定理 n阶行列式D 等于它的任一行(列)各元素与其对应的代数余子式乘积之和,即

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

$$(i = 1,2,\cdots n)$$

或

$$D = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

$$(j = 1,2,\cdots n)$$

证： （1）

$$D=\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=\sum(-1)^{N}a_{1j_1}a_{2j_2}\cdots a_{nj_n}$$

$$=\sum_{1j_2\cdots j_n}(-1)^{N}a_{11}a_{2j_2}\cdots a_{nj_n}$$

$$=a_{11}\sum_{j_2\cdots j_n}(-1)^{N}a_{2j_2}\cdots a_{nj_n}$$

$$=a_{11}M_{11}=a_{11}A_{11}$$

（2）

$$D=\begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & a_{1j} & a_{1,j+1} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1j} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ 0 & \cdots & 0 & a_{ij} & 0 & \cdots & 0 \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$$

$$=(-1)^{i+j-2}\begin{vmatrix} a_{ij} & 0 & \cdots & 0 \\ a_{1j} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{nj} & a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$=(-1)^{i+j-2}a_{ij}M_{ij} = a_{ij}A_{ij}$$

（3）　一般地

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1}+0+\cdots+0 & 0+a_{i2}+0+\cdots+0 & \cdots & 0+\cdots+0+a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{i2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in} \quad (i=1,2,\cdots n)$$

同理证明列的情形。

$$D_n = \begin{vmatrix} a & b & \cdots & b & b \\ b & a & \cdots & b & b \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & \cdots & b & a \end{vmatrix} = [a+(n-1)b](a-b)^{n-1}.$$

$$\text{求} D_n = \begin{vmatrix} x & a & a & \cdots & a \\ b & x & a & \cdots & a \\ b & b & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & x \end{vmatrix}.$$

推广　求$D_n=\begin{vmatrix} x & a & a & \cdots & a \\ b & x & a & \cdots & a \\ b & b & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & x \end{vmatrix}$.

解　若$a=b$，由2知$D_n=[x+(n-1)a](x-a)^{n-1}$;

若$a\neq b$，则有

$$D_n=\begin{vmatrix} (x-a)+a & 0+a & 0+a & \cdots & 0+a \\ b & x & a & \cdots & a \\ b & b & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & x \end{vmatrix}=\begin{vmatrix} x-a & 0 & 0 & \cdots & 0 \\ b & x & a & \cdots & a \\ b & b & x & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & x \end{vmatrix}+a\begin{vmatrix} 1 & 1 & \cdots & 1 \\ b & x & \cdots & a \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & x \end{vmatrix}$$

$$=(x-a)D_{n-1}+a\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x-b & \cdots & a-b \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & x-b \end{vmatrix}=(x-a)D_{n-1}+a(x-b)^{n-1}.$$

由a，b的对称性，知$D_n=(x-b)D_{n-1}+b(x-a)^{n-1}$.

$$\begin{cases} D_n=(x-a)D_{n-1}+a(x-b)^{n-1}, \\ D_n=(x-b)D_{n-1}+b(x-a)^{n-1}, \end{cases}$$

则 $D_n=\dfrac{a(x-b)^n-b(x-a)^n}{a-b}$，$(a\neq b)$.

$$\begin{vmatrix} & & & a_{1n} \\ & & a_{2(n-1)} & a_{2n} \\ & \iddots & & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix} = a_{1n}(-1)^{n+1} \begin{vmatrix} & & a_{2(n-1)} \\ & \iddots & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= a_{1n}(-1)^{n+1} a_{2(n-1)}(-1)^{n-1+1} \begin{vmatrix} & & a_{3(n-2)} \\ & \iddots & \vdots \\ a_{n1} & \cdots & a_{n(n-2)} \end{vmatrix}_{(n-2)\times(n-2)}$$

$$= (-1)^{(n+1)+n} a_{1n} a_{2(n-1)} \begin{vmatrix} & & a_{3(n-2)} \\ & \iddots & \vdots \\ a_{n1} & \cdots & a_{n(n-2)} \end{vmatrix}_{(n-2)\times(n-2)} = \cdots$$

$$= (-1)^{(n+1)+n+(n-1)+\cdots+4} a_{1n} a_{2(n-1)} \cdots a_{(n-2)3} \begin{vmatrix} & a_{(n-1)2} \\ a_{n1} & a_{n2} \end{vmatrix}_{2\times 2}$$

$$\begin{vmatrix} & & & a_{1n} \\ & & a_{2(n-1)} & a_{2n} \\ & \iddots & & \vdots \\ a_{n1} & \cdots & a_{n(n-1)} & a_{nn} \end{vmatrix} = (-1)^{(n+1)+n+(n-1)+\cdots+4} a_{1n}a_{2(n-1)}\cdots a_{(n-2)3} \begin{vmatrix} & a_{(n-1)2} \\ a_{n1} & a_{n2} \end{vmatrix}_{2\times 2}$$

$$= (-1)^{(n+1)+n+(n-1)+\cdots+4} a_{1n}a_{2(n-1)}\cdots a_{(n-2)3}(-1)a_{(n-1)2}a_{n1}$$

（注意：$(-1) = (-1)^5 = (-1)^{3+2}$）

$$= (-1)^{(n+1)+n+(n-1)+\cdots+4+3+2} a_{1n}a_{2(n-1)}\cdots a_{(n-2)3}a_{(n-1)2}a_{n1}$$

$$= (-1)^{2n+(n-1)+\cdots+3+2+1} a_{1n}a_{2(n-1)}\cdots a_{(n-2)3}a_{(n-1)2}a_{n1}$$

$$= (-1)^{(n-1)+\cdots+3+2+1} a_{1n}a_{2(n-1)}\cdots a_{(n-2)3}a_{(n-1)2}a_{n1}$$

$$= (-1)^{n(n-1)/2} a_{1n}a_{2(n-1)}\cdots a_{(n-2)3}a_{(n-1)2}a_{n1}$$