方阵的正整数幂

$$A^k = AA\cdots A \qquad A^0 = E \qquad A^{k+l} = A^k A^l$$

$$(AB)^k \neq A^k B^k$$

问题

$(AB)^k = A^k B^k$ 成立的条件?

AB*=*BA

矩阵的转置

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad A^T=\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

问题

$$\begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{pmatrix}^T = ?$$

矩阵的转置

运算规律

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(kA)^T = kA^T$$

问题

$$(AB)^T = ?$$

$$(AB)^T = B^T A^T$$

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矩阵的转置

自己证明

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(kA)^T = kA^T$$

$$(AB)^T = B^T A^T \qquad A = (a_{ij})_{m\times s} \qquad B = (b_{ij})_{s\times n}$$

$$A^T = (a_{ji})_{s\times m} \qquad B^T = (b_{ji})_{n\times s}$$

$$C = AB = (c_{ij})_{m\times n} \qquad B^T A^T = (d_{ij})_{n\times m}$$

$$c_{ji} = d_{ij}\ ?$$

$$c_{ji} = a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{js}b_{si}$$

$$d_{ij} = b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{si}a_{js}$$

$$\Rightarrow c_{ji} = d_{ij}$$

也就是 $(AB)^T = B^T A^T$

$$(ABC)^T = C^T B^T A^T$$

对称阵与反对称阵

对称阵：$A^T = A$

$a_{ij} = a_{ji}$

$AA^T, A^T A, A + A^T$

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

反对称阵：$A^T = -A$

$a_{ij} = -a_{ji}$ 且 $a_{ii} = 0$

$A - A^T$

$$(A - A^T)^T = A^T - (A^T)^T$$
$$= A^T - A = -(A - A^T).$$

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

任一方阵都可以分解成对称阵与反对称阵的和.

对称阵与反对称阵

问题

反对称阵：$A^T = -A$ 怎么说明

$$a_{ii} = 0?$$

练习

1. 设矩阵A与B为同阶对称阵，证明AB是对称阵的充要条件为$AB=BA$.

2. 矩阵$A=\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$的幂$A^n=?$

1.证：$\Rightarrow$：

$$\left.\begin{aligned}\because (AB)^T &= AB \\ \text{又}(AB)^T &= B^T A^T = BA\end{aligned}\right\} \Rightarrow AB = BA$$

$\Leftarrow$：

$$\because AB = BA$$

$$\therefore (AB)^T = B^T A^T = BA = AB$$

$\Rightarrow AB$为对称阵。

2.

$$A^2=\begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}$$

$$=\begin{pmatrix}\cos^2\varphi-\sin^2\varphi & -2\sin\varphi\cos\varphi \\ 2\sin\varphi\cos\varphi & \cos^2\varphi-\sin^2\varphi\end{pmatrix}$$

$$=\begin{pmatrix}\cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi\end{pmatrix}$$

$$\xrightarrow{\text{推测}} A^n=\begin{pmatrix}\cos n\phi & -\sin n\phi \\ \sin n\phi & \cos n\phi\end{pmatrix}.$$

设$A^{n-1}=\begin{pmatrix}\cos(n-1)\varphi & -\sin(n-1)\varphi \\ \sin(n-1)\varphi & \cos(n-1)\varphi\end{pmatrix}$

则$A^{n}=A^{n-1}A=$

$$\begin{pmatrix}\cos(n-1)\varphi & -\sin(n-1)\varphi \\ \sin(n-1)\varphi & \cos(n-1)\varphi\end{pmatrix}\begin{pmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{pmatrix}$$

$$=\begin{pmatrix}\cos n\varphi & -\sin n\varphi \\ \sin n\varphi & \cos n\varphi\end{pmatrix}$$

$$\therefore A^{n}=\begin{pmatrix}\cos n\varphi & -\sin n\varphi \\ \sin n\varphi & \cos n\varphi\end{pmatrix}$$