A Practical Introduction to Rigid Body Transformations using Lie Algebra for Robotics

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Abstract

This document is a comprehensive, self-contained and practical introduction to rotations and Rigid Body Transformations (RBT) in three dimensions. In addition, this document is intended to be a practical description of the *mrob* library for RBT and its usage on realistic applications. We will provide examples of each of the concepts in python code, binding to the library written in C++. The material is gathered from lecture notes on the course *Perception in Robotics* at Skoltech.

1 Rotations and Rigid Body Transformations

In this introductory section we will describe the mathematical properties for the groups of rotations and RBT. We will also consider some interesting properties and how we can actually use these groups in multiple ways, such as state variables, vector transformations or frame operations.

1.1 Rotations

All possible matrix rotations in 3D (generalizes to any dimension) are included in the special orthogonal group

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, RR^\top = I \,\wedge\, det(R) = 1\}, \tag{1}$$

where the binary operation between two elements of the group is matrix multiplication. Since matrix multiplication is *non-commutative*, the group is *non-commutative* as well.

Four axioms of groups:

- Closure: $R_1, R_2 \in SO(3) \Longrightarrow R_1 \cdot R_2 \in SO(3)$
- Associativity: $R_1(R_2R_3) = (R_1R_2)R_3$
- Identity element: $\exists! I \in SO(3) : RI = IR = R$. There exists a unique rotation I that satisfies this condition, and this element is the matrix identity.

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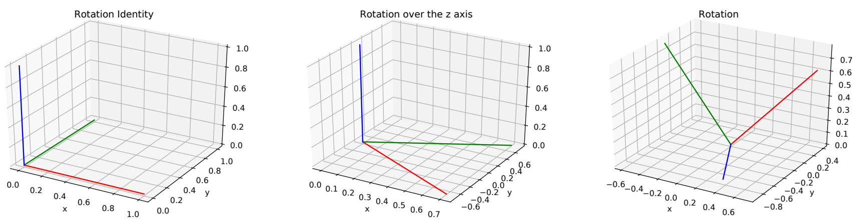

Figure 1: Examples of rotations: *Left*

- Inverse element: $\exists! R^{-1} \in SO(3) : RR^{-1} = I$. From the definition of the group one can derive the inverse element $R^{-1} = R^\top$.

The closure axiom implies that we can chain several different rotations and we will obtain a valid rotation as a result of this sequence of rotations. One has to be careful with the order, since the group operation is the matrix multiplication, meaning that in general

$$R_1 \cdot R_2 \neq R_2 \cdot R_1.$$

The group of rotations $SO(3)$ can be used to 1) transform vectors and rotate them into new reference frames; 2) to transform reference frames as well (with coincident origins); 3) another valuable application is to express orientations.

1.2 Rigid Body Transformations

Similarly to $SO(3)$, all possible rigid body transformation (RBT) matrices conform the Special Euclidean group,

$$SE(3) = \left\{ T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \mid R \in SO(3) \wedge p \in \mathbb{R}^3 \right\}, \tag{2}$$

which is the result of a rotation followed by a translation and the group operation is the matrix multiplication.

Four axioms of groups are also satisfied:

- Closure: $T_1, T_2 \in SE(3) \implies T_1 \cdot T_2 \in SE(3)$
$$T_1 \cdot T_2 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 \cdot R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix} \in SE(3)$$
- Associativity: $T_1(T_2 T_3) = (T_1 T_2) T_3$
- Identity element: $\exists! I \in SE(3) : TI = T$. There exists a unique identity element in the group which corresponds to a RBT. In particular, this is the 4×4 matrix identity.
- Inverse element: $\exists! T^{-1} \in SE(3) : TT^{-1} = I$. From the definition we can arrange terms such that the inverse corresponds to
$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 1 \end{bmatrix} \in SE(3). \tag{3}$$

As a result of the *closure* axiom one can chain a sequence of RBT and obtain a valid transformation, very similarly to rotations. The physical meaning is a sequence of different frames compose a general frame.

```
import mrob
import numpy as np
T = mrob.SE3()
```

For RBT the order matters as well,

$$T_1 \cdot T_2 \neq T_2 \cdot T_1,$$

where the left hand side and the right hand side are elements of the group, but in general they are not equal.

The potential uses of $SE(3)$ are very similar to rotations:

1. Transform points from one reference from to another.

$$^{W}p = {}^{W}T_{A} \cdot {}^{A}p$$

2. Transform reference frames:

$$^{W}T_{B} = {}^{W}T_{A} \cdot {}^{A}T_{B}$$

3. Express 3D poses (position and orientation). This is similar to the XYT parametrization for 2D poses ($SE(2)$) where 3 state variables $[x, y, \theta]$ completely define a RBT in 2D. However, we need to define which is the minimal representation for 3D poses which is the topic of the next section.

See seminar code for examples in BRT.

2 Lie Algebra for Rotations $SO(3)$

This section is devoted to explain Lie algebra for rotations. Later, the same intuition can be used to derive similar results to Rigid Body Transformations $SE(3)$ (Sec. 3)

2.1 Infinitesimal increments over Rotations $SO(3)$

First we need to understand the structure of infinitesimal variations in a rotation matrix.

As discussed before, R is orthonormal and has positive determinant, which constrains the space of solution in the differential form. A natural question arises regarding the group of rotations and RBT: What is the minimal representation? How many degrees of freedom?

To illustrate this, let's consider a rotation $R \in SO(3)$, and we are looking for a smooth rotation that provides an infinitesimal update to R in the following way:

$$\dot{R} = WR \qquad s.t. \quad RR^{\top} = I \tag{4}$$

$$\begin{aligned}\dot{R}R^\top + R(\dot{R})^\top = 0\\ W\underbrace{RR^\top}_{I} + \underbrace{RR^\top}_{I} W^\top = 0 \iff W = -W^\top.\end{aligned} \tag{5}$$

The group of matrices that satisfy (5) is known as *Skew symmetric matrices.* For 2D, the group looks like this $\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$ and for 3D rotations

$$W = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \boldsymbol{\omega}^\wedge \tag{6}$$

The hat operator $(\cdot)^\wedge$ denotes the construction of the skew symmetric matrix. In fact, the group is also a Lie Group, in particular $\boldsymbol{\omega}^\wedge \in \mathfrak{so}(3)$ around the identity element ($R = I$). The group operation is the Lie bracket operation, but we will not use it in this document. Lie groups in general need an additional property, which is smoothness (previously shown).

2.2 The Exponential Map

We have derived a differential form for rotations $\boldsymbol{\omega}^\wedge \in \mathfrak{so}(3)$ and the solution to the differential equation is of the form

$$R(t) = e^{\boldsymbol{\omega}^\wedge t} \cdot R(t_0). \tag{7}$$

Now the question is, can we solve this equation for matrices as well? There is an analogous derivation from kinematics, using the angular velocity of a frame [4]. In some sense the notation of $\boldsymbol{\omega}$ is drawn from here.

Actually this integration requires a constant $\boldsymbol{\omega}$ and a final time that we will set to $t = 1$, for instance. Some authors, in an abuse of notation notation, keep the angular velocity notation. We will follow a different convention, to distinguish between the derivative, with units $[rad/s]$ and simply an angle. Accordingly, $\boldsymbol{\theta}^\wedge = \boldsymbol{\omega}^\wedge t\Big|_{t=1}$ will corresponding to the skew-symmetric matrix of the "angle" $\boldsymbol{\theta}$.

Taylor expansion around the identity rotation.

$$\exp(\boldsymbol{\theta}^\wedge) = I + \boldsymbol{\theta}^\wedge + \frac{1}{2!}(\boldsymbol{\theta}^\wedge)^2 + \frac{1}{3!}(\boldsymbol{\theta}^\wedge)^3 + \ldots = \sum_n^\infty \frac{1}{n!}(\boldsymbol{\theta}^\wedge)^n \tag{8}$$

Skew symmetric matrices present a recursive property that turn out to be very useful, where $\theta = ||\boldsymbol{\theta}||_2$

$$(\boldsymbol{\theta}^\wedge)^2 = \omega \cdot \omega^\top - \theta^2 I, \qquad \theta^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \tag{9}$$

$$(\boldsymbol{\theta}^\wedge)^3 = (\omega\omega^\top - \theta^2 I)\,\boldsymbol{\theta}^\wedge = 0 - \theta^2\,\boldsymbol{\theta}^\wedge \tag{10}$$

and so forth. One can calculate the closed form for the series arising after the simplification given by skew symmetric matrices and will obtain the well known Rodrigues' formula:

$$R = \exp(\boldsymbol{\theta}^\wedge) = I + \frac{\sin(\theta)}{\theta}\,\boldsymbol{\theta}^\wedge + \frac{1-\cos(\theta)}{\theta^2}(\boldsymbol{\theta}^\wedge)^2 \tag{11}$$

An alternative interpretation of the Rodrigues' formula is drawn by using the angle-axis rotation:

$$R = I + \frac{\sin(\theta)}{\theta}a^\wedge + \frac{1-\cos(\theta)}{\theta^2}(a^\wedge)^2 = \cos(\theta)I + (1-\cos(\theta))aa^\top + \sin(\theta)a^\wedge \tag{12}$$

where $a = \frac{\boldsymbol{\theta}}{\theta}$ is a unit vector, the axis of rotation, and the angle of rotation around this axis is θ.

Problematic points are $\theta = 0$ and $\theta = \pm\pi$. The exponent is a surjective function, since a unique rotation can be obtained from different values of ω. The analogy with a 1D angle α is clear, where multiples values of $\alpha' = \alpha + i2\pi$, $\forall i \in \mathbb{Z}$ represent the same angle α.

This differential form is spanned by three variables:

$$\boldsymbol{\theta}^\wedge = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{G_1}\theta_1 + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{G_2}\theta_2 + \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{G_3}\theta_3 \tag{13}$$

The elements created by linear combinations of G_i span a vector space. The tangent space around the identity element in the Lie group is known as the manifold and it quite resembles a 3D Euclidean space $\mathbb{R}^3$.

There is a sequence of operations from rotations to the manifold:

$$(\cdot)^\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$$
$$\exp(\boldsymbol{\theta}^\wedge) : \mathfrak{so}(3) \to SO(3)$$

In an abuse of notation we can define the (capital) exponent as a composition of the functions above, which directly maps the manifold to rotations:

$$\mathrm{Exp}(\boldsymbol{\theta}) : \mathbb{R}^3 \to SO(3) \tag{14}$$

Useful properties of the exponent $R = \mathrm{Exp}(\omega)$

$$\mathrm{Exp}(-\boldsymbol{\theta}) = R^{-1} = R^\top \tag{15}$$
$$\mathrm{Exp}(\tau\boldsymbol{\theta}) = \mathrm{Exp}(\boldsymbol{\theta})^\tau \tag{16}$$

Is there an inverse solution? Yes, the logarithm, which can be easily obtained by writing the series of a rotation and its inverse (transpose):

$$\begin{aligned} R &= I + \frac{\sin(\theta)}{\theta}\,\boldsymbol{\theta}^\wedge + \frac{1-\cos(\theta)}{\theta^2}(\boldsymbol{\theta}^\wedge)^2 \\ R^{-1} = R^\top &= I - \frac{\sin(\theta)}{\theta}\,\boldsymbol{\theta}^\wedge + \frac{1-\cos(\theta)}{\theta^2}(\boldsymbol{\theta}^\wedge)^2 \\ \hline R - R^\top &= 0 + 2\cdot\frac{\sin(\theta)}{\theta}\,\boldsymbol{\theta}^\wedge + 0 \end{aligned}$$

$$
\begin{tikzcd}
SO(3) \arrow[r, bend left, "\ln()"] \arrow[rr, bend left=60, "\mathrm{Exp}()"] & \mathfrak{so}(3) \arrow[l, bend left, "\exp()"] \arrow[r, bend left, "(\cdot)^{\vee}"] & \mathbb{R}^3 \arrow[l, bend left, "(\cdot)^{\wedge}"] \arrow[ll, bend left=60, "\mathrm{Ln}()"]
\end{tikzcd}
$$

Figure 2: Mapping functions

which after some manipulation the following expression can be obtained:

$$\boldsymbol{\theta}^{\wedge} = \frac{\theta}{2\sin\theta}(R - R^{\top}). \tag{17}$$

The value of θ can be obtained similarly if we sum both expressions:

$$
\begin{aligned}
R + R^{\top} &= 2I + 0 + 2\frac{1-\cos(\theta)}{\theta^2}(\boldsymbol{\theta}^{\wedge})^2 \\
\mathrm{Tr}(R + R^{\top}) &= 2\,\mathrm{Tr}(I) + 2\frac{1-\cos(\theta)}{\theta^2}\,\mathrm{Tr}((\boldsymbol{\theta}^{\wedge})^2) \\
2\,\mathrm{Tr}(R) &= 2\cdot 3 + 2\frac{1-\cos(\theta)}{\theta^2}\,\mathrm{Tr}(\omega\omega^{\top} - \theta^2 I) \\
\mathrm{Tr}(R) &= 3 + \frac{1-\cos(\theta)}{\theta^2}(\theta^2 - 3\theta^2) \implies 2\cos(\theta) = \mathrm{Tr}(R) - 1
\end{aligned}
$$

which can be rearranged into the following equation to obtain θ:

$$\theta = arc\cos\left(\frac{\mathrm{Tr}(R) - 1}{2}\right). \tag{18}$$

The inverse operation is known as the logarithm, and it first maps a rotation to the Lie group:

$$\ln(R) : SO(3) \to \mathfrak{so}(3)$$

and then to map from the Lie group, to the manifold:

$$(\cdot)^{\vee} : \mathfrak{so}(3) \to \mathbb{R}^3.$$

Similarly as what we proposed to the Exp(), we can define a function that first maps a rotation to the Lie Algebra group and then to the manifold $\mathbb{R}^3$:

$$\mathrm{Ln}(R) : SO(3) \to \mathbb{R}^3.$$

3 Lie Algebra for RBT $SE(3)$

3.1 Infinitesimal increments over RBT $SE(3)$: Twists

A similar reasoning from Sec. 2 can be done, now for RBT

$$\dot{T} = \mathcal{W}T, \quad \text{s.t.} \quad T \in SE(3), \tag{19}$$

where $\mathcal{W}$ is a *Twist* of 3D poses. If we expand (19) further, we obtain

$$\mathcal{W} = \dot{T}T^{-1} = \begin{bmatrix} R^\top & -R^\top p \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^\top \dot{R} & R^\top \dot{p} \\ 0 & 0 \end{bmatrix} \tag{20}$$

One can identify the same result previously derived for $SO(3)$ plus a term related to the rotated derivative of the translation.

$$\mathcal{W} = \begin{bmatrix} \boldsymbol{\omega}^\wedge & \boldsymbol{v} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

We have denoted the components related to the rotation, representing an orientation, with $\boldsymbol{\omega}$ (angular velocities) and the components related to the translation vector as $\boldsymbol{v}$ (linear velocities).

This Twist can be integrated to obtain a RBT using the Taylor expansion

$$T(t) = e^{\mathcal{W}t} \cdot T(t_0) \tag{22}$$

As a consequence, the rotation is the integration of the *Twist*, composed of angular and linear velocities over a fixed amount of time

$$\mathcal{W}|_{t=1} = \xi^\wedge = \begin{bmatrix} \theta^\wedge & \rho \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 & \rho_1 \\ \theta_3 & 0 & -\theta_1 & \rho_2 \\ -\theta_2 & \theta_1 & 0 & \rho_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{23}$$

where there are 6 elements on the 4×4 matrix of generators.

The Lie Algebra $\mathfrak{se}(3)$ associated with the group of RBT $SE(3)$ represents the group infinitesimal RBT around the identity ($\mathcal{W} = \dot{T}$). There exist operators that relate both groups. In particular, the exponent operator $\exp : \mathfrak{se}(3) \to SE(3)$ and the logarithm $\ln : SE(3) \to \mathfrak{se}(3)$.

The vee $^\vee$ and hat $^\wedge$ operators simply encode (23) into a vector, whose space is called the manifold and from the manifold back to the Lie group. One can map a RBT $T \in SE(3)$ to $\xi \in \mathbb{R}^6$ by $\xi = \ln(T)^\vee$ and vice-versa $T = \exp(\xi^\wedge)$. In general, this mapping is surjective, but if $||w|| < \pi$, then we can consider it bijective.

The topic of Lie Algebra for RBT is vast and well documented. We just reviewed those concepts that are used on the sections below. For a more complete discussion on Lie algebra and its applications please check [3, 2, 5, 1].

4 Adjoint

4.1 $SO(3)$

$$R \cdot \text{Exp}(\theta) = \text{Exp}(Adj_R \cdot \theta) \cdot R \tag{24}$$

$$Adj_R = R \tag{25}$$

$$
\begin{tikzcd}
SE(3) \arrow[r, bend left, "\ln()"] \arrow[rr, bend left=60, "\mathrm{Exp}()"] & \mathfrak{se}(3) \arrow[l, bend left, "\exp()"] \arrow[r, bend left, "(\cdot)^{\vee}"] & \mathbb{R}^6 \arrow[l, bend left, "(\cdot)^{\wedge}"] \arrow[ll, bend left=60, "\mathrm{Ln}()"]
\end{tikzcd}
$$

Figure 3: Mapping functions

4.2 $SE(3)$

$$T \cdot \mathrm{Exp}(\xi) = \mathrm{Exp}(Adj_T \cdot \xi) \cdot T \tag{26}$$

Where ξ is a vector in the manifold and expresses a global transformation.

$$Adj_T = \begin{bmatrix} R & 0 \\ t^{\wedge}R & R \end{bmatrix}_{6\times 6} \tag{27}$$

The physical meaning of the adjoint for 3D RBT is a simple change of coordinates, however, these coordinates are expressed directly in the manifold.

5 Random Variables and Error Propagation

For state estimation, we are still interested in distributions, and that will be true for RBT as well. Then, the question is how to propose a distribution over a group of matrices which has some redundancies? In order to solve that, we need variables that are the minimal representation for $SE(3)$. We have discussed before that Lie algebra provides us the tools to do that. The peculiarity around the Lie groups is that we want to add some randomness into an element of the group and as a result mwe must obtain an element of the group too. For that, we can define a Gaussian random variable in the manifold and compose a *noisy* element. For $\bar{T} \in SE(3)$

$$T = \mathrm{Exp}(\delta) \cdot \bar{T}, \qquad \delta \sim \mathcal{N}(0, \Sigma_T) \tag{28}$$

We will follow a left-side expansion for adding small perturbations to the mean element $\bar{T}$.

References

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