Тройные интегралы

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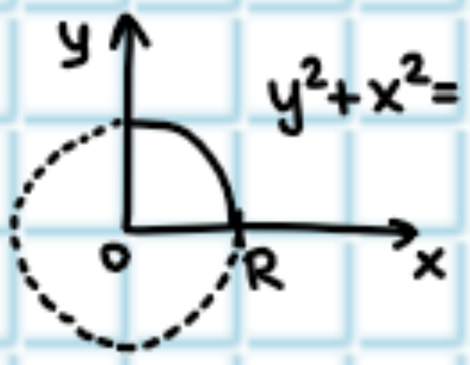
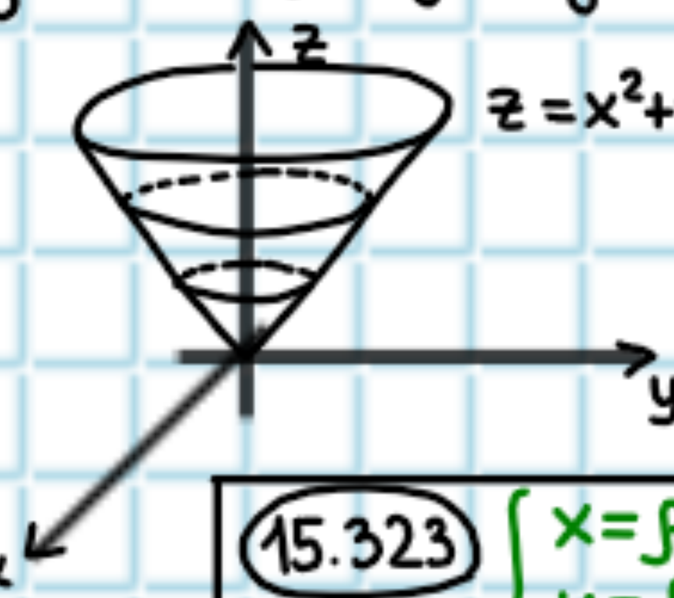
15.303 $\int_0^R dx\int_0^{\sqrt{R^2-x^2}}dy\int_0^{x^2+y^2}f\,dz=\int_0^R dy\int_0^{\sqrt{R^2-y^2}}dx\int_0^{x^2+y^2}f\,dz=\int_0^{R^2}dz\int_0^{\sqrt{z}}dy\int_{\sqrt{z-y^2}}^{\sqrt{R^2-y^2}}f\,dx+\int_0^{R^2}dz\int_{\sqrt{z}}^{R}dy\int_0^{\sqrt{R^2-y^2}}f\,dx=\int_0^R dx\int_{x^2}^{R^2}dz\int_{\sqrt{z-x^2}}^{\sqrt{R^2-x^2}}f\,dy+\int_0^R dx\int_0^{x^2}dz\int_0^{\sqrt{R^2-x^2}}f\,dy=$





$=\int_0^R dy\int_0^{y^2}dz\int_0^{\sqrt{R^2-y^2}}f\,dx+\int_0^R dy\int_{y^2}^{R^2}dz\int_{\sqrt{z-y^2}}^{\sqrt{R^2-y^2}}f\,dx=\int_0^{R^2}dz\int_0^{\sqrt{z}}dx\int_{\sqrt{z-x^2}}^{\sqrt{R^2-x^2}}f\,dy+\int_0^{R^2}dz\int_{\sqrt{z}}^{R}dx\int_0^{\sqrt{R^2-x^2}}f\,dy$

15.323 $\begin{cases}x=\rho\sin\theta\cos\varphi\\ y=\rho\sin\theta\sin\varphi\\ z=\rho\cos\theta;\ D=\rho^2\sin\theta\end{cases}$ $\rho^2\le 4R\rho\sin\theta$, $\rho^2\ge R\rho\sin\theta$, $\rho^2\sin^2\theta\le\frac{\rho^2\cos^2\theta}{3}\to tg^2\theta\le\frac{1}{3}\to\theta\in[-\frac{\pi}{6}+\pi k,\frac{\pi}{6}+\pi k]$

$\int_0^{2\pi}d\varphi\int_{R\sin\theta}^{4R\sin\theta}\rho^2d\rho\int_{-\pi/6}^{\pi/6}\cos\theta\,d\theta$

15.307 В цилиндрической системе координат расставить пределы интегрирования $\iiint_V f(x,y,z)\,dxdydz$

$\begin{cases}x=\rho\cos\varphi & \varphi\in[0;2\pi)\\ y=\rho\sin\varphi & \rho\in(0;R]\\ z=z & z\in[0;H]\end{cases}$ $x^2+y^2\le k^2z^2\longrightarrow\rho^2\le k^2z^2$

$\rho\in(0;R]\Rightarrow\rho\le|kz|$

тогда $V=\int_0^{2\pi}d\varphi\int_0^{kH}\rho\,d\rho\int_{\rho/k}^{H}dz$ ⊛

⊛ $\int_0^{2\pi}d\varphi\int_0^{H}dz\int_0^{|kz|}\rho\,d\rho$ $\int_0^{H}dz\int_0^{2\pi}d\varphi\int_0^{|kz|}\rho\,d\rho=\int_0^{H}dz\int_0^{|kz|}\rho\,d\rho\int_0^{2\pi}d\varphi$ и $\int_0^{|kH|}\rho\,d\rho\int_0^{2\pi}d\varphi\int_{\rho/k}^{H}dz=\int_0^{|kH|}\rho\,d\rho\int_{\rho/k}^{H}dz\int_0^{2\pi}d\varphi$

15.309 ... $x^2+y^2+z^2\le 2az\longrightarrow\rho^2+z^2\le 2az\longrightarrow(z-a)^2\le a^2-\rho^2$

$x^2+y^2\le z^2\longrightarrow\rho^2\le z^2$ т.е. $z\le a+\sqrt{a^2-\rho^2}$

$\begin{cases}x=\rho\cos\varphi & \varphi\in[0;2\pi)\\ y=\rho\sin\varphi & \rho\in(0;R]\\ z=z & z\in[0;H]\end{cases}$

$\int_0^{2\pi}d\varphi\int_0^{a}\rho\,d\rho\int_{\rho}^{a+\sqrt{a^2-\rho^2}}dz=\int_0^{2\pi}d\varphi\int_0^{a}dz\int_0^{z}\rho\,d\rho+\int_0^{2\pi}d\varphi\int_0^{2a}dz\int_0^{\sqrt{2az-z^2}}\rho\,d\rho$
(2) (1)

15.316 $x^2+y^2\le z$

$z\in[0;H]$

$\int_0^{2\pi}d\varphi\int_0^{H}dz\int_0^{\sqrt{z}}\rho\,d\rho$

15.317 $x\in[0;1]$, $y\in[0;1]$, $z\in[0;1]$ $0\le\rho\le\frac{1}{\cos\varphi}$ $\cos\varphi>\sin\varphi$ $\varphi\in[0;\frac{\pi}{4}]$, то есть $\rho\le\frac{1}{\cos\varphi}\le\frac{1}{\sin\varphi}$ $\varphi\in[0;\frac{\pi}{4}]$

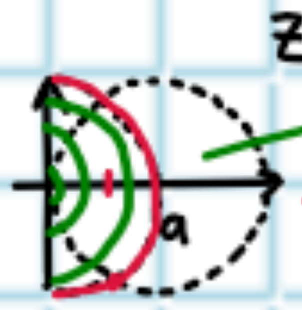
$0\le\rho\le\frac{1}{\sin\varphi}$ $\cos\varphi<\sin\varphi$ $\varphi\in[\frac{\pi}{4};\frac{\pi}{2}]$ ...

$z\in[0;1]$ $\int_0^{\pi/4}d\varphi\int_0^{1/\cos\varphi}\rho\,d\rho\int_0^1 f\,dz+\int_{\pi/4}^{\pi/2}d\varphi\int_0^{1/\sin\varphi}\rho\,d\rho\int_0^1 f\,dz$

15.319 $\begin{cases}\rho^2\le 2a\rho\cos\varphi\\ \rho^2\le a^2-az\end{cases}$ $\int_{-\pi/2}^{-\pi/3}d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho\int_0^{\frac{a^2-\rho^2}{a}}f\,dz+\int_{-\pi/3}^{\pi/3}d\varphi\int_0^{a}\rho\,d\rho\int_0^{\frac{a^2-\rho^2}{a}}f\,dz$

$z\ge 0,\ z\le a$

$+\int_{\pi/3}^{\pi/2}d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho\int_0^{\frac{a^2-\rho^2}{a}}f\,dz$



$z=0$ $\rho\le 2a\cos\varphi$, $\rho\le a$

$2a\cos\varphi=a\longrightarrow\varphi=\pm\frac{\pi}{3}+2\pi k,\ k\in\mathbb{Z}$

3.19. $\iint\limits_S yzdydz-x^2dxdz-y^2dxdy$, где S — часть поверхности конуса $x^2+z^2=y^2$ (нормальный вектор $\mathbf{n}$ которой образует тупой угол с ортом $\mathbf{j}$), отсекаемая плоскостями $y=0$, $y=1$. (*Ответ:* $\pi/4$.)

15.350 Решить $\int_1^2 dy\int_y^2 dx\int_0^{1/xy}\frac{1}{x(1+x^2y^2z^2)}dz=\frac{\pi}{48}$

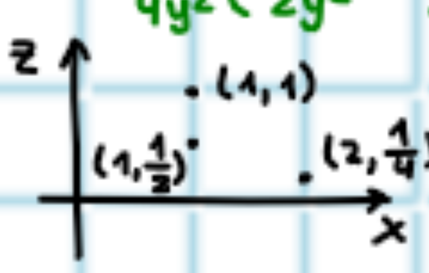
$\frac{1}{x^3y^2}\cdot arctg(x^2y^2z^2)\Big|_0^{\frac{1}{xy}}=$

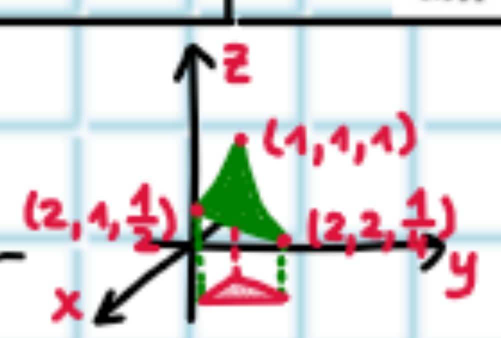
$=\frac{1}{x^3y^2}\cdot\frac{\pi}{4}$

$\int_y^2\frac{\pi}{4x^3y^2}dx=-\frac{1}{2x^2}\Big|_y^2\cdot\frac{\pi}{4y^2}=$

$=\frac{\pi}{4y^2}\left(\frac{1}{2y^2}-\frac{1}{8}\right)$

или $\int_{1/2}^{1}dz\int_1^{\frac{z}{3}-\frac{4z}{3}}dy\int_0^{1/yz}dx+\int_0^{1/2}dz\int_{-4z+3}^{\frac{z}{3}-\frac{4z}{3}}dy\int_0^{1/yz}dx$





$\iint\limits_S yzdydz-x^2dxdz-y^2dxdy$ $S\{x,y,z\}$
$S\{x,\sqrt{z^2+x^2},z\}$

тогда $\vec{r}'_x=\{\frac{\partial S_x}{\partial x},\frac{\partial S_y}{\partial x},\frac{\partial S_z}{\partial x}\}\longrightarrow\vec{r}'_x=(1,\frac{x}{\sqrt{z^2+x^2}},0)$

$\vec{r}'_z=(0,\frac{z}{\sqrt{z^2+x^2}},1)$

$\vec{n}=\vec{r}'_x\times\vec{r}'_z$

$\vec{n}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ 1&\frac{x}{\sqrt{x^2+z^2}}&0\\ 0&\frac{z}{\sqrt{x^2+z^2}}&1\end{vmatrix}\begin{matrix}\vec{i}&\vec{j}\\ 1&\\ 0&\frac{z}{\sqrt{x^2+z^2}}\end{matrix}=\vec{i}\underbrace{\left(\frac{x}{\sqrt{x^2+z^2}}\right)}_{\frac{\partial(y,z)}{\partial(x,z)}}+\vec{j}\underbrace{(-1)}_{\frac{\partial(z,x)}{\partial(x,z)}}+\vec{k}\underbrace{\left(\frac{z}{\sqrt{x^2+z^2}}\right)}_{\frac{\partial(x,y)}{\partial(x,z)}}$

$\iint\limits_S yzdydz-x^2dzdx-y^2dxdy=\iint\limits_D\left(\sqrt{z^2+x^2}\cdot z\cdot\frac{x}{\sqrt{z^2+x^2}}-x^2\cancel{\frac{\partial(z,x)}{\partial(z,x)}}+(x^2+z^2)\cdot\frac{z}{\sqrt{x^2+z^2}}\right)dxdz=$

$=\int_0^{2\pi}d\varphi\int_{\dots}^{\dots}\rho\cdot\rho^2(\sin\varphi\cos\varphi-\cos^2\varphi+\sin\varphi)\,d\rho=\frac{\rho^4}{4}\Big|_0^1\cdot\left(-\cos x-\frac{x}{2}-\frac{\sin 2x}{4}+\frac{\sin^2 x}{2}\right)\Big|_0^{2\pi}=$

zx $-x^2$ $\sqrt{x^2+z^2}\cdot z$

$=-\frac{\pi}{4}$ ⊛

якобиан перехода

$\begin{vmatrix}\frac{\partial x}{\partial\rho}&\frac{\partial x}{\partial\varphi}\\ \frac{\partial z}{\partial\rho}&\frac{\partial z}{\partial\varphi}\end{vmatrix}=\rho$

$x=\rho\cos\varphi$
$y=y$
$z=\rho\sin\varphi$

⊛ здесь должен стоять "+" по условию (угол тупой при $\vec{j}$, тогда "−" и "−" на "−" → "+")

Я устал решать одинаковые примеры, осталось разобрать что-нибудь интересное из поверхностных и криволинейных, например