RSA

Elements of Applied Data Security M

Alex Marchioni– alex.marchioni@unibo.it

Livia Manovi– livia.manovi@unibo.it

RSA

- RSA (Rivest-Shamir-Adleman, 1977) is a widely used asymmetric algorithm for secure communication
- The algorithm requires a pair of keys:
  - a **public key** k_{pub} used for encryption
  - a **private key** k_{priv} used for decryption.
- Based on **integer factorization**:
  - Plaintext x and ciphertext y are modeled as integers $x, y \in \mathbb{Z}_n$.
  - Security relies on the difficulty of factoring the product of two large prime numbers.

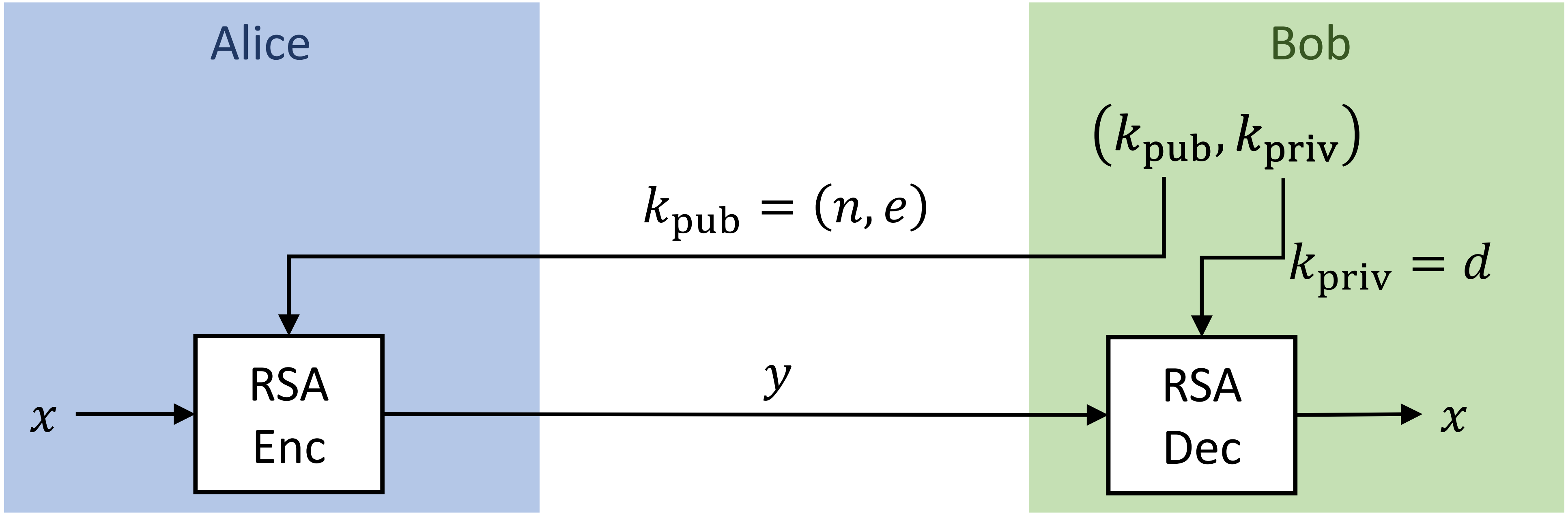
RSA – Encryption and Decryption

Encryption:

$$y = x^e \backslash n$$

Decryption:

$$x = y^d \backslash n$$



RSA – Key Generation

Key is generated through 5 steps:

1. Choose two prime numbers (p, q)
2. Compute $n = p \cdot q$
3. Compute $m = \phi(n) = (p-1)(q-1)$
4. Draw $e \in \mathbb{Z}_n$ such that $\gcd(e, m) = 1$
5. Compute d such that $d \cdot e \equiv 1 \pmod{m}$

Steps (4) and (5) require the Extended Euclidean Algorithm (EEA).

RSA – Practical issues

- <u>Exponentiation involving large numbers</u>
  - When you deal with very high numbers (a, b), (e.g., $a, b \in \mathbb{Z}_{2^{2048}}$), it is not trivial to compute a^b as it may require an unworkable amount of time.
  - To solve this problem, many algorithms for fast and efficient exponentiation have been studied. **Square-and-Multiply** is the base for most of them.

- <u>Generation of large prime numbers</u>
  - Testing for primality is a much easier task than integer factorization. Therefore, one can randomly draw a big number and then test for primality.
  - A well-known algorithm for testing primality is the **Miller Rabin Test**

Python integers

- RSA implementation must rely on Python integers `int`. Python lets integers be arbitrarily large (the only limit is the amount of memory available)
  - integers of thousands of bits
  - no overflow

- Conversely, NumPy integers have fixed limits defined by the `dtype`. Therefore, NumPy integers are limited in size and suffer from overflow.

Tasks

1. RSA implementation
   - Extended Euclidean Algorithm
   - Square and Multiply
   - Miller Rabin test
   - RSA class
2. RSA + AES

Task 1: RSA

Extended Euclidean Algorithm (EEA)

It computes the greatest common divisor (gcd) of two integers a and m.

Assuming $m > a$, if the gcd is 1, EEA also computes the inverse of a number a with respect to multiplication modulo m.

- **Input**:
  - a (int)
  - m (int)
- **Outputs**:
  - $\gcd(a, m)$
  - $s, t \in \mathbb{Z}$ such that $\gcd(a, m) = \mathrm{s} \cdot a + t \cdot m$. If $\gcd(a, m) = 1$ then $s \equiv a^{-1} \mathrm{mod}\, m$

Input a, m
$r_0, r_1 \leftarrow m, a$
$s_0, s_1, t_0, t_1 \leftarrow 0, 1, 1, 0$
$i \leftarrow 1$
while $\mathrm{r}_i \neq 0$
    $i \leftarrow i + 1$
    $r_i \leftarrow r_{i-2} \setminus r_{i-1}$
    $q_i \leftarrow (r_{i-2} - r_i) / r_{i-1}$
    $s_i \leftarrow s_{i-2} - q_i s_{i-1}$
    $t_i \leftarrow t_{i-2} - q_i t_{i-1}$
endwhile
Output $r_{i-1}, s_{i-1}, t_{i-1}$

Square-and-Multiply

Computes the exponentiation $x^e \backslash\, n$ by means of squaring and multiplication.

- **Input**:
  - base x (int)
  - exponent e (int)
  - modulo n (int)
- **Outputs**:
  - $y = x^e \backslash\, n$ (int)

$$
\begin{array}{l}
\textbf{Input } x, e = 0b e_{L-1} e_{L-2} \cdots e_1 e_0, n \\
L_{\max} = \max_i \{e_i = 1\} \\
y \leftarrow x \\
\textbf{For } i = L_{\max} - 1, \dots, 1, 0 \\
\quad y \leftarrow y^2 \backslash\, n \\
\quad \textbf{If } e_i = 1 \textbf{ then} \\
\quad\quad y \leftarrow y \cdot x \backslash\, n \\
\quad \textbf{endif} \\
\textbf{endfor} \\
\textbf{Output } y
\end{array}
$$

Miller-Rabin Primality Test

Determines whether a given number is likely to be prime or surely composite

- **Input**:
  - Candidate odd prime number p (int)
  - Number of trials N (int)
- **Outputs**:
  - Whether p is probably prime (True) or p is surely composite (False)

$\mathbf{Input}\ p = q \cdot 2^r + 1, N$
$\mathbf{For}\ i = 0, 1, \ldots, N-1$
    $\mathbf{draw}\ x \in \{2, 3, \ldots, p-2\}$
    $y \leftarrow x^q \setminus p$
    $\mathbf{If}\ (y = 1\ \mathbf{or}\ y = p-1)\ \mathbf{then}$
        $\mathbf{continue}$
    $\mathbf{endif}$
    $\mathbf{For}\ j = 0, 1, \ldots, r-1$
        $y \leftarrow y^2 \setminus p$
        $\mathbf{If}\ y = p-1\ \mathbf{then}$
            $\mathbf{continue}$ (main loop)
        $\mathbf{endif}$
    $\mathbf{endfor}$
    $\text{test} \leftarrow 1$
$\mathbf{endfor}$
$\text{test} \leftarrow 0$
$\mathbf{Output}\ \text{test}$

RSA class

RSA class must support both encryption and decryption

- **Decryption**: user must specify the length of the key (`length`) so that k_{pub} and k_{priv} can be generated.
- **Encryption**: user must provide the $k_{\text{pub}} = (n, e)$ so that a message can be encrypted.

```
class RSA:
  ''' class docstring '''

  def __init__(self, length=None, n=None, e=None):
    ''' constructor docstring '‘’
    self.length = ...
    self.n = ...
    self.e = ...
    ...

  def encrypt(self, plaintext):
    ...
    return ciphertext

  def decrypt(self , ciphertext):
    ...
    return plaintext
```

Task 1

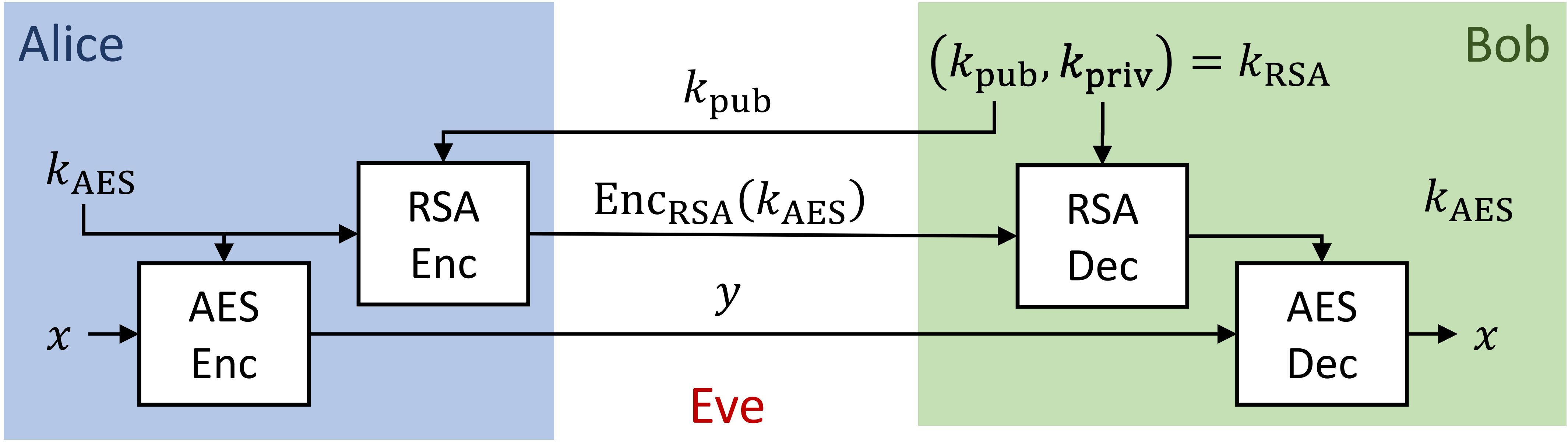
Implement and test:

- Extended Euclidean Algorithm
- Square-and-Multiply Algorithm
- Miller Rabin Test
- RSA class

Task 2: RSA + AES

RSA and AES

RSA is not suited to provide confidentiality in case of large messages. However, it can be exploited to establish a secure channel over which two entities (Alice and Bob) can exchange the key for a symmetric algorithm.



Task 2

Implementation of a secure communication channel:

- Create two instances of RSA and make them share a key $k_{\text{AES}} \in \mathbb{Z}_{2^{256}}$.
- Create two instances of AES sharing the same key and use them to encrypt/decrypt the message in file `lorem_ipsum.txt`.

Deadline

Tuesday, May 21st at 12PM (noon)