$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} \underline{a_{11}a_{22}a_{33}} + \underline{a_{12}a_{23}a_{31}} + \underline{a_{13}a_{21}a_{32}} \\ -\underline{a_{13}a_{22}a_{31}} - \underline{a_{11}a_{23}a_{32}} - \underline{a_{12}a_{21}a_{33}}. \end{array}$$

$$= a_{11}\underline{(a_{22}a_{33}-a_{23}a_{32})} + a_{12}\underline{(a_{23}a_{31}-a_{21}a_{33})}$$

$$+ a_{13}\underline{(a_{21}a_{32}-a_{22}a_{31})}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

记$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$,

$$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

$$D = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

类似地有

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3}, \quad i = 1,2,3.$$

或$D = a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}, \quad j = 1,2,3.$

A_{ij}称为元素a_{ij}的代数余子式。

计算3阶行列式

$$\begin{vmatrix} 1 & 0 & 0 \\ -5 & 2 & 3 \\ 3 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 4 & 3 \\ -5 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix}$$

用对角线法则做更简单。

$$\begin{vmatrix} 1 & 4 & -8 \\ 0 & 2 & 9 \\ 0 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 9 \\ 6 & 1 \end{vmatrix}.$$

一般地,对**n**阶行列式,

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

余子式与代数余子式

a_{ij} M_{ij} $(-1)^{i+j}M_{ij}$ A_{ij}

$$\boxed{A_{ij}=(-1)^{i+j}M_{ij}}$$

例如3阶行列式 $\begin{vmatrix} 1 & 4 & -8 \\ -5 & 2 & 9 \\ 3 & 6 & 1 \end{vmatrix}$

$$M_{23} = \begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix}, \qquad A_{23} = -\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix}.$$

一般地，余子式为

$$M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

n阶行列式的定义

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$D=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in} \quad (i=1,2,\cdots n)$$

或者

$$D=a_{1j}A_{1j}+a_{2j}A_{2j}+\cdots+a_{nj}A_{nj} \quad (j=1,2,\cdots,n)$$