Orbital Motion about a Black Hole

ACM40980 Presentation

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Agenda

- Introduction to General Relativity
- Schwarzschild Black Holes
- Kerr Black Holes
- Resonant Orbits

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Introduction to General Relativity

In the beginning...

Newton's Law of Universal Gravitation (1687)

- Every particle in the universe attracts every other particle.

Einstein's Theory of Special Relativity (1907)

- The motion of one object is always relative to the motion of another object.

Einstein's Theory of General Relativity (1915)

- Gravitation can be thought of as a curved field, instead of a force.

Introduction to General Relativity

Metrics and the Einstein Field Equations

From general relativity came the Einstein Field Equations

$$G_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}. \tag{1}$$

This relates the geometry of spacetime to the distribution of matter within it.

Exact solutions to these equations are called metrics.

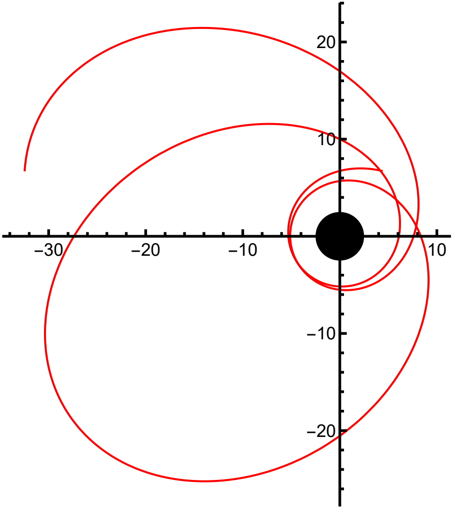
Schwarzschild Black Holes

History & Properties

The Schwarzschild metric was discovered in 1916 by Karl Schwarzschild.

A good starting point for understanding orbital motion, primarily because of the following two properties:

- No angular momentum
- No electrical charge



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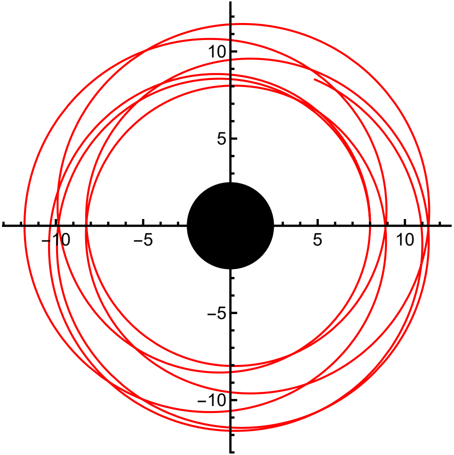


Schwarzschild Black Holes

The Schwarzschild Metric

Before we move on, we first define the Schwarzschild metric as

$$g_{\alpha\beta} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta) \end{pmatrix}, \tag{2}$$

where we note that only the non-zero entries are along the diagonal, also we define

$$f(r) = \frac{r - 2M}{r}. \tag{3}$$

Schwarzschild Black Holes

Orbital Motion: Two Ways

Euler-Lagrange Equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial u^\alpha}\right) = \frac{\partial \mathcal{L}}{\partial x^\alpha}$$

- We know there are constants because of Noether's Theorem

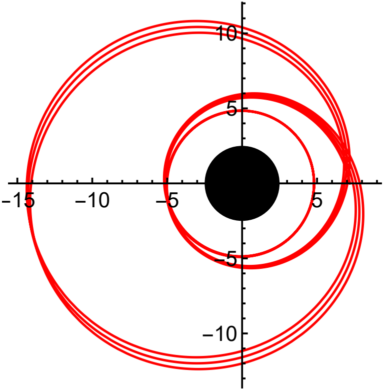
Geodesic Equation

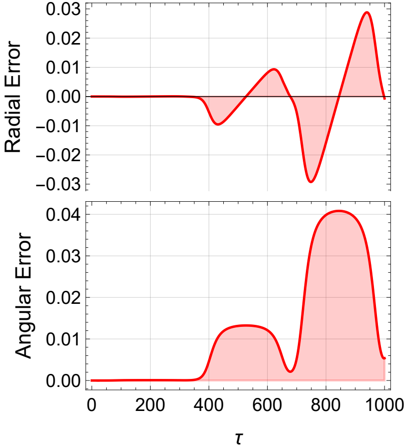
$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^t{}_{\beta\delta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\delta}{\mathrm{d}\lambda} = 0$$

- This is the most straightforward way to solve for the motion of a particle

Schwarzschild Black Holes

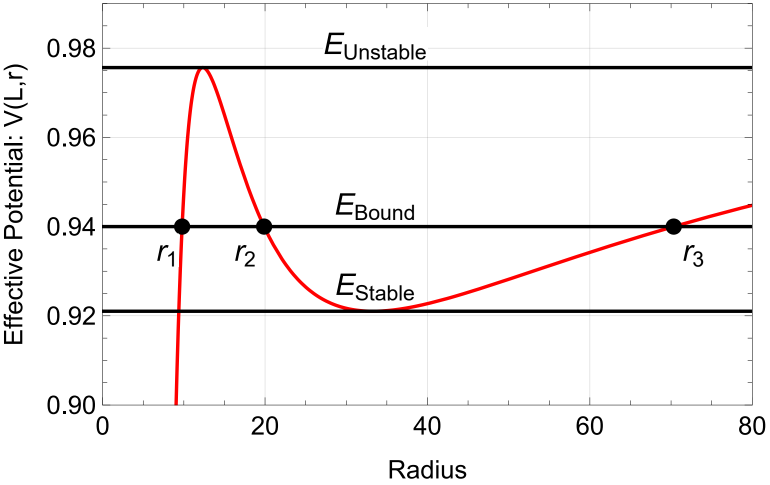
Orbital Motion: Results





Schwarzschild Black Holes

Bound Orbits



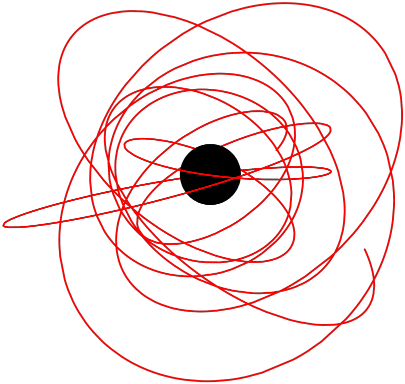
Kerr Black Holes

History & Properties

The Kerr metric was discovered in 1963 by Roy Kerr.

It describes the geometry about uncharged black holes with non-zero angular momentum.

We now take an interest in orbits outside the equatorial plane.



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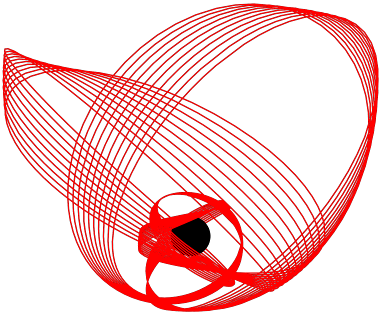

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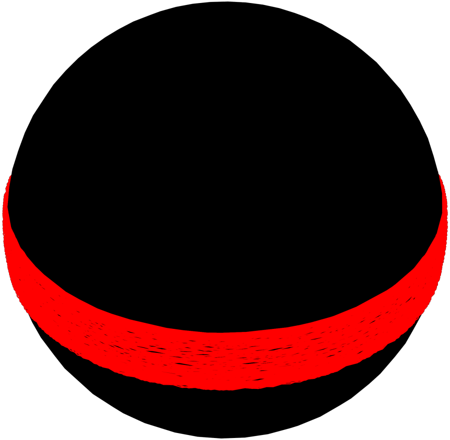

Kerr Black Holes

The Kerr Metric

Let us now define the line element of the Kerr metric:

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2(\theta)}{\Sigma}\right)\sin^2(\theta)\,\mathrm{d}\phi^2 \\ &- \frac{4Mar\sin^2(\theta)}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi, \end{aligned} \tag{4}$$

where

$$\Delta = r^2 - 2Mr + a^2, \tag{5}$$

$$\Sigma = r^2 + a^2\cos(\theta). \tag{6}$$

Kerr Black Holes

Orbital Frequencies

As bound Kerr orbits are triperiodic, there are three frequencies that we can examine.

- Υ_r: Radial oscillations
- Υ_θ: Polar oscillations
- Υ_ϕ: Rotations about the black hole's axis of spin

Kerr Black Holes

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Resonant Orbits
Definition

Resonant orbits occur when the following equation is satisfied

$$k\Upsilon_\theta = n\Upsilon_r, \tag{7}$$

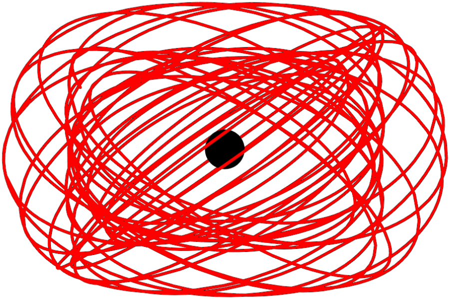
where k and n are two relatively prime integers.

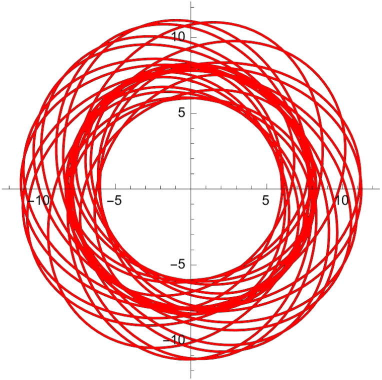
It says that it takes an equal amount of time for the polar motion to pass through k turning points as it does for the radial motion to pass through n turning points.

Resonant Orbits

Example

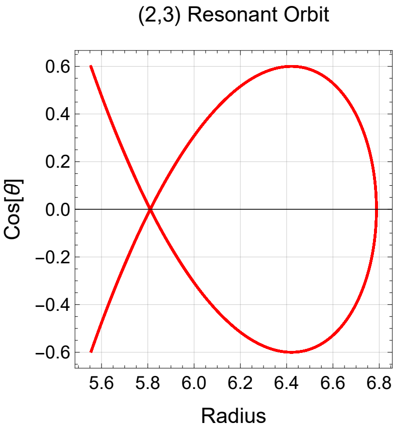
An example of a $(k, n) = (2, 3)$ resonant orbit is shown below

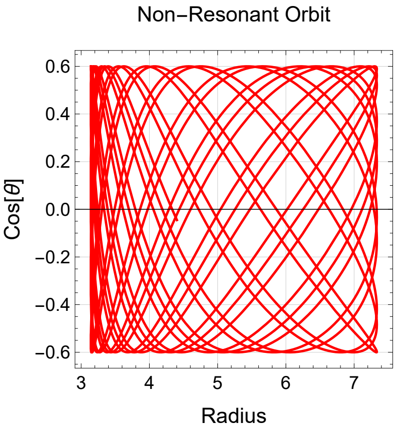




Resonant Orbits

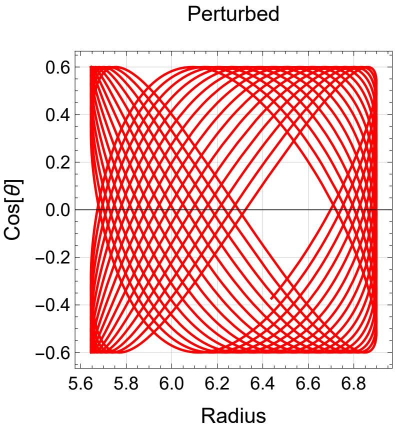
$(r, \cos(\theta))$ Plot

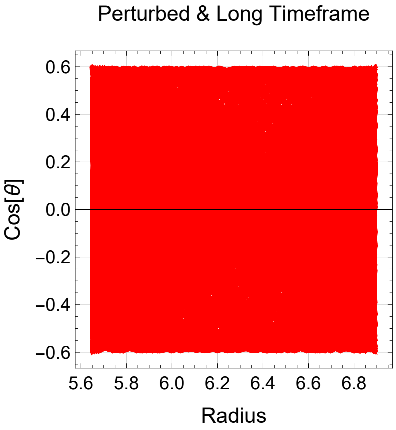




Resonant Orbits

Perturbations and Long Timeframes

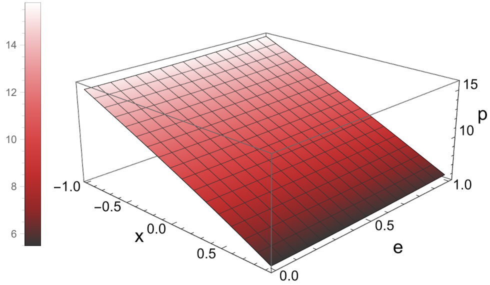




Resonant Orbits

A Strange Observation

Setting the spin of the black hole, a, to 0.9, the following plot was produced which shows the plane of all resonant (p, e, x)[1] values for $(2, 3)$-type resonances.



[1]**Note:** p is the size of the orbit, e is the eccentricity, and x is the inclination angle from the equatorial plane.

Resonant Orbits

Root-Finding: Creating a Plane of Initial Values

We want to develop a way to estimate these resonant p values, so we set $e = 0$ and solve the equation

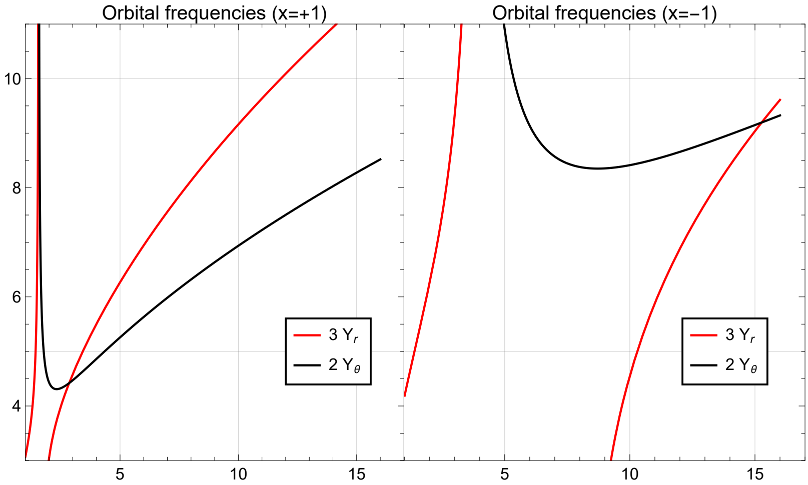
$$2\Upsilon_\theta = 3\Upsilon_r \tag{8}$$

for p when $x = +1$ and $x = -1$.

This will allow us to create a line in the (p, x) plane at $e = 0$ and then extrapolate out for $0 < e \leq 1$.

Resonant Orbits

Root-Finding: The Equations to be Solved



Resonant Orbits

Root-Finding: The Results

Solving for the points of intersection in the previous plots gives us

$$p_{x=+1} = 4.9626, \qquad p_{x=-1} = 15.2677. \tag{9}$$

We now calculate the equation for the line between the two points $(+1, 4.9626)$ and $(-1, 15.2677)$, which gives us

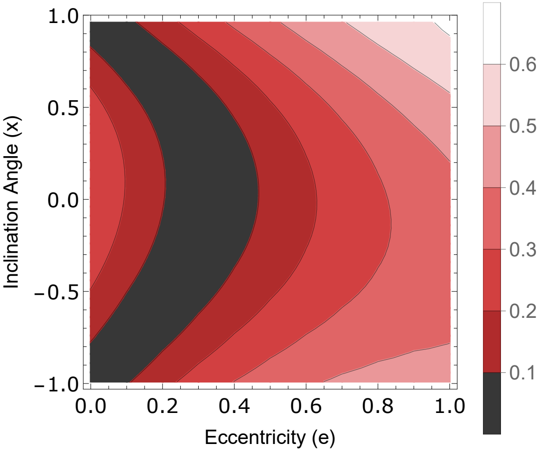
$$f(x) = 10.1152 - 5.1525x. \tag{10}$$

Finally, we extend this line out to a plane for $0 < e \leq 1$:

$$f(e, x) = 10.1152 - 5.1525x + e. \tag{11}$$

Resonant Orbits

Root-Finding: Error Compared to True Values



Thank you all for coming,
I am happy to answer any questions.