8-bit signed multiplier design using Modified (Radix-4) Booth Coding

EE577A

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Multiplication Representation

- Multiplicand: $X = -x_7 2^7 + \sum_{i=0}^{6} x_i\, 2^i$ 补码的位权展开式
- Multiplier: $Y = -y_7 2^7 + \sum_{i=0}^{6} y_i\, 2^i$
- Product: $Z = XY = \left(-x_7 2^7 + \sum_{i=0}^{6} x_i\, 2^i\right)\left(-y_7 2^7 + \sum_{i=0}^{6} y_i\, 2^i\right)$
- Partial product:
- $P_i = y_i\left(-x_7 2^7 + \sum_{i=0}^{6} x_i\, 2^i\right) = y_i X$
- $P_7 = -y_7\left(-x_7 2^7 + \sum_{i=0}^{6} x_i\, 2^i\right) = -y_7 \mathrm{X}$
- $Z = \sum_{i=0}^{7} P_i\, 2^i$

shift i bit

left

Radix-4 Booth Coding

- $Y = -y_7 2^7 + \sum_{i=0}^{6} y_i\, 2^i$
- $Y = -y_7 2^7 + y_6 2^6 + y_5 2^5 + y_4 2^4 + y_3 2^3 + y_2 2^2 + y_1 2^1 + y_0 2^0$
- $Y = (-2y_7 + y_6 + y_5)2^6 + (-2y_5 + y_4 + y_3)2^4 + (-2y_3 + y_2 + y_1)2^2 + (-2y_1 + y_0 + y_{-1})2^0$
- $Y = \sum_{i=0}^{3}(-2y_{2i+1} + y_{2i} + y_{2i-1})\, 4^i$
- Partial product:
- $P_i = (-2y_{2i+1} + y_{2i} + y_{2i-1})X$
- $Z = \sum_{i=0}^{3} P_i\, 4^i$ → PP的个数减半
- Number of partial products halves!

每个 P_i 左移2位

Booth Coding: Signed vs. Unsigned Numbers

- Case 1) X and Y are signed (two's complement) numbers
  - If Y has even bit width, the Booth coding is as shown before.
  - Otherwise, we should sign extend Y by one bit to make it have even bit width.
- Case 2) X and Y are unsigned numbers
  - If Y has even bit width, we should pad two zeros to the left of the MSB of Y.
  - Otherwise, we should pad one zero to the left of the MSB of Y.

Booth Coding

- -0 is implemented as 11111111+1

$S_i = Y_{2i+1}$

	y_{2i+1}	y_{2i}	y_{2i-1}	Partial Product	$K_{i,j}$	S_i
①	0	0	0	0X	0	0
	0	0	1	1X	X_j	0
②	0	1	0	1X	X_j	0
	0	1	1	2X	X_{j-1}	0
	1	0	0	-2X	X'_{j-1} 2X의 2의 보수	1
	1	0	1	-X	X'_j	1
	1	1	0	-X	X'_j	1
	1	1	1	-0	1	1

010 000 110 110 -2

$2^3 + -2 = +6$

$1 + 5 = 5$

Optimization : X is non-negative

								X_7	X_6	X_5	X_4	X_3	X_2	X_1	X_0		
							*	Y_7	Y_6	Y_5	Y_4	Y_3	Y_2	Y_1	Y_0		
S_0	S_0	S_0	S_0	S_0	S_0	S_0	$K_{0,8}$	$K_{0,7}$	$K_{0,6}$	$K_{0,5}$	$K_{0,4}$	$K_{0,3}$	$K_{0,2}$	$K_{0,1}$	$K_{0,0}$	$\longrightarrow$	P_0
S_1	S_1	S_1	S_1	S_1	$K_{1,8}$	$K_{1,7}$	$K_{1,6}$	$K_{1,5}$	$K_{1,4}$	$K_{1,3}$	$K_{1,2}$	$K_{1,1}$	$K_{1,0}$		S_0	$\longrightarrow$	P_1
S_2	S_2	S_2	$K_{2,8}$	$K_{2,7}$	$K_{2,6}$	$K_{2,5}$	$K_{2,4}$	$K_{2,3}$	$K_{2,2}$	$K_{2,1}$	$K_{2,0}$		S_1			$\longrightarrow$	P_2
S_3	$K_{3,8}$	$K_{3,7}$	$K_{3,6}$	$K_{3,5}$	$K_{3,4}$	$K_{3,3}$	$K_{3,2}$	$K_{3,1}$	$K_{3,0}$		S_2					$\longrightarrow$	P_3
									S_3								
Z_{15}	Z_{14}	Z_{13}	Z_{12}	Z_{11}	Z_{10}	Z_9	Z_8	Z_7	Z_6	Z_5	Z_4	Z_3	Z_2	Z_1	Z_0		

An optimization trick

- $S_0' = 1 - S_0$
- Overflow is ignored.

	1	1	1	1	1	1	1
+							S_0’
	S_0	S_0	S_0	S_0	S_0	S_0	S_0

Optimization : X is non-negative

						S_0'	if Xs ,								
1	1	1	1	1	1	1	$K_{0,8}$	$K_{0,7}$	$K_{0,6}$	$K_{0,5}$	$K_{0,4}$	$K_{0,3}$	$K_{0,2}$	$K_{0,1}$	$K_{0,0}$
				S_1'											
1	1	1	1	1	$K_{1,8}$	$K_{1,7}$	$K_{1,6}$	$K_{1,5}$	$K_{1,4}$	$K_{1,3}$	$K_{1,2}$	$K_{1,1}$	$K_{1,0}$		S_0
		S_2'													
1	1	1	$K_{2,8}$	$K_{2,7}$	$K_{2,6}$	$K_{2,5}$	$K_{2,4}$	$K_{2,3}$	$K_{2,2}$	$K_{2,1}$	$K_{2,0}$		S_1		
S_3'															
1	$K_{3,8}$	$K_{3,7}$	$K_{3,6}$	$K_{3,5}$	$K_{3,4}$	$K_{3,3}$	$K_{3,2}$	$K_{3,1}$	$K_{3,0}$		S_2				
									S_3						

Optimization : X is non-negative

						S_0’								
					1	1	$K_{0,8}$	$K_{0,7}$	$K_{0,6}$	$K_{0,5}$	$K_{0,4}$	$K_{0,3}$	$K_{0,2}$	$K_{0,1}$	$K_{0,0}$
				S_1’											
			1		$K_{1,8}$	$K_{1,7}$	$K_{1,6}$	$K_{1,5}$	$K_{1,4}$	$K_{1,3}$	$K_{1,2}$	$K_{1,1}$	$K_{1,0}$		S_0
		S_2’													
	1		$K_{2,8}$	$K_{2,7}$	$K_{2,6}$	$K_{2,5}$	$K_{2,4}$	$K_{2,3}$	$K_{2,2}$	$K_{2,1}$	$K_{2,0}$		S_1		
S_3’															
	$K_{3,8}$	$K_{3,7}$	$K_{3,6}$	$K_{3,5}$	$K_{3,4}$	$K_{3,3}$	$K_{3,2}$	$K_{3,1}$	$K_{3,0}$		S_2				
									S_3						

Optimization : X is non-negative

				S_0'	S_0	S_0	$K_{0,8}$	$K_{0,7}$	$K_{0,6}$	$K_{0,5}$	$K_{0,4}$	$K_{0,3}$	$K_{0,2}$	$K_{0,1}$	$K_{0,0}$
			1	S_1'	$K_{1,8}$	$K_{1,7}$	$K_{1,6}$	$K_{1,5}$	$K_{1,4}$	$K_{1,3}$	$K_{1,2}$	$K_{1,1}$	$K_{1,0}$		S_0
	1	S_2'	$K_{2,8}$	$K_{2,7}$	$K_{2,6}$	$K_{2,5}$	$K_{2,4}$	$K_{2,3}$	$K_{2,2}$	$K_{2,1}$	$K_{2,0}$		S_1		
S_3'	$K_{3,8}$	$K_{3,7}$	$K_{3,6}$	$K_{3,5}$	$K_{3,4}$	$K_{3,3}$	$K_{3,2}$	$K_{3,1}$	$K_{3,0}$		S_2				
									S_3						

Optimization : X is negative

				S_0	S_0'	S_0'	$K_{0,8}$	$K_{0,7}$	$K_{0,6}$	$K_{0,5}$	$K_{0,4}$	$K_{0,3}$	$K_{0,2}$	$K_{0,1}$	$K_{0,0}$
			1	S_1	$K_{1,8}$	$K_{1,7}$	$K_{1,6}$	$K_{1,5}$	$K_{1,4}$	$K_{1,3}$	$K_{1,2}$	$K_{1,1}$	$K_{1,0}$		S_0
	1	S_2	$K_{2,8}$	$K_{2,7}$	$K_{2,6}$	$K_{2,5}$	$K_{2,4}$	$K_{2,3}$	$K_{2,2}$	$K_{2,1}$	$K_{2,0}$		S_1		
S_3	$K_{3,8}$	$K_{3,7}$	$K_{3,6}$	$K_{3,5}$	$K_{3,4}$	$K_{3,3}$	$K_{3,2}$	$K_{3,1}$	$K_{3,0}$		S_2				
									S_3						

E_3'

S_{0-2} S_{0-0}

	S_{15}	S_{14}	S_{13}	S_{12}	S_{11}	S_{10}	S_9	S_8	S_7	S_6	S_5	S_4	S_3	S_2	S_1	S_0
C_{15}	C_{14}	C_{13}	C_{12}	C_{11}	C_{10}	C_9	C_8	C_7	C_6	C_5	C_4	C_3	C_2	C_1	C_0	
0	Z_{15}	Z_{14}	Z_{13}	Z_{12}	Z_{11}	Z_{10}	Z_9	Z_8	Z_7	Z_6	Z_5	Z_4	0	Z_2	0	
Z_{16}													Z_3		S_{0-1}	S_{0-0}
															Z_1	Z_0

Optimization : X

- $E_i = S_i \oplus X_7 = Y_{2i+1} \oplus X_7$

15 bit CSA

			E_0'	E_0	E_0	$K_{0,8}$	$K_{0,7}$	$K_{0,6}$	$K_{0,5}$	$K_{0,4}$	$K_{0,3}$	$K_{0,2}$	$K_{0,1}$	$K_{0,0}$
		1	E_1'	$K_{1,8}$	$K_{1,7}$	$K_{1,6}$	$K_{1,5}$	$K_{1,4}$	$K_{1,3}$	$K_{1,2}$	$K_{1,1}$	$K_{1,0}$		S_0
	1	E_2'	$K_{2,8}$	$K_{2,7}$	$K_{2,6}$	$K_{2,5}$	$K_{2,4}$	$K_{2,3}$	$K_{2,2}$	$K_{2,1}$	$K_{2,0}$		S_1	
	S_{16} 1	S_{13} E_2'	S_{12}	S_{11}	S_{10}	S_9	S_8	S_7	S_6	S_5	S_4	S_3	S_2	S_1 $= k_{0,1}$	S_0
E_3'	$K_{3,8}$	$K_{3,7}$	$K_{3,6}$	$K_{3,5}$	$K_{3,4}$	$K_{3,3}$	$K_{3,2}$	$K_{3,1}$	$K_{3,0}$		S_2				
C_{14} 0	C_{13} 0	C_{12}	C_{11}	C_{10}	C_9	C_8	C_7	C_6	C_5 S_3	C_4	C_3	C_2	C_1 0	C_0	0

$S_{1.15} = E_3'$
$C_{1.15} = 0$

E_3' $S_{1.15}$	$S_{1.14}$	$S_{1.13}$	$S_{1.12}$	$S_{1.11}$	$S_{1.10}$	$S_{1.9}$	$S_{1.8}$	$S_{1.7}$	$S_{1.6}$	$S_{1.5}$	$S_{1.4}$	$S_{1.3}$	$S_{1.2}$ $S_{0.2}$	$S_{1.1}$	$S_{1.0}$ $S_{0.0}$
$C_{1.14}$	$C_{1.13}$	$C_{1.12}$	$C_{1.11}$	$C_{1.10}$	$C_{1.9}$	$C_{1.8}$	$C_{1.7}$	$C_{1.6}$	$C_{1.5}$ S_3	$C_{1.4}$	$C_{1.3}$	$C_{1.2}$ 0	$C_{1.1}$	$C_{1.0}$ $= 0$	

$S_{2.15} = E_3'$
$C_{2.15} = 0$

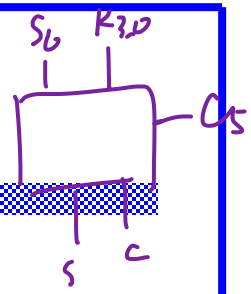
$S_{2.3} = S_{1.3}$
$C_{2.3} = 0$

$S_{2.1} = S_{1.1}$
$C_{2.1} = 0$

$S_{2.0} = S_{0.0}$
$C_{2.0} = 0$

$S_{2.15}$	$S_{2.14}$	$S_{2.13}$	$S_{2.12}$	$S_{2.11}$	$S_{2.10}$	$S_{2.9}$	$S_{2.8}$	$S_{2.7}$	$S_{2.6}$	$S_{2.5}$	$S_{2.4}$	$S_{2.3}$	$S_{2.2}$	$S_{2.1}$ $S_{1.1}$	$S_{2.0}$ $S_{0.0}$		
$C_{2.15}$ 0	$C_{2.14}$	$C_{2.13}$	$C_{2.12}$	$C_{2.11}$	$C_{2.10}$	$C_{2.9}$	$C_{2.8}$	$C_{2.7}$	$C_{2.6}$	$C_{2.5}$	$C_{2.4}$	$C_{2.3}$ $= 0$	$C_{2.2}$	$C_{2.1}$ $= 0$	$C_{2.0}$ $= 0$	0	
Z_{16}	Z_{15}	Z_{14}	Z_{13}	Z_{12}	Z_{11}	Z_{10}	Z_9	Z_8	Z_7	Z_6	Z_5	Z_4	Z_3	$S_{1.2}$ Z_2	$S_{1.1}$ Z_1	$S_{0.0}$ Z_0	

Example : X is positive and Y is positive



- $E_0 = 0 \oplus 0 = 0, E_1 = 0 \oplus 0 = 0, E_2 = 1 \oplus 0 = 1, E_3 = 0 \oplus 0 = 0$

								0	0	1	0	1	1	0	1	0		90
								*	0	0	1	1	0	1	0	1	0	* 53
						1	0	0	0	0	1	0	1	1	0	1	0	4770
					1	1	0	0	1	0	1	1	0	1	0		0	
		1	0	1	1	0	1	0	0	1	0	1			0			
			0	0	1	0	1	1	0	1	1	0	0	0	1	0		
1	0	0	1	0	1	1	0	1	0		1							
0	0	1	1	0	0	0	0	1	0	1	1	0	0	0				
1	1	1	0	1	1	0	1	0	1	0	0	0	0	1	0			
0	0	0	1	0	0	1	0	1	0	1	0	0	0	1	0			

E_0

E_2

(1x)

(1x)

(-1x)

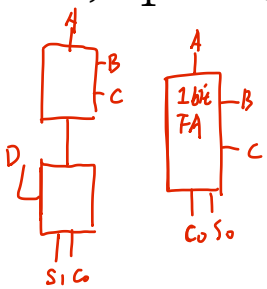
(1x)

Example : X is positive and Y is negative

- $E_0 = 0 \oplus 0 = 0, E_1 = 0 \oplus 0 = 0, E_2 = 1 \oplus 0 = 1, E_3 = 1 \oplus 0 = 1$

Example : X is negative and Y is positive

- $E_0 = 0 \oplus 1 = 1, E_1 = 0 \oplus 1 = 1, E_2 = 1 \oplus 1 = 0, E_3 = 0 \oplus 1 = 1$



								0	1	1	0	1	1	0	1	0			-38
							*	0	0	1	1	0	1	0	1	0		*	53
				0	1	1	1	1	1	0	1	1	0	1	0				-2014
			1	0	1	1	1	0	1	1	0	1	0		0				
	1	1	0	0	0	1	0	0	1	0	1		0						
0	1	1	1	0	1	1	0	1	0		1								
									0										
1	1	1	1	1	0	0	0	0	0	1	0	0	0	1	0				

HA

HA

2bit FA

3bit FA

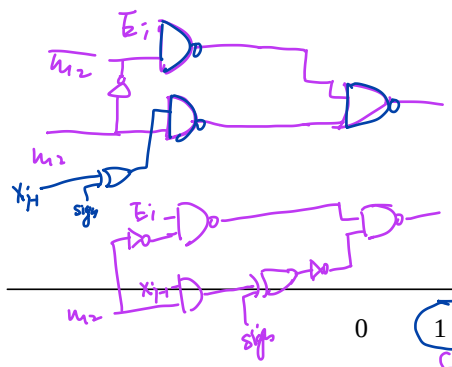
110

S=1
C=1

S=0
C0=1
C1=1

Example : X is negative and Y is negative

- $E_0 = 0 \oplus 1 = 1, E_1 = 0 \oplus 1 = 1, E_2 = 1 \oplus 1 = 0, E_3 = 1 \oplus 1 = 0$



$A + B$

$\overline{\bar{A} \cdot \bar{B}}$

$AB + CD$

$\overline{\overline{AB} \cdot \overline{CD}}$