Тригонометрия

Лопатенко Георгий М3102

1992

$$\int \sin^6 x\,dx = \frac{5}{6}\int \sin^4 x\,dx - \frac{1}{6}\sin^5 x\cos x = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \underbrace{\frac{5}{8}\int \sin^2 x\,dx}_{} + C,\ C\in\mathbb{R}$$

$$I_1 = \frac{5}{8}\int \frac{1-\cos 2x}{2}\,dx = \frac{5}{16}x - \frac{5}{32}\sin 2x + C_1,\ C_1\in\mathbb{R}$$

1994

$$\int \sin^2 x\cos^4 x\,dx = \int \sin^2 x\,dx - 2\int \sin^4 x\,dx + \int \sin^6 x\,dx = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{37}{8}\int \sin^2 x\,dx - \sin^2 x\cos x + C,\ C\in\mathbb{R}$$

1998

$$\int \frac{\cos^4 x}{\sin^3 x}\,dx = \int\left(\sin x - \frac{2}{\sin x} + \frac{1}{\sin^3 x}\right)dx = -\cos x - \frac{3}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right| - \frac{\cos x}{2\sin^2 x} + C,\ C\in\mathbb{R}$$

$$\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\left(\frac{x}{2}\right)\right|$$

$$\int \sin^{-1} x\,dx = -\frac{1}{-1}\cos x\cdot\sin^{-2} x + \frac{-1-1}{-1}\int \sin^{-3} x\,dx$$

$$\int \sin^{-3} x\,dx = \frac{\int \sin^{-1} x\,dx - \frac{\cos x}{\sin^2 x}}{2} = \frac{\ln\left|\operatorname{tg}\left(\frac{x}{2}\right)\right| - \frac{\cos x}{\sin^2 x}}{2}$$

2001

$$\int \frac{dx}{\sin^4 x\cos^4 x} = \int \frac{d(2x)}{2\left(\frac{\sin 2x}{2}\right)^4} = 8\int \frac{d(u)}{\sin^4 u} = \cancel{8}\int \frac{\cancel{2}(1+t^2)^3}{\cancel{16}t^4}\,dt$$

$$u = 2x;\ \sin u = \frac{2t}{1+t^2};\ t = \operatorname{tg}\frac{u}{2}$$

$$= \int \frac{1+3t^2+3t^4+t^6}{8t^4}\,dt = -\frac{1}{3t^3} - \frac{3}{t} + 3t + \frac{t^3}{3} + C,\ C\in\mathbb{R}$$

$$t = \operatorname{tg} x$$

2003

$$\int \frac{dx}{\sin x\cos^4 x} = \int \frac{\sin x\cdot dx}{(1-\cos^2 x)\cos^4 x} = \int \frac{dt}{t^4(t^2-1)} = \int\left(-\frac{1}{2(t+1)} - \frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{2(t-1)}\right)dt = \frac{\ln\left|\frac{1-\cos x}{1+\cos x}\right|}{2} + \frac{3\cos^2 x+1}{3\cos^3 x} + C,\ C\in\mathbb{R}$$

$$t = \cos x;\ dt = -\sin x\,dx$$

2004

$$\int \operatorname{tg}^5 x\,dx = \int \sin x\cdot\frac{(1-\cos^2 x)^2}{\cos^5 x}\,dx = \int\left(\frac{-1+2\cos^2 x-\cos^4 x}{\cos^5 x}\right)d(\cos x) = -\ln|\cos x| - \frac{1}{\cos^2 x} + \frac{1}{4\cos^4 x} + C,\ C\in\mathbb{R}$$

2006

$$\int \frac{\sin^4 x}{\cos^6 x}\,dx = \int \operatorname{tg}^4 x\,d(\operatorname{tg} x) = \frac{\operatorname{tg}^5 x}{5} + C,\ C\in\mathbb{R}$$

2011 а)

$$I_n = \int \sin^n x\,dx = \int \sin^{n-2} x\sin^2 x\,dx = I^{n-2} + \int -\sin^{n-2} x\cos^2 x\,dx = I^{n-2} - \cos x\cdot\frac{\sin^{n-1} x}{n-1} - \int \frac{\sin^n x}{n-1}\,dx$$

$$u = \cos x \quad d\upsilon = \sin^{n-2} x\,d(\sin x)$$

$$du = -\sin x\,dx \quad \upsilon = \frac{\sin^{n-1} x}{n-1}$$

$$I_n = \frac{I^{n-2} - \cos x\cdot\frac{\sin^{n-1} x}{n-1}}{\left(1+\frac{1}{n-1}\right)}$$

т.е.

$$\underbrace{\int \sin^n x\,dx}_{I_n} = \frac{n-1}{n}\underbrace{\int \sin^{n-2} x\,dx}_{I_{n-2}} - \frac{1}{n}\sin^{n-1} x\cos x + C,\ C\in\mathbb{R}$$

б)

$$K_n = \int \cos^n x\,dx = \int \cos^{n-2} x(1-\sin^2 x)\,dx$$

$$\| $$

$$K^{n-2} - \int \cos^{n-2} x\sin^2 x\,dx = K^{n-2} - \frac{\sin x}{n-1}\cdot\cos^{n-1} x + \frac{K^n}{n-1} \Rightarrow K^n = \frac{\sin x\cos^{n-1} x}{n} + \frac{n-1}{n}K^{n-2}$$

$$u = \sin x \quad du = \cos x\,dx$$

$$d\upsilon = \cos^{n-2} x\,d(\cos x)$$

$$\upsilon = \frac{\cos^{n-1} x}{n-1}$$

2014 $\displaystyle\int \cos x\cos 2x\cos 3x\,dx = \int\left(\frac{\cos x\,(4\cos^3x-3\cos x)}{2}+\underbrace{\frac{\cos^2 3x}{2}}\right)dx$

$\displaystyle 2\int\cos^4 x\,dx = 2\left(\frac34\underbrace{\int\cos^2x\,dx}_{\frac12 x+\frac{\sin 2x}{4}}+\frac{\sin x\cos^3 x}{4}\right)$

$\displaystyle Z_1=\frac16\int\left(\frac{1+\cos 6x}{2}\right)d(3x)=\frac14 x+\frac{\sin 6x}{24}+C,\ C\in\mathbb{R}$

2022 $\displaystyle\int\frac{dx}{\sin x-\sin\alpha}=\int\frac{dx}{2\cos(\frac{x+\alpha}{2})\sin(\frac{x-\alpha}{2})}=\frac{1}{\cos\alpha}\cdot\log\left|\frac{\sin(\frac{x-\alpha}{2})}{\cos(\frac{x+\alpha}{2})}\right|+C,\ C\in\mathbb{R}$

2026 $\displaystyle\int\frac{dx}{(2+\cos x)\sin x}=\int\frac{2}{1+t^2}\cdot\frac{1+t^2}{2t(2+\cos x)}dt=\int\frac{dt}{t\left(\frac{3+t^2}{1+t^2}\right)}=\int\frac{1+t^2}{t(3+t^2)}dt=\int\left(\frac{\frac23 t}{3+t^2}\right)dt+\int\frac{1/3}{t}dt=\frac13\ln|tg(\tfrac x2)|+\frac13\ln|3+tg^2\tfrac x2|+C,\ C\in\mathbb{R}$

$\displaystyle\int\frac{\frac{2}{3\cdot 2}\,d(3+t^2)}{3+t^2}=\frac13\ln|3+t^2|$

$x=2\,arctg\,t;\ dx=\frac{2}{1+t^2}dt$

$\displaystyle\frac{At+B}{3+t^2}+\frac{K}{t}=\frac{At^2+Bt+3K+Kt^2}{t(3+t^2)}=\frac{1+t^2}{t(3+t^2)}\longrightarrow\begin{cases}A+K=1\\ B=0\\ K=\frac13\end{cases}\longrightarrow (A,B,K)=\left(\tfrac23;0;\tfrac13\right)$

2027 $\displaystyle\int\frac{\sin^2x\,dx}{\sin x+2\cos x}=\int\frac{2}{\cancel{1+t^2}}\cdot\frac{\frac{4t^2}{(1+t^2)^2}}{\frac{2t+2-2t^2}{\cancel{1+t^2}}}dt=\int\frac{4t^2\,dt}{(1+t^2)^2(-t^2+t+1)}\ \dots$

$x=2\,arctg\,t$

2029 $\displaystyle\int\frac{\sin^2x}{1+\sin^2x}dx=\int\frac{\sin^2x+1-1}{1+\sin^2x}dx=x-\int\frac{dx}{\sin^2x+1}=x-\int\frac{1}{\sin^2x}\cdot\frac{dx}{1+\frac{1}{\sin^2x}}=x-\int\frac{dt}{t^2+2}=x-\frac{arctg\left(\frac{ctg\,x}{\sqrt2}\right)}{\sqrt2}+C,\ C\in\mathbb{R}$

$t=ctg\,x$, $dt=-\frac{1}{\sin^2x}dx$

2032 $\displaystyle\int\frac{\sin x\cos x}{\sin x+\cos x}dx=\int\frac{2}{\cancel{1+t^2}}\cdot\frac{\frac{2t(1-t^2)}{(1+t^2)^2}}{\frac{1+2t-t^2}{\cancel{1+t^2}}}dt=\int\frac{4t(1-t^2)}{(1+t^2)(1+2t-t^2)}dt=\int\left(\frac{1/4}{t^2-2t-1}-\underbrace{\frac{1/4}{t^2+1}}_{-\frac14 arctg\,t}+\underbrace{\frac{t+1}{2(t^2+1)^2}}_{\frac{arctg\,t}{2}+\frac{t-1}{2(t^2+1)}}\right)dt=\dots$

$\displaystyle\int\frac{1/4\,dt}{(t-1)^2-2}=\frac14\ln\left|\frac{t-1-1}{t-1+1}\right|=\frac14\ln\left|\frac{t-2}{t}\right|+C_1,\ C_1\in\mathbb{R}$

2035 $\displaystyle\int\frac{dx}{\sin^4x+\cos^4x}=\int\frac{1}{\cos^2x}\cdot\frac{1+tg^2x}{1+tg^4x}dx=\int\frac{1+t^2}{1+t^4}dt=\int\frac{\frac{1}{t^2}+1}{\frac{1}{t^2}+t^2}dt=\int\frac{\frac{1}{t^2}+1}{(\frac1t-t)^2+2}dt=\int\frac{-1}{(\frac1t-t)^2+2}d\left(\frac1t-t\right)=-\frac{arctg\left(\frac{\frac1t-t}{\sqrt2}\right)}{\sqrt2}+C,\ C\in\mathbb{R}$, $t=tg\,x$

$t=tg\,x$

$\displaystyle\frac{1}{\cos^4x}\cdot\frac{1}{1+tg^4x}=\frac{1}{\cos^2x}\cdot\frac{\sin^2x+\cos^2x}{\cos^2x}\cdot\frac{1}{1+tg^4x}$

2039 $\displaystyle\int\frac{dx}{\sin^6x+\cos^6x}=\int\frac{(1+t^2)^2}{t^6+1}dt=\int\frac{t^2+1}{t^4-t^2+1}dt=\int\left(\frac{1/2}{t^2+\sqrt3t+1}+\frac{1/2}{t^2-\sqrt3t+1}\right)dt=arctg(2t+\sqrt3)+arctg(2t-\sqrt3)+C,\ C\in\mathbb{R}$

$\displaystyle\frac{1}{\cos^2x}\cdot\frac{1}{tg^6x+1}\cdot\left(\frac{\cos^2x+\sin^2x}{\cos^2x}\right)^2$

$\displaystyle\int\frac{1/2\,dt}{(t+\frac{\sqrt3}{2})^2+(\frac12)^2}=\int\frac{2\,dt}{(2t+\sqrt3)^2+1}$

2040 $\displaystyle\int\frac{dx}{(\sin^2x+2\cos^3x)^2}$ – ?

2041 $\displaystyle\int\frac{dx}{a\sin x+b\cos x}=\int\frac{d(x+arctg\frac ba)}{\sqrt{a^2+b^2}\,\sin(x+arctg\frac ba)}=\frac{1}{\sqrt{a^2+b^2}}\ln\left|tg\frac k2\right|+C,\ C\in\mathbb{R}$

$k=x+arctg\frac ba$

2043.1 $\sin x=\frac{3}{10}(3\sin x+\cos x)+\frac{1}{10}(\sin x-3\cos x)$

$\displaystyle\int\frac{\sin x}{\sin x-3\cos x}dx=\int\frac{3(3\sin x+\cos x)}{10(\sin x-3\cos x)}dx+\frac{x}{10}=\frac{x}{10}+\frac{3}{10}\ln|\sin x-3\cos x|+C,\ C\in\mathbb{R}$

$\sin x-3\cos x\to t$