克莱姆法则

考虑方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

与二,三元方程组类似,***n***元方程组的解也可用行列式表示.

复习

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases} \qquad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{D_1}{D}, \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{D_2}{D}.$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

定理1 若方程组的系数行列式

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

则方程组有惟一解

$$x_1=\frac{D_1}{D}, x_2=\frac{D_2}{D}, \cdots, x_n=\frac{D_n}{D}.$$

其中

$$D_j = \begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

要证明这一定理,需证明三点.一是有解,二是解惟一,三是解的公式

$$x_j = \frac{D_j}{D} \qquad (j = 1,2,\cdots,n)$$

欲证 $$x_j = \frac{D_j}{D} \qquad (j = 1,2,\cdots,n)$$

是解,只需证明等式

$$a_{11}\frac{D_1}{D} + a_{12}\frac{D_2}{D} + \cdots + a_{1n}\frac{D_n}{D} = b_1$$

等n个式子成立.整理上式,得:

$$b_1 D - a_{11}D_1 - a_{12}D_2 - \cdots - a_{1n}D_n = 0.$$

分析这个式子可知应为n+1阶行列式。

这个n+1阶行列式是什么样的呢?

$$b_1 D - a_{11} D_1 - a_{12} D_2 - \cdots - a_{1n} D_n = 0.$$

$$D_{n+1} = \begin{vmatrix} b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

此行列式为零.将其按第一行展开,得

$$0 = D_{n+1} =$$

$$
b_1\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{11}\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} +
$$

$$
a_{12}\begin{vmatrix} b_1 & a_{11} & \cdots & a_{1n} \\ b_2 & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n1} & \cdots & a_{nn} \end{vmatrix} \begin{matrix} +\cdots+ \\ (-1)^{2+n} a_{1n} \end{matrix} \begin{vmatrix} b_1 & a_{11} & \cdots & a_{1,n-1} \\ b_2 & a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n1} & \cdots & a_{n,n-1} \end{vmatrix}
$$

$$
= b_1 D - a_{11} D_1 - a_{12} D_2 - \cdots - a_{1n} D_n = 0
$$

由 $D \neq 0$ 得证。

再证解是惟一的, 设$c_1, c_2, \cdots, c_n$为一组解, 即

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n = b_1, \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n = b_2, \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nn}c_n = b_n. \end{cases}$$

只需证

$$c_j = \frac{D_j}{D} \quad \text{即} \quad D \cdot c_j = D_j.$$

$$D \cdot c_j =$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1(j-1)} & a_{1j}c_j & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & a_{nj}c_j & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

$$= D_j$$

$$\begin{cases} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n = b_1, \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n = b_2, \\ \cdots\cdots\cdots\cdots \\ a_{n1}c_1 + a_{n2}c_2 + \cdots + a_{nn}c_n = b_n. \end{cases}$$

定理1 若方程组的系数行列式

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

则方程组有惟一解

$$x_1=\frac{D_1}{D}, x_2=\frac{D_2}{D}, \cdots, x_n=\frac{D_n}{D}.$$

定理2 若方程组的系数行列式不为 零,

则方程组有惟一解.

方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0, \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}$$

称为齐次线性方程组.

定理3 若齐次方程组的系数行列式

$D \neq 0$ 则方程组有惟一零解.

例1： λ为何值时，方程组有非零解？

$$\begin{cases} \lambda x + y - z = 0, \\ x + \lambda y - z = 0, \\ 2x - y + \lambda z = 0. \end{cases}$$

解 若方程组有非零解,则其系数行列式为零,即

$$D = \begin{vmatrix} \lambda & 1 & -1 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = \lambda^3 - 1 = 0 \Rightarrow \lambda = 1$$

故当 $\lambda = 1$ 时，方程组有非零解.

练习

叙述克莱姆法则。

用克莱姆法则求解下面方程组。

$$\begin{cases} x_1 + 2x_2 - x_3 = 1, \\ 2x_1 + 3x_2 + x_3 = 0, \\ 4x_1 + 7x_2 - 2x_3 = 2. \end{cases}$$

$$\begin{cases} x_1 = -3, \\ x_2 = 2, \\ x_3 = 0. \end{cases}$$