L09: Particle Filter and Monte-Carlo Localization

* Summary of L08 localization.

$$m = \left\{ \begin{bmatrix} m_{1,x} \\ m_{1,y} \end{bmatrix}, \begin{bmatrix} m_{2,x} \\ m_{2,y} \end{bmatrix}, \dots \begin{bmatrix} m_{j,x} \\ m_{j,y} \end{bmatrix}, \dots \right\}$$ map of known landmarks

The localization problem becomes a state estimation problem.

$bel(x_t) = p(x_t \mid \mathcal{U}, Z, m)$ $\Rightarrow$ EKF, UKF.

Assume (for now) $c_t^i = j$ $(z^i \to m_j)$ known correspondence

Particle filter (PF)

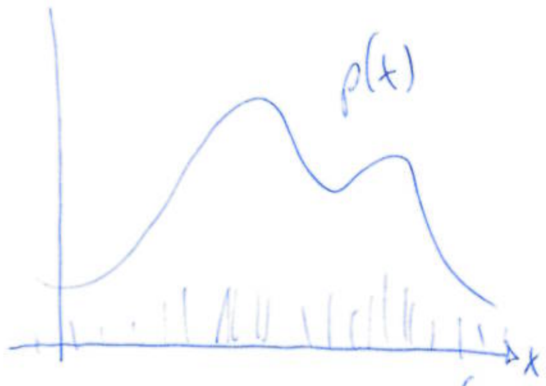
$$\begin{cases} \overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, dx_{t-1} \\ bel(x_t) = \eta\, p(z \mid x_t)\, \overline{bel}(x_t) \end{cases}$$

→ Gaussian filters: KF (linear functions); EKF, UKF (non-linear) } Unimodal distribution.

→ Non-parametric — Particle filter (PF)

Particle set $\mathcal{X}_t = \{ \langle x_t^{[1]}, w_t^{[1]} \rangle, \dots \langle x_t^{[M]}, w_t^{[M]} \rangle \}$

Ex:



(PS1)

$x^{[m]} \sim p(x)$

$w^{[m]} = p_t(x^{[m]})$

(Importance factors)
Weighted samples. Particles become a good representation of distributions (if M is large enough)

Q: weights on PS1? Sample mean, sample covariance?

1: Particle filter ($\mathcal{X}_{t-1}, u_t, z_t$):

2: for $m = 1:M$

3: $x_t^{[m]} \sim p(x_t \mid u_t, x_{t-1}^{[m]})$ propagation $\overline{bel}(x_t)$

4: $w_t^{[m]} = p(z_t \mid x_t^{[m]})$ correction

5: $\bar{\mathcal{X}}_t = \bar{\mathcal{X}}_t \cup \langle x_t^{[m]}, w_t^{[m]} \rangle$

6: $\mathcal{X}_t =$ resample* ($\bar{\mathcal{X}}_t$) (better correction)

$\mathcal{X}_t$ are "drawn" from $bel(x_t)$ and not $\overline{bel}$

[Gordon] reading introduces resampling as required for the PF to work properly.

* Bayes filter for full states

$$bel(x_{0:t}) = p(x_{0:t} \mid u_{1:t}, z_{1:t})$$

particles $x_{0:t}^{[m]} = x_0^{[m]}, x_1^{[m]}, \dots, x_t^{[m]}$ (new) Sequence of samples of states.

$$bel(x_{0:t}) = \eta\, p(z_t \mid x_{0:t}, z_{1:t-1}, u_{1:t})\, p(x_{0:t} \mid z_{1:t-1}, u_{1:t})$$

Markov + Bayes $= \eta\, p(z_t \mid x_t) \cdot p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})\, p(x_{0:t-1} \mid z_{1:t-1}, u_{1:t})$

$$= \eta\, p(z_t \mid x_t)\, p(x_t \mid x_{t-1}, u_t)\, \underbrace{p(x_{0:t-1} \mid z_{1:t-1}, u_{1:t-1})}_{bel(x_{0:t-1})}$$

$$\overline{bel}(x_{0:t}) = p(x_t \mid x_{t-1}, u_t)\, bel(x_{0:t-1})$$

$$bel(x_{0:t}) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_{0:t})$$

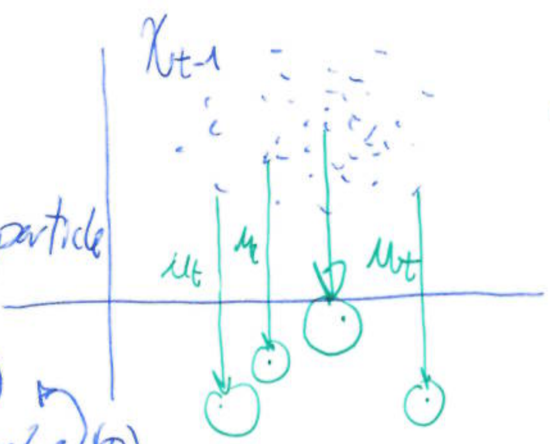
From this full state Bayes (no marginalization)

given a particle $x_{t-1}^{[m]} \sim bel(x_{0:t-1})$ (Q: $w_{t-1}^{[m]}$ on samp

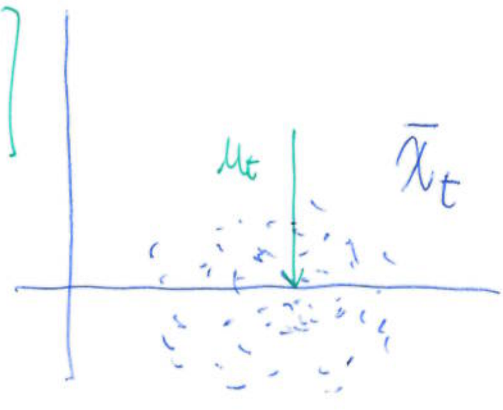
$\overline{bel}(x_{0:t})$ $\begin{cases} \boxed{\bar{x}_t^{[m]} \sim p(x_t \mid x_{t-1}^{[m]}, u_t)} \cdot (w_{t-1}^{[m]}) \\ \bar{w}_t^{[m]} = 1 \cdot w_{t-1}^{[m]} \end{cases}$ Sample.

importance factor from bel.

Ex:



$u_t = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$



for each particle we sample $p(x_t \mid x_t^{[m]}, u_t)$ Motion model (R)

* Importance sampling

$$E_p\{I(x\in A)\} = \int I(x\in A)\, p(x)\, dx$$

$$= \int I(x\in A)\, \frac{p(x)}{q(x)} \cdot q(x)\, dx$$

$$= E_q\{ I(x\in A)\cdot w(x)\}$$

$w(x) = \dfrac{p(x)}{q(x)}$ Importance factor.

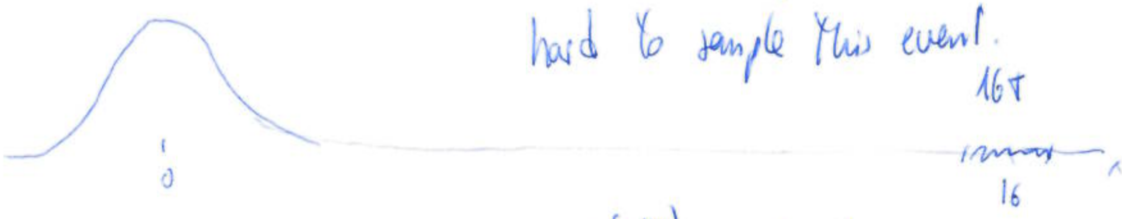
$p(x)$ target distribution (usually we can't use)

$q(x)$ proposal distribution.

Ex: Probability of sample a 1d r.v X in $[15, 17]$ if $p(x) \neq N(0,1)$? $A = \{x \mid 15 \leq x \leq 17\}$

1) $P(x\in A) = \sum I(x^m \in A)\cdot p(x^m)$, $x^m \sim p(x^m)$

hard to sample this event.



2) I.S.: $P(x\in A) = \sum I(x^m\in A) \underbrace{\frac{p(x^m)}{q(x^m)}}_{w^m} \cdot q(x^m)$, $x^m \sim q(x)$

for instance $q(x) = N(16, 1)$
(proposal distribution)

With this proposal distribution we don't need trillions of samples but only hundreds.

$$w^m = \frac{N(x^m; 0, 1)}{N(x^m; 16, 1)}$$

→ Correction step (line 4)

$$\underbrace{bel(x_{0:t})}_{\text{target distribution}} = \eta\, p(z_t | x_t) \cdot \underbrace{\overline{bel}(x_{0:t})}_{\bar{\mathcal{X}}_t}$$

Proposal distribution

$\bar{\mathcal{X}}_t$ particle set representing this belief.

In order to correctly characterize the posterior $bel(x_{0:t})$ we are going to weight the particles drawn (already) $\bar{\mathcal{X}}_t$ with proper weights or importance factors.

$x_t^{[m]} = \bar{x}_t^{[m]}$ (previously drawn) from proposal dist

$$w_t^{[m]} = \eta \cdot \frac{\text{target distribution}}{\text{proposal distribution}} = \cancel{\eta\, p(z_t | x_t^{[m]})}$$

$$= \frac{\eta\, p(z_t | x_t^{[m]}) \cdot \cancel{\overline{bel}(x_{0:t}^{[m]})}}{\cancel{\overline{bel}(x_{0:t}^{[m]})}\ (\bar{w}_t^{[m]})} = \boxed{\eta \cdot p(z_t | x_t^{[m]})}$$

Ex:

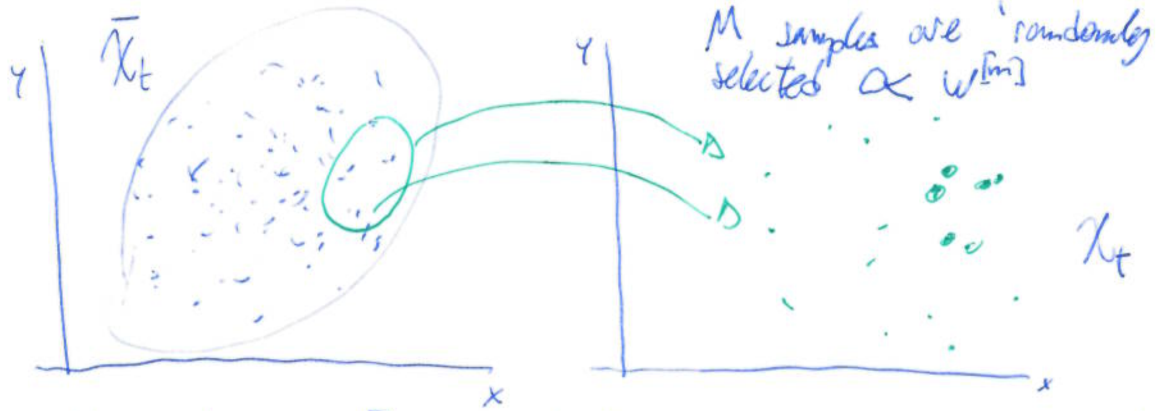
The size of the point corresponds to $\bar{w}_t^{[m]}$ new weight $w_t^{[m]}$

Problem: creates an almost empty set of particles with weights non-zero and many particles with low weights. ⇒ $\boxed{\text{Degeneracy over Time.}}$

* Resampling (The solution)

Idea: survival of the fittest. Only the most likely particles ($w^{[m]} \uparrow$) 'might' survive.

Ex:

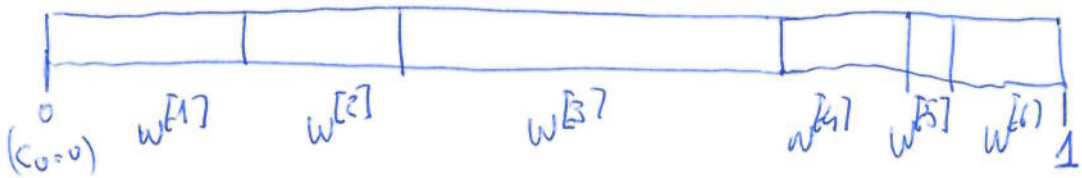


from M samples on $\bar{X}_t$ we get M samples on X_t (closer to true belief)

→ Independent Resampling. First solution

We create a cumulative distribution function:

$$C_m = C_{m-1} + w^{[m]} \qquad (+ \text{ normalization})$$

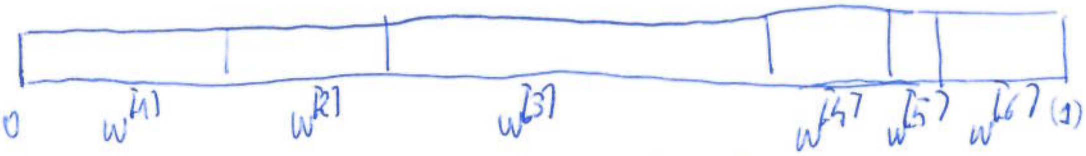


for $m = 1:M$
 $u \sim \mathcal{U}[0, 1]$ (uniform distribution)
 $j = \text{find}(c_m, u)$
 $X_t = X_t \cup \langle x_t^{[j]}, w_t^{[j]} \rangle$

⚠ Problem: over time, independent resampling induces a loss of diversity in the particle population X_t

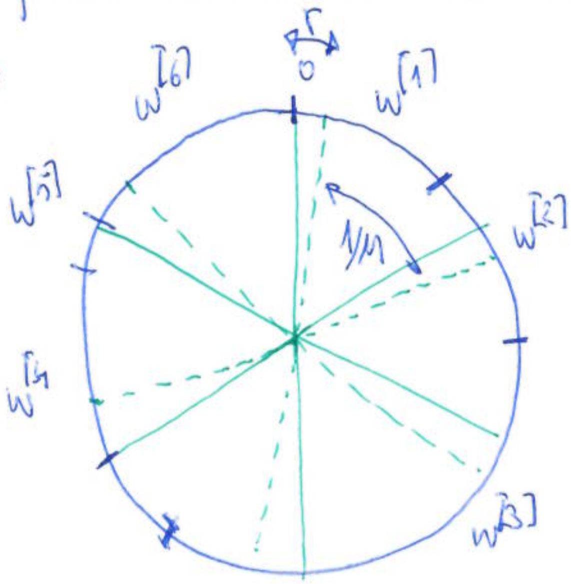
→ low-variance sampling

$$c_m = c_{m-1} + w^{[m]}$$



Idea: given an initial random configuration r, we add particles at intervals $1/M$ over the full set.

Ex:



$r = 0$

$\mathcal{X}_t = \{1, 2, 3, 3, 3, 5\}$

$r \sim \mathcal{U}[0, 1/M]$

$\mathcal{X}_t = \{$

Algorithm: low_variance_sampler $(\bar{\mathcal{X}}_t)$: (ProbRob 110)

$\mathcal{X}_t = \emptyset$, $c = w_t^{[1]}$, $i = 1$

$r \sim \mathcal{U}[0, 1/M]$

for $m = 1:M$

  $U = r + (m-1) \cdot \frac{1}{M}$

  while $U > c$

    $i{+}{+}$

    $c = c + \bar{w}^{[i]}$

  $\mathcal{X}_t = \mathcal{X}_t \cup \langle \bar{x}_t^{[i]}, \frac{1}{M} \rangle$

return $\mathcal{X}_t$.

* Monte-Carlo localization (MCL)

@ readings : Dellaert'99

We want to solve the Markov localization (L08) using PF.

$$bel(x_t) = p(x_t \mid u, z, m) \longrightarrow \mathcal{X}_t$$

map of landmarks

Particle set representing belief.

Algorithm: MCL $(\mathcal{X}_{t-1}, u_t, z_t, m)$:

1: $\bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$

2: for $m = 1 : M$

3: $x_t^{[m]} = \text{sample\_motion\_model}(u_t, x_{t-1}^{[m]})$ (L05)

4: $w_t^{[m]} = \text{measurement\_model}(z_t, x_t^{[m]}, m)$ (L08)

5: $\bar{\mathcal{X}}_t = \bar{\mathcal{X}}_t \cup \langle x_t^{[m]}, w_t^{[m]} \rangle$

6: $\mathcal{X}_t = \text{low\_variance\_sampling}(\bar{\mathcal{X}}_t)$

Next lecture SLAM: ProbRob Ch + readings @ canvas