Second order step response

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1   Step response

Note: These notes are to replace pages 17–19 in the supplemental notes on first- and second-order systems which have been distributed previously. Reference is made to the figures and equations in these notes.

Consider again the system shown in Figure 8. The differential equation for this system is given in equation (5). We now let the input force F_c take the form of a step $F_c(t) = F_0 u_s(t)$ with amplitude F_0, and further assume the system has zero initial conditions. (F_d remains zero as before.) The homogeneous part of the solution will take the forms that we have discussed previously, depending upon the damping ratio. The underdamped case is of interest here, ($0 \leq \zeta < 1$).

With the step input, we can calculate the response either by thinking in terms of a particular solution plus homogeneous solution, or via formal application of Laplace techniques. Adopting the first approach goes as follows: In steady-state, the input force has the constant value F_0. This force is balanced by the deflection of the spring such that the net force on the mass is zero. That is, we postulate the particular solution $x_p(t) = F_0/k$. As this solution has zero velocity and acceleration, the damper and mass do not contribute any term. To satisfy the condition of zero initial velocity and position, a suitable homogeneous response must be added to the particular response found above.

If a Laplace-based approach is taken, the calcuation goes as follows. Applying the Laplace transform to all the terms in the differential equation yields

$$(ms^2 + bs + k)X(s) = F(s). \tag{1}$$

Here we have applied the zero intial conditions as stated above, and also made the assumption that the input is zero before $t = 0$. Dividing through gives the system transfer function

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \tag{2}$$

The poles of this transfer function are the roots discussed earlier in the context of the homogeneous response. The DC gain is calculated by setting $s = 0$, which gives a value of $1/k$, consistent with the discussion of the particular solution above.

Recalling the definitions of natural frequency $\omega_n = \sqrt{k/m}$ and $\zeta = b/2\sqrt{km}$ lets us write this transfer function using a standard form as

$$\frac{X(s)}{F(s)} = \frac{1}{k} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{3}$$

We can view this as a gain $1/k$ multiplied by the standard unity-gain form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{4}$$

The transform of input step is

$$F_c(s) = \frac{F_0}{s}, \tag{5}$$

and thus the transform of the output position is given by

$$X(s) = \frac{1}{k} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{F_0}{s}. \tag{6}$$

Since the system is linear, the character of the response does not depend upon the driving amplitude, and we thus can normalize the problem by choosing $F_0 = k$ (or alternately plot all response magnitudes in units of F_0/k). With this choice, the output transform becomes

$$X(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}. \tag{7}$$

The Laplace transform time-domain pair with this $X(s)$ is

$$x(t) = 1 - e^{-\sigma t}(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t) \tag{8}$$

$$= 1 - \frac{\omega_n}{\omega_d}e^{-\sigma t}\sin(\omega_d t + \phi) \tag{9}$$

Here two equivalent forms of the solution are given, where we recall the definitions $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$, and where we define $\phi = \arctan(\sqrt{1 - \zeta^2}/\zeta)$.

A plot of this solution is shown in Figure 1 for various values of ζ. In this plot, we have normalized the amplitude as discussed above, and also normalized time to $t' = \omega_n t$. Note that in this plot, ζ affects both the overshoot (with a maximum of 100% overshoot for the case $\zeta = 0$) and the duration of the ringing in the response.

Second-order systems occur frequently in practice, and so standard parameters of this response have been defined. These include the maximum amount of overshoot M_p, the time at which this occurs t_p, the settling time t_s to within a specified tolerance band, and the 10-90% rise time t_r. These parameters are defined with respect to a standard second-order response as shown in Figure 2.
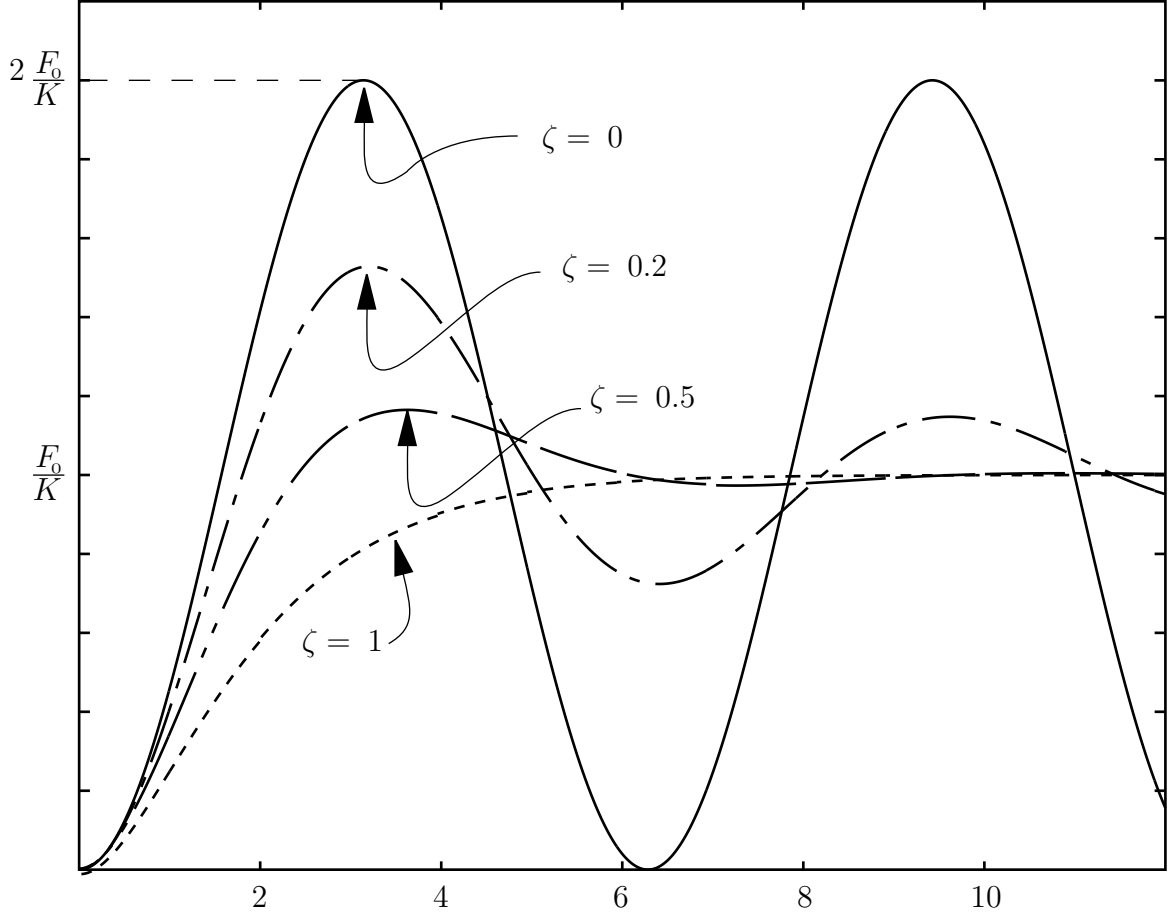
Figure 1: Second-order system response to a step in force of $F_0 u_s(t)$

1.1 Overshoot

Let's start with a discussion of the overshoot. The time at which the peak in (8) occurs can be determined by setting the derivative to zero. That is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma e^{-\sigma t}(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t) - e^{-\sigma t}(-\omega_d \sin \omega_d t + \sigma \cos \omega_d t) \qquad (10)$$

$$= e^{-\sigma t}(\frac{\sigma^2}{\omega_d} \sin \omega_d t + \omega_d \sin \omega_d t). \qquad (11)$$

This has a value of zero for $\omega_d t = \pi \pm n2\pi$; $n = 0, 1, 2, \ldots$ and thus the first peak in the response is found where $\omega_d t = \pi$, and thereby the peak time is given by

$$t_p = \frac{\pi}{\omega_d}. \qquad (12)$$

By definition, the overshoot M_p is the amount that the normalized response exceeds unity. That is, the peak value is $x_{peak} = 1 + M_p$. At the value of time t_p for the peak, we can calculate the
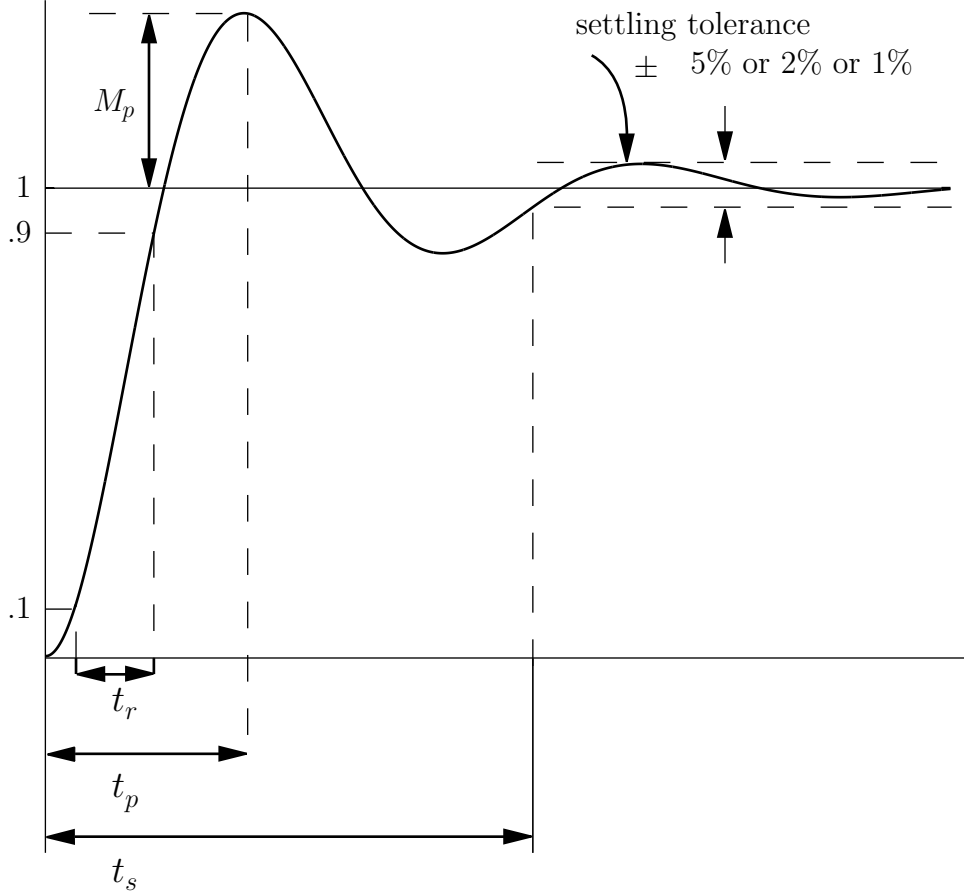
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Figure 2: Standard second-order step response parameters.

magnitude of the response as

$$x(t_p) = 1 - e^{-\sigma\pi/\omega_d}\left(\cos\pi + \frac{\sigma}{\omega_d}\sin\pi\right) \tag{13}$$

$$= 1 + e^{-\sigma\pi/\omega_d}. \tag{14}$$

With the definition of the peak value given above, we thus have

$$M_p = e^{-\sigma\pi/\omega_d} \tag{15}$$

$$= e^{-\pi\zeta/\sqrt{1-\zeta^2}};\ 0 \le \zeta < 1. \tag{16}$$

Recall that this formula only applies in the underdamped case, as indicated explicitly above.

While the result above is exact, it is a bit cumbersome to remember and to manipulate for design calculations. Thus, we introduce the approximation

$$M_p \approx 1 - \frac{\zeta}{0.6}. \tag{17}$$

This linear approximation is easier to compute with and for many design problems is sufficiently accurate.

1.2 Rise time

We define the rise time t_r as indicated in Figure 2 as the time required for the response to transit between 10% and 90% of the final value.[1] Inspection of Figure 1 shows that the rise time is not strondly dependent upon ζ, whereas it scale linearly with ω_n. For intermediate values of ζ, the rise time is approximately given by

$$t_r \approx \frac{1.8}{\omega_n}. \tag{18}$$

This approximation is very useful for determining the approximate natural frequency based upon step response measurements, or in sketching the step response of a known system.

1.3 Settling time

Finally, we define the settling time. As shown in Figure 2, the settling time is the time required for the response to stay within a defined error band $\pm\Delta$ of the final normalized value of unity. The exact criterion is difficult to express in analytic form, and so we use the conservative approximation of calculating the time at which the *envelope* of the response reaches values within the error band. This approximation is conservative, since the response itself never lies outside the envelope. Since the envelope decays to the final value of unity with an error magnitude of $e^{-\sigma t}$, it reaches the tolerance band for

$$e^{-\sigma t_s} = \Delta, \tag{19}$$

and thus we have $-\sigma t_s = \ln \Delta$, or

$$t_s = -\frac{1}{\sigma} \ln \Delta. \tag{20}$$

In order to calculate the settling time to a given value, it helps to remember that $e^{2.3} \approx 10$, and thus, for example $\ln 0.01 \approx -4.6$. With this insight, the 1% settling time is given by

$$t_s \approx \frac{4.6}{\zeta\omega_n}; \ (1\% \text{ settling}). \tag{21}$$

There is, of course, nothing special about settling to 1%. For example, if a large machine tool takes a move of 1 m, and if it can be modelled as a second order system, and if we require it to settle to 1 μm, then the settling time will be much longer. This specification requires settling to one part in $\Delta = 10^6$ of the move, and thus for this specification we will require a time of about $6 * 2.3/\zeta\omega_n$ to settle within tolerance.

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[1]Some texts define rise time as the time required to transit between 0% and 100%, since this is easier to calculate explicitly. However, in engineering practice, such a measure is difficult to implement, and the dependence on ζ is much stronger. Thus, in engineering practice we use the 10–90% rise time. Where these need to be distinguished, we can insert the subscript 10–90 or 0–100 to be clear which is being discussed.

1.4 Summary

The results developed above are summarized as:

$$
\begin{aligned}
t_r &\simeq \frac{1.8}{\omega_n} && \text{(medium values of } \zeta) \\
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} && \simeq \quad 1 - \frac{\zeta}{0.6} \\
t_p &= \frac{\pi}{\omega_d} \\
t_s &\simeq \frac{4.6}{\zeta\omega_n} && (\pm\ 1\%\ \text{error tolerance})
\end{aligned}
$$

1.5 Application to design

For the purposes of design, we can use the results derived above to place constraints on the location of the poles of a second order system. For example, if the rise time must be below a given required value t_{rr}, then we must have

$$
\omega_n > \frac{1.8}{t_{rr}}. \tag{22}
$$

Further, if we require that the overshoot be less than some required value M_{pr}, then we must have approximately

$$
\zeta > 0.6(1 - M_{pr}). \tag{23}
$$

Finally, if we require that the settling time be less than some required value t_{sr}, then we must have

$$
\zeta\omega_n > \frac{4.6}{t_{sr}}. \tag{24}
$$

Recall that $\zeta\omega_n = \sigma$ is the real part of the system poles.

These constraints can be expressed graphically by requiring that the system poles lie to the left of the shaded regions in the s-plane shown in Figure 3.
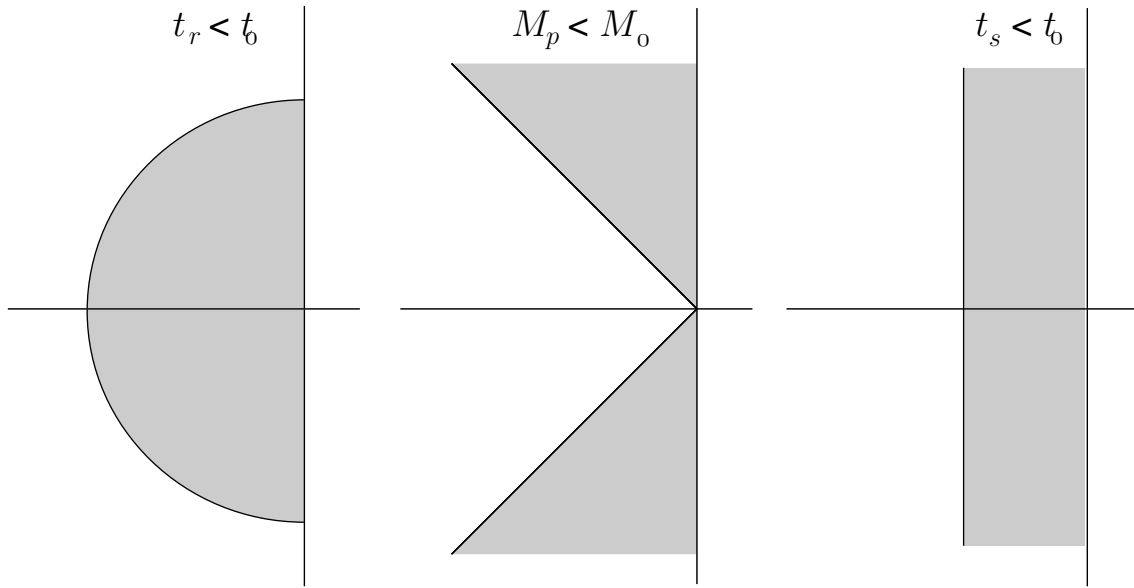
Figure 3: Grey areas show prohibited regions for poles to meet design specifications.