矩阵的运算

线性运算

1.相等:两个矩阵相等是指这两个矩阵有相同的行数与列数, 且对应元素相等.即

$$A=\left(a_{ij}\right)_{m\times n} = B=\left(b_{ij}\right)_{m\times n}$$

同型

型号相同

对应元素相等

$$a_{ij}=b_{ij}$$

2.加、减法

设矩阵 $A=\left(a_{ij}\right)_{m\times n}$ 与 $B=\left(b_{ij}\right)_{m\times n}$ 定义

$$A+B=\left(a_{ij}+b_{ij}\right)_{m\times n} \qquad A-B=\left(a_{ij}-b_{ij}\right)_{m\times n}$$

运算规律

$$A+B=B+A,\quad (A+B)+C=A+(B+C),$$

$$A+O=A=O+A,\quad A-A=O.$$

负矩阵 $A=\left(a_{ij}\right)_{m\times n}$ 的负矩阵为 $\left(-a_{ij}\right)_{m\times n}$

记作 $-A$,即 $-A=\left(-a_{ij}\right)_{m\times n}$

3.数乘

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

与数的乘法，简称为数乘。记作：kA

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

问题

注意：数*k*乘矩阵中的每一个元素。

$k=1,\ A.\quad k=-1,\ -A.\ 1A=A,(-1)A=-A.\quad 0A=?$

运算规律

$$k(A+B)=kA+kB,$$

$$k(lA)=(kl)A,(k+l)A=kA+lA.$$

矩阵的乘法

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 = \cdots \end{cases}$$

与

$$\begin{cases} x_1 = b_{11}t_1 + b_{12}t_2 \\ x_2 = b_{21}t_1 + b_{22}t_2 \\ x_3 = b_{31}t_1 + b_{32}t_2 \end{cases}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$\begin{cases} y_1 = (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31})t_1 + \\ \qquad (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32})t_2 \\ y_2 = (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31})t_1 + \\ \qquad (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32})t_2 \end{cases}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$$

一般地，有

$$A=(a_{ij})_{m\times s} \quad B=(b_{ij})_{s\times n} \quad C=AB=(c_{ij})_{m\times n}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ \vdots & \vdots & \vdots & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{a_{is}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{ms} \end{pmatrix}_{m\times s} \begin{pmatrix} b_{11} & \cdots & \boxed{b_{1j}} & \cdots & b_{1n} \\ b_{21} & \cdots & \boxed{b_{2j}} & \cdots & b_{2n} \\ \vdots & \vdots & \boxed{\vdots} & \vdots & \vdots \\ b_{s1} & \cdots & \boxed{b_{sj}} & \cdots & b_{sn} \end{pmatrix}_{s\times n} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ c_{i1} & \cdots & \boxed{c_{ij}} & \cdots & c_{in} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{pmatrix}_{m\times n}$$

$$c_{ij} \qquad (a_{i1} \quad a_{i2} \cdots a_{is}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{pmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{is}b_{sj}$$

$$C_{m\times n} = A_{m\times s} \; B_{s\times n}$$

A与B满足什么条件时能够相乘？

你记住了吗？

例1: $A=\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, B=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$AB=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O \qquad BA=\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix}$$

<u>显然</u> $AB \neq BA$

这正是矩阵与数的不同

例2: $A=\begin{pmatrix} 2 & 4 \\ -3 & -6 \end{pmatrix}, B=\begin{pmatrix} -1 & 4 \\ 2 & -1 \end{pmatrix}, C=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$AB=\begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix}, AC=\begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix}$$

$\Rightarrow AB = AC$ 但是 $B \neq C$

这又是矩阵与数的不同

请记住

1.矩阵乘法不满足交换率;

2.不满足消去率;

3.有非零的零因子。

运算规律

$$1.(AB)C = A(BC)$$

$$2.A(B+C) = AB + AC$$

$$(B+C)A = BA + CA$$

$$3.k(AB) = (kA)B = A(kB)$$

$$4.E_m A_{m\times n} = A = A_{m\times n} E_n$$

练习1

矩阵$A=\begin{pmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{pmatrix}$,

$AA=$ ____________________.

练习2

$$a_1 \cdots a_n \neq 0\text{，对角阵} A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix},\quad B = \begin{pmatrix} \frac{1}{a_1} & & \\ & \ddots & \\ & & \frac{1}{a_n} \end{pmatrix}.$$

$AB =$ ________________.

练习1答案

$$AA=\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$$

$$=\begin{pmatrix}\cos^2\varphi-\sin^2\varphi & -2\sin\varphi\cos\varphi\\ 2\sin\varphi\cos\varphi & \cos^2\varphi-\sin^2\varphi\end{pmatrix}$$

$$=\begin{pmatrix}\cos 2\varphi & -\sin 2\varphi\\ \sin 2\varphi & \cos 2\varphi\end{pmatrix}$$

练习2答案

$$AB=E.$$