Computer Project 1: Impact Absorption for Satellites

Out: 08/26/2022, Due: 09/26/2022

The computer project is assigned to a team of about 5 students. The team needs to submit <u>*one*</u> *report (electronically in PDF, scanned or typed) detailing (1) problem definition, (2) analytical solution, (3) numerical method, (4) results, (5) discussion, (6) conclusions, with the code attached. An example report is available on Canvas. The figures have to be generated using a computer language, with the axis labels, titles, and legends added. After the report submission, everyone will be asked to complete an online form that rates the contributions of all the team members, including themselves. The outcome of the online form will potentially result in differences in scores for different members of the same team; so that (1) those who do not contribute enough may end up with low scores even if the project is done well; (2) those who made sufficient contributions may end up with high scores even if the project is not done well.*

A company SpaceY plans to launch a satellite on their rocket Kestrel. There are many delicate devices on the satellite that are susceptible to accelerations, but strong impulsive loads are expected during the rocket launch. Therefore, it is desirable to mount the satellite on a vibration isolator to mitigate the impact of impulses. The CEO of SpaceY, Nolek Sum, asks for your help to design such a mount for their satellite on Kestrel.

After some simplification and modeling, the motion of a satellite on the mount during the rocket launch is governed by an ODE (all the quantities nondimensionalized),

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = f(t) \tag{1}$$

with the initial conditions

$$x(0) = x_o, \quad \dot{x}(0) = \dot{x}_o \tag{2}$$

where $\ddot{x}$, $\dot{x}$, and x are acceleration, velocity, and displacement of the satellite, respectively. The acceleration $\ddot{x}$ is effectively the force acting on the satellite, and ideally this acceleration should be as small as possible. The damping ratio ζ and natural frequency ω are two parameters of the satellite mount. The damping ratio determines how much impact is absorbed by the mount and hence is not passed to the satellite, while the natural frequency is a property of the mount and will be kept as constant (its value will be given below). The forcing term $f(t)$ is due to the acceleration of the rocket after ignition at $t = a$,

$$f = T_0 u(t - a) \tag{3}$$

where u is the unit step function and T_0 is its amplitude.

The goal of this project is to design the satellite mount, i.e. choose the damping ratio ζ, such that the maximum amplitude $A_{\max}$ of the satellite $\ddot{x}$ is less than a threshold A_c.

1 Analytical Solutions

(1) When $\zeta < 1$, and using Laplace Transform, show that the homogeneous solution (i.e. $f = 0$) to Eq. (1) is,

$$x(t) = e^{-\zeta\omega t}[c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)] \tag{4}$$

where $\omega_d = \omega\sqrt{1-\zeta^2}$.

(2) Next, using Laplace Transform, and assuming $x(0) = \dot{x}(0) = 0$, find the full solution to Eq. (1). When t is sufficiently large, what is the acceleration of the satellite, and why?

(3) Finally, in preparation for the numerical solution, derive the vector form of Eqs. (1) and (2),

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_o \tag{5}$$

where $\mathbf{y} = [y_1, y_2] = [x, \dot{x}]$.

2 Numerical Solutions

We start with developing a computer program that numerically solves Eq. (5), i.e. the response of the satellite given rocket vibration, using the RK2 algorithm presented in the class notes (NOT the improved Euler's method). We will verify the program by comparing the numerical and analytical solutions with the following numerical conditions,

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad \omega = 2, \quad \zeta = 0.05, \quad T_0 = 1.0, \quad a = 0.0 \tag{6}$$

(1) Using a step size of $\Delta t = 0.1$, generate the numerical solution over a time span of $t = [0, 8.0]$ using (1) the `ode45` from MATLAB® or `scipy.integrate.solve_ivp` from Python (or other language of your choice), and (2) your own program.

(2) Compare the numerical solutions to the analytical solution found in Question 1(2). The comparison should be visualized by *one figure* containing the time history of displacement and velocity from the three solutions. How does the three sets of solutions compare, e.g. in terms of accuracy? Hint: At least two sets of solutions should look almost identical.

(3) Verify the order of accuracy of the RK2 algorithm by checking the numerical errors associated with a few step sizes (e.g. $\Delta t = 0.1, 0.05, 0.025, \cdots$). The numerical error ϵ of the vectorized numerical solutions can be defined as,

$$\epsilon = ||\mathbf{y}_{exact} - \mathbf{y}_{numerical}|| \tag{7}$$

where $||\cdot||$ means the vector norm and $||\mathbf{y}|| = \sqrt{y_1^2 + y_2^2} = \sqrt{x^2 + \dot{x}^2}$. *One figure* should be provided to visualize the error v.s. step size plot, with a reference line of second-order accuracy. Hint: It would be better to plot in log-log scale, and you should be able to obtain a straight line.

3 Satellite Mount Design

Now that the computer program is ready, we proceed to design the vibration isolator for the satellite. SpaceY has given us the necessary parameters of their rocket and mount,

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad \omega = 100, \quad T_0 = 1.0, \quad a = 0.0, \quad A_c = 0.05 \tag{8}$$

Our goal is to find the range of ζ that reduces the maximum acceleration to less than $A_c = 0.05$.

(1) Develop another program that calls your RK2 program, such that it

   (a) Takes in a value of ζ;
   (b) Generates the numerical solution over a time span of your choice for Eqs. (5) and (8) using a step size of $\Delta t = 0.001$;
   (c) Returns the maximum acceleration $A_{\max}$ of the numerical solution;
   (d) As a sanity check, this program should return $A \approx 0.47$ when $\zeta = 0.05$ at $t = 0.01s$.

   Note that here you need to determine the time span of simulation for the satellite by your engineering intuition.

(2) Do a sweep of the damping ratio ζ over the range $[0.0, 2.0]$, with a step of $\Delta\zeta = 0.1$. Provide *one figure* of the ζ v.s. $t_{\max}$ plot. Identify the range of ζ that satisfies the design requirement $A_{\max} \leq A_c$ when $t \geq 0.05s$. What trend do you observe in this plot and what would be the physical reasons?

(3) **[Bonus, 10 pts]** In the actual operation, the satellite mount has some nonlinearities in its damping

$$\ddot{x} + 2\zeta\omega(1 - kx^2)\dot{x} + \omega^2 x = f(t) \tag{9}$$

   where $k = 5 \times 10^7$ and the initial conditions remain the same. Now for this nonlinear equation, we cannot solve analytically and have to rely on numerical solutions. Try to solve Eq. (9) using the RK2 program - you probably only need to change one line of code. Then determine the new range of ζ that satisfies the design requirement in acceleration. Does the range expand or shrink? Try to give an explanation.