

Deep Generative Models

Lecture 2

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Recap of previous lecture

We are given i.i.d. samples $\{\mathbf{x}_i\}_{i=1}^n \in X$ (e.g. $X = \mathbb{R}^m$) from unknown distribution $\pi(\mathbf{x})$.

Goal

We would like to learn a distribution $\pi(\mathbf{x})$ for

- ▶ evaluating $\pi(\mathbf{x})$ for new samples (how likely to get object \mathbf{x} ?);
- ▶ sampling from $\pi(\mathbf{x})$ (to get new objects $\mathbf{x} \sim \pi(\mathbf{x})$).

Challenge

Data is complex and high-dimensional.

MLE problem

Fix probabilistic model $p(\mathbf{x}|\boldsymbol{\theta})$ – the set of parameterized distributions, such that $p(\mathbf{x}|\boldsymbol{\theta}) \approx \pi(\mathbf{x})$.

$$\boldsymbol{\theta}^* = \arg \max_{\boldsymbol{\theta}} p(\mathbf{X}|\boldsymbol{\theta}) = \arg \max_{\boldsymbol{\theta}} \prod_{i=1}^n p(\mathbf{x}_i|\boldsymbol{\theta}) = \arg \max_{\boldsymbol{\theta}} \sum_{i=1}^n \log p(\mathbf{x}_i|\boldsymbol{\theta}).$$

Recap of previous lecture

Likelihood as product of conditionals

Let $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{x}_{1:i} = (x_1, \dots, x_i)$. Then

$$p(\mathbf{x}|\boldsymbol{\theta}) = \prod_{i=1}^m p(x_i|\mathbf{x}_{1:i-1}, \boldsymbol{\theta}); \quad \log p(\mathbf{x}|\boldsymbol{\theta}) = \sum_{i=1}^m \log p(x_i|\mathbf{x}_{1:i-1}, \boldsymbol{\theta}).$$

MLE problem for autoregressive model

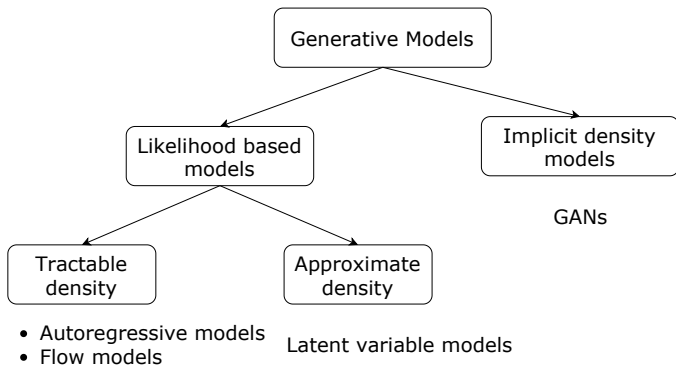
$$\boldsymbol{\theta}^* = \arg \max_{\boldsymbol{\theta}} p(\mathbf{X}|\boldsymbol{\theta}) = \arg \max_{\boldsymbol{\theta}} \sum_{i=1}^n \sum_{j=1}^m \log p(x_{ij}|\mathbf{x}_{i,1:j-1}\boldsymbol{\theta}).$$

Sampling

$$\hat{x}_1 \sim p(x_1|\boldsymbol{\theta}), \quad \hat{x}_2 \sim p(x_2|\hat{x}_1, \boldsymbol{\theta}), \dots, \quad \hat{x}_m \sim p(x_m|\hat{\mathbf{x}}_{1:m-1}, \boldsymbol{\theta})$$

New generated object is $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_m)$.

Generative models zoo



Bayesian framework

Bayes theorem

$$p(\mathbf{t}|\mathbf{x}) = \frac{p(\mathbf{x}|\mathbf{t})p(\mathbf{t})}{p(\mathbf{x})} = \frac{p(\mathbf{x}|\mathbf{t})p(\mathbf{t})}{\int p(\mathbf{x}|\mathbf{t})p(\mathbf{t})d\mathbf{t}}$$

- ▶ \mathbf{x} – observed variables, \mathbf{t} – unobserved variables (latent variables/parameters);
- ▶ $p(\mathbf{x}|\mathbf{t})$ – likelihood;
- ▶ $p(\mathbf{x}) = \int p(\mathbf{x}|\mathbf{t})p(\mathbf{t})d\mathbf{t}$ – evidence;
- ▶ $p(\mathbf{t})$ – prior distribution, $p(\mathbf{t}|\mathbf{x})$ – posterior distribution.

Meaning

We have the prior knowledge $p(\mathbf{t})$ of our unobserved variables \mathbf{t} . We have got the observed data \mathbf{x} . Posterior distribution $p(\mathbf{t}|\mathbf{x})$ summarizes what you know after the data has been observed.

Bayesian framework

Let consider the case, where the unobserved variables \mathbf{t} is our model parameters θ .

- ▶ $\mathbf{X} = \{\mathbf{x}_i\}_{i=1}^n$ – observed samples;
- ▶ $p(\theta)$ – prior parameters distribution (we treat model parameters θ as random variable).

Posterior distribution

$$p(\theta|\mathbf{X}) = \frac{p(\mathbf{X}|\theta)p(\theta)}{p(\mathbf{X})} = \frac{p(\mathbf{X}|\theta)p(\theta)}{\int p(\mathbf{X}|\theta)p(\theta)d\theta}$$

Bayesian inference

$$p(\mathbf{x}|\mathbf{X}) = \int p(\mathbf{x}|\theta)p(\theta|\mathbf{X})d\theta$$

Note the difference from

$$p(\mathbf{x}) = \int p(\mathbf{x}|\theta)p(\theta)d\theta.$$

Bayesian framework

Posterior distribution

$$p(\theta|\mathbf{X}) = \frac{p(\mathbf{X}|\theta)p(\theta)}{p(\mathbf{X})} = \frac{p(\mathbf{X}|\theta)p(\theta)}{\int p(\mathbf{X}|\theta)p(\theta)d\theta}$$

Bayesian inference

$$p(\mathbf{x}|\mathbf{X}) = \int p(\mathbf{x}|\theta)p(\theta|\mathbf{X})d\theta$$

If evidence $p(\mathbf{X})$ is intractable (due to multidimensional integral), we can't get posterior distribution and make the honest inference.

Maximum a posteriori (MAP) estimation

$$\theta^* = \arg \max_{\theta} p(\theta|\mathbf{X}) = \arg \max_{\theta} (\log p(\mathbf{X}|\theta) + \log p(\theta))$$

Bayesian framework

MAP estimation

$$\theta^* = \arg \max_{\theta} p(\theta|\mathbf{X}) = \arg \max_{\theta} (\log p(\mathbf{X}|\theta) + \log p(\theta))$$

Now θ^* is a deterministic variable, but we could treat it as a random variable with density $p(\theta|\mathbf{X}) = \delta(\theta - \theta^*)$.

Dirac delta function

$$\delta(x) = \begin{cases} +\infty, & x = 0; \\ 0, & x \neq 0; \end{cases} \quad \int f(x)\delta(x-y)dx = f(y).$$

MAP inference

$$p(\mathbf{x}|\mathbf{X}) = \int p(\mathbf{x}|\theta)p(\theta|\mathbf{X})d\theta \approx p(\mathbf{x}|\theta^*).$$

Latent variable models (LVM)

MLE problem

$$\theta^* = \arg \max_{\theta} p(\mathbf{X}|\theta) = \arg \max_{\theta} \prod_{i=1}^n p(\mathbf{x}_i|\theta) = \arg \max_{\theta} \sum_{i=1}^n \log p(\mathbf{x}_i|\theta).$$

Challenge

$p(\mathbf{x}|\theta)$ could be intractable.

Extend probabilistic model

Introduce latent variable \mathbf{z} for each sample \mathbf{x}

$$p(\mathbf{x}, \mathbf{z}|\theta) = p(\mathbf{x}|\mathbf{z}, \theta)p(\mathbf{z}); \quad \log p(\mathbf{x}, \mathbf{z}|\theta) = \log p(\mathbf{x}|\mathbf{z}, \theta) + \log p(\mathbf{z}).$$

$$p(\mathbf{x}|\theta) = \int p(\mathbf{x}, \mathbf{z}|\theta) d\mathbf{z} = \int p(\mathbf{x}|\mathbf{z}, \theta)p(\mathbf{z}) d\mathbf{z}.$$

Motivation

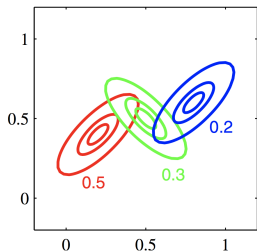
The distributions $p(\mathbf{x}|\mathbf{z}, \theta)$ and $p(\mathbf{z})$ could be quite simple.

Latent variable models (LVM)

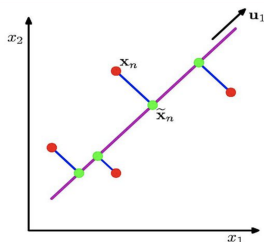
$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log \int p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})p(\mathbf{z})d\mathbf{z} \rightarrow \max_{\boldsymbol{\theta}}$$

Examples

Mixture of gaussians



PCA model

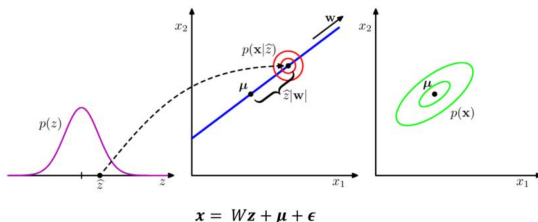


- ▶ $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_{\mathbf{z}}, \boldsymbol{\Sigma}_{\mathbf{z}})$
- ▶ $p(\mathbf{z}) = \text{Categorical}(\mathbf{z}|\boldsymbol{\pi})$
- ▶ $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) = \mathcal{N}(\mathbf{x}|\mathbf{W}\mathbf{z} + \boldsymbol{\mu}, \boldsymbol{\Sigma}_{\mathbf{z}})$
- ▶ $p(\mathbf{z}) = \mathcal{N}(\mathbf{z}|\mathbf{0}, \mathbf{I})$

Latent variable models (LVM)

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log \int p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})p(\mathbf{z})d\mathbf{z} \rightarrow \max_{\boldsymbol{\theta}}$$

PCA goal: Project original data \mathbf{X} onto low latent space while maximizing the variance of the projected data.



- ▶ $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) = \mathcal{N}(\mathbf{x}|\mathbf{W}\mathbf{z} + \boldsymbol{\mu}, \boldsymbol{\Sigma}_z)$
- ▶ $p(\mathbf{z}) = \mathcal{N}(\mathbf{z}|\mathbf{0}, \mathbf{I})$

Incomplete likelihood

MLE

$$\begin{aligned}\theta^* &= \arg \max_{\theta} p(\mathbf{X}, \mathbf{Z} | \theta) = \arg \max_{\theta} \prod_{i=1}^n p(\mathbf{x}_i, \mathbf{z}_i | \theta) = \\ &= \arg \max_{\theta} \sum_{i=1}^n \log p(\mathbf{x}_i, \mathbf{z}_i | \theta).\end{aligned}$$

Since \mathbf{Z} is unknown, maximize **incomplete likelihood**.

MILE problem

$$\begin{aligned}\theta^* &= \arg \max_{\theta} \log p(\mathbf{X} | \theta) = \arg \max_{\theta} \sum_{i=1}^n \log p(\mathbf{x}_i | \theta) = \\ &= \arg \max_{\theta} \sum_{i=1}^n \log \int p(\mathbf{x}_i, \mathbf{z}_i | \theta) d\mathbf{z}_i = \\ &= \arg \max_{\theta} \log \int p(\mathbf{x}_i | \mathbf{z}_i, \theta) p(\mathbf{z}_i) d\mathbf{z}_i.\end{aligned}$$

Variational lower bound

$$\begin{aligned}\log p(\mathbf{x}|\boldsymbol{\theta}) &= \log \frac{p(\mathbf{x}, \mathbf{z}|\boldsymbol{\theta})}{p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})} = \\&= \int q(\mathbf{z}) \log \frac{p(\mathbf{x}, \mathbf{z}|\boldsymbol{\theta})}{p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})} d\mathbf{z} = \int q(\mathbf{z}) \log \frac{p(\mathbf{x}, \mathbf{z}|\boldsymbol{\theta})q(\mathbf{z})}{p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})q(\mathbf{z})} d\mathbf{z} = \\&= \int q(\mathbf{z}) \log \frac{p(\mathbf{x}, \mathbf{z}|\boldsymbol{\theta})}{q(\mathbf{z})} d\mathbf{z} + \int q(\mathbf{z}) \log \frac{q(\mathbf{z})}{p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})} d\mathbf{z} = \\&= \mathcal{L}(q, \boldsymbol{\theta}) + KL(q(\mathbf{z})||p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})) \geq \mathcal{L}(q, \boldsymbol{\theta}).\end{aligned}$$

Kullback-Leibler divergence

- ▶ $KL(q||p) = \int q(\mathbf{z}) \log \frac{q(\mathbf{z})}{p(\mathbf{z})} d\mathbf{z};$
- ▶ $KL(q||p) \geq 0;$
- ▶ $KL(q||p) = 0 \Leftrightarrow q \equiv p.$

Variational lower bound

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \mathcal{L}(q, \boldsymbol{\theta}) + KL(q(\mathbf{z})||p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})) \geq \mathcal{L}(q, \boldsymbol{\theta}).$$

Evidence Lower Bound (ELBO)

$$\begin{aligned}\mathcal{L}(q, \boldsymbol{\theta}) &= \int q(\mathbf{z}) \log \frac{p(\mathbf{x}, \mathbf{z}|\boldsymbol{\theta})}{q(\mathbf{z})} d\mathbf{z} = \\ &= \int q(\mathbf{z}) \log p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) d\mathbf{z} + \int q(\mathbf{z}) \log \frac{p(\mathbf{z})}{q(\mathbf{z})} d\mathbf{z} \\ &= \mathbb{E}_q \log p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) - KL(q(\mathbf{z})||p(\mathbf{z}))\end{aligned}$$

Instead of maximizing incomplete likelihood, maximize ELBO (equivalently minimize KL)

$$\max_{\boldsymbol{\theta}} p(\mathbf{x}|\boldsymbol{\theta}) \quad \rightarrow \quad \max_{q, \boldsymbol{\theta}} \mathcal{L}(q, \boldsymbol{\theta}) \equiv \min_{q, \boldsymbol{\theta}} KL(q(\mathbf{z})||p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})).$$

EM-algorithm

$$\mathcal{L}(q, \theta) = \int q(\mathbf{z}) \log p(\mathbf{x}|\mathbf{z}, \theta) d\mathbf{z} + \int q(\mathbf{z}) \log \frac{p(\mathbf{z})}{q(\mathbf{z})} d\mathbf{z}.$$

Block-coordinate optimization

- ▶ Initialize θ^* ;
- ▶ E-step

$$q(\mathbf{z}) = \arg \max_q \mathcal{L}(q, \theta^*) = \arg \min_q KL(q||p) = p(\mathbf{z}|\mathbf{x}, \theta^*);$$

- ▶ M-step

$$\theta^* = \arg \max_{\theta} \mathcal{L}(q, \theta);$$

- ▶ Repeat E-step and M-step until convergence.

Ugly pic

Amortized variational inference

E-step

$$q(\mathbf{z}) = \arg \max_q \mathcal{L}(q, \boldsymbol{\theta}^*) = \arg \min_q KL(q||p) = p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta}^*).$$

- ▶ $p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta}^*)$ could be **intractable**;
- ▶ $q(\mathbf{z})$ is different for each object \mathbf{x} .

Idea

Restrict a family of all possible distributions $q(\mathbf{z})$ to a particular parametric class conditioned on sample: $q(\mathbf{z}|\mathbf{x}, \phi)$.

Variational Bayes

- ▶ E-step

$$\phi_k = \phi_{k-1} + \eta \nabla_{\phi} \mathcal{L}(\phi, \boldsymbol{\theta}_{k-1})|_{\phi=\phi_{k-1}}$$

- ▶ M-step

$$\boldsymbol{\theta}_k = \boldsymbol{\theta}_{k-1} + \eta \nabla_{\boldsymbol{\theta}} \mathcal{L}(\phi_k, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{k-1}}$$

Variational EM-algorithm

ELBO

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \mathcal{L}(q, \boldsymbol{\theta}) + KL(q(\mathbf{z})||p(\mathbf{z}|\mathbf{x}, \boldsymbol{\theta})) \geq \mathcal{L}(q, \boldsymbol{\theta}).$$

- ▶ E-step

$$\boldsymbol{\phi}_k = \boldsymbol{\phi}_{k-1} + \eta \nabla_{\boldsymbol{\phi}} \mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta}_{k-1})|_{\boldsymbol{\phi}=\boldsymbol{\phi}_{k-1}},$$

where $\boldsymbol{\phi}$ – parameters of variational distribution $q(\mathbf{z}|\mathbf{x}, \boldsymbol{\phi})$.

- ▶ M-step

$$\boldsymbol{\theta}_k = \boldsymbol{\theta}_{k-1} + \eta \nabla_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\phi}_k, \boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{k-1}},$$

where $\boldsymbol{\theta}$ – parameters of the generation function $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})$.

Now all we have to do is to obtain two gradients $\nabla_{\boldsymbol{\phi}} \mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta})$, $\nabla_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\theta})$.

Difficulty: number of samples n .

ELBO gradient (M-step, $\nabla_{\theta} \mathcal{L}(\phi, \theta)$)

$$\sum_{i=1}^n \mathcal{L}_i(\phi, \theta) = \sum_{i=1}^n \mathbb{E}_q \log p(\mathbf{x}_i | \mathbf{z}_i, \theta) - KL(q(\mathbf{z}_i | \mathbf{x}_i, \phi) || p(\mathbf{z}_i)) \rightarrow \max_{\phi, \theta}.$$

Optimization w.r.t. θ : **mini-batching** (1) + **Monte-Carlo** estimation (2)

$$\begin{aligned} \nabla_{\theta} \sum_{i=1}^n \mathcal{L}_i(\phi, \theta) &= \sum_{i=1}^n \int q(\mathbf{z}_i | \mathbf{x}_i, \phi) \nabla_{\theta} \log p(\mathbf{x}_i | \mathbf{z}_i, \theta) d\mathbf{z}_i \\ &\stackrel{(1)}{\approx} n \int q(\mathbf{z}_i | \mathbf{x}_i, \phi) \nabla_{\theta} \log p(\mathbf{x}_i | \mathbf{z}_i, \theta) d\mathbf{z}_i, \quad i \sim U[1, n] \\ &\stackrel{(2)}{\approx} n \nabla_{\theta} \log p(\mathbf{x}_i | \mathbf{z}_i^*, \theta), \quad \mathbf{z}_i^* \sim q(\mathbf{z}_i | \mathbf{x}_i, \phi). \end{aligned}$$

Monte-Carlo estimation (2):

$$\mathbb{E}_q f(\mathbf{z}) = \int q(\mathbf{z}) f(\mathbf{z}) d\mathbf{z} \approx f(\mathbf{z}^*), \text{ where } \mathbf{z}^* \sim q(\mathbf{z}).$$

ELBO gradient (E-step, $\nabla_{\phi} \mathcal{L}(\phi, \theta)$)

$$\sum_{i=1}^n \mathcal{L}_i(\phi, \theta) = \sum_{i=1}^n \mathbb{E}_q \log p(\mathbf{x}_i | \mathbf{z}_i, \theta) - KL(q(\mathbf{z}_i | \mathbf{x}_i, \phi) || p(\mathbf{z}_i)) \rightarrow \max_{\phi, \theta}.$$

Difference from M-step: density function $q(\mathbf{z} | \mathbf{x}, \phi)$ depends on the parameters ϕ , it is impossible to use the Monte-Carlo estimation:

$$\nabla_{\phi} \mathcal{L}(\phi, \theta) = \int \nabla_{\phi} q(\mathbf{z} | \mathbf{x}, \phi) \log p(\mathbf{x} | \mathbf{z}, \theta) d\mathbf{z} - \nabla_{\phi} KL$$

First step is not an expectation due to the gradient.

Possible solutions

- ▶ log-derivative trick;
- ▶ reparametrization trick.

ELBO gradient (E-step, $\nabla_{\phi} \mathcal{L}(\phi, \theta)$)

Log-derivative trick

$$\nabla_{\xi} q(\eta|\xi) = q(\eta|\xi) \left(\frac{\nabla_{\xi} q(\eta|\xi)}{q(\eta|\xi)} \right) = q(\eta|\xi) \nabla_{\xi} \log q(\eta|\xi).$$

$$\nabla_{\phi} q(\mathbf{z}|\mathbf{x}, \phi) = q(\mathbf{z}|\mathbf{x}, \phi) \nabla_{\phi} \log q(\mathbf{z}|\mathbf{x}, \phi).$$

ELBO

$$\begin{aligned} \nabla_{\phi} \sum_{i=1}^n \mathcal{L}_i(\phi, \theta) &= \sum_{i=1}^n \int \nabla_{\phi} q(\mathbf{z}_i|\mathbf{x}_i, \phi) \log p(\mathbf{x}_i|\mathbf{z}_i, \theta) d\mathbf{z}_i - \nabla_{\phi} KL \\ &= \sum_{i=1}^n \int q(\mathbf{z}_i|\mathbf{x}_i, \phi) [\nabla_{\phi} \log q(\mathbf{z}_i|\mathbf{x}_i, \phi) \log p(\mathbf{x}_i|\mathbf{z}_i, \theta)] d\mathbf{z}_i - \nabla_{\phi} KL \\ &\approx n \nabla_{\phi} \log q(\mathbf{z}_i^*|\mathbf{x}_i, \phi) \log p(\mathbf{x}_i|\mathbf{z}_i^*, \theta) - \nabla_{\phi} KL, \quad \mathbf{z}_i^* \sim q(\mathbf{z}_i|\mathbf{x}_i, \phi). \end{aligned}$$

Problem

Unstable solution with huge variance.

ELBO gradient (E-step, $\nabla_{\phi} \mathcal{L}(\phi, \theta)$)

Reparametrization trick

$$f(\xi) = \int q(\eta|\xi) h(\eta) d\eta$$

Let $\eta = g(\xi, \epsilon)$, where g is a deterministic function, ϵ is a random variable with a density function $r(\epsilon)$.

$$\begin{aligned} \nabla_{\xi} \int q(\eta|\xi) h(\eta) d\eta &= \nabla_{\xi} \int r(\epsilon) h(g(\xi, \epsilon)) d\epsilon \\ &\approx \nabla_{\xi} h(g(\xi, \epsilon^*)), \quad \epsilon^* \sim r(\epsilon). \end{aligned}$$

Example

$$q(\eta|\xi) = \mathcal{N}(\eta|\mu, \sigma^2), \quad r(\epsilon) = \mathcal{N}(\epsilon|0, 1), \quad \eta = \sigma \cdot \epsilon + \mu, \quad \xi = [\mu, \sigma].$$

ELBO gradient (E-step, $\nabla_{\phi} \mathcal{L}(\phi, \theta)$)

$$\begin{aligned}\nabla_{\phi} \sum_{i=1}^n \mathcal{L}_i(\phi, \theta) &= \sum_{i=1}^n \nabla_{\phi} \int q(\mathbf{z}_i | \mathbf{x}_i, \phi) \log p(\mathbf{x}_i | \mathbf{z}_i, \theta) d\mathbf{z}_i - \nabla_{\phi} KL \\ &\approx n \nabla_{\phi} \int r(\epsilon) \log p(\mathbf{x}_i | g(\mathbf{x}_i, \epsilon, \phi), \theta) d\epsilon - \nabla_{\phi} KL, \quad i \sim U[1, n] \\ &\approx n \nabla_{\phi} \log p(\mathbf{x}_i | g(\mathbf{x}_i, \epsilon^*, \phi), \theta) - \nabla_{\phi} KL, \quad \epsilon^* \sim r(\epsilon).\end{aligned}$$

Variational assumption

$$q(\mathbf{z} | \mathbf{x}, \phi) = \mathcal{N}(\mu(\mathbf{x}), \sigma(\mathbf{x})).$$

$$\mathbf{z} = g(\mathbf{x}, \epsilon, \phi) = \sigma(\mathbf{x}) \cdot \epsilon + \mu(\mathbf{x}).$$

$\nabla_{\phi} KL(q(\mathbf{z} | \mathbf{x}, \phi) || p(\mathbf{z}))$ has an analytical solution.

Variational autoencoder (VAE)

Final algorithm

- ▶ pick $i \sim U[1, n]$;
- ▶ compute stochastic gradient w.r.t. ϕ

$$\nabla_{\phi} \mathcal{L}(\phi, \theta) = n \nabla_{\phi} \log p(\mathbf{x}_i | g(\mathbf{x}_i, \epsilon^*, \phi), \theta) - \nabla_{\phi} KL(q(\mathbf{z}_i | \mathbf{x}_i, \phi) || p(\mathbf{z}_i)), \quad \epsilon^* \sim r(\epsilon);$$

- ▶ compute stochastic gradient w.r.t. θ

$$\nabla_{\theta} \mathcal{L}(\phi, \theta) = n \nabla_{\theta} \log p(\mathbf{x}_i | \mathbf{z}_i^*, \theta), \quad \mathbf{z}_i^* \sim q(\mathbf{z}_i | \mathbf{x}_i, \phi);$$

- ▶ update θ, ϕ according to the selected optimization method (SGD, Adam, RMSProp).

Variational autoencoder (VAE)

- ▶ Encoder $q(\mathbf{z}|\mathbf{x}, \phi) = \text{NN}_e(\mathbf{x}, \phi)$ outputs $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$.
- ▶ Decoder $p(\mathbf{x}|\mathbf{z}, \theta) = \text{NN}_d(\mathbf{z}, \theta)$ outputs parameters of the sample distribution.

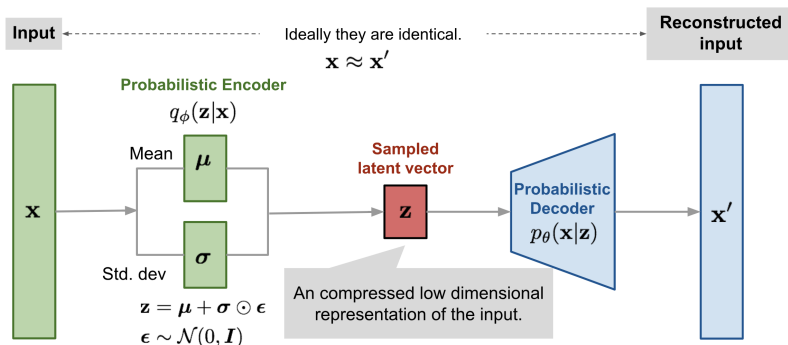
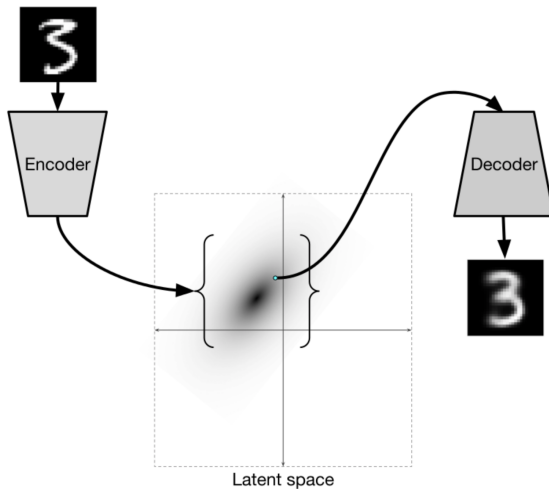


image credit:

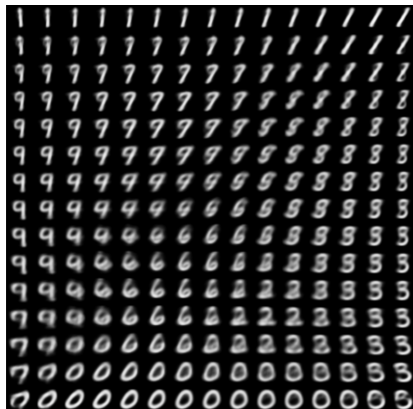
<https://lilianweng.github.io/lil-log/2018/08/12/from-autoencoder-to-beta-vae.html>

Variational Autoencoder



Variational Autoencoder

Generation objects by sampling the latent space $\mathbf{z} \sim \mathcal{N}(0, \mathbf{I})$



Summary

- ▶ Bayesian inference is a generalization of most common machine learning tasks. It allows to construct MLE, MAP and bayesian inference, to compare models complexity and many-many more cool stuff.
- ▶ Latent variable models introduce latent representation of observed samples to make model more interpretable.
- ▶ To find MLE of model parameters LVM maximizes variational evidence lower bound.
- ▶ Maximization of ELBO is performed by general variational EM algorithm.
- ▶ Amortized inference allows to efficiently compute stochastic gradients for ELBO and use deep neural network for $q(\mathbf{z}|\mathbf{x}, \phi)$ and $p(\mathbf{x}|\mathbf{z}_i, \theta)$.
- ▶ VAE model is a LVM model with encoder network for $q(\mathbf{z}|\mathbf{x}, \phi)$ and decoder network for $p(\mathbf{x}|\mathbf{z}_i, \theta)$.