** $v(t) = 3t^2 - 12$, $a(t) = 6t$, -25 m
14. 1 s, away
15. b. $v(0) = 5 - 3k, s(t - 3k) = -9k^3 + 30k^2 - 23k$
16. No.
17. b. $v(t) = 1, a(t) = 0$

Exercise 5.3

1. a. $\frac{dA}{dt} = 4$ m²/s **b.** $\frac{dS}{dt} = -3$m²/min **c.** $\frac{ds}{dt} = 70$ km/h, $t = .25$
d. $\frac{dx}{dt} = \frac{dy}{dt}$ **e.** $\frac{d\theta}{dt} = \frac{\pi}{10}$ rad/s
2. a. decreasing at 5.9ºC/s **b.** 0.577 m c. let $T''(x) = 0$.
3. 100 cm²/s, 20 cm/s
4. a. 100 cm³/s **b.** 336 cm²/s
5. 40 cm²/s
6. a. $\frac{5}{6\pi}$ m/s **b.** $\frac{5}{3\pi}$ m/s
7. $\frac{1}{\pi}$ km/h
8. $\frac{4}{9}$ m/s
9. 8 m/min
10. 214 m/s
11. $5\sqrt{13}$ km/h
12. a. $\frac{1}{72\pi}$ cm/s **b.** 0.01 cm/s **c.** 0.04 cm/s
13. $\frac{1}{2\pi}$ m/min, 94 min
15. 0.46 m³/a
16. $\frac{2}{\pi}$ cm/min
17. $V = \frac{5\sqrt{3}}{2}s^2$ (s-side of triangle)
18. $\frac{\sqrt{3}}{4}$ m/min
19. 144 m/min
20. 62.83 km/h
21. $\frac{4}{5\pi}$ cm/s, $\frac{8}{25\pi}$ cm/s
22. $x^2 + y^2 = \frac{l^2}{4}, \frac{x^2}{k^2} + \frac{y^2}{(l - k)^2} = 1$
23. 96 m/s

Exercise 5.4

1. a. Yes. The function is continuous.
b. No. There is a discontinuity at $x = 2$.
c. No. The left side of the domain is not defined.
d. Yes. The function is continuous on the domain given.

2.	**Absolute Maximum**	**Absolute Minimum**
a.	+8	−12
b.	+30	−5

c.	$+100$	-100
d.	30	-20

3. a. maximum 3 at $x = 0$, minimum -1 at $x = 2$
b. maximum 4 at $x = 0$, minimum 0 at $x = 2$
c. maximum 0 at $x = 0, 3$, minimum -4 at $x = -1, 2$
d. maximum 0 at $x = 0$, minimum -20 at $x = -2$
e. maximum 8 at $x = -1$, minimum -3 at $x = -2$
f. maximum $\frac{16}{3}$ at $x = 4$, minimum 0 at $x = 0$

4. a. maximum $\frac{52}{5}$ at $x = 10$, minimum $+4$ at $x = 2$
b. maximum 4 at $x = 4$, minimum 3 at $x = 9$ or $x = -1$
c. maximum 1 at $x = 1$, minimum $\frac{1}{2}$ at $x = 0$
d. maximum 47 at $x = -3$, minimum -169 at $x = 3$
e. maximum 2 at $x = 1$, minimum -2 at $x = -1$
f. maximum $\frac{8}{5}$ at $x = 2$, minimum $\frac{16}{17}$ at $x = 4$

5. a. minimum velocity $\frac{4}{5}$ m/s, maximum velocity $\frac{4}{3}$ m/s
b. minimum velocity of 4 as $t \to \infty$

6. 20
7. a. 80 km/h **b.** 50 km/h
8. maximum 0.0083, minimum 0.00625
9. 0.049 years
10. 70 km/h, \$31.50
11. 245
12. 300

Exercise 5.5

1. $L = W = 25$ cm
2. If the perimeter is fixed, then the figure will be a square.
3. 300 m × 150 m
4. $L = 82.4$ cm, $W = 22.4$ cm, $h = 8.8$ cm
5. 10 cm × 10 cm × 10 cm
6. 100 cm^2
7. a. $r = 5.4$ cm, $h = 10.8$ cm **b.** $h{:}d = 1{:}1$
8. a. 15 cm^2 **b.** 30 cm^2 **c.** The largest area occurs when the length and width are each equal to one-half of the sides adjacent to the right angle.
9. a. $AB = 20$ cm, $BC = AD = 20$ cm **b.** $15\sqrt{3} \times 10^4$ cm^3
10. a. $h = 1.085$ m, equal sides $= 0.957$ m **b.** Yes. All the wood would be used for the outer frame.
11. $t = 0.36$ h
14. a. $r = \frac{50}{\pi}$ cm and no square **b.** $r = 7$ cm, $w = 14$ cm
15. $\sqrt{17}$
16. Both slopes $= \frac{2}{a + b}$.
17. $\frac{25}{2}$
18. $\frac{4\sqrt{3}k^3}{9}$

Exercise 5.6

1. a. \$1.80/L **b.** \$1.07/L **c.** 5625 L
2. a. 15 terms **b.** 16 term/h **c.** 20 terms/h
3. a. $t = 1$ min **b.** 1.5 **d.** maximum **e.** decreasing
4. $h = 15\,000$ m, $C = \$6000/\text{h}$
5. 375 m × 250 m
6. $W = 24.0$ m, $L = 40.8$ m, $h = 20.4$ m
7. $r = 43$ mm, $h = 172$ mm
8. 10 586 m south of the power plant
9. \$22.50
10. 6 nautical miles/h
11. 139 km/h
12. a. \$15 **b.** \$12.50, \$825
14. $r = 2.285$ m, $h = 9.146$ m or 915 cm
15. 5.91 m from stronger light
16. $r = \frac{2}{3}r_0$, velocity $= \frac{4}{27}r_0{}^3A$

Review Exercise

1. a. $-\frac{3x^2}{5y^4}$ **b.** $-\frac{y^3}{x^3}$ **c.** $\frac{2}{3y^2(x + 1)^2}$ **d.** $\frac{2xy}{3x^2 + y^4}$
e. $\frac{14x^6y}{7y^2 - 10x^7}$, $y \neq 0$ **f.** $\frac{y^{\frac{2}{5}}\left(5x^{\frac{3}{5}} - 2\right)}{3x^{\frac{3}{5}}}$
2. a. $\frac{5}{4}$ **b.** 0
3. $\frac{4}{5}, -\frac{4}{5}$
4. $f'(x) = 4x^3 + 4x^{-5}$, $f''(x) = 12x^2 - 20x^{-6}$
5. $72x^7 - 42x$
7. $v(t) = 2t + \frac{1}{\sqrt{2t - 3}}$, $a(t) = 2 - \frac{1}{\sqrt{(2t - 3)^3}}$
8. $v(t) = 1 - \frac{5}{t^2}$, $a(t) = \frac{10}{t^3}$
9. $v(t) > 0$ for $0 \leq t \leq \frac{9}{2}$, $v\left(\frac{9}{2}\right) = 0$, $v(t) < 0$ for $t > \frac{9}{2}$
10. a. maximum 0, minimum -52 **b.** maximum 16, minimum -65
c. maximum 20, minimum 12
11. a. 62 m **b.** yes, 2 m beyond the stop sign
12. $x - y - 3 = 0$
13. maximum velocity $2 + 3\sqrt{3}$ at $t = \frac{\sqrt{3}}{3}$, minimum velocity 2 at $t = 0$
14. 250
15. a. i) \$2200 **ii)** \$5.50 **iii)** \$3.00, \$3.00 **b. i)** \$24 640 **ii)** \$61.60 **iii)** 43.21, \$43.21 **c. i)** \$5020 **ii)** \$12.55 **iii)** \$0.025, \$0.024 98 **d. i)** \$2705 **ii)** \$6.762 5 **iii)** \$4.993 75, \$4.993 76
16. 2000
17. a. Object is moving away from its starting position.
b. Object is moving towards its starting position.
18. a. $\frac{1}{4\pi}$ m/h **b.** $\frac{3}{50\pi}$ m/h
19. 2 cm^2/s
20. $2\sqrt{10}$ cm^3/s
21. $\frac{8\sqrt{5\pi}}{5}$
22. decreasing; -3.75 m/s
23. a. $t = \frac{2}{3}$ s **b.** maximum **c.** $a > 0$, accelerating
24. 27.14 cm × 27.14 cm × 13.57 cm
25. large: 189.9 m × 63.2 m; small 37.98 m × 63.2 m
26. base is 11.6 d × 31.6 d, $h = 4.2\ d$
27. $r = 4.3$ cm, $h = 8.6$ cm
28. Run the pipe 7199 m along the river from A, then cross to R.
29. 10:35
30. either \$204 or \$206
31. Run the pipe from P to a point 5669 m along the shore from A in the direction of the refinery. Then run the pipe along the shore.

Chapter 5 Test

1. $\frac{x + 2y}{y - 2x}$
2. $3x - 4y + 7 = 0$

3. a. 4 m/s **b.** 2 s, 4 s **c.** 12 m/s^2 **d.** towards

4. 240π m^2/s

5. a. 512 cm^3/min

b. Rate of change in volume depends on both $\frac{dr}{dt}$ and the radius. The larger the radius, the larger $\frac{dv}{dt}$ will be.

6. 1.6 cm^2/min

7. $\frac{9}{20\pi}$ m/min

8. minimum is -0.536, maximum is 1.6

9. 7.1 m/s

10. 250 m × 166.7 m

11. 162 mm × 324 m × 190 m

CHAPTER 6 THE EXPONENTIAL FUNCTION

Review of Prerequisite Skills

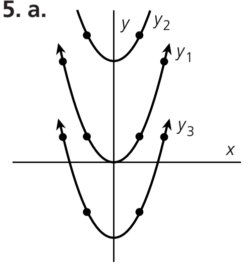
1. a. 64 **b.** 9 **c.** -27 **d.** $\frac{9}{16}$

2. a. $\frac{1}{x^7}$ **b.** $\frac{5}{m^2}$ **c.** $\frac{1}{27b^3}$ **d.** w^5 **e.** $\frac{9}{4}$

3. a. $\frac{1}{25}$ **b.** $\frac{3}{2}$ **c.** $\frac{64}{27}$ **d.** $\frac{9}{4}$

4. a. 2 **b.** 3 **c.** $\frac{1}{2}$

5. a.



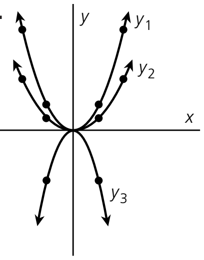
5. b. i) vertical shift of 4 units

ii) vertical shift of -3 units

c. vertical shift upwards of 4 units

d. A positive constant shifts graph upwards. A negative constant shifts graph downwards.

6. a.



6. b. i) compressed by $\frac{1}{2}$

ii) stretched by a factor of 2

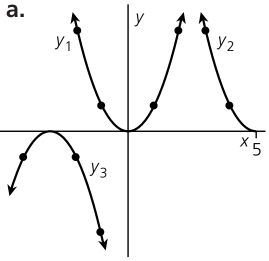
c. vertical stretch by factor of 3 and shifted upwards 25 units

d. $c < 0$, a reflection in the x-axis

$0 < c < 1$, a compression of a factor of c

$c > 1$, a stretch of a factor of c

7. a.



7. b. i) shift 5 units to the right

ii) shift 3 units left and reflected in the x-axis

c. shift 6 units to the left and 7 units down

d. A positive constant causes a shift to the left. A negative constant causes a shift to the right.

Exercise 6.1

1. a. 49 **b.** 0.16 **c.** 81 **d.** 125 **e.** 4 **f.** 64 **g.** $\frac{-1}{16}$ **h.** 1 **i.** $\frac{100}{9}$

j. 1 **k.** 6561 **l.** $\frac{1}{3}$ **m.** $-\frac{1}{2}$ **n.** $2^{-10} = \frac{1}{1024}$ **o.** 729

2. a. $\frac{x^2}{y^2}$ **b.** x^4y^4 **c.** $\frac{9a}{b^4}$ **d.** $\frac{8}{gh^2}$ **e.** x^3y^6 **f.** $\frac{b}{c}$ **g.** $\frac{5x^5}{2y^6}$ **h.** $\frac{\pi x}{4y^2}$

i. $\frac{1}{25x^4y^2}$ **j.** $\frac{a^6b^3}{c^3}$ **k.** $\frac{b^3}{a^6}$ **l.** b^8

3. a. $\frac{x^2}{3y^3}$ **b.** $\frac{\sqrt[3]{b^2}}{\sqrt{a}}$ **c.** x^4 **d.** $x^{\frac{2}{3}}y^{\frac{1}{12}}$ **e.** $\frac{2}{3a^3b}$ **f** $\frac{25}{x^7\sqrt{y}}$

4. a. $8x^2$ **b.** 8 **c.** 81 **d.** $8a^{\frac{5}{4}}$ **e.** $3p^2$ **f.** $2a^2$ **g.** a^6 **h.** $5^{\frac{-7}{12}}$ **i.** $t^{\frac{19}{6}}$

5. a. 12 **b.** $a + b$ **c.** $\frac{(p + q^2)^3}{q}$ **d.** $\frac{x - 1}{2x^4}$ **e.** $\frac{3t^2 - 2}{t^4}$ **f.** $\frac{3p^5 - 1}{p^7}$

6. a. $x^2 - x - x^{-\frac{1}{2}}$ **b.** $\frac{4\sqrt{x} - x}{x^2}$ **c.** $\sqrt{x} + 3$ **d.** $-\frac{\sqrt{x} + 1}{\sqrt{x}}$

7. By the law of exponents, $(a^m)^n = a^{mn}$, so $64^{\frac{1}{6}} = (8^2)^{\frac{1}{6}} = 8^{\frac{1}{3}}$.

Exercise 6.2

1.

	i)	ii)	iii)	iv)	v)
a.	1	decreasing	$0 < b < 1$	$x = 1, y = \frac{1}{2}$ $x = -1, y = 2$	$y = \left(\frac{1}{2}\right)^x$
b.	1	increasing	$b > 1$	$x = 1, y = 4$ $x = -1, y = \frac{1}{4}$	$y = 4^x$
c.	1	decreasing	$0 < b < 1$	$x = 1, y = \frac{1}{3}$ $x = -1, y = 3$	$y = \left(\frac{1}{3}\right)^x$
d.	1	increasing	$b > 1$	$x = 1, y = 8$ $x = -1, y = \frac{1}{8}$	$y = 8^x$

2. a. positive **b.** always increases **c.** 1

3. a. positive **b.** always decreases **c.** 1

4. Find b in the point $(1, b)$ on the graph

Exercise 6.3

1.

	Equation of Asymptote	Function Is	y-intercept
a.	$y = -5$	increasing	-4
b.	$y = 4$	increasing	5
c.	$y = 0$	decreasing	4
d.	$y = 2$	decreasing	3
e.	$y = -1$	increasing	1
f.	$y = 1$	decreasing	6

2. a. i) $y = 5$ **ii)** 8 **iii)** increasing **iv)** domain: $x \in R$, range: $y > 5$, $y \in R$

3. a. i) $y = -4$ **ii)** -2 **iii)** decreasing **iv)** domain: $x \in R$, range: $y > -4$ $y \in R$

4. The graph of $y = ab^x + c$ can be sketched with asymptote $y = c$ and y-intercept $y = a + c$, and if $b > 1$, it always increases or if $0 < b < 1$, it always decreases.

Exercise 6.4

1. 948 000

2. \$21 600

3. a. $P = 5000(1.07)^5$ **b. i)** 6125 **ii)** 13 800 **c.** $10\frac{1}{4}$ years

4. \$221 000

5. \$9500

6. \$0.65

7. 0.22 g

8. a. 20 days **b.** 5 days ago **c.** 10 days ago **d.** 25 days ago

9. a. \$4.14 **b. i)** 8 years **ii)** 35 years ago

10. a. 28 g **b.** 2 g **c.** 7 h

11. a. 15 h **b.** $A = 160\left(\frac{1}{2}\right)^{\frac{t}{15}}$ **c.** 174 mg **d.** 11.5 mg

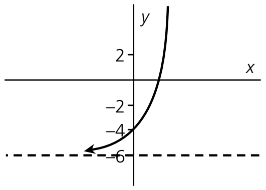
12. 5 h **b.** 320
14. a. 783 000 **b.** 2032
15. \$1075
16. B

Exercise 6.5

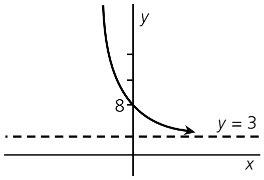
1. b. $y = 996.987(1.143)^x$ **c.** 3794 **d.** 17 h 15 min
2. a. $y = 0.660(1.462)^x$ **b.** 6.45 billion **c.** 2061
3. a. $y = 283.843(1.032)^x$ **b.** 317 348 **c.** 2062
4. a. $y = 9.277(2.539)^x$ **b.** 105 **c.** 1977
5. Answers will vary.
6. graphing, finite differences

Review Exercise

1. a. $\frac{72}{17}$ **b.** $\frac{1}{6}$ **c.** 27 **d.** 400
2. a. $\frac{1}{8}$ **b.** $\frac{9}{25}$ **c.** $\frac{1}{2}$ **d.** $\frac{25}{8}$
3. a. a^{2q^2} **b.** $x^{\frac{1}{18}}$ **c.** x^{-b^2} **d.** 2^{7p+q}
4. a. $\left(1 + \frac{5}{x}\right)\left(1 + \frac{3}{x}\right)$ or $x^{-2}(x + 5)(x + 3)$ **b.** $x^{\frac{1}{2}}(1 - x)(1 + x)$
c. $x^{-3}(x + 4)(x - 3)$ or $\frac{1}{x}\left(1 + \frac{4}{x}\right)(1 - \frac{3}{x})$ **d.** $x^{-\frac{1}{2}}(x + 5)(x - 5)$
5. a. $y = 8^x$ **b.** $y = \left(\frac{1}{3}\right)^x$
6. a. i) $y = -6$ **ii)** -4 **iii)** increasing **iv)** $x \in R, y > -6, y \in R$
b.



7. a. i) $y = 3$ **ii)** 8 **iii)** decreasing **iv)** domain: $x \in R$, range $y > 3, y \in R$
b.

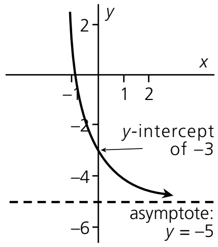


8. 1 638 400
9. 8 days
10. a. $y = 29040.595(1.0108)^x$ **b.** 34 487
c. 2011 **11. a. i)** 0.8 million/year **ii)** 3.79 million/year
iii) fivefold increase **b. i)** 0.38 million/year
ii) 2.77 million/year **iii)** sevenfold increase

Chapter 6 Test

1. a. 8 **b.** 25 **c.** $-\frac{5}{8}$ **d.** 16 **e.** 6 **f.** $-\frac{1}{5}$
2. a. a^3 **b.** $9x^4y^2$ **c.** x^6y^{-7} **d.** x^{2a} **e.** $x^{p^2-q^2-p-q}$ **f.** $x^{\frac{11}{12}}$
3. $x^{\frac{1}{2}} + 4$
4. positive, $b > 1$, increases; $0 < b < 1$, decreases; $b = 1$, constant

5. a. i) $y = -5$ **ii)** -3 **iii)** decreasing **iv)** domain: $x \in R$, range: $y > -5, y \in R$
b.



6. \$10 330
7. 2729
8. 3.5 min
9. a. $y = .660(1.462)^x$ **b.** 43 billion **c.** 4.65 m^2/person
d. Answers will vary.
10. a. $f(x) = 2^x + 3$

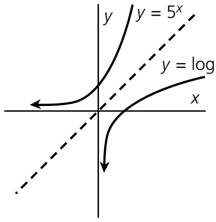
CHAPTER 7 THE LOGARITHMIC FUNCTION AND LOGARITHMS

Review of Prerequisite Skills

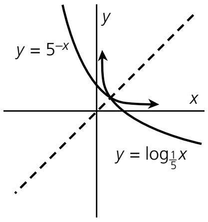
2. a. positive **b.** increasing **c.** 1
3. a. positive **b.** decreasing **c.** 1
4. approximately 7700
5. 32 h
6. a. 6.59 g **b.** 520 years

Exercise 7.1

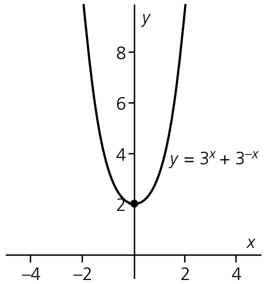
1. a. $\log_3 9 = 2$ **b.** $\log_9 1 = 0$ **c.** $\log_{\frac{1}{2}} \frac{1}{4} = 2$ **d.** $\log_{36} 5 = \frac{1}{2}$
e. $\log_{27} 9 = \frac{2}{3}$ **f.** $\log_2 \frac{1}{8} = -3$
2. a. $5^3 = 125$ **b.** $7^0 = 1$ **c.** $5^{-2} = \frac{1}{25}$ **d.** $7^{-1} = \frac{1}{7}$ **e.** $\left(\frac{1}{3}\right)^{-2} = 9$
f. $9^{\frac{3}{2}} = 27$
3. a. 1.5682 **b.** -0.6198 **c.** 3 **d.** 1.7160 **e.** 0.1303 **f.** 4.7214
4.



5.



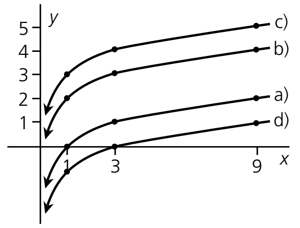
6. a. 3 **b.** 2 **c.** 4 **d.** 2 **e.** -3 **f.** -3 **g.** $\frac{1}{2}$ **h.** 4 **i.** $\frac{5}{4}$
7. a. 0 **b.** $-\frac{5}{2}$ **c.** $\frac{1}{2}$ **d.** $\frac{3}{4}$ **e.** $\frac{12}{5}$ **f.** $\frac{4}{3}$
8. a. 125 **b.** 16 **c.** 3 **d.** -3 **e.** $\frac{1}{3}$ **f.** 16
10.



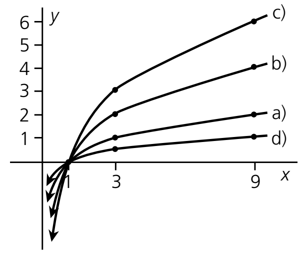
11. 23

Exercise 7.2

1. a. $\log_a x + \log_a y$ **b.** $\log_m p + \log_m q$
2. a. $\log_a(xw)$ **b.** $\log_a(sr)$
3. a. $\log_b x - \log_b y$ **b.** $\log_a r - \log_a s$
4. a. $4\log_6 13$ **b.** $-2\log_5 1.3$ **c.** $\frac{1}{3}\log_7 x$ **d.** $-\frac{3}{4}\log_a 6$
5. a. $\log_b x + \log_b y - \log_b z$ **b.** $\log_a x - \log_a y - \log_a z$
6.



7.



8. a. 3 **b.** 5 **c.** 2 **d.** 2 **e.** 5 **f.** 3
9. a. 1 **b.** 3 **c.** 2 **d.** 3 **e.** 4 **f.** 7 **g.** 3 **h.** 2 **i.** 3
10. a. $\frac{2}{3}\log_a x + \frac{4}{3}\log_a y$ **b.** $\frac{3}{2}\log_a x + \log_a y - \frac{1}{2}\log_a w$
c. $\frac{23}{8}\log_a x + \frac{11}{3}\log_a y$ **d.** $\frac{5}{4}\log_a x - \frac{3}{4}\log_a y$
11. a. 1.347 **b.** -0.1084 **c.** -1.4978 **d.** 1.8376 **e.** 0.1513 **f.** 2.0614
12. a. 2.5 **b.** -6.93 **c.** $-\frac{2}{3}$ **d.** 0.4889 **e.** -2.6178 **f.** -0.5831
13. a. -2.45 **b.** -0.83 **c.** 0.09 **d.** 0.59 **e.** 5.5 **f.** 2.4 **g.** 1.93 **h.** 0.64
14. a. $\log_a[\frac{\sqrt[3]{x}\sqrt[4]{y}}{\sqrt[5]{w^2}}]$ **b.** $\log_5\left(\frac{x^4}{y^2}\right) \div \log_5 w^3$
15. a. vertical translation of 1 unit up
b. vertical stretch of a factor of 2, vertical translation of 3 upwards **c.** vertical stretch of a factor of 3, upward vertical translation of 3 units
16. a. $\frac{77}{12}$ **b.** $\frac{23}{12}$
17. a. i) increases by 3 log2 or 0.9 **ii)** decreases by 3log2 or 0.9
b. i) increases by 5log4 or 3.01 **ii)** decreases by 5log5 or about 3.5

Exercise 7.3

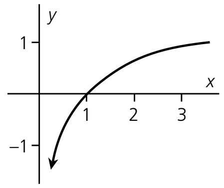
1. a. 16 **b.** 81 **c.** 6 **d.** 49
2. a. 1.46 **b.** 1.11 **c.** 2.32 **d.** 1.16
3. a. 72 **b.** $\frac{4}{3}$ **c.** $\pm$ 4 **d.** $\frac{1}{81}$ **e.** 64 **f.** $\frac{1 + \sqrt{61}}{2}$
4. a. 1 **b.** 5 **c.** 2 **d.** 2 **e.** 3
5. $y = \log_a x$ is defined only if $x > 0$ and $a > 0$.
6. 4 years, 3 months
7. 2400 years
8. 6 years
9. 1450; no.
10. 81
11. 2.23×10^{11}

Exercise 7.4

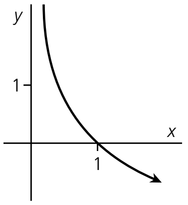
2. 10 times
3. 60 dB
4. 5.06
5. 100 times
6. 40 000 times
7. a. 5 times
8. 5 times
9. 32 000 times
10. 10 000
11. 10 000
12. 13
13. 3.2×10^{-7} mol/L
14. 3.2×10^{-7} mol/L

Exercise 7.5

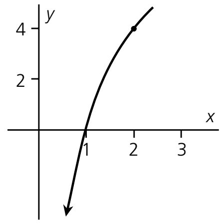
1. a. 1.892 **b.** 2.477 **c.** 0.656 **d.** 1.116
3. a.



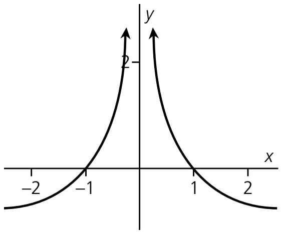
b.



c.



d.



4. Graph is reflected in y-axis.

Review Exercise

1. a. 3 **b.** -3 **c.** 2.5 **d.** $\frac{2}{3}$
2. a. 2 **b.** 6 **c.** $-\frac{1}{3}$ **d.** 5
3. a. $\frac{1}{8}$ **b.** 2 **c.** $\frac{11}{8}$ **d.** 3 or 2
4. twice as intense
5. 100 000 times
6. 2.4×10^{-6} mol/L
7. vertical stretch by a factor 2, translated 2 units up
8. a. 1.894 **b.** 2.202

Chapter 7 Test

1. a. 3 **b.** 3 **c.** -4 **d.** $\frac{1}{4}$ **e.** t **f.** $\frac{2}{3}$
2. a. 4 **b.** 2
3. vertically stretch by factor 2, translated 2 units up
4. a. 8 **b.** 4 **c.** 2 **d.** 3
5. Log of a negative number does not exist.
6. 16.87 h
7. 6.3 times
8. 1000 times
9. 4.90×10^{-9} mol/L

Cumulative Review—Chapters 5–7

1. a. $-\frac{x}{y}$ **b.** $\frac{x}{4y}$ **c.** $\frac{5 - 2x}{32y - 4}$ **d.** $\frac{4x - y}{x - 2}$ **e.** $-\frac{y^2}{x^2}$ **f.** $-\frac{2}{3}$
2. a. $2x - 3y + 13 = 0$ **b.** $27x + 11y - 59 = 0$

c. $x + y - 2 = 0$ **d.** $25x + 6y - 37 = 0$

3. a. $5x^4 - 15x^2 + 1; 20x^3 - 30x$ **b.** $\frac{4}{x^3}; -\frac{12}{x^4}$ **c.** $-2x^{-\frac{3}{2}}; 3x^{-\frac{5}{2}}$ **d.** $4x^3 + 4x^{-5}; 12x^2 - 20x^{-6}$

4. a. $20x^3 - 60x^2 + 42x - 6$ **b.** $-60x^3 - 72x + 2$

5. a. $s(t) = 3t^3 - 40.5t^2 + 162t; v(t) = 9t^2 - 81t + 162;$ $a(t) = 18t - 81$ **b.** stationary 3, 6; advancing $0 \le t < 3$, $6 < t \le 8$; retreating $3 < t < 6$ **c.** 4.5 **d.** $0 \le t < 4.5$ **e.** $4.5 < t \le 8$

6. a. $v(t) = 6t^2 + 6t - 36$ **b.** $a(t) = 12t + 6$ **c.** 61

7. i) a. \$4600 **b.** \$5.11 **c.** \$5.00 **ii) a.** \$8030 **b.** \$8.92 **c.** \$0.02

8. a. \$26, \$25, \$25.60, \$27

9. a. $-\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}$ **b.** 1000

10. a. 0.2 p.p.m./year **b.** 0.15 p.p.m.

11. Radius is decreasing at $\frac{5}{64\pi}$cm/min and surface area is decreasing at 2.5cm^2/min.

12. $\frac{1}{10\pi}$ m/h

14. a. 1 **b.** 2^{x+1} **c.** 1 **d.** 243 **e.** e^{3-x} **f.** e^{12x}

15. a. -3 **b.** 1, 3 **c.** 9 **d.** 2, 3 **e.** 0 **f.** 0

16. a. 19 940 **b.** 80 000

17. 26

18. a. $C(t) = P(1.05)^t, 0 \le t \le 10$ **b.** \$65.07 **c.** \$24.95

19. a. $V(t) = 30\,000(0.75)^t, t \ge 0$ **b.** \$16 875 **c.** 8 years

20. $y = 1200(0.6)^t, 0 \le t \le 4$

22. a. 2 **b.** 4 **c.** 5 **d.** -3 **e.** $\frac{1}{3}$ **f.** $-\frac{3}{2}$ **g.** 1 **h.** 0.342 **i.** -2 **j.** 7 **k.** $\frac{1}{3^{10}}$ **l.** a^4

23. a. $\log 2 - \log 3$ **b.** $\log x + \log y - \log z$ **c.** $-\log 5$ **d.** $\frac{1}{2}\log(x + 1) - \frac{1}{2}\log(x - 1)$ **e.** $4[\log(x^2 - 4) - 5\log x]$ **f.** $\log_a 4 + 5$

24. a. $\log\frac{x - 4}{3 - x}$ **b.** $\log_2\frac{x^3y^2}{z^4}$ **c.** $\log\frac{9}{\sqrt{x^2 + 1}}$ **d.** $\log \frac{x\sqrt[3]{x + 1}}{x - 5}$

25. a. 3.58 **b.** -0.63 **c.** 1.89 **d.** $0.\overline{6}$ **e.** 1.29 **f.** -3.91

26. a. 3 **b.** 2 **c.** 51 **d.** 10 **e.** 2 **f.** $\frac{24}{11}$ **g.** 2.8 **h.** -1 **i.** 14 **j.** $10^{-5}, 10^2$

27. a. 74 dB **b.** Yes. **c.** 1.0×10^7 W/m^2 **d.** 0.1 W/m^2

CHAPTER 8 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Review of Prerequisite Skills

1. a. $\frac{1}{9}$ **b.** 4 **c.** $\frac{1}{9}$ **d.** $\frac{9}{4}$.

2. a. $\log_5 625 = 4$ **b.** $\log_4\left(\frac{1}{16}\right) = -2$ **c.** $\log_x 3 = 3$ **d.** $\log_{10} 450 = W$ **e.** $\log_3 z = 8$ **f.** $\log_a T = b$

3. a. $11^2 = 121$ **b.** $125^{\frac{1}{3}} = x$ **c.** $a^4 = 1296$ **d.** $b^w = A$

4. a. 5 **b.** -4 **c.** 2 **d.** 2 **e.** 25 **f.** -6

5. a. 6.322 **b.** 2.397

Exercise 8.1

1. The graphs are identical.

2. The Power Rule is valid only when the function has the variable x in the base and a constant for the exponent.

3. a. $y' = 3e^{3x}$ **b.** $s' = 3e^{3t-5}$ **c.** $y' = 20e^{10t}$ **d.** $y' = -3e^{-3x}$ **e.** $y' = (2x - 6)e^{5-6x+x^2}$ **f.** $y' = \frac{1}{2}e^{\sqrt{x}}\, x^{-\frac{1}{2}}$

4. a. $y' = 6x^2e^{x^3}$ **b.** $y' = e^{3x}(1 + 3x)$ **c.** $f'(x) = x^{-1}e^{-x^3}(-3x^2 - x^{-1})$ **d.** $s' = 2e^{3t^2}\left(\frac{3t^2 - 1}{t^3}\right)$ **e.** $f'(x) = e^x\left[\frac{2x + 1}{2\sqrt{x}}\right]$ **f.** $h'(t) = 2te^{t^2} - 3e^{-t}$ **g.** $p' = (1 + e^w)e^{w+e^w}$ **h.** $g'(t) = \frac{2e^{2t}}{(1 + e^{2t})^2}$

5. a. $p'(1) = e^3 - e^{-3}$ **b.** $f'(0) = \frac{1}{e}$ **c.** $h'(-1) = -(2 + 3e)$

6. a. $x - 2y + 2 = 0$ **b.** $y = 0.499999958335x + 1$ **c.** Clearly the calculator is giving a 12 decimal place approximation to slope $\frac{1}{2}$, which is very awkward to use.

7. $x - 3y + (1 + \ln 3) = 0$

8. $y - e^{-1} = 0$

9. (0, 0) and $(2, 4e^{-2})$

11. a. $-3e^{-3x}, 9e^{-3x}, -27e^{-3x}$ **b.** $\frac{d^ny}{dx^n} = (-1)^n3^ne^{-3x}$

12. a. $x - y + 1 = 0$ **b.** $2x + y - 2 = 0$ **c.** In order to use the calculator, the equations must be reorganized to define y as a function of x. This is not easy to do with the relations given in this question.

13. a. 31 000 **b.** $\frac{dN}{dt} = -\frac{10^2}{3}e^{-\frac{t}{30}}$ **c.** decreasing at 17 per hour **d.** 31 000

14. a. $v = 40\left(1 - e^{-\frac{t}{4}}\right)$ **c.** 40 m/s **d.** $t = 4 \ln 20$, $s = 160(\ln 20 - 0.95)$

15. a. 1 **b.** e^2

16. $m = 2$ or -3

Exercise 8.2

1. A natural logarithm has base e; a common logarithm has base 10.

2.

n	100	10 000	100 000	10^9
$\left(n + \frac{1}{n}\right)^n$	2.70481	2.71815	2.71827	2.7182818

3. a. $\frac{5}{5x + 8}$ **b.** $\frac{2x}{x^2 + 1}$ **c.** $\frac{15}{t}$ **d.** $\frac{1}{2(x + 1)}$ **e.** $\frac{3t^2 - 4t}{t^3 - 2t^2 + 5}$ **f.** $\frac{2z + 3}{2(z^2 + 3z)}$

4. a. $1 + \ln x$ **b.** $\frac{1 - 2 \ln x}{x^3}$ **c.** 1 **d.** $\frac{3(\ln x)^2}{x}$ **e.** $e^t\left[\frac{1}{t} + \ln t\right]$ **f.** $\frac{-ze^{-z}}{e^{-z} + ze^{-z}}$ **g.** $\frac{e^t(t \ln t - 1)}{t(\ln t)^2}$ **h.** $\frac{e^{\sqrt{u}}}{2}\left(\frac{1}{u} + \frac{\ln\sqrt{u}}{\sqrt{u}}\right)$ **i.** $\frac{x^2 - 2x - 1}{(x^2 + 1)(x - 1)}$

5. a. $g'(1) = 2e$ **b.** $f'(5) = \frac{1}{10}$ **c.** $g'(1) = 2e \doteq 5.436563657$. The CALC button produces a value $g'(1) = 5.43657$, which is accurate to only 4 decimal places. For $f'(5)$, the CALC button produces in the first approximation $x = 5.042553$ and $f'(x) = 0.0983198$. The theoretical result is 0.1. The ZOOM must be used to improve the accuracy.

6. a. 0 **b.** no solution **c.** $0, \pm\sqrt{e - 1}$

7. a. $x - 3y - 1 = 0$ **c.** The first approximation answer on a window with domain $-1 \le x \le 4$ is $y \doteq 0.31286x - 0.31286$. This can be improved by using the ZOOM feature. Notice the equation is not as easy to use as the theoretical result.

8. $x - 2y + (2 \ln 2 - 4) = 0$

9. a. $(1, 0), (e^{-1}, e^{-2})$ **c.** The theoretical approach gives more accurate values in less time.

10. $x + 2y - 2 \ln 2 = 0$

11. a. 90 km/h **b.** $-\frac{90}{3t + 1}$ **c.** $\frac{-90}{7}$ km/h/s **d.** 6.36 s

12. a. 4.2 **b.** 1.16

14. a. $-\frac{(2 + \ln 2)}{2}$ **b.** -9

15. $\frac{1}{2}$

16. b. $S_3 = 2.5$, $S_4 = 2.\dot{6}$, $S_5 = 2.708\dot{3}$, $S_6 = 2.71\dot{6}$, $S_7 = 2.7180\dot{5}$

17. a. $\frac{1}{x}$ **b.** $\frac{2}{2x + 1}$ **c.** $2x \ln |x| + x$

Exercise 8.3

1. a. $3 \ln 2(2^{3x})$ **b.** $\ln 3.1(3.1^x) + 3x^2$ **c.** $3 \ln 10(10^{3t-5})$
d. $\ln 10(2n - 6)(10^{5-6n+n^2})$ **e.** $\frac{3x^2 - 4x}{\ln 5(x^3 - 2x^2 + 10)}$
f. $\frac{2}{\ln 10(1 - x^2)}$ **g.** $2(\ln 7)t(7^{t^2})$ **h.** $\frac{2t + 3}{2 \ln 2(t^2 + 3t)}$
i. $2(\ln 3)x(3^{x^2+3})$

2. a. $\frac{2^t(\ln 2t - 1)}{t^2}$ **b.** $\frac{2^{x+2}}{x \ln 2}\,[x\log_2 x(\ln 2)^2 + 1]$ **c.** $\frac{2 \ln 5 - \ln 4}{\ln 3}$
d. $2t \log_{10}(1 - t) - \frac{t^2}{(1 - t)\ln 10}\, x$ **e.** $\frac{3^{\frac{x}{2}}[x \ln 3 - 4]}{2x^3}$
f. $\left[\frac{2\sqrt{x + 1}}{x \ln 5} - \frac{\log(3x^2)}{2\sqrt{x + 1}}\right](x + 1)^{-1}$

3. a. $\frac{5}{52 \ln 2}$ **b.** $\frac{1}{24 \ln 2 \ln 3}$

4. $\frac{-3\ln 10}{4}$

5. a. $y = \left(20 \ln 10 + \frac{7}{\ln 10}\right)x + 10 - 5\left(20 \ln 10 + \frac{7}{\ln 10}\right)$
c. A first approximation, using the DRAW tool, gives $y = 53.05x - 255.3$. The theoretical calculation for the slope is $\left(20 \ln 10 + \frac{7}{\ln 10}\right) \doteq 49.091\,763$. To guarantee that the calculator is accurate to 3 decimal places, the ZOOM must be used until the x-coordinate value is accurate to 5 ± 0.0005.

6. $\sqrt{10} \ln 10x - 5y + 10\sqrt{10} = 0$

7. a. $x > 1$ **b.** At $x = 2$, $f'(2) = \frac{1}{2(\ln 2)^2}$. **c.** The calculator does not do base 2 logarithmic calculations. In this case, a double conversion will be required to convert the given function to base e.

8. a. 3.45 cm/m **b.** 10 min

9. a. As a ratio, $\frac{\text{Rate in 1978}}{\text{Rate in 1968}} = \frac{7.4}{1}$. **b.** The rate of increase for 1998 is 7.4 times larger than that for 1988.

10. b. 1.24 units/s

11. b. Rewrite 7^x as $e^{x \ln 7}$. **c.** The graph of $y = e^x$ is stretched vertically by a factor of ln 7.

12. c. The factor $\frac{1}{\ln 5}$ causes a vertical compression of the function $y = \ln x$.

Exercise 8.4

1. Calculator first approximations are

	Absolute Maximum	**Absolute Minimum**
a.	0.384 90	0
b.	46702.77	2.718 28
c.	10.043	−5961.9
d.	13.8355	2.804 40

2.

	Absolute Maximum	**Absolute Minimum**
a.	$\frac{2}{3\sqrt{3}} \doteq 0.3849$	0
b.	$\frac{e^{12}}{1 + \ln 12}$	e
c.	$\frac{e^3}{2}$	$-2e^8$
d.	6 ln 10 + ln 101 − ln 99	2.810 08

3. a. 5 **b.** 20 **c.** (54.9, 10) **e.** P grows exponentially to point I, then the growth rate decreases and the curve becomes concave down.

4. a. 1001 **b.** 500

5. five hundred units

6. 0.61

7. at $t = \frac{3}{4}h$

8. 47.25% when $t = 0.46h$

9. b. Growth rate in 1967 = 4.511 times growth rate in 1947.
c. Growth rate in 1967 is 7.5% of total invested.
d. total = \$59.537 billion, growth rate = 4.4849 billion per annum. **e.** \$62.5 billion, error was 3.5% **f.** Total = \$570.48959 billion and the rate of growth will be \$42.97498 billion.

10. $t = \frac{(\ln b - \ln a)}{(b - a)}$

11. a. 478 158 at $t = 38.2$ min **b.** 42.7 min

12. for course one, 10 h; for course two, 20 h

13. for course one, 8.2 h; for course two, 16.8 h

14. a. Graph $P = \frac{10000}{1 + 99e^{-t}}$. **b.** 4.595 days, $P = 5000$
c. At $t = 3$, growth rate is 1402 cells per day; at $t = 8$, the growth rate has slowed down to 311 cells per day.

Exercise 8.5

1. a. $\sqrt{10}x^{(\sqrt{10}-1)}$ **b.** $15\sqrt{2}x^{(3\sqrt{2}-1)}$ **c.** $\pi t^{(\pi-1)}$
d. $ex^{(e-1)} + e^x$

2. a. $\left(\frac{2 \ln x}{x}\right)x^{\ln x}$ **b.** $\frac{(x + 1)(x - 3)^2}{(x + 2)^3}\left[\frac{1}{x + 1} + \frac{2}{x - 3} - \frac{3}{x + 2}\right]$
c. $x^{\sqrt{x}}\left[\frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}\right]$ **d.** $\frac{1}{t^t}(-1 - \ln t)$

3. a. $2e^e$ **b.** $e(e + 2^{e-1})$ **c.** $-\frac{4}{27}$

4. $32(1 + 2 \ln 2)x - y - 16(3 + 8 \ln 2) = 0$

5. $-\frac{11}{36}$

6. $(e, e^{\frac{1}{e}})$

7. $(1, 1)$ and $(2, 4 + 4 \ln 2)$

8. $\frac{32(1 + \ln 4)^2}{(2 + \ln 4)}$

9. a. $v = \frac{t^{\frac{1}{t}}(1 - \ln t)}{t^2}$ $a = \frac{t^{\frac{1}{t}}}{t^4}(1 - \ln t)\left(1 - \ln t - 2t - \frac{t}{1 - \ln t}\right)$
b. $t = e$ and $a = -e^{\frac{1}{e} - 3})$

10. $e^{\pi} > \pi^e$

Review Exercise

1. a. $2e^{2x+3}$ **b.** $\frac{3t^2}{t^3 + 1}$ **c.** $\frac{3x^2 - 6x + 6}{x^3 - 3x^2 + 6x}$ **d.** $(5 - 6x)e^{-3x^2+5x}$
e. $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ **f.** $(\ln 2)e^x \times 2^{e^x}$

2. a. $e^x(x + 1)$ **b.** $\frac{x \ln x}{e^x}\left(\frac{1}{x} + \frac{1}{x \ln x} - 1\right)$ **c.** $\frac{\sqrt{2 + t^4}}{t} + \frac{2t^3 \ln (3t)}{\sqrt{2 + t^4}}$
d. $\frac{(x + 2)(x - 4)^5}{(2x^3 - 1)^2}\left[\frac{1}{x + 2} + \frac{5}{x - 4} - \frac{12x^2}{2x^3 - 1}\right]$ **e.** $\frac{2e^t}{(e^t + 1)^2}$
f. $e^x\left(\sqrt{x^2 + 3}\right)^{e^x}\left[\frac{x}{x^2 + 3} + \ln \sqrt{x^2 + 3}\right]$
g. $\left(\frac{30}{x}\right)^{2x}(2 \ln 30 - 2 - 2 \ln x)$ **h.** $\frac{[1 - y(x + y)e^{xy}]}{[x(x + y)e^{xy} - 1]}$

3. a. 1 **b.** $\frac{3 + 2\sqrt{3}}{3}, \frac{3 - 2\sqrt{3}}{3}$

4. a. $\left[\frac{1 + 10(\ln 10)^2}{\ln 10}\right] \times 10^9$ **b.** 0

5. a. $\frac{1}{t}$ **b.** $10e^{10x}(10x + 2)$

6. $(1 - \ln 4)x - 8y + (8 \ln 4 - 4) = 0$

8. a. 7 **b.** 4, -4 **c.** $-3, 0, 4$

9. $3x - y + \left(2 \ln 2 - 2\right) = 0$

10. $x = 1$

11. a. day 20 **b.** 42

12. 2.718 h

13. highest at 4 years, lowest at 0.368 years

14. a. c_2 **b.** c_1

15. a. $T'(x) = 10(0.9^{-x})\left(0.10536 + \frac{0.10536}{x} - \frac{1}{x^2}\right)$ **b.** 2.62

16. a. $\frac{\ln 2}{2}$ **b.** $2\ln 2 - 1$

17. a. 0 **b.** $C'(t) = k(5e^{-5t} - 2e^{-2t})$ **c.** 7.32 days

Chapter 8 Test

1. a. $-4xe^{-2x^2}$ **b.** $\frac{2x}{x^2 - 6}$ **c.** $(3^{x^2+3x})(\ln 3)(2x + 3)$

d. $\frac{1}{2}(3e^{3x} - 3e^{-3x})$ **e.** $\frac{8x^3 - 2x}{(2x - 1)\ln 10} + (12x^2 - 1)\log_{10}(2x - 1)$

f. $\frac{\frac{x}{x + 4} - 3 \ln(x + 4)}{x^4}$

2. $\frac{13}{14}$

3. 2

4. $\frac{2xy + 1 + \ln x}{3 - x^2}$

5. 1

6. $-2, -1$

7. $x + (1 + 28 \ln 3)y - (4 + 84 \ln 3) = 0$

8. b. 10 cm/s **c.** $t = \frac{\ln 2}{k}$, $a = -5k$ cm/s^2

9. a. \$87.70 **b.** \$9426.76

CHAPTER 9 CURVE SKETCHING

Review of Prerequisite Skills

1. a. $-\frac{3}{2}, 1$ **b.** $-2, 7$ **c.** $-\frac{5}{2}, -\frac{5}{2}$ **d.** $-2, -3 + 1$

2. a. $x < -\frac{7}{3}$ **b.** $x \leq 2$ **c.** $-1 < t < 3$ **d.** $x < -4$ or $x > 1$

4. a. 0 **c.** 0 **d.** 0

5. a. $x^3 + 6x^2 + x^{-2}$ **b.** $-\frac{(x^2 + 2x + 3)}{(x^2 - 3)^2}$ **c.** $-2xe^{-x^2}$

d. $x^4(5\ln x + 1)$

6. a. $x - 8 + \frac{28}{x + 3}$ **b.** $x + 7 - \frac{2}{x - 1}$

Exercise 9.1

1. a. $(0, 1), (-4, 33)$ **b.** $(0, 2)$ **c.** $\left(\frac{1}{2}, 0\right), (-2, -125), \left(\frac{9}{4}, -48.2\right)$

d. $(1,-3)$

2. Function is increasing when $f'(x) > 0$, whereas it is decreasing when $f'(x) < 0$.

3. a. rises up into quadrant I **b.** rises up into quadrant I

c. drops down into quadrant IV **d.** rises up into quadrant I

4.	**Increasing**	**Decreasing**	**Horizontal**
a.	OK	OK	$(-1, 4), (2, -1)$
b.			$(-1, 2), (1, 4)$
c.			none
d.			$(2, 3)$

5.	**Increasing**	**Decreasing**
a.	$x < -2$ or $x > 0$	$-2 < x < 0$
b.	$x < 0$ or $x > 4$	$0 < x < 4$
c.	$x < -1$ or $x > +1$	$-1 < x < 0$ or $0 < x < 1$
d.	$-1 < x < 3$	$x < -1$ or $x > 3$
e.	$x > \frac{1}{e}$	$0 < x < \frac{1}{e}$
f.	$x < 1$	$x > 1$

6. The function is increasing when $x < -3$ or $-2 < x < 1$ or $x > 1$. The function is decreasing when $-3 < x < -2$.

7. The function is increasing when $\frac{1}{2} < x < \frac{2}{3}$ or $x > 1$.

The function is decreasing when $x < \frac{1}{2}$ or $\frac{2}{3} < x < 1$.

9. $f(x) = x^3 + 3x^2 - 9x - 9$

11. a. $f(x)$ increases on $x < 4$, decreases for $x > 4$, $x = 4$

b. $f(x)$ increases when $-1 < x < 1$, $x = -1$ and 1

c. $f(x)$ decreases when $-2 < x < 3$, $x = -2$ and 3

14. strictly decreasing

Exercise 9.2

2. b. $(0, 0), (4, -32)$

3. a. $(-2, -16)$ is local minimum $(10, 0)$ is a local maximum, $(2, -16)$ is a local minimum

b. $\left(-3, -\frac{1}{3}\right)$ is local minimum, $\left(3, \frac{1}{3}\right)$ is local maximum

c. $\left(\frac{1}{4}, \frac{1}{4}e\right)$ is local maximum **d.** $\left(\frac{3}{2}, \ln\left(\frac{7}{4}\right)\right)$ is local minimum

4.	***x*-Intercept**	***y*-Intercept**
a.	$-2\sqrt{2}, 2\sqrt{2}, 0$	0
b.	0	0
c.	0	0
d.	none	ln 4

5. a. $(0, 3)$ is a local minimum, tangent parallel to x-axis; $(2, 27)$ is a local maximum, tangent parallel to x-axis

b. $(0, 0)$ is a local maximum, tangent parallel to t-axis; $\left(\frac{2}{3}, -\frac{4}{9e^2}\right)$ is a local minimum, tangent parallel to t-axis

c. $(5, 0)$ is neither **d.** $(0, -1)$ is a local minimum, tangent parallel to x-axis; $(-1, 0)$ has tangent parallel to y-axis $(1, 0)$ has tangent parallel to y-axis **e.** $(0, 0)$ is neither, tangent parallel **f.** $(0, 0)$ has tangent parallel to y-axis; $(1.516, -11.5)$ has tangent parallel to x-axis at a local minimum

7. a. $(2, 21)$ is a relative maximum

b. $(-3, 20)$ is a local maximum, $(3, -16)$ is a local minimum

c. $(-2, -4)$ is a local maximum, $(-1, -5)$ is a local minimum

d. no critical points **e.** $(1, 1)$ is a local minimum

f. $(0, 0)$ is neither, $(1, -1)$ is a local minimum

g. $(0, 1)$ is local maximum **h.** $\left(e^{-\frac{1}{2}}, -0.184\right)$ is local minimum

8. At $x = -6$, there is a local minimum. At $x = 2$, there is a local minimum. At $x = -1$ there is a local maximum.

10. $y = -\frac{11}{9}x^2 + \frac{22}{3}x + 1$

12. a. $y = 3x^4 - 4x^3 - 36x^2 - 9$ **b.** $(3, -198)$ **c.** local minima at $(-2, -73), (3, -198)$; local maxima at $(0, -9)$

13. a. local maximum at $(0, 4)$

b. local maximum at $(-\sqrt{2}, 28\sqrt{2})$; local minimum at $(\sqrt{2}, -28\sqrt{2})$

Exercise 9.3

1. a. vertical asymptotes $x = -2$, $x = 2$; horizontal asymptote $y = 1$ **b.** vertical asymptote $x = 0$; horizontal asymptote $y = 0$

3. a. 2 **b.** 5 **c.** $-\frac{5}{2}$ **d.** $\pm \infty$

4.	**Discontinuities**	**Vertical Asymptotes**
a.	$x = -5$	$x = -5$
b.	$x = 2$	$x = 2$
c.	$t = 3$	$t = 3$
d.	$x = 3$	none
e.	$x = \ln 2$	$x = \ln 2$
f.	$x \leq 0$	no asymptotes

5. a. $y = 1$ from below as $x \to \infty$, from above as $x \to -\infty$
b. $y = 0$ from above as $x \to \infty$, from below as $x \to -\infty$
c. $y = 3$ from above as $t \to \infty$, from below as $t \to -\infty$
d. no horizontal asymptote

7. a. $y = 3x + 7$ **b.** $y = x + 3$ **c.** $y = x - 2$ **d.** $y = x + 3$

8. a. As $x \to \infty$ $f(x)$ is above the line.
b. As $x \to -\infty$ $f(x)$ is below the line.

10. a. $y = \frac{a}{c}$ **b.** $x = -\frac{d}{c}$, $c \neq 0$ and $ax + b \neq k(cx + d)$

11. $a = \frac{9}{5}$ $b = \frac{3}{5}$

12. b. -2

14. $y = x + 1$

Exercise 9.4

1.	**Point A**	**Point B**	**Point C**	**Point D**
a.	negative	negative	positive	positive
b.	negative	negative	positive	negative
c.	negative	zero	negative	positive
d.	negative	zero	negative	positive

2. a. $(-1, 18)$ is a local maximum, $(5, -90)$ is a local minimum
b. $\left(0, \frac{25}{48}\right)$ is a local maximum
c. $(-1, -2)$ is a local maximum, $(1, 2)$ is a local minimum
d. neither

3. a. $(2, -36)$ **b.** $\left(-4, \frac{25}{64}\right), \left(4, \frac{25}{64}\right)$ **c.** no points **d.** $(3, 8)$

4. a. 24, curve is above **b.** 4, curve is above **c.** e, curve is above **d.** $-\frac{9\sqrt{10}}{1000}$, curve is below

5. b. i) 1 **ii)** 0, 2

6. For any $y = f(x)$
(1) evaluate $y = f'(x)$ and solve $f'(x) = 0$ to get at least one solution, x_1.
(2) evaluate $y = f''(x)$ and calculate $f''(x_1)$.
(3) if $f''(x_1) < 0$, then curve is concave down; if $f''(x_1) > 0$ then curve is concave up.

7. Step 4: Determine the type of critical point by using either the first derivative test or the second derivative test.

8. a. i) $(-2, -16), (0, 0)$ **b. i)** none **c. i)** none
d. i) $\left(\frac{-3}{\sqrt{2}}, \frac{-8\sqrt{2}}{9}\right), \left(\frac{3}{\sqrt{2}}, \frac{8\sqrt{2}}{9}\right)$

10. $f(x) = -3x^3 + 9x^2 - 1$

11. $\frac{27}{64}$

12. inflection points are $(0, 0)$, $\left(-\frac{b}{2a}, -\frac{b^4}{16a^3}\right)$

Exercise 9.5

2. $y = -\frac{1}{4}x^3 + 3x$

7. a. $y = 1$ as $x \to \infty$, $y = -1$ as $x \to -\infty$
b. $y = \frac{3}{2}$ as $x \to \infty$, $y = \frac{-3}{2}$ as $x \to -\infty$

Review Exercise

1. a. $y' = ne^{nx}, y'' = n^2e^{nx}$ **b.** $f'(x) = \frac{1}{2(x + 4)}$, $f''(x) = -\frac{1}{2(x + 4)^2}$ **c.** $s' = \frac{2e^t}{(e^t + 1)^2}, s'' = \frac{d^2S}{dt^2} = \frac{2e^t(1 - e^t)}{(e^t + 1)^3}$
d. $g'(t) = \frac{1}{\sqrt{1 + t^2}}, g''(t) = \frac{-t}{(1 + t^2)^{\frac{3}{2}}}$

2.	**Increasing**	**Decreasing**	**Derivative = 0**
a.	$x < 1$	$x > 1$	$x = 1$
b.	$x < -3$ or $x > 7$ or $-3 < x < 1$	$1 < x < 3$ or $3 < x < 7$	$x = 1, x = 7$

4. a. $(0, 20)$, is a local minimum; tangent is parallel to x-axis. $(3, 47)$ is a local maximum; tangent is parallel to x-axis.
b. $(-1, e^2)$ is a local minimum; tangent is parallel to x-axis.
c. $\left(-1, -\frac{1}{2}\right)$ is a local minimum; tangent is parallel to x-axis. $\left(7, \frac{1}{14}\right)$ is a local maximum; tangent is parallel to x-axis.
d. $(-1, \ln 5)$ is a local maximum; tangent is parallel to x-axis.

5. a. $a < x < b$ or $x > e$ **b.** $b < x < c$ **c.** $x < a$ or $d < x < e$
d. $c < x < d$

6.	**Discontinuity**	**Asymptote**	**Left Side**	**Right Side**
a.	at $x = 3$	$x = 3$	$y \to -\infty$	$y \to +\infty$
b.	at $x = -5$	$x = -5$	$g(x) \to +\infty$	$g(x) \to -\infty$
c.	at $x = \ln 4$	$x = \ln 4$	$s \to -\infty$	$s \to +\infty$
d.	at $x = -3$	none	$f(x) \to -8$	$f(x) \to -8$

7. a. $-e^{\frac{3}{2}}, -\frac{3}{e^{\frac{3}{2}}}$ **b.** $(-2, -2e^{-2})$

9. a. i) Concave up on $-1 < x < 3$, concave down on $x < -1$ or $x > 3$. **b.** Points of inflection when $x = -1$ or $x = 3$.
ii) a. Concave up on $-5 < x < 1$ or $x > 5$, concave down on $x < -5$ or $1 < x < 5$. **b.** Points of inflection when $x = -5, x = 1, x = 5$.

10. a. $a = 1, b = 0$

11. a. $y = x - 3$
b. $y = 4x + 11$ **13. a.** 18 994 when $t = 5$ **b.** when $t > 0$

15. a. $|k| \leq 2$ and $x \neq \pm k$

Chapter 9 Test

1. a. $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$
b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$
c. $(-9, 1), (-6, -2), (0, 1), (8, -2)$ **d.** $x = -3, x = 4$
e. $f''(x) > 0$ **f.** $-3 < x < 0$ or $4 < x < 8$ **g.** $(-8, 0), (10, -3)$

2. a. critical points: $\left(\frac{1}{2}, \frac{15}{8}\right), \left(-\frac{1}{2}, -\frac{17}{8}\right), (3, -45)$ **b.** $\left(\frac{1}{2}, \frac{15}{8}\right)$ is a local maximum; $\left(-\frac{1}{2}, -\frac{17}{8}\right)$ is a local minimum; $(3, -45)$ is a local minimum

4. discontinuities at $x = -2, x = 3$; vertical asymptote is $x = 3$; hole in the curve is at $\left(-2, -\frac{3}{5}\right)$

5. local minimum at $(1, -e^2)$, local maximum at $(-2, 2e^{-4})$

7. $k = \frac{11}{4}$

8. a. $f(x) = x^3 + 3x^2 + 2$

10. $k = -3$

Cumulative Review Chapters 3–9

1. a. $2 - \frac{1}{5}, 2 - \frac{1}{5^2}, 2 - \frac{1}{5^3}, 2 - \frac{1}{5^4}, \ldots 2$ **b.** $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}; 0$

2. a. $\frac{2}{3}$ **b.** $\frac{1}{2}$ **c.** 12 **d.** $-\frac{2}{5}$ **e.** 0 **f.** $\frac{1}{4}$ **g.** $-\frac{1}{3}$ **h.** $\frac{1}{2\sqrt{x}}$ **i.** $\frac{x}{2}$

3. $3x^2 - 10x + 10$

4. a. $2t + 10$ **b.** $\frac{x - 4}{x^3}$

5. a. $25t^4 - 100t^3 - 6t^2 + 34t - 35$
b. $\frac{x^4 + 4x^3 + 18x^2 - 15}{(x^2 + 2x + 5)^2}$ **c.** $e^w(2 + w)$ **d.** $\frac{4}{(e^t + e^{-t})^2}$
e. $e^x\left(\ln x + \frac{1}{x}\right)$ **f.** $(1 + \ln t) + e^t(1 + t)$
6. a. $(2t - 5)e^{(t^2 - 5t)}$ **b.** $\frac{2x + 1}{x^2 + x + 1}$ **c.** $\frac{3x^2 - 1}{2x^{\frac{3}{2}}\sqrt{3x^2 + 1}}$
d. $3\left(2 + \frac{1}{t}\right)e^{(2t+\ln t)}$ **e.** $\frac{1}{r \ln a} + \frac{3}{r}$ **f.** $\frac{y - e^{x+y}}{e^{x+y} - x}$
g. $x(a^2 - x^2)^{-\frac{3}{2}}$ **h.** $\frac{2y}{2xy - x}$
7. $-\frac{3}{4}$
8. a. $2^r(1 + r \ln 2) - 2e^{2r}(r^2 + r)$ **b.** $\frac{2a + 3bw}{2\sqrt{a + bw}}$
c. $\frac{18}{(2 + 3t)^{\frac{1}{2}}(2 - 3t)^{\frac{3}{2}}}$ **d.** $e^x + 2e^{-x}$ **e.** $-\frac{b^2x}{a^2y}$
f. $-\frac{x(x + 2y)}{(x^2 + y^2)}$
9. $e^{\pi^2}(1 + 2\pi^2)x - y - 2\pi^3e^{\pi^2} = 0$
10. a. 1, 2 **b.** 1, 2, −2
11. $\frac{10(y^2 - 6xy - x^2)}{(y - 3x)^3}$
12. e
13. $x - y - 12 = 0$ or $x - y + 12 = 0$
14. $10x - y - 32 = 0$ and $2x - y - 8 = 0$
15. $6x - 2y - (2 \ln 2 + 2) = 0$
16. a. 7 m **b.** 8.5 m/s, 9.3 m/s **c.** 1.5 m/s^2, 0.4 m/s^2 **d.** 10 m/s
17. a. −1 mm/s **b.** 0 **c.** 2 mm/s^2
18. a. 112π mm^2/s **b.** 56π mm/s^2
19. $\frac{3\sqrt{3}}{2}$ m/s
20. a. $\frac{dv}{dt}$ is rate of increase of volume; $\frac{dr}{dt}$ is rate of increase of radius; $\frac{dh}{dt}$ is rate of increase of height **b.** $V = \frac{5\pi r^3}{12}$
c. $\frac{1}{9\pi}$ cm/min
21. $a = \frac{1}{k(1 + 2\ln v)}$
22. 14:13
23. a. $(-3, 91), (2, -34); \left(-\frac{1}{2}, 28\frac{1}{2}\right)$ **b.** $(0, 3.6); \left(-\frac{5}{\sqrt{3}}, 1\right),$
$\left(\frac{5}{\sqrt{3}}, 1\right)$ **c.** $\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)\left(e^{-\frac{3}{2}}, -\frac{3e^{-3}}{2}\right)$ **d.** $\left(\frac{3}{2}, \frac{27e^{-3}}{8}\right); (0, 0),$
$\left(\frac{3 + \sqrt{3}}{2}, \frac{(3 + \sqrt{3})^3}{8}e^{-(3 + \sqrt{3})}\right),$
$\left(\frac{3 - \sqrt{3}}{2}, \frac{(3\ 3\ \sqrt{3})^3}{8}e^{\ \sqrt{3}-1}\right)$
e. $\left(-\sqrt{2}, -5\sqrt{2}e^{-\frac{1}{2}}\right), (\sqrt{2}, t\sqrt{2}e^{\frac{1}{2}}), (0, 0), \left(-\sqrt{6}, -5\sqrt{6}e^{-\frac{3}{2}}\right),$
$\left(\sqrt{6}, 5\sqrt{6}e^{-\frac{3}{2}}\right)$ **f.** $(1, 10e^{-1} + 2); (2, 20e^{-2} + 2)$
24. a. $x = -3, x = 3, y = 0; \left(0, -\frac{8}{9}\right)$ **b.** $x = -1, x = +1, y = 4x;$
$(0, 0), (-\sqrt{3}, -6\sqrt{3}), (\sqrt{3}, 6\sqrt{3})$
26. 14 062.5 m^2
27. $r = 4.3$ cm, $h = 8.6$ cm
28. $r = 6.8$ cm, $h = 27.5$ cm
29. a. $h = 140 - 2x$ **b.** $V = 101\ 629.5$ cm^3; $x = 46.7$ cm, $h = 46.6$ cm
30. $x \doteq 4.1$
31. a. 4000 **b.** 8 **d.** 6
32. $f(x) = 2x^3 - 12x^2 + 18x - 15$

34.	**Absolute Maximum**	**Absolute Minimum**
a.	82	2
b.	$9\frac{1}{3}$	2
c.	$\frac{e^4}{1 + e^4}$	$\frac{1}{2}$
d.	6.61	1

35. \$1140
36. $8x + y + 38 = 0$, $8x + 7y - 38 = 0$
38. −901 800 m^3/week
39. $f(x) = \frac{2}{9}x^3 + \frac{1}{3}x^2 - \frac{4}{3}x + \frac{7}{9}$
40. a. $x - 1\ 440\ 000 + 4y + 9 = 0$ **b.** $32x + 6y - 143 = 0$

APPENDIX A DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Review Exercise

1. a. $\frac{y}{r}$ **b.** $\frac{x}{r}$ **c.** $\frac{y}{x}$
2. a. 2π **b.** $\frac{\pi}{4}$ **c.** $-\frac{\pi}{2}$ **d.** $\frac{\pi}{6}$ **e.** $\frac{3\pi}{2}$ **f.** $-\frac{2\pi}{3}$ **g.** $\frac{5\pi}{4}$ **h.** $\frac{11\pi}{6}$
3. a. b **b.** $\frac{\mathrm{b}}{\mathrm{a}}$ **c.** a **d.** a **e.** b **f.** −b
4. a. $\cos\theta = -\frac{12}{13}, \tan\theta = -\frac{5}{12}$
b. $\sin\theta = -\frac{\sqrt{5}}{3}, \tan\theta = \frac{\sqrt{5}}{2}$
c. $\sin\theta = -\frac{2}{\sqrt{5}}, \cos\theta = \frac{1}{\sqrt{5}}$
d. $\cos\theta = 0$, $\tan\theta$ is undefined
5. a. per: π, amp: 1 **b.** per: 4π, amp: 2
c. per: 2, amp: 3 **d.** per: $\frac{\pi}{6}$, amp: $\frac{2}{7}$
e. per: 2π, amp: 5 **f.** per: π, amp: $\frac{3}{2}$
8. a. $\frac{\pi}{6}$ or $\frac{5\pi}{6}$ **b.** 0 or $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ or 2π
c. 0 or $\frac{\pi}{2}$ or π or $\frac{3\pi}{2}$ or 2π **d.** $\frac{\pi}{2}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$
e. $\frac{2\pi}{3}$ or $\frac{5\pi}{3}$ **f.** $\frac{\pi}{3}$ or $\frac{5\pi}{3}$

Exercise A1

2. a. $-\sin R$
3. a. $\frac{56}{65}$
5. c. $1 - 2\sin^2 A$
7. a. $\frac{\sqrt{3} - 1}{2\sqrt{2}}$ **b.** $\frac{\sqrt{3} - 1}{2\sqrt{2}}$ **c.** $\frac{1 - \sqrt{3}}{2\sqrt{2}}$ **d.** $\frac{-\sqrt{3} - 1}{2\sqrt{2}}$
8. a. $\frac{\sqrt{3}\cos x + \sin x}{2}$ **b.** $\frac{-\cos x - \sin x}{\sqrt{2}}$ **c.** $\frac{\cos x + \sin x}{\sqrt{2}}$
d. $-\sin x$
9. a. $\frac{-\sqrt{15} - 2\sqrt{2}}{12}$ **b.** $\frac{-2\sqrt{30} + 1}{12}$ **c.** $-\frac{7}{9}$ **d.** $\frac{-\sqrt{15}}{8}$
10. a. $\frac{3}{5}$ **b.** $\frac{4}{5}$
12. 2
13. 2
14. 2.65° and 5.2°
16. c. $\frac{2\tan A}{1 - \tan^2 A}$

Exercise A2

1. a. $2\cos 2x$ **b.** $2x - \sin x$ **c.** $-2\cos x \sin x$
d. $(3x^2 - 2)\cos(x^3 - 2x + 4)$ **e.** $8\sin(-4x)$ **f.** $\cos x - x\sin x$
g. $3\sec^2 3x$ **h.** $9\cos(3x + 2\pi)$ **i.** 0 **j.** $\frac{-1}{x^2}\cos\frac{1}{x}$
k. $-\frac{1}{2\sqrt{x}}\sin\sqrt{x}$ **l.** $6x^2\sin x + 2x^3\cos x - 3\cos x + 3x\sin x$
m. $2\cos 2x$ **n.** $\frac{-2x\sin 2x - \cos 2x}{x^2}$ **o.** $-2\cos 2x\sin(\sin 2x)$

p. $\frac{\cos x}{(1+\cos x)^2}$ **q.** $6x^2 \sec^2 x^3 \tan x^3$ **r.** $2e^x \cos x$

2. a. $y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$ **b.** $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ **c.** $y = 2x$

d. $y = -3\left(x - \frac{\pi}{2}\right)$ **e.** $y + \frac{\sqrt{3}}{2} = -\left(x - \frac{\pi}{4}\right)$

3. a. $\frac{1}{\cos y}$ **b.** -1 **c.** $\frac{-2\sec^2 2x}{3 \sin 3y}$ **d.** $\frac{-y \sin(xy)}{1 + x\sin(xy)}$

e. $-\frac{\sin y + \sin(x+y)}{x \cos y + \sin(x+y)}$

5. $-\csc x \cot x$; $\sec x \tan x$; $-\csc^2 x$

6. a. $\lim\limits_{x\to 0} \frac{\sin x}{x} = \frac{\pi}{180}$ **b.** $\frac{d}{dx} \sin x = \frac{\pi}{180} \cos x$

$\frac{d \cos x}{dx} = \frac{-\pi}{180} \sin x$

Exercise A3

1. $y = -\left(x - \frac{\pi}{2}\right)$

2. $y - \frac{5\pi}{6} = -\left(x - \frac{\pi}{3}\right)$

3. a. maximum $\sqrt{2}$; minimum $-\sqrt{2}$ **b.** maximum 2.26; minimum -5.14

4. $v = 80\pi \cos(10\pi t)$; $a = -800\pi^2 \sin(10\pi t)$

5. 5, 10, 20

6. $\frac{\pi}{3}$ rad

7. 4.5 m

8. $\frac{10\pi}{3}$ km/min

9. $\frac{3\sqrt{3}}{4}R^2$

10. 0.32 rad/s

11. 2.5 m

12. $\frac{\pi}{6}$ rad

APPENDIX B ANTIDERIVATIVES

Exercise B1

1. a. $2x + c$ **b.** $\frac{3}{2}x^2 - 4x + c$ **c.** $x^4 + \frac{1}{3}x^3 + c$ **d.** $2\ln x + c$

e. $\frac{3}{2}x^4 - \frac{2}{3}x^{\frac{3}{2}} + c$ **f.** $-\frac{1}{x} + \frac{1}{2x^2} - \frac{1}{3x^3} + c$ **g.** $-\cos 2x + c$

h. $\frac{1}{2}e^{x^2} + c$ **i.** $\frac{2}{9}(x^3 + 1)^{\frac{3}{2}} + c$ **j.** $\ln(\sin x) + c$

2. a. $4x^3 - 12x^2 + x + 5$ **b.** $2x^{\frac{3}{2}} + \cos x - 1$ **c.** $4x - \frac{3}{4}x^{\frac{4}{3}} - 8$

d. $\frac{1}{3}e^{3x} - \frac{1}{2}\ln x + \frac{2}{3}e^3$ **e.** $\frac{2}{3}\sqrt{x^3 + 1} + \frac{10}{3}$ **f.** $\frac{\sin^5 x}{5} - 1$

3. 10 051

4. 200 min

5. a. $10e^{-0.002t} - 9$ cm **b.** 0.94 cm **c.** 52.7 years

6. 8.75 m

Exercise B2

1. a. $\frac{8}{3}t^{\frac{3}{2}} + 4$ **b.** $3e^t - \ln(t + 1) - 1$ **c.** $2\left(t + \frac{1}{t+1}\right) - 2$

d. $\frac{3}{\pi} \sin(\pi t) - 1$

2. a. $-2t + 10$, $-t^2 + 10t$

b. $\frac{2}{9}(3t + 1)^{\frac{3}{2}} - \frac{2}{9}$, $\frac{4}{135}(3t + 1)^{\frac{5}{2}} - \frac{2}{9}t - \frac{4}{135}$

c. $\sin t - \cos t + 4$, $-\cos t - \sin t + 4t + 1$

d. $-\frac{2}{1 + 2t} + 2$, $-\ln(1 + 2t) + 2t + 8$

3. a. $-4.905t^2 + 450$ **b.** 9.58 s **c.** -94 m/s

4. a. $-4.905t^2 - 10t + 450$ **b.** 8.6 s **c.** -94.4 m/s

5. a. $-4.905t^2 + 10t + 450$ **b.** 10.7 s **c.** -95 m/s

6. 1.3 m/s^2

7. 1.1 m/s^2

8. 32 m/s

9. 127 m

Exercise B3

1. a. $200e^{2\cdot 2t}$ **b.** 416 **c.** 1.8 h

2. a. $150\,000e^{0.026t}$ **b.** 327 221

3. a. $200e^{-0.005t}$ **b.** 156 mg **c.** 738 days

4. 20 years

5. 14 296

6. $\frac{25000}{1 + 0.25e^{\frac{1}{3}\ln\left(\frac{6}{11}\right)t}}$

7. 9.5 min

8. 8 min