Real Analysis
Math 411 - Fall 2015
Taught by Professor Fioralba Cakoni

Lecture Notes by Ali Haider Ismail

Contents

0 Syllabus - Overiew Content

Course Real Analysis I

Time MTh 10:20 - 11:40 am

Room ARC 204

Term Fall 2015

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Required Text Principles of Mathematical Analysis by Walter Rudin

Recommended Additional Text Mathematical Analysis, Second Edition by Tom M. Apostol

Announcements and Advice

- Rudin's book is very terse with its proofs. It is strongly advised to do all the proofs by yourself to understand the missing steps
- Apostol's book has more description so may help when stuck
- Check out the full syllabus on Sakai, includes documents for how to write proofs and other recommended reading
- First Homework is due September 14th, Monday

If you are not Ali and are reading these notes, be sure to tell him if there are mathematical or typographical errors in this document. Talk to him in class or email him at ali.haider.ismail@rutgers.edu. If you see three large stars (like [[★★★]]) it is because the concept in question confuses me. If you can explain these be sure to let me know.

1 Foundations of Real Numbers

We will denote from now on the real numbers as $\mathbb{R}$.

Definition 1.1 (Inductive Set)**.** A subset, A, of real numbers that satisfies:

1. $1 \in A$
2. if $x \in A$ then $x + 1 \in A$

is called *inductive*. $\diamond$

Definition 1.2. The smallest inductive set is called the set of *positive integers* or *natural numbers* denoted by $\mathbb{N}$. $\diamond$

Note 1. Multiple definitons of integers (or in general defintions of any object) should be equivalent.

Definition 1.3. Integers are denoted $\mathbb{Z}$ and are equal to $\mathbb{N} \cup -\mathbb{N} \cup \{0\}$, where $-\mathbb{N} = \{-n | n \in \mathbb{N}\}$. $\diamond$

Principle 1.4 (Induction)**.** *To prove a statment P_n for all $n \in \mathbb{N}$ by the defintion of $\mathbb{N}$ we need to the following:*

1. *Verify that P_1 is true.*
2. *Assume P_{n-1} is true, then prove P_n is also true, for $n > 1$.*

Definition 1.5 (Properties)**.** The following are properties of nautral numbers.

1. $n = d * c$, then d *divides* n, i.e. is a divisor of n
2. n is *prime* if $n > 1$ and the only positive divisors are n and 1.

$\diamond$

Theorem 1.6 (Factorization in prime factors)**.** *Every postive integer $n > 1$ is either prime or can be represented as a product of primes. Furthermore this respresentation is unique up to reordering.*

Proof. We use induction to prove the first statement.

1. $n = 2$, is prime $\rightarrow$ true.
2. $2, \ldots, n-1$ is true using equivalent form induction to 1.4 where we assume P_k for $2 \leq k \leq n-1$.
   a. n is prime, we are done

b. Or it has a divisor $\neq 1$ and n. That is $n = c * d$ for $1 < c < n, 1 < d < n$. Hence c and d can be represented as a product of primes $\rightarrow$ so can n.

Now we prove uniquness. Assume that $n = d_1 d_2 \ldots d_s = c_1 c_2 \ldots c_t$ for d_i, c_j are primes. We want to show $s = t$ and that $\forall i, \exists! j, d_i = c_j$, We have

$$d_1 d_2 \ldots d_s = c_1 c_2 \ldots c_t.$$

d_1 divides the right hand side, hence it divides one of the factors because it is prime. Say after rearranging we call the factor that it divides c_1. Since d_1, c_1 are prime, $d_1 = c_1$. Divides both sides we get

$$\frac{n}{d_1} = d_2 d_3 \ldots d_s = c_2 c_3 \ldots c_t$$

Note that $2 < \frac{n}{d_1} < n$, therefore from our inductive assumption we conclude that $s = t$ and $d_i = c_j$ for $i = 2 \ldots s$ and $j = 2 \ldots t$. □

Theorem 1.7 (Infinite Primes - Euclid)**.** *There are an infinite number of primes.*

Proof. We prove this using contradiction. Assume to contrary that there are m primes $2, 3 \ldots n_m$ where n_m is the largest prime. Coniser $n = 2 * 3 \ldots n_m + 1$. n cannot be prime because $n > n_m$. Hence n is a product of primes, but n cannot be divides by an primes because it leaves a remainder of 1. Contradiction. □

Definition 1.8 (Rational Numbers)**.** *Rational Numbers* are quotients of integers $\frac{m}{n}, n \neq 0$ denoted as $\mathbb{Q}$. ◇

Corollary 1.9 (Properties)**.** *The following are properties of Rational Numbers.*

1. $\frac{a+b}{2}$ *is rational if* $a \in Q$ *and* $b \in Q$.

2. *between any two rational there are infinitely many rationals.*

Example 1.10
Show that there are no rational roots of $p^2 + 2$.

Solution: We solve $p^2 - 2 = 0$. Assume by contradiction that there is a rational root and let it be represented as $\frac{m}{n} \in \mathbb{Q}$ such that $\frac{m}{n}^2 = 2$. We can choose m, n such that they are coprime (i.e. they have no common factor). Then we get $m^2 = 2n^2 \Rightarrow m^2$ is even $\Rightarrow m$ is even. Thus it follows that $m = 2p \Rightarrow 4p^2 = 2n^2 \Rightarrow 2p^2 = n^2$. Hence n^2 is even $\Rightarrow n$ is even. Therefore we have that m and n are both even. Contradiction to our assumption that m, n are coprime.

Note 2. This shows that $\sqrt{2}$ is irrational

Exercise 1.1. Show that if m is not a perfect square $m \neq d * d$ then $\sqrt{m}$ is not rational. That is show that $p^2 = m, p \notin \mathbb{Q}$.

Example 1.11
Show that e is not in $\mathbb{Q}$.

Solution: We recall from Calc 2 that

$$e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

[[★★★ which leads to]]

$$e = \sum_{n=1}^{\infty} \frac{1}{n!}$$

. Thus it suffices to show that $e^{-1} \notin \mathbb{Q}$. Recall that

$$e^{-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$$

and define $S_m = \sum_{n=1}^{m} \frac{(-1)^n}{n!}$. It is clear that

$$0 < e^{-1} - S_{2k-1} < \frac{1}{(2k)!}$$

which leads us to

$$0 < (2k-1)!(e^{-1} - S_{2k-1}) < \frac{1}{2k} < \frac{1}{2}$$

$(2k-1)!S_{2k-1} \in \mathbb{Z}$, [[★★★ in fact it is positive because the error term is postive]]. If e^{-1} was rational, we can choose k large enough such that $(2k-1)!e^{-1}$ is in $\mathbb{N}$ because $e^{-1} = \frac{p}{q}, q \mid (2k-1)!$ if k is large enough. We conclude $(2k-1)!(e^{-1} - S_{2k-1}) \in \mathbb{Z}$ but this is not possible because there are no integers between 0 and $\frac{1}{2}$.

Note 3. Transcendental irrational numbers are uncountable which is not the case for algebraic irrational numbers

2 Ordered Sets, Fields, Ordered Fields

2.1 Ordered Sets

Example 2.1
Construct $A = \{p \in \mathbb{Q} | p^2 < 2\}$ and $B = \{p \in Q | p^2 > 2\}$. We will show that A contains no largest number; that is $\forall p \in A$, we can find a $q \in A$ such that $p < q$. We will also show B contains no smallest number; that is $\forall p \in B$, we can find a $q \in B$ such that $q < p$.

Take $q = p - \frac{p^2-2}{p+2}$ which we choose because we want to control the sign of q and of the quotient term. Its clear that after simplifying we get

$$q = \frac{2p+2}{p+2} \in Q \tag{2.2}$$

as the quotient of rational is rational. Now we compute

$$q^2 - 2 = \frac{4p^2+8p+4}{p^2+4p+4} - 2 = \frac{4p^2+8p+4-2p^2-8p-8}{(p+2)^2} = \frac{2p^2-4}{(p+2)^2} \tag{2.3}$$

1. If $p \in A, p^2 - 2 < 0$, this means $q^2 - 2 < 0 \to q \in A$. Further 2.2 implies $q > p$.
2. If $p \in B, p^2 - 2 > 0$, this means $q^2 - 2 > 0 \to q \in B$. Further 2.2 implies $p < q$.

Definition 2.4. Let S be a set. An *order* in S is a relation, denoted by "$<$" that satisfies

1. If $x \in S$ and $y \in S$ then only one of these statements holds true
   - $x < y$
   - $y < x$
   - $x = y$
2. If $x < y$ and $y < z$, then $x < z$, for $x, y, z \in S$

$\diamond$

Corollary 2.5. *1. We define "$>$" as the inverse relation of $<$. That is $x > y \iff y < x$.*

2. We define $\geq$ to represent $\not<$.

3. We define $\leq$ to represent $\not>$.

Example 2.6
Q is ordered. We get $p < q$ if $p - q < 0$.

Example 2.7
$S, \mathbb{P}(S) = \{A|A \subset S\}$. $A \subset B$ is like $A < B$. This is not ordered because two sets do not have to be subsets of each other.

Note 4. This last example, 2.7, is an example of a partially ordered set, often called a poset.

Definition 2.8. A ordered set S is a set when an order relation is defined for every element of S ◇

Definition 2.9. Let S be an ordered set and $E \subset S$

1. E is bounded above if $\exists \beta \in S$ such that $x \leq \beta$ for all $x \in E$. β is called an *upper bound* of E.

2. E is bounded below if $\exists \gamma \in S$ such that $\gamma \leq x$ for all $x \in E$. γ is called a *lower bound* of E.

◇

Note 5. $\mathbb{C}$ is a commonly used set (of numbers) that is not ordered

Definition 2.10. Let S be an ordered set and $E \subset S$ bounded above. Then $\alpha \in S$ is called the *least upper bound* of E if

a. α is an upper bound of E

b. If $\gamma < \alpha$ then γ is not an upper bound of E.

The least upper bound is denoted by $\sup E$ and is called the *supremum.* ◇

Exercise 2.1. Prove the supremum is unique.

Definition 2.11. Let S be an ordered set and $E \subset S$ bounded below. Then $\alpha \in S$ is called the *greatest lower bound* of E if

a. α is an lower bound of E

b. If $\gamma > \alpha$ then γ is not a lower bound of E.

The greatest lower bound is denoted by $\inf E$ and is called the *infimum.* ◇

Example 2.12
$E = \left\{\frac{1}{n}|n \in \mathbb{N}\right\}$. Then $\sup(E) = 1 \in E$ and $\inf(E) = 0 \notin E$, $E \subset Q$.

Example 2.13
$A = \{p > 0|p^2 < 2\}$ all positive elements of B are upper bounds, but $\sup(A)$ does not exist by 2.1

Example 2.14
$B = \{p > 0 | p^2 > 2\}$ all negative elements of A are lower bounds, but $\inf(B)$ does not exist by 2.1

Definition 2.15. We say that an ordered set S has the *least upper bound property* if any bounded above subset has the least upper bound in S. ◇

Example 2.16
$\mathbb{Q}$ does not have the least upper bound property.

Theorem 2.17. *Let S be an ordered set (o.s.) with the LUB property and $B \subset S$, $B \neq \mathbb{Q}$ and bounded below. Denote by L the set of all lower bounds of B. Then $\alpha = \sup(L)$ exists and $\alpha = \inf(B)$.*

Proof. Let $L = \{y \in S | y \leq x, \forall x \in B\}, L \neq \emptyset$ since B is bouded below (it must contatin at least one element). L is bounded above, since $\forall x \in B$ are upper bounds. Hence $\alpha = \sup L$ exists. We show that $\alpha = \inf B$. Indeed if $r < \alpha$ then r is not an upper bound of L. That is $\exists B \in L$ such that $r < B$. This means every $x \in B$ is $r < B \leq x$. That means $r \notin B$. Then $\alpha \in L$ i.e. $r \leq x, \forall x \in B$, r is a lower bound. If $B > \alpha$ then $B \notin L$ since α is an upper bound of L i.e. B is not a lower bound of B. □

Definition 2.18 (Fields)**.** The set F equipped with two operations "+" addition, "*" multiplication is called a *field* if:
Addition Axioms

1) If $x, y \in F$ then $x + y \in F$ (closure addition)
2) $x + y = y + x$ for all $x, y \in F$ (commnativity addition)
3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$ (associative addition)
4) F contains an element 0 such that $0 + x = x, \forall x \in F$ (identity)
5) To every $x \in F$ it corresponds an element $-x \in F$ such that $x + -x = 0$

Multiplicative Axioms

1) If $x, y \in F$, then $x * y \in F$ (multiplication closure)
2) $x * y = y * x, \forall x, y \in F$ (multiplication commutativity)
3) $(x * y) * z = x * (y * z), \forall x, y, z \in F$ (multiplication associative)
4) F contains an element $1 \neq 0$ such that $1 * x = x, \forall x \in F$ (multiplication identity)
5) $\forall x \in F, x \neq 0$ it corresponds an element $1/x \in F$ such that $x * 1/x = 1$ (mult inverse)

◇

Exercise 2.2. Do problems 1.14, 1.15, 1.16 in Rudin

Exercise 2.3. Show 0*x = 0

Proof. Find $x + y = x + z$, then $y = z$. Then we show

$$y = 0 + y = (x + (-x)) + y = -x + (x + y) = -x + (x + z) = z$$

Now using the above result we get

$$0 * x = (0 + 0)x = 0x + 0x \to 0x = 0$$

where we are using the above property with $z = 0$. So we get

$$x + y = x \to y = 0$$

□

2.2 Ordered Fields

Definition 2.19. Let F be a set which is an ordered set under "<" and a field under "+", "*" such that

1. $x + y < x + z$ if $x, y, z \in F$ and $y < z$
2. $xy > 0$ if $x, y \in F, x > 0, y > 0$

◇

Exercise 2.4. Prove if $x > 0$ then $(-x) < 0$

Theorem 2.20. *There exist and ordered field* $\mathbb{R}$ *which* $\mathbb{Q} \subset \mathbb{R}$ *and has the Least Upper Bound Property*

a) Dedekin (Appendix, Ch 1)

b) Completeness

Homework Hints - Prove the following Lemma

1. If $a + \varepsilon \leq b$ for every $\varepsilon > 0$ then $a \leq b$
2. If $a \leq b + \varepsilon$ for every $\varepsilon > 0$ then $a \leq b$
3. Restate inf,sup using limits

3 Archimedian Property

Problem 1. $A, B \subset \mathbb{R}$ bounded above. $A \neq \emptyset, B \neq \emptyset$. $C = \{a + b | a \in A, b \in B\}$. Prove that $\sup C = \sup A + \sup B$.

Proof. $C \neq \emptyset, \sup A = \alpha, \sup B = \beta$ and $\forall a \in A, \forall b \in B, a + b \leq \alpha + \beta$. C is bounded above then $\sup C = \gamma$ exists. Further

$$\gamma \leq \alpha + \beta \tag{3.1}$$

Take $\varepsilon > 0$. $\exists \tilde{a} \in A, \tilde{b} \in B$ such that $\alpha - \varepsilon < \tilde{a}$ and $\beta - \varepsilon < \tilde{b}$. Then

$$\alpha + \beta - 2\varepsilon < a + b \leq \alpha + \beta \rightarrow \alpha + \beta < \alpha + \beta + 2\varepsilon$$

Since $\tilde{a} + \tilde{b} \in C, \tilde{a} + \tilde{b} \leq \gamma$ so $\alpha + \beta \leq \gamma + 2\varepsilon$. Hence, $\forall \varepsilon > 0$, $\alpha + \beta < \gamma + 2\varepsilon$.

$$\alpha + \beta \leq \gamma \tag{3.2}$$

Then 3.1 and 3.2 gives us $\alpha + \beta = \gamma$. □

Problem 2 (from last lecture)**.** Prove $x \leq y + \varepsilon, \forall \varepsilon > 0 \rightarrow x \leq y$.

Proof. Suppose $y < x$, $x - y > 0$, $\varepsilon = \frac{x-y}{2}$. Then $y + \frac{x-y}{2} < x$ - Contradiction. □

Read Appendix to Ch 1 in Rudin for Dedekind Cuts, which are quite hard. The other way to prove completeness of $\mathbb{R}$ is using limits of cauchy sequences.

Proposition 3.3. *Prove that the set $\mathbb{N}$ is not bounded above*

Proof. If $\mathbb{N}$ was bounded above, then $\exists \alpha \in \mathbb{R}$ such that $\alpha = \sup \mathbb{N}$. Take $\alpha - 1$, there is $m \in \mathbb{N}$ such that $m > \alpha - 1$. Thi smeans $\alpha < m + 1 = \tilde{m} \in \mathbb{N}$ which contradicts that α is an upper bound. □

This means that for every real number $x \in \mathbb{R}$ there is a postive integer $n \in \mathbb{N}$ such that $n > x$.

Theorem 3.4 (Archmedian Property of Real Numbers)**.** *The following statements hold*

1. *If $x \in \mathbb{R}$ and $y \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $nx > y$.*
2. *If $x, y \in \mathbb{R}$, and $x < y$, then $\exists q \in \mathbb{Q}$ such that $x < q < y$.*

Proof of 1. Consider $A = \{nx | n \in \mathbb{N} \cup \{0\}\}$. Assume that $nx \leq y$, $\forall n$. This means A is bounded above and $A \neq \emptyset$. $\sup A = \alpha$ exists. $\alpha - x < \alpha$ since $x > 0$. Then there is a $m \in \mathbb{N} \cup \{0\}$ such that $\alpha - x < mx$. So $\alpha < (m + 1)x \in A$. Which contradicts α being an upperbound. □

Proof of 2. $x < y$ then $y - x > 0$. From part 1, $\exists n \in \mathbb{N}$ such that

$$n(y-x) > 1 \tag{3.5}$$

Furthermore,

$$\exists m_1 \in \mathbb{N} \text{ such that } m_1 > nx \tag{3.6}$$

$$\exists m_2 \in \mathbb{N} \text{ such that } m_2 > -nx \tag{3.7}$$

Then 3.6 and 3.7 gives us $-m_2 < mx < m_1$. Hence we choose $m \in \mathbb{Z}$ such that

$$m - 1 \leq nx < m \tag{3.8}$$

Through an inductive finite check, opptimize form right then -1. Then combining 3.5 and 3.8 we get

$$nx < m < 1 + nx < ny$$

Dividing by n, yield

$$x < \frac{m}{n} < y$$

□

Note 6. Can say nx, ny larger length than 1. So integer exists.

Assignment
Read in Rudin the following theorem, found on page 10, §1.2.1

Theorem 3.9. *$x \in \mathbb{R}, x > 0, \exists! y \in \mathbb{R}$ with $y > 0$ such that $y^n = x$ where $n \in \mathbb{N}$.*

Definition 3.10. A *finite decmial representation* of a $q \in \mathbb{Q}$ is

$$q = \pm\left(a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}\right)$$

with $a_0 \in \mathbb{N} \cup \{0\}, 0 \leq a_1 < 10$. ◇

Theorem 3.11. *Let $x \geq 0$, $x \in \mathbb{R}$ then for every $n \geq 1$ integer, there exists a finite decimal $r_n = a_0.a_1 \ldots a_n$ such that $r_n \leq x < r_n + \frac{1}{10^n}$.*

Proof. Let $S = PPallnonnegativeintegers \leq x$. $0 \in S \to S \neq \emptyset$. X is an upper bound. Let $a_0 = \sup S$ which exists. $a_0 = [x]$ the largest integer smaller then x, $a_0 \geq 0$. Then $10 > 10x - 10a_0 \geq 0$ because $x - a_0 < 1$ by def of a_0. We also get $0 \leq a_1 = [10x - 10a_0] \leq 9$. And so

$$a_0 + \frac{a_1}{10} \leq x < a_0 + \frac{a_1}{10} + \frac{1}{10}$$

Then we perform induction to finish the proof. □

Aside 3.12. $\frac{1}{10^n}, |x - r_n| < \frac{1}{10^n}$. There is significance to a similar yet slightly different statement $r_n < x \leq r_n + \frac{1}{10^n}$ [[★★★ infinite representation of rational?]] ◇

$\mathbb{R}$ extended real line is $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \rightarrow doesnotformafield$. It is used mainly as a convenience when down the road we will want to have limits to $+\infty$, $-\infty$ and neighborhoods of ∞. It is easy to say that all set bounded through this extension.

Notation 1. $[a, b], [a, b), [a, +\infty)$, and when ∞ is part of the set, $[a, +\infty]$

Definition 3.13. The set $\mathbb{C}$ is the set of *complex number* and is a field. Let $z \in \mathbb{C}$, $z = (a, b), i = (0, 1), z = a + ib,$ where $a = (a, 0), b = (0, b)$. $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$. Then $z_1 + z_2 = (a_1 + a_2, b_1 + b_2)$ and $z_1 * z_2 = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1)$ because $\mathbb{C}$ is isomorphic to matrix multiplicaton. See an algebra text. *** Rephrase this, it doesnt make sense now. $\diamond$

Note 7. $\mathbb{C}$ is not an ordered field

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