分块矩阵

一、分块矩阵的概念

$$A=\left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right)=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A=\left(\begin{array}{c|cc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right)=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}$$

定义：将矩阵用若干纵横直线分成若干个小块，每一小块称为矩阵的子块（或子阵），以子块为元素形成的矩阵称为分块矩阵。

二、分块矩阵的运算

1.线性运算　　加法与数乘

2.乘法运算　　符合乘法的要求

3.转置运算　　大块小块一起转

$$A^T=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix}^T=\begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \\ A_{13}^T & A_{23}^T \end{pmatrix}$$

三、几种特殊的分块阵

1.准对角阵

$$A=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix}$$

准对角阵或分块对角阵

$(A_i$为方阵，$i=1,2,\cdots,s)$

$$A=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix} \qquad B=\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_s \end{pmatrix}$$

$(A_i, B_i$为同阶方阵，$i=1,2,\cdots,s)$

则有

$$\boxed{A\pm B}=\begin{pmatrix} A_1\pm B_1 & & & \\ & A_2\pm B_2 & & \\ & & \ddots & \\ & & & A_s\pm B_s \end{pmatrix}$$

$$\boxed{kA}=\begin{pmatrix} kA_1 & & & \\ & kA_2 & & \\ & & \ddots & \\ & & & kA_s \end{pmatrix} \quad \boxed{AB}=\begin{pmatrix} A_1B_1 & & & \\ & A_2B_2 & & \\ & & \ddots & \\ & & & A_sB_s \end{pmatrix}$$

$$A^T=\begin{pmatrix} A_1{}^T & & & \\ & A_2{}^T & & \\ & & \ddots & \\ & & & A_s{}^T \end{pmatrix} \quad A^m=\begin{pmatrix} A_1{}^m & & & \\ & A_2{}^m & & \\ & & \ddots & \\ & & & A_s{}^m \end{pmatrix}$$

$$|A|=|A_1||A_2|\cdots|A_s|$$

A可逆 $\Leftrightarrow A_i$可逆

$(i=1,2,\cdots,s)$

$$\boxed{A^{-1}=\begin{pmatrix} A_1{}^{-1} & & & \\ & A_2{}^{-1} & & \\ & & \ddots & \\ & & & A_s{}^{-1} \end{pmatrix}}$$

$$\boxed{r(A)=r(A_1)+r(A_2)+\cdots+r(A_s)}$$

牢记这些公式！

例1

$$A=\left(\begin{array}{cc|cc}1 & 2 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 2 & 3\\ 0 & 0 & 1 & 3\end{array}\right)$$

求A的行列式，秩及逆。

解：将矩阵分块

$$A=\begin{pmatrix}A_1 & \\ & A_2\end{pmatrix}\Rightarrow |A|=|A_1||A_2|=3$$

$$r(A)=4$$

$$A^{-1}=\begin{pmatrix}A_1^{-1} & \\ & A_2^{-1}\end{pmatrix}=\begin{pmatrix}1 & -2 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & -\frac{1}{3} & \frac{2}{3}\end{pmatrix}$$

只须口算即可！

2.分块三角阵

$$A=\begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}$$

分块上三角阵
或准上三角阵

$(A_{ii}$为方阵，$i=1,2.)$

$$\boxed{|A|=|A_{11}||A_{22}|}$$

A可逆 $\Leftrightarrow$ A_{ii}可逆

$(i=1,2.)$

$$\boxed{A^{-1}=\begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{pmatrix}}$$

$$A=\begin{pmatrix} A_{11} & O \\ A_{21} & A_{22} \end{pmatrix}$$

$$\boxed{A^{-1}=\begin{pmatrix} A_{11}^{-1} & O \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{pmatrix}}$$

设$A^{-1}=\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ 则

$$AA^{-1}=\begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$=\begin{pmatrix} A_{11}X_{11}+A_{12}X_{21} & A_{11}X_{12}+A_{12}X_{22} \\ A_{22}X_{21} & A_{22}X_{22} \end{pmatrix}=\begin{pmatrix} E & O \\ O & E \end{pmatrix}$$

$$A_{22}X_{22}=E \Rightarrow X_{22}=A_{22}^{-1}$$

$$A_{22}X_{21}=O \Rightarrow X_{21}=O$$

$$A_{11}X_{11}+A_{12}X_{21}=E \Rightarrow X_{11}=A_{11}^{-1}$$

$$A_{11}X_{12}+A_{12}X_{22}=O \Rightarrow X_{12}=-A_{11}^{-1}A_{12}A_{22}^{-1}$$

测试

$$\begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}^{-1} = ?$$

$$\boxed{\begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{pmatrix}}$$

$$\begin{pmatrix} A_{11} & O \\ A_{21} & A_{22} \end{pmatrix}^{-1} = ?$$

$$\boxed{\begin{pmatrix} A_{11}^{-1} & O \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{pmatrix}}$$

你发现规律了吗？

例2.求矩阵的逆$A=\left(\begin{array}{cc|cc} 2 & 1 & 3 & 4 \\ 0 & 2 & 1 & 3 \\ \hline 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)$.

解：将矩阵分块

$$A=\begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix} \qquad A_{11}^{-1}=\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}=A_{22}^{-1}$$

$$A^{-1}=\begin{pmatrix} A_{11}{}^{-1} & -A_{11}{}^{-1}A_{12}A_{22}{}^{-1} \\ O & A_{22}{}^{-1} \end{pmatrix}$$

只须计算

$A_{11}{}^{-1}A_{12}A_{22}{}^{-1}$

$$A^{-1}=\begin{pmatrix} 1/2 & -1/4 & -5/8 & -5/16 \\ 0 & 1/2 & -1/4 & -5/8 \\ 0 & 0 & 1/2 & -1/4 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

3.分块斜对角阵 $M=\begin{pmatrix} O & A \\ B & O \end{pmatrix}$

$\boxed{M\text{可逆} \Leftrightarrow A,B\text{可逆}}$

猜一猜 $\begin{pmatrix} O & A \\ B & O \end{pmatrix}^{-1}=?$

$$\boxed{M^{-1}=\begin{pmatrix} O & B^{-1} \\ A^{-1} & O \end{pmatrix}}$$

例3.求矩阵的逆$M=\left(\begin{array}{cc|cc} 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 \\ \hline 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right)$

解：将矩阵分块

只须口算即可！

$$M=\begin{pmatrix} O & A \\ B & O \end{pmatrix}$$

$$M^{-1}=\begin{pmatrix} O & B^{-1} \\ A^{-1} & O \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 1/3 \\ 2 & -5 & 0 & 0 \\ -1 & 3 & 0 & 0 \end{pmatrix}$$