相关性判定定理4与5的证明

定理4: m个n维向量$\alpha_i = (a_{i1}, a_{i2}, \cdots, a_{in})\ (i = 1,2,\cdots m)$线性相关的充要条件是由$\alpha_i\,(i = 1,2,\cdots m)$构成的矩阵

$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

的秩$r(A) < m$.

证明定理4.

"$\Rightarrow$": $\alpha_1,\alpha_2,\cdots,\alpha_m$线性相关,

由定理1知，必有某个向量(不妨设α_m)可由其余$m-1$个向量线性表示, 即 $\alpha_m=k_1\alpha_1+\cdots+k_{m-1}\alpha_{m-1}$.

写成分量形式为

$$a_{mj}=k_1a_{1j}+k_2a_{2j}+\cdots+k_{m-1}a_{m-1,j}.$$

$$A=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\ a_{21}&a_{22}&\cdots&a_{2n}\\ \vdots&\vdots&\cdots&\vdots\\ a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix}$$

对A作初等变换

$$A=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_{m-1}\\ \alpha_m\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\ \vdots&\vdots&\cdots&\vdots\\ a_{m-1,1}&a_{m-1,2}&\cdots&a_{m-1,n}\\ a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix}\rightarrow\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\ \vdots&\vdots&\cdots&\vdots\\ a_{m-1,1}&a_{m-1,2}&\cdots&a_{m-1,n}\\ 0&0&\cdots&0\end{pmatrix}$$

$\Rightarrow r(A)<m.$

"$\Leftarrow$": $r(A)=r<m$,不妨设$r>0$, 且A的最左上角的r阶子式$D_r \neq 0$

考虑A的r+1阶子式

$$D_{r+1}=\begin{vmatrix} a_{11} & \cdots & a_{1r} & a_{1,j} \\ \vdots & \cdots & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{r,j} \\ a_{r+1,1} & \cdots & a_{r+1,r} & a_{r+1,j} \end{vmatrix}.$$

$$r(A)=r \Rightarrow D_{r+1}=0.$$

将D_j按最后一列展开，有：

$$a_{1j}A_1+a_{2j}A_2+\cdots+a_{rj}A_r+a_{r+1,j}D_r=0,\ j=1,2,\cdots,n.$$

$$a_{1j}A_1 + a_{2j}A_2 + \cdots + a_{rj}A_r + a_{r+1,j}D_r = 0, \qquad j = 1, 2, \cdots, n.$$

$$a_{11}A_1 + a_{21}A_2 + \cdots + a_{r1}A_r + a_{r+1,1}D_r = 0,$$

$$a_{12}A_1 + a_{22}A_2 + \cdots + a_{r2}A_r + a_{r+1,2}D_r = 0,$$

……….

$$a_{1n}A_1 + a_{2n}A_2 + \cdots + a_{rn}A_r + a_{r+1,n}D_r = 0.$$

按向量形式写，上式为：

$$\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_r A_r + \alpha_{r+1} D_r = \mathbf{0}.$$

$\because D_r \neq 0, \Rightarrow \alpha_1, \alpha_2, \cdots, \alpha_{r+1}$线性相关,

从而$\alpha_1, \alpha_2, \cdots, \alpha_m$线性相关。

定理5： 若 m 个 r 维向量

$$\alpha_i = (a_{i1}, a_{i2}, \cdots, a_{ir}) \quad (i = 1,2,\cdots,m)$$

线性无关，则对应的 m 个$r+1$ 维向量

$$\beta_i = (a_{i1}, a_{i2}, \cdots, a_{ir}, a_{i,r+1}) \quad (i = 1,2,\cdots,m)$$

也线性无关。

证明：设

$$A=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_m\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1r}\\ a_{21} & a_{22} & \cdots & a_{2r}\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mr}\end{pmatrix},$$

$$B=\begin{pmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_m\end{pmatrix}=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1r} & a_{1,r+1}\\ a_{21} & a_{22} & \cdots & a_{2r} & a_{2,r+1}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mr} & a_{m,r+1}\end{pmatrix}.$$

$$\Rightarrow m=r(A)\le r(B)\le m,\ \ \Rightarrow r(B)=m.$$

$\therefore \beta_1,\beta_2,\cdots,\beta_m$线性无关。