Semaine 14 – 16

Integration 积分

原函数：
$$\begin{cases} F'(x) = f(x). \\ d(F(x)) = f(x)\,dx \end{cases}$$

不定积分：原函数全体

$$F(x) \xrightarrow{d} f(x)\,dx$$
$$F(x)+C \xleftarrow{\int} f(x)\,dx$$

— **线性性**
$$\int [k_1 f(x) + k_2 g(x)]\,dx = k_1 \int f(x)\,dx + k_2 \int g(x)\,dx$$

— **公式表**
$$\tan x \leftrightarrow \sec x, \quad \begin{cases} \arcsin x \leftrightarrow \dfrac{1}{\sqrt{1-x^2}} \\ \arctan x \leftrightarrow \dfrac{1}{1+x^2} \end{cases}$$

* **换元积分法**

1. $$\int \boxed{f(x)}\,dx = \int \boxed{\ ? \cdot ?\ }\,dx = \int F(u)\,du + C$$
$$f(g(x))\ \ dg(x)$$

2. $x = \varphi(t)$
$$\int f(x)\,dx = \int \boxed{f(\varphi(t))\varphi'(t)}\,dt = \int F(t)\,dt + C$$

— **分部积分法**
$$u(x)\,v'(x) \longrightarrow u'(x)\,v(x)$$
$$\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx$$

— **有理函数积分**

1、拆成多项简单真分式，分母因式分解彻底
$$\frac{x^4}{x^3+1} = x - \frac{x}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1}$$

2. 求待定系数
- 留数法 $$\frac{1}{(x-1)(x-2)(x-3)}(x-1) = a + \left(\frac{b}{x-2} + \frac{c}{x-3}\right)\underline{(x-1)}_{\text{令}x=1}$$
- 极限法 $$\boxed{\frac{3x^2+1}{x(x+1)^2}}_{\to+\infty} = \boxed{\frac{x+1}{x}}_{\to+\infty} + \boxed{b} + \boxed{\frac{c}{x+1}}_{\to+\infty}$$
- 特殊值法

3. 万能公式 $\sin, \cos, \tan \leftrightarrow \tan\frac{x}{2}$

降次，降«项»

4. 根式换元（根式内可以三角换元）

5. 倒代换

换元积分法

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{dx}{1+(\frac{x}{a})^2} = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}\, d(\frac{x}{a})$

化一

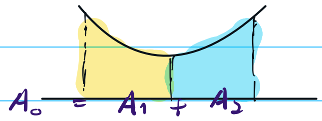
$= \frac{1}{a}\arctan(\frac{x}{a}) + C$

2. $\int \frac{dx}{\sqrt{x}(1+x)} = \int \frac{1}{1+x}\left(\frac{1}{\sqrt{x}}dx\right) = 2\int \frac{1}{1+x}\, d(\sqrt{x}) = 2\arctan\sqrt{x} + C$

一些基础的定理：

1 Relation de Chasles

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

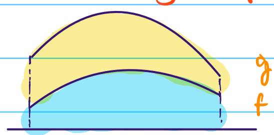


→ $\int_a^b f(x)\,dx \overset{Not}{=} -\int_b^a f(x)\,dx$

→ $\int_a^a f(x)\,dx \overset{Not}{=} 0$

2 Croissance de l'intégrale

$$f \leq g \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

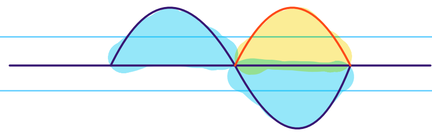


3 Linéarité de l'intégrale

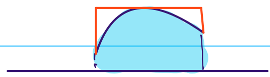
$$\int_a^b (\lambda f(x) + \mu g(x))\,dx = \lambda \int_a^b f(x)\,dx + \mu \int_a^b g(x)\,dx.$$

4 Majoration d'intégrales et conséquences

(1) $$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$



(2) $$\left|\int_a^b f(x)\,dx\right| \leq (b-a) \sup_{x\in[a,b]} |f(x)|$$



5. Inégalité de Cauchy-Schwarz

(3) $$\left[\int_a^b f(x)\,dx = 0\right] \Leftrightarrow [f = 0]$$

↓ toujours positive.

(4) $$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b f^2(x)\,dx}\,\sqrt{\int_a^b g^2(x)\,dx}$$

↓ Il y a égalité, alors $f = \lambda g$ ou $g = \lambda f$.

定积分的定义：

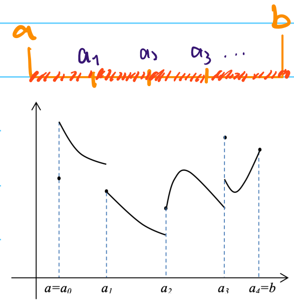
Fonction continue par morceaux :

$\exists$ une division de $[a,b]$

$(a_0 = a, a_1, a_2, \dots)$

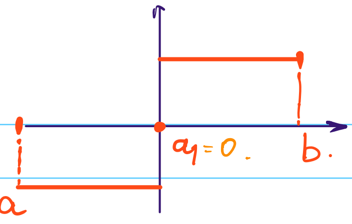
pour $\forall k \in [\![1, n]\!]$,

$$\begin{cases} g_k \text{ définie } \forall x \in]a_{k-1}, a_k[,\ f(x) = g_k(x) \\ \text{在点上的值，我们不关心。} \\ g_k \text{ continue sur } [a_{k-1}, a_k] \end{cases}$$



Fonction en escalier

Continue par morceaux 的特殊情况:
每个区间都是常数. (例: $[x]$)



Théorème

$\boxed{\begin{array}{c}\text{continue par morceaux sur } [a,b] : \\ \forall \varepsilon > 0,\ \exists \varphi \text{ en escalier},\ \forall x \in [a,b] \\ |f(x) - \varphi(x)| \leq \varepsilon .\end{array}}$

Théorème

$\boxed{\text{continue par morceaux} \Rightarrow f \text{ est intégrable}}$
sur $[a,b]$ sur $[a,b]$

Primitives

F primitive de f

$\boxed{\forall x \in I,\ F'(x) = f(x)}$

Il n'y a pas unicité.

On a des primitives célèbres à bien connaître (F désigne *une* primitive de f sur chaque intervalle où la fonction f est définie) :

dériver : $f(x)$

$f(x)$	$F(x)$
$x^\alpha\ (\alpha \neq -1)$	$\dfrac{x^{\alpha+1}}{\alpha+1}$
$\dfrac{1}{x}$	$\ln(\lvert x\rvert)$
e^x	e^x
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\dfrac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\tan(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\dfrac{1}{1+x^2}$	$\arctan(x)$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x)$
$\dfrac{1}{\sqrt{1+x^2}}$	$\ln\left(x + \sqrt{1+x^2}\right)$
$\dfrac{1}{\sqrt{x^2-1}}$	$\ln\left(\lvert x + \sqrt{x^2-1}\rvert\right)$

Primitives usuelles à connaître par ♥

PAUSE ^^

TFA.

$\boxed{\begin{array}{c} F : x \mapsto \int_a^x f(t)\,dt \text{ est la primitive de } f, \\ \text{qui s'annule en } a\end{array}}$

Crochet

牛顿-莱布尼茨公式 $[f(x)]_{x=a}^{x=b} = f(b) - f(a)$.

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = [f(x)]_{x=a}^{x=b}}$$

1. $\int_a^b e^x\,dx = [e^x]_a^b = e^b - e^a$
2. $\int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2$.
3.

$$\xrightarrow{x \to 0} \int_0^0$$

$$\lim_{x \to 0} \left[\frac{1}{x}\right] \int_0^x f(t)\,dt = f(0)$$

(IPP)

分部积分法

primitivation par parties

$$\boxed{\begin{gathered}\int u'v\,dx = uv - \int uv'\,dx \\ \left(\int u\,dv = uv - \int v\,du\right)\end{gathered}}$$

① $\int_1^2 x\ln x\,dx$.

$u = \ln x,\ v = x^2$. $\quad \frac{1}{2}\int_1^2 uv'\,dx = \frac{1}{2}uv - \frac{1}{2}\int_1^2 \underbrace{u'v}_{x^2 d(\ln x)}\,dx$

$= \left[\frac{1}{2}x^2\ln x\right]_1^2 - \frac{1}{2}\int_1^2 x\,dx$

$= 2\ln 2 - \frac{1}{2}\times\left[\frac{1}{2}x^2\right]_1^2$

$= 2\ln 2 - \frac{3}{4}$

② $\int_0^1 \arctan x\,dx$.

$u = \arctan x,\ v = x$

$v' = 1$

$\int_0^1 \arctan x\,dx = \left[x\arctan x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dt$

$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{d(1+x^2)}{1+x^2}$

$= \frac{\pi}{4} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$

$= \frac{\pi}{4} - \frac{\ln 2}{2}$

换元法

Changement de variable

$$\boxed{\begin{aligned}&\int_a^b f(\varphi(x))\varphi'(x)\,dx = \int_a^b f(t)\,dt\\&\int_{\varphi(a)}^{\varphi(b)} f[\varphi(x)]\,d[\varphi(x)] = \cdots\end{aligned}}$$

① $\int_0^1 \frac{e^x}{1+e^{2x}}\,dx$,

$$\int_0^1 \frac{e^x}{1+e^{2x}}\,dx = \int_0^1 \frac{1}{1+e^{2x}}\,d(\underbrace{e^x}_{u}) = \int_1^e \frac{1}{1+u^2}\,du.$$

$$= [\arctan x]_1^e$$

$$= \arctan e - \frac{\pi}{4}$$

② $\int_a^b \frac{1}{\sin x}\,dx$

$$\int_a^b \frac{1}{\sin x}\,dx = \int_a^b \frac{\sin x}{1-\cos^2 x}\,dx = \int_a^b \frac{1}{1-\cos^2 x}\,d(\underbrace{\cos x}_{u})$$

$$= \int_{\cos a}^{\cos b} \frac{1}{1-u^2}\,du$$

$$= \frac{1}{2}\left[\ln\left(\left|\frac{\cos b-1}{\cos b+1}\right|\right) - \ln\left|\frac{\cos a-1}{\cos a+1}\right|\right]$$