M2 Splines Package

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M2 Workshop
Boise State University

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Piecewise Polynomials

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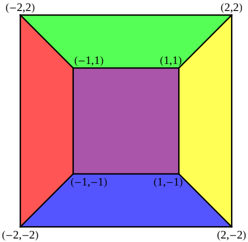

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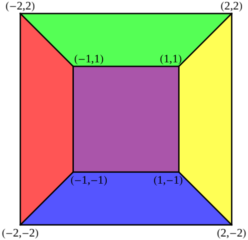

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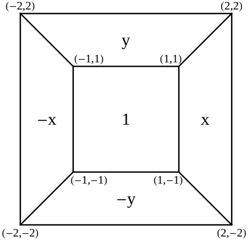

A Spline $F \in C^0_1(\mathcal{Q})$

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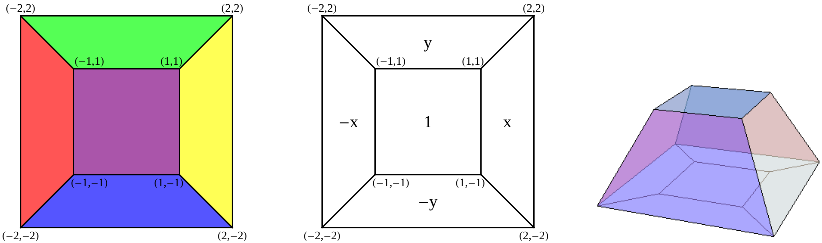
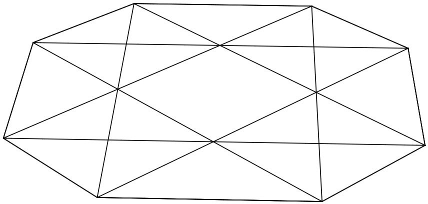


A Polytopal Complex $\mathcal{Q}$ A Spline $F \in C^0_1(\mathcal{Q})$ Graph of F

A More Interesting Example



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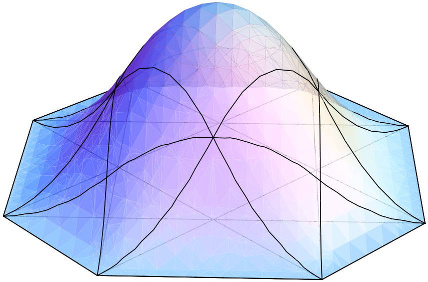

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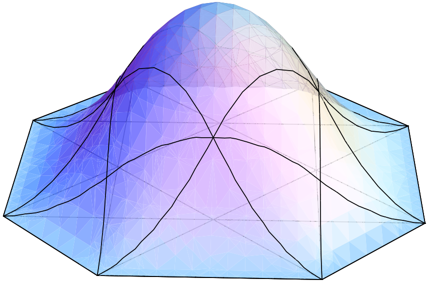

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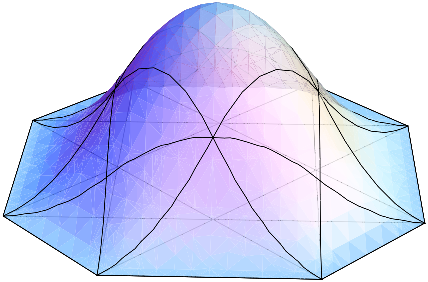

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- Graph of the **Zwart-Powell element**: a spline in $C^1_2(\mathcal{P})$.
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- This is an example of a **box spline**.

Polynomials of ZP Element

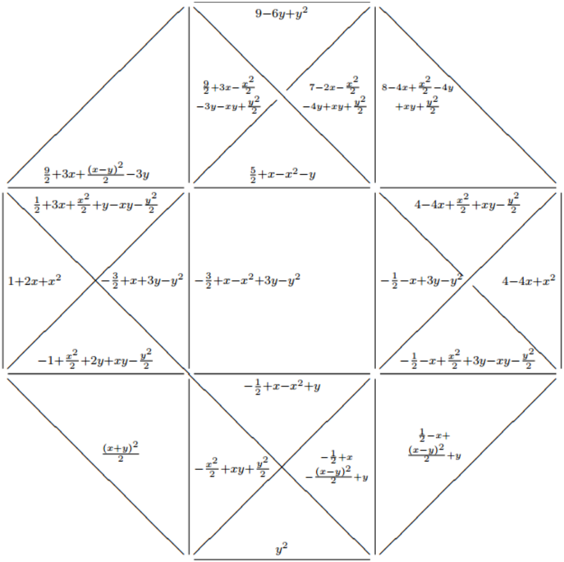
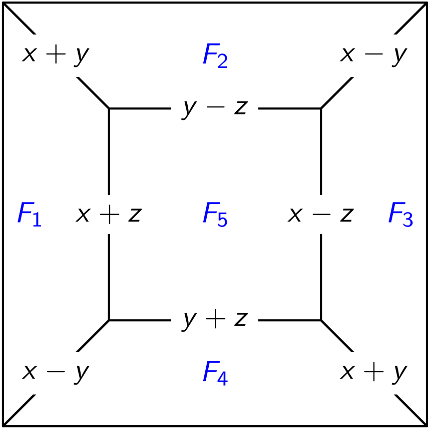


Image taken from C. Procesi, 'The algebra of the box-spline,' Temple University, 2006.

Algebraic Criterion



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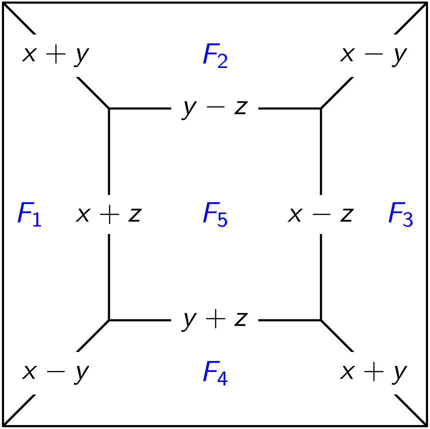
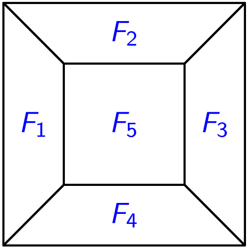


$(F_1, F_2, F_3, F_4, F_5) \in C^r(\widehat{\mathcal{Q}}) \iff$ there are $f_1, \ldots, f_8 \in S = \mathbb{R}[x, y, z]$ s.t.

$$\begin{aligned}
F_1 - F_2 &= f_1(x+y)^{r+1} \\
F_2 - F_3 &= f_2(x-y)^{r+1} \\
F_3 - F_4 &= f_3(x+y)^{r+1} \\
F_4 - F_1 &= f_4(x-y)^{r+1} \\
F_1 - F_5 &= f_5(x+z)^{r+1} \\
F_2 - F_5 &= f_6(y-z)^{r+1} \\
F_3 - F_5 &= f_7(x-z)^{r+1} \\
F_4 - F_5 &= f_8(y+z)^{r+1}
\end{aligned}$$

Spline Matrix for $C^r(\widehat{\mathcal{Q}})$



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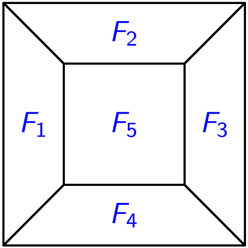

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$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & (x-y)^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & (x+y)^{r+1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & (x-y)^{r+1} & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & (x+y)^{r+1} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & (x+z)^{r+1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & (y-z)^{r+1} & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & (x-z)^{r+1} & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (x+z)^{r+1}
\end{pmatrix}
\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ -f_1 \\ -f_2 \\ -f_3 \\ -f_4 \\ -f_5 \\ -f_6 \\ -f_7 \\ -f_8 \end{pmatrix} = 0$$

The Dimension Question

Key Fact: $C^r_d(\mathcal{P})$ is a finite dimensional real vector space.

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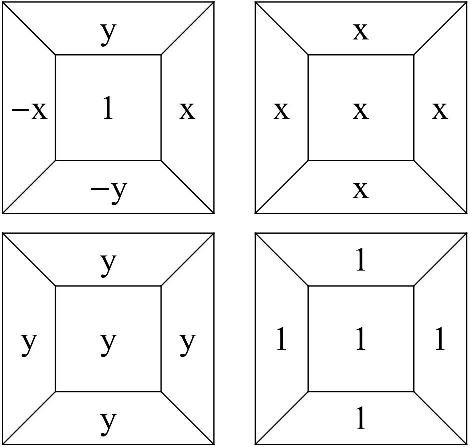
$\dim_{\mathbb{R}} C^0_1(\mathcal{Q}) = 4$



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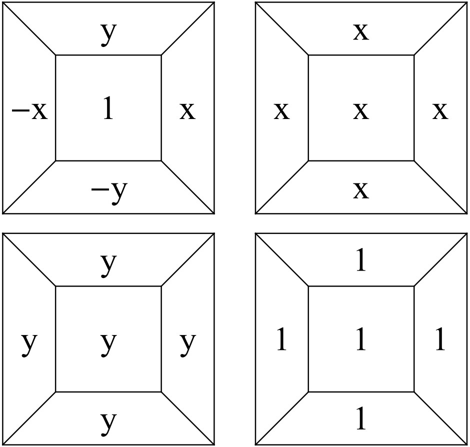

Two central problems in approximation theory:

1. Determine $\dim C^r_d(\mathcal{P})$
2. Construct a basis of $C^r_d(\mathcal{P})$ with 'good' properties