

# UR5 运动学正逆解

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## 连杆变换

标准的 DH 方法 (简称 stdDH) 建立 UR5 机械臂的模型中, 用  $a$  (连杆长度),  $d$  (连杆偏距),  $\alpha$  (连杆扭角),  $\theta$  (关节角度) 这四个量来建立机械臂的运动学模型, 需要注意在 stdDH 中, 第  $i-1$  坐标系建立在第  $i$  个关节轴上, 这一点对于编写递归的 Newton-Euler 算法很重要, 其中相邻连杆的齐次矩阵定义为:

$${}^{n-1}T_n = \left[ \begin{array}{ccc|c} \cos \theta_n & -\sin \theta_n \cos \alpha_n & \sin \theta_n \sin \alpha_n & a_n \cos \theta_n \\ \sin \theta_n & \cos \theta_n \cos \alpha_n & -\cos \theta_n \sin \alpha_n & a_n \sin \theta_n \\ 0 & \sin \alpha_n & \cos \alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{array} \right] = \left[ \begin{array}{c|c} \mathbf{R} & \mathbf{P} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \quad (1)$$

根据齐次变换矩阵的含义, 基坐标系到末端坐标系的齐次变换矩阵即机械臂的正运动学方程可表示成如下形式

$${}^0T_n = {}^0T_1 {}^1T_2 \dots {}^{n-1}T_n = \left[ \begin{array}{cccc} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (2)$$

UR5 机械臂的 DH 参数如下表所示

i	$a_i$	$\alpha_i$	$d_i$	$\theta_i$
1	0	$\frac{\pi}{2}$	$d_1$	$\theta_1$
2	$a_2$	0	0	$\theta_2$
3	$a_3$	0	0	$\theta_3$
4	0	$\frac{\pi}{2}$	$d_4$	$\theta_4$
5	0	$-\frac{\pi}{2}$	$d_5$	$\theta_5$
6	0	0	$d_6$	$\theta_6$

带入 DH 参数到式 (1) 中, 结合式 (2), 有

$${}^0T_1 = \left[ \begin{array}{cccc} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (3)$$

$${}^0T_2 = \left[ \begin{array}{cccc} c_1c_2 & -c_1s_2 & s_1 & a_2c_1c_2 \\ c_2s_1 & -s_1s_2 & -c_1 & a_2c_2s_1 \\ s_2 & c_2 & 0 & a_2s_2 + d_1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (4)$$

$${}^0T_3 = \left[ \begin{array}{cccc} c_1c_23 & -c_1s_23 & s_1 & c_1(a_2c_2 + a_3c_23) \\ c_23s_1 & -s_1s_23 & -c_1 & s_1(a_2c_2 + a_3c_23) \\ s_23 & c_23 & 0 & a_2s_2 + a_3s_23 + d_1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (5)$$

$${}^0T_4 = \left[ \begin{array}{cccc} c_1c_234 & s_1 & c_1s_234 & c_1(a_2c_2 + a_3c_23) + d_4s_1 \\ c_234s_1 & -c_1 & s_1s_234 & s_1(a_2c_2 + a_3c_23) - c_1d_4 \\ s_234 & 0 & -c_234 & a_2s_2 + a_3s_23 + d_1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (6)$$

$${}^0T_5 = \left[ \begin{array}{cccc} c_1c_5c_234 + s_1s_5 & -c_1s_234 & c_5s_1 - c_1c_234s_5 & c_1(a_2c_2 + a_3c_23 + d_5s_234) + d_4s_1 \\ c_5c_234s_1 - c_1s_5 & -s_1s_234 & -c_234s_1s_5 - c_1c_5 & s_1(a_2c_2 + a_3c_23 + d_5s_234) - c_1d_4 \\ c_5s_234 & c_234 & -s_5s_234 & a_2s_2 + a_3s_23 - c_234d_5 + d_1 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (7)$$

由式 (2) 知

$$\left\{ \begin{array}{l} n_x = c_6s_1s_5 + c_1(c_234c_5c_6 - s_234s_6), \\ n_y = c_6(c_234c_5s_1 - c_1s_5) - s_1s_234s_6, \\ n_z = c_5c_6s_234 + c_234s_6, \\ o_x = -s_1s_5s_6 - c_1(c_6s_234 + c_234c_5s_6), \\ o_y = -c_6s_1s_234 + (-c_234c_5s_1 + c_1s_5)s_6, \\ o_z = c_234c_6 - c_5s_234s_6, \\ a_x = c_5s_1 - c_1c_234s_5, \\ a_y = -c_1c_5 - c_234s_1s_5, \\ a_z = -s_234s_5, \\ p_x = (d_4 + c_5d_6)s_1 + c_1(a_2c_2 + a_3c_23 + d_5s_234 - c_234d_6s_5), \\ p_y = -c_1(d_4 + c_5d_6) + s_1(a_2c_2 + a_3c_23 + d_5s_234 - c_234d_6s_5), \\ p_z = d_1 - c_234d_5 + a_2s_2 + a_3s_23 - d_6s_234s_5, \end{array} \right. \quad (8)$$

在空间机械臂运动控制过程中, 对末端位姿的控制过程为: 首先, 给定目标位姿, 之后将目标位姿反解到关节空间, 得到各关节的目标位置, 继而通过控制各关节运动到目标位置, 使机械臂末端呈现目标位姿构型。

## 姿态与旋转矩阵的关系

令  $x, y, z$  分别表示末端坐标系原点相对于机械臂基坐标系的位置,  $\alpha, \beta, \gamma$  分别表示工具坐标系相对基坐标系的横滚、俯仰、偏转姿态 (RPY) 角, 也称作 ZYX 欧拉角, 即基坐标系先绕 Z 轴旋转  $\alpha$  角, 再绕 Y 轴旋转  $\beta$  角, 然后绕 X 轴旋转  $\gamma$  角. 不难得到, 从基坐标系 S 到工具坐标系 T 表示 RPY 姿态变换的矩阵如下

$$RPY[\alpha, \beta, \gamma] = \left[ \begin{array}{ccc} c_\alpha c_\beta & -c_\beta s_\alpha & s_\beta \\ c_\gamma s_\alpha + c_\alpha s_\beta s_\gamma & c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma & -c_\beta s_\gamma \\ s_\alpha s_\gamma - c_\alpha c_\gamma s_\beta & c_\gamma s_\alpha s_\beta + c_\alpha s_\gamma & c_\beta c_\gamma \end{array} \right] \quad (9)$$

式中,  $c_\alpha = \cos \alpha, s_\alpha = \sin \alpha$  以此类推.

从基座到末端执行器的变换矩阵记作如下形式:

$${}_T^S R_{xyz} = \left[ \begin{array}{ccc} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{array} \right] \quad (10)$$

观察式 (9) 和 (10) 可以得到:

如果  $\cos \beta \neq 0$  则

$$\left\{ \begin{array}{l} \beta = \text{atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}) \\ \alpha = \text{atan2}(r_{21}, r_{11}) \\ \gamma = \text{atan2}(r_{32}, r_{33}) \end{array} \right. \quad (11)$$

这样, 当知道末端的位姿, 即可确定末端相对与基座的齐次矩阵。

## 求解 $\theta_1$

逆解一般可用几何法与代数法, 两者有不同的优点, 具体就不展开说明了, 下面开始分析求逆解的过程. 由于 ur5 最后三轴交于一点, 可以通过手腕中心方便的求解出手腕中心点的位置, 进而通过几何位置关系方便的求出  $\theta_1$  的解, 设手腕中心点的位置为  $pw$ , 有

$$\left\{ \begin{array}{l} pw_1 = p_x - d_6a_x \\ pw_2 = p_y - d_6a_y \\ pw_3 = p_z - d_6a_z \end{array} \right. \quad (12)$$

通过几何观察有

$$\left\{ \begin{array}{l} \sin(\theta_1 + \phi_1) = d_4 / \sqrt{pw_1^2 + pw_2^2} \\ \cos(\theta_1 + \phi_1) = \pm \sqrt{1 - \sin^2(\theta_1 + \phi_1)} \\ \phi_1 = -\text{atan2}(pw_2, pw_1) \end{array} \right. \quad (13)$$

因此可得  $\theta_1$  的解为

$$\theta_1 = \text{atan2}(\sin(\theta_1 + \phi_1), \pm \cos(\theta_1 + \phi_1)) - \phi_1 \quad (14)$$

## 求解 $\theta_5$

求解  $\theta_5$ , 通过从方程中找到规律, 来依次求解各个关节角, 做如下变形

$$\begin{aligned} {}^1T_6 = {}^0T_1^{-1} {}^0T_6 &= \left[ \begin{array}{ccc} c_5c_6c_234 - s_6s_234 & -c_5c_234s_6 - c_6s_234 & -c_234s_5 \\ c_234s_6 + c_5c_6s_234 & c_6c_234 - c_5s_6s_234 & -s_5s_234 \\ c_6s_5 & -s_5s_6 & c_5 \end{array} \right] \begin{array}{l} a_2c_2 + a_3c_23 - c_234d_6s_5 + d_5s_234 \\ a_2s_2 + a_3s_23 - c_234d_5 - d_6s_5s_234 \\ c_5d_6 + d_4 \end{array} \\ = {}^0T_1^{-1} T_n &= \left[ \begin{array}{ccc} c_1nx + ny s_1 & c_1ox + oy s_1 & axc_1 + ays_1 \\ nz & oz & az \\ nx s_1 - c_1ny & ox s_1 - c_1oy & ax s_1 - ayc_1 \end{array} \right] \begin{array}{l} c_1px + py s_1 \\ pz - d_1 \\ px s_1 - c_1py \end{array} \end{aligned} \quad (15)$$

观察式 (15) 的矩阵的 (3,3) 项, 有

$$c_5 = -a_y c_1 + a_x s_1 \quad (16)$$

所以

$$\theta_5 = \text{atan2}(\pm \sqrt{1 - c_5^2}, c_5) \quad (17)$$

## 求解 $\theta_6$

观察式 (15) 矩阵的第 (3,1) , (3,2) 有

$$\left\{ \begin{array}{l} c_6s_5 = -n_y c_1 + n_x s_1 \\ -s_5s_6 = -o_y c_1 + o_x s_1 \end{array} \right. \quad (18)$$

不难得到

$$\theta_6 = \text{atan2}(-(-o_y c_1 + o_x s_1)/s_5, (-n_y c_1 + n_x s_1)/s_5) \quad (19)$$

## 求解 $\theta_2, \theta_3, \theta_4$

$\theta_2 + \theta_3 + \theta_4$  的求解

$$\begin{aligned} {}^2T_4 = {}^0T_1^{-1} {}^0T_4 &= \left[ \begin{array}{ccc} c_{234} & 0 & s_{234} \\ s_{234} & 0 & -c_{234} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right] \begin{array}{l} a_2c_2 + a_3c_23 \\ a_2s_2 + a_3s_23 \\ d_4 \\ 1 \end{array} \\ = {}^0T_1^{-1} T_n ({}^0T_4^{-1} {}^0T_6)^{-1} &= \left[ \begin{array}{ccc} -s_5(c_1o_x + s_1o_y) + c_5c_6(c_1n_x + s_1n_y) - c_5s_6(c_1o_x + o_1s_y) & c_5(c_1s_x + s_1s_y) + c_6s_5(c_1n_x + s_1n_y) - s_5s_6(c_1o_x + o_1s_y) & -c_5(c_1o_y + s_1o_z) - s_1(c_1o_0 + s_1o_1) \\ s_1(c_5(c_1n_x - s_1o_x) - s_5o_x) + c_1(s_1n_y + c_5(o_0s_6 - c_6s_5)) & s_1(c_1s_x + s_5(s_2s_5 - s_6s_4)) - c_1(c_1s_y + s_5(c_1n_y - o_2s_6)) & c_5(c_1o_0 + s_1o_1) - s_1(c_1o_x + s_1o_y) \\ s_1(c_5(c_1n_x - s_1o_x) - s_5o_x) & 0 & 0 \end{array} \right] \begin{array}{l} c_1(-d_6a_x + c_6d_5o_x) \\ c_1(-d_6a_y + c_6d_5o_y) \\ s_1(-d_6a_z + c_6d_5o_z) \end{array} \end{aligned} \quad (20)$$

观察式 (20) 中的第 (1,1), (2,1) 有

$$\left\{ \begin{array}{l} c_{234} = -s_5(c_1a_x + s_1a_y) + c_5c_6(c_1n_x + s_1n_y) - c_5s_6(c_1o_x + s_1o_y) \\ s_{234} = c_5(c_6n_z - s_6o_z) - s_5a_z \end{array} \right. \quad (21)$$

不难得到

$$\theta_2 + \theta_3 + \theta_4 = \text{atan2}(c_5(c_6n_z - s_6o_z) - s_5a_z, -s_5(c_1a_x + s_1a_y) + c_5c_6(c_1n_x + s_1n_y) - c_5s_6(c_1o_x + s_1o_y)) \quad (22)$$

求  $\theta_3$

观察等式 (20) 的第 (1,4) (2,4) 项, 为了书写的方便, 令

$$\left\{ \begin{array}{l} c1 = c_1(d_6(-a_x) + c_6d_5o_x + d_5s_6n_x + p_x) + s_1(d_6(-a_y) + c_6d_5o_y + d_5s_6n_y + p_y) \\ c2 = -d_6a_z + c_6d_5o_z + d_5s_6n_z - d_1 + p_z \end{array} \right. \quad (23)$$

则有

$$\left\{ \begin{array}{l} a_2c_2 + a_3c_23 = C1 \\ a_2s_2 + a_3s_23 = C2 \end{array} \right. \quad (24)$$

将式子 (24) 平方相加并化简可得

$$2a_3a_2c_3 + a_2^2 + a_3^2 = C1^2 + C2^2 \quad (25)$$

不难得到

$$c_3 = \frac{-(a_2^2 + a_3^2) + C1^2 + C2^2}{2a_2a_3} \quad (26)$$

因此可得  $\theta_3$  的解为

$$\theta_3 = \text{atan2}(\pm \sqrt{1 - c_3^2}, c_3) \quad (27)$$

利用三角函数展开 (24) 可得

$$\left\{ \begin{array}{l} a_2c_2 + a_3c_3c_2 - a_3s_2s_3 = C1 \\ a_3c_3s_2 + a_3c_2s_3 + a_2s_2 = C2 \end{array} \right. \quad (28)$$

可求出

$$\begin{aligned} c_2 &= \frac{C1(a_3c_3 + a_2) + a_3C2s_3}{-(a_3c_3 + a_2)^2 - a_3^2s_3^2} \\ s_2 &= \frac{-a_3c_3C2 + a_3C1s_3 + a_2(-C2)}{2a_3a_2c_3 + a_3^2c_3^2 + a_3^2s_3^2 + a_2^2} \end{aligned} \quad (29)$$

可得

$$\theta_2 = \text{atan2}(s_2, c_2) \quad (30)$$

因此  $\theta_4$  可由式 (22) 表示出来至此全部的关节角的解析表达式就求出来了

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