1 The Genesis of Fourier Analysis

1.1

(a) |z| is length of vector (x, y) in z-plane

(b) if |z|=0, then x = 0 and y = 0, therefore z=0

(c) $|\lambda z| = ((\lambda x)^2 + (\lambda y)^2)^{\frac{1}{2}} = |\lambda|(x^2 + y^2)^{\frac{1}{2}} = |\lambda||z|$

(d) $|z_1 z_2| = |r_1 r_2 e^{j\omega_1 + j\omega_2}| = r_1 r_2 = |z_1||z_2|$

$|z_1 + z_2| = |r_1 e^{j\omega_1} + r_2 e^{j\omega_2}| = |r_1 + r_2 e^{j(\omega_2 - \omega_1)}| \leq |r_1| + |r_2| = |z_1| + |z_2|$

(e) $|\frac{1}{z}| = |\frac{1}{re^{j\omega}}| = \frac{1}{r} = \frac{1}{|z|}$

1.2

(a) $\overline{z}$ is reflection symmetry about x axis.

(b) $|z|^2 = r^2 = re^{j\omega} \cdot re^{-j\omega} = z\overline{z}$

(c) if z belongs to the unit circle, then $|z| = 1$, $z = e^{j\omega}$, therefore $\frac{1}{z} = e^{-j\omega} = \overline{z}$

1.3

(a) if there are two different limits: ω_1, ω_2, then

$$\lim_{n\to\infty} |\omega_n - \omega_1| + |\omega_n - \omega_2| = 0$$

however,

$$\lim_{n\to\infty} |\omega_n - \omega_1| + |\omega_n - \omega_2| \geq \lim_{n\to\infty} |(\omega_n - \omega_1) - (\omega_n - \omega_2)| = \lim_{n\to\infty} |\omega_1 - \omega_2| > 0.$$

contrast with preceding results, therefore a converging sequence of complex numbers has a unique limit.

(b) **assume complex numbers converges, prove it is a Cauchy sequence.**

$\exists \omega \in \mathbb{C}$, $\forall \frac{\epsilon}{2} > 0$, $\exists N > 0$, when $n > N$, then $|\omega_n - \omega| < \frac{\epsilon}{2}$

$\therefore \forall n, m > N, |\omega_n - \omega_m| = |\omega_n - \omega + \omega - \omega_m| \leq |\omega_n - \omega| + |\omega_m - \omega| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\{\omega_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

assume sequence is a Cauchy sequence, prove it converges.

$\forall n, m > N$, $|\omega_n - \omega_m| = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2} < \epsilon$, which $x_{n,m}, y_{n,m} \in \mathbb{R}$

$\therefore \sqrt{(x_n - x_m)^2} < \epsilon$, $\sqrt{(y_n - y_m)^2} < \epsilon$

$\therefore \{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in $\mathbb{R}$, so do $\{y_n\}_{n=1}^{\infty}$.

$\therefore \exists x_0, y_0 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} |x_n - x_0| = 0$$

$$\lim_{n\to\infty} |y_n - y_0| = 0.$$

thus $\exists z_0 = x_0 + y_0 i \in \mathbb{C}$,

$$\lim_{n\to\infty} |z_n - z_0| = \lim_{n\to\infty} \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2} = \lim_{n\to\infty} \sqrt{0+0} = 0.$$

$\therefore \{z_n\}_{n=1}^{\infty}$ sequence converges.

(c) since $\sum_{n=1}^{\infty} a_n$ converges, set $A_N = \sum_{n=1}^{N} a_n$,

then according to the Cauchy criterion, $\forall \epsilon > 0$, there exists $N > 0$, such that

$$|A_n - A_m| < \epsilon, \text{whenever } n, m > N.$$

thus (assume $n > m$)

$$|S_n - S_m| = |z_n + z_{n-1} \ldots + z_m| \leq |z_n| + \ldots + |z_m| \leq a_n + \ldots + a_m = |A_n - A_m| < \epsilon.$$

$\therefore \{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{S_n\}_{n=1}^{\infty} = \sum_n z^n$ converges.

1.4

(a) **prove** $S_n = \sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ **converges**

since $a_n = \frac{|z|^n}{n!} \geq |z_n|$, $\sum_{n=0}^{\infty} a_n = e^{|z|}$ converges, from (3.c) result we know $\sum_{n=0}^{\infty} z_n$ converges.

prove $S_n = \sum_{n=0}^{\infty} z_n = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ **uniformly converges**