Sound and Complete Bidirectional Typechecking for Higher-Rank Polymorphism with Existentials and Indexed Types

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Bidirectional typechecking, in which terms either synthesize a type or are checked against a known type, has become popular for its applicability to a variety of type systems, its error reporting, and its ease of implementation. Following principles from proof theory, bidirectional typing can be applied to many type constructs. The principles underlying a bidirectional approach to indexed types (generalized algebraic datatypes) are less clear. Building on proof-theoretic treatments of equality, we give a declarative specification of typing based on focalization. This approach permits declarative rules for coverage of pattern matching, as well as support for first-class existential types using a focalized subtyping judgment. We use refinement types to avoid explicitly passing equality proofs in our term syntax, making our calculus similar to languages such as Haskell and OCaml. We also extend the declarative specification with an explicit rules for deducing when a type is principal, permitting us to give a complete declarative specification for a rich type system with significant type inference. We also give a set of algorithmic typing rules, and prove that it is sound and complete with respect to the declarative system. The proof requires a number of technical innovations, including proving soundness and completeness in a mutually recursive fashion.

Additional Key Words and Phrases: bidirectional typechecking, higher-rank polymorphism, indexed types, GADTs, equality types, existential types

1 INTRODUCTION

Consider a list type Vec with a numeric index representing its length, in Agda-like notation:

data Vec : Nat \rightarrow Type \rightarrow Type where
[] : $A \rightarrow$ Vec 0 A(::) : $A \rightarrow$ Vec $n A \rightarrow$ Vec (succ n) A

We can use this definition to write a head function that always gives us an element of type A when the length is at least one:

```
head: \forall n, A. Vec (succ n) A \rightarrow A head (x :: xs) = x
```

This clausal definition omits the clause for [], which has an index of 0. The type annotation tells us that head's argument has an index of succ(n) for some n. Since there is no natural number n such that 0 = succ(n), the nil case cannot occur and can be omitted.

This is an entirely reasonable explanation for programmers, but language designers and implementors will have more questions. First, designers of functional languages are accustomed to the

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benefits of the Curry–Howard correspondence, and expect a *logical* reading of type systems to accompany the operational reading. So what is the logical reading of GADTs? Second, how can we implement such a type system? Clearly we needed some equality reasoning to justify leaving off the nil case, which is not trivial in general.

Since we relied on equality information to omit the nil case, it seems reasonable to look to logical accounts of equality. In proof theory, it is possible to formulate equality in (at least) two different ways. The better-known is the *identity type* of Martin-Löf, but GADTs actually correspond best to the equality of Schroeder-Heister [1994] and Girard [1992]. The Girard-Schroeder-Heister (GSH) approach introduces equality via the reflexivity principle:

$$\Gamma \vdash t = t$$

The GSH elimination rule was originally formulated in a sequent calculus style, as follows:

for all
$$\theta$$
. if $\theta \in \text{csu}(s, t)$ then $\theta(\Gamma) \vdash \theta(C)$

$$\Gamma, (s = t) \vdash C$$

Here, we write csu(s, t) for a *complete set of unifiers* of s and t. So the rule says that we can eliminate an equality s = t if, for every substitution θ that makes s and t equal, we can prove the goal C. This rule has three important features, two good and one bad.

- (1) The GSH elimination rule is an invertible left rule. By "left rule", we mean that the rule decomposes the assumptions to the left of the turnstile (in this case, the assumption that s = t), and by "invertible", we mean the conclusion of the rule implies the premises. Invertible left rules are interesting, because they are known to correspond (via Curry–Howard) to the deconstruction steps carried out by pattern-matching rules [Krishnaswami 2009]. This is our first hint that the GSH rule has something to do with GADTs; programming languages like Haskell and OCaml indeed use pattern matching to propagate equality information.
- (2) Observe that if we have an inconsistent equation, we can immediately prove the goal. If we specialize the rule above to the equality 0 = 1, we get:

$$\overline{\Gamma$$
, $(0 = 1) \vdash C$

Because 0 and 1 have no unifiers, the complete set of unifiers is the empty set. As a result, the GSH rule for this instance has no premises, and the elimination rule for an absurd equation ends up looking exactly like the elimination rule for the empty type:

$$\overline{\Gamma, \bot \vdash C}$$

Moreover, recall that when we eliminate an empty type, we can view the eliminator abort(e) as a pattern match with no clauses. Together, these two features line up nicely with our definition of head, where the impossibility of the case for [] was indicated by the *absence* of a pattern clause. The use of equality in GADTs corresponds perfectly with the GSH equality.

(3) Alas, we cannot simply give a proof term assignment for first-order logic and call it a day. The third important feature of the GSH equality rule is its use of *unification*: it works by treating the free variables of the two terms as unification variables. But type inference algorithms also use unification, introducing unification variables to stand for unknown types. These two uses of unification are *entirely different!* Type inference introduces unification variables to stand for the specific instantiations of universal quantifiers. In contrast, the Girard–Schroeder-Heister rule uses unification to constrain the universal parameters. As a

¹The invertibility of equality elimination is certainly not obvious; see Schroeder-Heister [1994] and Girard [1992].

result, we need to understand how to integrate these two uses of unification, or at least how to keep them decently separated, in order to take this logical specification and implement type inference for it.

This problem—formulating indexed types in as logical a style as feasible, while retaining the ability to implement type inference algorithms for them—is the subject of this paper.

Contributions. It has long been known that GADTs are equivalent to the combination of existential types and equality constraints [Xi et al. 2003]. Our key contribution is to reduce GADTs to standard logical ingredients, while retaining the implementability of the type system. We manage this by formulating a system of indexed types in a bidirectional style (combining type synthesis with checking against a known type), which is both practically implementable and theoretically tidy.

- Our language supports implicit higher-rank polymorphism (in which quantifiers can be nested under arrows) including existential types. While algorithms for higher-rank universal polymorphism are well-known [Peyton Jones et al. 2007; Dunfield and Krishnaswami 2013], our approach to supporting existential types is novel.
 - We go beyond the standard practice of tying existentials to datatype declarations [Läufer and Odersky 1994], in favour of a first-class treatment of implicit existential types. This approach has historically been thought difficult, since treating existentials in a first-class way opens the door to higher-rank polymorphism that mixes universal and existential quantifiers.
 - Our approach extends existing bidirectional methods for handling higher-rank polymorphism, by adapting the proof-theoretic technique of *focusing* to give a novel *polarized subtyping judgment*, which lets us treat mixed quantifiers in a way that retains decidability while maintaining the essential properties of subtyping, such as stability under substitution and transitivity.
- Our language includes equality types in the style of Girard and Schroeder-Heister, but without an explicit introduction form for equality. Instead, we treat equalities as property types, in the style of intersection or refinement types: we do not write explicit equality proofs in our syntax, permitting us to more closely model how equalities are used in OCaml and Haskell.
- The use of focusing also lets us equip our calculus with nested pattern matching. This fits
 in neatly with our bidirectional framework, and permits us to give a formal specification
 of coverage checking with GADTs, which is easy to understand, easy to implement, and
 theoretically well-motivated.
- In contrast to systems which globally possess or lack principal types, our declarative system tracks whether or not a derivation has a principal type.
- Our system includes an unusual "higher-order principality" rule, which says that if only a single type can be synthesized for a term, then that type is principal. While this style of hypothetical reasoning is natural to explain to programmers, formalizing it requires giving an inference rule with universal quantification over possible typing derivations in the premise. This is an extremely non-algorithmic rule (even its well-foundedness is not immediate).
- As a result, the soundness and completeness proofs for our implementation have to be done in a new style. It is no longer possible to prove soundness and completeness independently, and instead we must prove them mutually recursively.
- We formulate an algorithmic type system (Section 5) for our declarative calculus, and prove that typechecking is decidable, deterministic (5.4), and sound and complete (Sections 6–7) with respect to the declarative system.
 - The resulting type system is relatively easy to implement (an undergraduate implemented most of it on his own from a draft of the paper, with minimal contact with the authors), and

is close in style to languages such as Haskell or OCaml. As a result, it seems like a reasonable basis for implementing new languages with expressive type systems.

Our algorithmic system (and, to a lesser extent, our declarative system) uses some techniques developed by Dunfield and Krishnaswami [2013], but we extend these to a far richer type language (existentials, indexed types, sums, products, equations over type variables), and we differ by supporting pattern matching, polarized subtyping, and principality tracking.

Supplementary material. The appendix contains rules omitted for space reasons, and full proofs.

2 OVERVIEW

To orient the reader, we give an overview and rationale of the novelties in our type system, before getting into the details of the typing rules and algorithm. As is well-known [Cheney and Hinze 2003; Xi et al. 2003], GADTs can be desugared into type expressions that use equality and existential types to express the return type constraints. These two features lead to difficulties in type-checking for GADTs.

Universal, existentials, and type inference. Practical typed functional languages must support some degree of type inference, most critically the inference of type arguments. That is, if we have a function f of type $\forall a.a \rightarrow a$, and we want to apply it to the argument 3, then we want to write f 3, and not f [Nat] 3 (as we would in pure System F). Even with a single type argument, the latter style is noisy, and programs using even moderate amounts of polymorphism rapidly become unreadable. However, omitting type arguments has significant metatheoretical implications. In particular, it forces us to include subtyping in our typing rules, so that (for instance) the polymorphic type $\forall a. \ a \rightarrow a$ is a subtype of its instantiations (like Nat \rightarrow Nat).

The subtype relation induced by System F-style polymorphism and function contravariance is already undecidable [Tiuryn and Urzyczyn 1996; Chrząszcz 1998], so even at the first step we must introduce restrictions on type inference to ensure decidability. In our case, matters are further complicated by the fact that we need to support *existential types* in addition to universal types.

Existentials are required to encode GADTs [Xi and Pfenning 1999], but programming languages have traditionally stringently restricted the use of existential types. Following the approach of Läufer and Odersky [1994], languages such as OCaml and Haskell tie existential introduction and elimination to datatype declarations, so that there is always a syntactic marker for when to introduce or eliminate existential types. This choice permits leaving existentials out of subtyping altogether, at the price of no longer permitting implicit subtyping (such as using $\lambda x. x + 1$ at type $\exists a. a \rightarrow a$).

While this is a practical solution, it increases the distance between a surface language and its type-theoretic core. Our goal is to give a *direct* type-theoretic account of as many features of our surface languages as possible. In addition to the theoretical tidiness, this also has practical language design benefits. By avoiding a complex elaboration step, we are forced to specify the type inference algorithm in terms of a language close to a programmer-visible surface language. This does increase the complexity of the approach, but in a productive way: we are forced to analyze and understand how type inference will look to the end programmer.

The key problem is that when both universal and existential quantifiers are permitted, the order in which to instantiate quantifiers when computing subtype entailments becomes unclear. For example, suppose we need to decide $\Gamma \vdash \forall a_1 . \exists a_2 . A(a_1, a_2) \leq \exists b_1 . \forall b_2 . B(b_1, b_2)$. An algorithm to solve this must either first introduce a unification variable for a_1 and a parameter for a_2 first, and only then introduce a unification variable for b_1 and a parameter for b_2 , or the other way around—and the order in which we make these choices matters! With the first order, the instantiation for

 b_1 may refer to a_2 , but the instantiation for a_1 cannot have b_2 as a free variable. With the second order, the instantiation for a_1 may have b_2 as a free variable, but b_1 may not refer to a_2 .

In some cases, depending on what $A(a_1, a_2)$ and $B(b_1, b_2)$ are, only one choice of order works. For example, if we are trying to decide $\Gamma \vdash \forall a_1$. $\exists a_2.\ a_1 \rightarrow a_2 \leq \exists b_1.\ \forall b_2.\ b_2 \rightarrow b_1$, we must choose the first order: we must pick an instantiation for a_1 , and then make a_2 into a parameter before we can instantiate b_1 as a_2 . The second order will not work, because b_1 must depend on a_2 . Conversely, if we are trying to solve $\Gamma \vdash \forall a_1.\ \exists a_2.\ a_1 \rightarrow a_2 \leq \exists b_1.\ \forall b_2.\ \exists b_3.\ b_1 \times b_2 \rightarrow b_3$, the first order will not work; we must instantiate b_1 (say, to int) and quantify over b_2 before instantiating a_1 as int $\times b_2$.

As a result, the outermost connectives do not reliably determine which side of a subtype judgement $\Gamma \vdash \forall a. A \leq \exists b. B$ to specialize first.

One implementation strategy is simply to give up determinism: an algorithm could backtrack when faced with deciding subtype entailments of this form. Unfortunately, backtracking is dangerous for a practical implementation, since it potentially causes type-checking to take exponential time. This tends to defeat the benefit of a complete declarative specification, since different implementations with different backtracking strategies could have radically different running times when checking the same program. So we may end up with an implementation that is theoretically complete, but incomplete in practice.

Instead, we turn to ideas from proof theory—specifically, polarized type theory. In the language of polarization, universals are a *negative* type, and existentials are a *positive* type. So we introduce two mutually recursive subtype relations: $\Gamma \vdash A \leq^+ B$ for positive types and $\Gamma \vdash A \leq^- B$ for negative types. The positive subtype relation only deconstructs existentials, and the negative subtype relation only deconstructs universals. This fixes the order in which quantifiers are instantiated, making the problem decidable (in fact, rather straightforward).

The price we pay is that fewer subtype entailments are derivable. Fortunately, any program typeable under a more liberal subtyping judgement can be made typable in our discipline by η -expanding it. Moreover, the lost subtype entailments seem to be rare in practice: most of the types we see in practice are of the form $\forall \vec{a}. \vec{A} \rightarrow \exists \vec{b}. B$, and this fits perfectly with the kinds of types our polarized subtyping judgement works best on. As a result, we keep fundamental expressivity, and also efficient decidability.

Equality as a property. The usual convention in Haskell and OCaml is to make equality proofs in GADT definitions implicit. We would like to model this feature directly, so that our calculus stays close to surface languages, without sacrificing the logical reading of the system. In this case, the appropriate logical concepts come from the theory of intersection types. A typing judgment such as $e: A \times B$ can be viewed as giving instructions on how to construct a value (pair an A with a B). But types can also be viewed as *properties*, where e: X is read "e has property X".

To model GADTs, we need both of these readings! For example, a term of vector type is constructed from nil and cons constructors, but also has a property governing its index. To support this combination, we introduce a type constructor $A \wedge P$, read "A with P", to model elements of type A satisfying the property (equation) P. (We also introduce $P \supset A$, read "P implies A", for its adjoint dual, consisting of terms which have the type A conditionally under the assumption that P holds.) Then we make equality t = t' into a property, and make use of standard rules for property types (which omit explicit proof terms) to type equality constraints [Dunfield 2007b, Section 2.4].

This gives us a logical account of how OCaml and Haskell avoid requiring explicit equality proofs in their syntax. The benefit of handling equality constraints through intersection types is that certain restrictions on typing that are useful for decidability, such as restricting property introduction to values, arise naturally from the semantic point of view—via the value restriction needed for soundly modeling intersection and union types [Davies and Pfenning 2000;

Dunfield and Pfenning 2003]. In addition, the appropriate approach to take when combining GADTs and effects is clear.²

Bidirectionality, pattern matching, and principality. Something that is not by itself novel in our approach is our decision to formulate both the declarative and algorithmic systems in a bidirectional style. Bidirectional checking [Pierce and Turner 2000] is a popular implementation choice for systems ranging from dependent types [Coquand 1996; Abel et al. 2008] and contextual types [Pientka 2008] to object-oriented languages [Odersky et al. 2001], but also has good proof-theoretic foundations [Watkins et al. 2004], making it useful both for specifying and implementing type systems. Bidirectional approaches make it clear to programmers where annotations are needed (which is good for specification), and can also remove unneeded nondeterminism from typing (which is good for both implementation and proving its correctness).

However, it is worth highlighting that because both bidirectionality and pattern matching arise from focalization, these two features fit together extremely well. In fact, by following the blueprint of focalization-based pattern matching, we can give a coverage-checking algorithm that explains when it is permissible to omit clauses in GADT pattern matching.

In the propositional case, the type synthesis judgment of a bidirectional type system generates principal types: if a type can be inferred for a term, that type is the most specific possible type for that term. (Indeed, in many cases, including the current system, the inferred type will even be unique.) This property is lost once quantifiers are introduced into the system, which is why it is not much remarked upon. However, prior work on GADTs, starting with Simonet and Pottier [2007], has emphasized the importance of the fact that handling equality constraints is much easier when the type of a scrutinee is principal. Essentially, this ensures that no existential variables can appear in equations, which prevents equation solving from interfering with unification-based type inference. The OutsideIn algorithm takes this consequence as a definition, permitting non-principal types just so long as they do not change the values of equations. However, Vytiniotis et al. [2011] note that while their system is sound, they no longer have a completeness result for their type system.

We use this insight to extend our bidirectional typechecking algorithm to track principality: The judgments we give track whether types are principal, and we use this to give a relatively simple specification for whether or not type annotations are needed. We are able to give a very natural spec to programmers—cases on GADTs must scrutinize terms with principal types, and an inferred type is principal just when it is the only type that can be inferred for that term—which soundly and completely corresponds to the implementation-side constraints: a type is principal when it contains no existential unification variables.

3 EXAMPLES

In this section, we give some examples of terms from our language, which illustrate the key features of our system and give a sense of how many type annotations are needed in practice. To help make this point clearly, all of the examples which follow are unsugared: they are the *actual* terms from our core calculus.

Mapping over lists. First, we begin with the traditional *map* function, which takes a function and applies it to every element of a list.

²The traditional eq GADT and its constructor ref1 can be encoded into our system as the type $1 \land (s = t)$, which which can be constructed as a unit value only under the constraint that s equals t.

rec
$$map. \lambda f. \lambda xs. case(xs, [] \Rightarrow []$$

 $\exists y :: ys \Rightarrow (f \ y) :: map \ f \ ys)$
 $: \forall n : \mathbb{N}. \ \forall \alpha : \star. \ \forall \beta : \star. \ (\alpha \rightarrow \beta) \rightarrow \text{Vec } n \ \alpha \rightarrow \text{Vec } n \ \beta$

This function case-analyzes its second argument xs. Given an empty xs, it returns the empty list; given a cons cell y: ys, it applies the argument function f to the head y and makes a recursive call on the tail ys.

In addition, we annotate the definition with a type. We have two type parameters α and β for the input and output element types. Since we are working with length-indexed lists, we also have a length index parameter n, which lets us show by typing that the input and output of map have the same length.

In our system, this type annotation is mandatory. Full type inference for definitions using GADTs requires polymorphic recursion, which is undecidable. As a result, this example also requires annotation in OCaml and GHC Haskell. However, Haskell and OCaml infer polymorphic types when no polymorphic recursion is needed. We adopt the simpler rule that *all* polymorphic definitions are annotated. This choice is motivated by Vytiniotis et al. [2010], who analyzed a large corpus of Haskell code and showed that implicit let-generalization was used primarily only for top-level definitions, and even then it is typically considered good practice to annotate top-level definitions for documentation purposes. Furthermore, experience with languages such as Agda and Idris (which do not implicitly generalize) show this is a modest burden in practice.

Nested patterns and GADTs. Now, we consider the *zip* function, which converts a pair of lists into a list of pairs. In ordinary ML or Haskell, we must consider what to do when the two lists are not the same length. However, with length-indexed lists, we can statically reject passing two lists of differing length:

This case expression has only two patterns, one for when both lists are empty and one for when both lists have elements, with the type annotation indicating that both lists must be of length n. Typing shows that the cases where one list is empty and the other is non-empty are impossible, so our coverage checking rules accept this as a complete set of patterns. This example also illustrates that we support nested pattern matching.

Existential Types. Now, we consider the *filter* function, which takes a predicate and a list, and returns a list containing the elements satisfying that predicate. This example makes a nice showcase for supporting existential types, since the size of the return value is not predictable statically.

```
\begin{split} \operatorname{rec} & \operatorname{filter.} \lambda p. \ \lambda xs. \ \operatorname{case} \big( xs, \ \ \big[ \ \big] \Rightarrow \big[ \ \big] \\ & | \ x :: xs \Rightarrow \operatorname{let} \ tl = \operatorname{filter} p \ xs \ \operatorname{in} \\ & \operatorname{case} \big( p \ xs, \\ & \operatorname{inj}_1 \ \_ \Rightarrow tl \\ & | \ \operatorname{inj}_2 \ \_ \Rightarrow x :: tl \big) \big) \\ & : \forall n : \mathbb{N}. \ \forall \alpha : \star. \ (\alpha \to 1+1) \to \operatorname{Vec} \ n \ \alpha \to \exists k : \mathbb{N}. \operatorname{Vec} \ k \ \alpha \end{split}
```

So, this function takes predicate and a vector of arbitrary size, and then returns a list of unknown size (represented by the existential type $\exists k : \mathbb{N}$. Vec $k \alpha$). Note that we did not need to package the existential in another datatype, as one would have to in OCaml or GHC Haskell—we are free to use existential types as "just another type constructor".

```
e ::= x \mid () \mid \lambda x. e \mid e s^{+} \mid rec x. v \mid (e : A)
Expressions
                                              |\langle e_1, e_2 \rangle| \operatorname{inj}_1 e | \operatorname{inj}_2 e | \operatorname{case}(e, \Pi)
                                              | [] | e_1 :: e_2
                                    v ::= x \mid () \mid \lambda x. e \mid rec x. v \mid (v : A)
Values
                                              |\langle v_1, v_2 \rangle| \operatorname{inj}_1 v | \operatorname{inj}_2 v | [] | v_1 :: v_2
                                     s := \cdot \mid e s
Spines
Nonempty spines s^+ := e s
Patterns
                                    \rho ::= x \mid \langle \rho_1, \rho_2 \rangle \mid \operatorname{inj}_1 \rho \mid \operatorname{inj}_2 \rho \mid [] \mid \rho_1 :: \rho_2
Branches
                                    \pi ::= \vec{\rho} \Rightarrow e
                                   \Pi ::= \cdot \mid (\pi \mid \Pi)
Branch lists
```

Fig. 1. Source syntax

```
Universal variables \alpha, \beta, \gamma
Sorts
                                        \kappa ::= \star \mid \mathbb{N}
Types
                                A, B, C ::= 1 \mid A \rightarrow B \mid A + B \mid A \times B
                                                |\alpha| \forall \alpha : \kappa. A \mid \exists \alpha : \kappa. A
                                                 | P \supset A | A \land P | \text{Vec } t A
Terms/monotypes
                                  t, \tau, \sigma ::= zero \mid succ(t) \mid 1 \mid \alpha
                                                | \tau \rightarrow \sigma | \tau + \sigma | \tau \times \sigma
                                    P,Q ::= t = t'
Propositions
Contexts
                                       \Psi ::= \cdot \mid \Psi, \alpha : \kappa \mid \Psi, x : Ap
Polarities
                                       \mathcal{P} ::= + | - | \circ
Binary connectives
                                     \oplus ::= \rightarrow |+| \times
                                     p, q ::= ! |
Principalities
                                                   sometimes omitted
```

Fig. 2. Syntax of declarative types and contexts

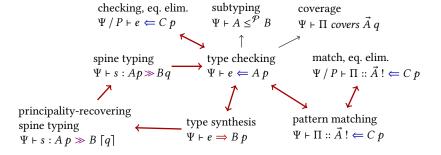


Fig. 3. Dependency structure of the declarative judgments

4 DECLARATIVE TYPING

Expressions. Expressions (Figure 1) are variables x; the unit value (); functions $\lambda x. e$; applications to a spine $e s^+$; fixed points rec x. v; annotations (e : A); pairs $\langle e_1, e_2 \rangle$; injections into a sum

Fig. 4. Subtyping in the declarative system

type $\operatorname{inj}_k e$; case expressions $\operatorname{case}(e,\Pi)$ where Π is a list of branches π , which can eliminate pairs and injections (see below); the empty vector []; and consing a head e_1 to a tail vector e_2 .

Values v are standard for a call-by-value semantics; the variables introduced by fixed points are considered values, because we only allow fixed points of values. A spine s is a list of expressions—arguments to a function. Allowing empty spines (written ·) is convenient in the typing rules, but would be strange in the source syntax, so (in the grammar of expressions e) we require a nonempty spine s^+ . We usually omit the empty spine ·, writing e_1 e_2 instead of e_1 e_2 ·. Since we use juxtaposition for both application e s^+ and spines, some strings are ambiguous; we resolve this ambiguity in favour of the spine, so e_1 e_2 e_3 is parsed as the application of e_1 to the spine e_2 e_3 , which is technically e_2 (e_3 ·). Patterns ρ consist of pattern variables, pairs, and injections. A branch π is a sequence of patterns $\vec{\rho}$ with a branch body e. We represent patterns as sequences, which enables us to deconstruct tuple patterns.

Types. We write types as A, B and C. We have the unit type 1, functions $A \to B$, sums A + B, and products $A \times B$. We have universal and existential types $\forall \alpha : \kappa.A$ and $\exists \alpha : \kappa.A$; these are predicative quantifiers over monotypes (see below). We write α , β , etc. for type variables; these are universal, except when bound within an existential type. We also have a *guarded type* $P \supset A$, read "P implies A". This implication corresponds to type A, provided P holds. Its dual is the *asserting type* $A \land P$, read "A with A", which witnesses the proposition A. In both, A has no runtime content.

Sorts, terms, monotypes, and propositions. Terms and monotypes t, τ , σ share a grammar but are distinguished by their *sorts* κ . Natural numbers zero and succ(t) are *terms* and have sort $\mathbb N$. Unit 1 has the sort \star of *monotypes.* A variable α stands for a term or a monotype, depending on the sort

 κ annotating its binder. Functions, sums, and products of monotypes are monotypes and have sort \star . We tend to prefer t for terms and σ , τ for monotypes.

A proposition P or Q is simply an equation t = t'. Note that terms, which represent runtime-irrelevant information, are distinct from expressions; however, an expression may include type annotations of the form $P \supset A$ and $A \land P$, where P contains terms.

Contexts. A declarative context Ψ is an ordered sequence of universal variable declarations $\alpha : \kappa$ and expression variable typings x : Ap, where p denotes whether the type A is principal (Section 4.2). A variable α can be free in a type A only if α was declared to the left: $\alpha : \star$, $x : \alpha p$ is well-formed, but $x : \alpha p$, $\alpha : \star$ is not.

4.1 Subtyping

We give our two subtyping relations, \leq^+ and \leq^- , in Figure 4. We treat the universal quantifier as a negative type (since it is a function in System F), and the existential as a positive type (since it is a pair in System F). We have two typing rules for each of these connectives, corresponding to the left and right rules for universals and existentials in the sequent calculus. We treat all other types as having no polarity. The positive and negative subtype judgments are mutually recursive, and the \leq^-_+ rule permits switching the polarity of subtyping from positive to negative when both of the types are non-positive, and conversely for \leq^+_- . When both types are neither positive nor negative, we require them to be equal ($\leq Refl P$).

In logical terms, functions and guarded types are negative; sums, products and assertion types are positive. We could potentially operate on these types in the negative and positive subtype relations, respectively. Leaving out (for example) function subtyping means that we will have to do some η -expansions to get programs to typecheck; we omit these rules to keep the implementation complexity low. (The idea that η -expansion can substitute for subsumption dates to Barendregt et al. [1983].)

This also illustrates a nice feature of bidirectional typing: we are relatively free to adjust the subtype relation to taste. Moreover, the structure of polarization makes it easy to work out just what the rules should be. E.g., to add function subtyping to our system, we would use the rule:

$$\frac{\Psi \vdash A' \leq^+ A \qquad \Psi \vdash B \leq^- B'}{\Psi \vdash A \longrightarrow B \leq^- A' \longrightarrow B'}$$

As polarized function types are a negative type of the form $X^+ \to Y^-$, we see (1) the rule as a whole lives in the negative subtyping judgement, (2) argument types compare in the positive judgement (with the usual contravariant twist), and (3) result types compare in the negative judgement.

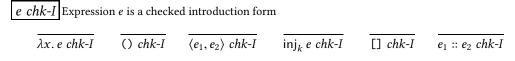


Fig. 5. "Checking intro form"

4.2 Typing judgments

Principality. Our typing judgments carry principalities: A! means that A is principal, and $A \not I$ means A is not principal. Note that a principality is part of a judgment, not part of a type. In the checking judgment $\Psi \vdash e \Leftarrow A p$ the type A is input; if p = !, we know that A is not the result of guessing. For example, the e in (e : A) is checked against A!. In the synthesis judgment

Fig. 6. Declarative typing, omitting rules for \times , +, and Vec

 $\Psi \vdash e \Rightarrow A p$, the type A is output, and p = ! means it is impossible to synthesize any other type, as in $\Psi \vdash (e : A) \Rightarrow A !$.

We sometimes omit a principality when it is I ("not principal"). We write $p \sqsubseteq q$, read "p at least as principal as q", for the reflexive closure of $I \subseteq I$.

Spine judgments. The ordinary form of spine judgment, $\Psi \vdash s : A p \gg C q$, says that if arguments s are passed to a function of type A, the function returns type C. For a function e applied to one argument e_1 , we write e e_1 as syntactic sugar for e $(e_1 \cdot)$. Supposing e synthesizes $A_1 \to A_2$, we apply $\text{Decl} \to \text{Spine}$, checking e_1 against A_1 and using DeclEmptySpine to derive $\Psi \vdash \cdot : A_2 p \gg A_2 p$.

Rule Decl \forall Spine does not decompose e s but instantiates a \forall . Note that, even if the given type $\forall \alpha : \kappa$. A is principal (p = !), the type $[\tau/\alpha]A$ in the premise is not principal—we could choose a different τ . In fact, the q in Decl \forall Spine is also always f, because no rule deriving the ordinary spine judgment can recover principality.

The *recovery spine judgment* $\Psi \vdash s : A p \gg C \lceil q \rceil$, however, can restore principality in situations where the choice of τ in DeclySpine cannot affect the result type C. If A is principal (p = !) but the ordinary spine judgment produces a non-principal C, we can try to recover principality with DeclSpineRecover. Its first premise is $\Psi \vdash s : A ! \gg C \ I$; its second premise (really, an infinite set of premises) quantifies over all derivations of $\Psi \vdash s : A ! \gg C' \ I$. If C' = C in all such derivations, then the ordinary spine rules erred on the side of caution: C is actually principal, so we can set q = ! in the conclusion of DeclSpineRecover.

If some $C' \neq C$, then C is certainly not principal, and we must apply DeclSpinePass, which simply transitions from the ordinary judgment to the recovery judgment.

Figure 3 shows the dependencies between the declarative judgments. Given the cycle containing the spine typing judgments, we need to stop and ask: Is DeclSpineRecover well-founded? For well-foundedness of type systems, we can often make a straightforward argument that, as we move from the conclusion of a rule to its premises, either the expression gets smaller, or the expression stays the same but the type gets smaller. In DeclSpineRecover, neither the expression nor the type get smaller. Fortunately, the rule that gives rise to the arrow from "spine typing" to "type checking" in Figure 3—Decl→Spine—does decompose its subject, and any derivations of a recovery judgment lurking within the second premise of DeclSpineRecover must be for a smaller spine. In the appendix (Lemma 1, p. 38), we prove that the recovery judgment, and all the other declarative judgments, are well-founded.

Example. In Section 5.1 we present some example derivations that illustrate how the spine typing rules work to recover principality.

Subtyping. Rule DeclSub invokes the subtyping judgment, at the join of the polarities of B (the type being checked against) and A (the type being synthesized). Using the join ensures that the polarity of B takes precedence over A's, which means the programmer control which subtyping mode to begin with via a type annotation.

Furthermore, the subtyping rule allows DeclSub to play the role of an existential introduction rule, by applying subtyping rule $\leq \exists R$ when B is an existential type.

Pattern matching. Rule DeclCase checks that the scrutinee has a type and principality, and then invokes the two main judgments for pattern matching. The $\Psi \vdash \Pi :: \vec{A} \ q \Leftarrow C \ p$ judgement checks that each branch in the list of branches Π is well-typed, taking a vector \vec{A} of pattern types to simplify the specification of coverage checking, as a well as a principality annotation covering all of the types (i.e., if any of the types in \vec{A} is non-principal, the whole vector is not principal).

The $\Psi \vdash \Pi$ covers \vec{A} q judgement does coverage checking for the list of branches. However, the DeclCase does not simply check that the patterns cover for the inferred type of the scrutinee—it checks that they cover for *every* possible type that could be inferred for the scrutinee. In the case that the scrutinee is principal, this is the same as checking coverage at the scrutinee's type, but when the scrutinee is not principal, this rule has the effect of preventing type inference from

$$\begin{array}{c} \Psi \vdash \Pi :: \vec{A} \ q \leftrightharpoons C \ p \end{array} \ \begin{array}{c} \text{Under context } \Psi, \\ \text{check branches } \Pi \ \text{ with patterns of type } \vec{A} \ \text{and bodies of type } C \\ \\ \hline \Psi \vdash :: \vec{A} \ q \leftrightharpoons C \ p \\ \hline \Psi \vdash :: \vec{A} \ q \leftrightharpoons C \ p \\ \hline \Psi \vdash (: \Rightarrow e) :: \cdot q \leftrightharpoons C \ p \end{array} \ \begin{array}{c} \Psi \vdash \pi :: \vec{A} \ q \leftrightharpoons C \ p \\ \hline \Psi \vdash (: \Rightarrow e) :: \cdot q \leftrightharpoons C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \leftrightharpoons C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \leftrightharpoons C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash C \ p \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash P \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash P \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash P \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash P \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash Q \vdash P \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash Q \vdash P \\ \hline \Psi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash Q \vdash P \end{array} \ \begin{array}{c} \Phi \vdash (x \vdash x) = p :: \cdot q \vdash Q \vdash Q \vdash Q \vdash Q \vdash Q \vdash Q \vdash$$

Fig. 7. Declarative pattern matching

using the shape of the patterns to infer a type, which is notoriously problematic with GADTs (e.g., whether a missing nil in a list match should be taken as evidence of coverage failure or that the length is non-zero). As with spine recovery, this rule is only well-founded because the universal quantification ranges over synthesized types over a subterm.

The $\Psi \vdash \Pi :: \vec{A} \ q \Leftarrow C \ p$ judgment (rules in Figure 7) systematically checks the typing of each branch in Π : rule <code>DeclMatchEmpty</code> succeeds on the empty list, and <code>DeclMatchSeq</code> checks one branch and recurs on the remaining branches. Rules for sums, units, and products break down patterns left to right, one constructor at a time. Products also extend the sequences of patterns and types, with <code>DeclMatch×</code> breaking down a pattern vector headed by a pair pattern $\langle p, p' \rangle, \vec{p}$

Fig. 8. Match coverage

into p, p', \vec{p} , and breaking down the type sequence $(A \times B), \vec{C}$ into A, B, \vec{C} . Once all the patterns are eliminated, the DeclMatchBase rule says that if the body typechecks, then the branch typechecks. For completeness, the variable and wildcard rules are restricted so that any top-level existentials and equations are eliminated before discarding the type.

The existential elimination rule DeclMatch∃ unpacks an existential type, and DeclMatch∧ breaks apart a conjunction by eliminating the equality using unification. The DeclMatch⊥ rule says that if the equation is false then typing succeeds, because this case is impossible. The DeclMatchUnify rule unifies the two terms of an equation and applies the substitution before continuing to check typing. Together, these two rules implement the Schroeder-Heister equality elimination rule. Because our language of terms has only simple first-order terms, either unification will fail, or there is a most general unifier. Note, however, that DeclMatch∧ only applies when the pattern type is principal. Otherwise, we use the DeclMatch∧⅓ rule, which throws away the equation and does not refine any types at all. In this way, we can ensure that we will only try to eliminate equations which are fully known (i.e., principal). Similar considerations apply to vectors, with length information being used to refine types only when the type of the scrutinee is principal.

Fig. 9. Pattern expansion

For sum types, we expand a list of branches into *two* lists, one for each injection. So $\Pi \stackrel{+}{\sim} \Pi_L \parallel \Pi_R$ will send all branches headed by $\operatorname{inj}_1 p$ into Π_L and all branches headed by $\operatorname{inj}_2 p$ into Π_R , with variables and wildcards being sent to both sides. Then DeclCovers+ checks the left and right branches independently.

As with typing, DeclCovers∃ just unpacks the existential type. Likewise, DeclCoversEqBot and DeclCoversEq handle the two cases arising from equations. If an equation is unsatisfiable, coverage succeeds since there are no possible values of that type. If it is satisfiable, we apply the substitution and continue coverage checking. Just as when typechecking patterns, we only use property types to refine coverage checking when the equations come from a principal type — the DeclCovers∧ \(\) rule simply throws away the equation when the type is not principal. (This is a sound approximation which ends up requiring more patterns when the type is not principal.)

So far, the coverage rules for pattern matching are almost purely type-directed. However, once recursive types like Vec n A enter the picture, matters become a little more subtle. The issue is

that if we split a wildcard _ of type Vec *n A*, the type *doesn't tell us when to stop*. That is, we could split a wildcard into a nil [] and cons _ :: _ pattern; or we could turn it into a nil [], singleton _ :: [] and two-or-longer _ :: _ :: _ pattern; and so on. The key issue is that the tail of a list has the same set of possible patterns as the list itself, and so blindly following the type structure will not ensure termination of coverage checking.

In this paper, we take the view that the patterns the programmer wrote should guide how much to split types when doing coverage checking for inductive types. In Figure 8, we introduce the Π guarded judgement, which checks to see if a constructor pattern is present in the leading column of patterns. If it is, then our algorithm will unfold the recursive type as part of type checking, and otherwise it will not. This is by no means a canonical choice: our choice is similar to the choice Agda makes, but other language implementations make other choices. In contrast, the OCaml coverage checking algorithm unfolds wildcard patterns at GADT type one step more than what the programmer wrote [Garrigue and Le Normand 2015]. (They also observe that precise exhaustiveness checking is undecidable, meaning that some choice of heuristic is unavoidable.)

4.2.1 Design Considerations for Pattern Matching.

Evaluation Order. Our typing and coverage checking rules are given assuming a call-by-value evaluation strategy. These coverage rules are not sound under a call-by-name evaluation order. Consider the following program, writing \bot for a looping term:

$$case(\bot : A \land (s = t), x \Rightarrow e)$$

When type-checking this program, the $DeclMatch \land$ and $DeclCovers \land$ rules are permitted to eliminate the equality s = t when checking e. However, one can use a looping program to inhabit $A \land (s = t)$ for any P, and so we have introduced a spurious equality into the context when checking e. In contrast, in a call-by-value language the scrutinee of a case will be reduced before the match proceeds, so this issue cannot arise. (In a total language such as Koka, these rules would be sound irrespective of evaluation order, since all evaluation strategies are indistinguishable.)

Redundant Patterns. These rules do not check for redundancy: DeclCoversEmpty applies even when branches are left over. When DeclCoversEmpty is applied, we could mark the $\cdot \Rightarrow e_1$ branch, and issue a warning for unmarked branches. This seems better as a warning than an error, since redundancy is not stable under substitution. For example, a case over (Vec n A) with [] and :: branches is not redundant—but if we substitute 0 for n, the :: branch becomes redundant.

Synthesis. Bidirectional typing is a form of partial type inference, which Pierce and Turner [2000] said should "eliminate especially those type annotations that are both *common* and *silly*". But our rules are rather parsimonious in what they synthesize; for instance, () does not synthesize 1, and so might need an annotation. Fortunately, it would be straightforward to add such rules, following the style of Dunfield and Krishnaswami [2013].

5 ALGORITHMIC TYPING

Our algorithmic rules closely mimic our declarative rules, except that whenever a declarative rule would make a guess, the algorithmic rule adds to the context an existential variable (written with a hat $\hat{\alpha}$). As typechecking proceeds, we add solutions to the existential variables, reflecting increasing knowledge. Hence, each declarative typing judgment has a corresponding algorithmic judgment with an output context as well as an input context. The algorithmic type checking judgment $\Gamma \vdash e \Leftarrow A p \dashv \Delta$ takes an input context Γ and yields an output context Δ that includes increased knowledge about what the types have to be. The notion of increasing knowledge is formalized by a judgment $\Gamma \longrightarrow \Delta$ (Section 5.3).

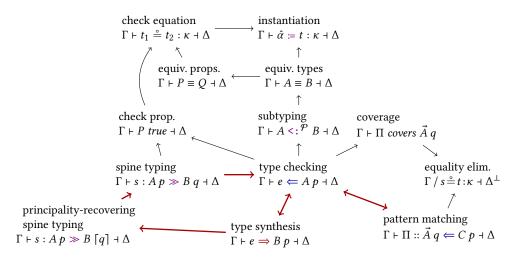


Fig. 10. Dependency structure of the algorithmic judgments

Figure 10 shows a dependency graph of the algorithmic judgments. Each declarative judgment has a corresponding algorithmic judgment, but the algorithmic system adds judgments such as type equivalence checking $\Gamma \vdash A \equiv B \dashv \Delta$ and variable instantiation $\Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta$. Declaratively, these judgments correspond to uses of reflexivity axioms; algorithmically, they correspond to solving existential variables to equate terms.

We give the algorithmic typing rules in Figure 14; rules for most other judgments are in the appendix. Our style of specification broadly follows Dunfield and Krishnaswami [2013]: we adapt their mechanisms of variable instantiation, context extension, and context application (to both types and other contexts). Our versions of these mechanisms, however, support indices, equations over universal variables, and the $\exists / \supset / \land$ connectives. We also differ in our formulation of spine typing, and by being able to track which types are principal.

5.1 Examples

To show how the spine typing rules recover principality, we present some example derivations.

Suppose we have an identity function id, defined in an algorithmic context Γ by the hypothesis $id: (\forall \alpha: \star. \alpha \to \alpha)!$. Since the hypothesis has!, the type of id is known to be principal. If we apply id to (), we expect to get something of unit type 1. Despite the \forall in the type of id, the resulting type should be principal, because no other type is possible. We can indeed derive that type:

$$\frac{(id:(\forall\alpha:\star.\alpha\to\alpha)!)\in\Gamma}{\Gamma\vdash id\Rightarrow(\forall\alpha:\star.\alpha\to\alpha)!+\Gamma} \text{ Var } \Gamma\vdash(()\cdot):(\forall\alpha:\star.\alpha\to\alpha)!\gg1\,\lceil!\rceil+\Gamma,\hat\alpha:\star=1}{\Gamma\vdash id\,(()\cdot)\Rightarrow1!+\Gamma,\hat\alpha:\star=1}\to E$$

(Here, we write the application id () as id (() ·), to show the structure of the spine as analyzed by the typing rules.) In the derivation of the second premise of $\rightarrow E$, shown below, we can follow

the evolution of the principality marker.

$$\frac{\Gamma, \hat{\alpha} : \star \vdash () \Leftarrow \hat{\alpha} \not f \dashv \Gamma, \hat{\alpha} : \star = 1}{\Gamma, \hat{\alpha} : \star \vdash () \cdot) : \hat{\alpha} \rightarrow \hat{\alpha} \not f} \xrightarrow{\Gamma, \hat{\alpha} : \star = 1 \vdash \cdot : 1 \not f \gg 1 \not f \dashv \Gamma, \hat{\alpha} : \star = 1} \xrightarrow{\Gamma, \hat{\alpha} : \star \vdash () \cdot) : \hat{\alpha} \rightarrow \hat{\alpha} \not f} \xrightarrow{\Gamma, \hat{\alpha} : \star \vdash () \vdash () \cdot : (\forall \alpha : \star \cdot \alpha \rightarrow \alpha)! \gg 1 \not f \dashv \Gamma, \hat{\alpha} : \star = 1} \xrightarrow{FEV(1) = \emptyset} SpineRecover$$

$$\frac{\Gamma \vdash (() \cdot) : (\forall \alpha : \star \cdot \alpha \rightarrow \alpha)! \gg 1 \not f \dashv \Gamma, \hat{\alpha} : \star = 1}{\Gamma \vdash () \cdot () : (\forall \alpha : \star \cdot \alpha \rightarrow \alpha)} \xrightarrow{I} \Rightarrow 1 \vdash [] \dashv \Gamma, \hat{\alpha} : \star = 1} \xrightarrow{SpineRecover}$$

- The input principality (marked "input") is !, because the input type $(\forall \alpha : \star. \alpha \to \alpha)$ was marked as principal in the hypothesis typing id.
- Rule SpineRecover begins by invoking the ordinary (non-recovering) spine judgment, passing all inputs unchanged, including the principality!.
- Rule \forall Spine adds an existential variable $\hat{\alpha}$ to represent the instantiation of the quantified type variable α , and substitutes $\hat{\alpha}$ for α . Since this instantiation is, in general, *not* principal, it replaces! with I (highlighted) in its premise. This marks the type $\hat{\alpha} \rightarrow \hat{\alpha}$ as non-principal.
- Rule \rightarrow Spine decomposes $\hat{\alpha} \rightarrow \hat{\alpha}$ and checks () against $\hat{\alpha}$, maintaining the principality I. Once principality is lost, it can only be recovered within the SpineRecover rule itself.
- Rule $11\hat{\alpha}$ notices that we are checking () against an unknown type $\hat{\alpha}$; since the expression is (), the type $\hat{\alpha}$ must be 1, so it adds that solution to its output context.
- Moving to the second premise of →Spine, we analyze the remaining part of the spine. That is
 just the empty spine ·, and rule EmptySpine passes its inputs along as outputs. In particular,
 the principality // is unchanged.
- The principalities are passed down to the conclusion of **∀Spine**, where *I* is highlighted.
- In SpineRecover, we notice that the output type 1 has no existential variables (FEV(1) = \emptyset), which allows us to recover principality of the output type: [!].

In the corresponding derivation in our declarative system, we have, instead, a check that no other types are derivable:

$$\frac{\Psi \vdash () \Leftarrow 1 \cancel{f}}{\Psi \vdash () \Leftrightarrow 1 \cancel{f}} \xrightarrow{\text{Decl11}} \frac{\Psi \vdash () \Rightarrow 1 \cancel{f}}{\Psi \vdash () \Rightarrow 1 \cancel{f}} \xrightarrow{\text{Decl} \to \text{Spine}} \text{ for all } C'. \text{ if } \Psi \vdash () \Rightarrow 1 \cancel{f} \Rightarrow 1 \cancel{$$

Here, we highlight the replacement in $Decl \forall Spine$ of the quantified type variable α by the "guessed" solution 1. The second premise of Decl Spine Recover checks that no other output type C' could have been produced, no matter what solution was chosen by $Decl \forall Spine$ for α .

Syntax. Expressions are the same as in the declarative system.

Existential variables. The algorithmic system adds existential variables $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ to types and terms/monotypes (Figure 11). We use the same meta-variables A, We write u for either a universal variable α or an existential variable $\hat{\alpha}$.

Contexts. An algorithmic context Γ is a sequence that, like a declarative context, may contain universal variable declarations $\alpha : \kappa$ and expression variable typings x : Ap. However, it may also have (1) *unsolved* existential variable declarations $\hat{\alpha} : \kappa$ (included in the $\Gamma, u : \kappa$ production); (2)

```
Universal variables
                                           \alpha, \beta, \gamma
Existential variables \hat{\alpha}, \hat{\beta}, \hat{\gamma}
Variables
                                                     u := \alpha \mid \hat{\alpha}
                                           A, B, C ::= 1 \mid A \rightarrow B \mid A + B \mid A \times B \mid \alpha \mid \hat{\alpha} \mid \forall \alpha : \kappa. A \mid \exists \alpha : \kappa. A
Types
                                                              | P \supset A | A \land P | \text{Vec } t A
                                               P,Q ::= t = t'
Propositions
                                                    \oplus ::= \rightarrow |+| \times
Binary connectives
                                            t, \tau, \sigma ::= \text{zero} \mid \text{succ}(t) \mid 1 \mid \alpha \mid \hat{\alpha} \mid \tau \rightarrow \sigma \mid \tau + \sigma \mid \tau \times \sigma
Terms/monotypes
                                          \Gamma, \Delta, \Theta ::= \cdot \mid \Gamma, u : \kappa \mid \Gamma, x : Ap \mid \Gamma, \hat{\alpha} : \kappa = \tau \mid \Gamma, \alpha = t \mid \Gamma, \blacktriangleright_u
Contexts
                                                    \Omega ::= \cdot \mid \Omega, \alpha : \kappa \mid \Omega, x : Ap \mid \Omega, \hat{\alpha} : \kappa = \tau \mid \Omega, \alpha = t \mid \Omega, \blacktriangleright_{u}
Complete contexts
                                            Possibly inconsistent contexts \Delta^{\perp} ::= \Delta \mid \perp
```

Fig. 11. Syntax of types, contexts, and other objects in the algorithmic system

```
 [\Gamma]\alpha \ = \ \begin{cases} [\Gamma]\tau \text{ when } (\alpha = \tau) \in \Gamma \\ \alpha \text{ otherwise} \end{cases} \qquad \begin{bmatrix} \Gamma[\hat{\alpha} : \kappa = \tau] \end{bmatrix} \hat{\alpha} \ = \ [\Gamma]\tau \\ [\Gamma[\hat{\alpha} : \kappa]] \hat{\alpha} \ = \ \hat{\alpha} \end{cases}   [\Gamma](P \supset A) \ = ([\Gamma]P) \supset ([\Gamma]A) \\ [\Gamma](A \land P) \ = ([\Gamma]A) \land ([\Gamma]P) \\ [\Gamma](A \oplus B) \ = ([\Gamma]A) \oplus ([\Gamma]B) \end{cases} \qquad [\Gamma](\forall \alpha : \kappa. \ A) \ = \ \forall \alpha : \kappa. \ [\Gamma]A \\ [\Gamma](\exists \alpha : \kappa. \ A) \ = \ \exists \alpha : \kappa. \ [\Gamma]A \\ [\Gamma](\exists \alpha : \kappa. \ A) \ = \ \exists \alpha : \kappa. \ [\Gamma]A \\ [\Gamma](\exists \alpha : \kappa. \ A) \ = \ \exists \alpha : \kappa. \ [\Gamma]A \end{cases}   [\Gamma](\exists \alpha : \kappa. \ A) \ = \ \exists \alpha : \kappa. \ [\Gamma]A \\ [\Gamma](\exists \alpha : \kappa. \ A) \ = \ \exists \alpha : \kappa. \ [\Gamma]A \end{cases}
```

Fig. 12. Applying a context, as a substitution, to a type

solved existential variable declarations $\hat{\alpha} : \kappa = \tau$; (3) equations over universal variables $\alpha = \tau$; and (4) markers \blacktriangleright_u . An equation $\alpha = \tau$ must appear to the right of the universal variable's declaration $\alpha : \kappa$. We use markers as delimiters within contexts. For example, rule \supset I adds \blacktriangleright_P , which tells it how much of its last premise's output context $(\Delta, \blacktriangleright_P, \Delta')$ should be dropped. (We abuse notation by writing \blacktriangleright_P rather than cluttering the context with a dummy α and writing $\blacktriangleright_\alpha$.)

A complete algorithmic context, denoted by Ω , is an algorithmic context with no unsolved existential variable declarations.

Assuming an equality can yield inconsistency: for example, zero = succ(zero). We write Δ^{\perp} for either a valid algorithmic context Δ or inconsistency \perp .

5.2 Context substitution $[\Gamma]A$ and hole notation $\Gamma[\Theta]$

An algorithmic context can be viewed as a substitution for its solved existential variables. For example, $\hat{\alpha} = 1$, $\hat{\beta} = \hat{\alpha} \rightarrow 1$ can be applied as if it were the substitution $1/\hat{\alpha}$, $(\hat{\alpha} \rightarrow 1)/\hat{\beta}$ (applied right to left), or the simultaneous substitution $1/\hat{\alpha}$, $(1\rightarrow 1)/\hat{\beta}$. We write $[\Gamma]A$ for Γ applied as a substitution (Figure 12).

Applying a complete context to a type A (provided it is well-formed: $\Omega \vdash A$ type) yields a type $[\Omega]A$ with no existentials. Such a type is well-formed under the *declarative* context obtained by dropping all the existential declarations and applying Ω to declarations x:A (to yield $x:[\Omega]A$). We can think of this context as the result of applying Ω to itself: $[\Omega]\Omega$. More generally, we can apply Ω to any context Γ that it extends: context application $[\Omega]\Gamma$ is given in Figure 13. The application $[\Omega]\Gamma$ is defined if and only if $\Gamma \longrightarrow \Omega$ (context extension; see Section 5.3), and applying Ω to any such Γ yields the same declarative context $[\Omega]\Omega$.

```
\begin{split} & [\cdot] \cdot & = \cdot \\ & [\Omega,x:Ap](\Gamma,x:A_{\Gamma}p) = [\Omega]\Gamma, \ x:[\Omega]Ap \ \text{if} \ [\Omega]A = [\Omega]A_{\Gamma} \\ & [\Omega,\alpha:\kappa](\Gamma,\alpha:\kappa) = [\Omega]\Gamma, \ \alpha:\kappa \\ & [\Omega,\blacktriangleright_u](\Gamma,\blacktriangleright_u) = [\Omega]\Gamma \\ & [\Omega,\alpha=t](\Gamma,\alpha=t') = \left[ [\Omega]t/\alpha \right] [\Omega]\Gamma \ \text{if} \ [\Omega]t = [\Omega]t' \\ & [\Omega,\hat{\alpha}:\kappa=t]\Gamma & = \begin{cases} [\Omega]\Gamma' \ \text{when} \ \Gamma = (\Gamma',\hat{\alpha}:\kappa=t') \\ [\Omega]\Gamma' \ \text{when} \ \Gamma = (\Gamma',\hat{\alpha}:\kappa) \end{cases} \end{split}
```

Fig. 13. Applying a complete context Ω to a context

In addition to appending declarations (as in the declarative system), we sometimes insert and replace declarations, so a notation for contexts with a hole is useful: $\Gamma = \Gamma_0[\Theta]$ means Γ has the form $(\Gamma_L, \Theta, \Gamma_R)$. For example, if $\Gamma = \Gamma_0[\hat{\beta}] = (\hat{\alpha}, \hat{\beta}, x : \hat{\beta})$, then $\Gamma_0[\hat{\beta} = \hat{\alpha}] = (\hat{\alpha}, \hat{\beta} = \hat{\alpha}, x : \hat{\beta})$.

We also use contexts with *two* ordered holes: if $\Gamma = \Gamma_0[\Theta_1][\Theta_2]$ then $\Gamma = (\Gamma_L, \Theta_1, \Gamma_M, \Theta_2, \Gamma_R)$.

5.3 The context extension relation $\Gamma \longrightarrow \Delta$

A context Γ is extended by a context Δ , written $\Gamma \longrightarrow \Delta$, if Δ has at least as much information as Γ , while conforming to the same declarative context—that is, $[\Omega]\Gamma = [\Omega]\Delta$ for some Ω . In a sense, $\Gamma \longrightarrow \Delta$ says that Γ is entailed by Δ : all positive information derivable from Γ can also be derived from Δ (which may have more information, say, that $\hat{\alpha}$ is equal to a particular type). We give the rules for extension in Figure 15.

The rules deriving the context extension judgment (Figure 15) say that the empty context extends the empty context (\longrightarrow Id); a term variable typing with A' extends one with A if applying the extending context Δ to A and A' yields the same type (\longrightarrow Var); universal variable declarations and equations must match (\longrightarrow Uvar, \longrightarrow Eqn); scope markers must match (\longrightarrow Marker); and, existential variables may either match (\longrightarrow Unsolved, \longrightarrow Solved), get solved by the extending context (\longrightarrow Solve), or be added by the extending context (\longrightarrow Add, \longrightarrow AddSolved).

Extension may change solutions, if information is preserved or increased: $(\hat{\alpha}: \star, \hat{\beta}: \star = \hat{\alpha}) \rightarrow (\hat{\alpha}: \star = 1, \hat{\beta}: \star = \hat{\alpha})$ directly increases information about $\hat{\alpha}$, and indirectly increases information about $\hat{\beta}$. More interestingly, if $\Delta = (\hat{\alpha}: \star = 1, \hat{\beta}: \star = \hat{\alpha})$ and $\Omega = (\hat{\alpha}: \star = 1, \hat{\beta}: \star = 1)$, then $\Delta \longrightarrow \Omega$: while the solution of $\hat{\beta}$ in Ω is different, in the sense that Ω contains $\hat{\beta}: \star = 1$ while Δ contains $\hat{\beta}: \star = \hat{\alpha}$, applying Ω to the solutions gives the same result: $[\Omega]\hat{\alpha} = [\Omega]1 = 1$, the same as $[\Omega]1 = 1$.

Extension is quite rigid, however, in two senses. First, if a declaration appears in Γ , it appears in all extensions of Γ . Second, *extension preserves order*. For example, if $\hat{\beta}$ is declared after $\hat{\alpha}$ in Γ , then $\hat{\beta}$ will also be declared after $\hat{\alpha}$ in every extension of Γ . This holds for every variety of declaration, including equations of universal variables. This rigidity aids in enforcing type variable scoping and dependencies, which are nontrivial in a setting with higher-rank polymorphism.

5.4 Determinacy

Given appropriate inputs (Γ, e, A, p) to the algorithmic judgments, only one set of outputs (C, q, Δ) is derivable (Theorem 5 in the supplementary material, p. 30). We use this property (for spine judgments) in the proof of soundness.

Fig. 14. Algorithmic typing, omitting rules for \times , +, and Vec

Fig. 15. Context extension

6 SOUNDNESS

We show that the algorithmic system is sound with respect to the declarative system. Soundness for the mutually recursive judgments depends on lemmas for the auxiliary judgments (instantiation, equality elimination, checkprop, algorithmic subtyping and match coverage), which are in Appendix J for space reasons. The main soundness result has mutually recursive parts for checking, synthesis, spines and matching—including the principality-recovering spine judgment.

Theorem 6.8 (Soundness of Algorithmic Typing). Given $\Delta \longrightarrow \Omega$:

```
(i) If Γ + e ← A p + Δ and Γ + A p type then [Ω]Δ + [Ω]e ← [Ω]A p.
(ii) If Γ + e ⇒ A p + Δ then [Ω]Δ + [Ω]e ⇒ [Ω]A p.
(iii) If Γ + s : A p ≫ B q + Δ and Γ + A p type then [Ω]Δ + [Ω]s : [Ω]A p ≫ [Ω]B q.
(iv) If Γ + s : A p ≫ B [q] + Δ and Γ + A p type then [Ω]Δ + [Ω]s : [Ω]A p ≫ [Ω]B [q].
(v) If Γ + Π :: A q ← C p + Δ and Γ + A q types and [Γ]A = A and Γ + C p type then [Ω]Δ + [Ω]Π :: [Ω]A q ← [Ω]C p.
(vi) If Γ / P + Π :: A! ← C p + Δ and Γ + P prop and FEV(P) = Ø and [Γ]P = P and Γ + A! types and Γ + C p type then [Ω]Δ / [Ω]P + [Ω]Π :: [Ω]A! ← [Ω]C p.
```

Much of this proof "turns the crank": apply the induction hypothesis to each premise, yielding derivations of corresponding declarative judgments (with Ω applied everywhere), then apply the corresponding declarative rule; for example, in the Sub case we finish by applying DeclSub. However, in the SpineRecover case we finish by applying DeclSpineRecover, but since DeclSpineRecover contains a premise that quantifies over all declarative derivations of a certain form, we must appeal to completeness! Consequently, soundness and completeness are really one theorem.

These parts are mutually recursive—later, we'll see that the DeclSpineRecover case of completeness must appeal to soundness (to show that the algorithmic type has no free existential variables). We cannot induct on the given derivation alone, because the derivations in the "for all" part of DeclSpineRecover are not subderivations. So we need a more involved induction measure that can make the leaps between soundness and completeness: lexicographic order with (1) the size of the subject term, (2) the judgment form, with ordinary spine judgments considered smaller than

recovering spine judgments, and (3) the height of the derivation:

$$\left\langle \begin{array}{ccc} & \text{ordinary spine judgment} \\ e/s/\Pi, & < & , & \text{height}(\mathcal{D}) \\ & \text{recovering spine judgment} \end{array} \right\rangle$$

Proof sketch—SpineRecover *case.* By i.h., $[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A ! \gg [\Omega]C q$. Our goal is to apply DeclSpineRecover, which requires that we show that for *all* C' such that $[\Omega]\Theta \vdash s : [\Omega]A ! \gg C' \c I$, we have $C' = [\Omega]C$. Suppose we have such a C'. By completeness (Theorem 12), $\Gamma \vdash s : [\Gamma]A ! \gg C'' \q \dashv \Delta''$ where $\Delta'' \longrightarrow \Omega''$. We already have (as a subderivation) $\Gamma \vdash s : A ! \gg C \c I \dashv \Delta$, so by determinacy, C'' = C and q = I and $\Delta'' = \Delta$. With the help of lemmas about context application, we can show $C' = [\Omega'']C'' = [\Omega'']C = [\Omega]C$. (Using completeness is permitted since our measure says a non-principality-restoring judgment is smaller.)

6.1 Auxiliary Soundness

For several auxiliary judgment forms, soundness is a matter of showing that, given two algorithmic terms, their declarative versions are equal. For example, for the instantiation judgment we have: **Lemma** (Soundness of Instantiation).

If $\Gamma \vdash \hat{\alpha} := \tau : \kappa \vdash \Delta$ and $\hat{\alpha} \notin FV([\Gamma]\tau)$ and $[\Gamma]\tau = \tau$ and $\Delta \longrightarrow \Omega$ then $[\Omega]\hat{\alpha} = [\Omega]\tau$.

We have similar lemmas for term equality $(\Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta)$, propositional equivalence $(\Gamma \vdash P \equiv Q \dashv \Delta)$ and type equivalence $(\Gamma \vdash A \equiv B \dashv \Delta)$.

Our eliminating judgments incorporate assumptions into the context Γ . We show that the algorithmic rules for these judgments just append equations over universal variables:

Lemma (Soundness of Equality Elimination). *If* $[\Gamma]\sigma = \sigma$ *and* $[\Gamma]t = t$ *and* $\Gamma \vdash \sigma : \kappa$ *and* $\Gamma \vdash t : \kappa$ *and* $FEV(\sigma) \cup FEV(t) = \emptyset$, *then*:

- (1) If $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa + \Delta$ then $\Delta = (\Gamma, \Theta)$ where $\Theta = (\alpha_1 = t_1, \dots, \alpha_n = t_n)$ and for all Ω such that $\Gamma \longrightarrow \Omega$ and all t' s.t. $\Omega \vdash t' : \kappa'$ we have $[\Omega, \Theta]t' = [\theta][\Omega]t'$ where $\theta = \text{mgu}(\sigma, t)$.
- (2) If $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot$ then no most general unifier exists.

The last lemmas for soundness move directly from an algorithmic judgment to the corresponding declarative judgment.

Lemma (Soundness of Checkprop). *If* $\Gamma \vdash P$ *true* $\dashv \Delta$ *and* $\Delta \longrightarrow \Omega$ *then* $\Psi \vdash [\Omega]P$ *true*. **Lemma** (Soundness of Match Coverage).

- (1) If $\Gamma \vdash \Pi$ covers \vec{A} q and $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash \vec{A}$! types and $[\Gamma]\vec{A} = \vec{A}$ then $[\Omega]\Gamma \vdash \Pi$ covers \vec{A} q.
- (2) If $\Gamma / P \vdash \Pi$ covers \vec{A} ! and $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash \vec{A}$! types and $[\Gamma]\vec{A} = \vec{A}$ and $[\Gamma]P = P$ then $[\Omega]\Gamma / P \vdash \Pi$ covers \vec{A} !.

Theorem 6.9 (Soundness of Algorithmic Subtyping). If $[\Gamma]A = A$ and $[\Gamma]B = B$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $\Delta \longrightarrow \Omega$ and $\Gamma \vdash A <: ^{\mathcal{P}}B + \Delta$ then $[\Omega]\Delta \vdash [\Omega]A \leq ^{\mathcal{P}}[\Omega]B$.

7 COMPLETENESS

We show that the algorithmic system is complete with respect to the declarative system. As with soundness, we need to show completeness of the auxiliary algorithmic judgments. We omit the full statements of these lemmas; as an example, if $[\Omega]\hat{\alpha} = [\Omega]\tau$ and $\hat{\alpha} \notin FV(\tau)$ then $\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$.

7.1 Separation

To show completeness, we will need to show that wherever the declarative rule DeclSpineRecover is applied, we can apply the algorithmic rule SpineRecover. Thus, we need to show that *semantic* principality—that no other type can be given—entails that a type has no free existential variables.

The principality-recovering rules are potentially applicable when we start with a principal type A! but produce C I, with DeclySpine changing! to I. Completeness (Thm. 12) will use the "for all" part of DeclSpineRecover, which quantifies over all types produced by the spine rules under a given declarative context $[\Omega]\Gamma$. By i.h. we get an algorithmic spine judgment $\Gamma \vdash s:A'! \gg C' I \vdash \Delta$. Since A' is principal, unsolved existentials in C' must have been introduced within this derivation—they can't be in Γ already. Thus, we might have $\hat{\alpha}: \star \vdash s:A'! \gg \hat{\beta}I \vdash \hat{\alpha}: \star, \hat{\beta}: \star$ where a DeclySpine subderivation introduced $\hat{\beta}$, but $\hat{\alpha}$ can't appear in C'. We also can't equate $\hat{\alpha}$ and $\hat{\beta}$ in Δ , which would be tantamount to $C' = \hat{\alpha}$. Knowing that unsolved existentials in C' are "new" and independent from those in Γ means we can argue that, if there were an unsolved existential in C', it would correspond to an unforced choice in a DeclySpine subderivation, invalidating the "for all" part of DeclSpineRecover. Formalizing "must have been introduced" requires several definitions.

Definition 7.1 (Separation). An algorithmic context Γ is *separable into* $\Gamma_L * \Gamma_R$ if (1) $\Gamma = (\Gamma_L, \Gamma_R)$ and (2) for all $(\hat{\alpha} : \kappa = \tau) \in \Gamma_R$ it is the case that $\mathsf{FEV}(\tau) \subseteq \mathsf{dom}(\Gamma_R)$.

If Γ is separable into $\Gamma_L * \Gamma_R$, then Γ_R is self-contained in the sense that all existential variables declared in Γ_R have solutions whose existential variables are themselves declared in Γ_R . Every context Γ is separable into $\cdot * \Gamma$ and into $\Gamma * \cdot$.

Definition 7.2 (Separation-Preserving Ext.). Separated context $\Gamma_L * \Gamma_R$ extends to $\Delta_L * \Gamma_R$, written $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$, if $(\Gamma_L, \Gamma_R) \longrightarrow (\Delta_L, \Delta_R)$ and $dom(\Gamma_L) \subseteq dom(\Delta_L)$ and $dom(\Gamma_R) \subseteq dom(\Delta_R)$.

Separation-preserving extension says that variables from one side of * haven't "jumped" to the other side. Thus, Δ_L may add existential variables to Γ_L , and Δ_R may add existential variables to Γ_R , but no variable from Γ_L ends up in Δ_R and no variable from Γ_R ends up in Δ_L . It is necessary to write $(\Gamma_L * \Gamma_R) \xrightarrow{} (\Delta_L * \Delta_R)$ rather than $(\Gamma_L * \Gamma_R) \xrightarrow{} (\Delta_L * \Delta_R)$, because only $\xrightarrow{}$ includes the domain conditions. For example, $(\hat{\alpha} * \hat{\beta}) \xrightarrow{} (\hat{\alpha}, \hat{\beta} = \hat{\alpha}) * \cdot$, but $\hat{\beta}$ has jumped to the left of * in the context $(\hat{\alpha}, \hat{\beta} = \hat{\alpha}) * \cdot$.

We prove many lemmas about separation, but use only one of them in the subsequent development (in the DeclSpineRecover case of typing completeness), and then only the part for spines. It says that if we have a spine whose type A mentions only variables in Γ_R , then the output context Δ extends Γ and preserves separation, and the output type C mentions only variables in Δ_R :

7.2 Completeness of typing

Like soundness, completeness has several mutually recursive parts (see the appendix, p. 36).

Theorem 7.11 (Completeness of Algorithmic Typing). Given $\Gamma \longrightarrow \Omega$ s.t. dom $(\Gamma) = \text{dom}(\Omega)$:

- (i) If $\Gamma \vdash A p$ type and $[\Omega]\Gamma \vdash [\Omega]e \Leftarrow [\Omega]A p$ and $p' \sqsubseteq p$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\operatorname{dom}(\Delta) = \operatorname{dom}(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Leftarrow [\Gamma]A p' \dashv \Delta$.
- (ii) If $\Gamma \vdash A \ p \ type \ and \ [\Omega] \Gamma \vdash [\Omega] e \Rightarrow A \ p \ then \ there \ exist \ \Delta, \ \Omega', A', \ and \ p' \sqsubseteq p \ such \ that \ \Delta \longrightarrow \Omega'$ and dom(\Delta) = dom(\Omega') \ and \Omega \lefta \Omega' \ and \Gamma \! \text{e} \Rightarrow A' \ p' \ \text{d} \ and \ A' = [\Delta] A' \ and \ A = [\Omega'] A'.
- (iii) If $\Gamma \vdash A \ p \ type \ and \ [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A \ p \gg B \ q \ and \ p' \sqsubseteq p \ then \ there \ exist \ \Delta, \ \Omega', B', \ and \ q' \sqsubseteq q \ such \ that \ \Delta \longrightarrow \Omega' \ and \ dom(\Delta) = dom(\Omega') \ and \ \Omega \longrightarrow \Omega' \ and \ \Gamma \vdash s : [\Gamma]A \ p' \gg B' \ q' \dashv \Delta \ and \ B' = [\Delta]B' \ and \ B = [\Omega']B'.$
- (iv) As part (iii), but with $\gg B \lceil q \rceil \cdots$ and $\gg B' \lceil q' \rceil \cdots$.

Proof sketch—DeclSpineRecover case. By i.h., $\Gamma \vdash s : [\Gamma]A ! \gg C' \not \vdash A$ where $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\operatorname{dom}(\Delta) = \operatorname{dom}(\Omega')$ and $C = [\Omega']C'$.

To apply SpineRecover, we need to show FEV($[\Delta]C'$) = \emptyset . Suppose, for a contradiction, that FEV($[\Delta]C'$) $\neq \emptyset$. Construct a variant of Ω' called Ω_2 that has a different solution for some $\hat{\alpha} \in \text{FEV}([\Delta]C')$. By soundness (Thm. 12), $[\Omega_2]\Gamma \vdash [\Omega_2]s : [\Omega_2]A ! \gg [\Omega_2]C' \not$. Using a separation lemma with the trivial $\Gamma = (\Gamma * \cdot)$ we get $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma * \cdot) \xrightarrow{*} (\Delta_L * \Delta_R)$ and FEV(C') $\subseteq \text{dom}(\Delta_R)$. That is, all existentials in C' were introduced within the derivation of the (algorithmic) spine judgment. Thus, applying Ω_2 to things gives the same result as Ω , except for C', giving $[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A ! \gg [\Omega_2]C' \not$. Now instantiate the "for all C_2 " premise with $C_2 = [\Omega_2]C'$, giving $C = [\Omega_2]C'$. But we chose Ω_2 to have a different solution for $\hat{\alpha} \in \text{FEV}(C')$, so we have $C \neq [\Omega_2]C'$: Contradiction. Therefore FEV($[\Delta]C'$) = \emptyset , so we can apply SpineRecover.

8 DISCUSSION AND RELATED WORK

A staggering amount of work has been done on GADTs and indexed types, and for space reasons we cannot offer a comprehensive survey of the literature. So we compare more deeply to fewer papers, to communicate our understanding of the design space.

Proof theory and type theory. As described in Section 1, there are two logical accounts of equality—the identity type of Martin-Löf and the equality type of Schroeder-Heister [1994] and Girard [1992]. The Girard/Schroeder-Heister equality has a more direct connection to pattern matching, which is why we make use of it. Coquand [1996] pioneered the study of pattern matching in dependent type theory. One perhaps surprising feature of Coquand's pattern-matching syntax is that it is strictly stronger than Martin-Löf's eliminators. His rules can derive the uniqueness of identity proofs as well as the disjointness of constructors. Constructor disjointness is also derivable from the Girard/Schroeder-Heister equality, because there is no unifier for two distinct constructors.

In future work, we hope to study the relation between these two notions of equality in more depth; richer equational theories (such as the theory of commutative rings or the $\beta\eta$ -theory of the lambda calculus) do not have decidable unification, but it seems plausible that there are hybrid approaches which might let us retain some of the convenience of the G/SH equality rule while retaining the decidability of Martin-Löf's J eliminator.

Indexed and refinement types. Dependent ML [Xi and Pfenning 1999] indexed programs with propositional constraints, extending the ML type discipline to maintain additional invariants. DML collected constraints from the program and passed them to a constraint solver, a technique used by systems like Stardust [Dunfield 2007a] and liquid types [Rondon et al. 2008].

From phantom types to GADTs. Leijen and Meijer [1999] introduced the term phantom type to describe a technique for programming in ML/Haskell where additional type parameters are used to constrain when values are well-typed. This idea proved to have many applications, ranging from foreign function interfaces [Blume 2001] to encoding Java-style subtyping [Fluet and Pucella 2006]. Phantom types allow constructing values with constrained types, but do not easily permit learning about type equalities by analyzing them, putting applications such as intensional type analysis [Harper and Morrisett 1995] out of reach. Both Cheney and Hinze [2003] and Xi et al. [2003] proposed treating equalities as a first-class concept, giving explicitly-typed calculi for equalities, but without studying algorithms for type inference.

Simonet and Pottier [2007] gave a constraint-based algorithm for type inference for GADTs. It is this work which first identified the potential intractibility of type inference arising from the interaction of hypothetical constraints and unification variables. To resolve this issue they introduce the notion of *tractable* constraints (i.e., constraints where hypothetical equations never contain existentials), and require placing enough annotations that all constraints are tractable. In general,

this could require annotations on case expressions, so subsequent work focused on relaxing this requirement. Though quite different in technical detail, stratified inference [Pottier and Régis-Gianas 2006] and *wobbly types* [Peyton Jones et al. 2006] both work by pushing type information from annotations to case expressions, with stratified type inference literally moving annotations around, and wobbly types tracking which parts of a type have no unification variables. Modern GHC uses the OutsideIn algorithm [Vytiniotis et al. 2011], which further relaxes the constraint: case analysis is permitted as long as it cannot modify what is known about an equation.

In our type system, the checking judgment of the bidirectional algorithm serves to propagate annotations; our requirement that the scrutinee of a case expression be principal ensures that no equations contain unification variables. The result is close in effect to stratified types, and is less expressive than OutsideIn. This is a deliberate design choice to keep the meaning of principality—that only a single type can be inferred for a term—clear and easy to understand.

To specify the OutsideIn approach, the case rule in our declarative system should permit scrutinizing an expression if all types that can be synthesized for it have exactly the same equations, even if they differ in their monotype parts. To achieve this, we would need to introduce a relation $C' \sim C$ which checks whether the equational constraints in C and C' are the same, and then modify the higher-order premise of the DeclSpineRecover rule to check that $C' \sim C$ (rather than C' = C, as it is currently). However, we thought such a spec is harder for programmers to develop an intuition for than simply saying that a scrutinee must synthesize a unique type.

Garrigue and Rémy [2013] proposed *ambivalent types*, which are a way of deciding when it is safe to generalize the type of a function using GADTs. This idea is orthogonal to our calculus, simply because we do no generalization at all: *every* polymorphic function takes an annotation. However, Garrigue and Rémy [2013] also emphasize the importance of *monotonicity*, which says that substitution should be stable under subtyping, that is, giving a more general type should not cause subtyping to fail. This condition is satisfied by our bidirectional system.

Karachalias et al. [2015] developed a coverage algorithm for GADTs that depends on external constraint solving; we offer a more self-contained but still logically-motivated approach.

Polarized subtyping. Barendregt et al. [1983] observed that a program which typechecks under a subtyping discipline can be checked without subtyping, provided that the program is sufficiently η -expanded. This idea of subtyping as η -expansion was investigated in a focused (albeit infinitary) setting by Zeilberger [2009]. Another notion of polarity arises from considering the (co-, contra-, in-)variance of type constructors. It is used by Abel [2006] to give a version of F^{ω} with subtyping, and Dolan and Mycroft [2017] apply this version of polarity to give a complete type inference algorithm for an ML-style language with subtyping. Our polarized subtyping judgment is closest in spirit to the work of Zeilberger [2009]. The restriction on our subtyping relation can be understood in terms of requiring the η expansions our subtyping relation infers to be in a focused normal form.

Extensions. To keep our formalization manageable, we left out some features that would be desirable in practice. In particular, we need (1) type constructors which take arguments and (2) recursive types [Pierce 2002, chapter 20]. The issue with both of these features is that they need to permit instantiating quantifiers with existentials and other binders, and our system relies upon monotypes (which do not contain such connectives). This limitation should create no difficulties in typical practice if we treat user-defined type constructors like List as monotypes, expanding the definition only as needed: when checking an expression against a user type constructor, and for pattern matching. Another extension, which we intend as future work, is to replace ordinary unification with pattern or nominal unification, to allow type instantiations containing binders.

Another extension is to increase the amount of type inference done. For instance, a natural question is whether we can extend the bidirectional approach to subsume the inference done by the

algorithm of Damas and Milner [1982]. On the implementation side, this seems easy—to support ML-style type inference, we can add rules to infer types for values:

$$\frac{\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e \Leftarrow \hat{\beta} \not \ \ \, | + \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta' \qquad \vec{\hat{\gamma}} = \mathsf{unsolved}(\Delta')}{\Gamma \vdash \lambda x. \ \, e \Rightarrow \forall \vec{\alpha}. \ \, [\vec{\alpha}/\vec{\hat{\gamma}}][\Delta'](\hat{\alpha} \to \hat{\beta}) \ \, ! \vdash \Delta}$$

This rule adds a marker $\blacktriangleright_{\hat{\alpha}}$ to the context, then checks the body e against the type $\hat{\beta}$. Our output type substitutes away all the solved existential variables to the right of $\blacktriangleright_{\hat{\alpha}}$, and generalizes over all unsolved variables to the right of the marker. Using an ordered context gives precise control over the scope of the existential variables, easily expressing polymorphic generalization.

However, in the presence of generalization, the declarative specification of type inference no longer strictly specifies the order of polymorphic quantifiers (i.e., $\forall \alpha, \beta$. $\alpha \to \beta \to (\alpha \times \beta)$ and $\forall \beta, \alpha. \alpha \to \beta \to (\alpha \times \beta)$ should be equivalent) and so our principal synthesis would no longer return types stable up to alpha-equivalence. Fixing this would be straightforward (by relaxing the definition of type equivalence), but we have not pursued this because we do not value let-generalization enough to pay the price of increased complexity in our proofs.

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REFERENCES

Andreas Abel. 2006. Towards Generic Programming with Sized Types. In *Mathematics of Program Construction (LNCS)*, Tarmo Uustalu (Ed.), Vol. 4014. Springer, 10–28.

Andreas Abel, Thierry Coquand, and Peter Dybjer. 2008. Verifying a Semantic $\beta\eta$ -Conversion Test for Martin-Löf Type Theory. In *Mathematics of Program Construction (MPC'08) (LNCS)*, Vol. 5133. Springer, 29–56.

Henk Barendregt, Mario Coppo, and Mariangiola Dezani-Ciancaglini. 1983. A Filter Lambda Model and the Completeness of Type Assignment. J. Symbolic Logic 48, 4 (1983), 931–940.

Matthias Blume. 2001. No-Longer-Foreign: Teaching an ML compiler to speak C "natively". *Electronic Notes in Theoretical Computer Science* 59, 1 (2001).

James Cheney and Ralf Hinze. 2003. First-Class Phantom Types. Technical Report CUCIS TR2003-1901. Cornell University. Jacek Chrząszcz. 1998. Polymorphic Subtyping Without Distributivity. In Mathematical Foundations of Computer Science (LNCS), Vol. 1450. Springer, 346–355.

Thierry Coquand. 1996. An Algorithm for Type-Checking Dependent Types. Science of Computer Programming 26, 1–3 (1996), 167–177.

Luis Damas and Robin Milner. 1982. Principal type-schemes for functional programs. In POPL. ACM Press, 207-212.

Rowan Davies and Frank Pfenning. 2000. Intersection Types and Computational Effects. In ICFP. ACM Press, 198-208.

Stephen Dolan and Alan Mycroft. 2017. Polymorphism, Subtyping, and Type Inference in MLsub. In *POPL*. ACM Press, 60–72.

Joshua Dunfield. 2007a. Refined typechecking with Stardust. In Programming Languages meets Programming Verification (PLPV '07). ACM Press, 21–32.

Joshua Dunfield. 2007b. A Unified System of Type Refinements. Ph.D. Dissertation. Carnegie Mellon University. CMU-CS-07-129.

Joshua Dunfield and Neelakantan R. Krishnaswami. 2013. Complete and Easy Bidirectional Typechecking for Higher-Rank Polymorphism. In *ICFP*. ACM Press, 429–442. arXiv:1306.6032 [cs.PL].

Joshua Dunfield and Frank Pfenning. 2003. Type Assignment for Intersections and Unions in Call-by-Value Languages. In FoSSaCS. Springer, 250–266.

Matthew Fluet and Riccardo Pucella. 2006. Phantom types and subtyping. (2006). arXiv:cs/0403034 [cs.PL].

Jacques Garrigue and Jacques Le Normand. 2015. GADTs and Exhaustiveness: Looking for the Impossible. In *Proceedings ML Family / OCaml Users and Developers workshops, ML Family/OCaml 2015 (EPTCS).* 23–35.

Jacques Garrigue and Didier Rémy. 2013. Ambivalent Types for Principal Type Inference with GADTs. In *APLAS*. Springer, 257–272.

Jean-Yves Girard. 1992. A Fixpoint Theorem in Linear Logic. (1992). Post to Linear Logic mailing list, http://www.seas.upenn.edu/~sweirich/types/archive/1992/msg00030.html.

Robert Harper and Greg Morrisett. 1995. Compiling polymorphism using intensional type analysis. In *POPL*. ACM Press, 130–141.

Georgios Karachalias, Tom Schrijvers, Dimitrios Vytiniotis, and Simon Peyton Jones. 2015. GADTs Meet Their Match: pattern-matching warnings that account for GADTs, guards, and laziness. In *ICFP*. ACM Press, 424–436.

Neelakantan R. Krishnaswami. 2009. Focusing on Pattern Matching. In POPL. ACM Press, 366-378.

Konstantin Läufer and Martin Odersky. 1994. Polymorphic type inference and abstract data types. *ACM Trans. Prog. Lang.* Sys. 16, 5 (1994), 1411–1430.

Daan Leijen and Erik Meijer. 1999. Domain specific embedded compilers. In *USENIX Conf. Domain-Specific Languages (DSL '99)*. ACM Press, 109–122.

Martin Odersky, Matthias Zenger, and Christoph Zenger. 2001. Colored Local Type Inference. In *POPL*. ACM Press, 41–53. Simon Peyton Jones, Dimitrios Vytiniotis, Stephanie Weirich, and Mark Shields. 2007. Practical type inference for arbitrary-rank types. *J. Functional Programming* 17, 1 (2007), 1–82.

Simon Peyton Jones, Dimitrios Vytiniotis, Stephanie Weirich, and Geoffrey Washburn. 2006. Simple unification-based type inference for GADTs. In *ICFP*. ACM Press, 50–61.

Brigitte Pientka. 2008. A type-theoretic foundation for programming with higher-order abstract syntax and first-class substitutions. In *POPL*. ACM Press, 371–382.

Benjamin C. Pierce. 2002. Types and Programming Languages. MIT Press.

Benjamin C. Pierce and David N. Turner. 2000. Local Type Inference. ACM Trans. Prog. Lang. Sys. 22 (2000), 1-44.

François Pottier and Yann Régis-Gianas. 2006. Stratified type inference for generalized algebraic data types. In *POPL*. ACM Press. 232–244.

Patrick Rondon, Ming Kawaguchi, and Ranjit Jhala. 2008. Liquid types. In PLDI. ACM Press, 159-169.

Peter Schroeder-Heister. 1994. Definitional reflection and the completion. In *Extensions of Logic Programming (LNCS)*. Springer, 333–347.

Vincent Simonet and François Pottier. 2007. A constraint-based approach to guarded algebraic data types. ACM Transactions on Programming Languages and Systems (TOPLAS) 29, 1 (2007), 1.

Jerzy Tiuryn and Paweł Urzyczyn. 1996. The Subtyping Problem for Second-Order Types is Undecidable. In *LICS*. IEEE Press.

Dimitrios Vytiniotis, Simon Peyton Jones, and Tom Schrijvers. 2010. Let should not be generalised. In *Workshop on Types in Language Design and Impl. (TLDI '10)*. ACM Press, 39–50.

Dimitrios Vytiniotis, Simon Peyton Jones, Tom Schrijvers, and Martin Sulzmann. 2011. OutsideIn(X): Modular type inference with local assumptions. *J. Functional Programming* 21, 4–5 (2011), 333–412.

Kevin Watkins, Iliano Cervesato, Frank Pfenning, and David Walker. 2004. A Concurrent Logical Framework: The Propositional Fragment. In *Types for Proofs and Programs*. Springer LNCS 3085, 355–377.

Hongwei Xi, Chiyan Chen, and Gang Chen. 2003. Guarded recursive datatype constructors. In *POPL*. ACM Press, 224–235. Hongwei Xi and Frank Pfenning. 1999. Dependent Types in Practical Programming. In *POPL*. ACM Press, 214–227.

Noam Zeilberger. 2009. Refinement Types and Computational Duality. In *Programming Languages meets Programming Verification (PLPV '09)*. ACM Press, 15–26.

Sound and Complete Bidirectional Typechecking for Higher-Rank Polymorphism with Existentials and Indexed Types: Full definitions, lemmas and proofs

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The first part (Sections 1–2) of this supplementary material contains rules, figures and definitions omitted in the main paper for space reasons, and a list of judgment forms (Section 2).

The remainder (Sections A–K') includes statements of all lemmas and theorems, along with full proofs. as well as statements of theorems and a few selected lemmas.

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1 Figures 8

1 Figures

We repeat some figures from the main paper. In Figures 6a and 14a, we include rules omitted from the main paper for space reasons.

Figure 6a: Declarative typing, including rules omitted from main paper

Figure 14a: Algorithmic typing, including rules omitted from main paper

 $\Psi \vdash t : \kappa$ Under context Ψ, term t has sort κ

$$\begin{split} \frac{(\alpha:\kappa) \in \Psi}{\Psi \vdash \alpha:\kappa} \; \mathsf{UvarSort} & \quad \frac{\Psi \vdash t_1:\star \quad \Psi \vdash t_2:\star}{\Psi \vdash t_1 \oplus t_2:\star} \; \mathsf{BinSort} \\ & \quad \frac{\Psi \vdash t:\mathbb{N}}{\Psi \vdash \mathsf{zero}:\mathbb{N}} \; \mathsf{ZeroSort} & \quad \frac{\Psi \vdash t:\mathbb{N}}{\Psi \vdash \mathsf{succ}(t):\mathbb{N}} \; \mathsf{SuccSort} \end{split}$$

 $\Psi \vdash P$ *prop* Under context Ψ , proposition P is well-formed

$$\frac{\Psi \vdash t : \mathbb{N} \qquad \Psi \vdash t' : \mathbb{N}}{\Psi \vdash t = t' prop}$$
 EqDeclProp

 $\Psi \vdash A \; \textit{type}$ Under context Ψ , type A is well-formed

$$\frac{(\alpha:\star) \in \Psi}{\Psi \vdash \alpha \ type} \ \ \text{DeclUvarWF} \qquad \frac{\Psi \vdash 1 \ type}{\Psi \vdash 1 \ type} \ \ \text{DeclUnitWF}$$

$$\frac{\Psi \vdash A \ type}{\Psi \vdash A \oplus B \ type} \ \ \oplus \in \{\rightarrow, \times, +\} \\ \frac{\Psi \vdash A \oplus B \ type}{\Psi \vdash (\forall \alpha: \kappa. \ A) \ type} \ \ \text{DeclBinWF} \qquad \frac{\Gamma \vdash t: \mathbb{N} \quad \Gamma \vdash A \ type}{\Gamma \vdash \text{Vec t A } \ type} \ \ \text{DeclVecWF}$$

$$\frac{\Psi, \alpha: \kappa \vdash A \ type}{\Psi \vdash (\exists \alpha: \kappa. \ A) \ type} \ \ \text{DeclExistsWF}$$

$$\frac{\Psi \vdash P \ prop \qquad \Psi \vdash A \ type}{\Psi \vdash P \supset A \ type} \ \ \text{DeclImpliesWF}$$

$$\frac{\Psi \vdash P \ prop \qquad \Psi \vdash A \ type}{\Psi \vdash A \land P \ type} \ \ \text{DeclWithWF}$$

 $\Psi dash ec{A}$ types Under context Ψ , types in $ec{A}$ are well-formed

for all
$$A \in \vec{A}$$
.

$$\frac{\Psi \vdash A \ type}{\Psi \vdash \vec{A} \ types}$$
 DeclTypevecWF

 Ψ ctx Declarative context Ψ is well-formed

$$\frac{\Psi \ ctx \qquad x \not\in \mathsf{dom}(\Psi) \qquad \Psi \vdash \mathsf{A} \ type}{\Psi, x : \mathsf{A} \ ctx} \ \mathsf{HypDeclCtx}$$

$$\frac{\Psi \ ctx \qquad \alpha \not\in \mathsf{dom}(\Psi)}{\Psi, \alpha : \kappa \ ctx} \ \mathsf{VarDeclCtx}$$

Figure 16: Sorting; well-formedness of propositions, types, and contexts in the declarative system

 $\Gamma \vdash \tau : \kappa$ Under context Γ, term τ has sort κ

$$\begin{array}{ll} \dfrac{(\mathfrak{u}:\kappa)\in\Gamma}{\Gamma\vdash\mathfrak{u}:\kappa}\,\mathsf{VarSort} & \dfrac{(\hat{\alpha}:\kappa=\tau)\in\Gamma}{\Gamma\vdash\hat{\alpha}:\kappa}\,\mathsf{SolvedVarSort} & \dfrac{}{\Gamma\vdash1:\star}\,\mathsf{UnitSort} \\ \\ \dfrac{\Gamma\vdash\tau_1:\star\quad\Gamma\vdash\tau_2:\star}{\Gamma\vdash\tau_1\oplus\tau_2:\star}\,\mathsf{BinSort} & \dfrac{}{\Gamma\vdash\mathsf{zero}:\mathbb{N}}\,\mathsf{ZeroSort} & \dfrac{}{\Gamma\vdash\mathsf{succ}(t):\mathbb{N}}\,\mathsf{SuccSort} \\ \end{array}$$

 $\Gamma \vdash P$ *prop* Under context Γ , proposition P is well-formed

$$\frac{\Gamma \vdash t : \mathbb{N} \qquad \Gamma \vdash t' : \mathbb{N}}{\Gamma \vdash t = t' prop}$$
 EqProp

 $\Gamma \vdash A \ type$ Under context Γ , type A is well-formed

$$\frac{(\mathfrak{u}:\star)\in\Gamma}{\Gamma\vdash\mathfrak{u}\;type}\;\mathsf{Var}\mathsf{WF}\qquad \frac{(\hat{\alpha}:\star=\tau)\in\Gamma}{\Gamma\vdash\hat{\alpha}\;type}\;\mathsf{Solved}\mathsf{Var}\mathsf{WF}\qquad \frac{\Gamma\vdash 1\;type}{\Gamma\vdash 1\;type}\;\mathsf{Unit}\mathsf{WF}$$

$$\frac{\Gamma\vdash A\;type}{\Gamma\vdash A\;\oplus B\;type} \quad \oplus\in\{\to,\times,+\} \\ \frac{\Gamma\vdash A\;\oplus B\;type}{\Gamma\vdash \forall\alpha:\kappa}\;\mathsf{Bin}\mathsf{WF}\qquad \frac{\Gamma\vdash t:\mathbb{N}}{\Gamma\vdash A\;type}\;\mathsf{Vec}\mathsf{WF}$$

$$\frac{\Gamma\vdash A\;\oplus B\;type}{\Gamma\vdash \forall\alpha:\kappa}\;\mathsf{Forall}\mathsf{WF}\qquad \frac{\Gamma,\alpha:\kappa\vdash A\;type}{\Gamma\vdash \exists\alpha:\kappa}\;\mathsf{Exists}\mathsf{WF}$$

$$\frac{\Gamma\vdash P\;prop}{\Gamma\vdash A\;type}\;\mathsf{Implies}\mathsf{WF}\qquad \frac{\Gamma\vdash P\;prop}{\Gamma\vdash A\;\wedge P\;type}\;\mathsf{With}\mathsf{WF}$$

 $\Gamma \vdash A \ p \ type$ Under context Γ , type A is well-formed and respects principality p

$$\frac{\Gamma \vdash A \; type \qquad \mathsf{FEV}([\Gamma]A) = \emptyset}{\Gamma \vdash A \; ! \; type} \; \mathsf{PrincipalWF} \qquad \qquad \frac{\Gamma \vdash A \; type}{\Gamma \vdash A \; ! \; type} \; \mathsf{NonPrincipalWF}$$

 $\Gamma \vdash \vec{A} \ [p] \ \textit{types}$ Under context Γ , types in \vec{A} are well-formed [with principality p]

$$\frac{\text{for all } A \in \vec{A}. \ \Gamma \vdash A \ type}{\Gamma \vdash \vec{A} \ types} \text{ TypevecWF} \qquad \frac{\text{for all } A \in \vec{A}. \ \Gamma \vdash A \ p \ type}{\Gamma \vdash \vec{A} \ p \ types} \text{ PrincipalTypevecWF}$$

 Γ ctx Algorithmic context Γ is well-formed

$$\frac{r \ ctx}{r} \ EmptyCtx \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ A \ type} \qquad \frac{r \ ctx}{r \ r \ a \ type} \qquad \frac{r \ r \ type}{r \ r \ a \ type} \qquad \frac{r \ r \ r \ type}{r \ r \$$

Figure 17: Well-formedness of types and contexts in the algorithmic system

 $\Gamma \vdash P$ *true* $\dashv \Delta$ Under context Γ , check P, with output context Δ

$$\frac{\Gamma \vdash t_1 \stackrel{\text{\tiny $}}{=} t_2 : \mathbb{N} \dashv \Delta}{\Gamma \vdash t_1 = t_2 \ \textit{true} \dashv \Delta} \ \mathsf{CheckpropEq}$$

$$\frac{\Gamma \ / \ t_1 \triangleq t_2 : \mathbb{N} \dashv \Delta^{\perp}}{\Gamma \ / \ t_1 = t_2 \dashv \Delta^{\perp}} \ \mathsf{ElimpropEq}$$

Figure 18: Checking and assuming propositions

$$\Gamma \vdash t_1 \stackrel{\circ}{=} t_2 : \kappa \dashv \Delta$$
 Check that t_1 equals t_2 , taking Γ to Δ

Figure 19: Checking equations

Figure 20: Head constructor clash

$$\boxed{\Gamma \ / \ \sigma \stackrel{\circ}{=} \tau : \kappa \dashv \Delta^{\perp}} \ \text{Unify σ and τ, taking Γ to Δ, or to inconsistency \bot}$$

$$\frac{\Gamma \, / \, \alpha \triangleq \alpha : \kappa \dashv \Gamma}{\Gamma \, / \, \text{zero} \triangleq \text{zero} : \mathbb{N} \dashv \Gamma} \, \text{ElimeqZero} \qquad \frac{\Gamma \, / \, \sigma \triangleq \tau : \mathbb{N} \dashv \Delta^{\perp}}{\Gamma \, / \, \text{succ}(\sigma) \triangleq \text{succ}(\tau) : \mathbb{N} \dashv \Delta^{\perp}} \, \text{ElimeqSucc}$$

$$\frac{\alpha \notin FV(\tau) \quad (\alpha = -) \notin \Gamma}{\Gamma \, / \, \alpha \triangleq \tau : \kappa \dashv \Gamma, \alpha = \tau} \, \text{ElimeqUvarL} \qquad \frac{\alpha \notin FV(\tau) \quad (\alpha = -) \notin \Gamma}{\Gamma \, / \, \tau \triangleq \alpha : \kappa \dashv \Gamma, \alpha = \tau} \, \text{ElimeqUvarR}$$

$$\frac{t \neq \alpha \quad \alpha \in FV(\tau)}{\Gamma \, / \, \alpha \triangleq \tau : \kappa \dashv \bot} \, \text{ElimeqUvarL} \qquad \frac{t \neq \alpha \quad \alpha \in FV(\tau)}{\Gamma \, / \, \tau \triangleq \alpha : \kappa \dashv \bot} \, \text{ElimeqUvarR}$$

$$\frac{t \neq \alpha \quad \alpha \in FV(\tau)}{\Gamma \, / \, \tau \triangleq \alpha : \kappa \dashv \bot} \, \text{ElimeqUvarR}$$

$$\frac{\Gamma \, / \, \tau_1 \triangleq \tau_1' : \star \dashv \Theta \quad \Theta \, / \, [\Theta] \tau_2 \triangleq [\Theta] \tau_2' : \star \dashv \Delta^{\perp}}{\Gamma \, / \, (\tau_1 \oplus \tau_2) \triangleq (\tau_1' \oplus \tau_2') : \star \dashv \Delta^{\perp}} \, \text{ElimeqBinBot}$$

$$\frac{\Gamma \, / \, \tau_1 \triangleq \tau_1' : \star \dashv \bot}{\Gamma \, / \, (\tau_1 \oplus \tau_2) \triangleq (\tau_1' \oplus \tau_2') : \star \dashv \bot} \, \text{ElimeqClash}$$

Figure 21: Eliminating equations

Figure 22: Algorithmic subtyping and equivalence

$$\begin{array}{c} \Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta \end{array} \begin{array}{c} \text{Under input context } \Gamma, \\ \text{instantiate } \hat{\alpha} \text{ such that } \hat{\alpha} = t \text{ with output context } \Delta \end{array} \\ & \frac{\Gamma_0 \vdash \tau : \kappa}{\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \dashv \Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1} \text{ InstSolve} \\ & \frac{\hat{\beta} \in \text{unsolved}(\Gamma[\hat{\alpha} : \kappa][\hat{\beta} : \kappa])}{\Gamma[\hat{\alpha} : \kappa][\hat{\beta} : \kappa] \vdash \hat{\alpha} := \hat{\beta} : \kappa \dashv \Gamma[\hat{\alpha} : \kappa][\hat{\beta} : \kappa = \hat{\alpha}]} \text{ InstReach} \\ & \frac{\Gamma[\hat{\alpha}_2 : \star, \hat{\alpha}_1 : \star, \hat{\alpha} : \star = \hat{\alpha}_1 \oplus \hat{\alpha}_2] \vdash \hat{\alpha}_1 := \tau_1 : \star \dashv \Theta}{\Gamma[\hat{\alpha} : \star] \vdash \hat{\alpha} := \tau_1 \oplus \tau_2 : \star \dashv \Delta} \text{ InstBin} \\ & \frac{\Gamma[\hat{\alpha}_2 : \star, \hat{\alpha}_1 : \star, \hat{\alpha} : \star = \hat{\alpha}_1 \oplus \hat{\alpha}_2] \vdash \hat{\alpha}_1 := \tau_1 : \star \dashv \Theta}{\Gamma[\hat{\alpha} : \star] \vdash \hat{\alpha} := \tau_1 \oplus \tau_2 : \star \dashv \Delta} \end{array} \text{ InstBin} \\ & \frac{\Gamma[\hat{\alpha}_1 : \mathbb{N}] \vdash \hat{\alpha} := \text{zero} : \mathbb{N} \dashv \Gamma[\hat{\alpha} : \mathbb{N} = \text{zero}]}{\Gamma[\hat{\alpha} : \mathbb{N}] \vdash \hat{\alpha} := \text{succ}(\hat{\alpha}_1)] \vdash \hat{\alpha}_1 := t_1 : \mathbb{N} \dashv \Delta}{\Gamma[\hat{\alpha} : \mathbb{N}] \vdash \hat{\alpha} := \text{succ}(t_1) : \mathbb{N} \dashv \Delta}} \text{ InstSucc} \end{array}$$

Figure 23: Instantiation

Figure 24: Algorithmic pattern matching

Under context Γ , patterns Π cover the types \vec{A}

 $\Gamma \vdash \Pi \ covers \ \vec{A} \ q$

Figure 25: Algorithmic match coverage

2 List of Judgments

For convenience, we list all the judgment forms:

Judgment	Description	Location
$\Psi \vdash t : \kappa$ $\Psi \vdash P \ prop$ $\Psi \vdash A \ type$ $\Psi \vdash \vec{A} \ types$ $\Psi \ ctx$	Index term/monotype is well-formed Proposition is well-formed Type is well-formed Type vector is well-formed Declarative context is well-formed	Figure 16 Figure 16 Figure 16 Figure 16
$\Psi \vdash A \leq^{\mathcal{P}} B$	Declarative subtyping	Figure 4
$\Psi \vdash P$ true	Declarative truth	Figure 6
$\begin{array}{l} \Psi \vdash e \Leftarrow A \ p \\ \Psi \vdash e \Rightarrow A \ p \\ \Psi \vdash s : A \ p \gg C \ q \\ \Psi \vdash s : A \ p \gg C \ \lceil q \rceil \end{array}$	Declarative checking Declarative synthesis Declarative spine typing Declarative spine typing, recovering principality	Figure 6 Figure 6 Figure 6 Figure 6
$\begin{array}{l} \Psi \vdash \Pi :: \vec{A} \; ! \Leftarrow C \; \mathfrak{p} \\ \Psi \; / \; P \vdash \Pi :: \vec{A} \; ! \Leftarrow C \; \mathfrak{p} \end{array}$	Declarative pattern matching Declarative proposition assumption	Figure 7 Figure 7
$\Psi \vdash \Pi \ \textit{covers} \ \vec{A} \ !$	Declarative match coverage	Figure 8
$\Gamma \vdash \tau : \kappa$ $\Gamma \vdash P \ prop$ $\Gamma \vdash A \ type$ $\Gamma \ ctx$	Index term/monotype is well-formed Proposition is well-formed Polytype is well-formed Algorithmic context is well-formed	Figure 17 Figure 17 Figure 17 Figure 17
$[\Gamma]A$	Applying a context, as a substitution, to a type	Figure 12
$\begin{array}{l} \Gamma \vdash P \; true \dashv \Delta \\ \Gamma \mathrel{/} P \dashv \Delta^{\perp} \\ \Gamma \vdash s \stackrel{\circ}{=} t : \kappa \dashv \Delta \\ s \; \# \; t \\ \Gamma \mathrel{/} s \stackrel{\circ}{=} t : \kappa \dashv \Delta^{\perp} \end{array}$	Check proposition Assume proposition Check equation Head constructors clash Assume/eliminate equation	Figure 18 Figure 19 Figure 20 Figure 21
$\Gamma \vdash A <: ^{\mathcal{P}} B \dashv \Delta$ $\Gamma / P \vdash A <: B \dashv \Delta$ $\Gamma \vdash P \equiv Q \dashv \Delta$ $\Gamma \vdash A \equiv B \dashv \Delta$ $\Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta$	Algorithmic subtyping Assume/eliminate proposition Equivalence of propositions Equivalence of types Instantiate	Figure 22 Figure 22 Figure 22 Figure 22 Figure 23
e chk-I	Checking intro form	Figure 5
$\begin{array}{l} \Gamma \vdash e \Leftarrow A \ p \dashv \Delta \\ \Gamma \vdash e \Rightarrow A \ p \dashv \Delta \\ \Gamma \vdash s : A \ p \gg C \ q \dashv \Delta \\ \Gamma \vdash s : A \ p \gg C \ \lceil q \rceil \dashv \Delta \end{array}$	Algorithmic checking Algorithmic synthesis Algorithmic spine typing Algorithmic spine typing, recovering principality	Figure 14 Figure 14 Figure 14 Figure 14
$\Gamma \vdash \Pi :: \vec{A} \ q \leftarrow C \ p \dashv \Delta$ $\Gamma / P \vdash \Pi :: \vec{A} ! \leftarrow C \ p \dashv \Delta$	Algorithmic pattern matching Algorithmic pattern matching (assumption)	Figure 24 Figure 24
$\Gamma \vdash \Pi \text{ covers } \vec{A} \text{ q}$	Algorithmic match coverage	Figure 25
$\Gamma \longrightarrow \Delta$	Context extension	Figure 15
$[\Omega]\Gamma$	Apply complete context	Figure 13

A Properties of the Declarative System

Lemma 1 (Declarative Well-foundedness). *Go to proof* The inductive definition of the following judgments is well-founded:

- (i) synthesis $\Psi \vdash e \Rightarrow B p$
- (ii) checking $\Psi \vdash e \Leftarrow A p$
- (iii) checking, equality elimination $\Psi / P \vdash e \Leftarrow C p$
- (iv) ordinary spine $\Psi \vdash s : A p \gg B q$
- (v) recovery spine $\Psi \vdash s : A p \gg B \lceil q \rceil$
- (vi) pattern matching $\Psi \vdash \Pi :: \vec{A} ! \Leftarrow C p$
- (vii) pattern matching, equality elimination $\Psi / P \vdash \Pi :: \vec{A} ! \Leftarrow C p$

Lemma 2 (Declarative Weakening). Go to proof

- (i) If $\Psi_0, \Psi_1 \vdash t : \kappa$ then $\Psi_0, \Psi, \Psi_1 \vdash t : \kappa$.
- (ii) If $\Psi_0, \Psi_1 \vdash P$ prop then $\Psi_0, \Psi, \Psi_1 \vdash P$ prop.
- (iii) If $\Psi_0, \Psi_1 \vdash P$ true then $\Psi_0, \Psi, \Psi_1 \vdash P$ true.
- (iv) If $\Psi_0, \Psi_1 \vdash A$ type then $\Psi_0, \Psi, \Psi_1 \vdash A$ type.

Lemma 3 (Declarative Term Substitution). *Go to proof* Suppose $\Psi \vdash t : \kappa$. Then:

- 1. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash t' : \kappa$ then $\Psi_0, [t/\alpha]\Psi_1 \vdash [t/\alpha]t' : \kappa$.
- 2. If Ψ_0 , $\alpha : \kappa$, $\Psi_1 \vdash P$ prop then Ψ_0 , $[t/\alpha]\Psi_1 \vdash [t/\alpha]P$ prop.
- 3. If Ψ_0 , $\alpha : \kappa, \Psi_1 \vdash A$ type then Ψ_0 , $[t/\alpha]\Psi_1 \vdash [t/\alpha]A$ type.
- 4. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash A <^{\mathcal{P}} B$ then $\Psi_0, [t/\alpha]\Psi_1 \vdash [t/\alpha]A <^{\mathcal{P}} [t/\alpha]B$.
- 5. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash P$ true then $\Psi_0, [t/\alpha]\Psi_1 \vdash [t/\alpha]P$ true.

Lemma 4 (Reflexivity of Declarative Subtyping). *Go to proof Given* $\Psi \vdash A$ *type, we have that* $\Psi \vdash A \stackrel{\mathcal{P}}{=} A$.

Lemma 5 (Subtyping Inversion). Go to proof

- If $\Psi \vdash \exists \alpha : \kappa$. $A \leq^+ B$ then Ψ , $\alpha : \kappa \vdash A \leq^+ B$.
- If $\Psi \vdash A <^- \forall \beta : \kappa$. B then $\Psi, \beta : \kappa \vdash A <^- B$.

Lemma 6 (Subtyping Polarity Flip). Go to proof

- If nonpos(A) and nonpos(B) and $\Psi \vdash A \leq^+ B$ then $\Psi \vdash A \leq^- B$ by a derivation of the same or smaller size.
- If nonneg(A) and nonneg(B) and Ψ ⊢ A ≤ B
 then Ψ ⊢ A < B by a derivation of the same or smaller size.
- If nonpos(A) and nonneg(A) and nonpos(B) and nonneg(B) and $\Psi \vdash A \leq^{\mathcal{P}} B$ then A = B.

Lemma 7 (Transitivity of Declarative Subtyping). *Go to proof* Given $\Psi \vdash A$ type and $\Psi \vdash B$ type and $\Psi \vdash C$ type:

(i) If
$$\mathcal{D}_1 :: \Psi \vdash A \leq^{\mathcal{P}} B$$
 and $\mathcal{D}_2 :: \Psi \vdash B \leq^{\mathcal{P}} C$ then $\Psi \vdash A \leq^{\mathcal{P}} C$.

Property 1. We assume that all types mentioned in annotations in expressions have no free existential variables. By the grammar, it follows that all expressions have no free existential variables, that is, $FEV(e) = \emptyset$.

B Substitution and Well-formedness Properties

Definition 1 (Softness). A context Θ is soft iff it consists only of $\hat{\alpha}$: κ and $\hat{\alpha}$: $\kappa = \tau$ declarations.

Lemma 8 (Substitution—Well-formedness). *Go to proof*

- (i) If $\Gamma \vdash A$ p type and $\Gamma \vdash \tau$ p type then $\Gamma \vdash [\tau/\alpha]A$ p type.
- (ii) If $\Gamma \vdash P$ prop and $\Gamma \vdash \tau$ p type then $\Gamma \vdash [\tau/\alpha]P$ prop. Moreover, if p = ! and $\mathsf{FEV}([\Gamma]P) = \emptyset$ then $\mathsf{FEV}([\Gamma][\tau/\alpha]P) = \emptyset$.

Lemma 9 (Uvar Preservation). *Go to proof If* $\Delta \longrightarrow \Omega$ *then:*

- (i) If $(\alpha : \kappa) \in \Omega$ then $(\alpha : \kappa) \in [\Omega]\Delta$.
- (ii) If $(x:Ap) \in \Omega$ then $(x:[\Omega]Ap) \in [\Omega]\Delta$.

Lemma 10 (Sorting Implies Typing). *Go to proof If* $\Gamma \vdash t : \star then \Gamma \vdash t type$.

Lemma 11 (Right-Hand Substitution for Sorting). *Go to proof* If $\Gamma \vdash t : \kappa$ then $\Gamma \vdash [\Gamma]t : \kappa$.

Lemma 12 (Right-Hand Substitution for Propositions). *Go to proof If* $\Gamma \vdash P$ *prop then* $\Gamma \vdash [\Gamma]P$ *prop.*

Lemma 13 (Right-Hand Substitution for Typing). *Go to proof* If $\Gamma \vdash A$ type then $\Gamma \vdash [\Gamma]A$ type.

Lemma 14 (Substitution for Sorting). *Go to proof* If $\Omega \vdash t : \kappa$ then $[\Omega]\Omega \vdash [\Omega]t : \kappa$.

Lemma 15 (Substitution for Prop Well-Formedness). *Go to proof If* $\Omega \vdash P$ *prop then* $[\Omega]\Omega \vdash [\Omega]P$ *prop.*

Lemma 16 (Substitution for Type Well-Formedness). *Go to proof* If $\Omega \vdash A$ type then $[\Omega]\Omega \vdash [\Omega]A$ type.

Lemma 17 (Substitution Stability). *Go to proof*

If (Ω, Ω_Z) is well-formed and Ω_Z is soft and $\Omega \vdash A$ type then $[\Omega]A = [\Omega, \Omega_Z]A$.

Lemma 18 (Equal Domains). Go to proof

If $\Omega_1 \vdash A$ type and $dom(\Omega_1) = dom(\Omega_2)$ then $\Omega_2 \vdash A$ type.

C Properties of Extension

Lemma 19 (Declaration Preservation). Go to proof If $\Gamma \longrightarrow \Delta$ and $\mathfrak u$ is declared in Γ , then $\mathfrak u$ is declared in Δ .

Lemma 20 (Declaration Order Preservation). *Go to proof* If $\Gamma \longrightarrow \Delta$ and $\mathfrak u$ is declared to the left of $\mathfrak v$ in Γ , then $\mathfrak u$ is declared to the left of $\mathfrak v$ in Δ .

Lemma 21 (Reverse Declaration Order Preservation). *Go to proof* If $\Gamma \longrightarrow \Delta$ and u and v are both declared in Γ and u is declared to the left of v in Δ , then u is declared to the left of v in Γ .

An older paper had a lemma

"Substitution Extension Invariance" If $\Theta \vdash A$ type and $\Theta \longrightarrow \Gamma$ then $[\Gamma]A = [\Gamma]([\Theta]A)$ and $[\Gamma]A = [\Theta]([\Gamma]A)$.

For the second part, $[\Gamma]A = [\Theta]([\Gamma]A)$, use Lemma 29 (Substitution Monotonicity) (i) or (iii) instead. The first part $[\Gamma]A = [\Gamma][\Theta]A$ hasn't been proved in this system.

Lemma 22 (Extension Inversion). Go to proof

- (i) If $\mathcal{D} :: \Gamma_0, \alpha : \kappa, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0 and Δ_1 such that $\Delta = (\Delta_0, \alpha : \kappa, \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$. Moreover, if Γ_1 is soft, then Δ_1 is soft.
- (ii) If $\mathcal{D} :: \Gamma_0, \blacktriangleright_{\mathfrak{u}}, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0 and Δ_1 such that $\Delta = (\Delta_0, \blacktriangleright_{\mathfrak{u}}, \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$. Moreover, if Γ_1 is soft, then Δ_1 is soft. Moreover, if $\mathsf{dom}(\Gamma_0, \blacktriangleright_{\mathfrak{u}}, \Gamma_1) = \mathsf{dom}(\Delta)$ then $\mathsf{dom}(\Gamma_0) = \mathsf{dom}(\Delta_0)$.
- (iii) If $\mathcal{D} :: \Gamma_0, \alpha = \tau, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0, τ' , and Δ_1 such that $\Delta = (\Delta_0, \alpha = \tau', \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ and $[\Delta_0]\tau = [\Delta_0]\tau'$ where $\mathcal{D}' < \mathcal{D}$.
- (iv) If $\mathcal{D}:: \Gamma_0, \hat{\alpha}: \kappa = \tau, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0, τ' , and Δ_1 such that $\Delta = (\Delta_0, \hat{\alpha}: \kappa = \tau', \Delta_1)$ and $\mathcal{D}':: \Gamma_0 \longrightarrow \Delta_0$ and $[\Delta_0]\tau = [\Delta_0]\tau'$ where $\mathcal{D}' < \mathcal{D}$.
- (v) If $\mathcal{D} :: \Gamma_0, x : A, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0, A' , and Δ_1 such that $\Delta = (\Delta_0, x : A', \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ and $[\Delta_0]A = [\Delta_0]A'$ where $\mathcal{D}' < \mathcal{D}$. Moreover, if Γ_1 is soft, then Δ_1 is soft. Moreover, if $\mathsf{dom}(\Gamma_0, x : A, \Gamma_1) = \mathsf{dom}(\Delta)$ then $\mathsf{dom}(\Gamma_0) = \mathsf{dom}(\Delta_0)$.
- (vi) If $\mathcal{D} :: \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \longrightarrow \Delta$ then either
 - there exist unique Δ_0 , τ' , and Δ_1 such that $\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$, or
 - there exist unique Δ_0 and Δ_1 such that $\Delta = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$.

Lemma 23 (Deep Evar Introduction). *Go to proof*

- (i) If Γ_0 , Γ_1 is well-formed and $\hat{\alpha}$ is not declared in Γ_0 , Γ_1 then Γ_0 , $\Gamma_1 \longrightarrow \Gamma_0$, $\hat{\alpha} : \kappa$, Γ_1 .
- (ii) If Γ_0 , $\hat{\alpha} : \kappa$, Γ_1 is well-formed and $\Gamma \vdash t : \kappa$ then Γ_0 , $\hat{\alpha} : \kappa$, $\Gamma_1 \longrightarrow \Gamma_0$, $\hat{\alpha} : \kappa = t$, Γ_1 .
- (iii) If Γ_0 , Γ_1 is well-formed and $\Gamma \vdash t : \kappa$ then Γ_0 , $\Gamma_1 \longrightarrow \Gamma_0$, $\hat{\alpha} : \kappa = t$, Γ_1 .

Lemma 24 (Soft Extension). Go to proof

If $\Gamma \longrightarrow \Delta$ and Γ, Θ ctx and Θ is soft, then there exists Ω such that $dom(\Theta) = dom(\Omega)$ and $\Gamma, \Theta \longrightarrow \Delta, \Omega$. **Definition 2** (Filling). The filling of a context $|\Gamma|$ solves all unsolved variables:

$$\begin{array}{lll} |\cdot| & = & \cdot \\ |\Gamma, x : A| & = & |\Gamma|, x : A \\ |\Gamma, \alpha : \kappa| & = & |\Gamma|, \alpha : \kappa \\ |\Gamma, \alpha = t| & = & |\Gamma|, \alpha = t \\ |\Gamma, \hat{\alpha} : \kappa = t| & = & |\Gamma|, \hat{\alpha} : \kappa = t \\ |\Gamma, \blacktriangleright_{\hat{\alpha}}| & = & |\Gamma|, \hbar_{\hat{\alpha}} : \kappa = 1 \\ |\Gamma, \hat{\alpha} : \star| & = & |\Gamma|, \hat{\alpha} : \star = 1 \\ |\Gamma, \hat{\alpha} : \mathbb{N}| & = & |\Gamma|, \hat{\alpha} : \mathbb{N} = \mathsf{zero} \end{array}$$

Lemma 25 (Filling Completes). If $\Gamma \longrightarrow \Omega$ and (Γ, Θ) is well-formed, then $\Gamma, \Theta \longrightarrow \Omega$, $|\Theta|$.

Proof. By induction on Θ , following the definition of |-| and applying the rules for \longrightarrow .

Lemma 26 (Parallel Admissibility). Go to proof

If $\Gamma_L \longrightarrow \Delta_L$ and $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$ then:

- (i) Γ_L , $\hat{\alpha}$: κ , $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}$: κ , Δ_R
- (ii) If $\Delta_L \vdash \tau' : \kappa$ then $\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$.
- (iii) If $\Gamma_L \vdash \tau : \kappa$ and $\Delta_L \vdash \tau'$ type and $[\Delta_L]\tau = [\Delta_L]\tau'$, then $\Gamma_L, \hat{\alpha} : \kappa = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$.

Lemma 27 (Parallel Extension Solution). Go to proof

If
$$\Gamma_L$$
, $\hat{\alpha}: \kappa$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}: \kappa = \tau'$, Δ_R and $\Gamma_L \vdash \tau: \kappa$ and $[\Delta_L]\tau = [\Delta_L]\tau'$ then Γ_L , $\hat{\alpha}: \kappa = \tau$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}: \kappa = \tau'$, Δ_R .

Lemma 28 (Parallel Variable Update). Go to proof

If
$$\Gamma_L$$
, $\hat{\alpha}: \kappa$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}: \kappa = \tau_0$, Δ_R and $\Gamma_L \vdash \tau_1: \kappa$ and $\Delta_L \vdash \tau_2: \kappa$ and $[\Delta_L]\tau_0 = [\Delta_L]\tau_1 = [\Delta_L]\tau_2$ then Γ_L , $\hat{\alpha}: \kappa = \tau_1$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}: \kappa = \tau_2$, Δ_R .

Lemma 29 (Substitution Monotonicity). *Go to proof*

- (i) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash t : \kappa$ then $[\Delta][\Gamma]t = [\Delta]t$.
- (ii) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash P$ prop then $[\Delta][\Gamma]P = [\Delta]P$.
- (iii) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash A$ type then $[\Delta][\Gamma]A = [\Delta]A$.

Lemma 30 (Substitution Invariance). Go to proof

- (i) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash t : \kappa$ and $\mathsf{FEV}([\Gamma]t) = \emptyset$ then $[\Delta][\Gamma]t = [\Gamma]t$.
- (ii) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash P$ prop and $\mathsf{FEV}([\Gamma]P) = \emptyset$ then $[\Delta][\Gamma]P = [\Gamma]P$.
- (iii) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash A$ type and $\mathsf{FEV}([\Gamma]A) = \emptyset$ then $[\Delta][\Gamma]A = [\Gamma]A$.

Definition 3 (Canonical Contexts). *A* (complete) context Ω is canonical iff, for all $(\hat{\alpha} : \kappa = t)$ and $(\alpha = t) \in \Omega$, the solution t is ground (FEV(t) = \emptyset).

Lemma 31 (Split Extension). Go to proof

```
If \Delta \longrightarrow \Omega

and \hat{\alpha} \in \text{unsolved}(\Delta)

and \Omega = \Omega_1[\hat{\alpha} : \kappa = t_1]

and \Omega is canonical (Definition 3)

and \Omega \vdash t_2 : \kappa

then \Delta \longrightarrow \Omega_1[\hat{\alpha} : \kappa = t_2].
```

C.1 Reflexivity and Transitivity

```
Lemma 32 (Extension Reflexivity). Go to proof If \Gamma ctx then \Gamma \longrightarrow \Gamma.
```

```
Lemma 33 (Extension Transitivity). Go to proof If \mathcal{D} :: \Gamma \longrightarrow \Theta and \mathcal{D}' :: \Theta \longrightarrow \Delta then \Gamma \longrightarrow \Delta.
```

C.2 Weakening

The "suffix weakening" lemmas take a judgment under Γ and produce a judgment under (Γ, Θ) . They do *not* require $\Gamma \longrightarrow \Gamma, \Theta$.

Lemma 34 (Suffix Weakening). *Go to proof* If $\Gamma \vdash t : \kappa$ then $\Gamma, \Theta \vdash t : \kappa$.

Lemma 35 (Suffix Weakening). *Go to proof* If $\Gamma \vdash A$ *type then* $\Gamma, \Theta \vdash A$ *type.*

The following proposed lemma is false.

```
"Extension Weakening (Truth)" If \Gamma \vdash P true \dashv \Delta and \Gamma \longrightarrow \Gamma' then there exists \Delta' such that \Delta \longrightarrow \Delta' and \Gamma' \vdash P true \dashv \Delta'.
```

Counterexample: Suppose $\hat{\alpha} \vdash \hat{\alpha} = 1$ true $\dashv \hat{\alpha} = 1$ and $\hat{\alpha} \longrightarrow (\hat{\alpha} = (1 \rightarrow 1))$. Then there does *not* exist such a Δ' .

Lemma 36 (Extension Weakening (Sorts)). *Go to proof* If $\Gamma \vdash t : \kappa$ and $\Gamma \longrightarrow \Delta$ then $\Delta \vdash t : \kappa$.

Lemma 37 (Extension Weakening (Props)). *Go to proof If* $\Gamma \vdash P$ *prop and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash P$ *prop.*

Lemma 38 (Extension Weakening (Types)). *Go to proof* If $\Gamma \vdash A$ type and $\Gamma \longrightarrow \Delta$ then $\Delta \vdash A$ type.

C.3 Principal Typing Properties

Lemma 39 (Principal Agreement). Go to proof

- (i) If $\Gamma \vdash A$! type and $\Gamma \longrightarrow \Delta$ then $[\Delta]A = [\Gamma]A$.
- (ii) If $\Gamma \vdash P$ prop and $FEV(P) = \emptyset$ and $\Gamma \longrightarrow \Delta$ then $[\Delta]P = [\Gamma]P$.

Lemma 40 (Right-Hand Subst. for Principal Typing). Go to proof If $\Gamma \vdash A$ p type then $\Gamma \vdash [\Gamma]A$ p type.

Lemma 41 (Extension Weakening for Principal Typing). *Go to proof* If $\Gamma \vdash A \mathfrak{p}$ type and $\Gamma \longrightarrow \Delta$ then $\Delta \vdash A \mathfrak{p}$ type.

Lemma 42 (Inversion of Principal Typing). Go to proof

- (1) If $\Gamma \vdash (A \rightarrow B)$ p type then $\Gamma \vdash A$ p type and $\Gamma \vdash B$ p type.
- (2) If $\Gamma \vdash (P \supset A)$ p type then $\Gamma \vdash P$ prop and $\Gamma \vdash A$ p type.
- (3) If $\Gamma \vdash (A \land P)$ p type then $\Gamma \vdash P$ prop and $\Gamma \vdash A$ p type.

C.4 Instantiation Extends

Lemma 43 (Instantiation Extension). *Go to proof If* $\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

C.5 Equivalence Extends

Lemma 44 (Elimeq Extension). Go to proof

If $\Gamma / s \stackrel{\circ}{=} t : \kappa \dashv \Delta$ then there exists Θ such that $\Gamma, \Theta \longrightarrow \Delta$.

Lemma 45 (Elimprop Extension). Go to proof

If $\Gamma / P \dashv \Delta$ then there exists Θ such that $\Gamma, \Theta \longrightarrow \Delta$.

Lemma 46 (Checkeq Extension). Go to proof

If $\Gamma \vdash A \equiv B \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

Lemma 47 (Checkprop Extension). Go to proof

If $\Gamma \vdash P$ *true* $\dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

Lemma 48 (Prop Equivalence Extension). Go to proof

If $\Gamma \vdash P \equiv Q \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

Lemma 49 (Equivalence Extension). Go to proof

If $\Gamma \vdash A \equiv B \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

C.6 Subtyping Extends

Lemma 50 (Subtyping Extension). *Go to proof* If $\Gamma \vdash A <: ^{\mp} B \dashv \Delta$ then $\Gamma \longrightarrow \Delta$.

C.7 Typing Extends

Lemma 51 (Typing Extension). Go to proof

If $\Gamma \vdash e \Leftarrow A p \dashv \Delta$ or $\Gamma \vdash e \Rightarrow A p \dashv \Delta$ or $\Gamma \vdash s : A p \gg B q \dashv \Delta$ or $\Gamma \vdash \Pi :: \vec{A} q \Leftarrow C p \dashv \Delta$ or $\Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta$ then $\Gamma \longrightarrow \Delta$.

C.8 Unfiled

Lemma 52 (Context Partitioning). Go to proof

If $\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \longrightarrow \Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z$ then there is a Ψ such that $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) = [\Omega]\Delta, \Psi$.

Lemma 53 (Softness Goes Away).

If $\Delta, \Theta \longrightarrow \Omega$, Ω_Z where $\Delta \longrightarrow \Omega$ and Θ is soft, then $[\Omega, \Omega_Z](\Delta, \Theta) = [\Omega]\Delta$.

Proof. By induction on Θ , following the definition of $[\Omega]\Gamma$.

Lemma 54 (Completing Stability). Go to proof

If $\Gamma \longrightarrow \Omega$ *then* $[\Omega]\Gamma = [\Omega]\Omega$.

Lemma 55 (Completing Completeness). Go to proof

- (i) If $\Omega \longrightarrow \Omega'$ and $\Omega \vdash t : \kappa$ then $[\Omega]t = [\Omega']t$.
- (ii) If $\Omega \longrightarrow \Omega'$ and $\Omega \vdash A$ type then $[\Omega]A = [\Omega']A$.
- (iii) If $\Omega \longrightarrow \Omega'$ then $[\Omega]\Omega = [\Omega']\Omega'$.

Lemma 56 (Confluence of Completeness). Go to proof

If $\Delta_1 \longrightarrow \Omega$ and $\Delta_2 \longrightarrow \Omega$ then $[\Omega]\Delta_1 = [\Omega]\Delta_2$.

Lemma 57 (Multiple Confluence). Go to proof

If $\Delta \longrightarrow \Omega$ and $\Omega \longrightarrow \Omega'$ and $\Delta' \longrightarrow \Omega'$ then $[\Omega]\Delta = [\Omega']\Delta'$.

Lemma 58 (Bundled Substitution for Sorting). *If* $\Gamma \vdash t : \kappa$ *and* $\Gamma \longrightarrow \Omega$ *then* $[\Omega]\Gamma \vdash [\Omega]t : \kappa$.

```
Proof.
```

```
\Gamma \vdash t : \kappa \qquad \text{Given}
\Omega \vdash t : \kappa \qquad \text{By Lemma 36 (Extension Weakening (Sorts))}
[\Omega]\Omega \vdash [\Omega]t : \kappa \qquad \text{By Lemma 14 (Substitution for Sorting)}
\Omega \longrightarrow \Omega \qquad \qquad \text{By Lemma 32 (Extension Reflexivity)}
[\Omega]\Omega = [\Omega]\Gamma \qquad \text{By Lemma 56 (Confluence of Completeness)}
[\Omega]\Gamma \vdash [\Omega]t : \kappa \qquad \text{By above equality}
```

Lemma 59 (Canonical Completion). Go to proof

```
If \Gamma \longrightarrow \Omega
```

then there exists Ω_{canon} such that $\Gamma \longrightarrow \Omega_{canon}$ and $\Omega_{canon} \longrightarrow \Omega$ and $dom(\Omega_{canon}) = dom(\Gamma)$ and, for all $\hat{\alpha} : \kappa = \tau$ and $\alpha = \tau$ in Ω_{canon} , we have $\mathsf{FEV}(\tau) = \emptyset$.

The completion Ω_{canon} is "canonical" because (1) its domain exactly matches Γ and (2) its solutions τ have no evars. Note that it follows from Lemma 57 (Multiple Confluence) that $[\Omega_{canon}]\Gamma = [\Omega]\Gamma$.

```
Lemma 60 (Split Solutions). Go to proof
```

```
If \Delta \longrightarrow \Omega and \hat{\alpha} \in \mathsf{unsolved}(\Delta)
```

then there exists $\Omega_1 = \Omega_1'[\hat{\alpha}: \kappa = t_1]$ such that $\Omega_1 \longrightarrow \Omega$ and $\Omega_2 = \Omega_1'[\hat{\alpha}: \kappa = t_2]$ where $\Delta \longrightarrow \Omega_2$ and $t_2 \neq t_1$ and Ω_2 is canonical.

D Internal Properties of the Declarative System

Lemma 61 (Interpolating With and Exists). Go to proof

```
(1) If \mathcal{D} :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow C p \text{ and } \Psi \vdash P_0 \text{ true }
then \mathcal{D}' :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow C \land P_0 p.
```

(2) If
$$\mathcal{D} :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow [\tau/\alpha] C_0 p \text{ and } \Psi \vdash \tau : \kappa$$
 then $\mathcal{D}' :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow (\exists \alpha : \kappa. C_0) p$.

In both cases, the height of \mathcal{D}' is one greater than the height of \mathcal{D} .

Moreover, similar properties hold for the eliminating judgment $\Psi / P \vdash \Pi :: \vec{A}! \Leftarrow C p$.

Lemma 62 (Case Invertibility). Go to proof

```
If \Psi \vdash \mathsf{case}(e_0, \Pi) \Leftarrow C p
```

then $\Psi \vdash e_0 \Rightarrow A$! and $\Psi \vdash \Pi :: A ! \Leftarrow C p$ and $\Psi \vdash \Pi$ covers A !

where the height of each resulting derivation is strictly less than the height of the given derivation.

E Miscellaneous Properties of the Algorithmic System

Lemma 63 (Well-Formed Outputs of Typing). *Go to proof*

(Spines) If
$$\Gamma \vdash s : A \neq \emptyset C p \dashv \Delta \text{ or } \Gamma \vdash s : A \neq \emptyset C \lceil p \rceil \dashv \Delta$$
 and $\Gamma \vdash A \neq type$ then $\Delta \vdash C p \text{ type}$.

(Synthesis) If
$$\Gamma \vdash e \Rightarrow A \ p \dashv \Delta$$
 then $A \vdash p$ type.

F Decidability of Instantiation

Lemma 64 (Left Unsolvedness Preservation). *Go to proof* If $\underline{\Gamma_0}, \hat{\alpha}, \underline{\Gamma_1} \vdash \hat{\alpha} := A : \kappa \dashv \Delta \ and \ \hat{\beta} \in \mathsf{unsolved}(\Gamma_0) \ then \ \hat{\beta} \in \mathsf{unsolved}(\Delta)$.

Lemma 65 (Left Free Variable Preservation). *Go to proof* If $\overbrace{\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1} \vdash \hat{\alpha} := t : \kappa \dashv \Delta$ and $\Gamma \vdash s : \kappa'$ and $\hat{\alpha} \notin FV([\Gamma]s)$ and $\hat{\beta} \in \text{unsolved}(\Gamma_0)$ and $\hat{\beta} \notin FV([\Gamma]s)$, then $\hat{\beta} \notin FV([\Delta]s)$.

Lemma 66 (Instantiation Size Preservation). *Go to proof* If $\widehat{\Gamma_0}$, $\widehat{\alpha}$, $\widehat{\Gamma_1} \vdash \widehat{\alpha} := \tau : \kappa \dashv \Delta$ and $\Gamma \vdash s : \kappa'$ and $\widehat{\alpha} \notin FV([\Gamma]s)$, then $|[\Gamma]s| = |[\Delta]s|$, where |C| is the plain size of the term C.

Lemma 67 (Decidability of Instantiation). *Go to proof* If $\Gamma = \Gamma_0[\hat{\alpha} : \kappa']$ and $\Gamma \vdash t : \kappa$ such that $[\Gamma]t = t$ and $\hat{\alpha} \notin FV(t)$, then:

(1) Either there exists Δ such that $\Gamma_0[\hat{\alpha}:\kappa'] \vdash \hat{\alpha} := t : \kappa \dashv \Delta$, or not.

G Separation

Definition 4 (Separation).

An algorithmic context Γ is separable and written $\Gamma_L * \Gamma_R$ if (1) $\Gamma = (\Gamma_L, \Gamma_R)$ and (2) for all $(\hat{\alpha} : \kappa = \tau) \in \Gamma_R$ it is the case that $\mathsf{FEV}(\tau) \subseteq \mathsf{dom}(\Gamma_R)$.

Any context Γ is separable into, at least, $\cdot * \Gamma$ and $\Gamma * \cdot \cdot$

Definition 5 (Separation-Preserving Extension).

The separated context $\Gamma_L * \Gamma_R$ extends to $\Delta_L * \Gamma_R$, written

$$(\Gamma_{\rm I} * \Gamma_{\rm R}) \xrightarrow{*} (\Delta_{\rm I} * \Delta_{\rm R})$$

if $(\Gamma_L, \Gamma_R) \longrightarrow (\Delta_L, \Delta_R)$ and $dom(\Gamma_L) \subseteq dom(\Delta_L)$ and $dom(\Gamma_R) \subseteq dom(\Delta_R)$.

Separation-preserving extension says that variables from one half don't "cross" into the other half. Thus, Δ_L may add existential variables to Γ_L , and Δ_R may add existential variables to Γ_R , but no variable from Γ_L ends up in Δ_R and no variable from Γ_R ends up in Δ_L .

It is necessary to write $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$ rather than $(\Gamma_L * \Gamma_R) \longrightarrow (\Delta_L * \Delta_R)$, because only $\xrightarrow{*}$ includes the domain conditions. For example, $(\hat{\alpha} * \hat{\beta}) \longrightarrow (\hat{\alpha}, \hat{\beta} = \hat{\alpha}) * \cdot$, but the variable $\hat{\beta}$ has "crossed over" to the left of * in the context $(\hat{\alpha}, \hat{\beta} = \hat{\alpha}) * \cdot$.

Lemma 68 (Transitivity of Separation). Go to proof

If
$$(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Theta_L * \Theta_R)$$
 and $(\Theta_L * \Theta_R) \xrightarrow{*} (\Delta_L * \Delta_R)$ then $(\Gamma_I * \Gamma_R) \xrightarrow{*} (\Delta_I * \Delta_R)$.

Lemma 69 (Separation Truncation). *Go to proof*

If H has the form $\alpha : \kappa$ or $\triangleright_{\hat{\alpha}}$ or \triangleright_{P} or x : Ap

and
$$(\Gamma_L * (\Gamma_R, H)) \xrightarrow{*} (\Delta_L * \Delta_R)$$

then
$$(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_0)$$
 where $\Delta_R = (\Delta_0, H, \Theta)$.

Lemma 70 (Separation for Auxiliary Judgments). Go to proof

- $$\begin{split} \textit{(i)} & \textit{ If } \Gamma_L * \Gamma_R \vdash \sigma \stackrel{\text{\tiny \circ}}{=} \tau : \kappa \dashv \Delta \\ & \textit{ and } \mathsf{FEV}(\sigma) \cup \mathsf{FEV}(\tau) \subseteq \mathsf{dom}(\Gamma_R) \\ & \textit{ then } \Delta = (\Delta_L * \Delta_R) \textit{ and } (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R). \end{split}$$
- (ii) If $\Gamma_L * \Gamma_R \vdash P$ true $\dashv \Delta$ and $FEV(P) \subseteq dom(\Gamma_R)$ then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

```
(iii) If \Gamma_L * \Gamma_R / \sigma \stackrel{\circ}{=} \tau : \kappa \dashv \Delta
                            and FEV(\sigma) \cup FEV(\tau) = \emptyset
                           then \Delta = (\Delta_L * (\Delta_R, \Theta)) and (\Gamma_L * (\Gamma_R, \Theta)) \xrightarrow{*} (\Delta_L * \Delta_R).
                  (iv) If \Gamma_L * \Gamma_R / P \dashv \Delta
                            and FEV(P) = \emptyset
                           then \Delta = (\Delta_L * (\Delta_R, \Theta)) and (\Gamma_L * (\Gamma_R, \Theta)) \xrightarrow{*} (\Delta_L * \Delta_R).
                    (v) If \Gamma_L * \Gamma_R \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta
                            and (FEV(\tau) \cup \{\hat{\alpha}\}) \subseteq dom(\Gamma_R)
                            then \Delta = (\Delta_I * \Delta_R) and (\Gamma_I * \Gamma_R) \xrightarrow{*} (\Delta_I * \Delta_R).
                  (vi) If \Gamma_L * \Gamma_R \vdash P \equiv Q \dashv \Delta
                           and FEV(P) \cup FEV(Q) \subseteq dom(\Gamma_R)
                           then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R).
                 (vii) If \Gamma_{I} * \Gamma_{R} \vdash A \equiv B \dashv \Delta
                           and FEV(A) \cup FEV(B) \subseteq dom(\Gamma_R)
                            then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R).
                 Lemma 71 (Separation for Subtyping). Go to proof
                       If \Gamma_L * \Gamma_R \vdash A <: \mathcal{P} B \dashv \Delta
                 and FEV(A) \subseteq dom(\Gamma_R)
                 and FEV(B) \subseteq dom(\Gamma_R)
                 then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R).
                 Lemma 72 (Separation—Main). Go to proof
          (Spines) If \Gamma_L * \Gamma_R \vdash s : A \mathfrak{p} \gg C \mathfrak{q} \dashv \Delta
                            or \Gamma_{L} * \Gamma_{R} \vdash s : A p \gg C \lceil q \rceil \dashv \Delta
                            and \Gamma_L * \Gamma_R \vdash A p type
                            and FEV(A) \subseteq dom(\Gamma_R)
                            then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R) and \mathsf{FEV}(C) \subseteq \mathsf{dom}(\Delta_R).
     (Checking) If \Gamma_{\rm I} * \Gamma_{\rm R} \vdash e \Leftarrow C \mathfrak{p} \dashv \Delta
                            and \Gamma_{I} * \Gamma_{R} \vdash C p type
                            and FEV(C) \subseteq dom(\Gamma_R)
                            then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R).
    (Synthesis) If \Gamma_L * \Gamma_R \vdash e \Rightarrow A \mathfrak{p} \dashv \Delta
                            then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R).
          (Match) If \Gamma_{I} * \Gamma_{R} \vdash \Pi :: \vec{A} \neq C \not = \Delta
                            and FEV(\vec{A}) = \emptyset
                            and FEV(C) \subseteq dom(\Gamma_R)
                            then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R).
(Match Elim.) If \Gamma_{I} * \Gamma_{R} / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta
                            and FEV(P) = \emptyset
                            and FEV(\vec{A}) = \emptyset
                            and FEV(C) \subseteq dom(\Gamma_R)
                            then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R).
```

H Decidability of Algorithmic Subtyping

Definition 6. The following connectives are large:



A type is large iff its head connective is large. (Note that a non-large type may contain large connectives, provided they are not in head position.)

The number of these connectives in a type A is denoted by #large(A).

H.1 Lemmas for Decidability of Subtyping

Lemma 73 (Substitution Isn't Large). Go to proof

For all contexts Θ , we have $\# large([\Theta]A) = \# large(A)$.

Lemma 74 (Instantiation Solves). Go to proof

If $\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$ and $[\Gamma]\tau = \tau$ and $\hat{\alpha} \notin FV([\Gamma]\tau)$ then $|\mathsf{unsolved}(\Gamma)| = |\mathsf{unsolved}(\Delta)| + 1$.

Lemma 75 (Checkeq Solving). *Go to proof* If $\Gamma \vdash s \stackrel{\circ}{=} t : \kappa \dashv \Delta$ then either $\Delta = \Gamma$ or $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)|$.

Lemma 76 (Prop Equiv Solving). Go to proof

If $\Gamma \vdash P \equiv Q \dashv \Delta$ then either $\Delta = \Gamma$ or $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)|$.

Lemma 77 (Equiv Solving). Go to proof

If $\Gamma \vdash A \equiv B \dashv \Delta$ then either $\Delta = \Gamma$ or $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)|$.

Lemma 78 (Decidability of Propositional Judgments). Go to proof

The following judgments are decidable, with Δ as output in (1)–(3), and Δ^{\perp} as output in (4) and (5).

We assume $\sigma = [\Gamma]\sigma$ and $t = [\Gamma]t$ in (1) and (4). Similarly, in the other parts we assume $P = [\Gamma]P$ and (in part (3)) $Q = [\Gamma]Q$.

- (1) $\Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta$
- (2) $\Gamma \vdash P true \dashv \Delta$
- (3) $\Gamma \vdash P \equiv Q \dashv \Delta$
- (4) $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta^{\perp}$
- (5) $\Gamma / P \dashv \Delta^{\perp}$

Lemma 79 (Decidability of Equivalence). Go to proof

Given a context Γ and types A, B such that $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists Δ such that $\Gamma \vdash A \equiv B \dashv \Delta$.

H.2 Decidability of Subtyping

Theorem 1 (Decidability of Subtyping). *Go to proof*

Given a context Γ and types A, B such that $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists Δ such that $\Gamma \vdash A <: \mathcal{P} B \dashv \Delta$.

H.3 Decidability of Matching and Coverage

Lemma 80 (Decidability of Guardedness Judgment). *Go to proof* For any set of branches Π , the relation Π guarded is decidable.

Lemma 81 (Decidability of Expansion Judgments). *Go to proof Given branches* Π , *it is decidable whether:*

- (1) there exists a unique Π' such that $\Pi \stackrel{\times}{\sim} \Pi'$;
- (2) there exist unique $\Pi_{\rm I}$ and $\Pi_{\rm R}$ such that $\Pi \stackrel{+}{\sim} \Pi_{\rm I} \parallel \Pi_{\rm R}$;
- (3) there exists a unique Π' such that $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$;

- (4) there exists a unique Π' such that $\Pi \stackrel{1}{\leadsto} \Pi'$.
- (5) there exist unique Π_{IJ} and $\Pi_{::}$ such that $\Pi \stackrel{\mathsf{Vec}}{\leadsto} \Pi_{IJ} \parallel \Pi_{::}$.

Lemma 82 (Expansion Shrinks Size). *Go to proof We define the size of a pattern* |p| *as follows:*

$$\begin{array}{lll} |x| & = & 0 \\ |_| & = & 0 \\ |\langle p, p' \rangle| & = & 1 + |p| + |p'| \\ |O| & = & 0 \\ |\text{inj}_1 p| & = & 1 + |p| \\ |\text{inj}_2 p| & = & 1 + |p| \\ |\mathcal{I}I| & = & 1 \\ |p :: p'| & = & 1 + |p| + |p'| \end{array}$$

We lift size to branches $\pi = \vec{p} \Rightarrow e$ as follows:

$$|\mathfrak{p}_1,\ldots,\mathfrak{p}_n\Rightarrow e|=|\mathfrak{p}_1|+\ldots+|\mathfrak{p}_n|$$

We lift size to branch lists $\Pi = \pi_1 \mid \dots \mid \pi_n$ as follows:

$$|\pi_1 \text{ I } \dots \text{ I } \pi_n| = |\pi_1| + \dots + |\pi_n|$$

Now, the following properties hold:

- 1. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ then $|\Pi| = |\Pi'|$.
- 2. If $\Pi \stackrel{1}{\sim} \Pi'$ then $|\Pi| = |\Pi'|$.
- 3. If $\Pi \stackrel{\times}{\leadsto} \Pi'$ then $|\Pi| \leq |\Pi'|$.
- 4. If $\Pi \stackrel{+}{\leadsto} \Pi_L \parallel \Pi_R$ then $|\Pi| \leq |\Pi_1|$ and $|\Pi| \leq |\Pi_2|$.
- 5. If $\Pi \overset{\mathsf{Vec}}{\leadsto} \Pi_{\mathit{I}\!\mathit{J}} \parallel \Pi_{::}$ then $|\Pi_{\mathit{I}\!\mathit{J}}| \leq |\Pi|$ and $|\Pi_{::}| \leq |\Pi|$.
- 6. If Π guarded and $\Pi \overset{\text{Vec}}{\leadsto} \Pi_{\Pi} \parallel \Pi_{::}$ then $|\Pi_{\Pi}| < |\Pi|$ and $|\Pi_{::}| < |\Pi|$.

Theorem 2 (Decidability of Coverage). Go to proof

Given a context Γ , branches Π and types \vec{A} , it is decidable whether $\Gamma \vdash \Pi$ covers \vec{A} q is derivable.

H.4 Decidability of Typing

 $C \mathfrak{p} \dashv \Delta$.

Theorem 3 (Decidability of Typing). Go to proof

- (i) Synthesis: Given a context Γ, a principality p, and a term e, it is decidable whether there exist a type A and a context Δ such that Γ ⊢ e ⇒ A p ⊢ Δ.
- (ii) Spines: Given a context Γ , a spine s, a principality $\mathfrak p$, and a type A such that $\Gamma \vdash A$ type, it is decidable whether there exist a type B, a principality $\mathfrak q$ and a context Δ such that $\Gamma \vdash s : A \mathfrak p \gg B \mathfrak q \dashv \Delta$.
- (iii) Checking: Given a context Γ , a principality p, a term e, and a type B such that $\Gamma \vdash B$ type, it is decidable whether there is a context Δ such that $\Gamma \vdash e \Leftarrow B \ p \dashv \Delta$.
- (iv) Matching: Given a context Γ , branches Π , a list of types \vec{A} , a type C, and a principality p, it is decidable whether there exists Δ such that $\Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta$.

 Also, if given a proposition P as well, it is decidable whether there exists Δ such that $\Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow$

I Determinacy 30

I Determinacy

Lemma 83 (Determinacy of Auxiliary Judgments). Go to proof

- (1) Elimeq: Given Γ , σ , t, κ such that $\mathsf{FEV}(\sigma) \cup \mathsf{FEV}(t) = \emptyset$ and $\mathcal{D}_1 :: \Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_1^{\perp}$ and $\mathcal{D}_2 :: \Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_2^{\perp}$, it is the case that $\Delta_1^{\perp} = \Delta_2^{\perp}$.
- (2) Instantiation: Given Γ , $\hat{\alpha}$, t, κ such that $\hat{\alpha} \in \text{unsolved}(\Gamma)$ and $\Gamma \vdash t : \kappa$ and $\hat{\alpha} \notin FV(t)$ and $\mathcal{D}_1 :: \Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta_2$ it is the case that $\Delta_1 = \Delta_2$.
- (3) Symmetric instantiation:

```
Given \Gamma, \hat{\alpha}, \hat{\beta}, \kappa such that \hat{\alpha}, \hat{\beta} \in \mathsf{unsolved}(\Gamma) and \hat{\alpha} \neq \hat{\beta} and \mathcal{D}_1 :: \Gamma \vdash \hat{\alpha} := \hat{\beta} : \kappa \dashv \Delta_1 and \mathcal{D}_2 :: \Gamma \vdash \hat{\beta} := \hat{\alpha} : \kappa \dashv \Delta_2 it is the case that \Delta_1 = \Delta_2.
```

- (4) Checkeq: Given Γ , σ , t, κ such that $\mathcal{D}_1 :: \Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_2$ it is the case that $\Delta_1 = \Delta_2$.
- (5) Elimprop: Given Γ , P such that $\mathcal{D}_1 :: \Gamma / P \dashv \Delta_1^{\perp}$ and $\mathcal{D}_2 :: \Gamma / P \dashv \Delta_2^{\perp}$ it is the case that $\Delta_1 = \Delta_2$.
- (6) Checkprop: Given Γ , P such that $\mathcal{D}_1 :: \Gamma \vdash P$ true $\dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash P$ true $\dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Lemma 84 (Determinacy of Equivalence). Go to proof

- (1) Propositional equivalence: Given Γ , P, Q such that $\mathcal{D}_1 :: \Gamma \vdash P \equiv Q \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash P \equiv Q \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.
- (2) Type equivalence: Given Γ , A, B such that $\mathcal{D}_1 :: \Gamma \vdash A \equiv B \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash A \equiv B \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Theorem 4 (Determinacy of Subtyping). Go to proof

(1) Subtyping: Given Γ , e, A, B such that $\mathcal{D}_1 :: \Gamma \vdash A <: \mathcal{P} B \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash A <: \mathcal{P} B \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Theorem 5 (Determinacy of Typing). *Go to proof*

- (1) Checking: Given Γ , e, A, p such that $\mathcal{D}_1 :: \Gamma \vdash e \Leftarrow A p \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash e \Leftarrow A p \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.
- (2) Synthesis: Given Γ , e such that $\mathcal{D}_1 :: \Gamma \vdash e \Rightarrow B_1 p_1 \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash e \Rightarrow B_2 p_2 \dashv \Delta_2$, it is the case that $B_1 = B_2$ and $p_1 = p_2$ and $\Delta_1 = \Delta_2$.
- (3) Spine judgments:

```
Given \Gamma, e, A, p such that \mathcal{D}_1 :: \Gamma \vdash e : A p \gg C_1 q_1 \dashv \Delta_1 and \mathcal{D}_2 :: \Gamma \vdash e : A p \gg C_2 q_2 \dashv \Delta_2, it is the case that C_1 = C_2 and q_1 = q_2 and \Delta_1 = \Delta_2.
```

The same applies for derivations of the principality-recovering judgments $\Gamma \vdash e : A p \gg C_k \lceil q_k \rceil \dashv \Delta_k$.

(4) Match judgments:

```
Given \Gamma, \Pi, \vec{A}, p, C such that \mathcal{D}_1 :: \Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta_1 and \mathcal{D}_2 :: \Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta_2, it is the case that \Delta_1 = \Delta_2.

Given \Gamma, P, \Pi, \vec{A}, p, C such that \mathcal{D}_1 :: \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta_1 and \mathcal{D}_2 :: \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta_2, it is the case that \Delta_1 = \Delta_2.
```

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J Soundness

J.1 Soundness of Instantiation

Lemma 85 (Soundness of Instantiation). *Go to proof* If $\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$ and $\hat{\alpha} \notin FV([\Gamma]\tau)$ and $[\Gamma]\tau = \tau$ and $\Delta \longrightarrow \Omega$ then $[\Omega]\hat{\alpha} = [\Omega]\tau$.

J.2 Soundness of Checkeq

Lemma 86 (Soundness of Checkeq). *Go to proof* If $\Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta$ where $\Delta \longrightarrow \Omega$ then $[\Omega]\sigma = [\Omega]t$.

J.3 Soundness of Equivalence (Propositions and Types)

Lemma 87 (Soundness of Propositional Equivalence). *Go to proof If* $\Gamma \vdash P \equiv Q \dashv \Delta$ *where* $\Delta \longrightarrow \Omega$ *then* $[\Omega]P = [\Omega]Q$.

Lemma 88 (Soundness of Algorithmic Equivalence). *Go to proof If* $\Gamma \vdash A \equiv B \dashv \Delta$ *where* $\Delta \longrightarrow \Omega$ *then* $[\Omega]A = [\Omega]B$.

J.4 Soundness of Checkprop

Lemma 89 (Soundness of Checkprop). *Go to proof If* $\Gamma \vdash P$ *true* $\dashv \Delta$ *and* $\Delta \longrightarrow \Omega$ *then* $\Psi \vdash [\Omega]P$ *true*.

J.5 Soundness of Eliminations (Equality and Proposition)

Lemma 90 (Soundness of Equality Elimination). *Go to proof* If $[\Gamma]\sigma = \sigma$ and $[\Gamma]t = t$ and $\Gamma \vdash \sigma : \kappa$ and $\Gamma \vdash t : \kappa$ and $F \vDash V(\sigma) \cup F \vDash V(t) = \emptyset$, then:

```
(1) If \Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta

then \Delta = (\Gamma, \Theta) where \Theta = (\alpha_1 = t_1, \dots, \alpha_n = t_n) and

for all \Omega such that \Gamma \longrightarrow \Omega

and all t' such that \Omega \vdash t' : \kappa',

it is the case that [\Omega, \Theta]t' = [\theta][\Omega]t', where \theta = \mathsf{mgu}(\sigma, t).
```

(2) If $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot$ then $mgu(\sigma, t) = \bot$ (that is, no most general unifier exists).

J.6 Soundness of Subtyping

Theorem 6 (Soundness of Algorithmic Subtyping). *Go to proof* If $[\Gamma]A = A$ and $[\Gamma]B = B$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $\Delta \longrightarrow \Omega$ and $\Gamma \vdash A <:^{\mathcal{P}} B \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]A \leq^{\mathcal{P}} [\Omega]B$.

J.7 Soundness of Typing

Theorem 7 (Soundness of Match Coverage). Go to proof

- 1. If $\Gamma \vdash \Pi$ covers \vec{A} q and $\Gamma \vdash \vec{A}$ q types and $[\Gamma] \vec{A} = \vec{A}$ and $\Gamma \longrightarrow \Omega$ then $[\Omega] \Gamma \vdash \Pi$ covers \vec{A} q.
- 2. If $\Gamma / P \vdash \Pi$ covers \vec{A} ! and $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash \vec{A}$! types and $[\Gamma]\vec{A} = \vec{A}$ and $[\Gamma]P = P$ then $[\Omega]\Gamma / P \vdash \Pi$ covers \vec{A} !.

Lemma 91 (Well-formedness of Algorithmic Typing). *Go to proof Given* Γ *ctx*:

- (i) If $\Gamma \vdash e \Rightarrow A p \dashv \Delta$ then $\Delta \vdash A p$ type.
- (ii) If $\Gamma \vdash s : A p \gg B q \dashv \Delta$ and $\Gamma \vdash A p$ type then $\Delta \vdash B q$ type.

Definition 7 (Measure). Let measure \mathcal{M} on typing judgments be a lexicographic ordering:

- 1. first, the subject expression e, spine s, or matches Π —regarding all types in annotations as equal in size;
- 2. second, the partial order on judgment forms where an ordinary spine judgment is smaller than a principality-recovering spine judgment—and with all other judgment forms considered equal in size; and,
- 3. third, the derivation height.

$$\left\langle \begin{array}{ccc} & \text{ordinary spine judgment} \\ e/s/\Pi, & < & , & \text{height}(\mathcal{D}) \\ & \text{recovering spine judgment} \end{array} \right. ,$$

Note that this definition doesn't take notice of whether a spine judgment is declarative or algorithmic.

This measure works to show soundness and completeness. We list each rule below, along with a 3-tuple. For example, for Sub we write $\langle =, =, < \rangle$, meaning that each judgment to which we need to apply the i.h. has a subject of the same size (=), a judgment form of the same size (=), and a smaller derivation height (<). We write "—" when a part of the measure need not be considered because a lexicographically more significant part is smaller, as in the Anno rule, where the premise has a smaller subject: $\langle <, -, - \rangle$.

Algorithmic rules (soundness cases):

- Var, 1l, 1l\hata, EmptySpine and Nil have no premises, or only auxiliary judgments as premises.
- Sub: $\langle =, =, < \rangle$
- Anno: $\langle <, -, \rangle$
- $\forall I, \forall Spine, \exists I, \land I: \langle =, =, < \rangle$
- \supset I: $\langle =, =, < \rangle$
- $\supset I \perp$ has only an auxiliary judgment, to which we need not apply the i.h., putting it in the same class as the rules with no premises.
- \supset Spine: $\langle =, =, < \rangle$
- \rightarrow I, \rightarrow I $\hat{\alpha}$, \rightarrow E, Rec: $\langle <$, -, \rangle
- SpineRecover: $\langle =, <, \rangle$
- SpinePass: $\langle =, <, \rangle$
- \rightarrow Spine, $+I_k$, $+I\hat{\alpha}_k$, $\times I$, $\times I\hat{\alpha}$, Cons: $\langle <, -, \rangle$
- $\hat{\alpha}$ Spine: $\langle =, =, < \rangle$
- Case: $\langle <, -, \rangle$

Declarative rules (completeness cases):

- DeclVar, DeclII, DeclEmptySpine and DeclNil have no premises, or only auxiliary judgments as premises.
- DeclSub: ⟨=,=,<⟩
- DeclAnno: ⟨<, -, -⟩

- Decl \forall I, Decl \forall Spine, Decl \exists I, Decl \land I, Decl \supset I, Decl \supset Spine: $\langle =, =, < \rangle$
- Decl \rightarrow I, Decl \rightarrow E, DeclRec: $\langle <, -, \rangle$
- DeclSpineRecover: $\langle =, <, \rangle$
- DeclSpinePass: $\langle =, <, \rangle$
- Decl \rightarrow Spine, Decl+I_k, Decl \times I, DeclCase, DeclCons, $\langle <, -, \rangle$

Definition 8 (Eagerness).

A derivation \mathcal{D} whose conclusion is \mathcal{J} is eager if:

- (i) J = Γ ⊢ e ← A p ⊢ Δ
 if Γ ⊢ A p type and A = [Γ]A implies that every subderivation of D is eager.
- (ii) $\mathcal{J} = \Gamma \vdash e \Rightarrow A p \dashv \Delta$ if $A = [\Delta]A$ and every subderivation of \mathcal{D} is eager.
- (iii) $\mathcal{J} = \Gamma \vdash s : A p \gg B q \dashv \Delta$ if $\Gamma \vdash A p$ type and $A = [\Gamma]A$ implies that $B = [\Delta]B$ and every subderivation of \mathcal{D} is eager.
- (iv) $\mathcal{J} = \Gamma \vdash s : A p \gg B \lceil q \rceil \dashv \Delta$ if $\Gamma \vdash A p$ type and $A = \lceil \Gamma \rceil A$ implies that $B = \lceil \Delta \rceil B$ and every subderivation of \mathcal{D} is eager.
- (v) $\mathcal{J} = \Gamma \vdash \Pi :: \vec{A} \neq C \neq C \neq \Delta$ if $\Gamma \vdash \vec{A} \neq T$ implies and $[\Gamma]\vec{A} = \vec{A}$ and $\Gamma \vdash C \neq T$ type and $C = [\Gamma]C$ implies that every subderivation of \mathcal{D} is eager.
- (vi) $\mathcal{J} = \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta$ if $\Gamma \vdash \vec{A} !$ types and $\Gamma \vdash P$ prop and $[\Gamma] \vec{A} = \vec{A}$ and $\Gamma \vdash C p$ type and $C = [\Gamma] C$ implies that every subderivation of \mathcal{D} is eager.

Theorem 8 (Eagerness of Types). Go to proof

- (i) If \mathcal{D} derives $\Gamma \vdash e \Leftarrow A p \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = [\Gamma]A$ then \mathcal{D} is eager.
- (ii) If \mathcal{D} derives $\Gamma \vdash e \Rightarrow A p \dashv \Delta$ then \mathcal{D} is eager.
- (iii) If \mathcal{D} derives $\Gamma \vdash s : A p \gg B q \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = [\Gamma]A$ then \mathcal{D} is eager.
- (iv) If \mathcal{D} derives $\Gamma \vdash s : A p \gg B \lceil q \rceil \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = [\Gamma]A$ then \mathcal{D} is eager.
- (v) If \mathcal{D} derives $\Gamma \vdash \Pi :: \vec{A} \neq C \neq C \neq \Delta$ and $\Gamma \vdash \vec{A} \neq D$ and $\Gamma \vdash \vec{A} \neq D$ and $\Gamma \vdash C \neq D$ type then \mathcal{D} is eager.

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(vi) If \mathcal{D} derives \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta and \Gamma \vdash P prop and FEV(P) = \emptyset and [\Gamma]P = P
          and \Gamma \vdash \vec{A}! types and \Gamma \vdash C p type
          then \mathcal{D} is eager.
Theorem 9 (Soundness of Algorithmic Typing). Go to proof
Given \Delta \longrightarrow \Omega:
   (i) If \Gamma \vdash e \Leftarrow A p \dashv \Delta and \Gamma \vdash A p type and A = [\Gamma]A then [\Omega]\Delta \vdash [\Omega]e \Leftarrow [\Omega]A p.
  (ii) If \Gamma \vdash e \Rightarrow A p \dashv \Delta then [\Omega]\Delta \vdash [\Omega]e \Rightarrow [\Omega]A p.
 (iii) If \Gamma \vdash s : A p \gg B q \dashv \Delta and \Gamma \vdash A p type and A = [\Gamma]A then [\Omega]\Delta \vdash [\Omega]s : [\Omega]A p \gg [\Omega]B q.
 (iv) If \Gamma \vdash s : A p \gg B \lceil q \rceil + \Delta and \Gamma \vdash A p type and A = \lceil \Gamma \mid A then \lceil \Omega \mid \Delta \vdash \lceil \Omega \mid s : \lceil \Omega \mid A p \gg \lceil \Omega \mid B \lceil q \rceil.
  (v) If \Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta and \Gamma \vdash \vec{A} ! types and [\Gamma] \vec{A} = \vec{A} and \Gamma \vdash C p type
          then \mathfrak{p} \vdash [\Omega]\Delta :: [\Omega]\Pi ! \Leftarrow [\Omega]\vec{A} \mathfrak{q}[\Omega]C.
 (vi) If \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta and \Gamma \vdash P prop and FEV(P) = \emptyset and [\Gamma]P = P
          and \Gamma \vdash \vec{A}! types and \Gamma \vdash C p type
          then [\Omega]\Delta / [\Omega]P \vdash [\Omega]\Pi :: [\Omega]\vec{A}! \Leftarrow [\Omega]C p.
         Completeness
K
K.1 Completeness of Auxiliary Judgments
Lemma 92 (Completeness of Instantiation). Go to proof
Given \Gamma \longrightarrow \Omega and dom(\Gamma) = dom(\Omega) and \Gamma \vdash \tau : \kappa and \tau = [\Gamma]\tau and \hat{\alpha} \in unsolved(\Gamma) and \hat{\alpha} \notin FV(\tau):
If [\Omega]\hat{\alpha} = [\Omega]\tau
then there are \Delta, \Omega' such that \Omega \longrightarrow \Omega' and \Delta \longrightarrow \Omega' and dom(\Delta) = dom(\Omega') and \Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta.
Lemma 93 (Completeness of Checkeq). Go to proof
Given \Gamma \longrightarrow \Omega and dom(\Gamma) = dom(\Omega)
and \Gamma \vdash \sigma : \kappa and \Gamma \vdash \tau : \kappa
and [\Omega]\sigma = [\Omega]\tau
then \Gamma \vdash [\Gamma] \sigma \stackrel{\circ}{=} [\Gamma] \tau : \kappa \dashv \Delta
where \Delta \longrightarrow \Omega' and dom(\Delta) = dom(\Omega') and \Omega \longrightarrow \Omega'.
Lemma 94 (Completeness of Elimeq). Go to proof
If [\Gamma]\sigma = \sigma and [\Gamma]t = t and \Gamma \vdash \sigma : \kappa and \Gamma \vdash t : \kappa and FEV(\sigma) \cup FEV(t) = \emptyset then:
(1) If mgu(\sigma, t) = \theta
        then \Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv (\Gamma, \Delta)
        where \Delta has the form \alpha_1 = t_1, \dots, \alpha_n = t_n
        and for all u such that \Gamma \vdash u : \kappa, it is the case that [\Gamma, \Delta]u = \theta([\Gamma]u).
(2) If mgu(\sigma, t) = \bot (that is, no most general unifier exists) then \Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot.
Lemma 95 (Substitution Upgrade). Go to proof
If \Delta has the form \alpha_1 = t_1, \ldots, \alpha_n = t_n
and, for all u such that \Gamma \vdash u : \kappa, it is the case that [\Gamma, \Delta]u = \theta([\Gamma]u),
then:
   (i) If \Gamma \vdash A type then [\Gamma, \Delta]A = \theta([\Gamma]A).
  (ii) If \Gamma \longrightarrow \Omega then [\Omega]\Gamma = \theta([\Omega]\Gamma).
```

(iii) If $\Gamma \longrightarrow \Omega$ then $[\Omega, \Delta](\Gamma, \Delta) = \theta([\Omega]\Gamma)$.

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(iv) If \Gamma \longrightarrow \Omega then [\Omega, \Delta]e = \theta([\Omega]e).

Lemma 96 (Completeness of Propequiv). Go to proof Given \Gamma \longrightarrow \Omega and \Gamma \vdash P prop and \Gamma \vdash Q prop and [\Omega]P = [\Omega]Q then \Gamma \vdash [\Gamma]P \equiv [\Gamma]Q \dashv \Delta where \Delta \longrightarrow \Omega' and \Omega \longrightarrow \Omega'.

Lemma 97 (Completeness of Checkprop). Go to proof If \Gamma \longrightarrow \Omega and \operatorname{dom}(\Gamma) = \operatorname{dom}(\Omega) and \Gamma \vdash P prop and [\Gamma]P = P and [\Omega]\Gamma \vdash [\Omega]P true then \Gamma \vdash P true \dashv \Delta where \Delta \longrightarrow \Omega' and \Omega \longrightarrow \Omega' and \operatorname{dom}(\Delta) = \operatorname{dom}(\Omega').
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K.2 Completeness of Equivalence and Subtyping

K.3 Completeness of Typing

Lemma 99 (Variable Decomposition). Go to proof If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$, then

```
1. if \Pi \stackrel{1}{\leadsto} \Pi'' then \Pi'' = \Pi'.
```

- 2. if $\Pi \stackrel{\times}{\sim} \Pi'''$ then there exists Π'' such that $\Pi''' \stackrel{\text{var}}{\sim} \Pi''$ and $\Pi'' \stackrel{\text{var}}{\sim} \Pi'$,
- 3. if $\Pi \stackrel{+}{\sim} \Pi_L \parallel \Pi_R$ then $\Pi_L \stackrel{\text{var}}{\sim} \Pi'$ and $\Pi_R \stackrel{\text{var}}{\sim} \Pi'$,
- 4. if $\Pi \stackrel{\mathsf{Vec}}{\leadsto} \Pi_{[]} \parallel \Pi_{::}$ then $\Pi' = \Pi_{[]}$.

Lemma 100 (Pattern Decomposition and Substitution). Go to proof

```
1. If \Pi \stackrel{\text{var}}{\leadsto} \Pi' then [\Omega]\Pi \stackrel{\text{var}}{\leadsto} [\Omega]\Pi'.
```

- 2. If $\Pi \stackrel{1}{\sim} \Pi'$ then $[\Omega]\Pi \stackrel{1}{\sim} [\Omega]\Pi'$.
- 3. If $\Pi \stackrel{\times}{\leadsto} \Pi'$ then $[\Omega]\Pi \stackrel{\times}{\leadsto} [\Omega]\Pi'$.
- 4. If $\Pi \stackrel{+}{\leadsto} \Pi_1 \parallel \Pi_2$ then $[\Omega] \Pi \stackrel{+}{\leadsto} [\Omega] \Pi_1 \parallel [\Omega] \Pi_2$.
- 5. If $\Pi \stackrel{\mathsf{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$ then $[\Omega] \Pi \stackrel{\mathsf{Vec}}{\leadsto} [\Omega] \Pi_1 \parallel [\Omega] \Pi_2$.
- 6. If $[\Omega] \prod \overset{\text{var}}{\leadsto} \Pi'$ then there is Π'' such that $[\Omega] \Pi'' = \Pi'$ and $\Pi \overset{\text{var}}{\leadsto} \Pi''$.

- 7. If $[\Omega] \prod \stackrel{1}{\sim} \prod'$ then there is \prod'' such that $[\Omega] \prod'' = \prod'$ and $\prod \stackrel{1}{\sim} \prod''$.
- 8. If $[\Omega] \Pi \stackrel{\times}{\leadsto} \Pi'$ then there is Π'' such that $[\Omega] \Pi'' = \Pi'$ and $\Pi \stackrel{\times}{\leadsto} \Pi''$.
- 9. If $[\Omega]\Pi \stackrel{+}{\sim} \Pi'_1 \parallel \Pi'_2$ then there are Π_1 and Π_2 such that $[\Omega]\Pi_1 = \Pi'_1$ and $[\Omega]\Pi_2 = \Pi'_2$ and $\Pi \stackrel{+}{\sim} \Pi_1 \parallel \Pi_2$.
- 10. If $[\Omega]\Pi \overset{\text{Vec}}{\leadsto} \Pi_1' \parallel \Pi_2'$ then there are Π_1 and Π_2 such that $[\Omega]\Pi_1 = \Pi_1'$ and $[\Omega]\Pi_2 = \Pi_2'$ and $\Pi \overset{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$.

Lemma 101 (Pattern Decomposition Functionality). Go to proof

- 1. If $\Pi \overset{\text{var}}{\leadsto} \Pi'$ and $\Pi \overset{\text{var}}{\leadsto} \Pi''$ then $\Pi' = \Pi''$.
- 2. If $\Pi \stackrel{1}{\leadsto} \Pi'$ and $\Pi \stackrel{1}{\leadsto} \Pi''$ then $\Pi' = \Pi''$.
- 3. If $\Pi \stackrel{\times}{\leadsto} \Pi'$ and $\Pi \stackrel{\times}{\leadsto} \Pi''$ then $\Pi' = \Pi''$.
- 4. If $\Pi \stackrel{+}{\sim} \Pi_1 \parallel \Pi_2$ and $\Pi \stackrel{+}{\sim} \Pi_1' \parallel \Pi_2'$ then $\Pi_1 = \Pi_1'$ and $\Pi_2 = \Pi_2'$.
- 5. If $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$ and $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$ then $\Pi_1 = \Pi'_1$ and $\Pi_2 = \Pi'_2$.

Lemma 102 (Decidability of Variable Removal). *Go to proof* For all Π , either there exists a Π' such that $\Pi \stackrel{\text{var}}{\rightarrow} \Pi'$ or there does not.

Lemma 103 (Variable Inversion). Go to proof

- 1. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ and $\Psi \vdash \Pi$ covers A, \vec{A} q then $\Psi \vdash \Pi'$ covers \vec{A} q.
- 2. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ and $\Gamma \vdash \Pi$ covers $A, \vec{A} \neq 0$, then $\Gamma \vdash \Pi'$ covers $\vec{A} \neq 0$.

Theorem 11 (Completeness of Match Coverage). Go to proof

- 1. If $\Gamma \vdash \vec{A}$ q types and $[\Gamma]\vec{A} = \vec{A}$ and (for all Ω such that $\Gamma \longrightarrow \Omega$, we have $[\Omega]\Gamma \vdash [\Omega]\Pi$ covers $[\Omega]\vec{A}$ q) then $\Gamma \vdash \Pi$ covers \vec{A} q.
- 2. If $[\Gamma]\vec{A} = \vec{A}$ and $[\Gamma]P = P$ and $\Gamma \vdash \vec{A}$! types and (for all Ω such that $\Gamma \longrightarrow \Omega$, we have $[\Omega]\Gamma / [\Omega]P \vdash [\Omega]\Pi$ covers $[\Omega]\vec{A}$!) then $\Gamma / P \vdash \Pi$ covers \vec{A} !.

Theorem 12 (Completeness of Algorithmic Typing). *Go to proof Given* $\Gamma \longrightarrow \Omega$ *such that* dom(Γ) = dom(Ω):

- (i) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]e \Leftarrow [\Omega]A$ p and $\mathfrak{p}' \sqsubseteq \mathfrak{p}$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Leftarrow [\Gamma]A$ $\mathfrak{p}' \dashv \Delta$.
- (ii) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]e \Rightarrow A$ p then there exist Δ , Ω' , A', and $\mathfrak{p}' \sqsubseteq \mathfrak{p}$ such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Rightarrow A' \mathfrak{p}' \dashv \Delta$ and $A' = [\Delta]A'$ and $A = [\Omega']A'$.
- (iii) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A$ p \gg B q and p' \sqsubseteq p then there exist Δ , Ω' , B' and q' \sqsubseteq q such that $\Delta \longrightarrow \Omega'$ and dom(Δ) = dom(Ω') and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash s : [\Gamma]A$ p' \gg B' q' $\dashv \Delta$ and B' = $[\Delta]B'$ and B = $[\Omega']B'$.
- (iv) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A$ p $\gg B$ $\lceil q \rceil$ and p' \sqsubseteq p then there exist Δ , Ω' , B', and $q' \sqsubseteq q$ such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash s : [\Gamma]A$ p' $\gg B'$ $\lceil q' \rceil \dashv \Delta$ and $B' = [\Delta]B'$ and $B = [\Omega']B'$.

- (v) If $\Gamma \vdash \vec{A}$! types and $\Gamma \vdash C$ p type and $[\Omega]\Gamma \vdash [\Omega]\Pi :: [\Omega]\vec{A}$ q $\Leftarrow [\Omega]C$ p and p' \sqsubseteq p then there exist Δ , Ω' , and C such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash \Pi :: [\Gamma]\vec{A}$ q $\Leftarrow [\Gamma]C$ p' $\dashv \Delta$.
- (vi) If $\Gamma \vdash \vec{A}$! types and $\Gamma \vdash P$ prop and $FEV(P) = \emptyset$ and $\Gamma \vdash C$ p type and $[\Omega]\Gamma / [\Omega]P \vdash [\Omega]\Pi :: [\Omega]\vec{A} ! \Leftarrow [\Omega]C$ p and $p' \sqsubseteq p$ then there exist Δ , Ω' , and C such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma / [\Gamma]P \vdash \Pi :: [\Gamma]\vec{A} ! \Leftarrow [\Gamma]C$ $p' \dashv \Delta$.

Proofs

In the rest of this document, we prove the results stated above, with the same sectioning.

A' Properties of the Declarative System

Lemma 1 (Declarative Well-foundedness).

The inductive definition of the following judgments is well-founded:

- (i) synthesis $\Psi \vdash e \Rightarrow B p$
- (ii) checking $\Psi \vdash e \Leftarrow A p$
- (iii) checking, equality elimination $\Psi / P \vdash e \Leftarrow C p$
- (iv) ordinary spine $\Psi \vdash s : A \mathfrak{p} \gg B \mathfrak{q}$
- (v) recovery spine $\Psi \vdash s : A \mathfrak{p} \gg B [\mathfrak{q}]$
- (vi) pattern matching $\Psi \vdash \Pi :: \vec{A} ! \Leftarrow C p$
- (vii) pattern matching, equality elimination $\Psi / P \vdash \Pi :: \vec{A} ! \Leftarrow C p$

Proof. Let |e| be the size of the expression e. Let |s| be the size of the spine s. Let $|\Pi|$ be the size of the branch list Π . Let # |arge(A)| be the number of "large" connectives \forall , \exists , \supset , \wedge in A.

First, stratify judgments by the size of the term (expression, spine, or branches), and say that a judgment is *at* n if it types a term of size n. Order the main judgment forms as follows:

```
synthesis judgment at n < checking judgments at n < ordinary spine judgment at n < recovery spine judgment at n < match judgments at n < synthesis judgment at n + 1 :
```

Within the checking judgment forms at n, we compare types lexicographically, first by the number of large connectives, and then by the ordinary size. Within the match judgment forms at n, we compare using a lexicographic order of, first, $\# | \text{large}(\vec{A}) |$; second, the judgment form, considering the match judgment to be smaller than the matchelim judgment; third, the size of \vec{A} . These criteria order the judgments as follows:

```
synthesis judgment at n  < \quad \text{(checking judgment at n with } \# | \text{large}(A) = 1 \\ < \text{ checkelim judgment at n with } \# | \text{large}(A) = 1 \\ < \text{ checking judgment at n with } \# | \text{large}(A) = 2 \\ < \text{ checkelim judgment at n with } \# | \text{large}(\vec{A}) = 2 \\ < \dots ) \\ < \quad \text{(match judgment at n with } \# | \text{large}(\vec{A}) = 1 \text{ and } \vec{A} \text{ of size } 1 \\ < \text{ match judgment at n with } \# | \text{large}(\vec{A}) = 1 \text{ and } \vec{A} \text{ of size } 2 \\ < \text{ match judgment at n with } \# | \text{large}(\vec{A}) = 1 \\ < \text{ match judgment at n with } \# | \text{large}(\vec{A}) = 2 \text{ and } \vec{A} \text{ of size } 1 \\ < \text{ match judgment at n with } \# | \text{large}(\vec{A}) = 2 \text{ and } \vec{A} \text{ of size } 2 \\ < \text{ matchelim judgment at n with } \# | \text{large}(\vec{A}) = 2 \\ < \text{ matchelim judgment at n with } \# | \text{large}(\vec{A}) = 2 \\ < \text{ ...} )
```

The class of ordinary spine judgments at 1 need not be refined, because the only ordinary spine rule applicable to a spine of size 1 is DeclEmptySpine, which has no premises; rules Decl \forall Spine, Decl \supset Spine, and Decl \rightarrow Spine are restricted to non-empty spines and can only apply to larger terms.

Similarly, the class of match judgments at 1 need not be refined, because only DeclMatchEmpty is applicable.

Note that we distinguish the "checkelim" form $\Psi / P \vdash e \Leftarrow A p$ of the checking judgment. We also define the size of an expression e to consider all types in annotations to be of the same size, that is,

$$|(e:A)| = |e| + 1$$

Thus, $|\theta(e)| = |e|$, even when e has annotations. This is used for DeclCheckUnify; see below.

We assume that coverage, which does not depend on any other typing judgments, is well-founded. We likewise assume that subtyping, $\Psi \vdash A$ *type*, $\Psi \vdash \tau : \kappa$, and $\Psi \vdash P$ *prop* are well-founded.

We now show that, for each class of judgments, every judgment in that class depends only on smaller judgments.

• Synthesis judgments

Claim: For all n, synthesis at n depends only on judgments at n-1 or less.

Proof. Rule DeclVar has no premises.

Rule DeclAnno depends on a premise at a strictly smaller term.

Rule Decl \rightarrow E depends on (1) a synthesis premise at a strictly smaller term, and (2) a recovery spine judgment at a strictly smaller term.

· Checking judgments

Claim: For all $n \ge 1$, the checking judgment over terms of size n with type of size m depends only on

- (1) synthesis judgments at size n or smaller, and
- (2) checking judgments at size n 1 or smaller, and
- (3) checking judgments at size n with fewer large connectives, and
- (4) checkelim judgments at size n with fewer large connectives, and
- (5) match judgments at size n-1 or smaller.

Proof. Rule DeclSub depends on a synthesis judgment of size n. (1)

Rule Decl11 has no premises.

Rule Declyl depends on a checking judgment at n with fewer large connectives. (3)

Rule Decl∃l depends on a checking judgment at n with fewer large connectives. (3)

Rule Decl \land I depends on a checking judgment at n with fewer large connectives. (3)

Rule $Decl \supset I$ depends on a checkelim judgment at n with fewer large connectives. (4)

Rules $Decl \rightarrow I$, DeclRec, $Decl+I_k$, $Decl \times I$, and DeclCons depend on checking judgments at size < n. (2)

Rule DeclNil depends only on an auxiliary judgment.

Rule DeclCase depends on:

- a synthesis judgment at size n (1),
- a match judgment at size < n (5), and
- a coverage judgment.

• Checkelim judgments

Claim: For all $n \ge 1$, the checkelim judgment $\Psi / P \vdash e \Leftarrow A p$ over terms of size n depends only on checking judgments at size n, with a type A' such that # | large(A') = # | large(A).

Proof. Rule DeclCheck⊥ has no nontrivial premises.

Rule DeclCheckUnify depends on a checking judgment: Since $|\theta(e)| = |e|$, this checking judgment is at n. Since the mgu θ is over monotypes, $\#large(\theta(A)) = \#large(A)$.

• Ordinary spine judgments

An ordinary spine judgment at 1 depends on no other judgments: the only spine of size 1 is the empty spine, so only DeclEmptySpine applies, and it has no premises.

Claim: For all $n \ge 2$, the ordinary spine judgment $\Psi \vdash s : A \ p \gg C \ q$ over spines of size n depends only on

- (a) checking judgments at size n-1 or smaller, and
- (b) ordinary spine judgments at size n-1 or smaller, and
- (c) ordinary spine judgments at size n with strictly smaller #large(A).

Proof. Rule Decl \forall Spine depends on an ordinary spine judgment of size n, with a type that has fewer large connectives. (c)

Rule $Decl \supset Spine$ depends on an ordinary spine judgment of size n, with a type that has fewer large connectives. (c)

Rule DeclEmptySpine has no premises.

Rule Decl \rightarrow Spine depends on a checking judgment of size n-1 or smaller (a) and an ordinary spine judgment of size n-1 or smaller (b).

• Recovery spine judgments

Claim: For all n, the recovery spine judgment at n depends only on ordinary spine judgments at n. *Proof.* Rules DeclSpineRecover and DeclSpinePass depend only on ordinary spine judgments at n.

Match judgments

Claim: For all $n \ge 1$, the match judgment $\Psi \vdash \Pi :: \vec{A} ! \Leftarrow C p$ over Π of size n depends only on

- (a) checking judgments at size n-1 or smaller, and
- (b) match judgments at size n-1 or smaller, and
- (c) match judgments at size n with smaller \vec{A} , and
- (d) matchelim judgments at size n with fewer large connectives in \vec{A} .

Proof. Rule DeclMatchEmpty has no premises.

Rule DeclMatchSeq depends on match judgments at n-1 or smaller (b).

Rule DeclMatchBase depends on a checking judgment at n-1 or smaller (a).

Rules DeclMatchUnit, DeclMatch \times , DeclMatch $+_k$, DeclMatchNeg, and DeclMatchWild depend on match judgments at n-1 or smaller (b).

Rule DeclMatch \exists depends on a match judgment at size n with smaller \vec{A} (c).

Rule DeclMatch \land depends on an matchelim judgment at n, with fewer large connectives in \vec{A} . (d)

• Matchelim judgments

Claim: For all $n \ge 1$, the matchelim judgment $\Psi / \Pi \vdash P :: \vec{A} ! \Leftarrow C p$ over Ψ of size n depends only on match judgments with the same number of large connectives in \vec{A} .

Proof. Rule DeclMatch⊥ has no nontrivial premises.

Rule DeclMatchUnify depends on a match judgment with the same number of large connectives (similar to DeclCheckUnify, considered above). \Box

Lemma 2 (Declarative Weakening).

- (i) If $\Psi_0, \Psi_1 \vdash t : \kappa$ then $\Psi_0, \Psi, \Psi_1 \vdash t : \kappa$.
- (ii) If $\Psi_0, \Psi_1 \vdash P$ prop then $\Psi_0, \Psi, \Psi_1 \vdash P$ prop.
- (iii) If $\Psi_0, \Psi_1 \vdash P$ true then $\Psi_0, \Psi, \Psi_1 \vdash P$ true.
- (iv) If $\Psi_0, \Psi_1 \vdash A$ type then $\Psi_0, \Psi, \Psi_1 \vdash A$ type.

Proof. By induction on the derivation.

Lemma 3 (Declarative Term Substitution). *Suppose* $\Psi \vdash t : \kappa$. *Then:*

- 1. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash t' : \kappa$ then $\Psi_0, [t/\alpha]\Psi_1 \vdash [t/\alpha]t' : \kappa$.
- 2. If Ψ_0 , $\alpha : \kappa, \Psi_1 \vdash P$ prop then Ψ_0 , $[t/\alpha]\Psi_1 \vdash [t/\alpha]P$ prop.
- 3. If Ψ_0 , $\alpha : \kappa, \Psi_1 \vdash A$ type then Ψ_0 , $[t/\alpha]\Psi_1 \vdash [t/\alpha]A$ type.
- 4. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash A \leq^{\mathcal{P}} B$ then $\Psi_0, [t/\alpha]\Psi_1 \vdash [t/\alpha]A \leq^{\mathcal{P}} [t/\alpha]B$.
- 5. If $\Psi_0, \alpha : \kappa, \Psi_1 \vdash P$ true then $\Psi_0, [t/\alpha]\Psi_1 \vdash [t/\alpha]P$ true.

Proof. By induction on the derivation of the substitutee.

Lemma 4 (Reflexivity of Declarative Subtyping).

Given $\Psi \vdash A$ *type, we have that* $\Psi \vdash A \leq^{\mathcal{P}} A$.

Proof. By induction on A, writing p for the sign of the subtyping judgment.

Our induction metric is the number of quantifiers on the outside of A, plus one if the polarity of A and the subtyping judgment do not match up (that is, if neg(A) and $\mathcal{P} = +$, or pos(A) and $\mathcal{P} = -$).

• **Case** *nonpos*(A), *nonneg*(A):

By rule $\leq \text{Refl}\mathcal{P}$.

• Case $A = \exists b : \kappa$. B and $\mathcal{P} = +$:

$$\begin{array}{ll} \Psi,b:\kappa\vdash B\leq^+B & \text{By i.h. (one less quantifier)}\\ \Psi,b:\kappa\vdash b:\kappa & \text{By rule UvarSort}\\ \Psi,b:\kappa\vdash B\leq^+\exists b:\kappa.\,B & \text{By rule}\leq\exists R\\ \Psi\vdash\exists b:\kappa.\,B\leq^+\exists b:\kappa.\,B & \text{By rule}\leq\exists L \end{array}$$

• Case $A = \exists b : \kappa$. B and $\mathcal{P} = -$:

$$\Psi \vdash \exists b : \kappa. B \leq^+ \exists b : \kappa. B$$
 By i.h. (polarities match) $\Psi \vdash \exists b : \kappa. B \leq^- \exists b : \kappa. B$ By \leq^+

• Case $A = \forall b : \kappa$. B and $\mathcal{P} = +$:

$$\Psi \vdash \forall b : \kappa. B \leq^- \forall b : \kappa. B$$
 By i.h. (polarities match) $\Psi \vdash \forall b : \kappa. B <^+ \forall b : \kappa. B$ By $<_{-}^{+}$

• Case $A = \forall b : \kappa$. B and P = -:

$$\begin{array}{ll} \Psi,b:\kappa\vdash B\leq^- B & \text{By i.h. (one less quantifier)} \\ \Psi,b:\kappa\vdash b:\kappa & \text{By rule UvarSort} \\ \Psi,b:\kappa\vdash \forall b:\kappa.\ B\leq^- B & \text{By rule}\leq\forall L \\ \Psi\vdash \forall b:\kappa.\ B\leq^- \forall b:\kappa.\ B & \text{By rule}\leq\forall R \end{array}$$

Lemma 5 (Subtyping Inversion).

- If $\Psi \vdash \exists \alpha : \kappa$. $A \leq^+ B$ then Ψ , $\alpha : \kappa \vdash A \leq^+ B$.
- If $\Psi \vdash A <^- \forall \beta : \kappa$. B then $\Psi, \beta : \kappa \vdash A <^- B$.

Proof. By a routine induction on the subtyping derivations.

Lemma 6 (Subtyping Polarity Flip).

- If nonpos(A) and nonpos(B) and $\Psi \vdash A \leq^+ B$ then $\Psi \vdash A \leq^- B$ by a derivation of the same or smaller size.
- If nonneg(A) and nonneg(B) and Ψ ⊢ A ≤⁻ B
 then Ψ ⊢ A ≤⁺ B by a derivation of the same or smaller size.
- If nonpos(A) and nonneg(A) and nonpos(B) and nonneg(B) and $\Psi \vdash A \leq^{\mathcal{P}} B$ then A = B.

Proof. By a routine induction on the subtyping derivations.

Lemma 7 (Transitivity of Declarative Subtyping). *Given* $\Psi \vdash A$ *type and* $\Psi \vdash B$ *type and* $\Psi \vdash C$ *type:*

(i) If
$$\mathcal{D}_1 :: \Psi \vdash A \leq^{\mathcal{P}} B$$
 and $\mathcal{D}_2 :: \Psi \vdash B \leq^{\mathcal{P}} C$ then $\Psi \vdash A \leq^{\mathcal{P}} C$.

Proof. By lexicographic induction on (1) the sum of head quantifiers in A, B, and C, and (2) the size of the derivation.

We begin by case analysis on the shape of B, and the polarity of subtyping:

• Case $B = \forall \beta : \kappa_2. B'$, polarity = -: We case-analyze \mathcal{D}_1 :

$$\begin{array}{ccc} \textbf{- Case} & \frac{\Psi \vdash \tau : \kappa_1 & \Psi \vdash [\tau/\alpha]A' \leq^- B}{\Psi \vdash \forall \alpha : \kappa_1.\, A' \leq^- B} \leq \forall L \end{array}$$

$$\begin{array}{ll} \Psi \vdash \tau : \kappa_1 & \text{Subderivation} \\ \Psi \vdash [\tau/\alpha] A' \leq^- B & \text{Subderivation} \\ \Psi \vdash B \leq^- C & \text{Given} \end{array}$$

$$\Psi \vdash [\tau/\alpha]A' \leq^- C$$
 By i.h. (A lost a quantifier)

$$\Psi \vdash A \leq^- C \qquad \qquad \text{By rule} \leq \forall L$$

$$\begin{array}{l} \textbf{- Case} \\ \frac{\Psi,\,\beta:\kappa_2\vdash A\leq^- B'}{\Psi\vdash A\leq^- \forall \beta:\kappa_2.\,B'}\leq \forall R \end{array}$$

We case-analyze \mathcal{D}_2 :

$$* \ \, \textbf{Case} \ \, \frac{\Psi \vdash \tau : \kappa_2 \qquad \Psi \vdash [\tau/\beta] B' \leq^- C}{\Psi \vdash \forall \beta : \kappa_2. \ \, B' \leq^- C} \leq \forall \mathsf{L}$$

$$\begin{array}{ll} \Psi,\beta:\kappa_2\vdash A\leq^- B' & \text{By Lemma 5 (Subtyping Inversion) on } \mathcal{D}_1 \\ \Psi\vdash\tau:\kappa_2 & \text{Subderivation} \end{array}$$

$$\Psi \vdash [\tau/\beta] B' \leq^- C$$
 Subderivation of \mathcal{D}_2 $\Psi \vdash A \leq^- [\tau/\beta] B'$ By Lemma 3 (Declarative Term Substitution)

$$\Psi \vdash A \leq^- C$$
 By i.h. (B lost a quantifier)

* Case
$$\frac{\Psi, c: \kappa_3 \vdash B \leq^- C'}{\Psi \vdash B \leq^- \forall c: \kappa_3. \ C'} \leq \forall \mathsf{R}$$

$$\Psi \vdash \mathsf{A} \leq^- \mathsf{B} \qquad \text{Given}$$

$$\Psi, c: \kappa_3 \vdash \mathsf{A} \leq^- \mathsf{B} \qquad \text{By Lemma 2 (Declarative Weakening)}$$

$$\Psi, c: \kappa_3 \vdash \mathsf{B} \leq^- C' \qquad \text{Subderivation}$$

$$\Psi, c: \kappa_3 \vdash \mathsf{A} \leq^- C' \qquad \text{By i.h. (C lost a quantifier)}$$

$$\Psi \vdash \mathsf{B} \leq^- \forall c: \kappa_3. \ C' \qquad \mathsf{By} \leq \forall \mathsf{R}$$

• Case *nonpos*(B), polarity = +:

Now we case-analyze \mathcal{D}_1 :

- Case
$$\frac{\Psi, \alpha : \tau \vdash A' \leq^{+} B}{\Psi \vdash \underbrace{\exists \alpha : \kappa_{1}. A'}_{A} \leq^{+} B} \leq \exists L$$

$$\begin{array}{ll} \Psi,\alpha:\tau\vdash A'\leq^+ B & \text{Subderivation} \\ \Psi,\alpha:\tau\vdash B\leq^+ C & \text{By Lemma 2 (Declarative Weakening) } (\mathcal{D}_2) \\ \Psi,\alpha:\tau\vdash A'\leq^+ C & \text{By i.h. (A lost a quantifier)} \\ \Psi\vdash \exists\alpha:\kappa_1.\ A'\leq^+ C & \text{By}\leq \exists L \end{array}$$

- Case
$$\frac{\Psi \vdash A \leq^- B \quad nonpos(A) \quad nonpos(B)}{\Psi \vdash A <^+ B} \leq^-_+$$

Now we case-analyze \mathcal{D}_2 :

$$* \ \, \textbf{Case} \ \, \underbrace{\frac{\Psi \vdash \tau : \kappa_3 \qquad \Psi \vdash B \leq^+ [\tau/c]C'}{\Psi \vdash B \leq^+ \underbrace{\exists c : \kappa_3. \ C'}_{C}} \leq \exists \mathsf{R}}$$

$$\begin{array}{ll} \Psi \vdash A \leq^+ B & \text{Given} \\ \Psi \vdash \tau : \kappa_3 & \text{Subderivation of } \mathcal{D}_2 \\ \Psi \vdash B \leq^+ [\tau/c]C' & \text{Subderivation of } \mathcal{D}_2 \\ \Psi \vdash A \leq^+ [\tau/c]C' & \text{By i.h. (C lost a quantifier)} \\ \Psi \vdash A \leq^+ \exists c : \kappa_3. C' & \text{By $\leq \exists R$} \end{array}$$

$$* \ \, \textbf{Case} \ \, \frac{\Psi \vdash B \leq^- C \quad \textit{nonpos}(B) \quad \textit{nonpos}(C)}{\Psi \vdash B \leq^+ C} \leq^+_+$$

$$\begin{array}{ll} \Psi \vdash A \leq^- B & \text{Subderivation of } \mathcal{D}_1 \\ \Psi \vdash B \leq^- C & \text{Subderivation of } \mathcal{D}_2 \\ \Psi \vdash A \leq^- C & \text{By i.h. } (\mathcal{D}_1 \text{ and } \mathcal{D}_2 \text{ smaller}) \\ & \textit{nonpos}(A) & \text{Subderivation of } \mathcal{D}_1 \\ & \textit{nonpos}(C) & \text{Subderivation of } \mathcal{D}_2 \\ \Psi \vdash A \leq^+ C & \text{By } \leq^-_+ \end{array}$$

• Case $B = \exists \beta : \kappa_2. B'$, polarity = +: Now we case-analyze \mathcal{D}_2 :

$$\begin{array}{ll} \Psi \vdash \tau : \kappa_3 & \quad \text{Subderivation of } \mathcal{D}_2 \\ \Psi \vdash B \leq^+ [\tau/\alpha] C' & \quad \text{Subderivation of } \mathcal{D}_2 \\ \Psi \vdash A \leq^+ B & \quad \text{Given} \end{array}$$

$$\Psi \vdash A <^+ B$$
 Given

$$\Psi \vdash A \leq^+ [\tau/\alpha]C' \quad \text{ By i.h. (C lost a quantifier)}$$

$$\Psi \vdash A \leq^+ C$$
 By rule $\leq \exists R$

$$\begin{array}{c} \textbf{- Case} \\ \frac{\Psi, \beta: \kappa_2 \vdash B' \leq^+ C}{\Psi \vdash \exists \beta: \kappa_2. \, B' \leq^+ C} \leq \exists \mathsf{L} \end{array}$$

Now we case-analyze \mathcal{D}_1 :

$$* \ \, \textbf{Case} \ \, \frac{\Psi \vdash \tau : \kappa_2 \qquad \Psi \vdash A \leq^+ [\tau/\beta]B'}{\Psi \vdash A \leq^+ \underbrace{\exists \beta : \kappa_2.\, B'}_{B}} \leq \exists \mathsf{R}$$

$$\begin{array}{ll} \Psi,\beta:\kappa_2\vdash B'\leq^+C & \text{Subderivation of }\mathcal{D}_2\\ \Psi\vdash\tau:\kappa_2 & \text{Subderivation of }\mathcal{D}_1\\ \Psi\vdash A\leq^+[\tau/\beta]B' & \text{Subderivation of }\mathcal{D}_1\\ \end{array}$$

$$\Psi \vdash [\tau/\beta]B' \leq^+ C$$
 By Lemma 3 (Declarative Term Substitution) $\Psi \vdash A <^+ C$ By i.h. (B lost a quantifier)

$$* \ \, \textbf{Case} \ \, \underbrace{\frac{\Psi,\,\alpha:\kappa_1\vdash A\,\leq^+\,B}{\Psi\vdash\underbrace{\exists\alpha:\kappa_1.\,A'}\,\leq^+\,B}}_{A} \leq \exists L$$

$$\Psi \vdash B \leq^+ C$$
 Given

$$\Psi, \alpha : \kappa_1 \vdash A' \leq^+ B$$
 Subderivation of \mathcal{D}_1

$$\begin{array}{ccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & & & \\ & & \\ & & & \\$$

$$\Psi, \alpha : \kappa_1 \vdash A' \leq^+ C$$
 By i.h. (A lost a quantifier)

$$\Psi \vdash \exists \alpha : \kappa_1 . A' \leq^+ C \quad \text{By } \leq \exists L$$

• Case nonneg(B), polarity = -:

We case-analyze \mathcal{D}_2 :

$$\begin{array}{c} \textbf{- Case} \\ \frac{\Psi, c: \kappa_3 \vdash B \leq^+ C'}{\Psi \vdash B \leq^+ \underbrace{\exists c: \kappa_3. \, C'}_{C}} \leq \forall \mathsf{R} \end{array}$$

$$\Psi$$
, $c : \kappa_3 \vdash B \leq^+ C'$ Subderivation of \mathcal{D}_2

$$\begin{array}{ll} \Psi,c:\kappa_3 \vdash B \leq^+ C' & \text{Subderivation of } \mathcal{D}_2 \\ \Psi,c:\kappa_3 \vdash A \leq^+ B & \text{By Lemma 2 (Declarative Weakening)} \\ \Psi,c:\kappa_3 \vdash A \leq^+ C' & \text{By i.h. (C lost a quantifier)} \end{array}$$

$$\Psi, c : \kappa_3 \vdash A \leq^+ C'$$
 By i.h. (C lost a quantifier)

$$\Psi \vdash A \leq^+ \forall c : \kappa_3. \; C' \quad \; By \leq \forall \mathsf{R}$$

We case-analyze \mathcal{D}_1 :

$$* \ \, \textbf{Case} \ \, \frac{\Psi \vdash \tau : \kappa_1 \qquad \Psi \vdash [\tau/\alpha] A^{\,\prime} \leq^- B}{\Psi \vdash \underbrace{\forall \alpha : \kappa_1.\,A^{\,\prime}}_{A} \leq^- B} \leq \forall L$$

$$\Psi \vdash B \leq^- C$$
 Given

$$\begin{array}{ll} \Psi \vdash B \leq^- C & \text{Given} \\ \Psi \vdash \tau : \kappa_1 & \text{Subderivation of } \mathcal{D}_1 \\ \Psi \vdash [\tau/\alpha] A' \leq^- B & \text{Subderivation of } \mathcal{D}_1 \\ \Psi \vdash [\tau/\alpha] A' \leq^- C & \text{By i.h. (A lost a quantifier)} \end{array}$$

$$\Psi \vdash [\tau/\alpha]A' \leq^- C$$
 By i.h. (A lost a quantifier)

$$\Psi \vdash \forall \alpha : \kappa_1. A' \leq^- C$$
 By $\leq \forall L$

$$* \ \, \textbf{Case} \ \, \frac{\Psi \vdash A \leq^+ B \quad \textit{nonpos}(A) \qquad \textit{nonpos}(B)}{\Psi \vdash A \leq^- B} \leq^+$$

$$\Psi \vdash A \leq^+ B$$
 Subderivation of \mathcal{D}

$$\Psi \vdash B \leq^+ C$$
 Subderivation of \mathcal{D}_2

$$\begin{array}{ll} \Psi \vdash A \leq^+ B & \quad \text{Subderivation of } \mathcal{D}_1 \\ \Psi \vdash B \leq^+ C & \quad \text{Subderivation of } \mathcal{D}_2 \\ \Psi \vdash A \leq^+ C & \quad \text{By i.h. (} \mathcal{D}_1 \text{ and } \mathcal{D}_2 \text{ smaller)} \end{array}$$

$$nonneg(A)$$
 Subderivation of \mathcal{D}_2 $nonneg(C)$ Subderivation of \mathcal{D}_2

$$\Psi \vdash A \leq^- C$$
 By \leq^+

Substitution and Well-formedness Properties

Lemma 8 (Substitution—Well-formedness).

- (i) If $\Gamma \vdash A$ p type and $\Gamma \vdash \tau$ p type then $\Gamma \vdash [\tau/\alpha]A$ p type.
- (ii) If $\Gamma \vdash P$ prop and $\Gamma \vdash \tau$ p type then $\Gamma \vdash [\tau/\alpha]P$ prop. Moreover, if p = ! and $FEV([\Gamma]P) = \emptyset$ then $FEV([\Gamma][\tau/\alpha]P) = \emptyset$.

Proof. By induction on the derivations of $\Gamma \vdash A$ p type and $\Gamma \vdash P$ prop.

Lemma 9 (Uvar Preservation).

If $\Delta \longrightarrow \Omega$ *then:*

(i) If
$$(\alpha : \kappa) \in \Omega$$
 then $(\alpha : \kappa) \in [\Omega]\Delta$.

(ii) If
$$(x:Ap) \in \Omega$$
 then $(x:[\Omega]Ap) \in [\Omega]\Delta$.

Proof. By induction on Ω , following the definition of context application (Figure 13).

Lemma 10 (Sorting Implies Typing). *If* $\Gamma \vdash t : \star then \Gamma \vdash t type$.

Proof. By induction on the given derivation. All cases are straightforward.

Lemma 11 (Right-Hand Substitution for Sorting). *If* $\Gamma \vdash t : \kappa$ *then* $\Gamma \vdash [\Gamma]t : \kappa$.

Proof. By induction on $|\Gamma \vdash t|$ (the size of t under Γ).

- Cases UnitSort: Here t=1, so applying Γ to t does not change it: $t=[\Gamma]t$. Since $\Gamma \vdash t:\kappa$, we have $\Gamma \vdash [\Gamma]t:\kappa$, which was to be shown.
- Case VarSort: If t is an existential variable $\hat{\alpha}$, then $\Gamma = \Gamma_0[\hat{\alpha}]$, so applying Γ to t does not change it, and we proceed as in the UnitSort case above.

If t is a universal variable α and Γ has no equation for it, then proceed as in the UnitSort case.

Otherwise, $t = \alpha$ and $(\alpha = \tau) \in \Gamma$:

$$\Gamma = (\Gamma_{\rm I}, \alpha : \kappa, \Gamma_{\rm M}, \alpha = \tau, \Gamma_{\rm R})$$

By the implicit assumption that Γ is well-formed, Γ_L , $\alpha : \kappa$, $\Gamma_M \vdash \tau : \kappa$.

By Lemma 34 (Suffix Weakening), $\Gamma \vdash \tau : \kappa$. Since $|\Gamma \vdash \tau| < |\Gamma \vdash \alpha|$, we can apply the i.h., giving

$$\Gamma \vdash [\Gamma]\tau : \kappa$$

By the definition of substitution, $[\Gamma]\tau = [\Gamma]\alpha$, so we have $\Gamma \vdash [\Gamma]\alpha : \kappa$.

- Case SolvedVarSort: In this case $t = \hat{\alpha}$ and $\Gamma = (\Gamma_L, \hat{\alpha} = \tau, \Gamma_R)$. Thus $[\Gamma]t = [\Gamma]\hat{\alpha} = [\Gamma_L]\tau$. We assume contexts are well-formed, so all free variables in τ are declared in Γ_L . Consequently, $|\Gamma_L \vdash \tau| = |\Gamma \vdash \tau|$, which is less than $|\Gamma \vdash \hat{\alpha}|$. We can therefore apply the i.h. to τ , yielding $\Gamma \vdash [\Gamma]\tau : \kappa$. By the definition of substitution, $[\Gamma]\tau = [\Gamma]\hat{\alpha}$, so we have $\Gamma \vdash [\Gamma]\hat{\alpha} : \kappa$.
- Case BinSort: In this case $t=t_1\oplus t_2$. By i.h., $\Gamma\vdash [\Gamma]t_1:\kappa$ and $\Gamma\vdash [\Gamma]t_2:\kappa$. By BinSort, $\Gamma\vdash ([\Gamma]t_1)\oplus ([\Gamma]t_2):\kappa$, which by the definition of substitution is $\Gamma\vdash [\Gamma](t_1\oplus t_2):\kappa$.

Lemma 12 (Right-Hand Substitution for Propositions). *If* $\Gamma \vdash P$ *prop then* $\Gamma \vdash [\Gamma]P$ *prop.*

Proof. Use inversion (EqProp), apply Lemma 11 (Right-Hand Substitution for Sorting) to each premise, and apply EqProp again. □

Lemma 13 (Right-Hand Substitution for Typing). If $\Gamma \vdash A$ type then $\Gamma \vdash [\Gamma]A$ type.

Proof. By induction on $|\Gamma \vdash A|$ (the size of A under Γ).

Several cases correspond to cases in the proof of Lemma 11 (Right-Hand Substitution for Sorting):

- the case for UnitWF is like the case for UnitSort;
- the case for SolvedVarSort is like the cases for VarWF and SolvedVarWF,
- the case for VarSort is like the case for VarWF, but in the last subcase, apply Lemma 10 (Sorting Implies Typing) to move from a sorting judgment to a typing judgment.
- the case for BinWF is like the case for BinSort.

Now, the new cases:

- Case ForallWF: In this case $A = \forall \alpha : \kappa$. A_0 . By i.h., $\Gamma, \alpha : \kappa \vdash [\Gamma, \alpha : \kappa]A_0$ *type*. By the definition of substitution, $[\Gamma, \alpha : \kappa]A_0 = [\Gamma]A_0$, so by ForallWF, $\Gamma \vdash \forall \alpha$. $[\Gamma]A_0$ *type*, which by the definition of substitution is $\Gamma \vdash [\Gamma](\forall \alpha. A_0)$ *type*.
- Case ExistsWF: Similar to the ForallWF case.
- Case ImpliesWF, WithWF: Use the i.h. and Lemma 12 (Right-Hand Substitution for Propositions), then apply ImpliesWF or WithWF.

Lemma 14 (Substitution for Sorting). *If* $\Omega \vdash t : \kappa$ *then* $[\Omega]\Omega \vdash [\Omega]t : \kappa$.

Proof. By induction on $|\Omega| + t$ (the size of t under Ω).

$$\bullet \ \, \textbf{Case} \ \, \frac{\mathfrak{u} : \kappa \in \Omega}{\Omega \vdash \mathfrak{u} : \kappa} \, \, \text{VarSort}$$

We have a complete context Ω , so u cannot be an existential variable: it must be some universal variable α .

If Ω lacks an equation for α , use Lemma 9 (Uvar Preservation) and apply rule UvarSort.

Otherwise, $(\alpha = \tau \in \Omega)$, so we need to show $\Omega \vdash [\Omega]\tau : \kappa$. By the implicit assumption that Ω is well-formed, plus Lemma 34 (Suffix Weakening), $\Omega \vdash \tau : \kappa$. By Lemma 11 (Right-Hand Substitution for Sorting), $\Omega \vdash [\Omega]\tau : \kappa$.

$$\begin{split} \hat{\alpha}: \kappa = & \tau \in \Omega & \text{Subderivation} \\ \Omega = & (\Omega_L, \hat{\alpha}: \kappa = \tau, \Omega_R) & \text{Decomposing } \Omega \\ \Omega_L \vdash \tau : \kappa & \text{By implicit assumption that } \Omega \text{ is well-formed} \\ \Omega_L, \hat{\alpha}: \kappa = & \tau, \Omega_R \vdash \tau : \kappa & \text{By Lemma 34 (Suffix Weakening)} \\ \Omega \vdash & [\Omega]\tau : \kappa & \text{By Lemma 11 (Right-Hand Substitution for Sorting)} \\ & [\Omega]\Omega \vdash [\Omega]\hat{\alpha}: \kappa & [\Omega]\tau = [\Omega]\hat{\alpha} \end{split}$$

$$\frac{}{\Omega \vdash 1: \star} \; \mathsf{UnitSort}$$

Since $1 = [\Omega]1$, applying UnitSort gives the result.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Omega \vdash \tau_1 : \star \qquad \Omega \vdash \tau_2 : \star}{\Omega \vdash \tau_1 \oplus \tau_2 : \star} \ \, \textbf{BinSort}$$

By i.h. on each premise, rule BinSort, and the definition of substitution.

Case

$$\frac{}{\Omega \vdash \mathsf{zero} : \mathbb{N}} \; \mathsf{ZeroSort}$$

Since zero = $[\Omega]$ zero, applying ZeroSort gives the result.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Omega \vdash t : \mathbb{N}}{\Omega \vdash \mathsf{succ}(t) : \mathbb{N}} \ \, \mathsf{SuccSort}$$

By i.h., rule SuccSort, and the definition of substitution.

Lemma 15 (Substitution for Prop Well-Formedness). *If* $\Omega \vdash P$ *prop then* $[\Omega]\Omega \vdash [\Omega]P$ *prop.*

Proof. Only one rule derives this judgment form:

$$\bullet \ \, \textbf{Case} \ \, \frac{\Omega \vdash t : \mathbb{N} \qquad \Omega \vdash t' : \mathbb{N}}{\Omega \vdash t = t' \ prop} \ \, \textbf{EqProp}$$

```
\begin{array}{lll} \Omega \vdash t : \mathbb{N} & \text{Subderivation} \\ [\Omega]\Omega \vdash [\Omega]t : \mathbb{N} & \text{By Lemma 14 (Substitution for Sorting)} \\ \Omega \vdash t' : \mathbb{N} & \text{Subderivation} \\ [\Omega]\Omega \vdash [\Omega]t' : \mathbb{N} & \text{By Lemma 14 (Substitution for Sorting)} \\ [\Omega]\Omega \vdash ([\Omega]t) = ([\Omega]t') \ \textit{prop} & \text{By EqProp} \\ \mathbb{F} & [\Omega]\Omega \vdash [\Omega](t = t') \ \textit{prop} & \text{By def. of subst.} \end{array}
```

Lemma 16 (Substitution for Type Well-Formedness). *If* $\Omega \vdash A$ *type then* $[\Omega]\Omega \vdash [\Omega]A$ *type.*

Proof. By induction on $|\Omega \vdash A|$.

Several cases correspond to those in the proof of Lemma 14 (Substitution for Sorting):

- the UnitWF case is like the UnitSort case (using DeclUnitWF instead of UnitSort);
- the VarWF case is like the VarSort case (using DeclUvarWF instead of UvarSort);
- the SolvedVarWF case is like the SolvedVarSort case.

However, uses of Lemma 11 (Right-Hand Substitution for Sorting) are replaced by uses of Lemma 13 (Right-Hand Substitution for Typing).

Now, the new cases:

$$\begin{array}{c} \bullet \ \, \mathbf{Case} \\ \hline & \frac{\Omega, \alpha: \kappa \vdash A_0 \ type}{\Omega \vdash \forall \alpha: \kappa. \ A_0 \ type} \ \, \mathbf{ForallWF} \\ \\ & \frac{\Omega, \alpha: \kappa \vdash A_0: \kappa'}{\Omega \vdash \alpha: \kappa. \ A_0 \ type} \ \, \mathbf{ForallWF} \\ \\ & \Omega, \alpha: \kappa \vdash [\Omega]A_0: \kappa' & \text{Subderivation} \\ & [\Omega]\Omega, \alpha: \kappa \vdash [\Omega]A_0: \kappa' & \text{By definition of completion} \\ & [\Omega]\Omega \vdash \forall \alpha: \kappa. \ [\Omega]A_0: \kappa' & \text{By DeclAllWF} \\ \\ & [\Omega]\Omega \vdash [\Omega](\forall \alpha: \kappa. \ A_0): \kappa' & \text{By def. of subst.} \\ \hline \end{array}$$

• Case ExistsWF: Similar to the ForallWF case, using DeclExistsWF instead of DeclAllWF.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Omega \vdash A_1 \ \textit{type} \qquad \Omega \vdash A_2 \ \textit{type}}{\Omega \vdash A_1 \oplus A_2 \ \textit{type}} \ \, \textbf{BinWF}$$

By i.h. on each premise, rule DeclBinWF, and the definition of substitution.

• Case VecWF: Similar to the BinWF case.

• Case
$$\frac{\Omega \vdash P \ prop}{\Omega \vdash A_0 \ \land P \ type} \ \ \text{WithWF}$$

Similar to the ImpliesWF case.

Lemma 17 (Substitution Stability).

If (Ω, Ω_Z) is well-formed and Ω_Z is soft and $\Omega \vdash A$ type then $[\Omega]A = [\Omega, \Omega_Z]A$.

Proof. By induction on Ω_Z .

Since Ω_Z is soft, either (1) $\Omega_Z = \cdot$ (and the result is immediate) or (2) $\Omega_Z = (\Omega', \hat{\alpha} : \kappa)$ or (3) $\Omega_Z = (\Omega', \hat{\alpha} : \kappa = t)$. However, according to the grammar for complete contexts such as Ω_Z , (2) is impossible. Only case (3) remains.

By i.h.,
$$[\Omega]A = [\Omega, \Omega']A$$
. Use the fact that $\Omega \vdash A$ *type* implies $FV(A) \cap dom(\Omega_Z) = \emptyset$.

Lemma 18 (Equal Domains).

If $\Omega_1 \vdash A$ type and $dom(\Omega_1) = dom(\Omega_2)$ then $\Omega_2 \vdash A$ type.

Proof. By induction on the given derivation.

C' Properties of Extension

Lemma 19 (Declaration Preservation). If $\Gamma \longrightarrow \Delta$ and u is declared in Γ , then u is declared in Δ .

Proof. By induction on the derivation of $\Gamma \longrightarrow \Delta$.

• Case
$$\xrightarrow{\cdot \longrightarrow \cdot}$$
 \longrightarrow Id

This case is impossible, since by hypothesis u is declared in Γ .

- Case u = x: Immediate.
- Case $u \neq x$: Since u is declared in $(\Gamma, x : A)$, it is declared in Γ . By i.h., u is declared in Δ , and therefore declared in $(\Delta, x : A')$.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \alpha : \kappa \longrightarrow \Delta, \alpha : \kappa} \longrightarrow \textbf{Uvar}$$

Similar to the \longrightarrow Var case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \, \hat{\alpha} : \kappa \longrightarrow \Delta, \, \hat{\alpha} : \kappa} \longrightarrow \text{Unsolved}$$

Similar to the \longrightarrow Var case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta \qquad [\Delta]t = [\Delta]t'}{\Gamma, \hat{\alpha}: \kappa = t \longrightarrow \Delta, \hat{\alpha}: \kappa = t'} \longrightarrow \text{Solved}$$

Similar to the \longrightarrow Var case.

• Case
$$\frac{\Gamma \longrightarrow \Delta \qquad [\Delta]t = [\Delta]t'}{\Gamma, \alpha = t \longrightarrow \Delta, \alpha = t'} \longrightarrow \mathsf{Eqn}$$

It is given that u is declared in $(\Gamma, \alpha = t)$. Since $\alpha = t$ is not a declaration, u is declared in Γ . By i.h., u is declared in Δ , and therefore declared in $(\Delta, \alpha = t')$.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}} \longrightarrow \mathsf{Marker}$$

Similar to the \longrightarrow Eqn case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \, \widehat{\beta} : \kappa' \longrightarrow \Delta, \, \widehat{\beta} : \kappa' = t} \longrightarrow \\ \textbf{Solve}$$

Similar to the \longrightarrow Var case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \hat{\alpha} : \kappa} \longrightarrow \! \mathsf{Add}$$

It is given that u is declared in Γ . By i.h., u is declared in Δ , and therefore declared in $(\Delta, \hat{\alpha} : \kappa)$.

$$\bullet \ \ \text{Case} \ \ \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \hat{\alpha} : \kappa = t} \longrightarrow \mathsf{AddSolved}$$

Similar to the \longrightarrow Add case.

Lemma 20 (Declaration Order Preservation). If $\Gamma \longrightarrow \Delta$ and $\mathfrak u$ is declared to the left of $\mathfrak v$ in Γ , then $\mathfrak u$ is declared to the left of $\mathfrak v$ in Δ .

Proof. By induction on the derivation of $\Gamma \longrightarrow \Delta$.

• Case
$$\longrightarrow \operatorname{Id}$$

This case is impossible, since by hypothesis u and v are declared in Γ .

• Case
$$\frac{\Gamma \longrightarrow \Delta \qquad [\Delta]A = [\Delta]A'}{\Gamma, x : A \longrightarrow \Delta, x : A'} \longrightarrow \mathsf{Var}$$

Consider whether v = x:

- Case v = x:

It is given that $\mathfrak u$ is declared to the left of $\mathfrak v$ in $(\Gamma, x : A)$, so $\mathfrak u$ is declared in Γ . By Lemma 19 (Declaration Preservation), $\mathfrak u$ is declared in Δ . Therefore $\mathfrak u$ is declared to the left of $\mathfrak v$ in $(\Delta, x : A')$.

- Case $v \neq x$:

Here, ν is declared in Γ . By i.h., μ is declared to the left of ν in Δ . Therefore μ is declared to the left of ν in $(\Delta, x : A')$.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \alpha : \kappa \longrightarrow \Delta, \alpha : \kappa} \longrightarrow \mathsf{Uvar}$$

Similar to the \longrightarrow Var case.

П

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \hat{\alpha} : \kappa \longrightarrow \Delta, \hat{\alpha} : \kappa} \longrightarrow \mathsf{Unsolved}$$

Similar to the \longrightarrow Var case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta \qquad [\Delta]t = [\Delta]t'}{\Gamma, \, \widehat{\alpha} : \kappa = t \longrightarrow \Delta, \, \widehat{\alpha} : \kappa = t'} \longrightarrow \text{Solved}$$

Similar to the \longrightarrow Var case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma,\, \hat{\beta}:\kappa' \longrightarrow \Delta,\, \hat{\beta}:\kappa' = t} \longrightarrow \\ \textbf{Solve}$$

Similar to the \longrightarrow Var case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta \qquad [\Delta]t = [\Delta]t'}{\Gamma, \, \alpha = t \longrightarrow \Delta, \, \alpha = t'} \longrightarrow \text{Eqn}$$

The equation $\hat{\alpha} = t$ does not declare any variables, so u and v must be declared in Γ . By i.h., u is declared to the left of v in Δ . Therefore u is declared to the left of v in Δ , $\hat{\alpha} : \kappa = t'$.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}} \longrightarrow \mathsf{Marker}$$

Similar to the \longrightarrow Eqn case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \, \hat{\alpha} : \kappa} \longrightarrow \! \mathsf{Add}$$

By i.h., u is declared to the left of v in Δ . Therefore u is declared to the left of v in $(\Delta, \hat{\alpha} : \kappa)$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \, \hat{\alpha} : \kappa = t} \longrightarrow \text{AddSolved}$$

Similar to the \longrightarrow Add case.

Lemma 21 (Reverse Declaration Order Preservation). *If* $\Gamma \longrightarrow \Delta$ *and* $\mathfrak u$ *and* $\mathfrak v$ *are both declared in* Γ *and* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Δ , *then* $\mathfrak u$ *is declared to the left of* $\mathfrak v$ *in* Γ .

Proof. It is given that $\mathfrak u$ and $\mathfrak v$ are declared in Γ . Either $\mathfrak u$ is declared to the left of $\mathfrak v$ in Γ , or $\mathfrak v$ is declared to the left of $\mathfrak u$. Suppose the latter (for a contradiction). By Lemma 20 (Declaration Order Preservation), $\mathfrak v$ is declared to the left of $\mathfrak u$ in Δ . But we know that $\mathfrak u$ is declared to the left of $\mathfrak v$ in Δ : contradiction. Therefore $\mathfrak u$ is declared to the left of $\mathfrak v$ in Γ .

Lemma 22 (Extension Inversion).

(i) If
$$\mathcal{D} :: \Gamma_0, \alpha : \kappa, \Gamma_1 \longrightarrow \Delta$$

then there exist unique Δ_0 and Δ_1
such that $\Delta = (\Delta_0, \alpha : \kappa, \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$.
Moreover, if Γ_1 is soft, then Δ_1 is soft.

- (ii) If $\mathcal{D} :: \Gamma_0, \blacktriangleright_{\mathfrak{u}}, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0 and Δ_1 such that $\Delta = (\Delta_0, \blacktriangleright_{\mathfrak{u}}, \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$. Moreover, if Γ_1 is soft, then Δ_1 is soft. Moreover, if $\mathsf{dom}(\Gamma_0, \blacktriangleright_{\mathfrak{u}}, \Gamma_1) = \mathsf{dom}(\Delta)$ then $\mathsf{dom}(\Gamma_0) = \mathsf{dom}(\Delta_0)$.
- (iii) If $\mathcal{D} :: \Gamma_0, \alpha = \tau, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0, τ' , and Δ_1 such that $\Delta = (\Delta_0, \alpha = \tau', \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ and $[\Delta_0]\tau = [\Delta_0]\tau'$ where $\mathcal{D}' < \mathcal{D}$.
- (iv) If $\mathcal{D}:: \Gamma_0, \hat{\alpha}: \kappa = \tau, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0, τ' , and Δ_1 such that $\Delta = (\Delta_0, \hat{\alpha}: \kappa = \tau', \Delta_1)$ and $\mathcal{D}':: \Gamma_0 \longrightarrow \Delta_0$ and $[\Delta_0]\tau = [\Delta_0]\tau'$ where $\mathcal{D}' < \mathcal{D}$.
- (v) If $\mathcal{D} :: \Gamma_0, x : A, \Gamma_1 \longrightarrow \Delta$ then there exist unique Δ_0, A' , and Δ_1 such that $\Delta = (\Delta_0, x : A', \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ and $[\Delta_0]A = [\Delta_0]A'$ where $\mathcal{D}' < \mathcal{D}$. Moreover, if Γ_1 is soft, then Δ_1 is soft. Moreover, if $\mathsf{dom}(\Gamma_0, x : A, \Gamma_1) = \mathsf{dom}(\Delta)$ then $\mathsf{dom}(\Gamma_0) = \mathsf{dom}(\Delta_0)$.
- (vi) If $\mathcal{D} :: \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \longrightarrow \Delta$ then either
 - there exist unique Δ_0 , τ' , and Δ_1 such that $\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$, or
 - there exist unique Δ_0 and Δ_1 such that $\Delta = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1)$ and $\mathcal{D}' :: \Gamma_0 \longrightarrow \Delta_0$ where $\mathcal{D}' < \mathcal{D}$.

Proof. In each part, we proceed by induction on the derivation of $\Gamma_0, \ldots, \Gamma_1 \longrightarrow \Delta$.

Note that in each part, the \longrightarrow Id case is impossible.

Throughout this proof, we shadow Δ so that it refers to the *largest proper prefix* of the Δ in the statement of the lemma. For example, in the \longrightarrow Var case of part (i), we really have $\Delta = (\Delta_{00}, x : A')$, but we call Δ_{00} " Δ ".

(i) We have Γ_0 , $\alpha : \kappa$, $\Gamma_1 \longrightarrow \Delta$.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \longrightarrow \Delta}{\Gamma, \beta : \kappa' \longrightarrow \Delta, \beta : \kappa'} \longrightarrow \text{Uvar}}_{\Gamma_0, \alpha : \kappa, \Gamma_1}$$

There are two cases:

- Case $\alpha : \kappa = \beta : \kappa'$:
 - $(\Gamma, \alpha : \kappa) = (\Gamma_0, \alpha : \kappa, \Gamma_1)$ where $\Gamma_0 = \Gamma$ and $\Gamma_1 = \cdot$
 - $(\Delta, \alpha : \kappa) = (\Delta_0, \alpha : \kappa, \Delta_1)$ where $\Delta_0 = \Delta$ and $\Delta_1 = \cdot$
 - if Γ_1 soft then Δ_1 soft since \cdot is soft
- Case $\alpha \neq \beta$:

$$\begin{array}{ll} (\Gamma,\beta:\kappa') = (\Gamma_0,\alpha:\kappa,\Gamma_1) & \text{Given} \\ & = (\Gamma_0,\alpha:\kappa,\Gamma_1',\beta:\kappa') & \text{Since the last element must be equal} \\ & \Gamma = (\Gamma_0,\alpha:\kappa,\Gamma_1') & \text{By injectivity of syntax} \end{array}$$

if Γ_1' soft then Δ_1 soft

- $(\Delta, \beta : \kappa') = (\Delta_0, \alpha : \kappa, \Delta_1, \beta : \kappa')$ By congruence
- if $\Gamma_1', \beta : \kappa'$ soft then $\Delta_1, \beta : \kappa'$ soft Since Γ'_1 , $\beta : \kappa'$ is not soft
- Case

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$$\underbrace{\frac{\Gamma \longrightarrow \Delta}{\Gamma_0, \alpha \colon \kappa / \Gamma_1} \longrightarrow \Delta, \hat{\alpha} \colon \kappa'}_{\Gamma_0, \alpha \colon \kappa, \Gamma_1} \longrightarrow \mathsf{Unsolved}$$

$$\begin{array}{ll} (\Gamma,\hat{\alpha}:\kappa') = (\Gamma_0,\alpha:\kappa,\Gamma_1) & \text{Given} \\ & = (\Gamma_0,\alpha:\kappa,\Gamma_1',\hat{\alpha}:\kappa') & \text{Since the last element must be equal} \\ & \Gamma = (\Gamma_0,\alpha:\kappa,\Gamma_1') & \text{By injectivity of syntax} \end{array}$$

$$\begin{array}{cccc} \Gamma \longrightarrow \Delta & & \text{Subderivation} \\ \Gamma_0, \alpha: \kappa, \Gamma_1' \longrightarrow \Delta & & \text{By equality} \\ \Delta = (\Delta_0, \alpha: \kappa, \Delta_1) & & \text{By i.h.} \\ \Gamma_0 \longrightarrow \Delta_0 & & '' \\ \text{if } \Gamma_1' \text{ soft then } \Delta_1 \text{ soft} & & '' \end{array}$$

$$(\Delta, \hat{\alpha} : \kappa') = (\Delta_0, \alpha : \kappa, \Delta_1, \hat{\alpha} : \kappa') \quad \text{By congruence}$$

Suppose Γ'_1 , $\hat{\alpha}$: κ' soft.

 Γ_1' soft By definition of softness Δ_1 soft By induction

 Δ_1 soft By definition of softness if Γ_1' , $\hat{\alpha}$: κ' soft then Δ_1 , $\hat{\alpha}$: κ' soft Implication introduction

 $\begin{array}{cccc} \bullet \ \, \textbf{Case} & \underbrace{\Gamma \longrightarrow \Delta \qquad [\Delta]t = [\Delta]t'}_{\clipt{\untering}} \longrightarrow \text{Solved} \\ & \underbrace{\Gamma, \hat{\alpha} : \kappa = t}_{\clipt{\untering}} \longrightarrow \Delta, \hat{\alpha} : \kappa = t' \end{array}$

Similar to the \longrightarrow Unsolved case.

$$\begin{array}{ccc} (\Delta,\beta=t')=(\Delta_0,\alpha:\kappa,\Delta_1,\beta=t') & \text{By congruence} \\ & \text{if } \Gamma_1',\beta=t \text{ soft then } \Delta_1,\beta=t' \text{ soft} & \text{Since } \Gamma_1',\beta=t \text{ is not soft} \end{array}$$

$$\begin{array}{c} \Gamma \longrightarrow \Delta \\ \hline \Gamma_0, \alpha : \kappa', \Gamma_1 \end{array} \longrightarrow \Delta, \hat{\alpha} : \kappa \longrightarrow \mathsf{Add} \\ \hline \qquad \qquad \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) \qquad \text{By i.h.} \\ \hline \qquad \qquad \Gamma_0 \longrightarrow \Delta_0 \qquad \qquad '' \\ \hline \qquad \qquad \text{if Γ_1 soft then Δ_1 soft} \qquad \qquad '' \\ \hline \qquad \qquad \Delta, \hat{\alpha} : \kappa' = (\Delta_0, \alpha : \kappa, \Delta_1, \hat{\alpha} : \kappa') \qquad \mathsf{By congruence of equality} \\ \hline \\ \text{Suppose Γ_1 soft.} \qquad \qquad \Delta_1 \text{ soft} \qquad \qquad \mathsf{By i.h.} \\ \hline \qquad \Delta_1, \hat{\alpha} : \kappa' \text{ soft} \qquad \qquad \mathsf{By definition of softnesss} \\ \hline \end{array}$$

if Γ_1 soft then Δ_1 , $\hat{\alpha}$: κ' soft

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Implication introduction

$$\begin{array}{c} \bullet \text{ Case} \\ \hline \begin{array}{c} \Gamma \longrightarrow \Delta \\ \hline \begin{array}{c} \Gamma \longrightarrow \Delta, \hat{\alpha} : \kappa' = t \end{array} \end{array} \longrightarrow \text{AddSolved} \\ \hline \\ \Delta = (\Delta_0, \alpha : \kappa, \Delta_1) & \text{By i.h.} \\ \hline \\ \Gamma_0 \longrightarrow \Delta_0 & \text{`''} \\ \hline \text{if Γ_1 soft then Δ_1 soft} & \text{`''} \\ \hline \\ \mathbb{S} & (\Delta, \hat{\alpha} : \kappa' = t) = (\Delta_0, \alpha : \kappa, \Delta_1, \hat{\alpha} : \kappa' = t) & \text{By congruence of equality} \\ \hline \\ \text{Suppose Γ_1 soft.} & \text{By i.h.} \\ \hline \\ \Delta_1 \text{ soft} & \text{By i.h.} \\ \hline \\ (\Delta_1, \hat{\alpha} : \kappa' = t) \text{ soft} & \text{By definition of softnesss} \\ \hline \\ \mathbb{S} & \text{if Γ_1 soft then $\Delta_1, \hat{\alpha} : \kappa' = t$ soft} & \text{Implication introduction} \\ \hline \end{array}$$

- (ii) We have $\Gamma_0, \blacktriangleright_\mathfrak{u}, \Gamma_1 \longrightarrow \Delta$. This part is similar to part (i) above, except for "if $\mathsf{dom}(\Gamma_0, \blacktriangleright_\mathfrak{u}, \Gamma_1) = \mathsf{dom}(\Delta)$ then $\mathsf{dom}(\Gamma_0) = \mathsf{dom}(\Delta_0)$ ", which follows by i.h. in most cases. In the \longrightarrow Marker case, either we have $\ldots, \blacktriangleright_\mathfrak{u'}$ where $\mathfrak{u'} = \mathfrak{u}$ —in which case the i.h. gives us what we need—or we have a matching $\blacktriangleright_\mathfrak{u}$. In this latter case, we have $\Gamma_1 = \cdot$. We know that $\mathsf{dom}(\Gamma_0, \blacktriangleright_\mathfrak{u}, \Gamma_1) = \mathsf{dom}(\Delta)$ and $\Delta = (\Delta_0, \blacktriangleright_\mathfrak{u})$. Since $\Gamma_1 = \cdot$, we have $\mathsf{dom}(\Gamma_0, \blacktriangleright_\mathfrak{u}) = \mathsf{dom}(\Delta_0, \blacktriangleright_\mathfrak{u})$. Therefore $\mathsf{dom}(\Gamma_0) = \mathsf{dom}(\Delta_0)$.
- (iii) We have Γ_0 , $\alpha = \tau$, $\Gamma_1 \longrightarrow \Delta$.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \longrightarrow \Delta}{\underset{\Gamma_0\,,\alpha\,=\,\tau,\Gamma_1}{\underbrace{\Gamma,\beta:\kappa'}} \longrightarrow \Delta,\beta:\kappa'}} \longrightarrow \textbf{Uvar}$$

$$\begin{split} (\Gamma_0,\alpha=\tau,\Gamma_1)&=(\Gamma,\beta:\kappa') & \text{Given} \\ &=(\Gamma_0,\alpha=\tau,\Gamma_1',\beta:\kappa') & \text{Since the final elements must be equal} \\ &\Gamma=(\Gamma_0,\alpha=\tau,\Gamma_1') & \text{By injectivity of context syntax} \\ &\Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{By i.h.} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Sin$$

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \begin{matrix} \Gamma \longrightarrow \Delta & [\Delta]A = [\Delta]A' \\ \hline \begin{matrix} \Gamma, \chi : A & \longrightarrow \Delta, \chi : A' \end{matrix} }_{\Gamma_0, \alpha = \tau, \Gamma_1} \longrightarrow \Delta, \chi : A'$$

Similar to the \longrightarrow Uvar case.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}} \longrightarrow \mathsf{Marker}$$

Similar to the \longrightarrow Uvar case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \hat{\alpha} : \kappa' \longrightarrow \Delta, \hat{\alpha} : \kappa'} \longrightarrow \text{Unsolved}$$

Similar to the \longrightarrow Uvar case.

• Case
$$\underbrace{ \begin{array}{ccc} \Gamma \longrightarrow \Delta & [\Delta]t = [\Delta]t' \\ \underline{\Gamma, \hat{\alpha} : \kappa' = t} \longrightarrow \Delta, \hat{\alpha} : \kappa' = t' \end{array} }_{\Gamma_0, \alpha = \tau, \Gamma_1} \longrightarrow \mathsf{Solved}$$

Similar to the \longrightarrow Uvar case.

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\underbrace{\Gamma, \hat{\beta} : \kappa' \longrightarrow \Delta, \hat{\beta} : \kappa' = t}} \longrightarrow \mathsf{Solve}$$

Similar to the \longrightarrow Uvar case.

$$\begin{array}{cccc} \bullet & \textbf{Case} & & & & [\Delta]t = [\Delta]t' \\ & & & & \underbrace{\Gamma, \beta = t}_{\Gamma_0, \alpha = \tau, \Gamma_1} & \longrightarrow \Delta, \beta = t' \end{array} \longrightarrow \mathsf{Eqn}$$

There are two cases:

- Case $\alpha \neq \beta$:

$$\begin{split} (\Gamma_0,\alpha=\tau,\Gamma_1)&=(\Gamma,\beta=t) & \text{Given} \\ &=(\Gamma_0,\alpha=\tau,\Gamma_1',\beta=t) & \text{Since the final elements must be equal} \\ &\Gamma=(\Gamma_0,\alpha=\tau,\Gamma_1') & \text{By injectivity of context syntax} \\ &\Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{By i.h.} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since the final elements must be equal} \\ & \Delta=(\Delta_0,\alpha=\tau',\Delta_1) & \text{Since$$

• Case
$$\frac{\Gamma \longrightarrow \Delta}{ \underset{\Gamma_0, \alpha = \tau, \Gamma_1}{ } \longrightarrow \Delta, \hat{\alpha} : \kappa'} \longrightarrow \mathsf{Add}$$
$$\Delta = (\Delta_0, \alpha = \tau', \Delta_1)$$

$$\begin{array}{cccc} \Delta = (\Delta_0, \alpha = \tau', \Delta_1) & \text{By i.h.} \\ & [\Delta_0]\tau = [\Delta_0]\tau' & \text{"} \\ & \Gamma_0 \longrightarrow \Delta_0 & \text{"} \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & & \\$$

$$(\Delta, \hat{\alpha}: \kappa') = (\Delta_0, \alpha = \tau', \Delta_1, \hat{\alpha}: \kappa') \quad \text{ By congruence of equality}$$

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \hline \begin{matrix} \Gamma \longrightarrow \Delta \\ \hline \begin{matrix} \Gamma_{\text{o}}, \alpha = \tau, \Gamma_{\text{I}} \end{matrix} & \longrightarrow \Delta, \hat{\alpha} : \kappa' = t \end{matrix} & \longrightarrow \mathsf{AddSolved} \\ \hline \end{matrix}$$

(iv) We have Γ_0 , $\hat{\alpha} : \kappa = \tau$, $\Gamma_1 \longrightarrow \Delta$.

• Case
$$\begin{array}{c} \Gamma \longrightarrow \Delta \\ \hline \Gamma_0, \hat{\alpha} : \kappa' \longrightarrow \Delta, \beta : \kappa' \end{array} \longrightarrow \text{Uvar} \\ \hline (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1) = (\Gamma, \beta : \kappa') & \text{Given} \\ \hline = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1') = (\Gamma, \beta : \kappa') & \text{Since the final elements must be equal} \\ \hline \Gamma = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1') & \text{By injectivity of context syntax} \\ \hline \Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) & \text{By i.h.} \\ \hline (\Delta_0]\tau = [\Delta_0]\tau' & " \\ \hline \Gamma_0 \longrightarrow \Delta_0 & " \\ \hline (\Delta, \beta : \kappa') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \beta : \kappa') & \text{By congruence of equality} \\ \hline \end{array}$$

$$\begin{array}{cccc} \bullet & \textbf{Case} & & & & [\Delta]A = [\Delta]A' \\ & & & \underbrace{\Gamma, \chi: A}_{\Gamma_0, \hat{\alpha}: \kappa = \tau, \Gamma_1} & \longrightarrow \Delta, \chi: A' \end{array} \longrightarrow \mathsf{Var}$$

Similar to the \longrightarrow Uvar case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \blacktriangleright_{\widehat{\beta}} \longrightarrow \Delta, \blacktriangleright_{\widehat{\beta}}} \longrightarrow \mathsf{Marker}$$

Similar to the \longrightarrow Uvar case.

• Case $\frac{\Gamma \longrightarrow \Delta}{\Gamma, \, \widehat{\beta} : \kappa' \longrightarrow \Delta, \, \widehat{\beta} : \kappa'} \longrightarrow \mathsf{Unsolved}$

Similar to the \longrightarrow Uvar case.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \longrightarrow \Delta}{\Gamma, \widehat{\beta} : \kappa' = t} \longrightarrow \Delta, \widehat{\beta} : \kappa' = t'}_{\Gamma_0, \widehat{\alpha} : \kappa = \tau, \Gamma_1} \longrightarrow \Delta, \widehat{\beta} : \kappa' = t'}_{} \longrightarrow \text{Solved}$$

There are two cases.

- Case
$$\hat{\alpha} = \hat{\beta}$$
:

$$\kappa' = \kappa \text{ and } t = \tau \text{ and } \Gamma_1 = \cdot \text{ and } \Gamma = \Gamma_0 \qquad \text{By injectivity of syntax} \\ \text{Where } \tau' = t' \text{ and } \Delta_1 = \cdot \text{ and } \Delta = \Delta_0 \\ \text{From subderivation } \Gamma \longrightarrow \Delta \\ \text{From premise } [\Delta_0]\tau = [\Delta_0]\tau' \qquad \qquad \text{From premise } [\Delta]t = [\Delta]t' \text{ and } x$$

- Case $\hat{\alpha} \neq \hat{\beta}$:

$$\begin{array}{ll} (\Gamma_0,\hat{\alpha}:\kappa\!=\!\tau,\Gamma_1)=(\Gamma,\hat{\beta}:\kappa'\!=\!t) & \text{Given} \\ =(\Gamma_0,\hat{\alpha}:\kappa\!=\!\tau,\Gamma_1',\hat{\beta}:\kappa'\!=\!t) & \text{Since the final elements must be equal} \\ \Gamma=(\Gamma_0,\hat{\alpha}:\kappa\!=\!\tau,\Gamma_1') & \text{By injectivity of context syntax} \end{array}$$

• Case
$$\frac{\Gamma \longrightarrow \Delta \qquad [\Delta]t = [\Delta]t'}{\underbrace{\Gamma, \beta = t}_{\Gamma : \beta : \nu = \sigma} \underbrace{\Gamma_{t}}_{\Gamma : \beta : \nu = \sigma} \underbrace{-\Delta, \beta = t'}_{\Gamma : \beta : \nu = \sigma} \underbrace{-\Delta}_{\Gamma : \nu}$$

$$\Gamma_0 \longrightarrow \Delta_0 \qquad \qquad ''$$

$$(\Delta, \beta = t') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \beta = t') \quad \text{By congruence of equality}$$

• Case
$$\frac{\Gamma \longrightarrow \Delta}{\overset{\Gamma}{\underset{\Gamma_0, \hat{\alpha}: \kappa = \tau, \Gamma_1}{\longleftarrow}} \Delta, \hat{\beta}: \kappa'} \longrightarrow \mathsf{Add}$$

$$\begin{array}{ccc} \Delta = (\Delta_0, \hat{\alpha}: \kappa \!=\! \tau', \Delta_1) & \text{By i.h.} \\ & [\Delta_0]\tau = [\Delta_0]\tau' & '' \\ & \Gamma_0 \longrightarrow \Delta_0 & '' \\ & & (\Delta, \hat{\beta}: \kappa') = (\Delta_0, \hat{\alpha}: \kappa \!=\! \tau', \Delta_1, \hat{\beta}: \kappa') & \text{By congruence of equality} \\ \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \longrightarrow \Delta}{\prod\limits_{\Gamma_0,\,\hat{\alpha}:\,\kappa \,=\, \tau,\,\Gamma_1} \longrightarrow \Delta,\,\hat{\beta}:\kappa'\,=\,t}}_{\Gamma_0,\,\hat{\alpha}:\,\kappa \,=\, \tau,\,\Gamma_1} \longrightarrow \mathsf{AddSolved}$$

• Case
$$\begin{array}{c} \Gamma \longrightarrow \Delta \\ \hline \Gamma, \widehat{\beta} : \kappa' \longrightarrow \Delta, \widehat{\beta} : \kappa' = t \end{array} \longrightarrow \text{Solve} \\ \hline (\Gamma, \widehat{\beta} : \kappa') \longrightarrow \Delta, \widehat{\beta} : \kappa' = t \end{array} \longrightarrow \text{Solve} \\ \hline (\Gamma, \widehat{\beta} : \kappa') = (\Gamma_0, \widehat{\alpha} : \kappa = \tau, \Gamma_1) & \text{Given} \\ = (\Gamma_0, \widehat{\alpha} : \kappa = \tau, \Gamma_1', \widehat{\beta} : \kappa') & \text{Since the last elements must be equal} \\ \Gamma = (\Gamma_0, \widehat{\alpha} : \kappa = \tau, \Gamma_1') & \text{By injectivity of syntax} \\ \hline \Gamma \longrightarrow \Delta & \text{Subderivation} \\ \Gamma_0, \widehat{\alpha} : \kappa = \tau, \Gamma_1' \longrightarrow \Delta & \text{By equality} \\ \Delta = (\Delta_0, \widehat{\alpha} : \kappa = \tau', \Delta_1) & \text{By i.h.} \\ \hline (\Delta_0)\tau = [\Delta_0]\tau' & " \\ \hline \Gamma_0 \longrightarrow \Delta_0 & " \\ \hline (\Delta, \widehat{\beta} : \kappa') = (\Delta_0, \widehat{\alpha} : \kappa = \tau', \Delta_1, \widehat{\beta} : \kappa') & \text{By congruence of equality} \\ \hline \end{array}$$

- (v) We have $\Gamma_0, x : A, \Gamma_1 \longrightarrow \Delta$. This proof is similar to the proof of part (i), except for the domain condition, which we handle similarly to part (ii).
- (vi) We have Γ_0 , $\hat{\alpha} : \kappa$, $\Gamma_1 \longrightarrow \Delta$.

• Case
$$\begin{array}{c} \Gamma \longrightarrow \Delta \\ \hline (\Gamma_0, \hat{\alpha} : \kappa') \longrightarrow \Delta, \beta : \kappa' \end{array} \longrightarrow \mathsf{Uvar} \\ (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1) = (\Gamma, \beta : \kappa') \qquad \qquad \mathsf{Given} \\ = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \beta : \kappa') \qquad \mathsf{Since the final elements must be equal} \\ \Gamma = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1') \qquad \qquad \mathsf{By injectivity of context syntax} \\ \end{array}$$

By induction, there are two possibilities:

- $\hat{\alpha}$ is not solved:

$$\begin{array}{ccc} \Delta = (\Delta_0, \hat{\alpha}: \kappa, \Delta_1) & \text{By i.h.} \\ & \Gamma_0 \longrightarrow \Delta_0 & '' \\ & & (\Delta, \beta: \kappa') = (\Delta_0, \hat{\alpha}: \kappa, \Delta_1, \beta: \kappa') & \text{By congruence of equality} \end{array}$$

- $\hat{\alpha}$ is solved:

$$\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) \qquad \text{By i.h.}$$

$$\Gamma_0 \longrightarrow \Delta_0 \qquad \qquad ''$$

$$(\Delta, \beta : \kappa') = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \beta : \kappa') \qquad \text{By congruence of equality}$$

$$\begin{array}{cccc} \bullet \ \, \textbf{Case} & \underline{\Gamma \longrightarrow \Delta} & [\Delta]A = [\Delta]A' \\ & \underline{\underbrace{\Gamma, \chi : A}} & \longrightarrow \Delta, \chi : A' \end{array} \longrightarrow \text{Var}$$

Similar to the \longrightarrow Uvar case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma \longrightarrow \Delta}{\Gamma, \blacktriangleright_{\widehat{B}} \longrightarrow \Delta, \blacktriangleright_{\widehat{B}}} \longrightarrow \mathsf{Marker}$$

Similar to the \longrightarrow Uvar case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Delta \qquad [\Delta]t = [\Delta]t'}{\Gamma, \beta = t \longrightarrow \Delta, \beta = t'} \longrightarrow \textbf{Eqn}$$

Similar to the \longrightarrow Uvar case.

$$\begin{array}{cccc} \bullet & \textbf{Case} & & & & & & & & & & & \\ & \underline{\Gamma, \hat{\beta}: \kappa' \! = \! t} & \longrightarrow \Delta, \hat{\beta}: \kappa' \! = \! t' & & \longrightarrow & \\ & & & & & & & & & \\ \hline \end{array}$$

Similar to the \longrightarrow Uvar case.

$$\begin{array}{c} \bullet \ \ \textbf{Case} \\ \underbrace{\frac{\Gamma \longrightarrow \Delta}{\Gamma_0, \hat{\alpha}: \kappa' \longrightarrow \Delta, \hat{\beta}: \kappa'}}_{\Gamma_0, \hat{\alpha}: \kappa, \Gamma_1} \longrightarrow \text{Unsolved} \end{array}$$

- Case
$$\hat{\alpha} \neq \hat{\beta}$$
:

$$\begin{split} (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1) &= (\Gamma, \hat{\beta} : \kappa') & \text{Given} \\ &= (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \hat{\beta} : \kappa') & \text{Since the final elements must be equal} \\ &\Gamma &= (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1') & \text{By injectivity of context syntax} \end{split}$$

By induction, there are two possibilities:

* $\hat{\alpha}$ is not solved:

$$\begin{array}{ccc} \Delta = (\Delta_0, \hat{\alpha}: \kappa, \Delta_1) & \text{By i.h.} \\ & \Gamma_0 \longrightarrow \Delta_0 & '' \\ & & (\Delta, \hat{\beta}: \kappa') = (\Delta_0, \hat{\alpha}: \kappa, \Delta_1, \hat{\beta}: \kappa') & \text{By congruence of equality} \end{array}$$

∗ α̂ is solved:

$$\begin{array}{ccc} \Delta = (\Delta_0, \hat{\alpha}: \kappa \!=\! \tau', \Delta_1) & \text{By i.h.} \\ & \Gamma_0 \longrightarrow \Delta_0 & '' \\ & & (\Delta, \hat{\beta}: \kappa') = (\Delta_0, \hat{\alpha}: \kappa \!=\! \tau', \Delta_1, \hat{\beta}: \kappa') & \text{By congruence of equality} \end{array}$$

- Case
$$\hat{\alpha} = \hat{\beta}$$
:

$$\kappa' = \kappa \text{ and } \Gamma_0 = \Gamma \text{ and } \Gamma_1 = \cdot \\ (\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1) \\ \Gamma_0 \longrightarrow \Delta_0$$
 By injectivity of syntax where $\Delta_0 = \Delta$ and $\Delta_1 = \cdot$ From premise $\Gamma \longrightarrow \Delta$

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \longrightarrow \Delta}{\underset{\Gamma_0,\hat{\alpha}:\kappa,\Gamma_1}{}{\Gamma} \longrightarrow \Delta,\hat{\beta}:\kappa'} \longrightarrow}_{\text{Add}} \longrightarrow \text{Add}$$

By induction, there are two possibilities:

- $\hat{\alpha}$ is not solved:

$$\begin{array}{ccc} \Delta = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1) & \text{By i.h.} \\ & \Gamma_0 \longrightarrow \Delta_0 & '' \\ & & (\Delta, \hat{\beta} : \kappa') = (\Delta_0, \hat{\alpha} : \kappa, \Delta_1, \hat{\beta} : \kappa') & \text{By congruence of equality} \end{array}$$

- $\hat{\alpha}$ is solved:

$$\begin{array}{ccc} \Delta = (\Delta_0, \hat{\alpha}: \kappa = \tau', \Delta_1) & \text{By i.h.} \\ & \Gamma_0 \longrightarrow \Delta_0 & '' \\ & & (\Delta, \hat{\beta}: \kappa') = (\Delta_0, \hat{\alpha}: \kappa = \tau', \Delta_1, \hat{\beta}: \kappa') & \text{By congruence of equality} \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \begin{array}{c} \Gamma \longrightarrow \Delta \\ \\ \overbrace{\Gamma_0\,,\hat{\alpha}:\kappa,\Gamma_1} \\ \end{array}}_{\Gamma_0,\hat{\alpha}:\kappa,\Gamma_1} \longrightarrow \Delta, \\ \widehat{\beta}:\kappa'=t \\ \end{array} \longrightarrow \mathsf{AddSolved}$$

By induction, there are two possibilities:

- $\hat{\alpha}$ is not solved:

$$\begin{array}{ccc} \Delta = (\Delta_0, \hat{\alpha}: \kappa, \Delta_1) & \text{By i.h.} \\ & \Gamma_0 \longrightarrow \Delta_0 & '' \\ & & (\Delta, \hat{\beta}: \kappa' = t) = (\Delta_0, \hat{\alpha}: \kappa, \Delta_1, \hat{\beta}: \kappa' = t) & \text{By congruence of equality} \end{array}$$

- $\hat{\alpha}$ is solved:

$$\begin{array}{ccc} \Delta=(\Delta_0,\hat{\alpha}:\kappa\!=\!\tau',\Delta_1) & \text{By i.h.} \\ & \Gamma_0\longrightarrow\Delta_0 & '' \\ & & (\Delta,\hat{\beta}:\kappa'\!=\!t)=(\Delta_0,\hat{\alpha}:\kappa\!=\!\tau',\Delta_1,\hat{\beta}:\kappa'\!=\!t) & \text{By congruence of equality} \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Gamma \longrightarrow \Delta}{\Gamma_0, \hat{\alpha}: \kappa' \longrightarrow \Delta, \hat{\beta}: \kappa' = t}}_{\Gamma_0, \hat{\alpha}: \kappa, \Gamma_1} \longrightarrow \mathsf{Solve}$$

 $\begin{array}{ll} \textbf{- Case } \hat{\alpha} \neq \hat{\beta} \colon \\ (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1) = (\Gamma, \hat{\beta} : \kappa') & \text{Given} \\ & = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \hat{\beta} : \kappa') & \text{Since the final elements must be equal} \\ & \Gamma = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1') & \text{By injectivity of context syntax} \end{array}$

By induction, there are two possibilities:

* $\hat{\alpha}$ is not solved:

* $\hat{\alpha}$ is solved:

$$\Delta = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1) \qquad \text{By i.h.}$$

$$\Gamma_0 \longrightarrow \Delta_0 \qquad \qquad ''$$

$$(\Delta, \hat{\beta} : \kappa' = t) = (\Delta_0, \hat{\alpha} : \kappa = \tau', \Delta_1, \hat{\beta} : \kappa' = t) \qquad \text{By congruence of equality}$$

- Case
$$\hat{\alpha} = \hat{\beta}$$
:

$$\begin{array}{ll} \Gamma = \Gamma_0 \text{ and } \kappa = \kappa' \text{ and } \Gamma_1 = \cdot \\ (\Delta, \widehat{\beta} : \kappa' = t) = (\Delta_0, \widehat{\alpha} : \kappa = \tau', \Delta_1) \end{array} \quad \begin{array}{ll} \text{By injectivity of syntax} \\ \text{where } \Delta_0 = \Delta \text{ and } \tau' = t \text{ and } \Delta_1 = \cdot \\ \hline \\ \text{From premise } \Gamma \longrightarrow \Delta \end{array}$$

Lemma 23 (Deep Evar Introduction). (i) If Γ_0 , Γ_1 is well-formed and $\hat{\alpha}$ is not declared in Γ_0 , Γ_1 then Γ_0 , $\Gamma_1 \longrightarrow \Gamma_0$, $\hat{\alpha} : \kappa$, Γ_1 .

- (ii) If Γ_0 , $\hat{\alpha}$: κ , Γ_1 is well-formed and $\Gamma \vdash t$: κ then Γ_0 , $\hat{\alpha}$: κ , $\Gamma_1 \longrightarrow \Gamma_0$, $\hat{\alpha}$: $\kappa = t$, Γ_1 .
- (iii) If Γ_0 , Γ_1 is well-formed and $\Gamma \vdash t : \kappa$ then Γ_0 , $\Gamma_1 \longrightarrow \Gamma_0$, $\hat{\alpha} : \kappa = t$, Γ_1 .

Proof.

- (i) Assume that Γ_0 , Γ_1 is well-formed. We proceed by induction on Γ_1 .
 - Case $\Gamma_1 = \cdot$:

$$\begin{array}{ccc} & \Gamma_0 \ ctx & Given \\ & \hat{\alpha} \not\in \mathsf{dom}(\Gamma_0) & Given \\ & \Gamma_0, \hat{\alpha} : \kappa \ ctx & By \ rule \ \mathsf{VarCtx} \\ & \Gamma_0 \longrightarrow \Gamma_0 & By \ \mathsf{Lemma} \ 32 \ (\mathsf{Extension} \ \mathsf{Reflexivity}) \\ & & \Gamma_0 \longrightarrow \Gamma_0, \hat{\alpha} : \kappa & By \ \mathsf{rule} \longrightarrow \mathsf{Add} \end{array}$$

• Case $\Gamma_1 = \Gamma_1', x : A$:

$$\begin{array}{cccc} \Gamma_0,\Gamma_1',x:A\ ctx & Given \\ \Gamma_0,\Gamma_1'\ ctx & By\ inversion \\ x\not\in dom(\Gamma_0,\Gamma_1') & By\ inversion\ (1) \\ \Gamma_0,\Gamma_1'\vdash A\ type & By\ inversion \\ \hat\alpha\not\in dom(\Gamma_0,\Gamma_1',x:A) & Given \\ \hat\alpha\not\neq x & By\ inversion\ (2) \\ \Gamma_0,\hat\alpha:\kappa,\Gamma_1'\ ctx & By\ i.h. \\ \Gamma_0,\Gamma_1'\longrightarrow \Gamma_0,\hat\alpha:\kappa,\Gamma_1' & "\\ \Gamma_0,\hat\alpha:\kappa,\Gamma_1'\vdash A\ type & By\ Lemma\ 36\ (Extension\ Weakening\ (Sorts)) \\ x\not\in dom(\Gamma_0,\hat\alpha:\kappa,\Gamma_1') & By\ (1)\ and\ (2) \\ \blacksquare & \Gamma_0,\Gamma_1',x:A\longrightarrow \Gamma_0,\hat\alpha:\kappa,\Gamma_1',x:A & By\longrightarrow Var \end{array}$$

```
• Case \Gamma_1 = \Gamma_1', \beta : \kappa':
                                                              \Gamma_0, \Gamma_1', \beta : \kappa' ctx
                                                                                                                           Given
                                                             \Gamma_0, \Gamma_1' ctx
                                                                                                                           By inversion
                                                              \beta \notin dom(\Gamma_0, \Gamma_1')
                                                                                                                          By inversion (1)
                                                              \hat{\alpha} \notin \text{dom}(\Gamma_0, \Gamma_1', \beta : \kappa')
                                                                                                                          Given
                                                              \hat{\alpha} \neq \beta
                                                                                                                           By inversion (2)
                                                              \Gamma_0, \hat{\alpha}: \kappa, \Gamma'_1 ctx
                                                                                                                           By i.h.
                                   \Gamma_0, \Gamma_1' \longrightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1'
                                                             \beta \notin dom(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1')
                                                                                                                          By (1) and (2)
        \Gamma_0, \Gamma_1', \beta : \kappa' \longrightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \beta : \kappa'
                                                                                                                          By \longrightarrow Uvar
• Case \Gamma_1 = \Gamma_1', \hat{\beta} : \kappa':
                                                             \Gamma_0, \Gamma_1', \hat{\beta} : \kappa' ctx
                                                                                                                           Given
                                                             \Gamma_0, \Gamma_1' ctx
                                                                                                                           By inversion
                                                              \hat{\beta} \notin \text{dom}(\Gamma_0, \Gamma_1')
                                                                                                                           By inversion (1)
                                                              \hat{\alpha} \notin \text{dom}(\Gamma_0, \Gamma_1', \hat{\beta} : \kappa')
                                                                                                                          Given
                                                                                                                          By inversion (2)
        \Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1} ctx
\Gamma_{0}, \Gamma'_{1} \longrightarrow \Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1}
\hat{\beta} \notin dom(\Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1})
\Gamma_{0}, \Gamma'_{1}, \hat{\beta} : \kappa' \longrightarrow \Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1}, \hat{\beta} : \kappa'
                                                                                                                          By i.h.
                                                                                                                          By (1) and (2)
                                                                                                                          By \longrightarrow Unsolved
• Case \Gamma_1 = (\Gamma_1', \hat{\beta} : \kappa' = t):
                                                                      \Gamma_0, \Gamma_1', \hat{\beta} : \kappa' = t \ ctx
                                                                                                                                            Given
                                                                      \Gamma_0, \Gamma'_1 ctx
                                                                                                                                            By inversion
                                                                      \hat{\beta} \notin dom(\Gamma_0, \Gamma_1')
                                                                                                                                            By inversion (1)
                                                   \Gamma_0, \Gamma_1' \vdash t : \kappa'
                                                                                                                                            By inversion
                                                                      \hat{\alpha} \notin dom(\Gamma_0, \Gamma_1', \hat{\beta} : \kappa' = t)
                                                                                                                                            Given
                                                                                                                                            By inversion (2)
                                                                      \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 ctx
                                                                                                                                            By i.h.
                                            \Gamma_0, \Gamma_1' \longrightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1'
                                     \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \vdash t : \kappa'
                                                                                                      By Lemma 36 (Extension Weakening (Sorts))
                                                                      \hat{\beta} \notin \text{dom}(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1')
                                                                                                                                           By (1) and (2)
        \label{eq:continuous_problem} \Gamma_0,\Gamma_1',\widehat{\beta}:\kappa'\!=\!t\longrightarrow\Gamma_0,\widehat{\alpha}:\kappa,\Gamma_1',\widehat{\beta}:\kappa'\!=\!t
                                                                                                                                           By \longrightarrow Solved
```

• Case $\Gamma_1 = (\Gamma_1', \beta = t)$:

```
\Gamma_0, \Gamma_1', \beta = t ctx
                                                                                            Given
                                          \Gamma_0, \Gamma_1' ctx
                                                                                            By inversion
                                          \beta \notin dom(\Gamma_0, \Gamma_1')
                                                                                            By inversion (1)
                           \Gamma_0, \Gamma_1' \vdash t : \mathbb{N}
                                                                                            By inversion
                                          \hat{\alpha} \notin dom(\Gamma_0, \Gamma_1', \beta = t)
                                                                                            Given
                                                                                            By inversion (2)
                                          \Gamma_0, \hat{\alpha}: \kappa, \Gamma'_1 ctx
                                                                                            By i.h.
                     \Gamma_0,\Gamma_1'\longrightarrow\Gamma_0,\hat\alpha:\kappa,\Gamma_1'
               \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \vdash t : \mathbb{N}
                                                                               By Lemma 36 (Extension Weakening (Sorts))
                                           \beta \notin dom(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1')
                                                                                           By (1) and (2)
\Gamma_0, \Gamma_1', \beta = t \longrightarrow \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \beta = t
                                                                                            By \longrightarrow Solved
```

• Case $\Gamma_1 = (\Gamma_1', \blacktriangleright_{\widehat{\beta}})$:

$$\begin{array}{cccc} \Gamma_0,\Gamma_1',\blacktriangleright_{\widehat{\beta}}\mathit{ctx} & \mathsf{Given} \\ \Gamma_0,\Gamma_1'\mathit{ctx} & \mathsf{By inversion} \\ \widehat{\beta} \notin \mathsf{dom}(\Gamma_0,\Gamma_1') & \mathsf{By inversion} \ (1) \\ \widehat{\alpha} \notin \mathsf{dom}(\Gamma_0,\Gamma_1',\blacktriangleright_{\widehat{\beta}}) & \mathsf{Given} \\ \widehat{\alpha} \neq \widehat{\beta} & \mathsf{By inversion} \ (2) \\ \Gamma_0,\widehat{\alpha}:\kappa,\Gamma_1'\mathit{ctx} & \mathsf{By i.h.} \\ \Gamma_0,\Gamma_1' & \longrightarrow \Gamma_0,\widehat{\alpha}:\kappa,\Gamma_1' & '' \\ \widehat{\beta} \notin \mathsf{dom}(\Gamma_0,\widehat{\alpha}:\kappa,\Gamma_1') & \mathsf{By} \ (1) \ \mathsf{and} \ (2) \\ \blacksquare & \Gamma_0,\Gamma_1',\blacktriangleright_{\widehat{\beta}} & \longrightarrow \Gamma_0,\widehat{\alpha}:\kappa,\Gamma_1',\blacktriangleright_{\widehat{\beta}} & \mathsf{By} & \longrightarrow \mathsf{Marker} \\ \end{array}$$

- (ii) Assume Γ_0 , $\hat{\alpha}$: κ , Γ_1 ctx. We proceed by induction on Γ_1 :
 - Case $\Gamma_1 = \cdot$:

$$\begin{array}{cccc} \Gamma_0 \vdash t : \kappa & & \text{Given} \\ & \Gamma_0, \Gamma_1 \ \text{ctx} & & \text{Given} \\ & \Gamma_0 \text{ ctx} & & \text{Since } \Gamma_1 = \cdot \\ & \Gamma_0 \longrightarrow \Gamma_0 & & \text{By Lemma 32 (Extension Reflexivity)} \\ & \Gamma_0, \hat{\alpha} : \kappa \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t & & \text{By rule} \longrightarrow \text{Solve} \\ & & & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 & & \text{Since } \Gamma_1 = \cdot \\ \end{array}$$

• Case $\Gamma_1 = (\Gamma_1', x : A)$:

$$\begin{array}{cccc} \Gamma_0 \vdash t : \kappa & & \text{Given} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', x : A \textit{ ctx} & & \text{Given} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \textit{ ctx} & & \text{By inversion} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \vdash A \textit{ type} & & \text{By inversion} \\ & \kappa \not\in \text{dom}(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1') & & \text{By inversion (1)} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' & \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 & & \text{By i.h.} \\ & \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 \vdash A \textit{ type} & & \text{By Lemma 36 (Extension Weakening (Sorts))} \\ & \kappa \not\in \text{dom}(\Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1') & & \text{since this is the same domain as (1)} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', x : A & \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1, x : A & & \text{By rule} & \longrightarrow \text{Var} \end{array}$$

• Case $\Gamma_1 = (\Gamma_1', \beta : \kappa')$:

```
\Gamma_0 \vdash t : \kappa
                                                                                                                                Given
                                                                  \Gamma_0, \hat{\alpha}: \kappa, \Gamma'_1, \beta: \kappa' ctx
                                                                                                                               Given
                                                                  \Gamma_0, \hat{\alpha}: \kappa, \Gamma'_1 ctx
                                                                                                                               By inversion
                                                         \beta \notin dom(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1')
                                                                                                                               By inversion (1)
                         \Gamma_0, \hat{\alpha}: \kappa, \Gamma_1' \longrightarrow \Gamma_0, \hat{\alpha}: \kappa = t, \Gamma_1
                                                                                                                                By i.h.
                                                         \beta \notin dom(\Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1')
                                                                                                                                since this is the same domain as (1)
        \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \beta : \kappa' \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1, \beta : \kappa'
                                                                                                                                By rule —→Uvar
• Case \Gamma_1 = (\Gamma_1', \hat{\beta} : \kappa'):
                                                                                                                                Given
                                                                  \Gamma_0, \hat{\alpha}: \kappa, \Gamma'_1, \hat{\beta}: \kappa' ctx
                                                                                                                               Given
                                                                   \Gamma_0, \hat{\alpha} : \kappa, \Gamma'_1 ctx
                                                                                                                                By inversion
        \widehat{\beta} \notin \mathsf{dom}(\Gamma_0, \widehat{\alpha} : \kappa, \Gamma_1')
\Gamma_0, \widehat{\alpha} : \kappa, \Gamma_1' \longrightarrow \Gamma_0, \widehat{\alpha} : \kappa = t, \Gamma_1
\widehat{\beta} \notin \mathsf{dom}(\Gamma_0, \widehat{\alpha} : \kappa = t, \Gamma_1')
\Gamma_0, \widehat{\alpha} : \kappa, \Gamma_1', \widehat{\beta} : \kappa' \longrightarrow \Gamma_0, \widehat{\alpha} : \kappa = t, \Gamma_1, \widehat{\beta} : \kappa'
                                                                                                                               By inversion (1)
                                                                                                                               By i.h.
                                                                                                                               since this is the same domain as (1)
                                                                                                                                By rule \longrightarrow Unsolved
• Case \Gamma_1 = (\Gamma_1', \hat{\beta} : \kappa' = t'):
                                                                                                                             Given
                                                    \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \hat{\beta} : \kappa' = t' ctx
                                                                                                                             Given
                 \Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1} \circ tx
\Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1} \circ tx
\Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1} \vdash t' : \kappa'
\hat{\beta} \notin dom(\Gamma_{0}, \hat{\alpha} : \kappa, \Gamma'_{1})
                                                                                                                             By inversion
                                                                                                                             By inversion
                                                                                                                             By inversion (1)
            \begin{array}{c} \Gamma_0, \hat{\alpha}: \kappa, \Gamma_1' & \longrightarrow \Gamma_0, \hat{\alpha}: \kappa = t, \Gamma_1 \\ \hat{\beta} \not\in dom(\Gamma_0, \hat{\alpha}: \kappa = t, \Gamma_1') \end{array}
                                                                                                                             By i.h.
                                                                                                                             since this is the same domain as (1)
          \Gamma_0, \hat{\alpha}: \kappa = t, \Gamma_1 \vdash t': \kappa'
                                                                                                      By Lemma 36 (Extension Weakening (Sorts))
           \Gamma_0, \hat{\alpha}: \kappa, \Gamma_1', \hat{\beta}: \kappa' = t' \longrightarrow \Gamma_0, \hat{\alpha}: \kappa = t', \Gamma_1, \hat{\beta}: \kappa' = t' By rule \longrightarrow Solved
• Case \Gamma_1 = (\Gamma_1', \beta = t'):
                                                                                                                                     Given
                                                                    \Gamma_0, \hat{\alpha}: \kappa, \Gamma'_1, \beta = t' ctx
                                                                                                                                    Given
                                                                    \Gamma_0, \hat{\alpha}: \kappa, \Gamma'_1 ctx
                                                                                                                                    By inversion
                                 \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \vdash t' : \mathbb{N}
                                                                                                                                    By inversion
                          \beta \notin dom(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1')
\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1
\beta \notin dom(\Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1')
                                                                                                                                    By inversion (1)
                                                                                                                                    By i.h.
                                                                                                                                    since this is the same domain as (1)
                         \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 \vdash t' : \mathbb{N}
                                                                                      By Lemma 36 (Extension Weakening (Sorts))
         \Gamma_0, \hat{\alpha}: \kappa, \Gamma_1', \beta = t' \longrightarrow \Gamma_0, \hat{\alpha}: \kappa = t', \Gamma_1, \beta = t' By rule \longrightarrow \text{Eqn}
• Case \Gamma_1 = (\Gamma_1', \blacktriangleright_{\widehat{G}}):
```

$$\begin{array}{ccc} \Gamma_0 \vdash t : \kappa & & \text{Given} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \blacktriangleright_{\widehat{\beta}} \textit{ctx} & & \text{Given} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \textit{ctx} & & \text{By inversion} \\ & \hat{\beta} \notin \text{dom}(\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1') & & \text{By inversion (1)} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1' \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1 & & \text{By i.h.} \\ & \hat{\beta} \notin \text{dom}(\Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1') & & \text{since this is the same domain as (1)} \\ & \Gamma_0, \hat{\alpha} : \kappa, \Gamma_1', \blacktriangleright_{\widehat{\beta}} \longrightarrow \Gamma_0, \hat{\alpha} : \kappa = t, \Gamma_1, \blacktriangleright_{\widehat{\beta}} & & \text{By rule} \longrightarrow \text{Unsolved} \\ \end{array}$$

(iii) Apply parts (i) and (ii) as lemmas, then Lemma 33 (Extension Transitivity).

Lemma 26 (Parallel Admissibility).

If $\Gamma_L \longrightarrow \Delta_L$ and $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$ then:

- (i) Γ_L , $\hat{\alpha}$: κ , $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha}$: κ , Δ_R
- (ii) If $\Delta_L \vdash \tau' : \kappa$ then $\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$.
- (iii) If $\Gamma_L \vdash \tau : \kappa$ and $\Delta_L \vdash \tau'$ type and $[\Delta_L]\tau = [\Delta_L]\tau'$, then $\Gamma_L, \hat{\alpha} : \kappa = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau', \Delta_R$.

Proof. By induction on Δ_R . As always, we assume that all contexts mentioned in the statement of the lemma are well-formed. Hence, $\hat{\alpha} \notin \mathsf{dom}(\Gamma_L) \cup \mathsf{dom}(\Gamma_R) \cup \mathsf{dom}(\Delta_L) \cup \mathsf{dom}(\Delta_R)$.

(i) We proceed by cases of Δ_R . Observe that in all the extension rules, the right-hand context gets smaller, so as we enter subderivations of Γ_L , $\Gamma_R \longrightarrow \Delta_L$, Δ_R , the context Δ_R becomes smaller.

The only tricky part of the proof is that to apply the i.h., we need $\Gamma_L \longrightarrow \Delta_L$. So we need to make sure that as we drop items from the right of Γ_R and Δ_R , we don't go too far and start decomposing Γ_L or Δ_L ! It's easy to avoid decomposing Δ_L : when $\Delta_R = \cdot$, we don't need to apply the i.h. anyway. To avoid decomposing Γ_L , we need to reason by contradiction, using Lemma 19 (Declaration Preservation).

- Case $\Delta_R = \cdot$: We have $\Gamma_L \longrightarrow \Delta_L$. Applying \longrightarrow Unsolved to that derivation gives the result.
- Case $\Delta_R = (\Delta_R', \hat{\beta})$: We have $\hat{\beta} \neq \hat{\alpha}$ by the well-formedness assumption. The concluding rule of $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R', \hat{\beta}$ must have been \longrightarrow Unsolved or \longrightarrow Add. In both cases, the result follows by i.h. and applying \longrightarrow Unsolved or \longrightarrow Add.

Note: In \longrightarrow Add, the left-hand context doesn't change, so we clearly maintain $\Gamma_L \longrightarrow \Delta_L$. In \longrightarrow Unsolved, we can correctly apply the i.h. because $\Gamma_R \neq \cdot$. Suppose, for a contradiction, that $\Gamma_R = \cdot$. Then $\Gamma_L = (\Gamma_L', \hat{\beta})$. It was given that $\Gamma_L \longrightarrow \Delta_L$, that is, $\Gamma_L', \hat{\beta} \longrightarrow \Delta_L$. By Lemma 19 (Declaration Preservation), Δ_L has a declaration of $\hat{\beta}$. But then $\Delta = (\Delta_L, \Delta_R', \hat{\beta})$ is not well-formed: contradiction. Therefore $\Gamma_R \neq \cdot$.

- Case $\Delta_R = (\Delta_R', \hat{\beta} : \kappa = t)$: We have $\hat{\beta} \neq \hat{\alpha}$ by the well-formedness assumption. The concluding rule must have been \longrightarrow Solved, \longrightarrow Solve or \longrightarrow AddSolved. In each case, apply the i.h. and then the corresponding rule. (In \longrightarrow Solved and \longrightarrow Solve, use Lemma 19 (Declaration Preservation) to show $\Gamma_R \neq \cdot$.)
- Case $\Delta_R = (\Delta_R', \alpha)$: The concluding rule must have been —>Uvar. The result follows by i.h. and applying —>Uvar.
- Case $\Delta_R = (\Delta_R', \alpha = \tau)$: The concluding rule must have been \longrightarrow Eqn. The result follows by i.h. and applying \longrightarrow Eqn.
- Case $\Delta_R = (\Delta'_R, \blacktriangleright_{\widehat{B}})$: Similar to the previous case, with rule \longrightarrow Marker.
- Case $\Delta_R = (\Delta_R', x : A)$: Similar to the previous case, with rule \longrightarrow Var.
- (ii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule \longrightarrow Solve.

(iii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule \longrightarrow Solved, using the given equality to satisfy the second premise.

Lemma 27 (Parallel Extension Solution).

If
$$\Gamma_L$$
, $\hat{\alpha} : \kappa$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha} : \kappa = \tau'$, Δ_R and $\Gamma_L \vdash \tau : \kappa$ and $[\Delta_L]\tau = [\Delta_L]\tau'$ then Γ_L , $\hat{\alpha} : \kappa = \tau$, $\Gamma_R \longrightarrow \Delta_L$, $\hat{\alpha} : \kappa = \tau'$, Δ_R .

Proof. By induction on Δ_R .

In the case where $\Delta_R = \cdot$, we know that rule \longrightarrow Solve must have concluded the derivation (we can use Lemma 19 (Declaration Preservation) to get a contradiction that rules out \longrightarrow AddSolved); then we have a subderivation $\Gamma_L \longrightarrow \Delta_L$, to which we can apply \longrightarrow Solved.

Lemma 28 (Parallel Variable Update).

If
$$\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau_0, \Delta_R \text{ and } \Gamma_L \vdash \tau_1 : \kappa \text{ and } \Delta_L \vdash \tau_2 : \kappa \text{ and } [\Delta_L] \tau_0 = [\Delta_L] \tau_1 = [\Delta_L] \tau_2$$
 then $\Gamma_L, \hat{\alpha} : \kappa = \tau_1, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} : \kappa = \tau_2, \Delta_R$.

Proof. By induction on Δ_R . Similar to the proof of Lemma 27 (Parallel Extension Solution), but applying \longrightarrow Solved at the end.

Lemma 29 (Substitution Monotonicity).

(i) If
$$\Gamma \longrightarrow \Delta$$
 and $\Gamma \vdash t : \kappa$ then $[\Delta][\Gamma]t = [\Delta]t$.

(ii) If
$$\Gamma \longrightarrow \Delta$$
 and $\Gamma \vdash P$ prop then $[\Delta][\Gamma]P = [\Delta]P$.

(iii) If
$$\Gamma \longrightarrow \Delta$$
 and $\Gamma \vdash A$ type then $[\Delta][\Gamma]A = [\Delta]A$.

Proof. We prove each part in turn; part (i) does not depend on parts (ii) or (iii), so we can use part (i) as a lemma in the proofs of parts (ii) and (iii).

• **Proof of Part (i):** By lexicographic induction on the derivation of $\mathcal{D}:: \Gamma \longrightarrow \Delta$ and $\Gamma \vdash t : \kappa$. We proceed by cases on the derivation of $\Gamma \vdash t : \kappa$.

$$\begin{array}{c} \textbf{- Case} \\ \frac{\hat{\alpha}: \kappa \in \Gamma}{\Gamma \vdash \hat{\alpha}: \kappa} \text{ VarSort} \end{array}$$

$$\begin{split} & [\Gamma] \hat{\alpha} = \hat{\alpha} & \text{Since } \hat{\alpha} \text{ is not solved in } \Gamma \\ & [\Delta] \hat{\alpha} = [\Delta] \hat{\alpha} & \text{Reflexivity} \\ & = [\Delta] [\Gamma] \hat{\alpha} & \text{By above equality} \end{split}$$

– Case
$$\frac{(\alpha:\kappa)\in\Gamma}{\Gamma\vdash\alpha:\kappa}\;\mathsf{VarSort}$$

Consider whether or not there is a binding of the form $(\alpha = \tau) \in \Gamma$.

* Case
$$(\alpha = \tau) \in \Gamma$$
:

* Case $(\alpha = \tau) \notin \Gamma$:

$$\begin{split} [\Gamma]\alpha &= \alpha & \text{By definition of substitution} \\ [\Delta][\Gamma]\alpha &= [\Delta]\alpha & \text{Apply } [\Delta] \text{ to both sides} \end{split}$$

- Case

$$\frac{}{\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1 \vdash \hat{\alpha} : \kappa}$$
 SolvedVarSort

Similar to the VarSort case.

- Case

$$\frac{}{\Gamma \vdash 1:\star} \; \mathsf{UnitSort}$$

$$[\Delta]1 = 1 = [\Delta][\Gamma]1$$
 Since $FV(1) = \emptyset$

$$\begin{array}{ccc} \textbf{- Case} & & & & \Gamma \vdash \tau_1 : \star & & \Gamma \vdash \tau_2 : \star \\ & & & & & & \Gamma \vdash \tau_1 \oplus \tau_2 : \star \end{array} \text{BinSort} \\ \end{array}$$

$$\begin{split} [\Delta][\Gamma]\tau_1 &= [\Delta]\tau_1 & \text{By i.h.} \\ [\Delta][\Gamma]\tau_2 &= [\Delta]\tau_2 & \text{By i.h.} \end{split}$$

$$\begin{array}{ll} [\Delta][\Gamma]\tau_1 \oplus [\Delta][\Gamma]\tau_2 = [\Delta]\tau_1 \oplus [\Delta]\tau_2 & \text{By congruence of equality} \\ [\Delta][\Gamma](\tau_1 \oplus \tau_2) = [\Delta](\tau_1 \oplus \tau_2) & \text{Definition of substitution} \end{array}$$

- Case

$$\frac{}{\Gamma \vdash \mathsf{zero} : \mathbb{N}}$$
 ZeroSort

$$[\Delta]$$
zero = zero = $[\Delta][\Gamma]$ zero Since $FV(zero) = \emptyset$

- Case $\frac{\Gamma \vdash t : \mathbb{N}}{\Gamma \vdash \mathsf{succ}(t) : \mathbb{N}} \mathsf{SuccSort}$

$$[\Delta][\Gamma]t = [\Delta]t$$
 By i.h.

$$succ([\Delta][\Gamma]t) = succ([\Delta]t)$$
 By congruence of equality $[\Delta][\Gamma]succ(t) = [\Delta]succ(t)$ By definition of substitution

• **Proof of Part (ii):** We have a derivation of $\Gamma \vdash P$ *prop*, and will use the previous part as a lemma.

• **Proof of Part (iii):** By induction on the derivation of $\Gamma \vdash A$ *type*, using the previous parts as lemmas.

- Case
$$\frac{(u:\star) \in \Gamma}{\Gamma \vdash u \; type} \; \mathsf{VarWF}$$

$$\Gamma \vdash u:\star \quad \mathsf{By \; rule \; VarSort}$$

$$[\Delta][\Gamma]u = [\Delta]u \quad \mathsf{By \; part \; (i)}$$

- Case
$$\frac{(\hat{\alpha}:\star\!=\!\tau)\in\Gamma}{\Gamma\vdash\hat{\alpha}\;type}\;\mathsf{SolvedVarWF}$$

$$\Gamma\vdash\hat{\alpha}:\star\quad\mathsf{By\;rule\;SolvedVarSort}$$

$$[\Delta][\Gamma]\hat{\alpha}=[\Delta]\hat{\alpha}\quad\mathsf{By\;part}\;(i)$$

$$\Gamma \vdash 1 : \star$$
 By rule UnitSort $[\Delta][\Gamma]1 = [\Delta]1$ By part (i)

- Case VecWF: Similar to the BinWF case.

- Case ExistsWF: Similar to the ForallWF case.

- Case
$$\frac{\Gamma \vdash P \textit{ prop} \qquad \Gamma \vdash A_0 \textit{ type}}{\Gamma \vdash A_0 \land P \textit{ type}} \; \text{WithWF}$$

Similar to the ImpliesWF case.

Lemma 30 (Substitution Invariance).

- (i) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash t : \kappa$ and $\mathsf{FEV}([\Gamma]t) = \emptyset$ then $[\Delta][\Gamma]t = [\Gamma]t$.
- (ii) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash P$ prop and $\mathsf{FEV}([\Gamma]P) = \emptyset$ then $[\Delta][\Gamma]P = [\Gamma]P$.
- (iii) If $\Gamma \longrightarrow \Delta$ and $\Gamma \vdash A$ type and $\mathsf{FEV}([\Gamma]A) = \emptyset$ then $[\Delta][\Gamma]A = [\Gamma]A$.

Proof. Each part is a separate induction, relying on the proofs of the earlier parts. In each part, the result follows by an induction on the derivation of $\Gamma \longrightarrow \Delta$.

The main observation is that Δ adds no equations for any variable of t, P, and A that Γ does not already contain, and as a result applying Δ as a substitution to $[\Gamma]$ t does nothing.

Lemma 24 (Soft Extension).

If $\Gamma \longrightarrow \Delta$ and Γ , Θ ctx and Θ is soft, then there exists Ω such that $dom(\Theta) = dom(\Omega)$ and Γ , $\Theta \longrightarrow \Delta$, Ω .

Proof. By induction on Θ .

- Case $\Theta = \cdot$: We have $\Gamma \longrightarrow \Delta$. Let $\Omega = \cdot$. Then $\Gamma, \Theta \longrightarrow \Delta, \Omega$.
- Case $\Theta = (\Theta', \hat{\alpha} : \kappa = t)$:

$$\begin{array}{ccc} \Gamma, \Theta' & \longrightarrow \Gamma, \Omega' & \text{By i.h.} \\ & \Gamma, \underbrace{\Theta', \hat{\alpha} : \kappa = t}_{\Theta} & \longrightarrow \Delta, \underbrace{\Omega', \hat{\alpha} : \kappa = t}_{\Omega} & \text{By rule} & \longrightarrow \text{Solved} \end{array}$$

• Case $\Theta = (\Theta', \hat{\alpha} : \kappa)$:

If
$$\kappa = \star$$
, let $t = 1$; if $\kappa = \mathbb{N}$, let $t = \mathsf{zero}$.

$$\begin{array}{ccc} & \Gamma, \Theta' \longrightarrow \Gamma, \Omega' & \text{By i.h.} \\ & \Gamma, \underbrace{\Theta', \hat{\alpha} : \kappa}_{\Theta} \longrightarrow \Delta, \underbrace{\Omega', \hat{\alpha} : \kappa = t}_{\Omega} & \text{By rule} \longrightarrow \mathsf{Solve} \end{array}$$

Lemma 31 (Split Extension).

$$\begin{split} & \text{If } \Delta \longrightarrow \Omega \\ & \text{and } \hat{\alpha} \in \mathsf{unsolved}(\Delta) \\ & \text{and } \Omega = \Omega_1[\hat{\alpha} : \kappa = t_1] \\ & \text{and } \Omega \text{ is canonical (Definition 3)} \\ & \text{and } \Omega \vdash t_2 : \kappa \\ & \text{then } \Delta \longrightarrow \Omega_1[\hat{\alpha} : \kappa = t_2]. \end{split}$$

Proof. By induction on the derivation of $\Delta \longrightarrow \Omega$. Use the fact that $\Omega_1[\hat{\alpha}: \kappa = t_1]$ and $\Omega_1[\hat{\alpha}: \kappa = t_2]$ agree on all solutions *except* the solution for $\hat{\alpha}$. In the \longrightarrow Solve case where the existential variable is $\hat{\alpha}$, use $\Omega \vdash t_2 : \kappa$.

C'.1 Reflexivity and Transitivity

Lemma 32 (Extension Reflexivity). *If* Γ *ctx then* $\Gamma \longrightarrow \Gamma$.

Proof. By induction on the derivation of Γ *ctx*.

• Case
$$\frac{}{\cdot ctx} \text{ EmptyCtx}$$

$$\cdot \longrightarrow \cdot \quad \text{By rule } \longrightarrow \text{Id}$$

• Case
$$\frac{\Gamma \ ctx \qquad x \not\in \text{dom}(\Gamma) \qquad \Gamma \vdash A \ type}{\Gamma, x : A \ ctx} \text{ HypCtx}$$

$$\Gamma \longrightarrow \Gamma \qquad \text{By i.h.}$$

$$[\Gamma]A = [\Gamma]A \qquad \text{By reflexivity}$$

$$\Gamma, x : A \longrightarrow \Gamma, x : A \qquad \text{By rule} \longrightarrow \text{Var}$$

• Case
$$\frac{\Gamma \ ctx \qquad \mathfrak{u} : \kappa \notin \mathsf{dom}(\Gamma)}{\Gamma, \mathfrak{u} : \kappa \ ctx} \ \mathsf{VarCtx}$$

$$\Gamma \longrightarrow \Gamma \qquad \mathsf{By i.h.}$$

$$\Gamma, \mathfrak{u} : \kappa \longrightarrow \Gamma, \mathfrak{u} : \kappa \qquad \mathsf{By rule} \longrightarrow \mathsf{Uvar or} \longrightarrow \mathsf{Unsolved}$$

• Case
$$\frac{\int ctx}{\int_{\Gamma} \mathbf{v}_{tt}} \frac{\mathbf{v}_{tt} \notin \Gamma}{\int_{\Gamma} \mathbf{v}_{tt}} \mathbf{M}$$
arkerCtx

$$\begin{array}{ccc} \Gamma &\longrightarrow \Gamma & & \text{By i.h.} \\ \Gamma, \blacktriangleright_{\mathfrak{u}} &\longrightarrow \Gamma, \blacktriangleright_{\mathfrak{u}} & & \text{By rule} \longrightarrow \mathsf{Marker} \end{array}$$

Lemma 33 (Extension Transitivity).

If
$$\mathcal{D} :: \Gamma \longrightarrow \Theta$$
 and $\mathcal{D}' :: \Theta \longrightarrow \Delta$ then $\Gamma \longrightarrow \Delta$.

Proof. By induction on \mathcal{D}' .

Case

$$\overline{\underbrace{\cdot}_{\Theta} \longrightarrow \underbrace{\cdot}_{\Delta}} \longrightarrow \mathsf{lc}$$

 $\Gamma = \cdot$ By inversion on \mathcal{D}

$$\cdot \longrightarrow \cdot \quad \text{ By rule } \longrightarrow \mathsf{Id}$$

$$\Gamma \longrightarrow \Delta$$
 Since $\Gamma = \Delta = \cdot$

 $\bullet \ \, \textbf{Case} \ \, \underbrace{ \frac{\Theta' \longrightarrow \Delta' \qquad [\Delta']A = [\Delta']A'}{\underbrace{\Theta', x:A} \longrightarrow \underbrace{\Delta', x:A'}_{\Delta}} \longrightarrow \textbf{Var}$

$$\Gamma = (\Gamma', x : A'')$$
 By inversion on \mathcal{D}

$$[\Theta]A'' = [\Theta]A$$
 By inversion on \mathcal{T}

$$\Gamma' \longrightarrow \Theta'$$
 By inversion on \mathcal{I}

$$\Gamma' \longrightarrow \Delta'$$
 By i.h

$$[\Delta'][\Theta']A'' = [\Delta'][\Theta']A$$
 By congruence of equality

$$[\Theta]A'' = [\Theta]A \qquad \text{By inversion on } \mathcal{D}$$

$$\Gamma' \longrightarrow \Theta' \qquad \text{By inversion on } \mathcal{D}$$

$$\Gamma' \longrightarrow \Delta' \qquad \text{By inversion on } \mathcal{D}$$

$$\Gamma' \longrightarrow \Delta' \qquad \text{By i.h.}$$

$$[\Delta'][\Theta']A'' = [\Delta'][\Theta']A \qquad \text{By congruence of equality}$$

$$[\Delta']A'' = [\Delta']A \qquad \text{By Lemma 29 (Substitution Monotonicity)}$$

$$= [\Delta']A' \qquad \text{By premise } [\Delta']A = [\Delta']A'$$

$$\Gamma', x : A'' \longrightarrow \Delta', x : A' \qquad \text{By } \longrightarrow \text{Var}$$

$$= [\Delta']A'$$
 By premise $[\Delta']A = [\Delta']A$

$$\Gamma', \chi : A'' \longrightarrow \Delta', \chi : A'$$
 By $\longrightarrow Var$

Case

$$\underbrace{ \begin{array}{c} \Theta' \longrightarrow \Delta' \\ \underline{\Theta', \alpha : \kappa} \longrightarrow \underline{\Delta', \alpha : \kappa} \end{array}}_{\Theta} \longrightarrow \mathsf{Uvar}$$

$$\Gamma = (\Gamma', \alpha : \kappa)$$
 By inversion on \mathcal{I}

$$\begin{array}{ccc} \Gamma = (\Gamma',\alpha:\kappa) & \text{By inversion on } \mathcal{D} \\ \Gamma' \longrightarrow \Theta' & \text{By inversion on } \mathcal{D} \\ \Gamma' \longrightarrow \Delta' & \text{By i.h.} \\ \Gamma',\alpha:\kappa \longrightarrow \Delta',\alpha:\kappa & \text{By } \longrightarrow \text{Uvar} \end{array}$$

$$\Gamma' \longrightarrow \Delta'$$
 By i.h

$$\Gamma', \alpha : \kappa \longrightarrow \Delta', \alpha : \kappa \qquad \text{By } \longrightarrow \text{Uvar}$$

• Case

$$\underbrace{\frac{\Theta' \longrightarrow \Delta'}{\underbrace{\Theta', \hat{\alpha} : \kappa} \longrightarrow \underbrace{\Delta', \hat{\alpha} : \kappa}}_{\Theta} \longrightarrow \mathsf{Unsolved}$$

Two rules could have concluded $\mathcal{D} :: \Gamma \longrightarrow (\Theta', \hat{\alpha} : \kappa)$:

$$\begin{array}{c} \textbf{- Case} & \underline{\Gamma' \longrightarrow \Theta'} \\ \underline{\underline{\Gamma', \alpha: \kappa} \longrightarrow \Theta', \alpha: \kappa} & \longrightarrow \text{Unsolved} \end{array}$$

$$\begin{array}{ccc} \Gamma' \longrightarrow \Delta' & \text{By i.h.} \\ \Gamma', \hat{\alpha} : \kappa \longrightarrow \Delta', \hat{\alpha} : \kappa & \text{By rule} \longrightarrow \mathsf{Add} \end{array}$$

- Case
$$\frac{\Gamma \longrightarrow \Theta'}{\Gamma \longrightarrow \Theta', \hat{\alpha} : \kappa} \longrightarrow \mathsf{Add}$$

$$\begin{array}{ll} \Gamma \longrightarrow \Delta' & \text{By i.h.} \\ \Gamma \longrightarrow \Delta', \hat{\alpha} : \kappa & \text{By rule} \longrightarrow \mathsf{Add} \end{array}$$

Two rules could have concluded $\mathcal{D} :: \Gamma \longrightarrow (\Theta', \hat{\alpha} : \kappa = t)$:

$$\begin{array}{ll} \Gamma' \longrightarrow \Delta' & \text{By i.h.} \\ [\Theta']t'' = [\Theta']t & \text{Premise} \\ [\Delta'][\Theta']t'' = [\Delta'][\Theta']t & \text{Applying } \Delta' \text{ to both sides} \\ [\Delta']t'' = [\Delta']t & \text{By Lemma 29 (Substitution Monotonicity)} \\ & = [\Delta']t' & \text{By premise } [\Delta']t = [\Delta']t' \\ \Gamma', \hat{\alpha}: \kappa = t'' \longrightarrow \Delta', \hat{\alpha}: \kappa = t' & \text{By rule } \longrightarrow \text{Solved} \end{array}$$

$$\label{eq:case_equation} \begin{array}{c} & \Gamma \longrightarrow \Theta' \\ \hline \Gamma \longrightarrow \Theta', \hat{\alpha} : \kappa = t \end{array} \longrightarrow \mathsf{AddSolved}$$

$$\begin{array}{ll} \Gamma \longrightarrow \Delta' & \text{By i.h.} \\ \Gamma \longrightarrow \Delta', \hat{\alpha} : \kappa \! = \! t' & \text{By rule} \longrightarrow \! \text{AddSolved} \end{array}$$

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Theta' \longrightarrow \Delta'}{\underbrace{\Theta', \alpha \! = \! t}} \longrightarrow \underbrace{\Delta', \alpha \! = \! t'}_{\Delta} \end{array} \longrightarrow \! \text{Eqn} \\ \end{array}$$

$$\begin{array}{ll} \Gamma = (\Gamma', \alpha \!=\! t'') & \text{By inversion on } \mathcal{D} \\ \Gamma' \longrightarrow \Theta' & \text{By inversion on } \mathcal{D} \\ [\Theta']t'' = [\Theta']t & \text{By inversion on } \mathcal{D} \\ [\Delta'][\Theta']t'' = [\Delta'][\Theta']t & \text{Applying } \Delta' \text{ to both sides} \\ \Gamma' \longrightarrow \Delta' & \text{By i.h.} \\ [\Delta']t'' = [\Delta']t & \text{By Lemma 29 (Substitution Monotonicity)} \\ &= [\Delta']t' & \text{By premise } [\Delta']t = [\Delta']t' \\ \Gamma', \alpha \!=\! t'' \longrightarrow \Delta', \alpha \!=\! t' & \text{By rule } \longrightarrow \text{Eqn} \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \, \Theta \longrightarrow \Delta' }_{\Theta \longrightarrow \underbrace{\Delta', \hat{\alpha} : \kappa}_{\Delta} } \longrightarrow \mathsf{Add}$$

$$\Gamma \longrightarrow \Delta'$$
 By i.h. $\Gamma \longrightarrow \Delta', \hat{\alpha} : \kappa$ By rule $\longrightarrow \mathsf{Add}$

$$\bullet \ \, \text{Case} \ \, \frac{\Theta \longrightarrow \Delta'}{\Theta \longrightarrow \underbrace{\Delta', \hat{\alpha} : \kappa = t}_{\Delta}} \longrightarrow \text{AddSolved}$$

$$\begin{array}{ll} \Gamma \longrightarrow \Delta' & \text{By i.h.} \\ \Gamma \longrightarrow \Delta', \hat{\alpha} : \kappa \! = \! t & \text{By rule} \longrightarrow \! \text{AddSolved} \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\Theta' \longrightarrow \Delta'}{\Theta', \blacktriangleright_{\mathfrak{U}}} \longrightarrow \underbrace{\Delta', \blacktriangleright_{\mathfrak{U}}}_{\Delta}}_{\Theta} \longrightarrow \mathsf{Marker}$$

$$\begin{array}{ccc} \Gamma = \Gamma', \blacktriangleright_{\mathfrak{u}} & \text{ By inversion on } \mathcal{D} \\ \Gamma' \longrightarrow \Theta' & \text{ By inversion on } \mathcal{D} \\ \Gamma' \longrightarrow \Delta' & \text{ By i.h.} \\ \Gamma', \blacktriangleright_{\mathfrak{u}} \longrightarrow \Delta', \blacktriangleright_{\mathfrak{u}} & \text{ By } \longrightarrow \mathsf{Uvar} \end{array}$$

C'.2 Weakening

Lemma 34 (Suffix Weakening). *If* $\Gamma \vdash t : \kappa$ *then* $\Gamma, \Theta \vdash t : \kappa$.

Proof. By induction on the given derivation. All cases are straightforward.

Lemma 35 (Suffix Weakening). *If* $\Gamma \vdash A$ *type then* $\Gamma, \Theta \vdash A$ *type.*

Proof. By induction on the given derivation. All cases are straightforward.

Lemma 36 (Extension Weakening (Sorts)). *If* $\Gamma \vdash t : \kappa$ *and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash t : \kappa$.

Proof. By a straightforward induction on $\Gamma \vdash t : \kappa$.

In the VarSort case, use Lemma 22 (Extension Inversion) (i) or (v). In the SolvedVarSort case, use Lemma 22 (Extension Inversion) (iv). In the other cases, apply the i.h. to all subderivations, then apply the rule. \Box

Lemma 37 (Extension Weakening (Props)). *If* $\Gamma \vdash P$ *prop and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash P$ *prop.*

Proof. By inversion on rule EqProp, and Lemma 36 (Extension Weakening (Sorts)) twice.

Lemma 38 (Extension Weakening (Types)). *If* $\Gamma \vdash A$ *type and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash A$ *type.*

Proof. By a straightforward induction on $\Gamma \vdash A$ *type*.

In the VarWF case, use Lemma 22 (Extension Inversion) (i) or (v). In the SolvedVarWF case, use Lemma 22 (Extension Inversion) (iv).

In the other cases, apply the i.h. and/or (for ImpliesWF and WithWF) Lemma 37 (Extension Weakening (Props)) to all subderivations, then apply the rule. \Box

C'.3 Principal Typing Properties

Lemma 39 (Principal Agreement).

- (i) If $\Gamma \vdash A$! type and $\Gamma \longrightarrow \Delta$ then $[\Delta]A = [\Gamma]A$.
- (ii) If $\Gamma \vdash P$ prop and $FEV(P) = \emptyset$ and $\Gamma \longrightarrow \Delta$ then $[\Delta]P = [\Gamma]P$.

Proof. By induction on the derivation of $\Gamma \longrightarrow \Delta$. Part (i):

$$\begin{array}{cccc} \bullet & \textbf{Case} & \underline{\Gamma_0 \longrightarrow \Delta_0} & & [\Delta_0]t = [\Delta_0]t' \\ \hline & \overline{\Gamma_0, \alpha \!=\! t \longrightarrow \underbrace{\Delta_0, \alpha \!=\! t'}_{\Delta}} & \longrightarrow \text{Eqn} \end{array}$$

If $\alpha \notin FV(A)$, then:

$$\begin{split} [\Gamma_0,\alpha=t]A &= [\Gamma_0]A & \text{By def. of subst.} \\ &= [\Delta_0]A & \text{By i.h.} \\ &= [\Delta_0,\alpha=t']A & \text{By def. of subst.} \end{split}$$

Otherwise, $\alpha \in FV(A)$.

 $\Gamma_0 \vdash t \ type \qquad \Gamma \ \text{is well-formed}$

 $\Gamma_0 \vdash [\Gamma_0]$ t type By Lemma 13 (Right-Hand Substitution for Typing)

Suppose, for a contradiction, that $FEV([\Gamma_0]t) \neq \emptyset$.

Since $\alpha \in FV(A)$, we also have $FEV([\Gamma]A) \neq \emptyset$, a contradiction.

```
FEV([\Gamma_0]t) \neq \emptyset
                                                            Assumption (for contradiction)
                [\Gamma_0]t = [\Gamma]\alpha
                                                            By def. of subst.
       \mathsf{FEV}([\Gamma]\alpha) \neq \emptyset
                                                            By above equality
                      \alpha \in FV(A)
                                                            Above
      FEV([\Gamma]A) \neq \emptyset
                                                            By a property of subst.
                      \Gamma \vdash A ! type
                                                            Given
      FEV([\Gamma]A) = \emptyset
                                                            By inversion
                   \Rightarrow \Leftarrow
   FEV([\Gamma_0]t) = \emptyset
                                                            By contradiction
                     \Gamma_0 \vdash t ! type
                                                            By PrincipalWF
                [\Gamma_0]t = [\Delta_0]t
                                                            By i.h.
                     \Gamma_0 \vdash [\Delta_0] t \ type
                                                            By above equality
     FEV([\Delta_0]t) = \emptyset
                                                            By above equality
                     \Gamma_0 \vdash [[\Delta_0]t/\alpha]A ! type
                                                            By Lemma 8 (Substitution—Well-formedness) (i)
[\Gamma_0] [\Delta_0] t/\alpha A = [\Delta_0] [\Delta_0] t/\alpha A
                                                            By i.h. (at [\Delta_0]t/\alpha]A)
     [\Gamma_0, \alpha = t]A = [\Gamma_0][\Gamma_0]t/\alpha]A
                                                            By def. of subst.
                       = [\Gamma_0][[\Delta_0]t/\alpha]A
                                                            By above equality
                       = \left[\Delta_0\right] \left[ [\Delta_0] t / \alpha \right] A
                                                            By above equality
                        = \left[\Delta_0\right] \left[\left[\Delta_0\right] t'/\alpha\right] A
                                                            By [\Delta_0]t = [\Delta_0]t'
                                                            By def. of subst.
                        = [\Delta]A
```

- $\bullet \ \textbf{Case} \longrightarrow \hspace{-0.5cm} \mathsf{Solved}, \longrightarrow \hspace{-0.5cm} \mathsf{Solve}, \longrightarrow \hspace{-0.5cm} \mathsf{Add}, \longrightarrow \hspace{-0.5cm} \mathsf{Solved} \text{:} \quad \mathsf{Similar} \ \mathsf{to} \ \mathsf{the} \longrightarrow \hspace{-0.5cm} \mathsf{Eqn} \ \mathsf{case}.$
- **Case** → Id, → Var, → Uvar, → Unsolved, → Marker: Straightforward, using the i.h. and the definition of substitution.

Part (ii): Similar to part (i), using part (ii) of Lemma 8 (Substitution—Well-formedness). □

Lemma 40 (Right-Hand Subst. for Principal Typing). If $\Gamma \vdash A$ p type then $\Gamma \vdash [\Gamma]A$ p type.

Proof. By cases of p:

• Case p = !:

• Case p = 1:

 $\Gamma \vdash A \ type$ By inversion $\Gamma \vdash [\Gamma]A \ type$ By Lemma 13 (Right-Hand Substitution for Typing) $\Gamma \vdash A \ / \ type$ By rule NonPrincipalWF

Lemma 41 (Extension Weakening for Principal Typing). If $\Gamma \vdash A$ p type and $\Gamma \longrightarrow \Delta$ then $\Delta \vdash A$ p type.

Proof. By cases of p:

• Case p = 1:

 $\Gamma \vdash A \ type$ By inversion

 $\Delta \vdash A \ type$ By Lemma 38 (Extension Weakening (Types))

 $\Delta \vdash A \not \mid type$ By rule NonPrincipalWF

• Case p = !:

```
\begin{array}{lll} \Gamma \vdash A \ type & \text{By inversion} \\ \text{FEV}([\Gamma]A) = \emptyset & \text{By inversion} \\ \Delta \vdash A \ type & \text{By Lemma 38 (Extension Weakening (Types))} \\ \Delta \vdash [\Delta]A \ type & \text{By Lemma 13 (Right-Hand Substitution for Typing)} \\ [\Delta]A = [\Gamma]A & \text{By Lemma 30 (Substitution Invariance)} \\ \text{FEV}([\Delta]A) = \emptyset & \text{By congruence of equality} \\ \Delta \vdash [\Delta]A \ ! \ type & \text{By rule PrincipalWF} \\ \end{array}
```

Lemma 42 (Inversion of Principal Typing).

- (1) If $\Gamma \vdash (A \rightarrow B)$ p type then $\Gamma \vdash A$ p type and $\Gamma \vdash B$ p type.
- (2) If $\Gamma \vdash (P \supset A)$ p type then $\Gamma \vdash P$ prop and $\Gamma \vdash A$ p type.
- (3) If $\Gamma \vdash (A \land P)$ p type then $\Gamma \vdash P$ prop and $\Gamma \vdash A$ p type.

Proof. Proof of part 1:

We have $\Gamma \vdash A \rightarrow B \ p \ type$.

• Case p = 1:

1
$$\Gamma \vdash A \rightarrow B \ type$$
 By inversion
 $\Gamma \vdash A \ type$ By inversion on 1
 $\Gamma \vdash B \ type$ By inversion on 1
 $\Gamma \vdash A \ f \ type$ By rule NonPrincipalWF
 $\Gamma \vdash B \ f \ type$ By rule NonPrincipalWF

• Case p = !:

$$\begin{array}{lll} & \Gamma \vdash A \to B \ type & \text{By inversion on } \Gamma \vdash A \to B \ ! \ type \\ \emptyset = \mathsf{FEV}([\Gamma](A \to B)) & " \\ & = \mathsf{FEV}([\Gamma]A \to [\Gamma]B) & \text{By definition of substitution} \\ & = \mathsf{FEV}([\Gamma]A) \cup \mathsf{FEV}([\Gamma]B) & \text{By definition of } \mathsf{FEV}(-) \\ \mathsf{FEV}([\Gamma]A) = \mathsf{FEV}([\Gamma]B) = \emptyset & \text{By properties of empty sets and unions} \\ & \Gamma \vdash A \ type & \text{By inversion on } 1 \\ & \Gamma \vdash B \ type & \text{By inversion on } 1 \\ & \Gamma \vdash A \ ! \ type & \text{By rule PrincipalWF} \\ & \Gamma \vdash B \ ! \ type & \text{By rule PrincipalWF} \end{array}$$

Part 2: We have $\Gamma \vdash P \supset A p$ *type*. Similar to Part 1.

Part 3: We have $\Gamma \vdash A \land P$ p *type*. Similar to Part 2.

C'.4 Instantiation Extends

Lemma 43 (Instantiation Extension).

If
$$\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$$
 then $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

Case

$$\underbrace{\frac{\Gamma_L \vdash \tau : \kappa}{\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R} \vdash \hat{\alpha} := \tau : \kappa \dashv \Gamma_L, \hat{\alpha} : \kappa = \tau, \Gamma_R}_{\Gamma} \text{ InstSolve}$$

Follows by Lemma 23 (Deep Evar Introduction) (ii).

Case

$$\underbrace{\frac{\hat{\beta} \in \mathsf{unsolved}(\Gamma_0[\hat{\alpha} : \kappa][\hat{\beta} : \kappa])}{\Gamma_0[\hat{\alpha} : \kappa][\hat{\beta} : \kappa]} \vdash \hat{\alpha} := \hat{\beta} : \kappa \dashv \Gamma_0[\hat{\alpha} : \kappa][\hat{\beta} : \kappa = \hat{\alpha}]}_{\Gamma} \; \mathsf{InstReach}$$

Follows by Lemma 23 (Deep Evar Introduction) (ii).

• Case
$$\frac{\Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]\vdash\hat{\alpha}_1:=\tau_1:\star\dashv\Theta\qquad\Theta\vdash\hat{\alpha}_2:=[\Theta]\tau_2:\star\dashv\Delta}{\Gamma_0[\hat{\alpha}:\star]\vdash\hat{\alpha}:=\tau_1\oplus\tau_2:\star\dashv\Delta} \text{ InstBin}$$

$$\begin{array}{ll} \Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]\vdash\hat{\alpha}_1:=\tau_1:\star\dashv\Theta & Subderivation \\ \Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]\longrightarrow\Theta & By i.h. \\ \Theta\vdash\hat{\alpha}_2:=[\Theta]\tau_2:\star\dashv\Delta & Subderivation \\ \Theta\longrightarrow\Delta & By i.h. \\ \Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]\longrightarrow\Delta & By i.h. \end{array}$$

$$\Gamma_0[\hat{\alpha}:\star] \longrightarrow \Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]$$
 By Lemma 23 (Deep Evar Introduction) (parts (i), (i), and (ii), using Lemma 33 (Extension Transitivity))

 $\Gamma_0[\hat{\alpha}:\star] \longrightarrow \Delta$ By Lemma 33 (Extension Transitivity)

• Case

$$\frac{}{\Gamma_0[\hat{\alpha}:\mathbb{N}]\vdash\hat{\alpha}:=\mathsf{zero}:\mathbb{N}\dashv\Gamma_0[\hat{\alpha}:\mathbb{N}=\mathsf{zero}]}\;\mathsf{InstZero}$$

Follows by Lemma 23 (Deep Evar Introduction) (ii).

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}_1:\mathbb{N},\hat{\alpha}:\mathbb{N} = \mathsf{succ}(\hat{\alpha}_1)] \vdash \hat{\alpha}_1 := t_1:\mathbb{N} \dashv \Delta}{\Gamma[\hat{\alpha}:\mathbb{N}] \vdash \hat{\alpha} := \mathsf{succ}(t_1):\mathbb{N} \dashv \Delta} \ \, \mathsf{InstSucc}$$

By reasoning similar to the InstBin case.

C'.5 Equivalence Extends

Lemma 44 (Elimeq Extension).

If $\Gamma / s \stackrel{\circ}{=} t : \kappa \dashv \Delta$ then there exists Θ such that $\Gamma, \Theta \longrightarrow \Delta$.

Proof. By induction on the given derivation. Note that the statement restricts the output to be a (consistent) context Δ .

Case

$$\frac{}{\Gamma \mathrel{/} \alpha \stackrel{\scriptscriptstyle \circ}{=} \alpha \mathrel{:} \kappa \mathrel{\dashv} \Gamma} \; \mathsf{ElimeqUvarRefl}$$

Since $\Delta = \Gamma$, applying Lemma 32 (Extension Reflexivity) suffices (let $\Theta = \cdot$).

Case

$$\Gamma$$
 / zero $\stackrel{\circ}{=}$ zero : $\mathbb{N} \dashv \Gamma$

Similar to the ElimeqUvarRefl case.

 $\begin{array}{c} \bullet \ \ \, \textbf{Case} \\ \hline \frac{\Gamma \ / \ \sigma \stackrel{\circ}{=} \ t : \mathbb{N} \ \neg \ \Delta}{\Gamma \ / \ \mathsf{succ}(\sigma) \ \stackrel{\circ}{=} \ \mathsf{succ}(t) : \mathbb{N} \ \neg \ \Delta} \ \ \mathsf{ElimeqSucc} \end{array}$

Follows by i.h.

 $\bullet \ \, \textbf{Case} \ \, \underbrace{ \frac{\Gamma_0[\hat{\alpha}:\kappa] \vdash \hat{\alpha} := t : \kappa \dashv \Delta}{\Gamma_0[\hat{\alpha}:\kappa] \mathrel{/} \hat{\alpha} \stackrel{\text{$\ \stackrel{\circ}{=}\ }}{t} : \kappa \dashv \Delta} }_{\Gamma} \ \, \text{ElimeqInstL}$

 $\bullet \ \, \textbf{Case} \ \, \frac{\alpha \notin FV([\Gamma] \texttt{t}) \qquad (\alpha = -) \notin \Gamma}{\Gamma \ \, / \ \, \alpha \stackrel{\text{$\, \stackrel{\circ}{=} \, $}}{\texttt{t}} : \ \, \mathsf{k} \dashv \Gamma, \, \alpha = \mathsf{t} } \ \, \textbf{ElimeqUvarL}$

Let Θ be $(\alpha = t)$.

$$\Gamma, \underbrace{\alpha = t}_{\Theta} \longrightarrow \Gamma, \alpha = t \quad \text{ By Lemma 32 (Extension Reflexivity)}$$

• Cases ElimegInstR, ElimegUvarR:

Similar to the respective L cases.

• Case $\frac{\sigma \# t}{\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot} \text{ ElimeqClash}$

The statement says that the output is a (consistent) context Δ , so this case is impossible.

Lemma 45 (Elimprop Extension).

If $\Gamma / P \dashv \Delta$ then there exists Θ such that $\Gamma, \Theta \longrightarrow \Delta$.

Proof. By induction on the given derivation. Note that the statement restricts the output to be a (consistent) context Δ .

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \ / \ \sigma \stackrel{\text{$\, \stackrel{\circ}{=}\, }}{} t : \mathbb{N} \ \dashv \Delta}{\Gamma \ / \ \sigma = t \ \dashv \Delta} \ \, \textbf{ElimpropEq}$$

$$\Gamma \ / \ \sigma \stackrel{\text{\tiny \circ}}{=} t : \mathbb{N} \dashv \Delta \quad \text{Subderivation}$$

$$\mathbb{F} \quad \Gamma, \Theta \longrightarrow \Delta \quad \text{By Lemma 44 (Elimeq Extension)}$$

Lemma 46 (Checkeq Extension).

If
$$\Gamma \vdash A \equiv B \dashv \Delta$$
 then $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

Case

$$\frac{\Gamma \vdash u \stackrel{\circ}{=} u : \kappa \dashv \Gamma}{\Gamma}$$
 CheckeqVar

Since $\Delta = \Gamma$, applying Lemma 32 (Extension Reflexivity) suffices.

• Cases CheckeqUnit, CheckeqZero: Similar to the CheckeqVar case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash \tau_1 \, \stackrel{\circ}{=}\, \tau_1' : \star \dashv \Theta \qquad \Theta \vdash [\Theta] \tau_2 \, \stackrel{\circ}{=}\, [\Theta] \tau_2' : \star \dashv \Delta}{\Gamma \vdash \tau_1 \, \oplus \, \tau_2 \, \stackrel{\circ}{=}\, \tau_1' \, \oplus \, \tau_2' : \star \dashv \Delta} \ \, \textbf{CheckeqBin}$$

$$\Gamma \longrightarrow \Theta$$
 By i.h.

$$\Theta \longrightarrow \Delta$$
 By i.h.

 $\Gamma \longrightarrow \Delta$ By Lemma 33 (Extension Transitivity)

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \\ \hline \Gamma \vdash \sigma \stackrel{\circ}{=} t : \mathbb{N} \dashv \Delta \\ \hline \Gamma \vdash \mathsf{succ}(\sigma) \stackrel{\circ}{=} \mathsf{succ}(t) : \mathbb{N} \dashv \Delta \end{array} \text{CheckeqSucc}$$

$$\Gamma \vdash \sigma \stackrel{\mathtt{e}}{=} t : \mathbb{N} \dashv \Delta \quad \text{Subderivation}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} := t : \kappa \dashv \Delta \qquad \hat{\alpha} \notin FV([\Gamma_0[\hat{\alpha}]]t)}{\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} \stackrel{\circ}{=} t : \kappa \dashv \Delta} \ \, \text{CheckeqInstL}$$

$$\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} := t : \kappa \dashv \Delta \quad \text{Subderivation}$$

$$\Gamma_0[\hat{\alpha}] \longrightarrow \Delta \quad \text{By Lemma 43 (Instantiation Extension)}$$

• Case CheckeqInstR: Similar to the CheckeqInstL case.

Lemma 47 (Checkprop Extension).

If
$$\Gamma \vdash P$$
 true $\dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash \sigma \stackrel{\circ}{=} t : \mathbb{N} \dashv \Delta}{\Gamma \vdash \sigma = t \ \textit{true} \dashv \Delta} \ \, \textbf{CheckpropEq}$$

Lemma 48 (Prop Equivalence Extension).

If
$$\Gamma \vdash P \equiv Q \dashv \Delta$$
 then $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

Lemma 49 (Equivalence Extension).

If
$$\Gamma \vdash A \equiv B \dashv \Delta$$
 then $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

Case

$$\frac{}{\Gamma \vdash \alpha \equiv \alpha \dashv \Gamma} \equiv \mathsf{Var}$$

Here $\Delta = \Gamma$, so Lemma 32 (Extension Reflexivity) suffices.

Case

$$\frac{}{\Gamma \vdash \hat{\alpha} \equiv \hat{\alpha} \dashv \Gamma} \equiv \mathsf{Exvar}$$

Similar to the $\equiv Var$ case.

• Case

$$\frac{}{\Gamma \vdash 1 \equiv 1 \dashv \Gamma} \equiv \! \mathsf{Unit}$$

Similar to the $\equiv Var$ case.

- Case \equiv Vec: Similar to the $\equiv \oplus$ case.
- Cases $\equiv \supset$, $\equiv \land$: Similar to the $\equiv \oplus$ case, but with Lemma 48 (Prop Equivalence Extension) on the first premise.

• Case
$$\frac{\Gamma, \alpha : \kappa \vdash A_0 \equiv B \dashv \Delta, \alpha : \kappa, \Delta'}{\Gamma \vdash \forall \alpha : \kappa. \ A_0 \equiv \forall \alpha : \kappa. \ B \dashv \Delta} \equiv \forall$$

$$\Gamma, \alpha : \kappa \vdash A_0 \equiv B \dashv \Delta, \alpha : \kappa, \Delta' \quad \text{Subderivation}$$

$$\Gamma, \alpha : \kappa \longrightarrow \Delta, \alpha : \kappa, \Delta' \quad \text{By i.h.}$$

$$\Gamma \longrightarrow \Delta \quad \text{By Lemma 22 (Extension Inversion) (i)}$$

• Case
$$\frac{\Gamma_0[\widehat{\alpha}] \vdash \widehat{\alpha} := \tau : \star \dashv \Delta \qquad \widehat{\alpha} \notin FV([\Gamma_0[\widehat{\alpha}]]\tau)}{\Gamma_0[\widehat{\alpha}] \vdash \widehat{\alpha} \equiv \tau \dashv \Delta} \equiv \text{InstantiateL}$$

$$\Gamma_0[\widehat{\alpha}] \vdash \widehat{\alpha} := \tau : \star \dashv \Delta \qquad \text{Subderivation}$$

$$\Gamma_0[\widehat{\alpha}] \longrightarrow \Delta \qquad \qquad \text{By Lemma 43 (Instantiation Extension)}$$

• Case ≡InstantiateR: Similar to the ≡InstantiateL case.

C'.6 Subtyping Extends

Lemma 50 (Subtyping Extension). *If* $\Gamma \vdash A <: \exists B \dashv \Delta \text{ then } \Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

• Case $<:\exists R$: Similar to the $<:\forall L$ case.

• Case
$$\frac{\Gamma, \alpha : \kappa \vdash A <: ^{-}B \dashv \Delta, \alpha : \kappa, \Theta}{\Gamma \vdash A <: ^{-}\forall \alpha : \kappa. B \dashv \Delta} <: \forall R$$

Similar to the <:∀L case, but using part (i) of Lemma 22 (Extension Inversion).

• Case <:∃L: Similar to the <:∀R case.

• Case
$$\frac{\Gamma \vdash A \equiv B \dashv \Delta}{\Gamma \vdash A <: \mathcal{P} \ B \dashv \Delta} <: Equiv$$

$$\Gamma \vdash A \equiv B \dashv \Delta \quad Subderivation$$

$$\Gamma \vdash \Delta \quad By \ Lemma \ 49 \ (Equivalence \ Extension)$$

C'.7 Typing Extends 83

C'.7 Typing Extends

Lemma 51 (Typing Extension). If $\Gamma \vdash e \Leftarrow A \ p \dashv \Delta$ or $\Gamma \vdash e \Rightarrow A \ p \dashv \Delta$ or $\Gamma \vdash s : A \ p \gg B \ q \dashv \Delta$ or $\Gamma \vdash \Pi :: \vec{A} \ q \Leftarrow C \ p \dashv \Delta$ or $\Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C \ p \dashv \Delta$ then $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

• Match judgments:

In rule MatchEmpty, $\Delta = \Gamma$, so the result follows by Lemma 32 (Extension Reflexivity).

Rules MatchBase, Match \times , Match $+_k$ and MatchWild each have a single premise in which the contexts match the conclusion (input Γ and output Δ), so the result follows by i.h. For rule MatchSeq, Lemma 33 (Extension Transitivity) is also needed.

In rule Match∃, apply the i.h., then use Lemma 22 (Extension Inversion) (i).

Match \wedge : Use the i.h.

MatchNeg: Use the i.h. and Lemma 22 (Extension Inversion) (v).

Match⊥: Immediate by Lemma 32 (Extension Reflexivity).

MatchUnify:

- Synthesis, checking, and spine judgments: In rules Var, 1I, EmptySpine, and $\supset I \perp$, the output context Δ is exactly Γ , so the result follows by Lemma 32 (Extension Reflexivity).
 - Case ∀I: Use the i.h. and Lemma 33 (Extension Transitivity).
 - **Case** \forall Spine, \exists I: By \longrightarrow Add, $\Gamma \longrightarrow \Gamma$, $\hat{\alpha} : \kappa$. The result follows by i.h. and Lemma 33 (Extension Transitivity).
 - Cases ∧I, ⊃Spine: Use Lemma 47 (Checkprop Extension), the i.h., and Lemma 33 (Extension Transitivity).
 - **Cases** Nil, Cons: Using reasoning found in the \land I and \supset I cases.
 - Case ⊃I:

$$\begin{array}{ccc} \Gamma, \blacktriangleright_P, \Theta' \longrightarrow \Theta & \text{By Lemma 45 (Elimprop Extension)} \\ \Theta \longrightarrow \Delta, \blacktriangleright_P, \Delta & \text{By i.h.} \\ \Gamma, \blacktriangleright_P, \Theta' \longrightarrow \Delta, \blacktriangleright_P, \Delta & \text{By Lemma 33 (Extension Transitivity)} \\ \blacksquare & \Gamma \longrightarrow \Delta & \text{By Lemma 22 (Extension Inversion)} \end{array}$$

- **Cases** \rightarrow I, Rec: Use the i.h. and Lemma 22 (Extension Inversion).
- **Cases** Sub, Anno, →E, →Spine, $+I_k$, ×I: Use the i.h., and Lemma 33 (Extension Transitivity) as needed.
- Case 11â: By Lemma 23 (Deep Evar Introduction) (ii).

- Case &Spine, +I&k, ×I&:
 Use Lemma 23 (Deep Evar Introduction) (i) twice, Lemma 23 (Deep Evar Introduction) (ii), the i.h., and Lemma 33 (Extension Transitivity).
- Case →Iâ: Use Lemma 23 (Deep Evar Introduction) (i) twice, Lemma 23 (Deep Evar Introduction) (ii), the i.h. and Lemma 22 (Extension Inversion) (v).
- Case Case: Use the i.h. on the synthesis premise and the match premise, and then Lemma 33 (Extension Transitivity). □

C'.8 Unfiled

Lemma 52 (Context Partitioning).

If $\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \longrightarrow \Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z$ then there is a Ψ such that $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) = [\Omega]\Delta, \Psi$.

Proof. By induction on the given derivation.

- **Case** \longrightarrow Id: Impossible: Δ , $\triangleright_{\hat{\alpha}}$, Θ cannot have the form \cdot .
- Case \longrightarrow Var: We have $\Omega_Z = (\Omega'_Z, x:A)$ and $\Theta = (\Theta', x:A')$. By i.h., there is Ψ' such that $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega'_Z](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta') = [\Omega]\Delta, \Psi'$. Then by the definition of context application, $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega'_Z, x:A](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta', x:A') = [\Omega]\Delta, \Psi', x:[\Omega']A$. Let $\Psi = (\Psi', x:[\Omega']A)$.
- Case \longrightarrow Uvar: Similar to the \longrightarrow Var case, with $\Psi = (\Psi', \alpha : \kappa)$.
- Cases \longrightarrow Eqn, \longrightarrow Unsolved, \longrightarrow Solved, \longrightarrow Add, \longrightarrow AddSolved, \longrightarrow Marker: Broadly similar to the \longrightarrow Uvar case, but the rightmost context element disappears in context application, so we let $\Psi = \Psi'$.

Lemma 54 (Completing Stability).

If $\Gamma \longrightarrow \Omega$ *then* $[\Omega]\Gamma = [\Omega]\Omega$.

Proof. By induction on the derivation of $\Gamma \longrightarrow \Omega$.

 $\bullet \ \ \text{Case} \\ \hline \\ \hline \\ \hline \\ \cdot \longrightarrow \cdot \\ } \longrightarrow \text{Id}$

Immediate.

 $\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma_0 \longrightarrow \Omega_0}{\Gamma_0, \, \alpha : \kappa \longrightarrow \Omega_0, \, \alpha : \kappa} \longrightarrow \textbf{Uvar} \end{array}$

Similar to \longrightarrow Var.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma_0 \longrightarrow \Omega_0}{\Gamma_0, \hat{\alpha}: \kappa \longrightarrow \Omega_0, \hat{\alpha}: \kappa} \longrightarrow \textbf{Unsolved}$$

Similar to \longrightarrow Var.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_0 \longrightarrow \Omega_0 \qquad [\Omega_0]t = [\Omega_0]t'}{\Gamma_0, \hat{\alpha}: \kappa \!=\! t \longrightarrow \Omega_0, \hat{\alpha}: \kappa \!=\! t'} \longrightarrow \! \mathsf{Solved}$$

Similar to \longrightarrow Var.

• Case
$$\frac{\Gamma_0 \longrightarrow \Omega_0}{\Gamma_0, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Omega_0, \blacktriangleright_{\hat{\alpha}}} \longrightarrow \mathsf{Marker}$$

Similar to \longrightarrow Var.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_0 \longrightarrow \Omega_0}{\Gamma_0, \, \widehat{\beta} : \kappa' \longrightarrow \Omega_0, \, \widehat{\beta} : \kappa' = t} \longrightarrow \! \mathsf{Solve}$$

Similar to \longrightarrow Var.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_0 \longrightarrow \Omega_0 \qquad [\Omega_0] t' = [\Omega_0] t}{\Gamma_0, \alpha \!=\! t' \longrightarrow \Omega_0, \alpha \!=\! t} \longrightarrow \! \mathsf{Eqn}$$

$$\begin{array}{ll} \Gamma_0 \longrightarrow \Omega_0 & \text{Subderivation} \\ [\Omega_0]t' = [\Omega_0]t & \text{Subderivation} \\ [\Omega_0]\Gamma_0 = [\Omega_0]\Omega_0 & \text{By i.h.} \\ [[\Omega_0]t/\alpha]([\Omega_0]\Gamma_0) = [[\Omega_0]t/\alpha]([\Omega_0]\Omega_0) & \text{By congruence of equality} \end{array}$$

$$[\Omega_0,\alpha=t](\Gamma_0,\alpha=t')=\Omega_0,\alpha=t\quad \text{ By definition of context substitution}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Omega_0}{\Gamma \longrightarrow \Omega_0, \hat{\alpha} : \kappa} \longrightarrow \! \mathsf{Add}$$

$$\begin{array}{ll} \Gamma \longrightarrow \Omega_0 & \text{Subderivation} \\ [\Omega_0] \Gamma = [\Omega_0] \Omega_0 & \text{By i.h.} \\ [\Omega_0, \hat{\alpha}: \kappa] \Gamma = [\Omega_0, \hat{\alpha}: \kappa] (\Omega_0, \hat{\alpha}: \kappa) & \text{By definition of context substitution} \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \longrightarrow \Omega_0}{\Gamma \longrightarrow \Omega_0, \hat{\alpha} : \kappa = t} \longrightarrow \text{AddSolved}$$

Similar to the \longrightarrow Add case.

Lemma 55 (Completing Completeness).

(i) If
$$\Omega \longrightarrow \Omega'$$
 and $\Omega \vdash t : \kappa$ then $[\Omega]t = [\Omega']t$.

(ii) If
$$\Omega \longrightarrow \Omega'$$
 and $\Omega \vdash A$ type then $[\Omega]A = [\Omega']A$.

(iii) If
$$\Omega \longrightarrow \Omega'$$
 then $[\Omega]\Omega = [\Omega']\Omega'$.

Proof.

• Part (i):

By Lemma 29 (Substitution Monotonicity) (i), $[\Omega']t = [\Omega'][\Omega]t$. Now we need to show $[\Omega'][\Omega]t = [\Omega]t$. Considered as a substitution, Ω' is the identity everywhere except existential variables $\hat{\alpha}$ and universal variables α . First, since Ω is complete, $[\Omega]t$ has no free existentials. Second, universal variables free in $[\Omega]t$ have no equations in Ω (if they had, their occurrences would have been replaced). But if Ω has no equation for α , it follows from $\Omega \longrightarrow \Omega'$ and the definition of context extension in Figure 15 that Ω' also lacks an equation, so applying Ω' also leaves α alone.

Transitivity of equality gives $[\Omega']t = [\Omega]t$.

- Part (ii): Similar to part (i), using Lemma 29 (Substitution Monotonicity) (iii) instead of (i).
- **Part (iii):** By induction on the given derivation of $\Omega \longrightarrow \Omega'$.

Only cases \longrightarrow Id, \longrightarrow Var, \longrightarrow Uvar, \longrightarrow Eqn, \longrightarrow Solved, \longrightarrow AddSolved and \longrightarrow Marker are possible. In all of these cases, we use the i.h. and the definition of context application; in cases \longrightarrow Var, \longrightarrow Eqn and \longrightarrow Solved, we also use the equality in the premise of the respective rule.

Lemma 56 (Confluence of Completeness).

If $\Delta_1 \longrightarrow \Omega$ and $\Delta_2 \longrightarrow \Omega$ then $[\Omega]\Delta_1 = [\Omega]\Delta_2$.

Proof.

$$\Delta_1 \longrightarrow \Omega \qquad \quad \text{Given}$$

$$[\Omega]\Delta_1 = [\Omega]\Omega$$
 By Lemma 54 (Completing Stability)

$$\Delta_2 \longrightarrow \Omega \qquad \quad \text{Given}$$

$$[\Omega]\Delta_2 = [\Omega]\Omega$$
 By Lemma 54 (Completing Stability)

$$[\Omega]\Delta_1 = [\Omega]\Delta_2$$
 By transitivity of equality

Lemma 57 (Multiple Confluence).

If
$$\Delta \longrightarrow \Omega$$
 and $\Omega \longrightarrow \Omega'$ and $\Delta' \longrightarrow \Omega'$ then $[\Omega]\Delta = [\Omega']\Delta'$.

Proof.

$$\Delta \longrightarrow \Omega$$
 Given

$$[\Omega]\Delta = [\Omega]\Omega$$
 By Lemma 54 (Completing Stability)

$$\Omega \longrightarrow \Omega'$$
 Given

$$[\Omega]\Omega = [\Omega']\Omega'$$
 By Lemma 55 (Completing Completeness) (iii)
= $[\Omega']\Delta'$ By Lemma 54 (Completing Stability) ($\Delta' \longrightarrow \Omega'$ given)

Lemma 59 (Canonical Completion).

If
$$\Gamma \longrightarrow \Omega$$

then there exists Ω_{canon} such that $\Gamma \longrightarrow \Omega_{canon}$ and $\Omega_{canon} \longrightarrow \Omega$ and $dom(\Omega_{canon}) = dom(\Gamma)$ and, for all $\hat{\alpha} : \kappa = \tau$ and $\alpha = \tau$ in Ω_{canon} , we have $\mathsf{FEV}(\tau) = \emptyset$.

Proof. By induction on Ω . In Ω_{canon} , make all solutions (for evars and uvars) canonical by applying Ω to them, dropping declarations of existential variables that aren't in dom(Γ).

Lemma 60 (Split Solutions).

```
If \Delta \longrightarrow \Omega and \hat{\alpha} \in \mathsf{unsolved}(\Delta)
```

then there exists $\Omega_1 = \Omega_1'[\hat{\alpha}: \kappa = t_1]$ such that $\Omega_1 \longrightarrow \Omega$ and $\Omega_2 = \Omega_1'[\hat{\alpha}: \kappa = t_2]$ where $\Delta \longrightarrow \Omega_2$ and $t_2 \neq t_1$ and Ω_2 is canonical.

Proof. Use Lemma 59 (Canonical Completion) to get Ω_{canon} such that $\Delta \longrightarrow \Omega_{canon}$ and $\Omega_{canon} \longrightarrow \Omega$, where for all solutions t in Ω_{canon} we have $\mathsf{FEV}(\mathsf{t}) = \emptyset$.

```
We have \Omega_{\text{canon}} = \Omega_1'[\hat{\alpha}: \kappa = t_1], where \mathsf{FEV}(t_1) = \emptyset. Therefore \blacksquare \square \Omega_1'[\hat{\alpha}: \kappa = t_1] \longrightarrow \Omega.
```

Now choose t_2 as follows:

- If $\kappa = \star$, let $t_2 = t_1 \rightarrow t_1$.
- If $\kappa = \mathbb{N}$, let $t_2 = \operatorname{succ}(t_1)$.

Thus, $r = t_2 \neq t_1$. Let $\Omega_2 = \Omega_1'[\hat{\alpha} : \kappa = t_2]$.

 $\longrightarrow \Omega_2$ By Lemma 31 (Split Extension)

D' Internal Properties of the Declarative System

Lemma 61 (Interpolating With and Exists).

- (1) If $\mathcal{D} :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow C p \text{ and } \Psi \vdash P_0 \text{ true}$ then $\mathcal{D}' :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow C \land P_0 p$.
- (2) If $\mathcal{D} :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow [\tau/\alpha] C_0 \ p \ and \ \Psi \vdash \tau : \kappa$ then $\mathcal{D}' :: \Psi \vdash \Pi :: \vec{A} ! \Leftarrow (\exists \alpha : \kappa. C_0) \ p$.

In both cases, the height of \mathcal{D}' is one greater than the height of \mathcal{D} .

Moreover, similar properties hold for the eliminating judgment $\Psi / P \vdash \Pi :: \tilde{A} ! \Leftarrow C p$.

Proof. By induction on the given match derivation.

In the DeclMatchBase case, for part (1), apply rule \land I. For part (2), apply rule \exists I.

In the DeclMatchNeg case, part (1), use Lemma 2 (Declarative Weakening) (iii). In part (2), use Lemma 2 (Declarative Weakening) (i). □

Lemma 62 (Case Invertibility).

If $\Psi \vdash \mathsf{case}(e_0, \Pi) \Leftarrow \mathsf{C} \mathsf{p}$

then $\Psi \vdash e_0 \Rightarrow A$! and $\Psi \vdash \Pi :: A! \Leftarrow C p$ and $\Psi \vdash \Pi$ covers A!

where the height of each resulting derivation is strictly less than the height of the given derivation.

Proof. By induction on the given derivation.

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \Psi \vdash \mathsf{case}(e_0, \Pi) \Rightarrow A \ q \qquad \Psi \vdash A \leq^{\mathsf{join}(\mathsf{pol}(B), \mathsf{pol}(A))} B}_{ \Psi \vdash \mathsf{case}(e_0, \Pi) \ \Leftarrow \ B \ p} \ \, \mathsf{DeclSub}$$

Impossible, because $\Psi \vdash \mathsf{case}(e_0, \Pi) \Rightarrow A \neq \emptyset$ is not derivable.

- Cases Decl∀I, Decl⊃I: Impossible: these rules have a value restriction, but a case expression is not a value.
- $\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash P \ \textit{true} \qquad \Psi \vdash \mathsf{case}(e_0,\Pi) \Leftarrow C_0 \ p}{\Psi \vdash \mathsf{case}(e_0,\Pi) \Leftarrow C_0 \land P \ p} \ \, \mathsf{Decl} \land \mathsf{I}$
 - $\begin{array}{ccc} & < n-1 & \Psi \vdash e_0 \Rightarrow A ! & \text{By i.h.} \\ & < n-1 & \Psi \vdash \Pi :: A \Leftarrow C_0 \text{ p} & & '' \end{array}$
 - $< n-1 \ \Psi \vdash \Pi \ covers \ A$ " $< n-1 \ \Psi \vdash P \ true$ Subderivation
 - $< n \ \forall \vdash \Pi :: A \Leftarrow C_0 \land P \ p$ By Lemma 61 (Interpolating With and Exists) (1)

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \tau : \kappa \qquad \Psi \vdash \mathsf{case}(e_0, \Pi) \Leftarrow [\tau/\alpha] C_0}{\Psi \vdash \mathsf{case}(e_0, \Pi) \Leftarrow \exists \alpha : \kappa. \ C_0 \ p} \ \, \mathsf{Decl} \exists \mathsf{I}$$

$$\Psi \vdash \Pi \text{ covers } A$$
 "

$$\Psi \vdash \tau : \kappa$$
 Subderivation

$$\Psi \vdash \Pi :: A \Leftarrow \exists \alpha : \kappa. C_0 p$$
 By Lemma 61 (Interpolating With and Exists) (2)

The heights of the derivations are similar to those in the Decl/I case.

Cases Decl1I, Decl→I, DeclRec, Decl+I_k, Decl×I, DeclNiI, DeclCons:
 Impossible, because in these rules *e* cannot have the form case(*e*₀, Π).

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Psi \vdash \mathsf{case}(e_0,\Pi) \Rightarrow A \; ! \qquad \Psi \vdash \Pi :: A \; ! \Leftarrow C \; p \qquad \Psi \vdash \Pi \; \textit{covers} \; A \; !}{\Psi \vdash \mathsf{case}(e_0,\Pi) \Leftarrow C \; p} \end{array} \\ \text{DeclCase}$$

Immediate.

E' Miscellaneous Properties of the Algorithmic System

Lemma 63 (Well-Formed Outputs of Typing).

(Spines) If
$$\Gamma \vdash s : A \neq D \subset P \dashv \Delta$$
 or $\Gamma \vdash s : A \neq D \subset P \dashv \Delta$ and $\Gamma \vdash A \neq D \subset P \in D$ type then $\Delta \vdash C \not = D \in D$ type.

(Synthesis) If
$$\Gamma \vdash e \Rightarrow A \ \mathfrak{p} \dashv \Delta$$
 then $A \vdash \mathfrak{p}$ type.

Proof. By induction on the given derivation.

- Case Anno: Use Lemma 51 (Typing Extension) and Lemma 41 (Extension Weakening for Principal Typing).
- Case ∀Spine: We have Γ ⊢ (∀α : κ. A₀) q type.
 By inversion, Γ, α : κ ⊢ A₀ q type.
 By properties of substitution, Γ, α̂ : κ ⊢ [α̂/α]A₀ q type.
 Now apply the i.h.
- Case Spine: Use Lemma 42 (Inversion of Principal Typing) (2), Lemma 47 (Checkprop Extension), and Lemma 41 (Extension Weakening for Principal Typing).
- Case SpineRecover:

By i.h.,
$$\Delta \vdash C \not V type$$
.
We have as premise $FEV(C) = \emptyset$.
Therefore $\Delta \vdash C ! type$.

- Case SpinePass: By i.h.
- Case EmptySpine: Immediate.
- Case → Spine: Use Lemma 42 (Inversion of Principal Typing) (1), Lemma 51 (Typing Extension), and Lemma 41 (Extension Weakening for Principal Typing).
- Case $\hat{\alpha}$ Spine: Show that $\hat{\alpha}_1 \to \hat{\alpha}_2$ is well-formed, then use the i.h.

F' Decidability of Instantiation

Lemma 64 (Left Unsolvedness Preservation).

If
$$\underline{\Gamma_0, \hat{\alpha}, \Gamma_1} \vdash \hat{\alpha} := A : \kappa \dashv \Delta \ and \ \hat{\beta} \in \mathsf{unsolved}(\Gamma_0) \ then \ \hat{\beta} \in \mathsf{unsolved}(\Delta)$$
.

Proof. By induction on the given derivation.

• Case

$$\underbrace{\frac{\Gamma_0 \vdash \tau : \kappa}{\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1} \vdash \hat{\alpha} := \tau : \kappa \dashv \underbrace{\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1}_{\Gamma}}_{\Gamma} \text{ InstSolve}$$

Immediate, since to the left of $\hat{\alpha}$, the contexts Δ and Γ are the same.

Case

$$\underbrace{\frac{\hat{\beta} \in \mathsf{unsolved}(\Gamma'[\hat{\alpha} : \kappa][\hat{\beta} : \kappa])}{\Gamma'[\hat{\alpha} : \kappa][\hat{\beta} : \kappa]} \vdash \hat{\alpha} := \hat{\beta} : \kappa \dashv \underbrace{\Gamma'[\hat{\alpha} : \kappa][\hat{\beta} : \kappa = \hat{\alpha}]}_{\Delta}}^{\mathsf{InstReach}}$$

Immediate, since to the left of $\hat{\alpha}$, the contexts Δ and Γ are the same.

• Case
$$\frac{\Gamma_0, \hat{\alpha}_2: \star, \hat{\alpha}_1: \star, \hat{\alpha}: \star = \hat{\alpha}_1 \oplus \hat{\alpha}_2, \Gamma_1 \vdash \hat{\alpha}_1:= \tau_1: \star \dashv \Theta \qquad \Theta \vdash \hat{\alpha}_2:= [\Theta]\tau_2: \star \dashv \Delta}{\Gamma_0, \hat{\alpha}: \star, \Gamma_1 \vdash \hat{\alpha}:= \tau_1 \oplus \tau_2: \star \dashv \Delta} \text{ InstBin}$$

We have $\hat{\beta} \in \text{unsolved}(\Gamma_0)$. Therefore $\hat{\beta} \in \text{unsolved}(\Gamma_0, \hat{\alpha}_2 : \star)$.

Clearly, $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma_0, \hat{\alpha}_2 : \star)$.

We have two subderivations:

$$\Gamma_0, \hat{\alpha}_2: \star, \hat{\alpha}_1: \star, \hat{\alpha}: \star = \hat{\alpha}_1 \oplus \hat{\alpha}_2, \Gamma_1 \vdash \hat{\alpha}_1:= A_1: \star \dashv \Theta$$
 (1)

$$\Theta \vdash \hat{\alpha}_2 := [\Theta] A_2 : \star \dashv \Delta \tag{2}$$

By induction on (1), $\hat{\beta} \in \mathsf{unsolved}(\Theta)$.

Also by induction on (1), with $\hat{\alpha}_2$ playing the role of $\hat{\beta}$, we get $\hat{\alpha}_2 \in \text{unsolved}(\Theta)$.

Since $\hat{\beta} \in \Gamma_0$, it is declared to the left of $\hat{\alpha}_2$ in $\Gamma_0, \hat{\alpha}_2 : \star, \hat{\alpha}_1 : \star, \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2, \Gamma_1$.

Hence by Lemma 20 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of $\hat{\alpha}_2$ in Θ . That is, $\Theta = (\Theta_0, \hat{\alpha}_2 : \star, \Theta_1)$, where $\hat{\beta} \in \mathsf{unsolved}(\Theta_0)$.

By induction on (2), $\hat{\beta} \in \mathsf{unsolved}(\Delta)$.

Case

$$\underbrace{\Gamma'[\hat{\alpha}:\mathbb{N}]}_{\Gamma} \vdash \hat{\alpha} := \mathsf{zero}: \mathbb{N} \dashv \underbrace{\Gamma'[\hat{\alpha}:\mathbb{N} = \mathsf{zero}]}_{\Delta} \ \mathsf{InstZero}$$

Immediate, since to the left of $\hat{\alpha}$, the contexts Δ and Γ are the same.

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma[\hat{\alpha}_1:\mathbb{N},\hat{\alpha}:\mathbb{N} = \mathsf{succ}(\hat{\alpha}_1)] \vdash \hat{\alpha}_1 := t_1:\mathbb{N} \dashv \Delta}{\Gamma[\hat{\alpha}:\mathbb{N}] \vdash \hat{\alpha} := \mathsf{succ}(t_1):\mathbb{N} \dashv \Delta} \ \mathsf{InstSucc} \end{array}$$

We have $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$. Therefore $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0, \hat{\alpha}_1 : \mathbb{N})$. By i.h., $\hat{\beta} \in \mathsf{unsolved}(\Delta)$.

Lemma 65 (Left Free Variable Preservation). If $\overbrace{\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1} \vdash \hat{\alpha} := t : \kappa \dashv \Delta \text{ and } \Gamma \vdash s : \kappa' \text{ and } \hat{\alpha} \notin FV([\Gamma]s)$ and $\hat{\beta} \in \text{unsolved}(\Gamma_0)$ and $\hat{\beta} \notin FV([\Gamma]s)$, then $\hat{\beta} \notin FV([\Delta]s)$.

Proof. By induction on the given instantiation derivation.

Case

$$\frac{\Gamma_0 \vdash \tau : \kappa}{\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \dashv \underbrace{\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1}_{\Lambda}} \text{ InstSolve}$$

We have $\hat{\alpha} \notin FV([\Gamma]\sigma)$. Since Δ differs from Γ only in $\hat{\alpha}$, it must be the case that $[\Gamma]\sigma = [\Delta]\sigma$. It is given that $\hat{\beta} \notin FV([\Gamma]\sigma)$, so $\hat{\beta} \notin FV([\Delta]\sigma)$.

Case

$$\frac{\hat{\gamma} \in \mathsf{unsolved}(\Gamma[\hat{\alpha} : \kappa][\hat{\gamma} : \kappa])}{\Gamma[\hat{\alpha} : \kappa][\hat{\gamma} : \kappa] \vdash \hat{\alpha} := \hat{\gamma} : \kappa \dashv \underbrace{\Gamma[\hat{\alpha} : \kappa][\hat{\gamma} : \kappa = \hat{\alpha}]}_{\Lambda}}^{\mathsf{InstReach}}$$

Since Δ differs from Γ only in solving $\hat{\gamma}$ to $\hat{\alpha}$, applying Δ to a type will not introduce a $\hat{\beta}$. We have $\hat{\beta} \notin FV([\Gamma]\sigma)$, so $\hat{\beta} \notin FV([\Delta]\sigma)$.

Case

$$\frac{\Gamma'}{\Gamma[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]\vdash\hat{\alpha}_1:=\tau_1:\star\dashv\Theta\quad\quad\Theta\vdash\hat{\alpha}_2:=[\Theta]\tau_2:\star\dashv\Delta}{\Gamma[\hat{\alpha}:\star]\vdash\hat{\alpha}:=\tau_1\oplus\tau_2:\star\dashv\Delta} \text{ InstBin}$$

We have $\Gamma \vdash \sigma$ type and $\hat{\alpha} \notin FV([\Gamma]\sigma)$ and $\hat{\beta} \notin FV([\Gamma]\sigma)$.

By weakening, we get $\Gamma' \vdash \sigma : \kappa'$; since $\hat{\alpha} \notin FV([\Gamma]\sigma)$ and Γ' only adds a solution for $\hat{\alpha}$, it follows that $[\Gamma']\sigma = [\Gamma]\sigma$.

Therefore $\hat{\alpha}_1 \notin FV([\Gamma']\sigma)$ and $\hat{\alpha}_2 \notin FV([\Gamma']\sigma)$ and $\hat{\beta} \notin FV([\Gamma']\sigma)$.

Since we have $\hat{\beta} \in \Gamma_0$, we also have $\hat{\beta} \in (\Gamma_0, \hat{\alpha}_2 : \star)$.

By induction on the first premise, $\hat{\beta} \notin FV([\Theta]\sigma)$.

Also by induction on the first premise, with $\hat{\alpha}_2$ playing the role of $\hat{\beta}$, we have $\hat{\alpha}_2 \notin FV([\Theta]\sigma)$.

Note that $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma_0, \hat{\alpha}_2 : \star)$.

By Lemma 64 (Left Unsolvedness Preservation), $\hat{\alpha}_2 \in \mathsf{unsolved}(\Theta)$.

Therefore Θ has the form $(\Theta_0, \hat{\alpha}_2 : \star, \Theta_1)$.

Since $\hat{\beta} \neq \hat{\alpha}_2$, we know that $\hat{\beta}$ is declared to the left of $\hat{\alpha}_2$ in $(\Gamma_0, \hat{\alpha}_2 : \star)$, so by Lemma 20 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of $\hat{\alpha}_2$ in Θ . Hence $\hat{\beta} \in \Theta_0$.

Furthermore, by Lemma 43 (Instantiation Extension), we have $\Gamma' \longrightarrow \Theta$.

Then by Lemma 36 (Extension Weakening (Sorts)), we have $\Delta \vdash \sigma : \kappa'$.

Using induction on the second premise, $\hat{\beta} \notin FV([\Delta]\sigma)$.

Case

$$\underbrace{\Gamma'[\hat{\alpha}:\mathbb{N}]}_{\Gamma} \vdash \hat{\alpha} := \mathsf{zero}: \mathbb{N} \dashv \underbrace{\Gamma'[\hat{\alpha}:\mathbb{N} = \mathsf{zero}]}_{\Lambda} \mathsf{InstZero}$$

We have $\hat{\alpha} \notin FV([\Gamma]\sigma)$. Since Δ differs from Γ only in $\hat{\alpha}$, it must be the case that $[\Gamma]\sigma = [\Delta]\sigma$. It is given that $\hat{\beta} \notin FV([\Gamma]\sigma)$, so $\hat{\beta} \notin FV([\Delta]\sigma)$.

Case

$$\frac{\overbrace{\Gamma'[\hat{\alpha}_1:\mathbb{N},\hat{\alpha}:\mathbb{N}=\mathsf{succ}(\hat{\alpha}_1)]}^{\Theta}\vdash\hat{\alpha}_1:=t_1:\mathbb{N}\dashv\Delta}{\underbrace{\Gamma'[\hat{\alpha}:\mathbb{N}]\vdash\hat{\alpha}:=\mathsf{succ}(t_1):\mathbb{N}\dashv\Delta}_{\Gamma}} \mathsf{InstSucc}$$

Γ⊢σ:κ' Θ⊢σ:κ'	Given By weakening
$\hat{\alpha} \notin FV([\Gamma]\sigma)$ $\hat{\alpha} \notin FV([\Theta]\sigma)$	Given $\hat{\alpha} \notin FV([\Gamma]\sigma)$ and Θ only solves $\hat{\alpha}$
$\begin{split} \Theta &= (\Gamma_0, \hat{\alpha}_1 : \mathbb{N}, \hat{\alpha} : \mathbb{N} = succ(\hat{\alpha}_1), \Gamma_1) \\ \hat{\beta} \not\in unsolved(\Gamma_0) \\ \hat{\beta} \not\in unsolved(\Gamma_0, \hat{\alpha}_1 : \mathbb{N}) \end{split}$	Given Given $\hat{\alpha}_1$ fresh
$\hat{\beta} \notin FV([\Gamma]\sigma)$ $\hat{\beta} \notin FV([\Theta]\sigma)$	Given $\hat{\alpha}_1$ fresh
$\hat{\beta} \notin FV([\Delta]\sigma)$	By i.h.

Lemma 66 (Instantiation Size Preservation). If Γ_0 , $\hat{\alpha}$, $\Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$ and $\Gamma \vdash s : \kappa'$ and $\hat{\alpha} \notin FV([\Gamma]s)$, then $|[\Gamma]s| = |[\Delta]s|$, where |C| is the plain size of the term C.

Proof. By induction on the given derivation.

Since Δ differs from Γ only in solving $\hat{\alpha}$, and we know $\hat{\alpha} \notin FV([\Gamma]\sigma)$, we have $[\Delta]\sigma = [\Gamma]\sigma$; therefore $[\Delta]\sigma = [\Gamma]\sigma$.

Case

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$$\underbrace{\overline{\Gamma'[\hat{\alpha}:\mathbb{N}]} \vdash \hat{\alpha} := \mathsf{zero}: \mathbb{N} \dashv \underbrace{\Gamma'[\hat{\alpha}:\mathbb{N} = \mathsf{zero}]}_{\Delta} \ \mathsf{InstZero}$$

Similar to the InstSolve case.

 $\bullet \ \, \textbf{Case} \ \, \underbrace{\frac{\widehat{\beta} \in \mathsf{unsolved}(\Gamma'[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa])}{\Gamma'[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa] \vdash \widehat{\alpha} := \widehat{\beta} : \kappa \dashv \underbrace{\Gamma'[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa = \widehat{\alpha}]}_{\Lambda}}^{\mathsf{InstReach}} \ \, \mathsf{InstReach}$

Here, Δ differs from Γ only in solving $\hat{\beta}$ to $\hat{\alpha}$. However, $\hat{\alpha}$ has the same size as $\hat{\beta}$, so even if $\hat{\beta} \in FV([\Gamma]\sigma)$, we have $|[\Delta]\sigma = [\Gamma]\sigma|$.

Case

$$\frac{\overbrace{\Gamma[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]}^{\Gamma'}\vdash\hat{\alpha}_1:=\tau_1:\star\dashv\Theta\qquad\Theta\vdash\hat{\alpha}_2:=[\Theta]\tau_2:\star\dashv\Delta}{\Gamma[\hat{\alpha}:\star]\vdash\hat{\alpha}:=\tau_1\oplus\tau_2:\star\dashv\Delta}\text{ InstBin}$$

We have $\Gamma \vdash \sigma : \kappa'$ and $\hat{\alpha} \notin FV([\Gamma]\sigma)$.

Since $\hat{\alpha}_1, \hat{\alpha}_2 \notin \text{dom}(\Gamma)$, we have $\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2 \notin FV([\Gamma]\sigma)$.

By Lemma 23 (Deep Evar Introduction), $\Gamma[\hat{\alpha}:\star] \longrightarrow \Gamma'$.

By Lemma 36 (Extension Weakening (Sorts)), $\Gamma' \vdash \sigma : \kappa'$.

Since $\hat{\alpha} \notin FV(\sigma)$, it follows that $[\Gamma']\sigma = [\Gamma]\sigma$, and so $|[\Gamma']\sigma| = |[\Gamma]\sigma|$.

By induction on the first premise, $|[\Gamma']\sigma| = |[\Theta]\sigma|$.

By Lemma 20 (Declaration Order Preservation), since $\hat{\alpha}_2$ is declared to the left of $\hat{\alpha}_1$ in Γ' , we have

that $\hat{\alpha}_2$ is declared to the left of $\hat{\alpha}_1$ in Θ .

By Lemma 64 (Left Unsolvedness Preservation), since $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma')$, it is unsolved in Θ : that is, $\Theta = (\Theta_0, \hat{\alpha}_2 : \star, \Theta_1)$.

By Lemma 43 (Instantiation Extension), we have $\Gamma' \longrightarrow \Theta$.

By Lemma 36 (Extension Weakening (Sorts)), $\Theta \vdash \sigma : \kappa'$.

Since $\hat{\alpha}_2 \notin FV([\Gamma']\sigma)$, Lemma 65 (Left Free Variable Preservation) gives $\hat{\alpha}_2 \notin FV([\Theta]\sigma)$.

By induction on the second premise, $|[\Theta]\sigma| = |[\Delta]\sigma|$, and by transitivity of equality, $|[\Gamma]\sigma| = |[\Delta]\sigma|$.

Case

$$\begin{split} \overbrace{\Gamma[\hat{\alpha}_1:\mathbb{N},\hat{\alpha}:\mathbb{N}=\text{succ}(\hat{\alpha}_1)]} \vdash \hat{\alpha}_1 := t_1:\mathbb{N} \dashv \Delta \\ \hline{\Gamma[\hat{\alpha}:\mathbb{N}]\vdash \hat{\alpha}:=\text{succ}(t_1):\mathbb{N} \dashv \Delta} \quad \text{InstSucc} \\ \hline{\Gamma[\hat{\alpha}:\star]\vdash \sigma:\kappa'} \qquad \text{Given} \\ \hline{\alpha\notin [\Gamma[\hat{\alpha}:\star]]\sigma} \qquad \text{Given} \\ \hline{\Gamma[\hat{\alpha}:\star]\longrightarrow \Gamma'} \qquad \text{By Lemma 23 (Deep Evar Introduction)} \\ \hline{\Gamma'\vdash \sigma:\kappa'} \qquad \text{By Lemma 36 (Extension Weakening (Sorts))} \\ \hline{[\Gamma']\sigma=[\Gamma[\hat{\alpha}:\star]]\sigma} \qquad \text{Since } \hat{\alpha}\notin FV([\Gamma[\hat{\alpha}:\star]]\sigma) \\ \hline{|[\Gamma']\sigma|=|[\Gamma[\hat{\alpha}:\star]]\sigma|} \qquad \text{By congruence of equality} \\ \hline{\alpha}_1\notin [\Gamma']\sigma \qquad \qquad \text{Since } [\Gamma']\sigma=[\Gamma[\hat{\alpha}:\star]]\sigma, \text{ and } \hat{\alpha}_1\notin \text{dom}(\Gamma[\hat{\alpha}:\star]) \\ \hline{|[\Gamma']\sigma|=|[\Theta]\sigma|} \qquad \text{By i.h.} \\ \hline{|[\Gamma[\hat{\alpha}:\star]]\sigma|=|[\Theta]\sigma|} \qquad \text{By transitivity of equality} \end{split}$$

Lemma 67 (Decidability of Instantiation). If $\Gamma = \Gamma_0[\hat{\alpha} : \kappa']$ and $\Gamma \vdash t : \kappa$ such that $[\Gamma]t = t$ and $\hat{\alpha} \notin FV(t)$, then:

(1) Either there exists Δ such that $\Gamma_0[\hat{\alpha}:\kappa'] \vdash \hat{\alpha} := t : \kappa \dashv \Delta$, or not.

Proof. By induction on the derivation of $\Gamma \vdash t : \kappa$.

 $\bullet \ \ \, \textbf{Case} \ \ \, \frac{(\mathfrak{u}:\kappa) \in \Gamma}{\Gamma_L,\hat{\alpha}:\kappa',\Gamma_R \vdash \mathfrak{u}:\kappa} \, \, \text{VarSort}$

If $\kappa \neq \kappa'$, no rule matches and no derivation exists.

Otherwise:

- If $(u : \kappa) \in \Gamma_L$, we can apply rule InstSolve.
- If $\mathfrak u$ is some unsolved existential variable $\widehat{\beta}$ and $(\widehat{\beta} : \kappa) \in \Gamma_R$, then we can apply rule InstReach.
- Otherwise, u is declared in Γ_R and is a universal variable; no rule matches and no derivation exists.

• Case $\frac{(\hat{\beta}:\kappa\!=\!\tau)\in\Gamma}{\Gamma\vdash\hat{\beta}:\kappa}\,\mathsf{SolvedVarSort}$

By inversion, $(\hat{\beta} : \kappa = \tau) \in \Gamma$, but $[\Gamma]\hat{\beta} = \hat{\beta}$ is given, so this case is impossible.

• Case UnitSort:

If $\kappa' = \star$, then apply rule InstSolve. Otherwise, no rule matches and no derivation exists.

 $\bullet \ \, \textbf{Case} \ \, \underbrace{ \begin{array}{ccc} \Gamma \vdash \tau_1 : \star & \Gamma \vdash \tau_2 : \star \\ \underline{\Gamma_L, \hat{\alpha} : \kappa', \Gamma_R} \vdash \tau_1 \oplus \tau_2 : \star \end{array}}_{\Gamma} \ \, \textbf{BinSort}$

If $\kappa' \neq \star$, then no rule matches and no derivation exists. Otherwise:

Given, $[\Gamma](\tau_1 \oplus \tau_2) = \tau_1 \oplus \tau_2$ and $\hat{\alpha} \notin FV([\Gamma](\tau_1 \oplus \tau_2))$.

If $\Gamma_{I} \vdash \tau_{1} \oplus \tau_{2} : \star$, then we have a derivation by InstSolve.

If not, the only other rule whose conclusion matches $\tau_1 \oplus \tau_2$ is InstBin.

First, consider whether Γ_L , $\hat{\alpha}_2 : \star$, $\hat{\alpha}_1 : \star$, $\hat{\alpha} : \star = \hat{\alpha}_1 \oplus \hat{\alpha}_2$, $\Gamma_R \vdash \hat{\alpha}_1 := t : \star \dashv -$ is decidable.

By definition of substitution, $[\Gamma](\tau_1 \oplus \tau_2) = ([\Gamma]\tau_1) \oplus ([\Gamma]\tau_2)$. Since $[\Gamma](\tau_1 \oplus \tau_2) = \tau_1 \oplus \tau_2$, we have $[\Gamma]\tau_1 = \tau_1$ and $[\Gamma]\tau_2 = \tau_2$.

By weakening, Γ_L , $\hat{\alpha}_2 : \star$, $\hat{\alpha}_1 : \star$, $\hat{\alpha} : \star = \hat{\alpha}_1 \oplus \hat{\alpha}_2$, $\Gamma_R \vdash \tau_1 \oplus \tau_2 : \star$.

Since $\Gamma \vdash \tau_1 : \star$ and $\Gamma \vdash \tau_2 : \star$, we have $\hat{\alpha}_1, \hat{\alpha}_2 \notin FV(\tau_1) \cup FV(\tau_2)$.

Since $\hat{\alpha} \notin FV(t) \supseteq FV(\tau_1)$, it follows that $[\Gamma']\tau_1 = \tau_1$.

By i.h., either there exists Θ s.t. Γ_L , $\hat{\alpha}_2:\star$, $\hat{\alpha}_1:\star$, $\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2$, $\Gamma_R\vdash\hat{\alpha}_1:=\tau_1:\star\dashv\Theta$, or not.

If not, then no derivation by InstBin exists.

Otherwise, there exists such a Θ . By Lemma 64 (Left Unsolvedness Preservation), we have $\hat{\alpha}_2 \in \text{unsolved}(\Theta)$.

By Lemma 65 (Left Free Variable Preservation), we know that $\hat{\alpha}_2 \notin FV([\Theta]\tau_2)$.

Substitution is idempotent, so $[\Theta][\Theta]\tau_2 = [\Theta]\tau_2$.

By i.h., either there exists Δ such that $\Theta \vdash \hat{\alpha}_2 := [\Theta]\tau_2 : \kappa \dashv \Delta$, or not.

If not, no derivation by InstBin exists.

Otherwise, there exists such a Δ . By rule InstBin, we have $\Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta$.

• Case

$$\frac{}{\Gamma \vdash \mathsf{zero} : \mathbb{N}} \; \mathsf{ZeroSort}$$

If $\kappa' \neq \mathbb{N}$, then no rule matches and no derivation exists. Otherwise, apply rule InstSolve.

• Case

$$\frac{\Gamma \vdash t_0 : \mathbb{N}}{\Gamma \vdash \mathsf{succ}(t_0) : \mathbb{N}} \mathsf{SuccSort}$$

If $\kappa' \neq \mathbb{N}$, then no rule matches and no derivation exists. Otherwise:

If $\Gamma_{I} \vdash \text{succ}(t_{0}) : \mathbb{N}$, then we have a derivation by InstSolve.

If not, the only other rule whose conclusion matches $succ(t_0)$ is InstSucc.

The remainder of this case is similar to the BinSort case, but shorter.

G' Separation

Lemma 68 (Transitivity of Separation).

If
$$(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Theta_L * \Theta_R)$$
 and $(\Theta_L * \Theta_R) \xrightarrow{*} (\Delta_L * \Delta_R)$ then $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

Proof.

$$\begin{array}{ccc} (\Gamma_L * \Gamma_R) & \xrightarrow{*} (\Theta_L * \Theta_R) & & \text{Given} \\ (\Gamma_L, \Gamma_R) & \longrightarrow (\Theta_L, \Theta_R) & & \text{By Definition 5} \\ \Gamma_L \subseteq \Theta_L \text{ and } \Gamma_R \subseteq \Theta_R & & '' \end{array}$$

$$\begin{array}{ccc} (\Gamma_L,\Gamma_R) \longrightarrow (\Delta_L,\Delta_R) & & \text{By Lemma 33 (Extension Transitivity)} \\ \Gamma_L \subseteq \Delta_L \text{ and } \Gamma_R \subseteq \Delta_R & & \text{By transitivity of } \subseteq \end{array}$$

$$(\Gamma_{\rm I} * \Gamma_{\rm R}) \xrightarrow{*} (\Delta_{\rm I} * \Delta_{\rm R})$$
 By Definition 5

G' Separation 94

Lemma 69 (Separation Truncation).

```
If H has the form \alpha : \kappa \text{ or } \blacktriangleright_{\widehat{\alpha}} \text{ or } \blacktriangleright_P \text{ or } x : A \text{ p} and (\Gamma_L * (\Gamma_R, H)) \xrightarrow{*} (\Delta_L * \Delta_R) then (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_0) where \Delta_R = (\Delta_0, H, \Theta).
```

Proof. By induction on Δ_R .

If $\Delta_R = (\dots, H)$, we have $(\Gamma_L * \Gamma_R, H) \xrightarrow{*} (\Delta_L * (\Delta, H))$, and inversion on \longrightarrow Uvar (if H is $(\alpha : \kappa)$, or the corresponding rule for other forms) gives the result (with $\Theta = \cdot$).

Otherwise, proceed into the subderivation of $(\Gamma_L, \Gamma_R, \alpha : \kappa) \longrightarrow (\Delta_L, \Delta_R)$, with $\Delta_R = (\Delta_R', \Delta')$ where Δ' is a single declaration. Use the i.h. on Δ_R' , producing some Θ' . Finally, let $\Theta = (\Theta', \Delta')$.

Lemma 70 (Separation for Auxiliary Judgments).

(i) If
$$\Gamma_L * \Gamma_R \vdash \sigma \stackrel{\circ}{=} \tau : \kappa \dashv \Delta$$

and $\mathsf{FEV}(\sigma) \cup \mathsf{FEV}(\tau) \subseteq \mathsf{dom}(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{} (\Delta_L * \Delta_R)$.

(ii) If
$$\Gamma_L * \Gamma_R \vdash P$$
 true $\dashv \Delta$
and $FEV(P) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

(iii) If
$$\Gamma_L * \Gamma_R / \sigma \stackrel{\circ}{=} \tau : \kappa \dashv \Delta$$

and $\mathsf{FEV}(\sigma) \cup \mathsf{FEV}(\tau) = \emptyset$
then $\Delta = (\Delta_L * (\Delta_R, \Theta))$ and $(\Gamma_L * (\Gamma_R, \Theta)) \xrightarrow{*} (\Delta_L * \Delta_R)$.

(iv) If
$$\Gamma_L * \Gamma_R / P \dashv \Delta$$

and $FEV(P) = \emptyset$
then $\Delta = (\Delta_L * (\Delta_R, \Theta))$ and $(\Gamma_L * (\Gamma_R, \Theta)) \xrightarrow{*} (\Delta_L * \Delta_R)$.

$$\begin{array}{l} \text{(v)} \ \, \textit{If} \ \, \Gamma_L * \Gamma_R \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta \\ \quad \textit{and} \ \, (\mathsf{FEV}(\tau) \cup \{\hat{\alpha}\}) \subseteq \mathsf{dom}(\Gamma_R) \\ \quad \textit{then} \ \, \Delta = (\Delta_L * \Delta_R) \ \, \textit{and} \ \, (\Gamma_L * \Gamma_R) \xrightarrow{\ \, \ \, } (\Delta_L * \Delta_R). \end{array}$$

(vii) If
$$\Gamma_L * \Gamma_R \vdash A \equiv B \dashv \Delta$$

and $\mathsf{FEV}(A) \cup \mathsf{FEV}(B) \subseteq \mathsf{dom}(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

Proof. Part (i): By induction on the derivation of the given checkeq judgment. Cases CheckeqVar, CheckeqUnit and CheckeqZero are immediate ($\Delta_L = \Gamma_L$ and $\Delta_R = \Gamma_R$). For case CheckeqSucc, apply the i.h. For cases CheckeqInstL and CheckeqInstR, use the i.h. (v). For case CheckeqBin, use reasoning similar to that in the \land I case of Lemma 72 (Separation—Main) (transitivity of separation, and applying Θ in the second premise).

Part (ii), checkprop: Use the i.h. (i).

Part (iv), elimprop: Use the i.h. (iii). Part (v), instjudg:

• Case InstSolve: Here, $\Gamma = (\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1)$ and $\Delta = (\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1)$. We have $\hat{\alpha} \in \mathsf{dom}(\Gamma_R)$, so the declaration $\hat{\alpha} : \kappa$ is in Γ_R . Since $\mathsf{FEV}(\tau) \subseteq \mathsf{dom}(\Gamma_R)$, the context Δ maintains the separation.

- Case InstReach: Here, $\Gamma = \Gamma_0[\hat{\alpha} : \kappa][\hat{\beta} : \kappa]$ and $\Delta = \Gamma_0[\hat{\alpha} : \kappa][\hat{\beta} : \kappa = \hat{\alpha}]$. We have $\hat{\alpha} \in \mathsf{dom}(\Gamma_R)$, so the declaration $\hat{\alpha} : \kappa$ is in Γ_R . Since $\hat{\beta}$ is declared to the right of $\hat{\alpha}$, it too must be in Γ_R , which can also be shown from $\mathsf{FEV}(\hat{\beta}) \subseteq \mathsf{dom}(\Gamma_R)$. Both declarations are in Γ_R , so the context Δ maintains the separation.
- Case InstZero: In this rule, Δ is the same as Γ except for a solution zero, which doesn't violate separation.
- Case InstSucc: The result follows by i.h., taking care to keep the declaration $\hat{\alpha}_1 : \mathbb{N}$ on the right when applying the i.h., even if $\hat{\alpha} : \mathbb{N}$ is the leftmost declaration in Γ_R , ensuring that succ($\hat{\alpha}_1$) does not violate separation.
- **Case** InstBin: As in the InstSucc case, the new declarations should be kept on the right-hand side of the separator. Otherwise the case is straightforward (using the i.h. twice and transitivity).

Part (vi), propequivjudg: Similar to the CheckeqBin case of part (i), using the i.h. (i). Part (vii), equivjudg:

- Cases \equiv Var, \equiv Exvar, \equiv Unit: Immediate ($\Delta_L = \Gamma_L$ and $\Delta_R = \Gamma_R$).
- Case ≡⊕: Similar to the case CheckeqBin in part (i).
- Case ≡Vec: Similar to the case CheckeqBin in part (i).
- Cases $\equiv \forall, \equiv \exists$: Similar to the case CheckeqBin in part (i).
- Cases $\equiv \supset$, $\equiv \land$: Similar to the case CheckeqBin in part (i), using the i.h. (vi).
- **Cases** ≡InstantiateL, ≡InstantiateR: Use the i.h. (v).

```
Lemma 71 (Separation for Subtyping). If \Gamma_L * \Gamma_R \vdash A <: ^{\mathcal{P}} B \dashv \Delta and \mathsf{FEV}(A) \subseteq \mathsf{dom}(\Gamma_R) and \mathsf{FEV}(B) \subseteq \mathsf{dom}(\Gamma_R) then \Delta = (\Delta_L * \Delta_R) and (\Gamma_L * \Gamma_R) \xrightarrow{\longrightarrow} (\Delta_L * \Delta_R).
```

Proof. By induction on the given derivation. In the <: Equiv case, use Lemma 70 (Separation for Auxiliary Judgments) (vii). Otherwise, the reasoning needed follows that used in the proof of Lemma 72 (Separation—Main). □

Lemma 72 (Separation—Main).

$$\begin{split} \textit{(Spines)} & \text{ If } \Gamma_L * \Gamma_R \vdash s : A \ p \gg C \ q \dashv \Delta \\ & \text{ or } \Gamma_L * \Gamma_R \vdash s : A \ p \gg C \ \lceil q \rceil \dashv \Delta \\ & \text{ and } \Gamma_L * \Gamma_R \vdash s : A \ p \gg C \ \lceil q \rceil \dashv \Delta \\ & \text{ and } \Gamma_L * \Gamma_R \vdash A \ p \ type \\ & \text{ and } \mathsf{FEV}(A) \subseteq \mathsf{dom}(\Gamma_R) \\ & \text{ then } \Delta = (\Delta_L * \Delta_R) \ \textit{ and } (\Gamma_L * \Gamma_R) \xrightarrow{} (\Delta_L * \Delta_R) \ \textit{ and } \mathsf{FEV}(C) \subseteq \mathsf{dom}(\Delta_R). \\ \textit{(Checking)} & \text{ If } \Gamma_L * \Gamma_R \vdash e \Leftarrow C \ p \dashv \Delta \\ & \text{ and } \Gamma_L * \Gamma_R \vdash C \ p \ type \\ & \text{ and } \mathsf{FEV}(C) \subseteq \mathsf{dom}(\Gamma_R) \\ & \text{ then } \Delta = (\Delta_L * \Delta_R) \ \textit{ and } (\Gamma_L * \Gamma_R) \xrightarrow{} (\Delta_L * \Delta_R). \end{split}$$

(Synthesis) If
$$\Gamma_L * \Gamma_R \vdash e \Rightarrow A p \dashv \Delta$$

then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

G' Separation 96

(Match Elim.) If
$$\Gamma_L * \Gamma_R / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta$$

and $FEV(P) = \emptyset$
and $FEV(\vec{A}) = \emptyset$
and $FEV(C) \subseteq dom(\Gamma_R)$
then $\Delta = (\Delta_L * \Delta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

Proof. By induction on the given derivation.

First, the (Match) judgment part, giving only the cases that motivate the side conditions:

- Case MatchBase: Here we use the i.h. (Checking), for which we need $FEV(C) \subseteq dom(\Gamma_R)$.
- Case Match \wedge : Here we use the i.h. (Match Elim.), which requires that $FEV(P) = \emptyset$, which motivates $FEV(\vec{A}) = \emptyset$.
- Case MatchNeg: In its premise, this rule appends a type $A \in \vec{A}$ to Γ_R and claims it is principal (z : A!), which motivates $FEV(\vec{A} = \emptyset)$.

Similarly, (Match Elim.):

• Case MatchUnify: Here we use Lemma 70 (Separation for Auxiliary Judgments) (iii), for which we need $FEV(\sigma) \cup FEV(\tau) = \emptyset$, which motivates $FEV(P) = \emptyset$.

Now, we show the cases for the (Spine), (Checking), and (Synthesis) parts.

- Cases Var, 1I, $\supset I \perp$: In all of these rules, the output context is the same as the input context, so just let $\Delta_L = \Gamma_L$ and $\Delta_R = \Gamma_R$.
- Case

$$\overline{\Gamma_L * \Gamma_R \vdash \cdot : A \; \mathfrak{p} \gg \underbrace{A}_{C} \underbrace{\mathfrak{p}}_{q} \dashv \Gamma_L * \Gamma_R} \; \mathsf{EmptySpine}$$

Let $\Delta_L = \Gamma_L$ and $\Delta_R = \Gamma_R$.

We have $\mathsf{FEV}(\mathsf{A}) \subseteq \mathsf{dom}(\Gamma_\mathsf{R})$. Since $\Delta_\mathsf{R} = \Gamma_\mathsf{R}$ and $C = \mathsf{A}$, it is immediate that $\mathsf{FEV}(C) \subseteq \mathsf{dom}(\Delta_\mathsf{R})$.

• Case
$$\frac{\Gamma_{L} * \Gamma_{R} \vdash e \Rightarrow A \ q \dashv \Theta \qquad \Theta \vdash A <: ^{\mathcal{P}} B \dashv \Delta}{\Gamma_{L} * \Gamma_{R} \vdash e \Leftarrow B \ p \dashv \Delta} \ \mathsf{Sub}$$

By i.h., $\Theta = (\Theta_L * \Theta_R)$ and $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Theta_L * \Theta_R)$.

By Lemma 71 (Separation for Subtyping), $\Delta = (\Delta_L * \Delta_R)$ and $(\Theta_L * \Theta_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

By Lemma 68 (Transitivity of Separation), $(\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)$.

• Case
$$\frac{\Gamma \vdash A! \ type \qquad \Gamma \vdash e \Leftarrow [\Gamma]A \ ! \dashv \Delta}{\Gamma \vdash (e : A) \Rightarrow [\Delta]A \ ! \dashv \Delta} \text{ Anno}$$

By i.h.; since $FEV(A) = \emptyset$, the condition on the (Checking) part is trivial.

Case

$$\overline{\Gamma[\boldsymbol{\hat{\alpha}}:\star] \vdash \textbf{()} \Leftarrow \boldsymbol{\hat{\alpha}} \ \dashv \Gamma[\boldsymbol{\hat{\alpha}}:\star=1]} \ \textbf{1}\boldsymbol{\hat{\alpha}}$$

Adding a solution with a ground type cannot destroy separation.

$$\bullet \ \, \textbf{Case} \ \, \frac{ \nu \ \textit{chk-I} \qquad \Gamma_L, \Gamma_R, \alpha : \kappa \vdash \nu \Leftarrow A_0 \ p \dashv \Delta, \alpha : \kappa, \Theta }{ \Gamma_L, \Gamma_R \vdash \nu \Leftarrow \forall \alpha : \kappa. \ A_0 \ p \dashv \Delta } \ \forall I$$

EF

 $(\Gamma_L * \Gamma_R) \xrightarrow{} (\Theta_L * \Theta_R)$

```
Given
            \mathsf{FEV}(\forall \alpha : \kappa. A_0) \subseteq \mathsf{dom}(\Gamma_R)
                         FEV(A_0) \subseteq dom(\Gamma_R, \alpha : \kappa)
                                                                                       From definition of FEV
                    (\Delta, \alpha : \kappa, \Theta) = (\Delta_L * \Delta'_R)
                                                                                       By i.h.
         (\Gamma_L * (\Gamma_R, \alpha : \kappa)) \xrightarrow{} (\Delta_L * \Delta'_R)
                       (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R)
                                                                                       By Lemma 69 (Separation Truncation)
                                    \Delta_R' = (\Delta_R, \alpha : \kappa, \Theta)
                    (\Delta, \alpha : \kappa, \Theta) = (\Delta_L * \Delta_R')
                                                                                      Above
                                            =(\Delta_{\mathsf{L}},\Delta_{\mathsf{R}}')
                                                                                       Definition of *
                                            = (\Delta_{L}, \Delta_{R}, \alpha : \kappa, \Theta)
                                                                                      By above equation
                                       \Delta = (\Delta_{\rm I}, \Delta_{\rm R})
                                                                                       α not multiply declared
EF
```

• Case
$$\frac{\Gamma_L, \Gamma_R, \hat{\alpha} : \kappa \vdash e \ s : [\hat{\alpha}/\alpha] A_0 \gg C \ q \dashv \Delta}{\Gamma_L, \Gamma_R \vdash e \ s : \forall \alpha : \kappa . A_0 \ p \gg C \ q \dashv \Delta} \ \forall \mathsf{Spine}$$

$$FEV(\forall \alpha : \kappa A_0.) \subseteq \mathsf{dom}(\Gamma_R) \qquad \mathsf{Given}$$

$$FEV([\hat{\alpha}/\alpha] A_0) \subseteq \mathsf{dom}(\Gamma_R, \hat{\alpha} : \kappa) \qquad \mathsf{From \ definition \ of \ FEV}$$

$$\Delta = (\Delta_L * \Delta_R) \qquad \mathsf{By \ i.h.}$$

$$(\Gamma_L * (\Gamma_R, \hat{\alpha} : \kappa)) \xrightarrow{\to} (\Delta_L * \Delta_R) \qquad "$$

$$\mathsf{FEV}(C) \subseteq \mathsf{dom}(\Delta_R) \qquad "$$

$$\mathsf{dom}(\Gamma_L) \subseteq \mathsf{dom}(\Delta_L) \qquad \mathsf{By \ Definition \ 5}$$

$$\mathsf{dom}(\Gamma_R, \hat{\alpha} : \kappa) \subseteq \mathsf{dom}(\Delta_R) \qquad \mathsf{By \ Definition \ 5}$$

$$\mathsf{dom}(\Gamma_R, \hat{\alpha} : \kappa) \subseteq \mathsf{dom}(\Delta_R) \qquad \mathsf{By \ Definition \ 5}$$

$$\mathsf{dom}(\Gamma_R) \cup \{\hat{\alpha}\} \subseteq \mathsf{dom}(\Delta_R) \qquad \mathsf{By \ Definition \ of \ dom}(-)$$

$$\mathsf{dom}(\Gamma_R) \subseteq \mathsf{dom}(\Delta_R) \qquad \mathsf{Property \ of \ } \subseteq$$

$$(\Gamma_L, \Gamma_R) \longrightarrow (\Delta_L, \Delta_R) \qquad \mathsf{By \ Lemma \ 51 \ (Typing \ Extension)}$$

• Case
$$\frac{e \text{ not a case} \qquad \Gamma_L * \Gamma_R \vdash P \text{ true} \dashv \Theta \qquad \Theta \vdash e \Leftarrow [\Theta] A_0 \text{ p} \dashv \Delta}{\Gamma_L * \Gamma_R \vdash e \Leftarrow (A_0 \land P) \text{ p} \dashv \Delta} \land I$$

$$\Gamma_L * \Gamma_R \vdash (A_0 \land P) \text{ p} \text{ type} \qquad \text{Given}$$

$$\Gamma_L * \Gamma_R \vdash P \text{ prop} \qquad \text{By inversion}$$

$$\Gamma_L * \Gamma_R \vdash A_0 \text{ p} \text{ type} \qquad \text{By inversion}$$

$$FEV(A_0 \land P) \subseteq \text{dom}(\Gamma_R) \qquad \text{Given}$$

$$FEV(P) \subseteq \text{dom}(\Gamma_R) \qquad \text{By def. of FEV}$$

$$FEV(A_0) \subseteq \text{dom}(\Gamma_R) \qquad W$$

$$\Theta = (\Theta_L * \Theta_R) \qquad \text{By Lemma 70 (Separation for Auxiliary Judgments) (i)}$$

By Definition 5

```
\mathsf{FEV}(A_0) \subseteq \mathsf{dom}(\Gamma_R)
                                                       Above
     dom(\Gamma_R) \subseteq dom(\Theta_R)
                                                       By Definition 5
                                                       By previous line
    FEV(A_0) \subseteq dom(\Theta_R)
\mathsf{FEV}([\Theta]A_0) \subseteq \mathsf{dom}(\Theta_R)
                                                       Previous line and (\Gamma_I * \Gamma_R) \xrightarrow{*} (\Theta_I * \Theta_R)
         \Gamma_{L} * \Gamma_{R} \vdash (A_0 \land P) p type
                                                       Given
         \Gamma_L * \Gamma_R \vdash A_0 p type
                                                       By inversion
                 \Theta \vdash A_0 p type
                                                       By Lemma 41 (Extension Weakening for Principal Typing)
                                                       By Lemma 13 (Right-Hand Substitution for Typing)
                 \Theta \vdash [\Theta] A_0 p type
                 \Delta = (\Delta_{L} * \Delta_{R})
(\Theta_L * \Theta_R) \xrightarrow{*} (\Delta_L * \Delta_R)
  (\Gamma_{\rm I} * \Gamma_{\rm R}) \xrightarrow{*} (\Delta_{\rm I} * \Delta_{\rm R})
                                                       By Lemma 68 (Transitivity of Separation)
```

- **Case** Nil: Similar to a section of the ∧I case.
- Case Cons: Similar to the ∧I case, with an extra use of the i.h. for the additional second premise.

• Case
$$\frac{v \, chk \cdot I \quad \Gamma_L * (\Gamma_R, \blacktriangleright_P) \ / \ P + \Theta \quad \Theta \vdash \nu \Leftarrow [\Theta] A_0 \ ! + \Delta, \blacktriangleright_P, \Delta'}{\Gamma_L * \Gamma_R \vdash \nu \Leftarrow P \supset A_0 \ ! + \Delta} \supset I$$

$$\Gamma_L * \Gamma_R \vdash (P \supset A_0) \ ! \ type \quad \text{Given}$$

$$\Gamma_L * \Gamma_R \vdash P \supset A_0 \ prop \quad \text{By inversion}$$

$$FEV(P) = \emptyset \quad \text{By def. of FEV}$$

$$\Gamma_L * (\Gamma_R, \blacktriangleright_P) \ / \ P + \Theta \quad \text{Subderivation}$$

$$\Theta = (\Theta_L * (\Theta_R, \Theta_Z)) \quad \text{By Lemma 70 (Separation for Auxiliary Judgments) (iv)}$$

$$(\Gamma_L * (\Gamma_R, \blacktriangleright_P, \Theta_Z)) \xrightarrow{\longrightarrow} (\Theta_L * (\Theta_R, \Theta_Z)) \quad "$$

$$\Gamma_L * \Gamma_R \vdash (P \supset A_0) \ ! \ type \quad \text{By Lemma 42 (Inversion of Principal Typing) (2)}$$

$$\Gamma_L, \Gamma_R, \blacktriangleright_P, \Theta_Z \vdash A_0 \ ! \ type \quad \text{By Lemma 35 (Suffix Weakening)}$$

$$\Theta \vdash [\Theta] A_0 \ ! \ type \quad \text{By Lemmas 41 and 40}$$

$$FEV(A_0) = \emptyset \quad \text{Above and def. of FEV}$$

$$FEV(A_0) \subseteq \text{dom}(\Theta_R, \Theta_Z) \quad \text{Immediate}$$

$$(\Delta, \blacktriangleright_P, \Delta') = (\Delta_L * \Delta'_R) \quad \text{By i.h.}$$

$$(\Theta_L * (\Theta_R, \Theta_Z)) \xrightarrow{\longrightarrow} (\Delta_L * \Delta'_R) \quad \text{By Lemma 68 (Transitivity of Separation)}$$

$$(\Gamma_L * (\Gamma_R, \blacktriangleright_P)) \xrightarrow{\longrightarrow} (\Delta_L * \Delta'_R) \quad \text{By Lemma 69 (Separation Truncation)}$$

$$\Delta'_R = (\Delta_R, \blacktriangleright_P, \dots) \quad "$$

$$\Delta = (\Delta_L, \Delta_R) \quad \text{Similar to the } \forall I \text{ case}$$

• Case $\exists I$: Similar to the $\forall Spine$ case.

$$\begin{array}{lll} \Gamma_L * \Gamma_R \vdash (P \supset A_0) \ p \ type & Given \\ \Gamma_L * \Gamma_R \vdash P \ prop & By \ inversion \\ \Gamma_L, \Gamma_R \vdash P \ true \dashv \Theta & Subderivation \\ \Theta = (\Theta_L * \Theta_R) & By \ Lemma \ 70 \ (Separation \ for \ Auxiliary \ Judgments) \ (i) \\ (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Theta_L * \Theta_R) & " \\ \Theta \vdash e \ s : [\Theta] A_0 \ p \gg C \ q \dashv \Delta & Subderivation \\ (\Delta, \blacktriangleright_P, \Delta') = (\Delta_L * \Delta'_R) & By \ i.h. \\ (\Theta_L * \Theta_R) \xrightarrow{*} (\Delta_L * \Delta'_R) & " \\ FEV(C) \subseteq dom(\Delta_R) & " \\ \blacksquare & \Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R) & By \ Lemma \ 68 \ (Transitivity \ of \ Separation) \end{array}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_L, \Gamma_R, x : C \, \mathfrak{p} \vdash \nu \Leftarrow C \, \mathfrak{p} \dashv \Delta, x : C \, \mathfrak{p}, \Theta}{\Gamma_L, \Gamma_R \vdash \text{rec } x. \nu \Leftarrow C \, \mathfrak{p} \dashv \Delta} \, \, \text{Rec}$$

$$\begin{array}{ll} \Gamma_L * \Gamma_R \vdash C \ p \ type & Given \\ FEV(C) \subseteq dom(\Gamma_R) & Given \\ \Gamma_L * (\Gamma_R, x \colon C \ p) \vdash C \ p \ type & By \ weakening \ and \ Definition \ 4 \\ \Gamma_L, \Gamma_R, x \colon C \ p \vdash \nu \Leftarrow C \ p \dashv \Delta, x \colon C \ p, \Theta & Subderivation \\ (\Delta, x \colon C \ p, \Theta) = (\Delta_L, \Delta_R') & By \ i.h. \\ (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R') & " \\ (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Delta_L * \Delta_R) & By \ Lemma \ 69 \ (Separation \ Truncation) \\ \Delta_R' = (\Delta_R, x \colon C \ p, \ldots) & " \\ \Delta = (\Delta_L, \Delta_R) & Similar \ to \ the \ \forall I \ case \\ \end{array}$$

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma_L, \Gamma_R, x \colon\! A \:\! p \vdash e \Leftarrow B \: p \dashv \Delta, x \colon\! A \:\! p, \Theta}{\Gamma_L, \Gamma_R \vdash \lambda x. \:\! e \Leftarrow A \to B \: p \dashv \Delta} \to \!\! I \end{array}$$

• Case
$$\frac{\Gamma_{0}[\hat{\alpha}_{1}:\star,\hat{\alpha}_{2}:\star,\hat{\alpha}:\star=\hat{\alpha}_{1}\to\hat{\alpha}_{2}],x:\hat{\alpha}_{1}\vdash e_{0}\Leftarrow\hat{\alpha}_{2}\dashv\Delta,x:\hat{\alpha}_{1},\Delta'}{\underbrace{\Gamma_{0}[\hat{\alpha}:\star]\vdash\lambda x.\,e_{0}\Leftarrow\hat{\alpha}\dashv\Delta}}\to I\hat{\alpha}$$

We have $(\Gamma_L * \Gamma_R) = \Gamma_0[\hat{\alpha} : \star]$. We also have $\mathsf{FEV}(\hat{\alpha}) \subseteq \mathsf{dom}(\Gamma_R)$. Therefore $\hat{\alpha} \in \mathsf{dom}(\Gamma_R)$ and

$$\Gamma_0[\hat{\alpha}:\star] = \Gamma_1, \Gamma_2, \hat{\alpha}:\star, \Gamma_3$$

where
$$\Gamma_R = (\Gamma_2, \hat{\alpha} : \star, \Gamma_3)$$
.

Then the input context in the premise has the following form:

$$\Gamma_0[\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\rightarrow\hat{\alpha}_2], x:\hat{\alpha}_1 = \Gamma_L, \Gamma_2,\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\rightarrow\hat{\alpha}_2,\Gamma_3,x:\hat{\alpha}_1$$

Let us separate this context at the same point as $\Gamma_0[\hat{\alpha}:\star]$, that is, after Γ_L and before Γ_2 , and call the resulting right-hand context Γ_R' . That is,

$$\begin{split} &\Gamma_0[\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\to\hat{\alpha}_2],x:\hat{\alpha}_1 &= \Gamma_L * \left(\underbrace{\Gamma_2,\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\to\hat{\alpha}_2,\Gamma_3,x:\hat{\alpha}_1}_{\Gamma_R'}\right) \\ & \text{FEV}(\hat{\alpha})\subseteq \text{dom}(\Gamma_R) & \text{Given} \\ &\Gamma_L*\Gamma_R'\vdash e_0 \Leftarrow \hat{\alpha}_2\dashv \Delta,x:\hat{\alpha}_1,\Delta' & \text{Subderivation} \\ &\Gamma_L*\Gamma_R'\vdash \hat{\alpha}_2 \not\text{! type} & \hat{\alpha}_2 \in \text{dom}(\Gamma_R') \\ &\text{FEV}(\hat{\alpha}_2)\subseteq \text{dom}(\Gamma_R') & \hat{\alpha}_2 \in \text{dom}(\Gamma_R') \\ &(\Delta,x:\hat{\alpha}_1,\Delta') = (\Delta_L,\Delta_R') & \text{By i.h.} \\ &(\Gamma_L*\Gamma_R')\xrightarrow{*} (\Delta_L*\Delta_R') & \text{Similar to the } \forall I \text{ case} \\ &(\Gamma_L*\Gamma_R)\xrightarrow{*} (\Delta_L*\Delta_R) & \text{"} \end{split}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash e \Rightarrow A \ p \dashv \Theta \qquad \Theta \vdash s : [\Theta] A \ p \gg C \ \lceil q \rceil \dashv \Delta}{\Gamma \vdash e \, s \Rightarrow C \ q \dashv \Delta} \rightarrow \! \mathsf{E}$$

Use the i.h. and Lemma 68 (Transitivity of Separation), with Lemma 91 (Well-formedness of Algorithmic Typing) and Lemma 13 (Right-Hand Substitution for Typing).

Use the i.h.

₽

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash s : A \ \mathfrak{p} \gg C \ \mathfrak{q} \dashv \Delta \qquad \left((\mathfrak{p} = \cancel{\mathcal{Y}}) \ \text{or} \ (\mathfrak{q} = \cancel{!}) \ \text{or} \ (\mathsf{FEV}([\Delta]C) \neq \emptyset) \right)}{\Gamma \vdash s : A \ \mathfrak{p} \gg C \ \lceil \mathfrak{q} \rceil \dashv \Delta} \ \, \mathsf{SpinePass}$$

Use the i.h.

$$\bullet \ \, \textbf{Case} \ \, \frac{ \Gamma_L * \Gamma_R \vdash e \Leftarrow A_1 \ p \dashv \Theta \qquad \Theta \vdash s : [\Theta] A_2 \ p \gg C \ q \dashv \Delta }{ \Gamma_L * \Gamma_R \vdash e \ s : A_1 \rightarrow A_2 \ p \gg C \ q \dashv \Delta } \rightarrow \, \textbf{Spine}$$

• Case
$$\frac{\Gamma \vdash e \Leftarrow A_k p \dashv \Delta}{\Gamma \vdash \mathsf{inj}_k e \Leftarrow A_1 + A_2 p \dashv \Delta} + \mathsf{I}_k$$

Use the i.h. (inverting $\Gamma \vdash (A_1 + A_2) p$ *type*).

• Case
$$\frac{\Gamma \vdash e_1 \Leftarrow A_1 \ p \dashv \Theta \qquad \Theta \vdash e_2 \Leftarrow [\Theta]A_2 \ p \dashv \Delta}{\Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow A_1 \times A_2 \ p \dashv \Delta} \times I$$

$$\Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow A_1 \times A_2 \ p \dashv \Delta$$

$$\Gamma \vdash \langle e_1, e_2 \rangle \Leftrightarrow A_1 \times A_2 \ p \dashv \Delta$$

$$\Gamma \vdash \langle e_1, e_2 \rangle \Leftrightarrow A_1 \times A_2 \ p \dashv \Delta$$
 Siven
$$\Gamma \vdash \langle e_1 + e_1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_5$$

• Case
$$\frac{\Gamma[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\times\hat{\alpha}_2]\vdash e_1\Leftarrow\hat{\alpha}_1\dashv\Theta\qquad\Theta\vdash e_2\Leftarrow[\Theta]\hat{\alpha}_2\dashv\Delta}{\Gamma[\hat{\alpha}:\star]\vdash\langle e_1,e_2\rangle\Leftarrow\hat{\alpha}\dashv\Delta}\times I\hat{\alpha}$$

We have $(\Gamma_I * \Gamma_R) = \Gamma_0[\hat{\alpha} : \star]$. We also have $\mathsf{FEV}(\hat{\alpha}) \subseteq \mathsf{dom}(\Gamma_R)$. Therefore $\hat{\alpha} \in \mathsf{dom}(\Gamma_R)$ and

$$\Gamma_0[\hat{\alpha}:\star] = \Gamma_L, \Gamma_2, \hat{\alpha}:\star, \Gamma_3$$

where $\Gamma_R = (\Gamma_2, \hat{\alpha}: \star, \Gamma_3)$.

Then the input context in the premise has the following form:

$$\Gamma_0[\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\times\hat{\alpha}_2] = (\Gamma_L,\Gamma_2,\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\times\hat{\alpha}_2,\Gamma_3)$$

Let us separate this context at the same point as $\Gamma_0[\hat{\alpha}:\star]$, that is, after Γ_L and before Γ_2 , and call the resulting right-hand context Γ_P' :

$$\Gamma_0[\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\times\hat{\alpha}_2] \ = \ \Gamma_L \ * \ \left(\underbrace{\Gamma_2,\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\times\hat{\alpha}_2,\Gamma_3}_{\Gamma'_R}\right)$$

By Lemma 23 (Deep Evar Introduction) (i), (ii), (ii) and the definition of separation, we can show

$$(\Gamma_L * (\Gamma_2, \hat{\alpha}: \star, \Gamma_3)) \xrightarrow{*} (\Gamma_L * (\Gamma_2, \hat{\alpha}_1: \star, \hat{\alpha}_2: \star, \hat{\alpha}: \star = \hat{\alpha}_1 \times \hat{\alpha}_2, \Gamma_3))$$

$$\begin{array}{ccc} (\Gamma_L * \Gamma_R) \xrightarrow{*} (\Gamma_L * \Gamma_R') & \text{ By above equalities} \\ & & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & \\ & & & \\ & &$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1+\hat{\alpha}_2] \vdash e \Leftarrow \hat{\alpha}_k \ \, \dashv \Delta}{\Gamma[\hat{\alpha}:\star] \vdash \mathsf{inj}_k \ \, e \Leftarrow \hat{\alpha} \ \, \dashv \Delta} + \mathsf{I}\hat{\alpha}_k$$

Similar to the $\times I\hat{\alpha}$ case, but simpler.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\to\hat{\alpha}_2]\vdash e\ s_0:(\hat{\alpha}_1\to\hat{\alpha}_2)\ \gg C\ \dashv \Delta}{\Gamma[\hat{\alpha}:\star]\vdash e\ s_0:\hat{\alpha}\ \gg C\ \dashv \Delta}\ \, \hat{\alpha} \mathsf{Spine}$$

Similar to the $\times I\hat{\alpha}$ and $+I\hat{\alpha}_k$ cases, except that (because we're in the spine part of the lemma) we have to show that $FEV(C)\subseteq dom(\Delta_R)$. But we have the same C in the premise and conclusion, so we get that by applying the i.h.

• Case
$$\frac{\Gamma \vdash e \Rightarrow A \mathrel{!} \dashv \Theta \qquad \Theta \vdash \Pi :: A \mathrel{q} \Leftarrow [\Theta] C \mathrel{p} \dashv \Delta \qquad \Pi \vdash [\Delta] A \textit{ covers } \Delta}{\Gamma \vdash \mathsf{case}(e,\Pi) \Leftarrow C \mathrel{p} \dashv \Delta} \mathsf{Case}$$

Use the i.h. and Lemma 68 (Transitivity of Separation).

H' Decidability of Algorithmic Subtyping

H'.1 Lemmas for Decidability of Subtyping

Lemma 73 (Substitution Isn't Large).

For all contexts Θ , we have $\# large([\Theta]A) = \# large(A)$.

Proof. By induction on A, following the definition of substitution.

Lemma 74 (Instantiation Solves).

If
$$\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$$
 and $[\Gamma]\tau = \tau$ and $\hat{\alpha} \notin FV([\Gamma]\tau)$ then $|\mathsf{unsolved}(\Gamma)| = |\mathsf{unsolved}(\Delta)| + 1$.

Proof. By induction on the given derivation.

$$\bullet \ \ \textbf{Case} \ \ \frac{\Gamma_L \vdash \tau : \kappa}{\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R \vdash \hat{\alpha} := \tau : \kappa \dashv \Gamma_L, \hat{\alpha} : \kappa = \tau, \Gamma_R} \ \, \text{InstSolve}$$

It is evident that $|\mathsf{unsolved}(\Gamma_L, \hat{\alpha} : \kappa, \Gamma_R)| = |\mathsf{unsolved}(\Gamma_L, \hat{\alpha} : \kappa = \tau, \Gamma_R)| + 1$.

$$\bullet \ \, \textbf{Case} \ \, \frac{ \widehat{\beta} \in \mathsf{unsolved}(\Gamma[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa]) }{ \Gamma[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa] \vdash \widehat{\alpha} := \underbrace{\widehat{\beta}}_{\mathtt{T}} : \kappa \dashv \Gamma[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa = \widehat{\alpha}] } \ \, \mathsf{InstReach}$$

Similar to the previous case.

• Case
$$\frac{\Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]\vdash\hat{\alpha}_1:=\tau_1:\star\dashv\Theta\quad\quad\Theta\vdash\hat{\alpha}_2:=[\Theta]\tau_2:\star\dashv\Delta}{\Gamma_0[\hat{\alpha}:\star]\vdash\hat{\alpha}:=\tau_1\oplus\tau_2:\star\dashv\Delta} \text{ InstBin}}$$

$$|\text{unsolved}(\Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}=\hat{\alpha}_1\oplus\hat{\alpha}_2])|=|\text{unsolved}(\Gamma_0[\hat{\alpha}])|+1\quad\text{Immediate}}$$

$$|\text{unsolved}(\Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}=\hat{\alpha}_1\oplus\hat{\alpha}_2])|=|\text{unsolved}(\Theta)|+1\quad\quad\text{By i.h.}}$$

$$|\text{unsolved}(\Gamma)|=|\text{unsolved}(\Theta)|\quad\quad\text{Subtracting 1}$$

$$=|\text{unsolved}(\Delta)|+1\quad\quad\text{By i.h.}}$$

Case

$$\frac{}{\Gamma[\hat{\alpha}:\mathbb{N}]\vdash\hat{\alpha}:=\mathsf{zero}:\mathbb{N}\dashv\Gamma[\hat{\alpha}:\mathbb{N}=\mathsf{zero}]}\;\mathsf{InstZero}$$

Similar to the InstSolve case.

• Case
$$\frac{\Gamma_0[\hat{\alpha}_1:\mathbb{N},\hat{\alpha}:\mathbb{N}=\mathsf{succ}(\hat{\alpha}_1)]\vdash\hat{\alpha}_1:=t_1:\mathbb{N}\dashv\Delta}{\Gamma_0[\hat{\alpha}:\mathbb{N}]\vdash\hat{\alpha}:=\mathsf{succ}(t_1):\mathbb{N}\dashv\Delta} \ \mathsf{InstSucc}$$

$$|\mathsf{unsolved}(\Delta)|+1=|\mathsf{unsolved}(\Gamma_0[\hat{\alpha}_1:\mathbb{N},\hat{\alpha}:\mathbb{N}=\mathsf{succ}(\hat{\alpha}_1)])| \quad \text{By i.h.}$$

$$= |\mathsf{unsolved}(\Gamma_0[\hat{\alpha}:\mathbb{N}])| \quad \text{By definition of unsolved}(-)$$

Lemma 75 (Checkeq Solving). *If* $\Gamma \vdash s \stackrel{\circ}{=} t : \kappa \dashv \Delta$ *then either* $\Delta = \Gamma$ *or* $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)|$. *Proof.* By induction on the given derivation.

Case

$$\frac{}{\Gamma \vdash \mathfrak{u} \stackrel{\circ}{=} \mathfrak{u} : \kappa \dashv \underbrace{\Gamma}_{\Lambda}} \text{ CheckeqVar}$$

Here $\Delta = \Gamma$.

• Cases CheckeqUnit, CheckeqZero: Similar to the CheckeqVar case.

• Case
$$\frac{\Gamma \vdash \sigma \stackrel{\circ}{=} t : \mathbb{N} \dashv \Delta}{\Gamma \vdash \mathsf{succ}(\sigma) \stackrel{\circ}{=} \mathsf{succ}(t) : \mathbb{N} \dashv \Delta} \mathsf{CheckeqSucc}$$

Follows by i.h.

$$\bullet \ \, \textbf{Case} \ \, \frac{ \Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} := t : \kappa \dashv \Delta \qquad \hat{\alpha} \notin \mathit{FV}(t) }{ \underbrace{\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} \, \mathring{=} \, t : \kappa \dashv \Delta }_{\Gamma} } \ \, \text{CheckeqInstL}$$

$$\begin{split} \Gamma_0[\widehat{\alpha}] \vdash \widehat{\alpha} &:= t : \kappa \dashv \Delta & \text{Subderivation} \\ \Gamma \vdash \widehat{\alpha} &:= t : \kappa \dashv \Delta & \Gamma = \Gamma_0[\widehat{\alpha}] \\ \Delta &= \Gamma \text{ or } |\mathsf{unsolved}(\Delta)| = |\mathsf{unsolved}(\Gamma)| - 1 & \text{By Lemma 74 (Instantiation Solves)} \\ & \Delta = \Gamma \text{ or } |\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)| \end{split}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}:\kappa] \vdash \hat{\alpha} := t : \kappa \dashv \Delta \qquad \hat{\alpha} \notin \mathit{FV}(t)}{\Gamma[\hat{\alpha}:\kappa] \vdash t \stackrel{\circ}{=} \hat{\alpha} : \kappa \dashv \Delta} \ \, \text{CheckeqInstR}$$

Similar to the CheckeqInstL case.

$$\begin{split} \Gamma \vdash \sigma_1 \, \stackrel{\circ}{=} \, \tau_1 : \star \dashv \Theta & \text{Subderivation} \\ \Theta = \Gamma \text{ or } |\mathsf{unsolved}(\Theta)| < |\mathsf{unsolved}(\Gamma)| & \text{ By i.h.} \end{split}$$

$$\begin{split} -\Theta &= \Gamma \colon \\ \Theta &\vdash [\Theta] \sigma_2 \stackrel{\circ}{=} [\Theta] \tau_2 : \star \dashv \Delta & \text{Subderivation} \\ \Gamma &\vdash [\Gamma] \sigma_2 \stackrel{\circ}{=} [\Gamma] \tau_2 : \star \dashv \Delta & \text{By } \Theta = \Gamma \\ & \qquad \qquad \Delta = \Gamma \text{ or } |\mathsf{unsolved}(\Gamma)| = |\mathsf{unsolved}(\Delta)| + 1 & \text{By i.h.} \end{split}$$

$$\begin{split} &- |\mathsf{unsolved}(\Theta)| < |\mathsf{unsolved}(\Gamma)| \text{:} \\ &\Theta \vdash [\Theta]\sigma_2 \triangleq [\Theta]\tau_2 : \star \dashv \Delta & \text{Subderivation} \\ &\Delta = \Theta \text{ or } |\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Theta)| & \text{By i.h.} \end{split}$$

If $\Delta = \Theta$ then substituting Δ for Θ in $|\mathsf{unsolved}(\Theta)| < |\mathsf{unsolved}(\Gamma)|$ gives $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)|$. If $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Theta)|$ then transitivity of < gives $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)$. \square

Lemma 76 (Prop Equiv Solving).

If $\Gamma \vdash P \equiv Q \dashv \Delta$ then either $\Delta = \Gamma$ or $|unsolved(\Delta)| < |unsolved(\Gamma)|$.

Proof. Only one rule can derive the judgment:

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash \sigma_1 \stackrel{\text{\tiny \circ}}{=} t_1 : \mathbb{N} \dashv \Theta \qquad \Theta \vdash [\Theta] \sigma 2 \stackrel{\text{\tiny \circ}}{=} [\Theta] t_2 : \mathbb{N} \dashv \Delta}{\Gamma \vdash (\sigma_1 = \sigma 2) \equiv (t_1 = t_2) \dashv \Delta} \equiv \hspace{-0.5cm} \mathsf{PropEq}$$

By Lemma 75 (Checkeq Solving) on the first premise,

either $\Theta = \Gamma$ or $|\mathsf{unsolved}(\Theta)| < |\mathsf{unsolved}(\Gamma)|$.

In the former case, the result follows from Lemma 75 (Checkeq Solving) on the second premise.

In the latter case, applying Lemma 75 (Checkeq Solving) to the second premise either gives $\Delta = \Theta$, and therefore

$$|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)|$$

or gives $|unsolved(\Delta)| < |unsolved(\Theta)|$, which also leads to $|unsolved(\Delta)| < |unsolved(\Gamma)|$.

Lemma 77 (Equiv Solving).

If $\Gamma \vdash A \equiv B \dashv \Delta$ then either $\Delta = \Gamma$ or $|unsolved(\Delta)| < |unsolved(\Gamma)|$.

Proof. By induction on the given derivation.

Case

$$\frac{}{\Gamma \vdash \alpha \equiv \alpha \dashv \Gamma} \equiv \mathsf{Var}$$

Here $\Delta = \Gamma$.

• Cases ≡Exvar, ≡Unit: Similar to the ≡Var case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash A_1 \equiv B_1 \dashv \Theta \qquad \Theta \vdash [\Theta] A_2 \equiv [\Theta] B_2 \dashv \Delta}{\Gamma \vdash (A_1 \oplus A_2) \equiv (B_1 \oplus B_2) \dashv \Delta} \equiv \oplus$$

By i.h., either $\Theta = \Gamma$ or $|\mathsf{unsolved}(\Theta)| < |\mathsf{unsolved}(\Gamma)|$.

In the former case, apply the i.h. to the second premise. Now either $\Delta = \Theta$ —and therefore $\Delta = \Gamma$ —or $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Theta)|$. Since $\Theta = \Gamma$, we have $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma)|$.

In the latter case, we have $|\operatorname{unsolved}(\Theta)| < |\operatorname{unsolved}(\Gamma)|$. By i.h. on the second premise, either $\Delta = \Theta$, and substituting Δ for Θ gives $|\operatorname{unsolved}(\Delta)| < |\operatorname{unsolved}(\Gamma)|$ —or $|\operatorname{unsolved}(\Delta)| < |\operatorname{unsolved}(\Theta)|$, which combined with $|\operatorname{unsolved}(\Theta)| < |\operatorname{unsolved}(\Gamma)|$ gives $|\operatorname{unsolved}(\Delta)| < |\operatorname{unsolved}(\Gamma)|$.

• Case \equiv Vec: Similar to the $\equiv \oplus$ case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma\!\!\!\!/ \, \alpha: \kappa \vdash A_0 \equiv B_0 \dashv \Delta, \alpha: \kappa, \Delta'}{\Gamma \vdash \forall \alpha: \kappa. \, A_0 \equiv \forall \alpha: \kappa. \, B_0 \dashv \Delta} \equiv \forall$$

By i.h., either $(\Delta, \alpha : \kappa, \Delta') = (\Gamma, \alpha : \kappa)$, or $|\mathsf{unsolved}(\Delta, \alpha : \kappa, \Delta')| < |\mathsf{unsolved}(\Gamma, \alpha : \kappa)|$.

In the former case, Lemma 22 (Extension Inversion) (i) tells us that $\Delta' = \cdot$. Thus, $(\Delta, \alpha : \kappa) = (\Gamma, \alpha : \kappa)$, and so $\Delta = \Gamma$.

In the latter case, we have $|\operatorname{unsolved}(\Delta, \alpha : \kappa, \Delta')| < |\operatorname{unsolved}(\Gamma, \alpha : \kappa)|$, that is:

$$|\mathsf{unsolved}(\Delta)| + 0 + |\mathsf{unsolved}(\Delta')| < |\mathsf{unsolved}(\Gamma)| + 0$$

Since $|unsolved(\Delta')|$ cannot be negative, we have $|unsolved(\Delta)| < |unsolved(\Gamma)|$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash P \equiv Q \dashv \Theta \qquad \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta}{\Gamma \vdash P \supset A_0 \equiv Q \supset B_0 \dashv \Delta} \equiv \supset$$

Similar to the $\equiv \oplus$ case, but using Lemma 76 (Prop Equiv Solving) on the first premise instead of the i.h.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash P \equiv Q \dashv \Theta \qquad \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta}{\Gamma \vdash A_0 \land P \equiv B_0 \land Q \dashv \Delta} \equiv \land$$

Similar to the $\equiv \land$ case.

$$\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma_0[\widehat{\alpha}] \vdash \widehat{\alpha} := \tau : \star \dashv \Delta \qquad \widehat{\alpha} \notin FV(\tau)}{\underbrace{\Gamma_0[\widehat{\alpha}] \vdash \widehat{\alpha} \equiv \tau \dashv \Delta}} \equiv \\ \\ \textbf{InstantiateL} \end{array}$$

By Lemma 74 (Instantiation Solves), $|unsolved(\Delta)| = |unsolved(\Gamma)| - 1$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_0[\widehat{\alpha}] \vdash \widehat{\alpha} := \tau : \star \dashv \Delta \qquad \widehat{\alpha} \notin \mathit{FV}(\tau)}{\Gamma_0[\widehat{\alpha}] \vdash \tau \equiv \widehat{\alpha} \dashv \Delta} \equiv \mathsf{InstantiateR}$$

Similar to the \equiv InstantiateL case.

Lemma 78 (Decidability of Propositional Judgments).

The following judgments are decidable, with Δ as output in (1)–(3), and Δ^{\perp} as output in (4) and (5).

We assume $\sigma = [\Gamma]\sigma$ and $t = [\Gamma]t$ in (1) and (4). Similarly, in the other parts we assume $P = [\Gamma]P$ and (in part (3)) $Q = [\Gamma]Q$.

- (1) $\Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta$
- (2) $\Gamma \vdash P true \dashv \Delta$
- (3) $\Gamma \vdash P \equiv O \dashv \Delta$
- (4) $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta^{\perp}$
- (5) $\Gamma / P \dashv \Delta^{\perp}$

Proof. Since there is no mutual recursion between the judgments, we can prove their decidability in order, separately.

- (1) Decidability of $\Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta$: By induction on the sizes of σ and t.
 - Cases CheckeqVar, CheckeqUnit, CheckeqZero: No premises.
 - Case CheckeqSucc: Both σ and t get smaller in the premise.
 - Cases CheckegInstL, CheckegInstR: Follows from Lemma 67 (Decidability of Instantiation).
- (2) *Decidability of* $\Gamma \vdash P$ *true* $\dashv \Delta$: By induction on σ and t. But we have only one rule deriving this judgment form, CheckpropEq, which has the judgment in (1) as a premise, so decidability follows from part (1).
- (3) *Decidability of* $\Gamma \vdash P \equiv Q \dashv \Delta$: By induction on P and Q. But we have only one rule deriving this judgment form, \equiv PropEq, which has two premises of the form (1), so decidability follows from part (1).
- (4) Decidability of $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta^{\perp}$: By lexicographic induction, first on the number of unsolved variables (both universal and existential) in Γ , then on σ and t. We also show that the number of unsolved variables is nonincreasing in the output context (if it exists).

- Cases ElimeqUvarRefl, ElimeqZero: No premises, and the output is the same as the input.
- Case ElimeqClash: The only premise is the clash judgment, which is clearly decidable. There is no output.
- Case ElimeqBin: In the first premise, we have the same Γ but both σ and t are smaller. By i.h., the first premise is decidable; moreover, either some variables in Θ were solved, or no additional variables were solved.

If some variables in Θ were solved, the second premise is smaller than the conclusion according to our lexicographic measure, so by i.h., the second premise is decidable.

If no additional variables were solved, then $\Theta = \Gamma$. Therefore $[\Theta]\tau_2 = [\Gamma]\tau_2$. It is given that $\sigma = [\Gamma]\sigma$ and $t = [\Gamma]t$, so $[\Gamma]\tau_2 = \tau_2$. Likewise, $[\Theta]\tau_2' = [\Gamma]\tau_2' = \tau_2'$, so we aremaking a recursive call on a strictly smaller subterm.

Regardless, Δ^{\perp} is either \perp , or is a Δ which has no more unsolved variables than Θ , which in turn has no more unsolved variables than Γ .

Case ElimeqBinBot:

The premise is invoked on subterms, and does not yield an output context.

- ullet Case ElimeqSucc: Both σ and t get smaller. By i.h., the output context has fewer unsolved variables, if it exists.
- Cases ElimeqInstL, ElimeqInstR: Follows from Lemma 67 (Decidability of Instantiation). Furthermore, by Lemma 74 (Instantiation Solves), instantiation solves a variable in the output.
- Cases ElimeqUvarL, ElimeqUvarR: These rules have no nontrivial premises, and α is solved in the output context.
- Cases ElimeqUvarL \perp , ElimeqUvarR \perp : These rules have no nontrivial premises, and produce the output context \perp .
- (5) *Decidability of* $\Gamma / P \dashv \Delta^{\perp}$: By induction on P. But we have only one rule deriving this judgment form, ElimpropEq, for which decidability follows from part (4).

Lemma 79 (Decidability of Equivalence).

Given a context Γ and types A, B such that $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists Δ such that $\Gamma \vdash A \equiv B \dashv \Delta$.

Proof. Let the judgment $\Gamma \vdash A \equiv B \dashv \Delta$ be measured lexicographically by

- (E1) #large(A) + #large(B);
- (E2) |unsolved(Γ)|, the number of unsolved existential variables in Γ ;
- (E3) |A| + |B|.
 - Cases ≡Var, ≡Exvar, ≡Unit: No premises.

$$\begin{array}{cccc} \bullet & \textbf{Case} & \\ & \frac{\Gamma \vdash A_1 \equiv B_1 \dashv \Theta & \Theta \vdash [\Theta] A_2 \equiv [\Theta] B_2 \dashv \Delta}{\Gamma \vdash A_1 \oplus A_2 \equiv B_1 \oplus B_2 \dashv \Delta} \equiv \oplus \\ \end{array}$$

In the first premise, part (E1) either gets smaller (if A_2 or B_2 have large connectives) or stays the same. Since the first premise has the same input context, part (E2) remains the same. However, part (E3) gets smaller.

In the second premise, part (E1) either gets smaller (if A_1 or B_1 have large connectives) or stays the same.

• Case \equiv Vec: Similar to a special case of $\equiv \oplus$, where two of the types are monotypes.

• Case
$$\frac{\Gamma\!\!,\,\alpha:\kappa\vdash A_0\equiv B_0\dashv \Delta,\alpha:\kappa,\Delta'}{\Gamma\vdash \underbrace{\forall\alpha:\kappa.A_0}_A\equiv \underbrace{\forall\alpha:\kappa.B_0}_B\dashv \Delta}\equiv \forall$$

Since $\# large(A_0) + \# large(B_0) = \# large(A) + \# large(B) - 2$, the first part of the measure gets smaller.

• Case
$$\frac{\Gamma \vdash P \equiv Q \dashv \Theta \qquad \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta}{\Gamma \vdash \underbrace{P \supset A_0}_{A} \equiv \underbrace{Q \supset B_0}_{B} \dashv \Delta} \equiv \supset$$

The first premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (3).

For the second premise, by Lemma 73 (Substitution Isn't Large), $\#large([\Theta]A_0) = \#large(A_0)$ and $\#large([\Theta]B_0) = \#large(B_0)$. Since $\#large(A) = \#large(A_0) + 1$ and $\#large(B) = \#large(B_0) + 1$, we have

$$\# large([\Theta]A_0) + \# large([\Theta]B_0) < \# large(A) + \# large(B)$$

which makes the first part of the measure smaller.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash P \equiv Q \dashv \Theta \quad \ \, \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta}{\Gamma \vdash A_0 \land P \equiv B_0 \land Q \dashv \Delta} \equiv \land$$

Similar to the $\equiv \supset$ case.

$$\bullet \ \ \, \textbf{Case} \ \ \, \frac{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} := \tau : \star \dashv \Delta \qquad \hat{\alpha} \notin \mathit{FV}(\tau)}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \equiv \tau \dashv \Delta} \equiv \mathsf{InstantiateL}$$

Follows from Lemma 67 (Decidability of Instantiation).

• Case ≡InstantiateR: Similar to the ≡InstantiateL case.

H'.2 Decidability of Subtyping

Theorem 1 (Decidability of Subtyping).

Given a context Γ and types A, B such that $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $[\Gamma]A = A$ and $[\Gamma]B = B$, it is decidable whether there exists Δ such that $\Gamma \vdash A <: \mathcal{P} B \dashv \Delta$.

Proof. Let the judgments be measured lexicographically by #large(A) + #large(B).

For each subtyping rule, we show that every premise is smaller than the conclusion, or already known to be decidable. The condition that $[\Gamma]A = A$ and $[\Gamma]B = B$ is easily satisfied at each inductive step, using the definition of substitution.

Now, we consider the rules deriving $\Gamma \vdash A <: \mathcal{P} B \dashv \Delta$.

• Case A not headed by
$$\forall /\exists$$

$$\frac{B \text{ not headed by } \forall /\exists}{\Gamma \vdash A <: ^{\mathcal{P}} B \dashv \Delta} <: \mathsf{Equiv}$$

In this case, we appeal to Lemma 79 (Decidability of Equivalence).

• Case B not headed by
$$\forall$$

$$\frac{\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\hat{\alpha}/\alpha]A <: ^{-}B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta}{\Gamma \vdash \forall \alpha : \kappa \mid A <: ^{-}B \dashv \Delta} <: \forall L$$

The premise has one fewer quantifier.

• Case
$$\frac{\Gamma, \beta : \kappa \vdash A <: ^{-} B \dashv \Delta, \beta : \kappa, \Theta}{\Gamma \vdash A <: ^{-} \forall \beta : \kappa. \ B \dashv \Delta} <: \forall R$$

The premise has one fewer quantifier.

• Case
$$\frac{\Gamma, \alpha : \kappa \vdash A <: ^+ B \dashv \Delta, \alpha : \kappa, \Theta}{\Gamma \vdash \exists \alpha : \kappa, A <: ^+ B \dashv \Delta} <: \exists L$$

The premise has one fewer quantifier.

• Case A not headed by \exists $\frac{\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} : \kappa \vdash A <: ^{+} [\hat{\beta}/\beta]B \dashv \Delta, \blacktriangleright_{\hat{\beta}}, \Theta}{\Gamma \vdash A <: ^{+} \exists \beta : \kappa, B \dashv \Delta} <: \exists R$

The premise has one fewer quantifier.

• Case $\frac{neg(A)}{\Gamma \vdash A <: ^- B \dashv \Delta \quad nonpos(B)} <: ^-_+ L$

Consider whether B is negative.

- Case neg(B): $B = \forall \beta : \kappa. B' \qquad \text{Definition of } neg(B)$ $\Gamma, \beta : \kappa \vdash A <: ^-B' \dashv \Delta, \beta : \kappa, \Theta \qquad \text{Inversion on the premise}$

There is one fewer quantifier in the subderivation.

Case nonneg(B):
 In this case, B is not headed by a ∀.

 $A = \forall \alpha : \kappa. \ A' \qquad \qquad \text{Definition of } \textit{neg}(A) \\ \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\hat{\alpha}/\alpha] A' <: ^- ' \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \qquad \text{Inversion on the premise}$

There is one fewer quantifier in the subderivation.

• Case $\frac{nonpos(A)}{\Gamma \vdash A <: ^- B \dashv \Delta \quad neg(B)}{\Gamma \vdash A <: ^+ B \dashv \Delta} <: ^-_+ R$

 $B = \forall \beta : \kappa. \ B' \qquad \qquad \text{Definition of } \textit{neg}(B) \\ \Gamma, \beta : \kappa \vdash A <: ^- B' \dashv \Delta, \beta : \kappa, \Theta \qquad \text{Inversion on the premise}$

There is one fewer quantifier in the subderivation.

• Case $\frac{pos(A)}{\Gamma \vdash A <: ^+ B \dashv \Delta \quad nonneg(B)} <: ^+ _- L$

This case is similar to the $<: _{+}^{-}R$ case.

Case

$$\frac{\textit{nonneg}(A)}{\Gamma \vdash A <: ^{+}B \dashv \Delta} \frac{\textit{nonneg}(A)}{\textit{pos}(B)} <: ^{+}_{-}R$$

This case is similar to the <: L case.

H'.3 Decidability of Matching and Coverage

Lemma 80 (Decidability of Guardedness Judgment).

For any set of branches Π , the relation Π guarded is decidable.

Proof. This follows via a routine induction on Π , counting the number of branch lists.

Lemma 81 (Decidability of Expansion Judgments).

Given branches Π , it is decidable whether:

- (1) there exists a unique Π' such that $\Pi \stackrel{\times}{\leadsto} \Pi'$;
- (2) there exist unique Π_L and Π_R such that $\Pi \stackrel{+}{\leadsto} \Pi_L \parallel \Pi_R$;
- (3) there exists a unique Π' such that $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$;
- (4) there exists a unique Π' such that $\Pi \stackrel{1}{\leadsto} \Pi'$.
- (5) there exist unique Π_{IJ} and $\Pi_{::}$ such that $\Pi \stackrel{\mathsf{Vec}}{\leadsto} \Pi_{IJ} \parallel \Pi_{::}$.

Proof. In each part, by induction on Π : Every rule either has no premises, or breaks down Π in its nontrivial premise.

Lemma 82 (Expansion Shrinks Size).

We define the size of a pattern |p| as follows:

$$\begin{array}{lll} |x| & = & 0 \\ |_| & = & 0 \\ |\langle p, p' \rangle| & = & 1 + |p| + |p'| \\ |O| & = & 0 \\ |\text{inj}_1 p| & = & 1 + |p| \\ |\text{inj}_2 p| & = & 1 + |p| \\ |\mathcal{L}I| & = & 1 \\ |p :: p'| & = & 1 + |p| + |p'| \end{array}$$

We lift size to branches $\pi = \vec{p} \Rightarrow e$ as follows:

$$|\mathfrak{p}_1,\ldots,\mathfrak{p}_n\Rightarrow e|=|\mathfrak{p}_1|+\ldots+|\mathfrak{p}_n|$$

We lift size to branch lists $\Pi = \pi_1 \mid \dots \mid \pi_n$ as follows:

$$|\pi_1| \dots |\pi_n| = |\pi_1| + \dots + |\pi_n|$$

Now, the following properties hold:

- 1. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ then $|\Pi| = |\Pi'|$.
- 2. If $\Pi \stackrel{1}{\leadsto} \Pi'$ then $|\Pi| = |\Pi'|$.
- 3. If $\Pi \stackrel{\times}{\leadsto} \Pi'$ then $|\Pi| \leq |\Pi'|$.

- 4. If $\Pi \stackrel{+}{\leadsto} \Pi_L \parallel \Pi_R$ then $|\Pi| \leq |\Pi_1|$ and $|\Pi| \leq |\Pi_2|$.
- 5. If $\Pi \stackrel{\mathsf{Vec}}{\leadsto} \Pi_{\Pi} \parallel \Pi_{::}$ then $|\Pi_{\Pi}| \leq |\Pi|$ and $|\Pi_{::}| \leq |\Pi|$.
- 6. If Π guarded and $\Pi \overset{\text{Vec}}{\leadsto} \Pi_{IJ} \parallel \Pi_{::}$ then $|\Pi_{IJ}| < |\Pi|$ and $|\Pi_{::}| < |\Pi|$.

Proof. Properties 1-5 follow by a routine induction on the derivation of the expansion judgement. Since expansion only ever removes pattern constructors, and only ever adds wildcards, it never increases the size of the resulting branch list.

Case 6 for vectors proceeds by induction on the derivation of Π guarded, and furthermore depends upon the proof for case 5.

1. Case

$$[], \vec{p} \Rightarrow e \mid \Pi \text{ guarded}$$

By inversion on the expansion derivation, we know $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_{\Pi} \parallel \Pi_{::}$.

By part 5, we know that $|\Pi_{\square}| \leq |\Pi|$ and $|\Pi_{::}| \leq |\Pi|$.

By the definition of size, we know that $|\vec{p} \Rightarrow e| < |[], \vec{p} \Rightarrow e|$.

Hence $|\vec{p} \Rightarrow e \mid \Pi_{[]}| < |[], \vec{p} \Rightarrow e \mid \Pi|$.

By the definition of size, we know that $|\Pi| < |[], \vec{p} \Rightarrow e |\Pi|$.

Hence $|\Pi_{::}| < |[], \vec{p} \Rightarrow e |\Pi|$.

2. Case

$$p :: p', \vec{p} \Rightarrow e \mid \Pi \text{ guarded}$$

By inversion on the expansion derivation, we know $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_{\Pi} \parallel \Pi_{::}$.

By part 5, we know that $|\Pi_{\Pi}| \leq |\Pi|$ and $|\Pi_{::}| \leq |\Pi|$.

By the definition of size, we know that $|p, p', \vec{p} \Rightarrow e| < |p :: p', \vec{p} \Rightarrow e|$.

Hence $|p, p', \vec{p} \Rightarrow e \mid \Pi_{::}| < |p :: p', \vec{p} \Rightarrow e \mid \Pi|$.

By the definition of size, we know that $|\Pi| < |\mathfrak{p} :: \mathfrak{p}', \vec{\mathfrak{p}} \Rightarrow e \mid \Pi|$.

Hence $|\Pi_{\Pi}| < |[], \vec{p} \Rightarrow e |\Pi|$.

3. Case

$$\frac{\Pi \text{ guarded}}{\vec{p} \Rightarrow e \mid \Pi \text{ guarded}}$$

By inversion on the expansion derivation, we know $\Pi \stackrel{\mathsf{Vec}}{\leadsto} \Pi_{\square} \parallel \Pi_{::}$.

By induction, $|\Pi_{[]}| < |\Pi|$ and $|\Pi_{::}| < |\Pi|$.

- By the definition of size, $|_, \vec{p} \Rightarrow e \mid \Pi_{\square}| < |_, \vec{p} \Rightarrow e \mid \Pi|$
- By the definition of size, $| , \vec{p} \Rightarrow e \mid \Pi_{::} | < | , \vec{p} \Rightarrow e \mid \Pi |$

4. Case

$$\frac{\Pi \text{ guarded}}{x, \vec{p} \Rightarrow e \mid \Pi \text{ guarded}}$$

Similar to previous case.

Theorem 2 (Decidability of Coverage).

Given a context Γ , branches Π and types \vec{A} , it is decidable whether $\Gamma \vdash \Pi$ covers \vec{A} q is derivable.

Proof. By induction on, lexicographically, (1) the size $|\Pi|$ of the branch list Π and then (2) the number of \wedge connectives in \vec{A} , and then (3) the size of \vec{A} , considered to be the sum of the sizes |A| of each type A in \vec{A} (treating constraints s = t as size 1).

(For CoversVar, CoversVec, CoversVec, and Covers+, we also use the appropriate part of Lemma 81 (Decidability of Expansion Judgments), as well as Lemma 82 (Expansion Shrinks Size).)

- Case CoversEmpty: No premises.
- Case CoversVar: The number of \land connectives does not grow, and the size of the branch list stays the same, and \vec{A} gets smaller.
- Case Covers1: The number of \wedge connectives and the size of the branch list stays the same, and \vec{A} gets smaller.
- Case Covers \land : The size of the branch list stays the same, and the number of \land connectives in \vec{A} goes down. This lets us analyze the two possibilities for the coverage-with-assumptions judgement:
 - **Case** CoversEq: The first premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (4). The number of \wedge connectives in \vec{A} gets smaller (note that applying Δ as a substitution cannot add \wedge connectives).
 - Case CoversEqBot: The premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (4).
- Case Covers \land !: The size of the branch list stays the same, and the number of \land connectives in \vec{A} goes down
- Case Covers×: The size of the branch list does not grow, the number of \land connectives stays the same, and \vec{A} gets smaller, since $|A_1| + |A_2| < |A_1 \times A_2|$.
- Case Covers+: Here we have $\vec{A} = (A_1 + A_2, \vec{B})$. In the first premise, we have (A_1, \vec{B}) , which is smaller than \vec{A} , and in the second premise we have (A_2, \vec{B}) , which is likewise smaller. (In both premises, the size of the branch list does not grow, and the number of \land connectives stays the same.)
- Case CoversVec:

Since Π guarded is decidable, and $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_{\Pi} \parallel \Pi_{::}$ is decidable, then we know that the size of the branch lists Π_{Π} and $\Pi_{::}$ is strictly smaller than Π .

This lets us analyze the two cases for each premise, noting that the assumption is decidable by Lemma 78 (Decidability of Propositional Judgments) (4).

- Case CoversEq: The first premise (that t= zero) is decidable by Lemma 78 (Decidability of Propositional Judgments) (4). The size of Π_{\square} is strictly smaller than Π 's size, so we can still appeal to induction (note Δ as a substitution cannot add change the size of a branch list).
- Case CoversEqBot: Decidable by Lemma 78 (Decidability of Propositional Judgments) (4).

The cons case is nearly identical:

- **Case** CoversEq: The first premise (that t = succ(n)) is decidable by Lemma 78 (Decidability of Propositional Judgments) (4). The size of Π_{\square} is strictly smaller than Π 's size, so we can still appeal to induction (note Δ as a substitution cannot add change the size of a branch list).
- Case CoversEqBot: Decidable by Lemma 78 (Decidability of Propositional Judgments) (4).
- Case CoversVec!:

Since Π guarded is decidable, and $\Pi \overset{\text{Vec}}{\leadsto} \Pi_{\Pi} \parallel \Pi_{::}$ is decidable, then we know that the size of the branch lists Π_{Π} and $\Pi_{::}$ is strictly smaller than Π .

• Case Covers \exists : The size of the branch list does not grow, and \vec{A} gets smaller.

- Case CoversEq: The first premise is decidable by Lemma 78 (Decidability of Propositional Judgments)
 (4). The number of ∧ connectives in A gets smaller (note that applying Δ as a substitution cannot add ∧ connectives).
- Case CoversEqBot: Decidable by Lemma 78 (Decidability of Propositional Judgments) (4). □

H'.4 Decidability of Typing

Theorem 3 (Decidability of Typing).

- (i) Synthesis: Given a context Γ , a principality $\mathfrak p$, and a term $\mathfrak e$, it is decidable whether there exist a type A and a context Δ such that $\Gamma \vdash \mathfrak e \Rightarrow A \mathfrak p \dashv \Delta$.
- (ii) Spines: Given a context Γ , a spine s, a principality $\mathfrak p$, and a type A such that $\Gamma \vdash A$ type, it is decidable whether there exist a type B, a principality $\mathfrak q$ and a context Δ such that $\Gamma \vdash s : A \mathfrak p \gg B \mathfrak q \dashv \Delta$.
- (iii) Checking: Given a context Γ , a principality p, a term e, and a type B such that $\Gamma \vdash B$ type, it is decidable whether there is a context Δ such that $\Gamma \vdash e \Leftarrow B \ p \dashv \Delta$.
- (iv) Matching: Given a context Γ , branches Π , a list of types \vec{A} , a type C, and a principality p, it is decidable whether there exists Δ such that $\Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta$.

Also, if given a proposition P as well, it is decidable whether there exists Δ such that $\Gamma / P \vdash \Pi :: \vec{A} ! \leftarrow C \not p \dashv \Delta$.

Proof. For rules deriving judgments of the form

$$\begin{array}{l} \Gamma \vdash e \Rightarrow -- \dashv -\\ \Gamma \vdash e \Leftarrow B p \dashv -\\ \Gamma \vdash s : B p \gg -- \dashv -\\ \Gamma \vdash \Pi :: \vec{A} \ q \Leftarrow C p \dashv -\\ \end{array}$$

(where we write "—" for parts of the judgments that are outputs), the following induction measure on such judgments is adequate to prove decidability:

where $\langle ... \rangle$ denotes lexicographic order, and where (when comparing two judgments typing terms of the same size) the synthesis judgment (top line) is considered smaller than the checking judgment (second line). That is,

$$\Rightarrow \prec \leftarrow / \gg / Match$$

Two match judgments are compared according to, first, the list of branches Π (which is a subterm of the containing case expression, allowing us to invoke the i.h. for the Case rule), then the size of the list of types \vec{A} (considered to be the sum of the sizes |A| of each type A in \vec{A}), and then, finally, whether the judgment is $\Gamma/P \vdash \ldots$ or $\Gamma \vdash \ldots$, considering the former judgment ($\Gamma/P \vdash \ldots$) to be larger.

Note that this measure only uses the input parts of the judgments, leading to a straightforward decidability argument.

We will show that in each rule deriving a synthesis, checking, spine or match judgment, every premise is smaller than the conclusion.

• Case EmptySpine: No premises.

- Case \rightarrow Spine: In each premise, the expression/spine gets smaller (we have e s in the conclusion, e in the first premise, and s in the second premise).
- Case Var: No nontrivial premises.
- **Case** Sub: The first premise has the same subject term *e* as the conclusion, but the judgment is smaller because our measure considers synthesis to be smaller than checking.
 - The second premise is a subtyping judgment, which by Theorem 1 is decidable.
- **Case** Anno: It is easy to show that the judgment $\Gamma \vdash A$! *type* is decidable. The second premise types e, but the conclusion types (e : A), so the first part of the measure gets smaller.
- Cases 11, $11\hat{\alpha}$: No premises.
- Case $\forall I$: Both the premise and conclusion type e, and both are checking; however, $\# | arge(A_0) < \# | arge(\forall \alpha : \kappa, A_0)$, so the premise is smaller.
- **Case** ∀Spine: Both the premise and conclusion type *e* s, and both are spine judgments; however, #large(−) decreases.
- Case \land I: By Lemma 78 (Decidability of Propositional Judgments) (2), the first premise is decidable. For the second premise, $\# | arge([\Theta]A_0) = \# | arge(A_0) < \# | arge(A_0 \land P)$.
- Case \exists I: Both the premise and conclusion type e, and both are checking; however, #large(-) decreases so the premise is smaller.
- **Case** ⊃I: For the first premise, use Lemma 78 (Decidability of Propositional Judgments) (5). In the second premise, #large(−) gets smaller (similar to the ∧I case).
- Case ⊃I⊥: The premise is decidable by Lemma 78 (Decidability of Propositional Judgments) (5).
- Case \supset Spine: Similar to the \land I case.
- Cases \rightarrow I, \rightarrow I $\hat{\alpha}$: In the premise, the term is smaller.
- Cases \rightarrow E: In all premises, the term is smaller.
- Cases $+I_k$, $+I\hat{\alpha}_k$, $\times I$, $\times I\hat{\alpha}$: In all premises, the term is smaller.
- **Case** Case: In the first premise, the term is smaller. In the second premise, we have a list of branches that is a proper subterm of the case expression. The third premise is decidable by Theorem 2.

We now consider the match rules:

- Case MatchEmpty: No premises.
- Case MatchSeq: In each premise, the list of branches is properly contained in Π , making each premise smaller by the first part (" $e/s/\Pi$ ") of the measure.
- **Case** MatchBase: The term e in the premise is properly contained in Π .
- Cases Match \exists , Match \times , Match $+_k$, MatchNeg, MatchWild: Smaller by part (2) of the measure.
- Case Match \(: \): The premise has a smaller \vec{A} , so it is smaller by the \vec{A} part of the measure. (The premise is the other judgment form, so it is *larger* by the "match judgment form" part, but \vec{A} lexicographically dominates.)
- Case Match \(\pm\$: For the premise, use Lemma 78 (Decidability of Propositional Judgments) (4).
- Case MatchUnify:

Lemma 78 (Decidability of Propositional Judgments) (4) shows that the first premise is decidable. The second premise has the same (single) branch and list of types, but is smaller by the "match judgment form" part of the measure.

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I' Determinacy

Lemma 83 (Determinacy of Auxiliary Judgments).

- (1) Elimeq: Given Γ , σ , t, κ such that $\mathsf{FEV}(\sigma) \cup \mathsf{FEV}(t) = \emptyset$ and $\mathcal{D}_1 :: \Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_1^{\perp}$ and $\mathcal{D}_2 :: \Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_2^{\perp}$, it is the case that $\Delta_1^{\perp} = \Delta_2^{\perp}$.
- (2) Instantiation: Given Γ , $\hat{\alpha}$, t, κ such that $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ and $\Gamma \vdash t : \kappa$ and $\hat{\alpha} \notin FV(t)$ and $\mathcal{D}_1 :: \Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash \hat{\alpha} := t : \kappa \dashv \Delta_2$ it is the case that $\Delta_1 = \Delta_2$.
- (3) Symmetric instantiation:

```
Given \Gamma, \hat{\alpha}, \hat{\beta}, \kappa such that \hat{\alpha}, \hat{\beta} \in \mathsf{unsolved}(\Gamma) and \hat{\alpha} \neq \hat{\beta} and \mathcal{D}_1 :: \Gamma \vdash \hat{\alpha} := \hat{\beta} : \kappa \dashv \Delta_1 and \mathcal{D}_2 :: \Gamma \vdash \hat{\beta} := \hat{\alpha} : \kappa \dashv \Delta_2 it is the case that \Delta_1 = \Delta_2.
```

- (4) Checkeq: Given Γ , σ , t, κ such that $\mathcal{D}_1 :: \Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta_2$ it is the case that $\Delta_1 = \Delta_2$.
- (5) Elimprop: Given Γ , P such that $\mathcal{D}_1 :: \Gamma / P \dashv \Delta_1^{\perp}$ and $\mathcal{D}_2 :: \Gamma / P \dashv \Delta_2^{\perp}$ it is the case that $\Delta_1 = \Delta_2$.
- (6) Checkprop: Given Γ , P such that $\mathcal{D}_1 :: \Gamma \vdash P$ true $\dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash P$ true $\dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Proof.

Proof of Part (1) (Elimeq).

Rule ElimeqZero applies if and only if $\sigma = t = zero$.

Rule ElimeqSucc applies if and only if σ and t are headed by succ.

Now suppose $\sigma = \alpha$.

• Rule ElimeqUvarRefl applies if and only if $t = \alpha$. (Rule ElimeqClash cannot apply; rules ElimeqUvarL and ElimeqUvarR have a free variable condition; rules ElimeqUvarL \perp and ElimeqUvarR \perp have a condition that $\sigma \neq t$.)

In the remainder, assume $t \neq alpha$.

• If $\alpha \in FV(t)$, then rule ElimeqUvarL \perp applies, and no other rule applies (including ElimeqUvarR \perp and ElimeqClash).

In the remainder, assume $\alpha \notin FV(t)$.

• Consider whether ElimeqUvarR \perp applies. The conclusion matches if we have $t=\beta$ for some $\beta \neq \alpha$ (that is, $\sigma=\alpha$ and $t=\beta$). But ElimeqUvarR \perp has a condition that $\beta \in FV(\sigma)$, and $\sigma=\alpha$, so the condition is not satisfied.

In the symmetric case, use the reasoning above, exchanging L's and R's in the rule names.

Proof of Part (2) (Instantiation).

Rule InstBin applies if and only if t has the form $t_1 \oplus t_2$.

Rule InstZero applies if and only if t has the form zero.

Rule InstSucc applies if and only if t has the form $succ(t_0)$.

If t has the form $\hat{\beta}$, then consider whether $\hat{\beta}$ is declared to the left of $\hat{\alpha}$ in the given context:

- If $\hat{\beta}$ is declared to the left of $\hat{\alpha}$, then rule InstReach cannot be used, which leaves only InstSolve.
- If $\hat{\beta}$ is declared to the right of $\hat{\alpha}$, then InstSolve cannot be used because $\hat{\beta}$ is not well-formed under Γ_0 (the context to the left of $\hat{\alpha}$ in InstSolve). That leaves only InstReach.
- $\hat{\alpha}$ cannot be $\hat{\beta}$, because it is given that $\hat{\alpha} \notin FV(t) = FV(\hat{\beta}) = {\hat{\beta}}.$

Proof of Part (3) (Symmetric instantiation).

InstBin, InstZero and InstSucc cannot have been used in either derivation.

Suppose that InstSolve concluded \mathcal{D}_1 . Then Δ_1 is the same as Γ with $\hat{\alpha}$ solved to $\hat{\beta}$. Moreover, $\hat{\beta}$ is declared to the left of $\hat{\alpha}$ in Γ . Thus, InstSolve cannot conclude \mathcal{D}_2 . However, InstReach can conclude \mathcal{D}_2 , but produces a context Δ_2 which is the same as Γ but with $\hat{\alpha}$ solved to $\hat{\beta}$. Therefore $\Delta_1 = \Delta_2$.

The other possibility is that InstReach concluded \mathcal{D}_1 . Then Δ_1 is the same as Γ with $\hat{\beta}$ solved to $\hat{\alpha}$, with $\hat{\alpha}$ declared to the left of $\hat{\beta}$ in Γ . Thus, InstReach cannot conclude \mathcal{D}_2 . However, InstSolve can conclude \mathcal{D}_2 , producing a context Δ_2 which is the same as Γ but with $\hat{\beta}$ solved to $\hat{\alpha}$. Therefore $\Delta_1 = \Delta_2$.

Proof of Part (4) (Checkeq).

Rule CheckeqVar applies if and only if $\sigma=t=\hat{\alpha}$ or $\sigma=t=\alpha$ (note the free variable conditions in CheckeqInstL and CheckeqInstR).

Rule CheckeqUnit applies if and only if $\sigma = t = 1$.

Rule CheckeqBin applies if and only if σ and t are both headed by the same binary connective.

Rule CheckegZero applies if and only if $\sigma = t = zero$.

Rule CheckeqSucc applies if and only if σ and t are headed by succ.

Now suppose $\sigma = \hat{\alpha}$. If t is not an existential variable, then CheckeqInstR cannot be used, which leaves only CheckeqInstL. If t is an existential variable, that is, some $\hat{\beta}$ (distinct from $\hat{\alpha}$), and is unsolved, then both CheckeqInstL and CheckeqInstR apply, but by part (3), we get the same output context from each.

The $t = \hat{\alpha}$ subcase is similar.

Proof of Part (5) (Elimprop). There is only one rule deriving this judgment; the result follows by part (1).

Proof of Part (6) (Checkprop). There is only one rule deriving this judgment; the result follows by part (4).

Lemma 84 (Determinacy of Equivalence).

- (1) Propositional equivalence: Given Γ , P, Q such that $\mathcal{D}_1 :: \Gamma \vdash P \equiv Q \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash P \equiv Q \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.
- (2) Type equivalence: Given Γ , A, B such that $\mathcal{D}_1 :: \Gamma \vdash A \equiv B \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash A \equiv B \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Proof.

Proof of Part (1) (propositional equivalence). Only one rule derives judgments of this form; the result follows from Lemma 83 (Determinacy of Auxiliary Judgments) (4).

Proof of Part (2) (type equivalence). If neither A nor B is an existential variable, they must have the same head connectives, and the same rule must conclude both derivations.

If A and B are the same existential variable, then only $\equiv Exvar$ applies (due to the free variable conditions in $\equiv InstantiateL$ and $\equiv InstantiateR$).

If A and B are different unsolved existential variables, the judgment matches the conclusion of both \equiv InstantiateL and \equiv InstantiateR, but by part (3) of Lemma 83 (Determinacy of Auxiliary Judgments), we get the same output context regardless of which rule we choose.

Theorem 4 (Determinacy of Subtyping).

(1) Subtyping: Given Γ , e, A, B such that $\mathcal{D}_1 :: \Gamma \vdash A <: \mathcal{P} B \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash A <: \mathcal{P} B \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Proof. First, we consider whether we are looking at positive or negative subtyping, and then consider the outermost connective of A and B:

• If $\Gamma \vdash A <: ^+ B \dashv \Delta_1$ and $\Gamma \vdash A <: ^+ B \dashv \Delta_2$, then we know the last rule ending the derivation of \mathcal{D}_1 and \mathcal{D}_2 must be:

The only case in which there are two possible final rules is in the \forall/\forall case. In this case, regardless of the choice of rule, by inversion we get subderivations $\Gamma \vdash A <:= B \dashv \Delta_1$ and $\Gamma \vdash A <:= B \dashv \Delta_2$.

• If $\Gamma \vdash A <: ^- B \dashv \Delta_1$ and $\Gamma \vdash A <: ^- B \dashv \Delta_2$, then we know the last rule ending the derivation of \mathcal{D}_1 and \mathcal{D}_2 must be:

The only case in which there are two possible final rules is in the \forall/\forall case. In this case, regardless of the choice of rule, by inversion we get subderivations $\Gamma \vdash A <: ^+B \dashv \Delta_1$ and $\Gamma \vdash A <: ^+B \dashv \Delta_2$.

As a result, the result follows by a routine induction.

Theorem 5 (Determinacy of Typing).

- (1) Checking: Given Γ , e, A, p such that $\mathcal{D}_1 :: \Gamma \vdash e \Leftarrow A p \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash e \Leftarrow A p \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.
- (2) Synthesis: Given Γ , e such that $\mathcal{D}_1 :: \Gamma \vdash e \Rightarrow B_1 p_1 \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash e \Rightarrow B_2 p_2 \dashv \Delta_2$, it is the case that $B_1 = B_2$ and $p_1 = p_2$ and $p_2 = D_2$.
- (3) Spine judgments:

Given Γ , e, A, p such that $\mathcal{D}_1 :: \Gamma \vdash e : A p \gg C_1 q_1 \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash e : A p \gg C_2 q_2 \dashv \Delta_2$, it is the case that $C_1 = C_2$ and $q_1 = q_2$ and $\Delta_1 = \Delta_2$.

The same applies for derivations of the principality-recovering judgments $\Gamma \vdash e : A p \gg C_k \lceil q_k \rceil \dashv \Delta_k$.

(4) Match judgments:

Given Γ , Π , \vec{A} , p, C such that $\mathcal{D}_1 :: \Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Given
$$\Gamma$$
, P , Π , \vec{A} , p , C such that $\mathcal{D}_1 :: \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta_1$ and $\mathcal{D}_2 :: \Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta_2$, it is the case that $\Delta_1 = \Delta_2$.

Proof.

Proof of Part (1) (checking).

The rules with a checking judgment in the conclusion are: 1I, $1I\hat{\alpha}$, $\forall I$, $\land I$, $\exists I$, $\supset I$, $\rightarrow I$, $\rightarrow I\hat{\alpha}$, Rec, $+I_k$, $+I\hat{\alpha}_k$, $\times I$, $\times I\hat{\alpha}$, Case, NiI, Cons.

The table below shows which rules apply for given e and A. The extra "chk-I?" column highlights the role of the "chk-I" ("check-intro") category of syntactic forms: we restrict the introduction rules for \forall and \supset to

type only these forms. For example, given e = x and $A = (\forall \alpha : \kappa. A_0)$, we need not choose between Sub and $\forall I$: the latter is ruled out by its *chk-I* premise.

						A							
		chk-I?	\forall	<i>Note 1</i> ⊃	3	\wedge	\rightarrow	+	×	1	â	α	Vec
	$\lambda x. e_0$	chk-I	$\forall I$	⊃I/⊃I⊥	∃I	\wedge I	\rightarrow l	Ø	Ø	Ø	$\!$	Ø	Ø
	rec x.v	Note 2	Rec	Rec	Rec	Rec	Rec	Rec	Rec	Rec	Rec	Rec	Ø
	$\operatorname{inj}_k e_0$	chk-I	$\forall I$	\supset I $/\supset$ I \perp	∃I	\wedge I	Ø	$+I_k$	Ø	Ø	$+I\hat{\alpha}_k$	Ø	Ø
	$\langle e_1, e_2 \rangle$	chk-I	$\forall I$	\supset I $/\supset$ I \perp	∃I	\wedge I	Ø	Ø	$\times I$	Ø	$\times I \hat{\alpha}$	Ø	Ø
	()	chk-I	$\forall I$	\supset I $/\supset$ I \perp	∃I	\wedge I	Ø	Ø	Ø	11	1Iâ	Ø	Ø
e	[]	chk-I	$\forall I$	\supset I $/\supset$ I \perp	∃I	\wedge I	Ø	Ø	Ø	Ø	Ø	Ø	Nil
	$e_1 :: e_2$	chk-I	$\forall I$	\supset I $/\supset$ I \perp	∃I	\wedge I	Ø	Ø	Ø	Ø	Ø	Ø	Cons
	$case(e_0,\Pi)$	Note 3	Case	Case	Case	Case	Case	Case	Case	Case	Case	Case	Case
	χ		Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub
	$(e_0:A)$		Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub
	e ₁ s		Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub	Sub

Notes:

- Note 1: The choice between ⊃I and ⊃I⊥ is resolved by Lemma 83 (Determinacy of Auxiliary Judgments)
 (5).
- *Note 2:* Fixed points are a checking form, but not an introduction form. So if e is rec x. v, we need not choose between an introduction rule for a large connective and the Rec rule: only the Rec rule is viable. Large connectives must, therefore, be introduced *inside* the typing of the body v.
- *Note 3:* Case expressions are a checking form, but not an introduction form. So if *e* is a case expression, we need not choose between an introduction rule for a large connective and the Case rule: only the Case rule is viable. Large connectives must, therefore, be introduced *inside* the branches.

Proof of Part (2) (synthesis). Only four rules have a synthesis judgment in the conclusion: Var, Anno and \rightarrow E Rule Var applies if and only if e has the form x. Rule Anno applies if and only if e has the form $(e_0 : A)$. Otherwise, the judgment can be derived only if e has the form $e_1 e_2$, by \rightarrow E.

Proof of Part (3) (spine judgments). For the ordinary spine judgment, rule EmptySpine applies if and only if the given spine is empty. Otherwise, the choice of rule is determined by the head constructor of the input type: \rightarrow/\rightarrow Spine; \forall/\forall Spine; \supset/\supset Spine; $\hat{\alpha}/\hat{\alpha}$ Spine.

For the principality-recovering spine judgment: If p=1, only rule SpinePass applies. If p=1 and q=1, only rule SpinePass applies. If p=1 and q=1, then the rule is determined by FEV(C): if $FEV(C)=\emptyset$ then only SpineRecover applies; otherwise, $FEV(C)\neq\emptyset$ and only SpinePass applies.

Proof of Part (4) (matching). First, the elimination judgment form $\Gamma / P \vdash \ldots$: It cannot be the case that both $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot$ and $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Theta$, so either Match \bot concludes both \mathcal{D}_1 and \mathcal{D}_2 (and the result follows), or MatchUnify concludes both \mathcal{D}_1 and \mathcal{D}_2 (in which case, apply the i.h.).

Now the main judgment form, without "/ P": either Π is empty, or has length one, or has length greater than one. MatchEmpty applies if and only if Π is empty, and MatchSeq applies if and only if Π has length greater than one. So in the rest of this part, we assume Π has length one.

Moreover, MatchBase applies if and only if \vec{A} has length zero. So in the rest of this part, we assume the length of \vec{A} is at least one.

Let A be the first type in \vec{A} . Inspection of the rules shows that given particular A and ρ , where ρ is the first pattern, only a single rule can apply, or no rule (" \emptyset ") can apply, as shown in the following table:

			A				
	∃	\wedge	+	×	Vec	other	
$inj_k \rho_0$	Match∃	$Match \land$	$Match+_k$	Ø	Ø	Ø	
ρ $\langle \rho_1, \rho_2 \rangle$	$Match \exists$	$Match \land$	Ø	$Match \times$	Ø	Ø	
z	$Match \exists$	$Match \land$	MatchNeg	MatchNeg	MatchNeg	MatchNeg	
	$Match \exists$	$Match \land$	MatchWild	MatchWild	MatchWild	MatchWild	
<u>[]</u>	$Match \exists$	$Match \wedge$	Ø	Ø	MatchNil	Ø	
$\rho_1 :: \rho_2$	$Match \exists$	$Match \land$	Ø	Ø	MatchCons	Ø	

J' Soundness

J'.1 Instantiation

Lemma 85 (Soundness of Instantiation).

If
$$\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$$
 and $\hat{\alpha} \notin FV([\Gamma]\tau)$ and $[\Gamma]\tau = \tau$ and $\Delta \longrightarrow \Omega$ then $[\Omega]\hat{\alpha} = [\Omega]\tau$.

Proof. By induction on the derivation of $\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_0 \vdash \tau : \kappa}{\Gamma_0, \hat{\alpha} : \kappa, \Gamma_1 \vdash \hat{\alpha} := \tau : \kappa \dashv \underbrace{\Gamma_0, \hat{\alpha} : \kappa = \tau, \Gamma_1}_{\Lambda}} \ \, \textbf{InstSolve}$$

$$[\Delta] \hat{\alpha} = [\Delta] \tau$$
 By definition

$$\square$$
 $[\Omega] \hat{\alpha} = [\Omega] \tau$ By Lemma 29 (Substitution Monotonicity) to each side

$$\begin{array}{c} \bullet \ \ \, \mathbf{Case} \\ \frac{\widehat{\beta} \in \mathsf{unsolved}(\Gamma[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa])}{\Gamma[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa] \vdash \widehat{\alpha} := \underbrace{\widehat{\beta}}_{\tau} : \kappa \dashv \underbrace{\Gamma[\widehat{\alpha} : \kappa][\widehat{\beta} : \kappa = \widehat{\alpha}]}_{\Delta} } \ \, \mathsf{InstReach} \\ \end{array}$$

$$\begin{split} [\Delta] \widehat{\beta} &= [\Delta] \widehat{\alpha} & \text{By definition} \\ [\Omega] [\Delta] \widehat{\beta} &= [\Omega] [\Delta] \widehat{\alpha} & \text{Applying } \Omega \text{ to each side} \\ & \\ \mathbb{G} &= [\Omega] \underbrace{\widehat{\beta}} &= [\Omega] \widehat{\alpha} & \text{By Lemma 29 (Substitution Monotonicity) to each side} \end{split}$$

$$\bullet \ \, \textbf{Case} \underbrace{\frac{\Gamma'}{\Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\oplus\hat{\alpha}_2]}}_{\Gamma_0[\hat{\alpha}:\star]\vdash\hat{\alpha}:=\tau_1:\star\dashv\Theta} \underbrace{\quad \Theta\vdash\hat{\alpha}_2:=[\Theta]\tau_2:\star\dashv\Delta}_{\text{InstBin}}$$

Case

$$\frac{}{\Gamma_0[\hat{\alpha}:\mathbb{N}]\vdash\hat{\alpha}:=\mathsf{zero}:\mathbb{N}\dashv\Gamma_0[\hat{\alpha}:\mathbb{N}=\mathsf{zero}]}\;\mathsf{Inst}\mathsf{Zero}$$

Similar to the InstSolve case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma_0[\hat{\alpha}_1:\mathbb{N},\hat{\alpha}:\mathbb{N} = \mathsf{succ}(\hat{\alpha}_1)] \vdash \hat{\alpha}_1 := t_1:\mathbb{N} \dashv \Delta}{\Gamma_0[\hat{\alpha}:\mathbb{N}] \vdash \hat{\alpha} := \mathsf{succ}(t_1) : \mathbb{N} \dashv \Delta} \ \, \mathsf{InstSucc}$$

Similar to the InstBin case, but simpler.

Lemma 86 (Soundness of Checkeg).

If
$$\Gamma \vdash \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta$$
 where $\Delta \longrightarrow \Omega$ then $[\Omega]\sigma = [\Omega]t$.

Proof. By induction on the given derivation.

Case

$$\frac{}{\Gamma \vdash u \stackrel{{}_\circ}{=} u : \kappa \dashv \Gamma} \text{ CheckeqVar}$$

- $[\Omega]u = [\Omega]u$ By reflexivity of equality
- Cases CheckeqUnit, CheckeqZero: Similar to the CheckeqVar case.
- Case $\frac{\Gamma \vdash \sigma_0 \stackrel{\mathfrak{s}}{=} t_0 : \mathbb{N} \dashv \Delta}{\Gamma \vdash \mathsf{succ}(\sigma_0) \stackrel{\mathfrak{s}}{=} \mathsf{succ}(t_0) : \mathbb{N} \dashv \Delta} \ \mathsf{CheckeqSucc}$ $\Gamma \vdash \sigma_0 \stackrel{\text{e}}{=} t_0 : \mathbb{N} \dashv \Delta$ Subderivation

$$[\Omega]\sigma_0=[\Omega]t_0$$
 By i.h.

$$\begin{split} [\Omega]\sigma_0 &= [\Omega]t_0 & \text{By i.h.} \\ \text{succ(}[\Omega]\sigma_0\text{)} &= \text{succ(}[\Omega]t_0\text{)} & \text{By congruence} \end{split}$$

 $[\Omega](\mathsf{succ}(\sigma_0)) = [\Omega](\mathsf{succ}(t_0))$ By definition of substitution

$$\begin{array}{lll} \bullet & \textbf{Case} \\ & & \frac{\Gamma \vdash \sigma_0 \stackrel{\circ}{=} t_0 : \star \dashv \Theta \quad \Theta \vdash [\Theta] \sigma_1 \stackrel{\circ}{=} [\Theta] t_1 : \star \dashv \Delta}{\Gamma \vdash \sigma_0 \oplus \sigma_1 \stackrel{\circ}{=} t_0 \oplus t_1 : \star \dashv \Delta} \quad \text{CheckeqBin} \\ & & & \Gamma \vdash \sigma_0 \stackrel{\circ}{=} t_0 : \mathbb{N} \dashv \Delta \quad \text{Subderivation} \\ & & \Theta \vdash [\Theta] \sigma_1 \stackrel{\circ}{=} [\Theta] t_1 : \star \dashv \Delta \quad \text{Subderivation} \\ & & \Delta \longrightarrow \Omega \quad & \text{Given} \\ & & \Theta \longrightarrow \Delta \quad & \text{By Lemma 46 (Checkeq Extension)} \\ & & \Theta \longrightarrow \Omega \quad & \text{By Lemma 33 (Extension Transitivity)} \\ & & [\Omega] \sigma_0 = [\Omega] t_0 \quad & \text{By i.h. on first subderivation} \\ & & [\Omega] [\Theta] \sigma_1 = [\Omega] [\Theta] t_1 \quad & \text{By i.h. on second subderivation} \\ & & [\Omega] [\Theta] \sigma_1 = [\Omega] \sigma_1 \quad & \text{By Lemma 29 (Substitution Monotonicity)} \\ & & [\Omega] [\Theta] t_1 = [\Omega] t_1 \quad & \text{By Lemma 29 (Substitution Monotonicity)} \\ & & [\Omega] \sigma_0 \oplus [\Omega] \sigma_1 = [\Omega] t_0 \oplus [\Omega] t_1 \quad & \text{By transitivity of equality} \\ & & [\Omega] \sigma_0 \oplus [\Omega] \sigma_1 = [\Omega] t_0 \oplus [\Omega] t_1 \quad & \text{By congruence of equality} \\ & & [\Omega] (\sigma_0 \oplus \sigma_1) = [\Omega] (t_0 \oplus t_1) \quad & \text{By definition of substitution} \end{array}$$

• Case
$$\frac{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} := t : \kappa \dashv \Delta \qquad \hat{\alpha} \notin FV(t)}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \stackrel{\circ}{=} t : \kappa \dashv \Delta} \text{ CheckeqInstL}$$

$$\Gamma[\hat{\alpha}] \vdash \hat{\alpha} := t : \kappa \dashv \Delta \qquad \text{Subderivation}$$

$$\hat{\alpha} \notin FV(t) \qquad \text{Premise}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}:\kappa] \vdash \hat{\alpha} := \, \sigma : \kappa \dashv \Delta \qquad \hat{\alpha} \notin \mathit{FV}(t)}{\Gamma[\hat{\alpha}:\kappa] \vdash \sigma \stackrel{\circ}{=} \hat{\alpha} : \kappa \dashv \Delta} \ \, \text{CheckeqInstR}$$

Similar to the CheckeqInstL case.

 $[\Omega]\hat{\alpha} = [\Omega]t$

Lemma 87 (Soundness of Propositional Equivalence). *If* $\Gamma \vdash P \equiv Q \dashv \Delta$ *where* $\Delta \longrightarrow \Omega$ *then* $[\Omega]P = [\Omega]Q$.

Proof. By induction on the given derivation.

• Case
$$\frac{\Gamma \vdash \sigma_1 \stackrel{\circ}{=} t_1 : \mathbb{N} \dashv \Theta \qquad \Theta \vdash [\Theta] \sigma_2 \stackrel{\circ}{=} [\Theta] t_2 : \mathbb{N} \dashv \Delta}{\Gamma \vdash (\sigma_1 = \sigma_2) \equiv (t_1 = t_2) \dashv \Delta} \equiv \mathsf{PropEq}$$

$$\frac{\Delta \longrightarrow \Omega}{\Theta \longrightarrow \Delta} \qquad \qquad \mathsf{Given}$$

$$\Theta \longrightarrow \Delta \qquad \qquad \mathsf{By Lemma 46 (Checkeq Extension) (on 2nd premise)}$$

$$\Theta \longrightarrow \Omega \qquad \qquad \mathsf{By Lemma 33 (Extension Transitivity)}$$

$$\Gamma \vdash \sigma_1 \stackrel{\circ}{=} t_1 : \mathbb{N} \dashv \Theta \qquad \mathsf{Given}$$

$$[\Omega] \sigma_1 = [\Omega] t_1 \qquad \qquad \mathsf{By Lemma 86 (Soundness of Checkeq)}$$

$$\Theta \vdash [\Theta] \sigma_2 \stackrel{\circ}{=} [\Theta] t_2 : \mathbb{N} \dashv \Delta \qquad \mathsf{Given}$$

$$[\Omega] [\Theta] \sigma_2 = [\Omega] [\Theta] t_2 \qquad \qquad \mathsf{By Lemma 86 (Soundness of Checkeq)}$$

$$[\Omega] [\Theta] \sigma_2 = [\Omega] \sigma_2 \qquad \qquad \mathsf{By Lemma 29 (Substitution Monotonicity)}$$

$$[\Omega] [\Theta] t_2 = [\Omega] t_2 \qquad \qquad \mathsf{By Lemma 29 (Substitution Monotonicity)}$$

$$[\Omega] \sigma_2 = [\Omega] t_2 \qquad \qquad \mathsf{By Lemma 29 (Substitution Monotonicity)}$$

$$[\Omega] \sigma_2 = [\Omega] t_2 \qquad \qquad \mathsf{By Lemma 29 (Substitution Monotonicity)}$$

$$[\Omega] \sigma_1 = [\Omega] \sigma_2) = ([\Omega] t_1 = [\Omega] t_2) \qquad \qquad \mathsf{By congruence of equality}$$

$$[\Omega] (\sigma_1 = \sigma_2) = [\Omega] (t_1 = t_2) \qquad \qquad \mathsf{By definition of substitution}$$

$$\square$$

By Lemma 85 (Soundness of Instantiation)

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Lemma 88 (Soundness of Algorithmic Equivalence). *If* $\Gamma \vdash A \equiv B \dashv \Delta$ *where* $\Delta \longrightarrow \Omega$ *then* $[\Omega]A = [\Omega]B$.

Proof. By induction on the given derivation.

Case

$$\frac{}{\Gamma \vdash \alpha \equiv \alpha \dashv \Gamma} \equiv \!\! \mathsf{Var}$$

- \square $[\Omega]\alpha = [\Omega]\alpha$ By reflexivity of equality
- Cases ≡Exvar, ≡Unit: Similar to the ≡Var case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash A_1 \equiv B_1 \dashv \Theta \qquad \Theta \vdash [\Theta] A_2 \equiv [\Theta] B_2 \dashv \Delta}{\Gamma \vdash A_1 \oplus A_2 \equiv B_1 \oplus B_2 \dashv \Delta} \equiv \oplus$$

$$[\Omega]A_2 = [\Omega]B_2$$
 By Lemma 29 (Substitution Monotonicity)

 $([\Omega]A_1) \oplus ([\Omega]A_2) = ([\Omega]B_1) \oplus ([\Omega]B_2)$ By above equations

$$\begin{array}{ll} \Gamma\!,\alpha:\kappa\vdash A_0\equiv B_0\dashv \Delta,\alpha:\kappa,\Delta' & \text{Subderivation} \\ \Delta\longrightarrow\Omega & \text{Given} \end{array}$$

$$\Gamma$$
, α : κ, · \longrightarrow Δ, α : κ, Δ' By Lemma 49 (Equivalence Extension)

П

• Case
$$\frac{\Gamma \vdash P \equiv Q \dashv \Theta \qquad \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta}{\Gamma \vdash P \supset A_0 \equiv Q \supset B_0 \dashv \Delta} \equiv \supset$$

$$\begin{array}{lll} \Delta \longrightarrow \Omega & \text{Given} \\ & \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta & \text{Subderivation} \\ & \Theta \longrightarrow \Delta & \text{By Lemma 49 (Equivalence Extension)} \\ & \Theta \longrightarrow \Omega & \text{By Lemma 33 (Extension Transitivity)} \\ & \Gamma \vdash P \equiv Q \dashv \Theta & \text{Subderivation} \\ & [\Omega] P = [\Omega] Q & \text{By Lemma 87 (Soundness of Propositional Equivalence)} \\ & \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta & \text{Subderivation} \\ & [\Omega] [\Theta] A_0 = [\Omega] [\Theta] B_0 & \text{By i.h.} \\ & [\Omega] A_0 = [\Omega] B_0 & \text{By Lemma 29 (Substitution Monotonicity)} \end{array}$$

• Case
$$\frac{\Gamma \vdash P \equiv Q \dashv \Theta \qquad \Theta \vdash [\Theta] A_0 \equiv [\Theta] B_0 \dashv \Delta}{\Gamma \vdash A_0 \land P \equiv B_0 \land Q \dashv \Delta} \equiv \land$$

Similar to the $\equiv \supset$ case.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} := \tau : \star \dashv \Delta \qquad \hat{\alpha} \not \in \mathit{FV}(\tau)}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \equiv \underbrace{\tau}_A \dashv \Delta} \equiv \mathsf{InstantiateL}$$

$$\Gamma[\hat{\alpha}] \vdash \hat{\alpha} := \tau : \star \dashv \Delta \quad \text{Subderivation}$$

$$[\Omega] \hat{\alpha} = [\Omega] \tau \qquad \qquad \text{By Lemma 85 (Soundness of Instantiation)}$$

• Case ≡InstantiateR: Similar to the ≡InstantiateL case.

J'.2 Soundness of Checkprop

Lemma 89 (Soundness of Checkprop). *If* $\Gamma \vdash P$ *true* $\neg \Delta$ *and* $\Delta \longrightarrow \Omega$ *then* $\Psi \vdash [\Omega]P$ *true*.

Proof. By induction on the derivation of $\Gamma \vdash P$ *true* $\dashv \Delta$.

• Case
$$\frac{\Gamma \vdash \sigma \stackrel{\circ}{=} t : \mathbb{N} \dashv \Delta}{\Gamma \vdash \sigma \stackrel{\circ}{=} t : \mathbb{N} \dashv \Delta} \text{ CheckpropEq}$$

$$\Gamma \vdash \sigma \stackrel{\circ}{=} t : \mathbb{N} \dashv \Delta \qquad \text{Subderivation}$$

$$[\Omega]\sigma = [\Omega]t \qquad \qquad \text{By Lemma 86 (Soundness of Checkeq)}$$

$$\Psi \vdash [\Omega]\sigma = [\Omega]t \ \textit{true} \qquad \text{By DeclCheckpropEq}$$

$$\Psi \vdash [\Omega](\sigma = t) \ \textit{true} \qquad \text{By def. of subst.}$$

J'.3 Soundness of Eliminations (Equality and Proposition)

Lemma 90 (Soundness of Equality Elimination).

 $\Psi \vdash [\Omega] P true$

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If $[\Gamma]\sigma = \sigma$ and $[\Gamma]t = t$ and $\Gamma \vdash \sigma$: κ and $\Gamma \vdash t$: κ and $FEV(\sigma) \cup FEV(t) = \emptyset$, then:

By $P = (\sigma = t)$

- (1) If $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta$ then $\Delta = (\Gamma, \Theta)$ where $\Theta = (\alpha_1 = t_1, \dots, \alpha_n = t_n)$ and for all Ω such that $\Gamma \longrightarrow \Omega$ and all t' such that $\Omega \vdash t' : \kappa'$, it is the case that $[\Omega, \Theta]t' = [\theta][\Omega]t'$, where $\theta = \mathsf{mgu}(\sigma, t)$.
- (2) If $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot$ then $\mathsf{mgu}(\sigma, t) = \bot$ (that is, no most general unifier exists).

Proof. First, we need to recall a few properties of term unification.

- (i) If σ is a term, then $mgu(\sigma, \sigma) = id$.
- (ii) If f is a unary constructor, then $mgu(f(\sigma), f(t)) = mgu(\sigma, t)$, supposing that $mgu(\sigma, t)$ exists.
- (iii) If f is a binary constructor, and $\sigma = \mathsf{mgu}(f(\sigma_1, \sigma_2), f(t_1, t_2))$ and $\sigma_1 = \mathsf{mgu}(\sigma_1, t_1)$ and $\sigma_2 = \mathsf{mgu}([\sigma_1]\sigma_2, [\sigma_1]t_2)$, then $\sigma = \sigma_2 \circ \sigma_1 = \sigma_1 \circ \sigma_2$.
- (iv) If $\alpha \notin FV(t)$, then $mgu(\alpha, t) = (\alpha = t)$.
- (v) If f is an n-ary constructor, and σ_i and t_i (for $i \le n$) have no unifier, then $f(\sigma_1, \ldots, \sigma_n)$ and $f(t_1, \ldots, t_n)$ have no unifier.

We proceed by induction on the derivation of Γ / $\sigma \stackrel{\circ}{=} t : \kappa \dashv \Delta^{\perp}$, proving both parts with a single induction.

Case

$$\frac{}{\Gamma \mathrel{/} \alpha \triangleq \alpha \mathrel{:} \kappa \dashv \Gamma} \; \mathsf{ElimeqUvarRefl}$$

Here we have $\Delta = \Gamma$, so we are in part (1). Let $\theta = id$ (which is $mgu(\sigma, \sigma)$). We can easily show $[id][\Omega]\alpha = [\Omega, \alpha] = [\Omega, \cdot]\alpha$.

Case

$$\overline{\Gamma \ / \ \mathsf{zero} \stackrel{\circ}{=} \mathsf{zero} : \mathbb{N} \dashv \Gamma} \ \mathsf{ElimeqZero}$$

Similar to the ElimeqUvarRefl case.

$$\begin{array}{c} \bullet \ \, \mathsf{Case} \\ \\ \frac{\Gamma \, / \, \mathsf{t}_1 \, \stackrel{\circ}{=} \, \mathsf{t}_2 : \mathbb{N} \, \dashv \Delta^{\perp}}{\Gamma \, / \, \mathsf{succ}(\mathsf{t}_1) \, \stackrel{\circ}{=} \, \mathsf{succ}(\mathsf{t}_2) : \mathbb{N} \, \dashv \Delta^{\perp}} \ \, \mathsf{ElimeqSucc} \\ \end{array}$$

We distinguish two subcases:

– Case $\Delta^{\perp} = \Delta$:

Since we have the same output context in the conclusion and premise, the "for all t'..." part follows immediately from the i.h. (1).

The i.h. also gives us $\theta_0 = mgu(t_1, t_2)$.

Let $\theta = \theta_0$. By property (ii), $mgu(t_1, t_2) = mgu(succ(t_1), succ(t_2)) = \theta$.

- Case $\Delta^{\perp} = \bot$:

$$\begin{array}{ccc} & \Gamma \ / \ t_1 \stackrel{\circ}{=} t_2 : \mathbb{N} \ \dashv \bot & Subderivation \\ & \mathsf{mgu}(t_1,t_2) = \bot & \mathsf{By i.h. (2)} \\ & & \mathsf{mgu}(\mathsf{succ}(t_1),\mathsf{succ}(t_2)) = \bot & \mathsf{By contrapositive of property (ii)} \end{array}$$

 $\bullet \ \, \textbf{Case} \ \, \frac{\alpha \notin \mathit{FV}(t) \qquad (\alpha \! = \! -) \notin \Gamma}{\Gamma \; / \; \alpha \stackrel{\circ}{=} t : \kappa \dashv \Gamma, \alpha \! = \! t} \; \mathsf{ElimeqUvarL}$

Here $\Delta \neq \bot$, so we are in part (1).

$$\begin{split} [\Omega,\alpha=t]t' &= \begin{bmatrix} [\Omega]t/\alpha \end{bmatrix}[\Omega]t' & \text{By a property of substitution} \\ &= [\Omega][t/\alpha][\Omega]t' & \text{By a property of substitution} \\ &= [\Omega][\theta][\Omega]t' & \text{By mgu}(\alpha,t) = (\alpha/t) \\ &= [\theta][\Omega]t' & \text{By a property of substitution (θ creates no evars)} \end{split}$$

• Case $\frac{\alpha \notin \mathit{FV}(\mathsf{t}) \qquad (\alpha \! = \! -) \notin \Gamma}{\Gamma \ / \ \mathsf{t} \stackrel{\circ}{=} \alpha : \kappa \dashv \Gamma, \alpha \! = \! \mathsf{t}} \ \mathsf{ElimeqUvarR}$

Similar to the ElimeqUvarL case.

Case

$$\frac{}{\Gamma \mathrel{/} 1 \stackrel{\circ}{=} 1 : \star \dashv \Gamma} \mathsf{ElimeqUnit}$$

Similar to the ElimegUvarRefl case.

Either Δ^{\perp} is some Δ , or it is \perp .

- Case
$$\Delta^{\perp} = \Delta$$
:

Suppose $\Omega \vdash \mathfrak{u} : \kappa'$.

$$\begin{split} [\Omega, \Delta_1, \Delta_2] \mathfrak{u} &= [\theta_2] [\Omega, \Delta_1] \mathfrak{u} & \text{By (IH-2nd), with } \mathfrak{u}_2 = \mathfrak{u} \\ &= [\theta_2] [\theta_1] [\Omega] \mathfrak{u} & \text{By (IH-1st), with } \mathfrak{u}_1 = \mathfrak{u} \\ &= [\Omega] [\theta_2 \circ \theta_1] \mathfrak{u} & \text{By a property of substitution} \end{split}$$

- Case $\Delta^{\perp} = \bot$:

Use the i.h. (2) on the second premise to show $\mathsf{mgu}(\tau_2,\tau_2') = \bot$, then use property (v) of unification to show $\mathsf{mgu}((\tau_1 \oplus \tau_2), (\tau_1' \oplus \tau_2')) = \bot$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \, / \, \tau_1 \, \stackrel{\circ}{=} \, \tau_1' : \star \dashv \bot}{\Gamma \, / \, \tau_1 \oplus \tau_2 \, \stackrel{\circ}{=} \, \tau_1' \oplus \tau_2' : \star \dashv \bot} \, \, \text{ElimeqBinBot}$$

Similar to the \perp subcase for ElimeqSucc, but using property (v) instead of property (ii).

• Case
$$\frac{\sigma \ \# \ t}{\Gamma \ / \ \sigma \ \mathring{=} \ t : \kappa \dashv \bot} \ \mathsf{ElimeqClash}$$

Since $\sigma \# t$, we know σ and t have different head constructors, and thus no unifier. \square

Theorem 6 (Soundness of Algorithmic Subtyping).

If $[\Gamma]A = A$ and $[\Gamma]B = B$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $\Delta \longrightarrow \Omega$ and $\Gamma \vdash A <: \mathcal{P} B \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]A \leq^{\mathcal{P}} [\Omega]B$.

Proof. By induction on the given derivation.

$$\bullet \ \, \textbf{Case} \ \, \frac{ B \ \, \text{not headed by} \ \, \forall \qquad \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\hat{\alpha}/\alpha] A_0 <: ^- B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta }{ \Gamma \vdash \forall \alpha : \kappa. \, A_0 <: ^- B \dashv \Delta } <: \forall L$$

Let
$$\Omega' = (\Omega, |\triangleright_{\hat{\alpha}}, \Theta|)$$
.

$$\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\hat{\alpha}/\alpha] A_0 <: ^- B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta$$
 Subderivation
$$\Delta \longrightarrow \Omega$$
 Given

$$(\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \longrightarrow \Omega'$$
 By Lemma 25 (Filling Completes)
$$\Gamma \vdash \forall \alpha : \kappa. \ A_0 \ type$$
 Given

$$\Gamma, \alpha : \kappa \vdash A_0 \ type \qquad \qquad \text{By inversion (ForallWF)} \\ \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\hat{\alpha}/\alpha]A_0 \ type \qquad \qquad \text{By a property of substitution} \\ \Gamma \vdash B \ type \qquad \qquad \text{Given}$$

$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash [\Omega'][\hat{\alpha}/\alpha] A_0 \leq^- [\Omega'] B \qquad \text{By i.h.}$$

$$\Omega \vdash B \ type \qquad \qquad \text{By Lemma 36 (Extension Weakening (Sorts))}$$

$$[\Omega']B = [\Omega]B$$
 By Lemma 17 (Substitution Stability)
$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash [\Omega'][\hat{\alpha}/\alpha]A_0 \leq^- [\Omega]B$$
 By above equality

$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash \left[[\Omega'] \hat{\alpha} / \alpha \right] [\Omega'] A_0 \leq^- [\Omega] B \qquad \text{By distributivity of substitution}$$

$$\begin{array}{ccc} \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash \hat{\alpha} : \kappa & \text{By VarSort} \\ \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta & \text{By Lemma 50 (Subtyping Extension)} \\ \Theta \text{ is soft} & \text{By Lemma 22 (Extension Inversion) (ii)} \end{array}$$

 $\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \vdash \hat{\alpha} : \kappa$ By Lemma 36 (Extension Weakening (Sorts))

 $\begin{array}{ll} (\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \longrightarrow \Omega' & \text{Above} \\ [\Omega']\Omega' \vdash [\Omega']\hat{\alpha} : \kappa & \text{By Lemma 14 (Substitution for Sorting)} \\ [\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash [\Omega']\hat{\alpha} : \kappa & \text{By Lemma 54 (Completing Stability)} \end{array}$

$$\begin{split} [\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash \forall \alpha : \kappa. & [\Omega']A_0 \leq^- [\Omega]B & \text{By } \leq \forall L \\ [\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash \forall \alpha : \kappa. & [\Omega, \alpha : \kappa]A_0 \leq^- [\Omega]B & \text{By Lemma 17 (Substitution Stability)} \\ & [\Omega]\Delta \vdash \forall \alpha : \kappa. & [\Omega, \alpha : \kappa]A_0 \leq^- [\Omega]B & \text{By Lemma 52 (Context Partitioning)} + \text{thinning} \\ & [\Omega]\Delta \vdash \forall \alpha : \kappa. & [\Omega]A_0 \leq^- [\Omega]B & \text{By def. of substitution} \\ & [\Omega]\Delta \vdash [\Omega](\forall \alpha : \kappa. A_0) \leq^- [\Omega]B & \text{By def. of substitution} \end{split}$$

• Case $<:\exists R$: Similar to the $<:\forall L$ case.

$$\begin{array}{c} \bullet \;\; \textbf{Case} \\ \frac{\Gamma, \, \beta : \kappa \vdash A <: ^{-} \, B_{0} \dashv \Delta, \, \beta : \kappa, \Theta}{\Gamma \vdash A <: ^{-} \, \forall \beta : \kappa. \, B_{0} \dashv \Delta} <: \forall R \end{array}$$

• Case $<:\exists L:$ Similar to the $<:\forall R$ case.

• Case
$$\frac{\Gamma \vdash A \equiv B \dashv \Delta}{\Gamma \vdash A <: \mathcal{P} B \dashv \Delta} <: \mathsf{Equiv}$$

$$\Gamma \vdash A <: ^{\mathcal{P}} B \dashv \Delta$$

$$\Gamma \vdash A \equiv B \dashv \Delta$$

$$\Delta \longrightarrow \Omega$$

$$[\Omega]A = [\Omega]B$$

$$\Gamma \longrightarrow \Delta$$

$$\Gamma \vdash A type$$

$$[\Omega]\Omega \vdash [\Omega]A type$$

$$[\Omega]\Delta \vdash [\Omega]A type$$

$$[\Omega]A type$$

$$[\Omega]A \vdash [\Omega]A type$$

$$[\Omega]A ty$$

Case

$$\frac{\Gamma \vdash A <: ^- B \dashv \Delta \quad nonpos(B)}{\Gamma \vdash A <: ^+ B \dashv \Delta} <: ^+_+ L$$

$$\Gamma \vdash A <: ^- B \dashv \Delta \quad \text{By inversion}$$

$$neg(A) \quad \text{By inversion}$$

$$nonpos(B) \quad \text{By inversion}$$

$$nonpos(A) \quad \text{since } neg(A)$$

$$[\Omega]\Gamma \vdash [\Omega]A \leq^- [\Omega]B \quad \text{By induction}$$

$$[\Omega]\Gamma \vdash [\Omega]A \leq^+ [\Omega]B \quad \text{By} \leq^+_+$$

Case

$$\frac{\textit{nonpos}(A)}{\Gamma \vdash A <:^{-} B \dashv \Delta} \frac{\textit{nonpos}(A)}{\textit{neg}(B)} <:^{-}_{+} R$$

Similar to the <: $_{+}^{-}L$ case.

Case

$$\frac{\Gamma \vdash A <: ^{+} B \dashv \Delta \quad \begin{array}{c} \textit{pos}(A) \\ \textit{nonneg}(B) \\ \Gamma \vdash A <: ^{-} B \dashv \Delta \end{array} <: ^{+} L$$

Similar to the $<: _{\perp}^{-}L$ case.

Case

$$\frac{\textit{nonneg}(A)}{\Gamma \vdash A <: ^{+}B \dashv \Delta \qquad \textit{pos}(B)} <: \dot{_}R$$

Similar to the $<: _{\perp}^{-}L$ case.

J'.4 Soundness of Typing

Theorem 7 (Soundness of Match Coverage).

- 1. If $\Gamma \vdash \Pi$ covers \vec{A} q and $\Gamma \vdash \vec{A}$ q types and $[\Gamma]\vec{A} = \vec{A}$ and $\Gamma \longrightarrow \Omega$ then $[\Omega]\Gamma \vdash \Pi$ covers \vec{A} q.
- 2. If $\Gamma / P \vdash \Pi$ covers \vec{A} ! and $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash \vec{A}$! types and $[\Gamma]\vec{A} = \vec{A}$ and $[\Gamma]P = P$ then $[\Omega]\Gamma / P \vdash$ Π covers \vec{A} !.

Proof. By mutual induction on the given algorithmic coverage derivation.

1. Case

$$\frac{}{\cdot \Rightarrow e_1 \text{ I } \ldots \vdash \cdot \textit{covers } \Gamma} \text{ CoversEmpty}$$

$$[\Omega] \Gamma \vdash \cdot \Rightarrow e_1 \text{ I } \ldots \textit{covers} \cdot \quad \text{By DeclCoversEmpty}$$

- Cases CoversVar, Covers1, Covers×, Covers+, Covers∃, Covers√, CoversVec, Covers∧ ¼, CoversVec ½:
- Use the i.h. and apply the corresponding declarative rule.

2.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \ / \ [\Gamma]t_1 \ \mathring{=} \ [\Gamma]t_2 : \kappa \dashv \bot}{\Gamma \ / \ t_1 = t_2 \vdash \Pi \ \textit{covers} \ \vec{A} \ !} \ \, \text{CoversEqBot}$$

$$\Gamma / [\Gamma]t_1 \stackrel{\text{\tiny $}}{=} [\Gamma]t_2 : \kappa \dashv \bot$$
 Subderivation

$$mgu([\Gamma]t_1, [\Gamma]t_2) = \bot$$
 By Lemma 90 (Soundness of Equality Elimination) (2) $mgu(t_1, t_2) = \bot$ By given equality

$$\square$$
 $[\Omega]\Gamma / t_1 = t_2 \vdash \Pi \text{ covers } \vec{A}$ By DeclCoversEqBot

Lemma 91 (Well-formedness of Algorithmic Typing).

Given Γ ctx:

(i) If
$$\Gamma \vdash e \Rightarrow A p \dashv \Delta$$
 then $\Delta \vdash A p$ type.

(ii) If
$$\Gamma \vdash s : A p \gg B q \dashv \Delta$$
 and $\Gamma \vdash A p$ type then $\Delta \vdash B q$ type.

Proof. 1. Suppose $\Gamma \vdash e \Rightarrow A p \dashv \Delta$:

• Case
$$\frac{(x:A\ p)\in \Gamma}{\Gamma\vdash x\Rightarrow [\Gamma]A\ p\dashv \Gamma}\ \mathsf{Var}$$

$$\Gamma = (\Gamma_0, x : A p, \Gamma_1) \quad (x : A p) \in \Gamma$$

$$\Gamma \vdash A p type \qquad Follows from \Gamma ctx$$

• Case
$$\frac{\Gamma \vdash A! \; type \qquad \Gamma \vdash e \Leftarrow [\Gamma]A \; ! \; \dashv \Delta}{\Gamma \vdash (e : A) \Rightarrow [\Delta]A \; ! \; \dashv \Delta} \; \mathsf{Anno}$$

$$\Gamma \vdash A ! type$$
 By inversion

$$\Gamma \longrightarrow \Delta$$
 By Lemma 51 (Typing Extension)

$$\begin{array}{ll} \Gamma \vdash A \ ! \ type & \text{By inversion} \\ \Gamma \longrightarrow \Delta & \text{By Lemma 51 (Typing Extension)} \\ \Delta \vdash A \ ! \ type & \text{By Lemma 41 (Extension Weakening for Principal Typing)} \end{array}$$

$$\Delta \vdash [\Delta]A$$
! *type* By Lemma 39 (Principal Agreement) (i)

Case

$$\frac{\Gamma \vdash e \Rightarrow A \ p \dashv \Theta \qquad \Theta \vdash s : [\Theta] A \ p \gg C \ q \dashv \Delta \qquad \begin{array}{c} p = \cancel{y} \ \text{or} \ q = \cancel{!} \\ \text{or} \ \mathsf{FEV}([\Delta]C) \neq \emptyset \\ \hline \Gamma \vdash e \ s \Rightarrow C \ q \dashv \Delta \end{array} \rightarrow \mathsf{E}$$

$$\begin{array}{lll}
\Gamma \vdash e \Rightarrow A \ p \dashv \Theta & \text{By inversion} \\
\Theta \vdash A \ p \ type & \text{By induction} \\
\Theta \vdash [\Theta] A \ p \ type & \text{By Lemma 40} \\
\Theta \ ctx & \text{By implicit as}
\end{array}$$

$$\Theta \vdash [\Theta] A p \text{ type}$$
 By Lemma 40 (Right-Hand Subst. for Principal Typing)

$$\Theta$$
 ctx By implicit assumption

$$\Theta \vdash s : [\Theta] \land p \gg C \neq \Delta$$
 By inversion

$$\triangle \vdash C$$
 q *type* By mutual induction

2. Suppose $\Gamma \vdash s : A p \gg B q \dashv \Delta$ and $\Gamma \vdash A p$ type:

• Case

3

$$\frac{}{\Gamma \vdash \cdot : A p \gg A p \dashv \Gamma} \text{ EmptySpine}$$

$$\Gamma \vdash A p \text{ type} \quad \text{Given}$$

 $\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash e \Leftarrow A \ \mathfrak{p} \dashv \Theta \qquad \Theta \vdash s : [\Theta] B \ \mathfrak{p} \gg C \ q \dashv \Delta}{\Gamma \vdash e \ s : A \to B \ \mathfrak{p} \gg C \ q \dashv \Delta} \to \mathsf{Spine}$

 $\Gamma \vdash A \rightarrow B \ p \ type$ Given

By Lemma 42 (Inversion of Principal Typing) $\Gamma \vdash B p type$

By Lemma 41 (Extension Weakening for Principal Typing) $\Theta \vdash B p type$ $\Theta \vdash [\Theta] B p type$ By Lemma 40 (Right-Hand Subst. for Principal Typing)

By induction $\Delta \vdash C \neq type$

 $\begin{array}{c} \bullet \ \, \textbf{Case} \\ \frac{\Gamma, \, \hat{\alpha} : \kappa \vdash e \; s : [\hat{\alpha}/\alpha] A \; \gg C \; q \dashv \Delta}{\Gamma \vdash e \; s : \forall \alpha : \kappa. \, A \; p \gg C \; q \dashv \Delta} \; \forall \mathsf{Spine} \\ \end{array}$

 $\Gamma \vdash \forall \alpha : \kappa. \ A \ p \ type \ Given$ $\Gamma \vdash \forall \alpha : \kappa$. A *type* By inversion $\begin{array}{ccc} \Gamma, \alpha: \kappa \vdash A \ type & \text{By inversion} \\ \Gamma, \hat{\alpha}: \kappa, \alpha: \kappa \vdash A \ type & \text{By weakening} \\ \Gamma, \hat{\alpha}: \kappa \vdash [\hat{\alpha}/\alpha]A \ type & \text{By substitution} \\ \Delta \vdash C \ q \ type & \text{By induction} \end{array}$

 $\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash P \ \textit{true} \dashv \Theta \qquad \Theta \vdash e \ s : [\Theta] A \ \mathfrak{p} \gg C \ \mathfrak{q} \dashv \Delta}{\Gamma \vdash e \ s : P \supset A \ \mathfrak{p} \gg C \ \mathfrak{q} \dashv \Delta} \supset \\ \mathsf{Spine}$

 $\Gamma \vdash P \supset A \ p \ type$ Given

 $\Gamma \vdash P \ prop$ By Lemma 42 (Inversion of Principal Typing)

 $\Gamma \vdash A p type$

By Lemma 47 (Checkprop Extension)

 $\Gamma \longrightarrow \Theta$ $\Theta \vdash A \mathfrak{p} type$ By Lemma 41 (Extension Weakening for Principal Typing) $\Theta \vdash [\Theta] A p type$ By Lemma 40 (Right-Hand Subst. for Principal Typing)

 $\Delta \vdash C \neq type$ By induction **3**

 $\frac{\overbrace{\Gamma[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]}^{\Theta}\vdash e\ s:(\hat{\alpha}_1\rightarrow\hat{\alpha}_2)\ \gg C\ \dashv \Delta}{\Gamma[\hat{\alpha}:\star]\vdash e\ s:\hat{\alpha}\ \gg C\ \dashv \Delta}\ \hat{\alpha} \mathsf{Spine}$ Case

 $\Theta \vdash \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \ type$ By rules $\triangle \vdash C$ q type By induction

Theorem 8 (Eagerness of Types).

(i) If \mathcal{D} derives $\Gamma \vdash e \Leftarrow A p \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = [\Gamma]A$ then \mathcal{D} is eager.

- (ii) If \mathcal{D} derives $\Gamma \vdash e \Rightarrow A p \dashv \Delta$ then \mathcal{D} is eager.
- (iii) If \mathcal{D} derives $\Gamma \vdash s : A p \gg B q \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = [\Gamma]A$ then \mathcal{D} is eager.
- (iv) If \mathcal{D} derives $\Gamma \vdash s : A p \gg B \lceil q \rceil \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = \lceil \Gamma \rceil A$ then \mathcal{D} is eager.
- (vi) If $\mathcal D$ derives $\Gamma \ / \ P \vdash \Pi :: \vec A \ ! \Leftarrow C \ p \dashv \Delta$ and $\Gamma \vdash P$ prop and $\mathsf{FEV}(P) = \emptyset$ and $[\Gamma]P = P$ and $\Gamma \vdash \vec A \ !$ types and $\Gamma \vdash C \ p$ type then $\mathcal D$ is eager.

Proof. By induction on the given derivation.

Part (i), checking

- Case Rec: By i.h. (i).
- Case Sub: By i.h. (ii) and (i).
- **Case** ∀I, ∃I: By i.h. (i).
- Case **△**I:

Substitution is idempotent, so in the last premise $[\Theta][\Theta]A_0 = [\Theta]A_0$ and we can use the i.h. (i).

- Case \supset I: Similar to the \land I case.
- Case ⊃I⊥: This rule has no subderivations of the relevant form, so the case is trivial.
- Case \rightarrow I: By i.h. (i).
- Case $\rightarrow |\hat{\alpha}|$:

In the premise, $\lceil \Gamma_0[\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1\rightarrow\hat{\alpha}_2], x:\hat{\alpha}_1 \rceil = \hat{\alpha}_2$ so we can use the i.h. (i).

- Case $+I_k$: By i.h. (i).
- Case $+I\hat{\alpha}_k$: Similar to the $\rightarrow I\hat{\alpha}$ case.
- Case ×I:

By i.h. (i) on the first subderivation, then i.h. (i) on the second subderivation (using the fact that $[\Theta][\Theta]A_2 = [\Theta]A_2$).

- Case $\times I \hat{\alpha}$: Similar to the $\to I \hat{\alpha}$ case.
- Case Nil: This rule has no subderivations of the relevant form, so the case is trivial.
- Case Cons:

By i.h. (i) on the subderivations typing e_1 and e_2 , using $[\Gamma'][\Gamma']A_0 = [\Gamma']A_0$ and $[\Theta][\Theta](\text{Vec } \hat{\alpha} A_0) = [\Theta](\text{Vec } \hat{\alpha} A_0)$.

$$\begin{array}{c} \bullet \;\; \textbf{Case} \\ & \Theta \vdash \Pi :: [\Theta] B \; q \Leftarrow [\Theta] A \; p \dashv \Delta \\ \hline \frac{\Gamma \vdash e \Rightarrow B \; q \dashv \Theta \qquad \Delta \vdash \Pi \; covers \; [\Delta] B \; q}{\Gamma \vdash \mathsf{case}(e,\Pi) \Leftarrow A \; p \dashv \Delta} \;\; \mathsf{Case} \end{array}$$

By Definition 8, the given derivation is eager.

Part (ii), synthesis

• Case Var: Substitution is idempotent: $[\Gamma][\Gamma]A_0 = [\Gamma]A_0$.

By inversion, $\Delta = \Gamma$ and $A = [\Gamma]A_0$ where $(x : A_0p) \in \Gamma$.

Using the above equations, we have

$$[\Gamma][\Gamma]A_0 = [\Gamma]A_0$$
$$[\Gamma]A = A$$
$$[\Delta]A = A$$

This rule has no subderivations, so there is nothing else to show.

• **Case** Anno: By inversion, $A = [\Delta]A_0$.

Substitution is idempotent, so $[\Gamma][\Gamma]A_0 = [\Gamma]A_0$ and we can use the i.h. (i) to show that the checking subderivation is eager.

The type in the conclusion is $[\Delta]A_0$, which by idempotence is equal to $[\Delta][\Delta]A_0$. Since $A = [\Delta]A_0$, we have $A = [\Delta]A$.

• Case
$$\frac{\Gamma \vdash e \Rightarrow B \ p \dashv \Theta \qquad \Theta \vdash s : B \ p \gg A \ \lceil q \rceil \dashv \Delta}{\Gamma \vdash e \ s \Rightarrow A \ q \dashv \Delta} \rightarrow \!\! E$$

$$\begin{array}{ccc} \mathcal{D}_1 :: & \Gamma \vdash e \Rightarrow B \ p \dashv \Theta & Subderivation \\ & B = [\Theta]B \ and \ \mathcal{D}_1 \ eager & By \ i.h. \ (ii) \ on \ \mathcal{D}_1 \end{array}$$

$$\begin{array}{ccc} \mathcal{D}_2 :: & \Theta \vdash s : B \; p \gg A \; \lceil q \rceil \dashv \Delta & \text{Subderivation} \\ & B = [\Theta]B & \text{Above} \\ & A = [\Theta]A \; \text{and} \; \mathcal{D}_2 \; \text{eager} & \text{By i.h. (iv) on} \; \mathcal{D}_2 \end{array}$$

$$A = [\Theta]A$$
 Above
$$\mathcal{D}_1 \text{ eager}$$
 Above
$$\mathcal{D}_2 \text{ eager}$$
 Above

Parts (iii) and (iv), spines

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma, \hat{\alpha} : \kappa \vdash e \ s_0 : [\hat{\alpha}/\alpha] A_0 \ \cancel{Y} \gg C \ q \dashv \Delta}{\Gamma \vdash e \ s_0 : \forall \alpha : \kappa. \ A_0 \ p \gg C \ q \dashv \Delta} \ \, \forall \mathsf{Spine}$$

It is given that $[\Gamma](\forall \alpha : \kappa. A_0) = (\forall \alpha : \kappa. A_0)$.

Therefore, $[\Gamma]A_0 = A_0$.

Since $\hat{\alpha}$ is not solved in Γ , $\hat{\alpha}$: κ , we also have

$$[\Gamma, \hat{\alpha} : \kappa][\hat{\alpha}/\alpha]A_0 = [\hat{\alpha}/\alpha]A_0$$

By i.h., $C = [\Delta]C$ and all subderivations are eager. Since the output type and output context of the conclusion are C and Δ , the same as the premise, we have $C = [\Delta]C$.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash P \ \textit{true} \dashv \Theta \qquad \Theta \vdash e \ s_0 : [\Theta] A_0 \ p \gg C \ q \dashv \Delta}{\Gamma \vdash e \ s_0 : P \supset A_0 \ p \gg C \ q \dashv \Delta} \supset \\ \, \textbf{Spine}$$

Substitution is idempotent, so $[\Theta][\Theta]A_0 = [\Theta]A_0$, and we can apply the i.h. showing $C = [\Delta]C$ and that all subderivations are eager. Since the output type and output context of the conclusion are C and Δ , the same as the premise, we have $C = [\Delta]C$.

- Case SpineRecover: By i.h. (iii).
- Case SpinePass: By i.h. (iii).
- Case

$$\overline{\Gamma \vdash \cdot : A p \gg \underbrace{A}_{C} \underbrace{p}_{q} \dashv \underbrace{\Gamma}_{\Delta} } \text{ EmptySpine}$$

We have $[\Gamma]A = A$. Since C = A, we also have $[\Gamma]C = C$; since $\Gamma = \Delta$, we also have $[\Delta]C = C$, which was to be shown.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash e \Leftarrow A_1 \ p \dashv \Theta \qquad \Theta \vdash s : [\Theta] A_2 \ p \gg C \ q \dashv \Delta}{\Gamma \vdash e \ s : A_1 \longrightarrow A_2 \ p \gg C \ q \dashv \Delta} \rightarrow \, \textbf{Spine}$$

We have $[\Gamma](A_1 \to A_2) = A_1 \to A_2$. Therefore, $[\Gamma]A_1 = A_1$. By i.h. on the first subderivation, its subderivations are eager.

Substitution is idempotent, so $[\Theta][\Theta]A_2 = [\Theta]A_2$. By i.h. on the second subderivation, $[\Delta]C = C$ (and its subderivations are eager).

Since the output type and output context of the conclusion are C and Δ , the same as the premise, we have $C = [\Delta]C$; we also showed that all subderivations are eager.

• Case
$$\frac{\Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\to\hat{\alpha}_2]\vdash e\ s_0:(\hat{\alpha}_1\to\hat{\alpha}_2)\ \gg C\ \dashv \Delta}{\Gamma_0[\hat{\alpha}:\star]\vdash e\ s_0:\hat{\alpha}\ \gg C\ \dashv \Delta}\ \text{$\hat{\alpha}$Spine}$$

By definition of substitution,

$$\left[\Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\rightarrow\hat{\alpha}_2]\right](\hat{\alpha}_1\rightarrow\hat{\alpha}_2)=(\hat{\alpha}_1\rightarrow\hat{\alpha}_2)$$

Therefore, we can apply the i.h.

Since the output type and output context of the conclusion are C and Δ , the same as the premise, we have $C = [\Delta]C$; we also showed that all subderivations are eager.

Parts (v) and (vi), pattern matching

Part (v), rules MatchEmpty, etc.: By i.h. (v) and, in MatchBase, i.h. (i). MatchSeq: By i.h. (v), using idempotency of substitution for \vec{A} .

Part (vi), rule Match \perp : trivial. Part (vi), rule MatchUnify: by the assumption $\Gamma \vdash \vec{A}$! types, the vector \vec{A} has no existential variables at all, so in the second premise, $\vec{A} = [\Gamma]\vec{A}$ and we can apply the i.h. (v).

Theorem 9 (Soundness of Algorithmic Typing).

Given $\Delta \longrightarrow \Omega$:

- (i) If $\Gamma \vdash e \Leftarrow A p \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = [\Gamma]A$ then $[\Omega]\Delta \vdash [\Omega]e \Leftarrow [\Omega]A p$.
- (ii) If $\Gamma \vdash e \Rightarrow A p \dashv \Delta$ then $[\Omega]\Delta \vdash [\Omega]e \Rightarrow [\Omega]A p$.
- (iii) If $\Gamma \vdash s : A p \gg B q \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = [\Gamma]A$ then $[\Omega]\Delta \vdash [\Omega]s : [\Omega]A p \gg [\Omega]B q$.
- (iv) If $\Gamma \vdash s : A p \gg B \lceil q \rceil \dashv \Delta$ and $\Gamma \vdash A p$ type and $A = \lceil \Gamma \rceil A$ then $\lceil \Omega \rceil \Delta \vdash \lceil \Omega \rceil s : \lceil \Omega \rceil A p \gg \lceil \Omega \rceil B \lceil q \rceil$.
- (v) If $\Gamma \vdash \Pi :: \vec{A} \neq C p \dashv \Delta$ and $\Gamma \vdash \vec{A} !$ types and $[\Gamma]\vec{A} = \vec{A}$ and $\Gamma \vdash C p$ type then $p \vdash [\Omega]\Delta :: [\Omega]\Pi ! \Leftarrow [\Omega]\vec{A} \neq [\Omega]C$.
- (vi) If $\Gamma / P \vdash \Pi :: \vec{A} ! \Leftarrow C p \dashv \Delta$ and $\Gamma \vdash P$ prop and $FEV(P) = \emptyset$ and $[\Gamma]P = P$ and $\Gamma \vdash \vec{A} !$ types and $\Gamma \vdash C p$ type then $[\Omega]\Delta / [\Omega]P \vdash [\Omega]\Pi :: [\Omega]\vec{A} ! \Leftarrow [\Omega]C p$.

Proof. By induction, using the measure in Definition 7.

Where the i.h. is used, we elide the reasoning establishing the condition $[\Gamma]A = A$ for parts (i), (iii), (iv), (v) and (vi): this condition follows from Theorem 8, which ensures that the appropriate condition holds for all subderivations.

• Case
$$\frac{(x:A\,p)\in\Gamma}{\Gamma\vdash x\Rightarrow [\Gamma]A\,p\dashv\Gamma} \, \text{Var}$$

$$(x:A\,p)\in\Gamma \qquad \text{Premise}$$

$$(x:A\,p)\in\Delta \qquad \qquad \Gamma=\Delta$$

$$\Delta\longrightarrow\Omega \qquad \qquad \text{Given}$$

$$(x:[\Omega]A\,p)\in[\Omega]\Gamma \qquad \qquad \text{By Lemma 9 (Uvar Preservation) (ii)}$$

$$[\Omega]\Gamma\vdash[\Omega]x\Rightarrow [\Omega]A\,p \qquad \qquad \text{By DeclVar}$$

$$\Delta\longrightarrow\Omega \qquad \qquad \text{Given}$$

$$\Gamma\longrightarrow\Omega \qquad \qquad \Gamma=\Delta$$

$$[\Omega]A=[\Omega][\Gamma]A \qquad \qquad \text{By Lemma 29 (Substitution Monotonicity) (iii)}$$

$$[\Omega]\Gamma\vdash[\Omega]x\Rightarrow [\Omega][\Gamma]A\,p \qquad \text{By above equality}$$

• Case
$$\frac{\Gamma \vdash e \Rightarrow A \neq \neg \Theta \qquad \Theta \vdash A < :^{join(pol(B),pol(A))} \; B \dashv \Delta}{\Gamma \vdash e \Leftarrow B \; p \dashv \Delta} \; Sub}$$

$$\Gamma \vdash e \Rightarrow A \neq \neg \Theta \qquad \qquad Subderivation$$

$$\Theta \vdash A < :^{\mathcal{P}} \; B \dashv \Delta \qquad \qquad Subderivation$$

$$\Theta \rightarrow \Delta \qquad \qquad By \; Lemma \; 51 \; (Typing \; Extension)$$

$$\Delta \rightarrow \Omega \qquad \qquad Given$$

$$\Theta \rightarrow \Omega \qquad \qquad By \; Lemma \; 33 \; (Extension \; Transitivity)$$

$$[\Omega]\Theta \vdash [\Omega]e \Rightarrow [\Omega]A \; q \qquad \qquad By \; Lemma \; 56 \; (Confluence \; of \; Completeness)$$

$$[\Omega]\Theta \vdash [\Omega]e \Rightarrow [\Omega]A \; q \qquad \qquad By \; Lemma \; 56 \; (Confluence \; of \; Completeness)$$

$$[\Omega]\Delta \vdash [\Omega]e \Rightarrow [\Omega]A \; q \qquad \qquad By \; above \; equality$$

$$\Theta \vdash A < :^{join(pol(B),pol(A))} \; B \dashv \Delta \qquad Subderivation$$

$$[\Omega]\Delta \vdash [\Omega]A \leq^{join(pol(B),pol(A))} \; [\Omega]B \quad By \; Theorem \; 6$$

$$[\Omega]\Delta \vdash [\Omega]e \Leftarrow [\Omega]B \; p \qquad By \; DeclSub$$

• Case
$$\frac{\Gamma \vdash A_0! \ type \qquad \Gamma \vdash e_0 \Leftarrow [\Gamma] A_0 \ ! \dashv \Delta}{\Gamma \vdash (e_0 : A_0) \Rightarrow [\Delta] A_0 \ ! \dashv \Delta} \ \text{Anno}$$

$$\Gamma \vdash e_0 \Leftarrow [\Gamma] A_0 \ ! \dashv \Delta \qquad \text{Subderivation}$$

$$[\Omega] \Delta \vdash [\Omega] e_0 \Leftarrow [\Omega] [\Gamma] A_0 \ ! \qquad \text{By i.h.}$$

$$\Gamma \vdash A_0! \ type \qquad \qquad \text{Subderivation}$$

$$\Gamma \vdash A_0 \ type \qquad \qquad \text{By inversion}$$

$$\Gamma \vdash A_0 \ type \qquad \qquad \text{By inversion}$$

$$\Gamma \vdash A_0 \ type \qquad \qquad \text{By inversion}$$

$$\Gamma \vdash A_0 \ type \qquad \qquad \text{By inversion}$$

 $\Gamma \longrightarrow \Delta$ By Lemma 51 (Typing Extension) $\Delta \longrightarrow \Omega$ Given $\Gamma \longrightarrow \Omega$ By Lemma 33 (Extension Transitivity) $\Omega \vdash A_0$ type By Lemma 36 (Extension Weakening (Sorts)) $[\Omega]\Omega \vdash [\Omega]A_0$ type By Lemma 16 (Substitution for Type Well-Formedness) $[\Omega]\Omega = [\Omega]\Delta$ By Lemma 54 (Completing Stability) $[\Omega]\Delta \vdash [\Omega]A_0$ type By above equality $[\Omega][\Gamma]A_0 = [\Omega]A_0$ By Lemma 29 (Substitution Monotonicity) (iii) $[\Omega]\Delta \vdash [\Omega]e_0 \Leftarrow [\Omega]A_0$! By above equality $[\Omega]\Delta \vdash ([\Omega]e_0 : [\Omega]A_0) \Rightarrow [\Omega]A_0$! By DeclAnno $[\Omega] A_0 = A_0$ From definition of substitution $[\Omega]\Delta \vdash [\Omega](e_0:A_0) \Rightarrow [\Omega]A_0$! By above equality

Case

13

$$\overline{\Gamma \vdash ()} \Leftarrow 1 \, p \dashv \underbrace{\Gamma}_{\Delta} 1 I$$

$$\begin{array}{ll} [\Omega]\Delta \vdash () \Leftarrow 1 \ p & \text{By Decl1I} \\ & \\ \square \cap [\Omega]\Delta \vdash [\Omega]() \Leftarrow [\Omega]1 \ p & \text{By definition of substitution} \end{array}$$

• Case

E P

$$\overline{\Gamma_0[\hat{\alpha}:\star] \vdash ()} \Leftarrow \hat{\alpha} \not\vdash \exists \underbrace{\Gamma_0[\hat{\alpha}:\star=1]}_{\Delta} \ ^{1}\hat{\alpha}$$

$$\begin{split} \Gamma_0[\widehat{\alpha}:\star=1] &\longrightarrow \Omega & \text{Given} \\ [\Omega]\widehat{\alpha} &= [\Omega][\Delta]\widehat{\alpha} & \text{By Lemma 29 (Substitution Monotonicity) (i)} \\ &= [\Omega]1 & \text{By definition of context application} \\ &= 1 & \text{By definition of context application} \\ [\Omega]\Delta \vdash () &\Leftarrow 1 \not \text{I} & \text{By Decl1I} \\ [\Omega]\Delta \vdash [\Omega]() &\Leftarrow [\Omega]\widehat{\alpha} \not \text{I} & \text{By above equality} \end{split}$$

• Case $\frac{ \nu \ chk\text{-}I \qquad \Gamma, \alpha : \kappa \vdash \nu \Leftarrow A_0 \ p \dashv \Delta, \alpha : \kappa, \Theta }{ \Gamma \vdash \nu \Leftarrow \forall \alpha : \kappa, A_0 \ p \dashv \Delta } \ \forall I$

```
\Gamma, \alpha \vdash \nu \Leftarrow A_0 p \dashv \Delta'
                                                                   Subderivation
           [\Omega']\Delta' \vdash [\Omega]\nu \Leftarrow [\Omega']A_0 p By i.h.
         [\Omega']A_0 = [\Omega]A_0
                                                                   By Lemma 17 (Substitution Stability)
           [\Omega']\Delta' \vdash [\Omega]v \Leftarrow [\Omega]A_0 p
                                                                   By above equality
                 \underbrace{\Delta,\alpha,\Theta}_{\Delta'} \longrightarrow \underbrace{\Omega,\alpha,|\Theta|}_{\Omega'}
                                                                                         Above
                                                                                         Above
                      [\Omega']\Delta' = ([\Omega]\Delta, \alpha)
                                                                                         By Lemma 53 (Softness Goes Away)
                      [\Omega]\Delta, \alpha \vdash [\Omega]\nu \Leftarrow [\Omega]A_0 p
                                                                                         By above equality
                           [\Omega]\Delta \vdash [\Omega]\nu \Leftarrow \forall \alpha. [\Omega]A_0 p
                                                                                         By Decl∀I
                           [\Omega]\Delta \vdash [\Omega]\nu \Leftarrow [\Omega](\forall \alpha. A_0) p
                                                                                        By definition of substitution
         13
 \begin{array}{c} \bullet \  \, \textbf{Case} \\ \frac{\Gamma, \hat{\alpha} : \kappa \vdash e \ s_0 : [\hat{\alpha}/\alpha] A_0 \ \cancel{I} \gg C \ q \dashv \Delta}{\Gamma \vdash e \ s_0 : \forall \alpha : \kappa. \ A_0 \ p \gg C \ q \dashv \Delta} \ \forall \mathsf{Spine} \end{array} 
       \Gamma, \hat{\alpha} : \kappa \vdash e \ s_0 : [\hat{\alpha}/\alpha] A_0 \not I \gg C \ q \dashv \Delta
                                                                                                         Subderivation
          [\Omega]\Delta \vdash [\Omega](e s_0) : [\Omega][\hat{\alpha}/\alpha]A_0 \not I \gg [\Omega]C q
                                                                                                         By i.h.
          [\Omega]\Delta \vdash [\Omega](e s_0) : [[\Omega]\hat{\alpha}/\alpha][\Omega]A_0 \not I \gg [\Omega]C q By a property of substitution
              \Gamma, \hat{\alpha} : \kappa \vdash \hat{\alpha} : \kappa
                                                   By VarSort
         \Gamma, \hat{\alpha} : \kappa \longrightarrow \Delta
                                                   By Lemma 51 (Typing Extension)
                       \Delta \vdash \hat{\alpha} : \kappa
                                                   By Lemma 36 (Extension Weakening (Sorts))
                  \Delta \longrightarrow \Omega
                                                   Given
                 [\Omega]\Delta \vdash [\Omega]\hat{\alpha}: \kappa By Lemma 58 (Bundled Substitution for Sorting)
                  [\Omega]\Delta \vdash [\Omega](e s_0) : \forall \alpha : \kappa. [\Omega]A_0 p \gg [\Omega]C q
                                                                                                                By Decl∀Spine
         \square [\Omega]\Delta \vdash [\Omega](e s_0) : [\Omega](\forall \alpha : \kappa. A_0) p \gg [\Omega]C q By def. of subst.
• Case e chk-I
                                     \frac{\Gamma \vdash P \text{ true} \dashv \Theta \qquad \Theta \vdash e \Leftarrow [\Theta] A_0 \ p \dashv \Delta}{\Gamma \vdash e \Leftarrow A_0 \land P \ p \dashv \Delta} \land I
                                 \Gamma \vdash P \ true \dashv \Theta
                                                                                                     Subderivation
                           \Delta \longrightarrow \Omega
                                                                                                     Given
                           \Theta \longrightarrow \Delta
                                                                                                     By Lemma 51 (Typing Extension)
                           \Theta \longrightarrow \Omega
                                                                                                     By Lemma 33 (Extension Transitivity)
                          [\Omega]\Theta \vdash [\Omega]P true
                                                                                                     By Lemma 89 (Soundness of Checkprop)
                                                                                                     By Lemma 56 (Confluence of Completeness)
                          [\Omega]\Delta \vdash [\Omega]P true
                                \Theta \vdash e \Leftarrow [\Theta] A_0 \mathfrak{p} \dashv \Delta
                                                                                                     Subderivation
                          [\Omega]\Delta \vdash [\Omega]e \Leftarrow ([\Omega][\Theta]A_0) p
                                                                                                     By i.h.
                          [\Omega]\Delta \vdash [\Omega]e \Leftarrow ([\Omega][\Theta]A_0) \land [\Omega]Pp
                                                                                                     By Decl∧I
                 [\Omega][\Theta]A_0 = [\Omega]A_0
                                                                                                     By Lemma 29 (Substitution Monotonicity) (iii)
                          [\Omega]\Delta \vdash [\Omega]e \Leftarrow ([\Omega]A_0) \land [\Omega]P p
                                                                                                     By above equality
                          [\Omega]\Delta \vdash [\Omega]e \Leftarrow [\Omega](A_0 \land P) p
                                                                                                     By def. of substitution
       3
```

```
Case
                  \frac{\Gamma \vdash t = \mathsf{zero} \; true \; \exists \; \Delta}{\Gamma \vdash \Gamma \; \vdash \; \vdash \; (\mathsf{Vec} \; t \; A) \; n \; \exists \; \Delta} \; \mathsf{Nil}
                         \Gamma \vdash t = \text{zero } true \dashv \Delta
                                                                                                 Subderivation
                   \Delta \longrightarrow \Omega
                                                                                                 Given
                  [\Omega]\Delta \vdash [\Omega](t = zero) true
                                                                                                By Lemma 89 (Soundness of Checkprop)
                 [\Omega]\Delta \vdash [\Omega]t = \mathsf{zero}\ \mathit{true}
                                                                                                By def. of substitution
                [\Omega]\Delta \vdash [\Omega][] \Leftarrow (\text{Vec } [\Omega]t [\Omega]A) p
                                                                                                Bv DeclNil
Case
                                                                                                      \Gamma' \vdash e_1 \leftarrow [\Gamma'] A_0 p \dashv \Theta
                 \frac{\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \mathbb{N} \vdash t = \mathsf{succ}(\hat{\alpha}) \ true \dashv \Gamma' \qquad \Theta \vdash e_2 \Leftarrow [\Theta](\mathsf{Vec} \ \hat{\alpha} \ \mathsf{A}_0) \ \cancel{y} \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta'}{\Gamma \vdash e_1 :: e_2 \Leftarrow (\mathsf{Vec} \ t \ \mathsf{A}_0) \ p \dashv \Delta} \ \mathsf{Cons}
                      \Gamma, \triangleright_{\hat{\alpha}}, \hat{\alpha} : \mathbb{N} \vdash \mathbf{t} = \mathsf{succ}(\hat{\alpha}) \ true \dashv \Gamma'
                                                                                                             Subderivation
                                   \Delta \longrightarrow \Omega
                                                                                                             Given
                                  \Gamma' \longrightarrow \Theta
                                                                                                             By Lemma 51 (Typing Extension)
                                  \Theta \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta'
                                                                                                             By Lemma 51 (Typing Extension)
                   \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta' \longrightarrow \Omega'
                                                                                                             By Lemma 25 (Filling Completes)
                                                                                                             By Lemma 33 (Extension Transitivity)
                               [\Omega']\Gamma' \vdash [\Omega'](t = \operatorname{succ}(\hat{\alpha})) true
                                                                                                             By Lemma 89 (Soundness of Checkprop)
            [\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Delta') \vdash [\Omega'](t = \mathsf{succ}(\hat{\alpha})) \ true
                                                                                                             By Lemma 56 (Confluence of Completeness)
            [\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Delta') \vdash [\Omega](t = \mathsf{succ}(\hat{\alpha})) \ true
                                                                                                             By Lemma 17 (Substitution Stability)
                                                                                                             By Lemma 52 (Context Partitioning) + thinning
                                  [\Omega]\Delta \vdash [\Omega](t = \mathsf{succ}(\hat{\alpha})) true
       1
                                  [\Omega]\Delta \vdash ([\Omega]t) = \mathsf{succ}([\Omega]\hat{\alpha}) true
                                                                                                            By def. of substitution
                                 \Gamma' \vdash e_1 \Leftarrow [\Gamma'] A_0 \mathfrak{p} \dashv \Theta
                                                                                                    Subderivation
                         [\Omega']\Theta \vdash [\Omega']e_1 \Leftarrow ([\Omega'][\Gamma']A_0) p
                                                                                                    By i.h.
             [\Omega'][\Gamma']A_0 = [\Omega']A_0
                                                                                                    By Lemma 29 (Substitution Monotonicity) (iii)
                         [\Omega']\Theta \vdash [\Omega']e_1 \leftarrow [\Omega']A_0 p
                                                                                                    By above equality
         2
                           [\Omega]\Delta \vdash [\Omega]e_1 \Leftarrow [\Omega]A_0 p
                                                                                                    Similar to above
                                               \Theta \vdash e_2 \Leftarrow [\Theta](\text{Vec } \hat{\alpha} A_0) \not ! \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta'
                                                                                                                                              Subderivation
                   [\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Delta') \vdash [\Omega']e_2 \Leftarrow [\Omega'][\Theta](\text{Vec } \hat{\alpha} \land A_0) \not Y
                                                                                                                                              By i.h.
                                        [\Omega]\Delta \vdash [\Omega]e_2 \Leftarrow [\Omega](\mathsf{Vec}\ \widehat{\alpha}\ \mathsf{A}_0) \not Y
                                                                                                                                              Similar to above
             3
                                        [\Omega]\Delta \vdash [\Omega]e_2 \Leftarrow (\text{Vec}([\Omega]\hat{\alpha})[\Omega]A_0)p
                                                                                                                                              By def. of substitution
                                        [\Omega]\Delta \vdash ([\Omega]e_1) :: [\Omega]e_2 \Leftarrow \text{Vec}([\Omega]t) [\Omega]A_0 p
                                                                                                                                              By DeclCons (premises: 1, 2, 3)
                                        [\Omega]\Delta \vdash [\Omega](e_1 :: e_2) \Leftarrow [\Omega](\text{Vec t } A_0) p
                                                                                                                                              By def. of substitution
         ₽
```

```
• Case \frac{e \; \mathit{chk-I} \qquad \Gamma, \hat{\alpha} : \kappa \vdash e \Leftarrow [\hat{\alpha}/\alpha] A_0 \; \dashv \Delta}{\Gamma \vdash e \Leftarrow \exists \alpha : \kappa. \; A_0 \; p \dashv \Delta} \; \exists \mathsf{I}
                                                                                                      Subderivation
                        \Gamma, \hat{\alpha} : \kappa \vdash e \Leftarrow [\hat{\alpha}/\alpha]A_0 \dashv \Delta
                           [\Omega]\Delta \vdash [\Omega]e \Leftarrow [\Omega][\hat{\alpha}/\alpha]A_0
                                                                                                      By i.h.
                           [\Omega]\Delta \vdash [\Omega]e \Leftarrow [[\Omega]\hat{\alpha}/\alpha][\Omega]A_0
                                                                                                       By a property of substitution
                        \Gamma, \hat{\alpha} : \kappa \vdash \hat{\alpha} : \kappa
                                                                                                      By VarSort
                  \Gamma, \hat{\alpha} : \kappa \longrightarrow \Delta
                                                                                                       By Lemma 51 (Typing Extension)
                                 \Delta \vdash \hat{\alpha} : \kappa
                                                                                                       By Lemma 36 (Extension Weakening (Sorts))
                            \Delta \longrightarrow \Omega
                                                                                                       Given
                           [\Omega]\Delta \vdash [\Omega]\hat{\alpha} : \kappa
                                                                                                      By Lemma 58 (Bundled Substitution for Sorting)
                           [\Omega]\Delta \vdash [\Omega]e \Leftarrow \exists \alpha : \kappa. [\Omega]A_0 p
                                                                                                      By Decl∃I
                           [\Omega]\Delta \vdash [\Omega]e \Leftarrow [\Omega](\exists \alpha : \kappa. A_0) p
                                                                                                     By def. of subst.
        ₽
• Case v chk-I
                                      \frac{\Gamma, \blacktriangleright_{P} / P \dashv \Theta^{+} \qquad \Theta^{+} \vdash \nu \Leftarrow [\Theta^{+}] A_{0} ! \dashv \Delta, \blacktriangleright_{P}, \Delta'}{\Gamma \vdash \nu \Leftarrow P \supset A_{0} ! \dashv \Delta} \supset I
                                                       \Gamma \vdash A ! type
                                  FEV([\Gamma]A) = \emptyset
                                                                                                    By inversion on rule PrincipalWF
                                   FEV([\Gamma]P) = \emptyset
                                                                                                   A = (P \supset A_0)
                                              \Gamma_{\bullet P} / P \dashv \Theta^+
                                                                                                   Subderivation
                                              \Gamma, \blacktriangleright_P / \sigma \stackrel{\circ}{=} t : \kappa \dashv \Theta^+
                                                                                                   By inversion
         \mathsf{FEV}(\lceil \Gamma \rceil \sigma) \cup \mathsf{FEV}(\lceil \Gamma \rceil t) = \emptyset
                                                                                                    By FEV([\Gamma]P) = \emptyset above
                                                \Theta^+ = (\Gamma, \blacktriangleright_P, \Theta)
                                                                                                    By Lemma 90 (Soundness of Equality Elimination)
                                      [\Omega',\Theta]t'=[\theta][\Gamma,\blacktriangleright_P]t'
                                                                                                    " (for all \Omega' extending (\Gamma, \triangleright_P) and t' s.t. \Omega' \vdash t' : \kappa')
                                                     \theta = mgu(\sigma, t)
                                                \Delta \longrightarrow \Omega
                                                                                                    Given
                                             \Theta^+ \longrightarrow \Delta, \triangleright_P, \Delta'
                                                                                                    By Lemma 51 (Typing Extension)
                                    \Gamma, \blacktriangleright_{P}, \Theta \longrightarrow \Delta, \blacktriangleright_{P}, \Delta'
                                                                                                   By above equalities
                                       Let \Omega^+ = (\Omega, \blacktriangleright_P, \Delta').
                                   \Delta, \blacktriangleright_{P}, \Theta \longrightarrow \Omega, \blacktriangleright_{P}, \Delta'
                                                                                                    By repeated \longrightarrow Eqn
                                            \Theta^+ \longrightarrow \Omega^+
                                                                                                    By Lemma 33 (Extension Transitivity)
          [\Omega', \Theta]B = [\theta][\Gamma, \blacktriangleright_P]B By Lemma 95 (Substitution Upgrade) (i)
                                                                    (for all \Omega' extending (\Gamma, \blacktriangleright_P \text{ and B s.t. } \Omega' \vdash B : \kappa')
```

```
\Theta^+ \vdash \nu \Leftarrow [\Theta^+] A_0 ! \dashv \Delta, \triangleright_P, \Delta'
                                                                                                    Subderivation
                   [\Omega^+](\Delta, \blacktriangleright_P, \Delta') \vdash [\Omega]v \Leftarrow [\Omega^+][\Theta^+]A_0!
                                                                                                    By i.h.
                            \Gamma, \triangleright_{P}, \Theta \longrightarrow \Omega, \triangleright_{P}, \Delta'
                                                                                                   By Lemma 33 (Extension Transitivity)
                                      \Gamma \longrightarrow O
                                                                                                   By Lemma 22 (Extension Inversion)
                       [\Omega^+][\Theta^+]A_0=[\Omega^+]A_0
                                                                                                   By Lemma 29 (Substitution Monotonicity)
                                             = [\theta][\Omega, \triangleright_{P}]A_0
                                                                                                   Above, with (\Omega, \triangleright_P) as \Omega' and A_0 as B
                                              = [\theta][\Omega]A_0
                                                                                                   By def. of substitution
         [\Omega, \blacktriangleright_{P}, \Theta](\Delta, \blacktriangleright_{P}, \Delta') = [\theta][\Omega]\Delta
                                                                                                   By Lemma 95 (Substitution Upgrade) (iii)
                                       [\theta][\Omega]\Delta \vdash [\Omega][\theta]\nu \Leftarrow [\theta][\Omega]A_0!
                                                                                                                       By above equalities
         [\Omega^+](\Delta, \triangleright_P, \Delta') / (\sigma = t) \vdash [\Omega] \nu \Leftarrow [\Omega] A_0 !
                                                                                                                       By DeclCheckUnify
                        [\Omega^+](\Delta, \triangleright_P, \Delta') = [\Omega]\Delta
                                                                                                                       From def. of context application
                           [\Omega]\Delta / (\sigma = t) \vdash [\Omega]\nu \Leftarrow [\Omega]A_0!
                                                                                                                       By above equality
                                            [\Omega]\Delta \vdash [\Omega]\nu \Leftarrow (\sigma = t) \supset [\Omega]A_0!
                                                                                                                       By Decl⊃I
                                            [\Omega]\Delta \vdash [\Omega]\nu \leftarrow ([\Omega]\sigma = [\Omega]t) \supset [\Omega]A_0! By FEV condition above
• Case \frac{\nu \ chk \cdot I \qquad \Gamma, \blacktriangleright_P \ / \ P \dashv \bot}{\Gamma \vdash \nu \Leftarrow P \supset A_0 \ ! \dashv \underbrace{\Gamma}_{\Delta} } \supset I \bot
                                     \Gamma, \triangleright_{P} / P \dashv \bot
                                                                              Subderivation
                                     \Gamma, \blacktriangleright_P / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot By inversion
                                          P = (\sigma = t)
       \mathsf{FEV}([\Gamma]\sigma) \cup \mathsf{FEV}([\Gamma]t) = \emptyset
                                                                              As in ⊃I case (above)
                                                                               By Lemma 90 (Soundness of Equality Elimination)
                             mgu(\sigma, t) = \bot
                [\Omega]\Delta / (\sigma = t) \vdash [\Omega]\nu \Leftarrow [\Omega]A_0!
                                                                                                          By DeclCheck⊥
                                 [\Omega]\Delta \vdash [\Omega]\nu \Leftarrow (\sigma = t) \supset [\Omega]A_0!
                                                                                                          By Decl⊃l
                                 [\Omega]\Delta \vdash [\Omega]\nu \Leftarrow ([\Omega](\sigma = t)) \supset [\Omega]A_0! By above FEV condition
                                 [\Omega]\Delta \vdash [\Omega]\nu \Leftarrow [\Omega](P \supset A_0)!
                                                                                                          By def. of subst.
        137
                            Let \Omega' = \Omega.
                                \Omega \longrightarrow \Omega'
                                                                                                          By Lemma 32 (Extension Reflexivity)
        37
                                  \Delta \longrightarrow \Omega'
                                                                                                          Given
        EF
\Theta \vdash e \ s_0 : [\Theta] A_0 \ p \gg C \ q \dashv \Delta Subderivation
       \Theta \longrightarrow \Delta
                                                                    By Lemma 51 (Typing Extension)
       \Delta \longrightarrow \Omega
                                                                     Given
       \Theta \longrightarrow \Omega
                                                                     By Lemma 33 (Extension Transitivity)
```

 $[\Omega]\Delta \vdash [\Omega](e \; s_0) : [\Omega][\Theta]A_0 \; \mathfrak{p} \gg [\Omega]C \; \mathfrak{q}$ By i.h. $[\Omega][\Theta]A_0 = [\Omega]A_0$ By Lemma 29 (Substitution Monotonicity) (iii) $[\Omega]\Delta \vdash [\Omega](e s_0) : [\Omega]A_0 p \gg [\Omega]C q$ By above equality $\Gamma \vdash P \ true \dashv \Theta$ Subderivation $[\Omega]\Theta \vdash [\Omega]P$ true By Lemma 97 (Completeness of Checkprop) $[\Omega]\Theta = [\Omega]\Delta$ By Lemma 56 (Confluence of Completeness) $[\Omega]\Delta \vdash [\Omega]P$ true By above equality $[\Omega]\Delta \vdash [\Omega](e \; s_0) : ([\Omega]P) \supset [\Omega]A_0 \; p \gg [\Omega]C \; q$ By Decl⊃Spine $[\Omega]\Delta \vdash [\Omega](e\ s_0): [\Omega](P \supset A_0)\ p \gg [\Omega]C\ q$ By def. of subst.

• Case
$$\frac{\Gamma, x : A_1 p \vdash e_0 \Leftarrow A_2 p \dashv \Delta, x : A_1 p, \Theta}{\Gamma \vdash \lambda x. e_0 \Leftarrow A_1 \rightarrow A_2 p \dashv \Delta} \rightarrow I$$

$$\Delta \longrightarrow \Omega$$

$$\Delta, x : A_1 p \longrightarrow \Omega, x : [\Omega] A_1 p \qquad \text{By} \longrightarrow \text{Var}$$

$$\Gamma, x : A_1 p \longrightarrow \Delta, x : A_1 p, \Theta \qquad \text{By Lemma 51 (Typing Extension)}$$

$$\Theta \text{ soft} \qquad \text{By Lemma 22 (Extension Inversion) (v)}$$

$$(\text{with } \Gamma_R = \cdot, \text{ which is soft)}$$

$$By \text{Lemma 25 (Filling Completes)}$$

$$\Gamma, x : A_1 p \mapsto \Theta_0 \Leftarrow A_2 p \dashv \Delta' \qquad \text{Subderivation}$$

$$[\Omega']\Delta' \vdash [\Omega]e_0 \Leftarrow [\Omega']A_2 p \qquad \text{By i.h.}$$

$$[\Omega']A_2 = [\Omega]A_2 \qquad \text{By Lemma 17 (Substitution Stability)}$$

$$[\Omega']\Delta' \vdash [\Omega]e_0 \Leftarrow [\Omega]A_2 p \qquad \text{By above equality}$$

$$\Delta, x : A_1 p, \Theta \longrightarrow \Omega, x : [\Omega]A_1 p, |\Theta| \qquad \text{Above}$$

$$[\Omega']\Delta' = ([\Omega]\Delta, x : [\Omega]A_1 p, |\Theta| \qquad \text{Above}$$

$$[\Omega']\Delta' = ([\Omega]\Delta, x : [\Omega]A_1 p) \qquad \text{By Lemma 53 (Softness Goes Away)}$$

$$[\Omega]\Delta, x : [\Omega]A_1 p \vdash [\Omega]e_0 \Leftarrow [\Omega]A_2 p \qquad \text{By above equality}$$

$$[\Omega]\Delta \vdash \lambda x : [\Omega]e_0 \Leftarrow ([\Omega]A_1) \rightarrow ([\Omega]A_2) p \qquad \text{By Decl} \rightarrow l$$

$$[\Omega]\Delta \vdash [\Omega](\lambda x : e_0) \Leftarrow [\Omega](A_1 \rightarrow A_2) p \qquad \text{By definition of substitution}$$

$$\bullet \ \, \textbf{Case} \ \, \frac{ \nu \ \textit{chk-I} \qquad \Gamma, x : A \ p \vdash \nu \Leftarrow A \ p \dashv \Delta, x : A \ p, \Theta}{\Gamma \vdash \text{rec} \ x. \nu \Leftarrow A \ p \dashv \Delta} \ \, \text{Rec}$$

Similar to the \rightarrow I case, applying DeclRec instead of Decl \rightarrow I.

$$[\Omega]\Theta \vdash [\Omega]s_0 : [\Omega]A \neq [\Omega]C \mid p \mid By i.h.$$

$$[\Omega]\Delta \vdash [\Omega](e_0 s_0) \Rightarrow [\Omega]C \neq By rule Decl \rightarrow E$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash s : A \ ! \gg C \not ! \dashv \Delta \qquad \text{FEV}(C) = \emptyset}{\Gamma \vdash s : A \ ! \gg C \ \lceil ! \rceil \dashv \Delta} \ \, \text{SpineRecover}$$

$$\begin{array}{ll} \Gamma \vdash s : A ! \gg C \not \! I \dashv \Delta & \text{Subderivation} \\ [\Omega] \Gamma \vdash [\Omega] s : [\Omega] A ! \gg [\Omega] C \not q & \text{By i.h.} \end{array}$$

We show the quantified premise of DeclSpineRecover, namely,

for all
$$C'$$
.
if $[\Omega]\Theta \vdash s : [\Omega]A ! \gg C' \not$ then $C' = [\Omega]C$

Suppose we have C' such that $[\Omega]\Gamma \vdash s : [\Omega]A ! \gg C' \not$. To apply DeclSpineRecover, we need to show $C' = [\Omega]C$.

We have thus shown the above "for all C'...." statement.

$$\lceil \Omega \rceil \Gamma \vdash \lceil \Omega \rceil s : \lceil \Omega \rceil A ! \gg \lceil \Omega \rceil C \lceil ! \rceil$$
 By DeclSpineRecover

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash s : A \ \mathfrak{p} \gg C \ \mathfrak{q} \dashv \Delta \qquad \left((\mathfrak{p} = \cancel{I}) \ \text{or} \ (\mathfrak{q} = !) \ \text{or} \ (\mathsf{FEV}(C) \neq \emptyset) \right)}{\Gamma \vdash s : A \ \mathfrak{p} \gg C \ \lceil \mathfrak{q} \rceil \dashv \Delta} \ \, \mathsf{SpinePass}$$

$$\Gamma \vdash s : A p \gg C q \dashv \Delta$$
 Subderivation $[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A p \gg [\Omega]C q$ By i.h.

$$[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A \mathfrak{p} \gg [\Omega]C [\mathfrak{q}]$$
 By DeclSpinePass

Case

$$\frac{}{\Gamma \vdash \cdot : A \mathrel{\mathfrak{p}} \gg A \mathrel{\mathfrak{p}} \dashv \Gamma} \mathrel{\mathsf{EmptySpine}}$$

$$\mathbb{F} [\Omega] \Gamma \vdash \cdot : [\Omega] A \mathfrak{p} \gg [\Omega] A \mathfrak{p}$$
 By DeclEmptySpine

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash e_0 \Leftarrow A_1 \ \mathfrak{p} \dashv \Theta \qquad \Theta \vdash s_0 : [\Theta] A_2 \ \mathfrak{p} \gg C \ \mathfrak{q} \dashv \Delta}{\Gamma \vdash e_0 \ s_0 : A_1 \longrightarrow A_2 \ \mathfrak{p} \gg C \ \mathfrak{q} \dashv \Delta} \to \mathsf{Spine}$$

$$\Delta \longrightarrow \Omega$$
 Given

$$\Theta \longrightarrow \Delta$$
 By Lemma 51 (Typing Extension) $\Theta \longrightarrow \Omega$ By Lemma 33 (Extension Transitivity)

$$\Gamma \vdash e_0 \Leftarrow A_1 p \dashv \Theta$$
 Subderivation

$$[\Omega]\Theta \vdash [\Omega]e_0 \Leftarrow [\Omega]A_1 p$$
 By i.h.

$$[\Omega]\Theta = [\Omega]\Delta$$
 By Lemma 56 (Confluence of Completeness)

$$[\Omega]\Delta \vdash [\Omega]e_0 \Leftarrow [\Omega]A_1 p$$
 By above equality

$$\begin{array}{ll} \Theta \vdash s_0 : [\Theta] A_2 \ p \gg C \ q \dashv \Delta & Subderivation \\ [\Omega] \Delta \vdash [\Omega] s_0 : [\Omega] [\Theta] A_2 \ p \gg [\Omega] C \ q & By \ i.h. \\ [\Omega] [\Theta] A_2 = [\Omega] A_2 & By \ Lemma \ 29 \ (Substitution \ Monotonicity) \\ [\Omega] \Delta \vdash [\Omega] s_0 : [\Omega] A_2 \ p \gg [\Omega] C \ q & By \ above \ equality \end{array}$$

$$\begin{split} [\Omega]\Delta \vdash [\Omega](e_0 \ s_0) : ([\Omega]A_1) \to [\Omega]A_2 \ p \gg [\Omega]C \ q & \text{By Decl} \to \text{Spine} \\ [\Omega]\Delta \vdash [\Omega](e_0 \ s_0) : [\Omega](A_1 \to A_2) \ p \gg [\Omega]C \ q & \text{By def. of subst.} \end{split}$$

• Case
$$\frac{\Gamma \vdash e_0 \Leftarrow A_k \ p \dashv \Delta}{\Gamma \vdash \inf_k e_0 \Leftarrow A_1 + A_2 \ p \dashv \Delta} + I_k$$

$$\begin{array}{ll} \Gamma \vdash e_0 \Leftarrow A_k \ p \dashv \Delta & \text{Subderivation} \\ [\Omega] \Delta \vdash [\Omega] e_0 \Leftarrow [\Omega] A_k \ p & \text{By i.h.} \\ [\Omega] \Delta \vdash \mathsf{inj}_k \ [\Omega] e_0 \Leftarrow ([\Omega] A_1) + ([\Omega] A_2) \ p & \text{By Decl} + \mathsf{I}_k \end{array}$$

$$\square$$
 $[\Omega]\Delta \vdash [\Omega](\mathsf{inj}_k e_0) \Leftarrow [\Omega](A_1 + A_2) p$ By def. of substitution

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma[\hat{\alpha}_1:\star,\hat{\alpha}_2:\star,\hat{\alpha}:\star=\hat{\alpha}_1+\hat{\alpha}_2] \vdash e_0 \Leftarrow \hat{\alpha}_k \not \! I \dashv \Delta}{\Gamma[\hat{\alpha}:\star] \vdash \mathsf{inj}_k e_0 \Leftarrow \hat{\alpha} \not \! I \dashv \Delta} + \mathsf{I}\hat{\alpha}_k$$

```
\Gamma[\ldots,\hat{\alpha}:\star=\hat{\alpha}_1+\hat{\alpha}_2]\vdash e_0 \Leftarrow \hat{\alpha}_k \not I \dashv \Delta
                                                                                                                                            Subderivation
                                                  [\Omega]\Delta \vdash [\Omega]e_0 \Leftarrow [\Omega]\hat{\alpha}_k !
                                                                                                                                            By i.h.
                                                 [\Omega]\Delta \vdash \mathsf{inj}_k [\Omega]e_0 \Rightarrow ([\Omega]\hat{\alpha}_1) + ([\Omega]\hat{\alpha}_2) \not Y
                                                                                                                                            By Decl+I_k
                     ([\Omega]\hat{\alpha}_1) + ([\Omega]\hat{\alpha}_2) = [\Omega]\hat{\alpha}
                                                                                                                                            Similar to the \rightarrow l\hat{\alpha} case (above)
                                                 By above equality / def. of subst.
        3
 \begin{array}{cccc} \bullet \text{ Case } & & \\ & \frac{\Gamma \vdash e_1 \Leftarrow A_1 \ \mathfrak{p} \dashv \Theta & \Theta \vdash e_2 \Leftarrow [\Theta] A_2 \ \mathfrak{p} \dashv \Delta}{\Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow A_1 \times A_2 \ \mathfrak{p} \dashv \Delta} \times I \end{array} 
                        \Theta \vdash e_2 \Leftarrow [\Theta]A_2 \mathfrak{p} \dashv \Delta
                                                                                                                   Subderivation
                   \Theta \longrightarrow \Delta
                                                                                                                  By Lemma 51 (Typing Extension)
                   \Theta \longrightarrow \Omega
                                                                                                                  By Lemma 33 (Extension Transitivity)
                         \Gamma \vdash e_1 \Leftarrow A_1 \mathfrak{p} \dashv \Theta
                                                                                                                   Subderivation
                  [\Omega]\Theta \vdash [\Omega]e_1 \Leftarrow [\Omega]A_1 p
                                                                                                                   By i.h.
                  [\Omega]\Delta \vdash [\Omega]e_1 \Leftarrow [\Omega]A_1 p
                                                                                                                   By Lemma 56 (Confluence of Completeness)
                        \Theta \vdash e_2 \Leftarrow [\Theta] A_2 p \dashv \Delta
                                                                                                                   Subderivation
                  [\Omega]\Delta \vdash [\Omega]e_2 \Leftarrow [\Omega][\Theta]A_2 p
                                                                                                                   By i.h.
                         \Gamma \vdash A_1 \times A_2 \ type
                                                                                                                   Given
                         \Gamma \vdash A_2 type
                                                                                                                   By inversion
                    \Gamma \longrightarrow \Theta
                                                                                                                   By Lemma 51 (Typing Extension)
                        \Theta \vdash A_2 type
                                                                                                                   By Lemma 38 (Extension Weakening (Types))
                  [\Omega]\Delta \vdash [\Omega]e_2 \Leftarrow [\Omega]A_2 p
                                                                                                                   By Lemma 29 (Substitution Monotonicity)
                 [\Omega]\Delta \vdash \langle [\Omega]e_1, [\Omega]e_2 \rangle \Leftarrow ([\Omega]A_1) \times [\Omega]A_2 p By Decl×I
       \square \square \Delta \vdash [\Omega] \langle e_1, e_2 \rangle \Leftarrow [\Omega] (A_1 \times A_2) \mathfrak{p}
                                                                                                                  By def. of substitution
• Case \frac{\Gamma[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\times\hat{\alpha}_2]\vdash e_1\Leftarrow\hat{\alpha}_1\not\vdash\vdash\Theta\quad\quad\Theta\vdash e_2\Leftarrow[\Theta]\hat{\alpha}_2\not\vdash\vdash\Delta}{\Gamma[\hat{\alpha}:\star]\vdash\langle e_1,e_2\rangle\Leftarrow\hat{\alpha}\not\vdash\vdash\Delta}\times\mathsf{I}\hat{\alpha}
                                          \Delta \longrightarrow \Omega
                                          \Theta \longrightarrow \Delta
                                                                                                        By Lemma 51 (Typing Extension)
                                          \Theta \longrightarrow \Omega
                                                                                                        By Lemma 33 (Extension Transitivity)
        \Gamma[\ldots,\hat{\alpha}:\star=\hat{\alpha}_1\times\hat{\alpha}_2]\vdash e_1\Leftarrow\hat{\alpha}_1\not\vdash\neg\Theta
                                                                                                        Subderivation
                                        [\Omega]\Theta \vdash [\Omega]e_1 \Leftarrow [\Omega]\hat{\alpha}_1 \not
                                                                                                        By i.h.
                                       [\Omega]\Theta = [\Omega]\Delta
                                                                                                        By Lemma 56 (Confluence of Completeness)
                                         [\Omega]\Delta \vdash [\Omega]e_1 \Leftarrow [\Omega]\hat{\alpha}_1 \not
                                                                                                       By above equality
                                               \Theta \vdash e_2 \Leftarrow [\Theta] \hat{\alpha}_2 \not ! \dashv \Delta
                                                                                                        Subderivation
                                         By i.h.
                                [\Omega][\Theta]\hat{\alpha}_2 = [\Omega]\hat{\alpha}_2
                                                                                                        By Lemma 29 (Substitution Monotonicity)
                                         [\Omega]\Delta \vdash [\Omega]e_2 \Leftarrow [\Omega]\hat{\alpha}_2 !
                                                                                                       By above equality
                                            [\Omega]\Delta \vdash \langle [\Omega]e_1, [\Omega]e_2 \rangle \leftarrow ([\Omega]\hat{\alpha}_1) \times [\Omega]\hat{\alpha}_2 \not I By Decl×I
                   ([\Omega]\hat{\alpha}_1) \times [\Omega]\hat{\alpha}_2 = [\Omega]\hat{\alpha}
                                                                                                                                            Similar to the \rightarrow l\hat{\alpha} case (above)
                                            [\Omega]\Delta \vdash [\Omega]\langle e_1, e_2 \rangle \Leftarrow [\Omega]\hat{\alpha} \not
                                                                                                                                            By above equality
         ₽
```

• Case
$$\frac{\Gamma[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star=\hat{\alpha}_1\to\hat{\alpha}_2]\vdash e_0\ s_0:(\hat{\alpha}_1\to\hat{\alpha}_2)\ \not\!\!\!/\gg C\ \not\!\!\!/ +\Delta}{\Gamma[\hat{\alpha}:\star]\vdash e_0\ s_0:\hat{\alpha}\ \not\!\!\!/\gg C\ \not\!\!\!/ +\Delta}\ \hat{\alpha} \text{Spine}$$

$$\Gamma[\ldots,\hat{\alpha}:\star=\hat{\alpha}_1\to\hat{\alpha}_2]\vdash e_0\ s_0:(\hat{\alpha}_1\to\hat{\alpha}_2)\ \not\!\!/\gg C\ \not\!\!\!/ +\Delta \qquad \text{Subderivation}$$

$$[\Omega]\Delta\vdash [\Omega](e_0\ s_0):[\Omega](\hat{\alpha}_1\to\hat{\alpha}_2)\ \not\!\!/\gg [\Omega]C\ \not\!\!/ \qquad \text{By i.h.}$$

$$[\Omega](\hat{\alpha}_1\to\hat{\alpha}_2)=[\Omega]\hat{\alpha} \qquad \qquad \text{Similar to the \to} 1\hat{\alpha} \text{ case}$$

$$[\Omega]\Delta\vdash [\Omega](e_0\ s_0):[\Omega]\hat{\alpha}\ \not\!\!/\gg [\Omega]C\ \not\!\!/ \qquad \text{By above equality}$$

$$\begin{array}{lll} \Gamma \vdash e_0 \Rightarrow B \mathrel{!} \vdash \Theta & \text{Subderivation} \\ \Theta \longrightarrow \Delta & \text{By Lemma 51 (Typing Extension)} \\ \Theta \longrightarrow \Omega & \text{By Lemma 33 (Extension Transitivity)} \\ [\Omega]\Theta \vdash [\Omega]e_0 \Rightarrow [\Omega]B \mathrel{!} & \text{By i.h.} \\ [\Omega]\Delta \vdash [\Omega]e_0 \Rightarrow [\Omega]B \mathrel{!} & \text{By Lemma 56 (Confluence of Completeness)} \\ \Theta \vdash \Pi :: [\Theta]B \Leftarrow [\Theta]C p \vdash \Delta & \text{Subderivation} \\ \Gamma \vdash e_0 \Rightarrow B \mathrel{!} \vdash \Theta & \text{Subderivation} \\ \Theta \vdash B \mathrel{!} type & \text{By Lemma 63 (Well-Formed Outputs of Typing) (Synthesis)} \\ \Gamma \vdash C p type & \text{Given} \\ \Gamma \longrightarrow \Theta & \text{By Lemma 51 (Typing Extension)} \\ \Theta \vdash C p type & \text{By Lemma 41 (Extension Weakening for Principal Typing)} \\ \Theta \vdash [\Theta]C p type & \text{By Lemma 40 (Right-Hand Subst. for Principal Typing)} \\ [\Omega]\Delta \vdash [\Omega]\Pi :: [\Omega]B \Leftarrow [\Omega][\Theta]C p & \text{By i.h. (v)} \\ [\Omega]\Theta \mid C = [\Omega]C & \text{By Lemma 29 (Substitution Monotonicity)} \\ [\Omega]\Delta \vdash [\Omega]\Pi :: [\Omega]B \Leftarrow [\Omega]C p & \text{By above equalities} \\ \end{array}$$

Assume Ω such that $\Delta \longrightarrow \Omega$.

Assume D such that $[\Omega]\Delta \vdash e \Rightarrow D q$.

Hence $[\Omega]\Gamma \vdash e \Rightarrow D q$.

By Theorem 12, there exist B' and Θ' such that $\Gamma \vdash e_0 \Rightarrow B' \neq Q'$ and $\Omega \longrightarrow \Omega'$ and $D = [\Omega']B'$ and $B' = [\Theta']B'$.

By Lemma 5 (Determinacy of Typing), we know $\Theta' = \Theta$ and B' = B, which means $D = [\Omega][\Delta]B$.

By Lemma 7 (Soundness of Match Coverage), $[\Omega]\Delta \vdash [\Omega]\Pi$ *covers* $[\Omega][\Delta]B$ q.

Hence $[\Omega]\Delta \vdash [\Omega]\Pi$ covers D q.

By rule DeclCase, $[\Omega]\Delta \vdash [\Omega]$ case $(e_0, \Pi) \Leftarrow [\Omega]C \mathfrak{p}$

Part (v):

• Case MatchEmpty: Apply rule DeclMatchEmpty.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash e \Leftarrow C \ \mathfrak{p} \dashv \Delta}{(\cdot \Rightarrow e) \vdash \cdot :: C \ \mathfrak{p} \Leftarrow \Delta \ \Gamma \dashv} \ \, \textbf{MatchBase}$$

Apply the i.h. and DeclMatchBase.

• Case MatchUnit: Apply the i.h. and DeclMatchUnit.

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash \pi : \vec{A} \ q \Leftarrow C \ p \dashv \Theta \qquad \Theta \vdash \Pi' :: \vec{A} \ q \Leftarrow C \ p \dashv \Delta}{\Gamma \vdash \pi \ \textbf{I} \ \Pi' :: \vec{A} \ q \Leftarrow C \ p \dashv \Delta} \ \, \text{MatchSeq}$$

Apply the i.h. to each premise, using lemmas for well-formedness under Θ ; then apply DeclMatchSeq.

- Cases Match∃, Match∧, MatchWild, MatchNil, MatchCons:
 Apply the i.h. and the corresponding declarative match rule.
- Cases Match \times , Match $+_k$:

We have $\Gamma \vdash \vec{A}$! *types*, so the first type in \vec{A} has no free existential variables. Apply the i.h. and the corresponding declarative match rule.

Construct Ω' and show $\Delta, z : A !, \Delta' \longrightarrow \Omega'$ as in the \rightarrow I case. Use the i.h., then apply rule DeclMatchNeg.

Part (vi):

• Case
$$\frac{\Gamma \ / \ \sigma \stackrel{\text{\tiny \circ}}{=} \tau : \kappa \dashv \bot}{\Gamma \ / \ \sigma = \tau \vdash \vec{\rho} pe :: \vec{A} \ ! \Leftarrow C \ p \dashv \Gamma} \ \mathsf{Match} \bot$$

$$\text{ } \quad [\Omega]\Gamma \mathbin{/} [\Omega](\sigma = \tau) \vdash [\Omega](\vec{\rho}pe) :: [\Omega]\vec{A} \Leftarrow [\Omega]C \ p \quad \text{ By DeclMatch} \bot$$

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma, \blacktriangleright_P \ / \ \sigma \stackrel{\circ}{=} \tau : \kappa \dashv \Gamma' \qquad \Gamma' \vdash \vec{\rho} \Rightarrow e :: \vec{A} \ q \Leftarrow C \ p \dashv \Delta, \blacktriangleright_P, \Delta'}{\Gamma \ / \ \sigma = \tau \vdash \vec{\rho} \Rightarrow e :: \vec{A} \ ! \Leftarrow C \ p \dashv \Delta} \ \, \text{MatchUnify}$$

$$\begin{array}{ll} \Gamma, \blacktriangleright_P \ / \ \sigma \stackrel{\circ}{=} \tau : \kappa \dashv \Gamma' & \text{Subderivation} \\ (\sigma = \tau) = [\Gamma](\sigma = \tau) & \text{Given} \\ = [\Omega](\sigma = \tau) & \text{By Lemma 29 (Substitution Monotonicity) (i)} \\ \Gamma' = (\Gamma, \blacktriangleright_P, \Theta) & \text{By Lemma 90 (Soundness of Equality Elimination)} \\ \Theta = ((\alpha_1 = t_1), \ldots, (\alpha_n = t_n)) & '' \\ \theta = \mathsf{mgu}([\Omega]\sigma, [\Omega]\tau) & '' \\ [\Omega, \blacktriangleright_P, \Theta]t' = [\theta][\Omega, \blacktriangleright_P]t' & '' \ \text{for all } \Omega, \blacktriangleright_P \vdash t' : \kappa' \end{array}$$

$$\Gamma, \blacktriangleright_{P}, \Theta \vdash \vec{\rho} \Rightarrow e :: \vec{A} \Leftarrow C \ p \dashv \Delta, \blacktriangleright_{P}, \Delta'$$
Subderivation
$$[\Omega, \blacktriangleright_{P}, \Theta](\Delta, \blacktriangleright_{P}, \Delta') \vdash [\Omega, \blacktriangleright_{P}, \Theta](\vec{\rho} \Rightarrow e) :: [\Omega, \blacktriangleright_{P}, \Theta]\vec{A} \Leftarrow [\Omega, \blacktriangleright_{P}, \Theta]C \ p$$
By i.h.

```
(\Omega, \blacktriangleright_{P}, \Theta) = [\theta](\Omega, \blacktriangleright_{P}) \qquad \text{By Lemma 95 (Substitution Upgrade) (iii)} [\Omega, \blacktriangleright_{P}, \Theta] \vec{A} = [\theta][\Omega, \blacktriangleright_{P}] \vec{A} \qquad \text{By Lemma 95 (Substitution Upgrade) (i)} [\Omega, \blacktriangleright_{P}, \Theta] C = [\theta][\Omega, \blacktriangleright_{P}] C \qquad \text{By Lemma 95 (Substitution Upgrade) (i)} [\Omega, \blacktriangleright_{P}, \Theta](\vec{\rho} \Rightarrow e) = [\theta][\Omega](\vec{\rho} \Rightarrow e) \qquad \text{By Lemma 95 (Substitution Upgrade) (iv)} \theta([\Omega, \blacktriangleright_{P}] \Gamma) \vdash [\theta][\Omega](\vec{\rho} \Rightarrow e) :: \theta([\Omega, \blacktriangleright_{P}] \vec{A}) \Leftarrow \theta([\Omega, \blacktriangleright_{P}] C) p \qquad \text{By above equalities} \theta([\Omega] \Gamma) \vdash [\theta][\Omega](\vec{\rho} \Rightarrow e) :: \theta([\Omega] \vec{A}) \Leftarrow \theta([\Omega] C) p \qquad \text{Subst. not affected by } \blacktriangleright_{P} \blacksquare \square \qquad [\Omega] \Gamma / [\Omega](\sigma = \tau) \vdash [\Omega](\vec{\rho} \Rightarrow e) :: [\Omega] \vec{A} \Leftarrow [\Omega] C p \qquad \text{By DeclMatchUnify}
```

K' Completeness

K'.1 Completeness of Auxiliary Judgments

Lemma 92 (Completeness of Instantiation).

Given $\Gamma \longrightarrow \Omega$ and $\operatorname{dom}(\Gamma) = \operatorname{dom}(\Omega)$ and $\Gamma \vdash \tau : \kappa$ and $\tau = [\Gamma]\tau$ and $\hat{\alpha} \in \operatorname{unsolved}(\Gamma)$ and $\hat{\alpha} \notin FV(\tau)$: If $[\Omega]\hat{\alpha} = [\Omega]\tau$

then there are Δ , Ω' such that $\Omega \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$.

Proof. By induction on τ .

We have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq^{\mathcal{P}} [\Omega]A$. We now case-analyze the shape of τ .

• Case $\tau = \hat{\beta}$:

$$\hat{\alpha} \notin FV(\hat{\beta})$$
 Given $\hat{\alpha} \neq \hat{\beta}$ From definition of $FV(-)$ $\hat{\beta} \in \mathsf{unsolved}(\Gamma)$ From $[\Gamma]\hat{\beta} = \hat{\beta}$ Let $\Omega' = \Omega$. By Lemma 32 (Extension Reflexivity)

Now consider whether $\hat{\alpha}$ is declared to the left of $\hat{\beta}$, or vice versa.

- Case $(\Gamma = \Gamma_0[\hat{\beta} : \kappa][\hat{\alpha} : \kappa]$: Similar, but using InstSolve instead of InstReach.
- Case $\tau = \alpha$:

We have $[\Omega] \hat{\alpha} = \alpha$, so (since Ω is well-formed), α is declared to the left of $\hat{\alpha}$ in Ω . We have $\Gamma \longrightarrow \Omega$.

By Lemma 21 (Reverse Declaration Order Preservation), we know that α is declared to the left of $\hat{\alpha}$ in Γ ; that is, $\Gamma = \Gamma_{\Gamma} [\alpha : \kappa] [\hat{\alpha} : \kappa]$.

Let $\Delta = \Gamma_L[\alpha:\kappa][\hat{\alpha}:\kappa=\alpha]$ and $\Omega' = \Omega$. By InstSolve, $\Gamma_L[\alpha:\kappa][\hat{\alpha}:\kappa] \vdash \hat{\alpha} := \alpha:\kappa \dashv \Delta$. By Lemma 27 (Parallel Extension Solution), $\Gamma_L[\alpha:\kappa][\hat{\alpha}:\kappa=\alpha] \longrightarrow \Omega$. We have $\mathsf{dom}(\Delta) = \mathsf{dom}(\Gamma)$ and $\mathsf{dom}(\Omega') = \mathsf{dom}(\Omega)$; therefore, $\mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')$.

• Case $\tau = 1$:

Similar to the $\tau = \alpha$ case, but without having to reason about where α is declared.

• Case $\tau = zero$:

Similar to the $\tau = 1$ case.

• Case $\tau = \tau_1 \oplus \tau_2$:

$$\begin{split} [\Omega] & \hat{\alpha} = [\Omega](\tau_1 \oplus \tau_2) & \text{Given} \\ & = ([\Omega]\tau_1) \oplus ([\Omega]\tau_2) & \text{By definition of substitution} \\ & \tau_1 \oplus \tau_2 = [\Gamma](\tau_1 \oplus \tau_2) & \text{Given} \\ & \tau_1 = [\Gamma]\tau_1 & \text{By definition of substitution and congruence} \\ & \tau_2 = [\Gamma]\tau_2 & \text{Similarly} \\ & \hat{\alpha} \notin FV(\tau_1 \oplus \tau_2) & \text{Given} \\ & \hat{\alpha} \notin FV(\tau_1) & \text{From definition of } FV(-) \\ & \hat{\alpha} \notin FV(\tau_2) & \text{Similarly} \\ & \Gamma = \Gamma_0[\hat{\alpha}:\star] & \text{By } \hat{\alpha} \in \text{unsolved}(\Gamma) \\ & \Gamma \longrightarrow \Omega & \text{Given} \\ & \Gamma_0[\hat{\alpha}:\star] \longrightarrow \Gamma_0[\hat{\alpha}_2:\star,\hat{\alpha}_1:\star,\hat{\alpha}:\star] & \text{By Lemma 23 (Deep Evar Introduction) (i) twice} \\ & \dots,\hat{\alpha}_2,\hat{\alpha}_1 \vdash \hat{\alpha}_1 \oplus \hat{\alpha}_2:\star & \text{Straightforward} \\ & \Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1 \oplus \hat{\alpha}_2] & \text{By Lemma 23 (Deep Evar Introduction) (ii)} \\ & \Gamma_0[\hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}=\hat{\alpha}_1 \oplus \hat{\alpha}_2] & \text{By Lemma 23 (Extension Transitivity)} \end{split}$$

(In the last few lines above, and the rest of this case, we omit the ": ★" annotations in contexts.)

Since $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ and $\Gamma \longrightarrow \Omega$, we know that Ω has the form $\Omega_0[\hat{\alpha} = \tau_0]$.

To show that we can extend this context, we apply Lemma 23 (Deep Evar Introduction) (iii) twice to introduce $\hat{\alpha}_2 = \tau_2$ and $\hat{\alpha}_1 = \tau_1$, and then Lemma 28 (Parallel Variable Update) to overwrite τ_0 :

$$\underbrace{\Omega_0[\hat{\alpha}\!=\!\tau_0]}_{\Omega}\ \longrightarrow\ \Omega_0[\hat{\alpha}_2\!=\!\tau_2,\hat{\alpha}_1\!=\!\tau_1,\hat{\alpha}\!=\!\hat{\alpha}_1\oplus\hat{\alpha}_2]$$

We have $\Gamma \longrightarrow \Omega$, that is,

$$\Gamma_0[\hat{\alpha}] \longrightarrow \Omega_0[\hat{\alpha} = \tau_0]$$

By Lemma 26 (Parallel Admissibility) (i) twice, inserting unsolved variables $\hat{\alpha}_2$ and $\hat{\alpha}_1$ on both contexts in the above extension preserves extension:

$$\underbrace{\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha}] \longrightarrow \Omega_0[\hat{\alpha}_2 = \tau_2,\hat{\alpha}_1 = \tau_1,\hat{\alpha} = \tau_0]}_{\Gamma_1} \quad \text{By Lemma 26 (Parallel Admissibility) (ii) twice} \\ \underbrace{\Gamma_0[\hat{\alpha}_2,\hat{\alpha}_1,\hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2]}_{\Gamma_1} \longrightarrow \underbrace{\Omega_0[\hat{\alpha}_2 = \tau_2,\hat{\alpha}_1 = \tau_1,\hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2]}_{\Omega_1} \quad \text{By Lemma 28 (Parallel Variable Update)}$$

Since $\hat{\alpha} \notin FV(\tau)$, it follows that $[\Gamma_1]\tau = [\Gamma]\tau = \tau$.

Therefore $\hat{\alpha}_1 \notin FV(\tau_1)$ and $\hat{\alpha}_1, \hat{\alpha}_2 \notin FV(\tau_2)$.

By Lemma 55 (Completing Completeness) (i) and (iii), $[\Omega_1]\Gamma_1 = [\Omega]\Gamma$ and $[\Omega_1]\hat{\alpha}_1 = \tau_1$.

By i.h., there are Δ_2 and Ω_2 such that $\Gamma_1 \vdash \hat{\alpha}_1 := \tau_1 : \kappa \dashv \Delta_2$ and $\Delta_2 \longrightarrow \Omega_2$ and $\Omega_1 \longrightarrow \Omega_2$.

Next, note that $[\Delta_2][\Delta_2]\tau_2 = [\Delta_2]\tau_2$.

By Lemma 64 (Left Unsolvedness Preservation), we know that $\hat{\alpha}_2 \in \mathsf{unsolved}(\Delta_2)$.

By Lemma 65 (Left Free Variable Preservation), we know that $\hat{\alpha}_2 \notin FV([\Delta_2]\tau_2)$.

By Lemma 33 (Extension Transitivity), $\Omega \longrightarrow \Omega_2$.

We know $[\Omega_2]\Delta_2 = [\Omega]\Gamma$ because:

$$[\Omega_2]\Delta_2 = [\Omega_2]\Omega_2$$
 By Lemma 54 (Completing Stability)
= $[\Omega]\Omega$ By Lemma 55 (Completing Completeness) (iii)
= $[\Omega]\Gamma$ By Lemma 54 (Completing Stability)

By Lemma 55 (Completing Completeness) (i), we know that $[\Omega_2]\hat{\alpha}_2 = [\Omega_1]\hat{\alpha}_2 = \tau_2$.

By Lemma 55 (Completing Completeness) (i), we know that $[\Omega_2]\tau_2 = [\Omega]\tau_2$.

Hence we know that $[\Omega_2]\Delta_2 \vdash [\Omega_2]\hat{\alpha}_2 \leq^{\mathcal{P}} [\Omega_2]\tau_2$.

By i.h., we have Δ and Ω' such that $\Delta_2 \vdash \hat{\alpha}_2 := [\Delta_2]\tau_2 : \kappa \dashv \Delta$ and $\Omega_2 \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$.

By rule InstBin, $\Gamma \vdash \hat{\alpha} := \tau : \kappa \dashv \Delta$.

By Lemma 33 (Extension Transitivity), $\Omega \longrightarrow \Omega'$.

• Case $\tau = \operatorname{succ}(\tau_0)$:

Similar to the $\tau = \tau_1 \oplus \tau_2$ case, but simpler.

Lemma 93 (Completeness of Checkeq).

Given $\Gamma \longrightarrow \Omega$ and $dom(\Gamma) = dom(\Omega)$

and $\Gamma \vdash \sigma : \kappa$ and $\Gamma \vdash \tau : \kappa$

and $[\Omega]\sigma = [\Omega]\tau$

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then $\Gamma \vdash [\Gamma] \sigma \stackrel{\circ}{=} [\Gamma] \tau : \kappa \dashv \Delta$

where $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$.

Proof. By mutual induction on the sizes of $[\Gamma]\sigma$ and $[\Gamma]\tau$.

We distinguish cases of $[\Gamma]\sigma$ and $[\Gamma]\tau$.

• Case $[\Gamma]\sigma = [\Gamma]\tau = 1$:

$$\Gamma \vdash 1 \stackrel{\circ}{=} 1 : \star \dashv \underbrace{\Gamma}_{\Delta} \qquad \text{By CheckeqUnit}$$
 Let $\Omega' = \Omega$.
$$\Gamma \longrightarrow \Omega \qquad \qquad \text{Given}$$

$$\Delta \longrightarrow \Omega' \qquad \qquad \Delta = \Gamma \text{ and } \Omega' = \Omega$$

$$\text{Given}$$

$$\text{dom}(\Gamma) = \text{dom}(\Omega) \qquad \qquad \text{Given}$$

• Case $[\Gamma]\sigma = [\Gamma]t = \mathsf{zero}$:

 $\Omega \longrightarrow \Omega'$

Similar to the case for 1, applying CheckeqZero instead of CheckeqUnit.

• Case $[\Gamma]\sigma = [\Gamma]t = \alpha$:

Similar to the case for 1, applying CheckeqVar instead of CheckeqUnit.

- Case $[\Gamma]\sigma = \hat{\alpha}$ and $[\Gamma]t = \hat{\beta}$:
 - If $\hat{\alpha} = \hat{\beta}$: Similar to the case for 1, applying CheckeqVar instead of CheckeqUnit.

By Lemma 32 (Extension Reflexivity)

- If $\hat{\alpha} \neq \hat{\beta}$:

• Case $[\Gamma]\sigma = \hat{\alpha}$ and $[\Gamma]t = 1$ or zero or α : Similar to the previous case, except:

$$\hat{\alpha} \not\in \mathit{FV}(\underbrace{1}_{\lceil \Gamma \rceil t}) \quad \text{ By definition of } \mathit{FV}(-)$$

and similarly for 1 and α .

- Case $[\Gamma]t = \hat{\alpha}$ and $[\Gamma]\sigma = 1$ or zero or α : Symmetric to the previous case.
- Case $[\Gamma]\sigma = \hat{\alpha}$ and $[\Gamma]t = \text{succ}([\Gamma]t_0)$:

If $\hat{\alpha} \notin FV([\Gamma]t_0)$, then $\hat{\alpha} \notin FV([\Gamma]t)$. Proceed as in the previous several cases.

The other case, $\hat{\alpha} \in FV([\Gamma]t_0)$, is impossible:

We have $\hat{\alpha} \leq [\Gamma] t_0$.

Therefore $\hat{\alpha} \prec \text{succ}([\Gamma]t_0)$, that is, $\hat{\alpha} \prec [\Gamma]t$.

By a property of substitutions, $[\Omega] \hat{\alpha} \prec [\Omega] [\Gamma] t$.

Since $\Gamma \longrightarrow \Omega$, by Lemma 29 (Substitution Monotonicity) (i), $[\Omega][\Gamma]t = [\Omega]t$, so $[\Omega]\hat{\alpha} \prec [\Omega]t$. But it is given that $[\Omega]\hat{\alpha} = [\Omega]t$, a contradiction.

- Case $[\Gamma]t = \hat{\alpha}$ and $[\Gamma]\sigma = \text{succ}([\Gamma]\sigma_0)$: Symmetric to the previous case.
- Case $[\Gamma]\sigma = [\Gamma]\sigma_1 \oplus [\Gamma]\sigma_2$ and $[\Gamma]t = [\Gamma]t_1 \oplus [\Gamma]t_2$:

- Case $[\Gamma]\sigma = \hat{\alpha}$ and $[\Gamma]t = t_1 \oplus t_2$: Similar to the $\hat{\alpha}/\text{succ}(-)$ case, showing the impossibility of $\hat{\alpha} \in FV([\Gamma]t_k)$ for k = 1 and k = 2.
- Case $[\Gamma]t = \hat{\alpha}$ and $[\Gamma]\sigma = \sigma_1 \oplus \sigma_2$: Symmetric to the previous case.

Lemma 94 (Completeness of Elimeq).

If $[\Gamma]\sigma = \sigma$ and $[\Gamma]t = t$ and $\Gamma \vdash \sigma$: κ and $\Gamma \vdash t$: κ and $FEV(\sigma) \cup FEV(t) = \emptyset$ then:

- (1) If $\mathsf{mgu}(\sigma,t) = \theta$ then $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv (\Gamma,\Delta)$ where Δ has the form $\alpha_1 = t_1, \ldots, \alpha_n = t_n$ and for all u such that $\Gamma \vdash u : \kappa$, it is the case that $[\Gamma,\Delta]u = \theta([\Gamma]u)$.
- (2) If $mgu(\sigma, t) = \bot$ (that is, no most general unifier exists) then $\Gamma / \sigma \stackrel{\circ}{=} t : \kappa \dashv \bot$.

Proof. By induction on the structure of $[\Gamma]\sigma$ and $[\Gamma]t$.

• Case $[\Omega]\sigma = t = zero$:

$$\begin{array}{lll} \text{mgu}(\mathsf{zero},\mathsf{zero}) = \cdot & \text{By properties of unification} \\ & \Gamma \ / \ \mathsf{zero} \stackrel{\circ}{=} \ \mathsf{zero} : \mathbb{N} \dashv \Gamma & \text{By rule ElimeqZero} \\ & \Gamma \ / \ \mathsf{zero} \stackrel{\circ}{=} \ \mathsf{zero} : \mathbb{N} \dashv \Gamma, \Delta & \text{where } \Delta = \cdot \\ & \mathbb{Suppose} \ \Gamma \vdash \mathfrak{u} : \kappa'. \\ & [\Gamma, \Delta]\mathfrak{u} = [\Gamma]\mathfrak{u} & \text{where } \Delta = \cdot \\ & = \theta([\Gamma]\mathfrak{u}) & \text{where } \theta \text{ is the identity} \\ \end{array}$$

- Case $\sigma = \operatorname{succ}(\sigma')$ and $t = \operatorname{succ}(t')$:
 - Case $mgu(succ(\sigma'), succ(t')) = \theta$:

```
mgu(\sigma', t') = mgu(succ(\sigma'), succ(t')) = \theta
                                                                                        By properties of unification
            succ(\sigma') = [\Gamma] succ(\sigma')
                                                                                        Given
                            = \operatorname{succ}([\Gamma]\sigma')
                                                                                        By definition of substitution
                       \sigma' = \lceil \Gamma \rceil \sigma'
                                                                                        By injectivity of successor
             succ(t') = [\Gamma] succ(t')
                                                                                        Given
                            =\operatorname{succ}([\Gamma]t')
                                                                                        By definition of substitution
                        t' = [\Gamma]t'
                                                                                        By injectivity of successor
                        \Gamma \ / \ \sigma' \stackrel{\text{\tiny $\circ$}}{=} t' : \mathbb{N} \dashv \Gamma, \Delta
                                                                                        By i.h.
                [\Gamma, \Delta]u = \theta([\Gamma]u) for all u such that ...
₽
                        \Gamma / \operatorname{succ}(\sigma') \stackrel{\circ}{=} \operatorname{succ}(t') : \mathbb{N} \dashv \Gamma, \Delta
                                                                                       By rule ElimeqSucc
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```

- Case $mgu(succ(\sigma'), succ(t')) = \bot$:

```
mgu(\sigma', t') = mgu(succ(\sigma'), succ(t')) = \bot
                                                                         By properties of unification
   succ(\sigma') = [\Gamma] succ(\sigma')
                                                                          Given
                  = \operatorname{succ}([\Gamma]\sigma')
                                                                          By definition of substitution
              \sigma' = [\Gamma]\sigma'
                                                                         By injectivity of successor
    succ(t') = [\Gamma] succ(t')
                  =\operatorname{succ}([\Gamma]t')
                                                                         By definition of substitution
               t' = [\Gamma]t'
                                                                         By injectivity of successor
               \Gamma / \sigma' \stackrel{\circ}{=} t' : \mathbb{N} \dashv \bot
                                                                         By i.h.
               \Gamma / \operatorname{succ}(\sigma') \stackrel{\circ}{=} \operatorname{succ}(t') : \mathbb{N} \dashv \bot
                                                                         By rule ElimeqSucc
```

```
• Case \sigma = \sigma_1 \oplus \sigma_2 and t = t_1 \oplus t_2:
    First we establish some properties of the subterms:
               \sigma_1 \oplus \sigma_2 = [\Gamma](\sigma_1 \oplus \sigma_2)
                                                               Given
                              = [\Gamma]\sigma_1 \oplus [\Gamma]\sigma_2 By definition of substitution
                    [\Gamma]\sigma_1 = \sigma_1
                                                               By injectivity of \oplus
      REF
                    [\Gamma]\sigma_2 = \sigma_2
                                                               By injectivity of \oplus
      RF.
                t_1 \oplus t_2 = [\Gamma](t_1 \oplus t_2)
                                                               Given
                              = [\Gamma]t_1 \oplus [\Gamma]t_2
                                                               By definition of substitution
                     [\Gamma]t_1=t_1
                                                               By injectivity of \oplus
      EF
                     [\Gamma]t_2 = t_2
                                                               By injectivity of \oplus
      B
        - Subcase mgu(\sigma, t) = \bot:
                * Subcase mgu(\sigma_1, t_1) = \bot:
                       \Gamma / \sigma_1 \stackrel{\circ}{=} t_1 : \kappa \dashv \bot
                                                                                   By i.h.
                       \Gamma / \sigma_1 \oplus \sigma_2 \stackrel{\circ}{=} t_1 \oplus t_2 : \kappa \dashv \bot By rule ElimegBinBot
                * Subcase mgu(\sigma_1, t_1) = \theta_1 and mgu(\theta_1(\sigma_2), \theta_1(t_2)) = \bot:
                                 \Gamma / \sigma_1 \stackrel{\circ}{=} t_1 : \kappa \dashv \Gamma, \Delta_1
                                                                                                     By i.h.
                       [\Gamma, \Delta_1]u = \theta_1([\Gamma]u) for all u such that ...
                                                   [\Gamma, \Delta_1]\sigma_2 = \theta_1([\Gamma]\sigma_2) Above line with \sigma_2 as u
                                                                   =\theta_1(\sigma_2)
                                                                                            [\Gamma]\sigma_2 = \sigma_2
                                                                                            Above line with t2 as u
                                                   [\Gamma, \Delta_1] \mathbf{t}_2 = \theta_1([\Gamma] \mathbf{t}_2)
                                                                                             Since [\Gamma]\sigma_2 = \sigma_2
                                                                   =\theta_1(t_2)
                                                                                            By transitivity of equality
                        mgu([\Gamma, \Delta_1]\sigma_2, [\Gamma, \Delta_1]t_2) = \theta_2
                        [\Gamma, \Delta_1][\Gamma, \Delta_1]\sigma_2 = [\Gamma, \Delta_1]\sigma_2 By Lemma 29 (Substitution Monotonicity)
                        [\Gamma, \Delta_1][\Gamma, \Delta_1]t_2 = [\Gamma, \Delta_1]t_2
                                                                          By Lemma 29 (Substitution Monotonicity)
                                [\Gamma, \Delta_1] / [\Gamma, \Delta_1] \sigma_2 \stackrel{\text{def}}{=} [\Gamma, \Delta_1] t_2 : \kappa \dashv \bot By i.h.
                                  \Gamma \ / \ \sigma_1 \oplus \sigma_2 \stackrel{\mathtt{e}}{=} t_1 \oplus t_2 : \kappa \dashv \bot
                                                                                                   By rule ElimeqBin
        - Subcase mgu(\sigma, t) = \theta:
                     \mathsf{mgu}(\sigma_1 \oplus \sigma_2, \mathsf{t}_1 \oplus \mathsf{t}_2) = \theta = \theta_2 \circ \theta_1
                                                                                                                              By properties of unifiers
                                        mgu(\sigma_1, t_1) = \theta_1
                                                                                                                              "
                         mgu(\theta_1(\sigma_2), \theta_1(t_2)) = \theta_2
                                                          \Gamma / \sigma_1 \stackrel{\circ}{=} t_1 : \kappa \dashv \Gamma, \Delta_1
                                                                                                                              By i.h.
                                                [\Gamma, \Delta_1]u = \theta_1([\Gamma]u) for all u such that ...
                                                                                                              Above line with \sigma_2 as \mathfrak u
                                              [\Gamma, \Delta_1]\sigma_2 = \theta_1([\Gamma]\sigma_2)
```

 $[\Gamma]\sigma_2 = \sigma_2$

 $[\Gamma]\sigma_2 = \sigma_2$

Above line with t2 as u

By transitivity of equality

 $=\theta_1(\sigma_2)$

 $=\theta_1(t_2)$

 $[\Gamma,\Delta_1]t_2=\theta_1([\Gamma]t_2)$

 $\mathsf{mgu}([\Gamma, \Delta_1]\sigma_2, [\Gamma, \Delta_1]t_2) = \theta_2$

П

$$[\Gamma, \Delta_1][\Gamma, \Delta_1]\sigma_2 = [\Gamma, \Delta_1]\sigma_2$$
 By Lemma 29 (Substitution Monotonicity) $[\Gamma, \Delta_1][\Gamma, \Delta_1]t_2 = [\Gamma, \Delta_1]t_2$ By Lemma 29 (Substitution Monotonicity)

Suppose
$$\Gamma \vdash u : \kappa'$$
.
$$[\Gamma, \Delta_1, \Delta_2] u = \theta_2([\Gamma, \Delta_1] u) \quad \text{By **}$$

$$= \theta_2(\theta_1([\Gamma] u)) \quad \text{By *}$$

$$= \theta([\Gamma] u) \quad \theta = \theta_2 \circ \theta_1$$

- Case $\sigma = \alpha$:
 - **–** Subcase α ∈ FV(t):

$$\begin{array}{ll} \mathsf{mgu}(\alpha,t) = \bot & \quad \text{By properties of unification} \\ & \quad \Gamma \ / \ \alpha \stackrel{\circ}{=} t : \kappa \dashv \bot & \quad \text{By rule ElimeqUvarL} \\ \end{array}$$

- Subcase α ∉ FV(t):

$$\begin{array}{ll} \mathsf{mgu}(\alpha,t) = [t/\alpha] & \mathsf{By} \ \mathsf{properties} \ \mathsf{of} \ \mathsf{unification} \\ (\alpha = t') \not \in \Gamma & [\Gamma] \alpha = \alpha \\ \hline \qquad \qquad \qquad \Gamma \ / \ \alpha \stackrel{\circ}{=} t : \kappa \dashv \Gamma, \alpha = t & \mathsf{By} \ \mathsf{rule} \ \mathsf{ElimeqUvarL} \end{array}$$

Suppose
$$\Gamma \vdash u : \kappa'$$
.
 $[\Gamma, \alpha = t]u = [\Gamma]([t/\alpha]u)$ By definition of substitution $= [[\Gamma]t/\alpha][\Gamma]u$ By properties of substitution $= [t/\alpha][\Gamma]u$ $[\Gamma]t = t$

• Case $t = \alpha$: Similar to previous case.

Lemma 95 (Substitution Upgrade).

If Δ has the form $\alpha_1=t_1,\ldots,\alpha_n=t_n$ and, for all u such that $\Gamma\vdash u:\kappa$, it is the case that $[\Gamma,\Delta]u=\theta([\Gamma]u)$, then:

(i) If
$$\Gamma \vdash A$$
 type then $[\Gamma, \Delta]A = \theta([\Gamma]A)$.

(ii) If
$$\Gamma \longrightarrow \Omega$$
 then $[\Omega]\Gamma = \theta([\Omega]\Gamma)$.

(iii) If
$$\Gamma \longrightarrow \Omega$$
 then $[\Omega, \Delta](\Gamma, \Delta) = \theta([\Omega]\Gamma)$.

(iv) If
$$\Gamma \longrightarrow \Omega$$
 then $[\Omega, \Delta]e = \theta([\Omega]e)$.

Proof. Part (i): By induction on the given derivation, using the given "for all" at the leaves.

Part (ii): By induction on the given derivation, using part (i) in the →Var case.

Part (iii): By induction on Δ . In the base case ($\Delta = \cdot$), use part (ii). Otherwise, use the i.h. and the definition of context substitution.

Part (iv): By induction on e, using part (i) in the
$$e = (e_0 : A)$$
 case.

```
Lemma 96 (Completeness of Propequiv).
```

```
Given \Gamma \longrightarrow \Omega
and \Gamma \vdash P prop and \Gamma \vdash Q prop
and [\Omega]P = [\Omega]Q
then \Gamma \vdash [\Gamma]P \equiv [\Gamma]Q \dashv \Delta
where \Delta \longrightarrow \Omega' and \Omega \longrightarrow \Omega'.
```

Proof. By induction on the given derivations. There is only one possible case:

```
 \bullet \  \, \textbf{Case} \  \, \frac{\Gamma \vdash \sigma_1 : \mathbb{N} \qquad \Gamma \vdash \sigma_2 : \mathbb{N}}{\Gamma \vdash \sigma_1 = \sigma_2 \ prop} \  \, \textbf{EqProp} \quad \frac{\Gamma \vdash \tau_1 : \mathbb{N} \qquad \Gamma \vdash \tau_2 : \mathbb{N}}{\Gamma \vdash \tau_1 = \tau_2 \ prop} \  \, \textbf{EqProp} 
                  [\Omega](\sigma_1 = \sigma_2) = [\Omega](\tau_1 = \tau_2)
                                  [\Omega]\sigma_1 = [\Omega]\tau_1
                                                                                                                     Definition of substitution
                                  [\Omega]\sigma_2 = [\Omega]\tau_2
                                              \Gamma \vdash \sigma_1 : \mathbb{N}
                                                                                                                      Subderivation
                                              \Gamma \vdash \tau_1 : \mathbb{N}
                                                                                                                      Subderivation
                                              \Gamma \vdash [\Gamma]\sigma_1 \triangleq [\Gamma]\sigma_2 : \mathbb{N} \dashv \Theta
                                                                                                                     By Lemma 93 (Completeness of Checkeg)
                                      \Theta \longrightarrow \Omega_0
                                      \Omega \longrightarrow \Omega_0
                                              \Gamma \vdash \sigma_2 : \mathbb{N}
                                                                                                                      Subderivation
                                            \Theta \vdash \sigma_2 : \mathbb{N}
                                                                                                                     By Lemma 36 (Extension Weakening (Sorts))
                                            \Theta \vdash \tau_2 : \mathbb{N}
                                            \Theta \vdash [\Theta]\tau_1 \stackrel{\circ}{=} [\Theta]\tau_2 : \mathbb{N} \dashv \Delta
                                                                                                                      By Lemma 93 (Completeness of Checkeq)
                                       \Delta \longrightarrow \Omega_0
        3
                                    \Omega_0 \longrightarrow \Omega'
                                   [\Theta]\tau_1 = [\Theta][\Gamma]\tau_1
                                                                                                                      By Lemma 29 (Substitution Monotonicity) (i)
                                   [\Theta]\tau_2 = [\Theta][\Gamma]\tau_2
                                                                                                                     By above equalities
                                            \Theta \vdash [\Theta][\Gamma]\tau_1 \stackrel{\text{\tiny $}}{=} [\Theta][\Gamma]\tau_2 : \mathbb{N} \dashv \Delta
                                      \Omega \longrightarrow \Omega'
                                                                                                                      By Lemma 33 (Extension Transitivity)
        13
                    \Gamma \vdash ([\Gamma]\sigma_1 = [\Theta]\sigma_2) \equiv ([\Gamma]\tau_1 = [\Theta]\tau_2) \dashv \Gamma
                                                                                                                  By \equiv PropEq
                  \Gamma \vdash ([\Gamma]\sigma_1 = [\Gamma]\sigma_2) \equiv ([\Gamma]\tau_1 = [\Gamma]\tau_2) \dashv \Gamma
                                                                                                                   By above equalities
```

Lemma 97 (Completeness of Checkprop).

```
\begin{split} & \text{If } \Gamma \longrightarrow \Omega \text{ and } \mathsf{dom}(\Gamma) = \mathsf{dom}(\Omega) \\ & \text{and } \Gamma \vdash P \text{ } \textit{prop} \\ & \text{and } [\Gamma]P = P \\ & \text{and } [\Omega]\Gamma \vdash [\Omega]P \text{ } \textit{true} \\ & \text{then } \Gamma \vdash P \text{ } \textit{true} \dashv \Delta \\ & \text{where } \Delta \longrightarrow \Omega' \text{ and } \Omega \longrightarrow \Omega' \text{ and } \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega'). \end{split}
```

Proof. Only one rule, DeclCheckpropEq, can derive $[\Omega]\Gamma \vdash [\Omega]P$ true, so by inversion, P has the form $(t_1 = t_2)$ where $[\Omega]t_1 = [\Omega]t_2$.

```
By inversion on \Gamma \vdash (t_1 = t_2) prop, we have \Gamma \vdash t_1 : \mathbb{N} and \Gamma \vdash t_2 : \mathbb{N}.
```

Then by Lemma 93 (Completeness of Checkeq), $\Gamma \vdash [\Gamma]t_1 \stackrel{\circ}{=} [\Gamma]t_2 : \mathbb{N} \dashv \Delta$ where $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$. By CheckpropEq, $\Gamma \vdash (t_1 = t_2)$ true $\dashv \Delta$.

K'.2 Completeness of Equivalence and Subtyping

Lemma 98 (Completeness of Equiv).

If $\Gamma \longrightarrow \Omega$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $[\Omega]A = [\Omega]B$

then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A \equiv [\Gamma]B \dashv \Delta$.

Proof. By induction on the derivations of $\Gamma \vdash A$ *type* and $\Gamma \vdash B$ *type*.

We distinguish cases of the rule concluding the first derivation. In the first four cases (ImpliesWF, WithWF, ForallWF, ExistsWF), it follows from $[\Omega]A = [\Omega]B$ and the syntactic invariant that Ω substitutes terms t (rather than types A) that the second derivation is concluded by the *same* rule. Moreover, if none of these three rules concluded the first derivation, the rule concluding the second derivation must *not* be ImpliesWF, WithWF, ForallWF or ExistsWF either.

Because Ω is predicative, the head connective of $[\Gamma]A$ must be the same as the head connective of $[\Omega]A$. We distinguish cases that are *imposs*. (impossible), **fully written out**, and similar to fully-written-out cases. For the lower-right case, where both $[\Gamma]A$ and $[\Gamma]B$ have a binary connective \oplus , it must be the same connective.

The Vec type is omitted from the table, but can be treated similarly to \supset and \land .

						[Γ]B			
		\supset	\wedge	∀β.Β′	∃β. B′	1	α	β	$B_1 \oplus B_2$
	\supset	Implies	imposs.	imposs.	imposs.	imposs.	imposs.	imposs.	imposs.
	\wedge	imposs.	With	imposs.	imposs.	imposs.	imposs.	imposs.	imposs.
,	∀α. A′	imposs.	imposs.	Forall	imposs.	imposs.	imposs.	imposs.	imposs.
	∃α. A′	imposs.	imposs.	imposs.	Exists	imposs.	imposs.	imposs.	imposs.
	1	imposs.	imposs.	imposs.	imposs.	2.Units	imposs.	2.BEx.Unit	imposs.
$[\Gamma]A$	α	imposs.	imposs.	imposs.	imposs.	imposs.	2.Uvars	2.BEx.Uvar	imposs.
	â	imposs.	imposs.	imposs.	imposs.	2.AEx.Unit	2.AEx.Uvar	2.AEx.SameEx 2.AEx.OtherEx	2.AEx.Bin
A	$A_1 \oplus A_2$	imposs.	imposs.	imposs.	imposs.	imposs.	imposs.	2.BEx.Bin	2.Bins

$$\bullet \ \, \textbf{Case} \ \, \frac{\Gamma \vdash P \ \textit{prop} \qquad \Gamma \vdash A_0 \ \textit{type}}{\Gamma \vdash P \supset A_0 \ \textit{type}} \ \, \textbf{ImpliesWF}$$

This case of the rule concluding the first derivation coincides with the Implies entry in the table.

We have $[\Omega]A = [\Omega]B$, that is, $[\Omega](P \supset A_0) = [\Omega]B$.

Because Ω is predicative, B must have the form $Q \supset B_0$, where $[\Omega]P = [\Omega]Q$ and $[\Omega]A_0 = [\Omega]B_0$.

```
\Gamma \vdash P \ prop
                                                      Subderivation
           \Gamma \vdash A_0 type
                                                      Subderivation
           \Gamma \vdash Q \supset B_0 type
                                                      Given
           \Gamma \vdash Q \; \textit{prop}
                                                      By inversion on rule ImpliesWF
           \Gamma \vdash B_0 \ type
           \Gamma \vdash [\Gamma] P \equiv [\Gamma] Q \dashv \Theta
                                                      By Lemma 96 (Completeness of Propequiv)
      \Theta \longrightarrow \Omega_0
     \Omega \longrightarrow \Omega_0
       \Gamma \longrightarrow \Theta
                                                      By Lemma 48 (Prop Equivalence Extension)
           \Gamma \vdash A_0 \ type
           \Gamma \vdash B_0 \ type
                                                     Above
  [\Omega]A_0 = [\Omega]B_0
                                                     Above
[\Omega_0]A_0 = [\Omega_0]B_0
                                                      By Lemma 55 (Completing Completeness) (ii) twice
           \Gamma \vdash [\Gamma] A_0 \equiv [\Gamma] B_0 \dashv \Delta
                                                     By i.h.
      \Delta \longrightarrow \Omega'
    \Omega_0 \longrightarrow \Omega'
\Omega \longrightarrow \Omega'
                                                                               By Lemma 33 (Extension Transitivity)
     \Gamma \vdash ([\Gamma]P \supset [\Gamma]A_0) \equiv ([\Gamma]Q \supset [\Gamma]B_0) \dashv \Delta \quad \text{By} \equiv \supset
                                                                               By definition of substitution
     \Gamma \vdash [\Gamma](P \supset A_0) \equiv [\Gamma](Q \supset B_0) \dashv \Delta
```

• Case WithWF: Similar to the ImpliesWF case, coinciding with the With entry in the table.

• Case
$$\frac{\Gamma, \alpha : \kappa \vdash A_0 \ type}{\Gamma \vdash \forall \alpha : \kappa . \ A_0 \ type} \text{ For all WF}$$

This case coincides with the Forall entry in the table.

- Case ExistsWF: Similar to the ForallWF case. (This is the Exists entry in the table.)
- Case BinWF: If BinWF also concluded the second derivation, then the proof is similar to the ImpliesWF case, but on the first premise, using the i.h. instead of Lemma 96 (Completeness of Propequiv). This is the 2.Bins entry in the lower right corner of the table.

If BinWF did not conclude the second derivation, we are in the 2.AEx.Bin or 2.BEx.Bin entries; see below.

In the remainder, we cover the 4×4 region in the lower right corner, starting from **2.Units**. We already handled the **2.Bins** entry in the extreme lower right corner. At this point, we split on the forms of $[\Gamma]A$ and $[\Gamma]B$ instead; in the remaining cases, one or both types is atomic (e.g. **2.Uvars**, **2.AEx.Bin**) and we will not need to use the induction hypothesis.

• Case 2.Units: $[\Gamma]A = [\Gamma]B = 1$

$$\begin{array}{lll} & \Gamma \vdash 1 \equiv 1 \dashv \Gamma & \text{By} \equiv \text{Unit} \\ & \Gamma \longrightarrow \Omega & \text{Given} \\ & \text{Let } \Omega' = \Omega'. \\ & & \Delta \longrightarrow \Omega & \Delta = \Gamma \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & &$$

• Case 2.Uvars: $[\Gamma]A = [\Gamma]B = \alpha$

• Case 2.AExUnit: $[\Gamma]A = \hat{\alpha}$ and $[\Gamma]B = 1$

```
\Gamma \longrightarrow \Omega
                                                Given
             1 = [\Omega]1
                                                By definition of substitution
                                                By definition of FV(-)
              \hat{\alpha} \notin FV(1)
       [\Omega]\hat{\alpha} = [\Omega]1
                                                Given
              \Gamma \vdash \hat{\alpha} := 1 : \star \dashv \Delta By Lemma 92 (Completeness of Instantiation) (1)
         \Omega \longrightarrow \Omega'
ESF
                                                 "
         \Delta \longrightarrow \Omega'
             1 = \lceil \Gamma \rceil 1
                                                By definition of substitution
              \hat{\alpha} \notin FV(1)
                                                By definition of FV(-)
E P
              \Gamma \vdash \hat{\alpha} \equiv 1 \dashv \Delta
                                                By ≡InstantiateL
```

- Case 2.BExUnit: $[\Gamma]A = 1$ and $[\Gamma]B = \hat{\alpha}$ Symmetric to the 2.AExUnit case.
- Case 2.AEx.Uvar: $[\Gamma]A = \hat{\alpha}$ and $[\Gamma]B = \alpha$ Similar to the 2.AEx.Unit case, using $\beta = [\Omega]\beta = [\Gamma]\beta$ and $\hat{\alpha} \notin FV(\beta)$.
- Case 2.BExUvar: $[\Gamma]A = 1$ and $[\Gamma]B = \hat{\alpha}$ Symmetric to the 2.AExUvar case.
- Case 2.AEx.SameEx: $[\Gamma]A = \hat{\alpha} = \hat{\beta} = [\Gamma]B$

• Case 2.AEx.OtherEx: $[\Gamma]A = \hat{\alpha}$ and $[\Gamma]B = \hat{\beta}$ and $\hat{\alpha} \neq \hat{\beta}$

Either $\hat{\alpha} \in FV([\Gamma]\hat{\beta})$, or $\hat{\alpha} \notin FV([\Gamma]\hat{\beta})$.

- $\hat{\alpha}$ ∈ $FV([\Gamma]\hat{\beta})$:

We have $\hat{\alpha} \leq [\Gamma] \hat{\beta}$.

Therefore $\hat{\alpha} = [\Gamma]\hat{\beta}$, or $\hat{\alpha} \prec [\Gamma]\hat{\beta}$.

But we are in Case 2.AEx.OtherEx, so the former is impossible.

Therefore, $\hat{\alpha} \prec [\Gamma] \hat{\beta}$.

By a property of substitutions, $[\Omega] \hat{\alpha} \prec [\Omega] [\Gamma] \hat{\beta}$.

Since $\Gamma \longrightarrow \Omega$, by Lemma 29 (Substitution Monotonicity) (iii), $[\Omega][\Gamma]\hat{\beta} = [\Omega]\hat{\beta}$, so $[\Omega]\hat{\alpha} \prec [\Omega]\hat{\beta}$.

But it is given that $[\Omega]\hat{\alpha} = [\Omega]\hat{\beta}$, a contradiction.

 $- \hat{\alpha} \notin FV([\Gamma]\hat{\beta})$:

$$\Gamma \vdash \hat{\alpha} := [\Gamma] \hat{\beta} : \star \dashv \Delta$$
 By Lemma 92 (Completeness of Instantiation)

$$\Gamma \vdash \hat{\alpha} \equiv [\Gamma] \hat{\beta} \dashv \Delta \qquad \text{By} \equiv \text{InstantiateL}$$

$$\quad \square \quad \Omega \longrightarrow \Omega'$$

• Case 2.AEx.Bin: $[\Gamma]A = \hat{\alpha}$ and $[\Gamma]B = B_1 \oplus B_2$

Since $[\Gamma]B$ is an arrow, it cannot be exactly $\hat{\alpha}$. By the same reasoning as in the previous case (2.AEx.OtherEx), $\hat{\alpha} \notin FV([\Gamma]\hat{\beta})$.

$$\Gamma \vdash \hat{\alpha} := [\Gamma]B : \star \dashv \Delta \qquad \text{By Lemma 92 (Completeness of Instantiation)}$$

$$\begin{array}{ccc} & \Delta \longrightarrow \Omega' & & \\ & & \\ & & \Omega \longrightarrow \Omega' & & \\ & & \end{array}$$

$$\Gamma \vdash [\Gamma] A \equiv [\Gamma] B \rightarrow \Delta \quad \text{By} \equiv \text{InstantiateL}$$

• Case 2.BEx.Bin: $[\Gamma]A = A_1 \oplus A_2$ and $[\Gamma]B = \hat{\beta}$

Symmetric to the **2.AEx.Bin** case, applying \equiv InstantiateR instead of \equiv InstantiateL.

Theorem 10 (Completeness of Subtyping).

If $\Gamma \longrightarrow \Omega$ and $\operatorname{dom}(\Gamma) = \operatorname{dom}(\Omega)$ and $\Gamma \vdash A$ type and $\Gamma \vdash B$ type and $[\Omega]\Gamma \vdash [\Omega]A \leq^{\mathcal{P}} [\Omega]B$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $\operatorname{dom}(\Delta) = \operatorname{dom}(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash [\Gamma]A <:^{\mathcal{P}} [\Gamma]B \dashv \Delta$.

Proof. By induction on the number of \forall/\exists quantifiers in $[\Omega]A$ and $[\Omega]B$.

It is straightforward to show $dom(\Delta) = dom(\Omega')$; for examples of the necessary reasoning, see the proof of Theorem 12.

We have $[\Omega]\Gamma \vdash [\Omega]A <^{\text{join}(\text{pol}(B),\text{pol}(A))} [\Omega]B$.

• Case
$$\frac{[\Omega]\Gamma \vdash [\Omega]A \ type \qquad nonpos([\Omega]A)}{[\Omega]\Gamma \vdash [\Omega]A \leq^{-}\underbrace{[\Omega]A}_{[\Omega]B}} \leq \mathsf{Refl} -$$

First, we observe that, since applying Ω as a substitution leaves quantifiers alone, the quantifiers that head A must also head B. For convenience, we alpha-vary B to quantify over the same variables as A.

– If A is headed by
$$\forall$$
, then $[\Omega]A = (\forall \alpha : \kappa. [\Omega]A_0) = (\forall \alpha : \kappa. [\Omega]B_0) = [\Omega]B$.
Let $\Gamma_0 = (\Gamma, \alpha : \kappa, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa)$.
Let $\Omega_0 = (\Omega, \alpha : \kappa, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa = \alpha)$.

* If $pol(A_0) \in \{-, 0\}$, then:

(We elide the straightforward use of lemmas about context extension.)

$$\begin{split} & [\Omega_0]\Gamma_0 \vdash [\Omega]A_0 \leq^- [\Omega]A_0 & \text{By} \leq \text{Refl} - \\ & [\Omega_0]\Gamma_0 \vdash [\Omega_0][\hat{\alpha}/\alpha]A_0 \leq^- A_0 & \text{By def. of subst.} \\ & \Delta_0 \longrightarrow \Omega_0' & \text{By i.h. (fewer quantifiers)} \\ & \Omega_0 \longrightarrow \Omega_0' & \text{"} \\ & \Gamma_0 \vdash [\Gamma_0][\hat{\alpha}/\alpha]A_0 <:^- [\Gamma]B_0 \dashv \Delta_0 & \text{"} \\ & \Gamma_0 \vdash [\hat{\alpha}/\alpha][\Gamma_0]A_0 <:^- [\Gamma]B_0 \dashv \Delta_0 & \hat{\alpha} \text{ unsolved in } \Gamma_0 \\ & \Gamma_0 \vdash [\hat{\alpha}/\alpha][\Gamma]A_0 <:^- [\Gamma]B_0 \dashv \Delta_0 & \Gamma_0 \text{ substitutes as } \Gamma \\ & \Gamma, \alpha : \kappa \vdash \forall \alpha : \kappa. [\Gamma]A_0 <:^- [\Gamma]B_0 \dashv \Delta, \alpha : \kappa, \Theta & \text{By} <: \forall L \\ & \Gamma \vdash \forall \alpha : \kappa. [\Gamma]A_0 <:^- \forall \alpha : \kappa. [\Gamma]B_0 \dashv \Delta & \text{By} <: \forall R \\ & \Gamma \vdash [\Gamma](\forall \alpha : \kappa. A_0) <:^- [\Gamma](\forall \alpha : \kappa. B_0) \dashv \Delta & \text{By def. of subst.} \\ & \blacksquare & \Delta \longrightarrow \Omega & \text{By lemma} \\ & \blacksquare & \Omega \longrightarrow \Omega_0' & \text{By lemma} \\ & \blacksquare & \Omega \longrightarrow \Omega_0' & \text{By lemma} \end{split}$$

- * If $pol(A_0) = +$, then proceed as above, but apply $\leq Refl+$ instead of $\leq Refl-$, and apply $<:_L^+L$ after applying the i.h. (Rule $<:_L^+R$ also works.)
- If A is not headed by \forall :

We have $nonneg([\Omega]A)$. Therefore nonneg(A), and thus A is not headed by \exists . Since the same quantifiers must also head B, the conditions in rule <: Equiv are satisfied.

• Case \leq Refl+: Symmetric to the \leq Refl- case, using <:_L (or <:_R), and <: \exists R/<: \exists L instead of <: \forall L/<: \forall R.

• Case
$$\frac{[\Omega]\Gamma \vdash \tau : \kappa \qquad [\Omega]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq^- [\Omega]B}{[\Omega]\Gamma \vdash \underbrace{\forall \alpha : \kappa.}_{[\Omega]A_0} \leq^- [\Omega]B} \leq \forall L$$

We begin by considering whether or not $[\Omega]B$ is headed by a universal quantifier.

-
$$[\Omega]B = (\forall \beta : \kappa'. B')$$
:

$$[\Omega]\Gamma, \beta : \kappa' \vdash [\Omega]A \leq^- B'$$
 By Lemma 5 (Subtyping Inversion)

The remaining steps are similar to the $\leq \forall R$ case.

– [Ω]B not headed by ∀:

$$\begin{array}{cccc} [\Omega]\Gamma \vdash \tau : \kappa & Subderivation \\ \Gamma \longrightarrow \Omega & Given \\ \Gamma, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Omega, \blacktriangleright_{\hat{\alpha}} & By \longrightarrow \mathsf{Marker} \\ \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \longrightarrow \underbrace{\Omega, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa = \tau}_{\Omega_0} & By \longrightarrow \mathsf{Solve} \\ [\Omega]\Gamma = [\Omega_0](\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa) & By definition of context application (lines 16, 13) \end{array}$$

$$\begin{split} [\Omega]\Gamma &\vdash [\tau/\alpha][\Omega]A_0 \leq^- [\Omega]B & \text{Subderivation} \\ [\Omega_0](\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa) &\vdash [\tau/\alpha][\Omega]A_0 \leq^- [\Omega]B & \text{By above equality} \\ [\Omega_0](\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa) &\vdash [[\Omega_0]\hat{\alpha}/\alpha][\Omega]A_0 \leq^- [\Omega]B & \text{By definition of substitution} \\ [\Omega_0](\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa) &\vdash [[\Omega_0]\hat{\alpha}/\alpha][\Omega_0]A_0 \leq^- [\Omega_0]B & \text{By definition of substitution} \\ [\Omega_0](\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa) &\vdash [\Omega_0][\hat{\alpha}/\alpha]A_0 \leq^- [\Omega_0]B & \text{By distributivity of substitution} \\ \end{split}$$

$$\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\Gamma][\hat{\alpha}/\alpha] A_0 <: ^-[\Gamma] B \dashv \Delta_0 \quad \text{ By definition of substitution}$$

$$\begin{array}{ll} \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\Gamma][\hat{\alpha}/\alpha] A_0 <: ^- [\Gamma] B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta & \text{By above equality } \Delta_0 = (\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \\ \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\hat{\alpha}/\alpha][\Gamma] A_0 <: ^- [\Gamma] B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta & \text{By def. of subst. } ([\Gamma] \hat{\alpha} = \hat{\alpha} \text{ and } [\Gamma] \alpha = \alpha) \\ [\Gamma] B \text{ not headed by } \forall & \text{From the case assumption} \\ \Gamma \vdash \forall \alpha : \kappa. [\Gamma] A_0 <: ^- [\Gamma] B \dashv \Delta & \text{By } <: \forall L \\ \Gamma \vdash [\Gamma] (\forall \alpha : \kappa. A_0) <: ^- [\Gamma] B \dashv \Delta & \text{By definition of substitution} \end{array}$$

• Case
$$\frac{[\Omega]\Gamma, \beta: \kappa \vdash [\Omega]A \leq^{-} [\Omega]B_{0}}{[\Omega]\Gamma \vdash [\Omega]A \leq^{-} \underbrace{\forall \beta: \kappa. [\Omega]B_{0}}_{[\Omega]B}} \leq \forall \mathsf{R}$$

```
B = \forall \beta : \kappa. B_0
                                                                                         \Omega predicative
                      [\Omega]\Gamma \vdash [\Omega]A \leq^- [\Omega]B
                                                                                         Given
                                                                                         By above equality
                      [\Omega]\Gamma \vdash [\Omega]A \leq^- \forall \beta. [\Omega]B_0
             [\Omega]\Gamma, \beta : \kappa \vdash [\Omega]A \leq^{-} [\Omega]B_0
                                                                                         Subderivation
[\Omega, \beta : \kappa](\Gamma, \beta : \kappa) \vdash [\Omega, \beta : \kappa]A \leq^{-} [\Omega, \beta : \kappa]B_0
                                                                                         By definitions of substitution
                   \Gamma, \beta : \kappa \vdash [\Gamma, \beta : \kappa] A <: ^- [\Gamma, \beta : \kappa] B_0 \dashv \Delta'
                                                                                         By i.h. (B lost a quantifier)
           \Omega,\beta:\kappa\longrightarrow\Omega_0'
                   \Gamma, \beta : \kappa \vdash [\Gamma] A <: ^- [\Gamma] B_0 \dashv \Delta'
                                                                                         By definition of substitution
               \Gamma, \beta : \kappa \longrightarrow \Delta'
                                                                    By Lemma 43 (Instantiation Extension)
                        \Delta' = (\Delta, \beta : \kappa, \Theta)
                                                                   By Lemma 22 (Extension Inversion) (i)
                        \Gamma \longrightarrow \Delta
         Ω_0'=(Ω', β: κ, Ω_R) By Lemma 22 (Extension Inversion) (i) Δ \longrightarrow Ω'
 34
                \Gamma, \beta : \kappa \vdash [\Gamma] A <: \Gamma B_0 \dashv \Delta, \beta : \kappa, \Theta By above equality
         \Omega, \beta : \kappa \longrightarrow \Omega', \beta : \kappa, \Omega_R
                                                                                       By above equality
               \Omega \longrightarrow \Omega'
                                                                                       By Lemma 33 (Extension Transitivity)
                         \Gamma \vdash [\Gamma]A <: ^- \forall \beta : \kappa. [\Gamma]B_0 \dashv \Delta
                                                                                       By <: \forall R
                         \Gamma \vdash [\Gamma]A <: ^-[\Gamma](\forall \beta : \kappa. B_0) \dashv \Delta By definition of substitution
 EF
        \frac{[\Omega]\Gamma,\alpha:\kappa\vdash[\Omega]A_0\leq^+[\Omega]B}{[\Omega]\Gamma\vdash\underbrace{\exists\alpha:\kappa.}[\Omega]A_0}\leq^+[\Omega]B}\leq\exists L
                         A = \exists \alpha : \kappa. A_0
                                                                                         \Omega predicative
                      [\Omega]\Gamma \vdash [\Omega]A) \leq^+ [\Omega]B
                                                                                         Given
                      [\Omega]\Gamma \vdash [\Omega]\exists \alpha : \kappa. A_0 \leq^+ [\Omega]B
                                                                                         By above equality
             [\Omega]\Gamma, \alpha : \kappa \vdash [\Omega]A_0 \leq^+ [\Omega]B
                                                                                         Subderivation
[\Omega, \alpha : \kappa](\Gamma, \alpha : \kappa) \vdash [\Omega, \alpha : \kappa]A_0 \leq^+ [\Omega, \alpha : \kappa]B
                                                                                         By definitions of substitution
                   \Gamma, \alpha : \kappa \vdash [\Gamma, \beta : \kappa] A_0 <: ^+ [\Gamma, \beta : \kappa] B \dashv \Delta'
                                                                                         By i.h. (A lost a quantifier)
                    \Delta' \longrightarrow \Omega'_0
           \Omega, \alpha : \kappa \longrightarrow \Omega'_0
                   \Gamma, \alpha : \kappa \vdash [\Gamma]A <: ^+ [\Gamma]B_0 \dashv \Delta'
                                                                                         By definition of substitution
               \Gamma, \alpha : \kappa \longrightarrow \Delta'
                                                                   By Lemma 43 (Instantiation Extension)
                         \Delta' = (\Delta, \alpha : \kappa, \Theta)
                                                                   By Lemma 22 (Extension Inversion) (i)
                        \Gamma \longrightarrow \Delta
         \Delta,\alpha:\kappa,\Theta\longrightarrow\Omega_0'
                                              By \Delta' \longrightarrow \Omega'_0 and above equality
                        \Omega_0' = (\Omega', \alpha : \kappa, \Omega_R) By Lemma 22 (Extension Inversion) (i)
                        \Delta \longrightarrow \Omega'
```

$$\bullet \ \, \textbf{Case} \ \, \frac{\Psi \vdash \tau : \kappa \qquad \Psi \vdash [\Omega] A \leq^+ [\tau/\beta] B_0}{\Psi \vdash [\Omega] A \leq^+ \underbrace{\exists \beta : \kappa. \, B_0}_{[\Omega] B}} \leq \exists \mathsf{R}$$

We consider whether $[\Omega]A$ is headed by an existential.

If
$$[\Omega]A = \exists \alpha : \kappa' . A'$$
:

$$[\Omega]\Gamma, \alpha : \kappa' \vdash A' \leq^+ [\Omega]B$$
 By Lemma 5 (Subtyping Inversion)

The remaining steps are similar to the $\leq \exists L$ case.

If $[\Omega]A$ not headed by \exists :

$$\begin{array}{cccc} [\Omega]\Gamma \vdash \tau : \kappa & Subderivation \\ \Gamma \longrightarrow \Omega & Given \\ \Gamma, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Omega, \blacktriangleright_{\hat{\alpha}} & By \longrightarrow \mathsf{Marker} \\ \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \longrightarrow \underbrace{\Omega, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa = \tau}_{\Omega_0} & By \longrightarrow \mathsf{Solve} \\ & [\Omega]\Gamma = [\Omega_0](\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa) & By \ definition \ of \ context \ application \ (lines 16, 13) \end{array}$$

$$\begin{split} [\Omega]\Gamma \vdash [\Omega]A \leq^+ [\tau/\beta][\Omega]B_0 & \text{Subderivation} \\ [\Omega_0](\Gamma,\blacktriangleright_{\hat{\alpha}},\hat{\alpha}:\kappa) \vdash [\Omega]A \leq^+ [\tau/\beta][\Omega]B_0 & \text{By above equality} \\ [\Omega_0](\Gamma,\blacktriangleright_{\hat{\alpha}},\hat{\alpha}:\kappa) \vdash [\Omega]A \leq^+ [[\Omega_0]\hat{\alpha}/\beta][\Omega]B_0 & \text{By definition of substitution} \\ [\Omega_0](\Gamma,\blacktriangleright_{\hat{\alpha}},\hat{\alpha}:\kappa) \vdash [\Omega_0]A \leq^+ [[\Omega_0]\hat{\alpha}/\beta][\Omega_0]B_0 & \text{By definition of substitution} \\ [\Omega_0](\Gamma,\blacktriangleright_{\hat{\alpha}},\hat{\alpha}:\kappa) \vdash [\Omega_0]A \leq^+ [\Omega_0][\hat{\alpha}/\beta]B_0 & \text{By distributivity of substitution} \end{split}$$

$$\begin{array}{ll} \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\Gamma]A <: ^+ [\Gamma][\hat{\alpha}/\beta]B_0 \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta & \text{By above equality } \Delta_0 = (\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \\ \Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} : \kappa \vdash [\Gamma]A <: ^+ [\hat{\alpha}/\beta][\Gamma]B_0 \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta & \text{By def. of subst. } ([\Gamma]\hat{\alpha} = \hat{\alpha} \text{ and } [\Gamma]\beta = \beta) \\ \Gamma \mid A \text{ not headed by } \exists & \text{From the case hypothesis} \\ \Gamma \vdash [\Gamma]A <: ^+ \exists \beta : \kappa. [\Gamma]B_0 \dashv \Delta & \text{By definition of substitution} \\ \Gamma \vdash [\Gamma]A <: ^+ [\Gamma](\exists \beta : \kappa. B_0) \dashv \Delta & \text{By definition of substitution} \end{array}$$

K'.3 Completeness of Typing

Lemma 99 (Variable Decomposition). *If* $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$, *then*

- 1. if $\Pi \stackrel{1}{\leadsto} \Pi''$ then $\Pi'' = \Pi'$.
- 2. if $\Pi \stackrel{\times}{\sim} \Pi'''$ then there exists Π'' such that $\Pi''' \stackrel{\text{var}}{\sim} \Pi''$ and $\Pi'' \stackrel{\text{var}}{\sim} \Pi'$,
- 3. if $\Pi \stackrel{+}{\sim} \Pi_{I} \parallel \Pi_{R}$ then $\Pi_{I} \stackrel{\text{var}}{\sim} \Pi'$ and $\Pi_{R} \stackrel{\text{var}}{\sim} \Pi'$,
- 4. if $\Pi \stackrel{\mathsf{Vec}}{\leadsto} \Pi_{\Pi} \parallel \Pi_{::}$ then $\Pi' = \Pi_{\Pi}$.

Proof. Each of these follows by induction on Π and decomposition of the two input derivations.

Lemma 100 (Pattern Decomposition and Substitution).

- 1. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ then $[\Omega] \Pi \stackrel{\text{var}}{\leadsto} [\Omega] \Pi'$.
- 2. If $\Pi \stackrel{1}{\sim} \Pi'$ then $[\Omega]\Pi \stackrel{1}{\sim} [\Omega]\Pi'$.
- 3. If $\Pi \stackrel{\times}{\leadsto} \Pi'$ then $[\Omega]\Pi \stackrel{\times}{\leadsto} [\Omega]\Pi'$.
- 4. If $\Pi \stackrel{+}{\sim} \Pi_1 \parallel \Pi_2$ then $[\Omega]\Pi \stackrel{+}{\sim} [\Omega]\Pi_1 \parallel [\Omega]\Pi_2$.
- 5. If $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$ then $[\Omega] \Pi \stackrel{\text{Vec}}{\leadsto} [\Omega] \Pi_1 \parallel [\Omega] \Pi_2$.
- 6. If $[\Omega]\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ then there is Π'' such that $[\Omega]\Pi'' = \Pi'$ and $\Pi \stackrel{\text{var}}{\leadsto} \Pi''$.
- 7. If $[\Omega] \Pi \stackrel{1}{\leadsto} \Pi'$ then there is Π'' such that $[\Omega] \Pi'' = \Pi'$ and $\Pi \stackrel{1}{\leadsto} \Pi''$.
- 8. If $[\Omega]\Pi \stackrel{\times}{\leadsto} \Pi'$ then there is Π'' such that $[\Omega]\Pi'' = \Pi'$ and $\Pi \stackrel{\times}{\leadsto} \Pi''$.
- 9. If $[\Omega]\Pi \stackrel{+}{\leadsto} \Pi'_1 \parallel \Pi'_2$ then there are Π_1 and Π_2 such that $[\Omega]\Pi_1 = \Pi'_1$ and $[\Omega]\Pi_2 = \Pi'_2$ and $\Pi \stackrel{+}{\leadsto} \Pi_1 \parallel \Pi_2$.
- 10. If $[\Omega]\Pi \overset{\text{Vec}}{\leadsto} \Pi_1' \parallel \Pi_2'$ then there are Π_1 and Π_2 such that $[\Omega]\Pi_1 = \Pi_1'$ and $[\Omega]\Pi_2 = \Pi_2'$ and $\Pi \overset{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$.

Proof. Each case is proved by induction on the relevant derivation.

Lemma 101 (Pattern Decomposition Functionality).

- 1. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ and $\Pi \stackrel{\text{var}}{\leadsto} \Pi''$ then $\Pi' = \Pi''$.
- 2. If $\Pi \stackrel{1}{\sim} \Pi'$ and $\Pi \stackrel{1}{\sim} \Pi''$ then $\Pi' = \Pi''$.
- 3. If $\Pi \stackrel{\times}{\leadsto} \Pi'$ and $\Pi \stackrel{\times}{\leadsto} \Pi''$ then $\Pi' = \Pi''$.
- 4. If $\Pi \stackrel{+}{\sim} \Pi_1 \parallel \Pi_2$ and $\Pi \stackrel{+}{\sim} \Pi_1' \parallel \Pi_2'$ then $\Pi_1 = \Pi_1'$ and $\Pi_2 = \Pi_2'$.
- 5. If $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$ and $\Pi \stackrel{\text{Vec}}{\leadsto} \Pi_1 \parallel \Pi_2$ then $\Pi_1 = \Pi'_1$ and $\Pi_2 = \Pi'_2$.

Proof. By induction on the derivation of $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$.

Lemma 102 (Decidability of Variable Removal). For all Π , either there exists a Π' such that $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ or there does not.

Proof. This follows from an induction on the structure of Π .

Lemma 103 (Variable Inversion).

- 1. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ and $\Psi \vdash \Pi$ covers $A, \vec{A} \neq 0$, then $\Psi \vdash \Pi'$ covers $\vec{A} \neq 0$.
- 2. If $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$ and $\Gamma \vdash \Pi$ covers $A, \vec{A} \neq 0$, then $\Gamma \vdash \Pi'$ covers $\vec{A} \neq 0$.

Proof. This follows by induction on the relevant derivations.

Theorem 11 (Completeness of Match Coverage).

- 1. If $\Gamma \vdash \vec{A}$ q types and $[\Gamma]\vec{A} = \vec{A}$ and (for all Ω such that $\Gamma \longrightarrow \Omega$, we have $[\Omega]\Gamma \vdash [\Omega]\Pi$ covers $[\Omega]\vec{A}$ q) then $\Gamma \vdash \Pi$ covers \vec{A} q.
- 2. If $[\Gamma]\vec{A} = \vec{A}$ and $[\Gamma]P = P$ and $\Gamma \vdash \vec{A}$! types and (for all Ω such that $\Gamma \longrightarrow \Omega$, we have $[\Omega]\Gamma / [\Omega]P \vdash [\Omega]\Pi$ covers $[\Omega]\vec{A}$!) then $\Gamma / P \vdash \Pi$ covers \vec{A} !.

Proof. By mutual induction, with the induction metric lexicographically ordered on the number of pattern constructor symbols in the branches of Π , the number of connectives in \vec{A} , and 1 if P is present/0 if it is absent

- 1. Assume $\Gamma \vdash \vec{A}$ q types and $[\Gamma]\vec{A} = \vec{A}$ and (for all Ω such that $\Gamma \longrightarrow \Omega$, we have $[\Omega]\Gamma \vdash [\Omega]\Pi$ covers $[\Omega]\vec{A}$ q)
 - Case $\vec{A} = \cdot$:

Choose a completing substitution Ω .

Then we have $[\Omega]\Gamma \vdash [\Omega]\Pi$ covers \cdot q.

By inversion, we see that DeclCoversEmpty was the last rule, and that we have $[\Omega]\Gamma \vdash [\Omega] \cdot \Rightarrow e_1 \mid \dots covers \cdot q$. Hence by CoversEmpty, we have $\Gamma \vdash \cdot \Rightarrow e_1 \mid \dots covers \cdot q$.

• Case $\vec{A} = A, \vec{B}$:

By Lemma 102 (Decidability of Variable Removal) either

- Case $\Pi \stackrel{\text{var}}{\leadsto} \Pi'$:

Assume Ω such that $\Gamma \longrightarrow \Omega$.

By assumption, $[\Omega]\Gamma \vdash [\Omega]\Pi$ covers $[\Omega](A, \vec{B})$ q.

By Lemma 100 (Pattern Decomposition and Substitution), $[\Omega]\Pi \stackrel{\text{var}}{\leadsto} [\Omega]\Pi'$.

By Lemma 103 (Variable Inversion), $[\Omega]\Gamma \vdash [\Omega]\Pi'$ covers $[\Omega]\vec{B}$ q.

So for all Ω such that $\Gamma \longrightarrow \Omega$, $[\Omega]\Gamma \vdash [\Omega]\Pi'$ covers $[\Omega]\vec{B}$ q.

By induction, $\Gamma \vdash \Pi'$ covers \vec{B} q.

- By rule CoversVar, $\Gamma \vdash \Pi$ covers A, \vec{B} q.
- Case $\forall \Pi'. \neg (\Pi \stackrel{\text{var}}{\sim} \Pi')$:
 - * Case $\hat{\alpha}$, \vec{B} :

This case is impossible. Choose a completing substitution Ω such that $[\Omega] \hat{\alpha} = 1 \to 1$, and then by assumption we have $[\Omega] \Gamma \vdash [\Omega] \Pi$ covers $1 \to 1$, $[\Omega] \vec{B}$ q. By inversion we have that $[\Omega] \Pi \overset{\text{var}}{\leadsto} \Pi'$. By Lemma 100 (Pattern Decomposition and Substitution), we have a Π'' such that $[\Omega] \Pi'' = \Pi'$, and $\Pi \overset{\text{var}}{\leadsto} \Pi''$. This yields the contradiction.

- * Case $C \rightarrow D, \vec{B}$:
- * Case $\forall \alpha : \kappa. A, \vec{B}$:
- * Case α , \vec{B} :

Similar to the $\hat{\alpha}$ case.

```
* Case \vec{A} = 1, \vec{B}:
   Choose an arbitrary \Omega such that \Gamma \longrightarrow \Omega.
   By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](1, \overline{B}) q.
   By inversion, we know the rule DeclCovers1 applies (since the variable case has been ruled out).
   Hence [\Omega]\Pi \stackrel{1}{\sim} \Pi'' and [\Omega]\Gamma \vdash \Pi'' covers [\Omega]\vec{B} q.
   By Lemma 100 (Pattern Decomposition and Substitution), there is a \Pi' such that
     [\Omega]\Pi' = \Pi'' and \Pi \stackrel{1}{\sim} \Pi'.
   Assume \Omega such that \Gamma \longrightarrow \Omega.
          By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](1, \vec{B}) q.
          By inversion, we know the rule DeclCovers1 applies (since the variable case has been ruled out).
          Hence [\Omega]\Pi \stackrel{1}{\sim} \Pi'' and [\Omega]\Gamma \vdash \Pi'' covers [\Omega]\vec{B} q.
          By Lemma 100 (Pattern Decomposition and Substitution),
             there is a \hat{\Pi}'' such that \Pi'' = [\Omega] \hat{\Pi}'' and \Pi \stackrel{1}{\leadsto} \hat{Pi}'.
          By Lemma 101 (Pattern Decomposition Functionality), we know \hat{\Pi}' = \Pi'.
   So for all \Omega such that \Gamma \longrightarrow \Omega, [\Omega]\Gamma \vdash [\Omega]\Pi' covers [\Omega]\vec{B} q.
   By induction, \Gamma \vdash \Pi' covers \vec{B} q.
   By rule Covers 1, \Gamma \vdash \Pi covers A, \vec{B} q.
* Case C \times D, \vec{B}:
   Choose an arbitrary \Omega such that \Gamma \longrightarrow \Omega.
   By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](C \times D, \vec{B}) q.
   By inversion, we know the rule DeclCovers× applies (since the variable case has been ruled out).
   Hence [\Omega]\Pi \stackrel{\times}{\leadsto} \Pi'' and [\Omega]\Gamma \vdash \Pi'' covers [\Omega](C, D, \vec{B}) q.
   By Lemma 100 (Pattern Decomposition and Substitution), there is a \Pi' such that
     [\Omega]\Pi' = \Pi'' and \Pi \stackrel{\times}{\sim} \Pi'.
   Assume \Omega such that \Gamma \longrightarrow \Omega.
          By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](C \times D, \vec{B}) q.
          By inversion, we know the rule DeclCovers× applies (since the variable case has been ruled out).
          Hence [\Omega]\Pi \stackrel{\times}{\leadsto} \Pi'' and [\Omega]\Gamma \vdash \Pi'' covers [\Omega](C, D, \vec{B}) q.
          By Lemma 100 (Pattern Decomposition and Substitution),
             there is a \hat{\Pi}'' such that \Pi'' = [\Omega] \hat{\Pi}'' and \Pi \stackrel{\times}{\leadsto} \hat{\Pr}'.
          By Lemma 101 (Pattern Decomposition Functionality), we know \hat{\Pi}' = \Pi'.
   So for all \Omega such that \Gamma \longrightarrow \Omega, [\Omega]\Gamma \vdash [\Omega]\Pi' covers [\Omega](C, D, \vec{B}) q.
   By induction, \Gamma \vdash \Pi' covers C, D, \vec{B} q.
   By rule Covers \times, \Gamma \vdash \Pi covers C \times D, \vec{B} q.
* Case C + D, \vec{B}:
   Choose an arbitrary \Omega such that \Gamma \longrightarrow \Omega.
   By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](C \times D, \vec{B}) q.
   By inversion, we know the rule DeclCovers+ applies (since the variable case has been ruled out).
   Hence [\Omega]\Pi \stackrel{+}{\sim} \Pi'_1 \parallel \Pi'_2 and [\Omega]\Gamma \vdash \Pi'_1 covers [\Omega](C, \vec{B}) q and [\Omega]\Gamma \vdash \Pi'_2 covers [\Omega](D, \vec{B}) q.
   By Lemma 100 (Pattern Decomposition and Substitution), there is a \Pi_1 and \Pi_2 such that
     [\Omega]\Pi_1 = \Pi_1' and [\Omega]\Pi_2 = \Pi_2' and \Pi \stackrel{+}{\leadsto} \Pi_1 \parallel \Pi_2.
   Assume \Omega such that \Gamma \longrightarrow \Omega.
          By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](C \times D, B) q.
          By inversion, we know the rule DeclCovers+ applies (since the variable case has been ruled out).
          Hence [\Omega]\Pi \stackrel{+}{\sim} \hat{\Pi}'_1 \parallel \hat{\Pi}'_2 and [\Omega]\Gamma \vdash \hat{\Pi}'_1 covers [\Omega](C, \vec{B}) q and [\Omega]\Gamma \vdash \hat{\Pi}'_2 covers [\Omega](D, \vec{B}) q.
```

By Lemma 100 (Pattern Decomposition and Substitution),

```
there is a \hat{\Pi}_1' such that \hat{\Pi}_1' = [\Omega]\hat{\Pi}_1 and \hat{\Pi}_2' = [\Omega]\hat{\Pi}_2 and \Pi \stackrel{+}{\sim} \hat{Pi}_1 \parallel \hat{\Pi}_2.
                                 By Lemma 101 (Pattern Decomposition Functionality), we know \hat{\Pi}_i = \Pi_i.
                          So for all \Omega such that \Gamma \longrightarrow \Omega, [\Omega]\Gamma \vdash [\Omega]\Pi_1 covers [\Omega](C, \vec{B}) q.
                          So for all \Omega such that \Gamma \longrightarrow \Omega, [\Omega]\Gamma \vdash [\Omega]\Pi_2 covers [\Omega](D, \vec{B}) q.
                          By induction, \Gamma \vdash \Pi_1 covers C, \vec{B} q.
                          By induction, \Gamma \vdash \Pi_2 covers D, \vec{B} q.
                          By rule Covers+, \Gamma \vdash \Pi covers C + D, \vec{B} q.
                      * Case Vec n A, \vec{B}:
                          Similar to the previous case.
                      * Case \exists \alpha : \kappa. C, \vec{B}:
                          Assume \Omega such that \Gamma \longrightarrow \Omega.
                                 By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](\exists \alpha : \kappa. C, \vec{B}) q.
                                 By inversion, we know the rule DeclCovers∃ applies.
                                 Hence [\Omega]\Gamma, \alpha : \kappa \vdash [\Omega]\Pi covers [\Omega](C, \vec{B}) q.
                          So for all \Omega such that \Gamma \longrightarrow \Omega, [\Omega](\Gamma, \alpha : \kappa) \vdash [\Omega]\Pi covers [\Omega](C, \vec{B}) q.
                          By induction, \Gamma, \alpha : \kappa \vdash \Pi covers C, \vec{B} q.
                          By rule Covers \exists \alpha : \kappa \in C, \vec{B} \neq \emptyset.
                      * Case C \wedge P, \vec{B}:
                          · Case q = 1: Similar to the previous case.
                          · Case q = !:
                             Assume \Omega such that \Gamma \longrightarrow \Omega.
                                    By assumption, [\Omega]\Gamma \vdash [\Omega]\Pi covers [\Omega](C \land P, \vec{B}) q.
                                    By inversion, we know the rule DeclCovers∧ applies.
                                    Hence [\Omega]\Gamma / [\Omega]P \vdash [\Omega]\Pi covers [\Omega](C, \vec{B})!.
                             So for all \Omega such that \Gamma \longrightarrow \Omega, [\Omega](\Gamma, \alpha : \kappa) / [\Omega]P \vdash [\Omega]\Pi covers [\Omega](C, \vec{B})!.
                             By mutual induction, \Gamma / P \vdash \Pi \text{ covers } C, \vec{B} !.
                             By rule Covers \land, \Gamma \vdash \Pi covers C \land P, \vec{B}!.
2. Assume [\Gamma]\vec{A} = \vec{A} and [\Gamma]P = P and \Gamma \vdash \vec{A}! types and (for all \Omega such that \Gamma \longrightarrow \Omega, we have
    [\Omega]\Gamma / [\Omega]P \vdash [\Omega]\Pi \text{ covers } [\Omega]\vec{A}!).
    Let (t_1 = t_2) be P.
    Consider whether the mgu(t_1, t_2) exists
         • Case \theta = mgu(t_1, t_2):
                      mgu(t_1, t_2) = \theta
                                                            Premise
               \Gamma / t_1 \stackrel{\circ}{=} t_2 : \kappa \dashv \Gamma, \Theta
                                                            By Lemma 94 (Completeness of Elimeq) (1)
               \Gamma / [\Gamma]t_1 \stackrel{\circ}{=} [\Gamma]t_2 : \kappa \dashv \Gamma, \Theta Follows from given assumption
            Assume \Omega such that \Gamma, \Theta \longrightarrow \Omega.
            By Lemma 59 (Canonical Completion), there is a \Omega' such that [\Omega]\Gamma = [\Omega']\Gamma and dom(\Gamma) =
            dom(\Gamma').
            Moreover, by Lemma 22 (Extension Inversion), we can construct a \Omega'' such that \Omega' = \Omega'', \Theta and
            \Gamma \longrightarrow \Omega'.
            By assumption, [\Omega'']\Gamma / [\Omega''](t_1 = t_2) \vdash [\Omega'']\Pi covers \vec{A}!.
            There is only one way this derivation could be constructed:
```

$$\label{eq:case_equation} \begin{array}{l} \textbf{- Case} \\ \frac{\theta = \mathsf{mgu}(t_1, t_2) \qquad [\theta][\Omega'']\Gamma \vdash [\theta][\Omega'']\Pi \ \textit{covers} \ [\theta][\Omega'']\vec{A} \,!}{[\Omega'']\Gamma \, / \ [\Omega''](t_1 = t_2) \vdash [\Omega'']\Pi \ \textit{covers} \ [\Omega'']\vec{A}!} \ \text{DeclCoversEq} \end{array}$$

```
 \begin{split} [\theta][\Omega'']\Gamma \vdash [\theta][\Omega'']\Pi \ covers \ ([\theta][\Omega'']\vec{A}) & \text{Subderivation} \\ [\theta][\Omega'']\Gamma = [\Omega'',\Theta](\Gamma,\Theta) & \text{By Lemma 95 (Substitution Upgrade) (iii)} \\ [\theta][\Omega'']\Pi = [\Omega'',\Theta]\Pi & \text{By Lemma 95 (Substitution Upgrade) (iv)} \\ ([\theta][\Omega'']\vec{A}) = ([\Omega,\Theta][\Gamma,\Theta]\vec{A}) & \text{By Lemma 95 (Substitution Upgrade) (i)} \\ [\Omega'',\Theta](\Gamma,\Theta) \vdash [\Omega'',\Theta]\Pi \ covers \ [\Omega'',\Theta][\Gamma,\Theta]\vec{A} & \text{By above equalities} \\ [\Omega'](\Gamma,\Theta) \vdash [\Omega']\Pi \ covers \ [\Omega'][\Gamma,\Theta]\vec{A} & \text{By above equalities} \\ [\Omega](\Gamma,\Theta) \vdash [\Omega]\Pi \ covers \ [\Omega][\Gamma,\Theta]\vec{A} & \text{By above equalities} \\ \end{split}
```

So we know by induction that $\Gamma, \Theta \vdash [\Gamma, \Theta] \Pi$ *covers* $[\Gamma, \Theta] \vec{A}$!.

Hence by CoversEq we have $\Gamma / t_1 = t_2 \vdash \Pi$ covers \vec{A} !.

• Case $mgu(t_1, t_2) = \bot$:

$$\begin{array}{ccc} \text{mgu}(t_1,t_2) = \bot & \text{Premise} \\ \Gamma \ / \ t_1 \stackrel{\circ}{=} t_2 : \kappa \dashv \bot & \text{By Lemma 94 (Completeness of Elimeq) (2)} \\ \Gamma \ / \ [\Gamma]t_1 \stackrel{\circ}{=} [\Gamma]t_2 : \kappa \dashv \bot & \text{Follows from given assumption} \\ & & & & & & & & & & & & & \\ \mathbb{R} \ \hline & \Gamma \ / \ t_1 = t_2 \vdash \Pi \ \textit{covers} \ \vec{A} & & & & & & & & & & \\ \mathbb{R} \ \hline \end{array}$$

Theorem 12 (Completeness of Algorithmic Typing). *Given* $\Gamma \longrightarrow \Omega$ *such that* dom(Γ) = dom(Ω):

- (i) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]e \Leftarrow [\Omega]A$ p and $\mathfrak{p}' \sqsubseteq \mathfrak{p}$ then there exist Δ and Ω' such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Leftarrow [\Gamma]A$ $\mathfrak{p}' \dashv \Delta$.
- (ii) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]e \Rightarrow A$ p then there exist Δ , Ω' , A', and $\mathfrak{p}' \sqsubseteq \mathfrak{p}$ such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash e \Rightarrow A'$ $\mathfrak{p}' \dashv \Delta$ and $A' = [\Delta]A'$ and $A = [\Omega']A'$.
- (iii) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A$ p \gg B q and p' \sqsubseteq p then there exist Δ , Ω' , B' and q' \sqsubseteq q such that $\Delta \longrightarrow \Omega'$ and dom(Δ) = dom(Ω') and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash s : [\Gamma]A$ p' \gg B' q' $\dashv \Delta$ and B' = $[\Delta]B'$ and B = $[\Omega']B'$.
- (iv) If $\Gamma \vdash A$ p type and $[\Omega]\Gamma \vdash [\Omega]s : [\Omega]A$ p $\gg B$ $\lceil q \rceil$ and p' \sqsubseteq p then there exist Δ , Ω' , B', and $q' \sqsubseteq q$ such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash s : [\Gamma]A$ p' $\gg B'$ $\lceil q' \rceil \dashv \Delta$ and $B' = [\Delta]B'$ and $B = [\Omega']B'$.
- (v) If $\Gamma \vdash \vec{A}$! types and $\Gamma \vdash C$ p type and $[\Omega]\Gamma \vdash [\Omega]\Pi :: [\Omega]\vec{A} \neq [\Omega]C$ p and $p' \sqsubseteq p$ then there exist Δ , Ω' , and C such that $\Delta \longrightarrow \Omega'$ and $dom(\Delta) = dom(\Omega')$ and $\Omega \longrightarrow \Omega'$ and $\Gamma \vdash \Pi :: [\Gamma]\vec{A} \neq [\Gamma]C$ $p' \dashv \Delta$.

```
(vi) If \Gamma \vdash \vec{A}! types and \Gamma \vdash P prop and FEV(P) = \emptyset and \Gamma \vdash C p type and [\Omega]\Gamma / [\Omega]P \vdash [\Omega]\Pi :: [\Omega]\vec{A} ! \Leftarrow [\Omega]C p and p' \sqsubseteq p then there exist \Delta, \Omega', and C such that \Delta \longrightarrow \Omega' and dom(\Delta) = dom(\Omega') and \Omega \longrightarrow \Omega' and \Gamma / [\Gamma]P \vdash \Pi :: [\Gamma]\vec{A} ! \Leftarrow [\Gamma]C p' \dashv \Delta.
```

Proof. By induction, using the measure in Definition 7.

 $\Gamma \vdash e \Leftarrow A p \dashv \Delta$

• Case
$$\frac{(x:A\,\mathfrak{p})\in [\Omega]\Gamma}{[\Omega]\Gamma\vdash x\Rightarrow A\,\mathfrak{p}} \ \operatorname{DeclVar}$$

$$(x:A\,\mathfrak{p})\in [\Omega]\Gamma \qquad \operatorname{Premise}$$

$$\Gamma\longrightarrow \Omega \qquad \operatorname{Given}$$

$$(x:A'\,\mathfrak{p})\in \Gamma \ \text{where} \ [\Omega]A'=A \qquad \operatorname{From} \ \operatorname{definition} \ \operatorname{of} \ \operatorname{context} \ \operatorname{application}$$

$$\operatorname{Let} \ \Delta=\Gamma.$$

$$\operatorname{Let} \ \Omega'=\Omega.$$

$$\Gamma\longrightarrow \Omega \qquad \operatorname{Given}$$

$$\Gamma\longrightarrow \Omega \qquad \operatorname{By} \ \operatorname{Lemma} \ 32 \ (\operatorname{Extension} \ \operatorname{Reflexivity})$$

$$\Gamma\vdash x\Rightarrow [\Gamma]A'\,\mathfrak{p}\dashv\Gamma \qquad \operatorname{By} \ \operatorname{Var}$$

$$\Gamma \cap A' = \Gamma \cap A' \qquad \operatorname{By} \ \operatorname{Idempotence} \ \operatorname{of} \ \operatorname{substitution}$$

$$\Gamma\longrightarrow \Omega \qquad \operatorname{Given}$$

$$\Gamma\longrightarrow \Omega \qquad \operatorname{Given}$$

$$\Gamma\longrightarrow \Omega \qquad \operatorname{Given}$$

$$\Gamma\longrightarrow \Omega \qquad \operatorname{Given}$$

$$\Gamma \cap \Omega \qquad \operatorname{Given}$$

By Sub

• Case
$$\frac{[\Omega]\Gamma \vdash [\Omega]A \ type \qquad [\Omega]\Gamma \vdash [\Omega]e_0 \Leftarrow [\Omega]A \ !}{[\Omega]\Gamma \vdash [\Omega](e_0 : A) \Rightarrow A \ !} \text{ DeclAnno}$$

$$\frac{[\Omega]\Gamma \vdash [\Omega]e_0 \Leftarrow [\Omega]A \ !}{[\Omega]A = [\Omega][\Gamma]A} \text{ Subderivation}$$

$$[\Omega]A = [\Omega][\Gamma]A \text{ By Lemma 29 (Substitution Monotonicity)}$$

$$[\Omega]\Gamma \vdash [\Omega]e_0 \Leftarrow [\Omega][\Gamma]A \ ! \text{ By above equality}$$

$$\Gamma \vdash e_0 \Leftarrow [\Gamma]A \ ! \vdash A \text{ By i.h.}$$

$$\Gamma \vdash e_0 \Rightarrow \Gamma = \Gamma = \Gamma \text{ Monotonicity}$$

$$\Gamma \vdash A \Rightarrow \Gamma = \Gamma \text{ Monotonicity}$$

$$\Gamma \vdash A \Rightarrow \Gamma = \Gamma \text{ Monotonicity}$$

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$$\Gamma \vdash A \Rightarrow \Gamma = \Gamma \text{ Monotonicity}$$

Case

$$\overline{[\Omega]\Gamma \vdash \textbf{()} \Leftarrow 1\, \textbf{p}} \,\, ^{\mathsf{Decl1I}}$$

We have $[\Omega]A = 1$. Either $[\Gamma]A = 1$, or $[\Gamma]A = \hat{\alpha}$ where $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$.

In the former case:

In the latter case, since $A = \hat{\alpha}$ and $\Gamma \vdash \hat{\alpha}$ p *type* is given, it must be the case that p = I.

$$\bullet \ \, \textbf{Case} \ \, \frac{\nu \ \textit{chk-I} \qquad [\Omega]\Gamma, \alpha : \kappa \vdash [\Omega]\nu \Leftarrow A_0 \ p}{[\Omega]\Gamma \vdash [\Omega]\nu \Leftarrow \forall \alpha : \kappa. \ A_0 \ p} \ \, \text{Decl}\forall I$$

```
[\Omega]A = \forall \alpha : \kappa. A_0
                                                                                          Given
                                   = \forall \alpha : \kappa. [\Omega] A'
                                                                                          By def. of subst. and predicativity of \Omega
                            A_0 = [\Omega]A'
                                                                                          Follows from above equality
                [\Omega]\Gamma, \alpha : \kappa \vdash [\Omega]\nu \Leftarrow [\Omega]A'p
                                                                                          Subderivation and above equality
                           \Gamma \longrightarrow \Omega
                                                                                          Given
                \Gamma, \alpha : \kappa \longrightarrow \Omega, \alpha : \kappa
                                                                                          Bv \longrightarrow Uvar
              [\Omega]\Gamma, \alpha : \kappa = [\Omega, \alpha : \kappa](\Gamma, \alpha : \kappa)
                                                                                          By definition of context substitution
[\Omega, \alpha : \kappa](\Gamma, \alpha : \kappa) \vdash [\Omega]\nu \Leftarrow [\Omega]A' p
                                                                                          By above equality
[\Omega, \alpha : \kappa](\Gamma, \alpha : \kappa) \vdash [\Omega]v \Leftarrow [\Omega, \alpha : \kappa]A'p
                                                                                          By definition of substitution
                      \Gamma, \alpha : \kappa \vdash \nu \Leftarrow [\Gamma, \alpha : \kappa] A' p \dashv \Delta'
                                                                                                           By i.h.
                        \Delta' \longrightarrow \Omega'_0
              \Omega, \alpha : \kappa \longrightarrow \Omega_0^{\prime\prime}
                                                                                                            "
                \mathsf{dom}(\Delta') = \mathsf{dom}(\Omega'_0)
                \Gamma, \alpha : \kappa \longrightarrow \Delta'
                                                                                                           By Lemma 51 (Typing Extension)
                             \Delta' = (\Delta, \alpha : \kappa, \Theta)
                                                                                                           By Lemma 22 (Extension Inversion) (i)
         \Delta, \alpha : \kappa, \Theta \longrightarrow \Omega'_0
                                                                                                           By above equality
                           \Omega_0' = (\Omega', \alpha : \kappa, \Omega_Z)
                                                                                                           By Lemma 22 (Extension Inversion) (i)
                          \Delta \longrightarrow \Omega'
13
                  \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
ESF
                         \Omega \longrightarrow \Omega'
                                                                           By Lemma 22 (Extension Inversion) on \Omega, \alpha : \kappa \longrightarrow \Omega'_0
                      \Gamma, \alpha : \kappa \vdash \nu \Leftarrow [\Gamma, \alpha : \kappa] A' p \dashv \Delta, \alpha : \kappa, \Theta
                                                                                                           By above equality
                      \Gamma, \alpha : \kappa \vdash \nu \leftarrow [\Gamma] A' p \dashv \Delta, \alpha : \kappa, \Theta
                                                                                                           By definition of substitution
                                \Gamma \vdash \nu \Leftarrow \forall \alpha : \kappa. [\Gamma] A' p \dashv \Delta
                                                                                                           By \forall I
                                 \Gamma \vdash \nu \Leftarrow [\Gamma](\forall \alpha : \kappa. A') p \dashv \Delta
                                                                                                           By definition of substitution
3
         \frac{[\Omega]\Gamma \vdash \tau : \kappa \qquad [\Omega]\Gamma \vdash [\Omega](e\ s_0) : [\tau/\alpha][\Omega]A_0\ \cancel{I} \gg B\ q}{[\Omega]\Gamma \vdash [\Omega](e\ s_0) : \forall \alpha : \kappa.\ [\Omega]A_0\ p \gg B\ q} \ \mathsf{Decl} \forall \mathsf{Spine}
               [\Omega]\Gamma \vdash \tau : \kappa
                                                                                                    Subderivation
                \Gamma \longrightarrow \Omega
                                                                                                    Given
      \Gamma, \hat{\alpha} : \kappa \longrightarrow \Omega, \hat{\alpha} : \kappa = \tau
                                                                                                    By \longrightarrow Solve
              [\Omega]\Gamma \vdash [\Omega](e s_0) : [\tau/\alpha][\Omega]A_0 \not I \gg B q
                                                                                                   Subderivation
                     \tau = [\Omega]\tau
                                                                                                    FEV(\tau) = \emptyset
[\tau/\alpha][\Omega]A_0 = [\tau/\alpha][\Omega, \hat{\alpha}: \kappa = \tau]A_0
                                                                                                    By def. of subst.
                        = [\Omega]\tau/\alpha][\Omega, \hat{\alpha}: \kappa = \tau]A_0
                                                                                                    By above equality
                         = [\Omega, \hat{\alpha}: \kappa = \tau][\hat{\alpha}/\alpha]A_0
                                                                                                    By distributivity of substitution
              [\Omega]\Gamma = [\Omega, \hat{\alpha} : \kappa = \tau](\Gamma, \hat{\alpha} : \kappa)
                                                                                                    By definition of context application
```

$$\begin{split} [\Omega, \hat{\alpha} : \kappa = \tau] (\Gamma, \hat{\alpha} : \kappa) &\vdash [\Omega] (e \ s_0) : \left[\Omega, \hat{\alpha} : \kappa = \tau \right] [\hat{\alpha}/\alpha] A_0 \not I \gg B \ q & \text{By above equalities} \\ \Gamma, \hat{\alpha} : \kappa \vdash e \ s_0 : [\Gamma, \hat{\alpha} : \kappa] [\hat{\alpha}/\alpha] A_0 \not I \gg B' \ q \dashv \Delta & \text{By i.h.} \\ B &= [\Omega, \hat{\alpha} : \kappa = \tau] B' & " \\ \Delta &\longrightarrow \Omega' & " \\ \text{dom}(\Delta) &= \text{dom}(\Omega') & " \\ \Omega &\longrightarrow \Omega' & " \\ B' &\longrightarrow [\Delta] B' & " \\ B &\longrightarrow [\Omega'] B' & " \\ \end{split}$$

$$[\Gamma, \hat{\alpha} : \kappa] [\hat{\alpha}/\alpha] A_0 & \text{By def. of context application} \\ &= [\hat{\alpha}/\alpha] [\Gamma] A_0 & \Gamma \ \text{does not subst. for } \alpha \\ \Gamma, \hat{\alpha} : \kappa \vdash e \ s_0 : [\hat{\alpha}/\alpha] [\Gamma] A_0 \not I \gg B' \ q \dashv \Delta & \text{By above equality} \\ \Gamma \vdash e \ s_0 : \forall \alpha : \kappa. \ [\Gamma] A_0 \ p \gg B' \ q \dashv \Delta & \text{By def. of subst.} \end{split}$$

• Case
$$\frac{\nu \ chk \text{-} I \qquad [\Omega]\Gamma \ / \ [\Omega]P \vdash [\Omega]\nu \Leftarrow [\Omega]A_0 \ !}{[\Omega]\Gamma \vdash [\Omega]\nu \Leftarrow ([\Omega]P) \supset [\Omega]A_0 \ !} \text{ Decl} \supset I$$
$$[\Omega]\Gamma \ / \ [\Omega]P \vdash [\Omega]\nu \Leftarrow [\Omega]A_0 \ ! \qquad \text{Subderivation}$$

The concluding rule in this subderivation must be DeclCheck \bot or DeclCheckUnify. In either case, $[\Omega]P$ has the form $(\sigma' = \tau')$ where $\sigma' = [\Omega]\sigma$ and $\tau' = [\Omega]\tau$.

$$\begin{array}{c} \textbf{- Case} & \text{mgu}([\Omega]\sigma,[\Omega]\tau) = \bot \\ \hline [\Omega]\Gamma \mathbin{/} [\Omega](\sigma = \tau) \vdash [\Omega]\nu \leftarrow [\Omega]A_0 \ ! \end{array} \text{DeclCheck} \bot$$

We have $\mathsf{mgu}([\Omega]\sigma, [\Omega]\tau) = \bot$. To apply Lemma 94 (Completeness of Elimeq) (2), we need to show conditions 1–5.

```
\Gamma \vdash (\sigma = \tau) \supset A_0! type
                                                                              Given
        [\Omega]((\sigma=\tau)\supset A_0)=[\Gamma]((\sigma=\tau)\supset A_0)
                                                                              By Lemma 39 (Principal Agreement) (i)
                             [\Omega]\sigma=[\Gamma]\sigma
                                                                              By a property of subst.
                             [\Omega]\tau = [\Gamma]\tau
                                                                              Similar
                                    \Gamma \vdash \sigma : \kappa
                                                               By inversion
                                    \Gamma \vdash [\Gamma]\sigma : \kappa
                                                              By Lemma 11 (Right-Hand Substitution for Sorting)
     3
                                    \Gamma \vdash [\Gamma]\tau : \kappa
                                                               Similar
               mgu([\Omega]\sigma, [\Omega]\tau) = \bot
                                                         Given
                 mgu([\Gamma]\sigma, [\Gamma]\tau) = \bot
                                                         By above equalities
                                                         By inversion on ***
             \mathsf{FEV}(\sigma) \cup \mathsf{FEV}(\tau) = \emptyset
   \mathsf{FEV}([\Omega]\sigma) \cup \mathsf{FEV}([\Omega]\tau) = \emptyset
                                                         By a property of complete contexts
5 \mathsf{FEV}([\Gamma]\sigma) \cup \mathsf{FEV}([\Gamma]\tau) = \emptyset
                                                         By above equalities
1
                               [\Gamma][\Gamma]\sigma = [\Gamma]\sigma
                                                         By idempotence of subst.
2
                               [\Gamma][\Gamma]\tau = [\Gamma]\tau
                                                         By idempotence of subst.
```

```
\Gamma / [\Gamma] \sigma \stackrel{\circ}{=} [\Gamma] \tau : \kappa \dashv \bot By Lemma 94 (Completeness of Elimeq) (2)
        \Gamma_{\bullet P} / [\Gamma] \sigma = [\Gamma] \tau \dashv \bot
                                                           By ElimpropEq
                            \Gamma \vdash \nu \Leftarrow ([\Gamma]\sigma = [\Gamma]\tau) \supset [\Gamma]A_0! \dashv \Gamma
                                                                                             By \supset I \perp
                            \Gamma \vdash \nu \Leftarrow [\Gamma]((\sigma = \tau) \supset A_0) ! \dashv \Gamma
                                                                                              By def. of subst.
                       \Gamma \longrightarrow \Omega
                                                                                              Given
                      \Omega \longrightarrow \Omega
                                                                                              By Lemma 32 (Extension Reflexivity)
                dom(\Gamma) = dom(\Omega)
                                                                                              Given
 \begin{array}{c} \textbf{- Case} \\ \frac{\mathsf{mgu}([\Omega]\sigma, [\Omega]\tau) = \theta}{[\Omega]\Gamma \, / \, (([\Omega]\sigma) = [\Omega]\tau) \vdash [\Omega]e \Leftarrow [\Omega]A_0 \, !} \, \, \mathsf{DeclCheckUnify} \end{array} 
    We have mgu([\Omega]\sigma, [\Omega]\tau) = \theta, and will need to apply Lemma 94 (Completeness of Elimeq) (1).
    That lemma has five side conditions, which can be shown exactly as in the DeclCheck⊥ case above.
         mgu(\sigma, \tau) = \theta
                                                    Premise
             Let \Omega_0 = (\Omega, \triangleright_P).
                    \Gamma \longrightarrow \Omega
                                                    Given
              \Gamma, \triangleright_P \longrightarrow \Omega_0
                                                    By —→Marker
             \mathsf{dom}(\Gamma) = \mathsf{dom}(\Omega)
                                                    Given
       dom(\Gamma, \triangleright_P) = dom(\Omega_0) By def. of dom(-)
        \Gamma, \triangleright_{P} / [\Gamma] \sigma \stackrel{\circ}{=} [\Gamma] \tau : \kappa \dashv \Gamma, \triangleright_{P}, \Theta By Lemma 94 (Completeness of Elimeq) (1)
                                        \Gamma, \blacktriangleright_{P} / [\Gamma]\sigma = [\Gamma]\tau \dashv \Gamma, \blacktriangleright_{P}, \Theta
                                                                                                    By ElimpropEq
        EQ0 for all \Gamma, \triangleright_P \vdash u : \kappa. [\Gamma, \triangleright_P, \Theta]u = \theta([\Gamma, \triangleright_P]u)
                           \Gamma \vdash P \supset A_0! type Given
                           \Gamma \vdash A_0! type
                                                              By inversion
                      \Gamma \longrightarrow \Omega
                                                              Given
        EQa [\Gamma]A_0 = [\Omega]A_0
                                                              By Lemma 39 (Principal Agreement) (i)
             Let \Omega_1 = (\Omega, \triangleright_P, \Theta).
              \theta([\Omega]\Gamma) \vdash \theta(e) \Leftarrow \theta([\Omega]A_0)! Subderivation
          \Gamma, \triangleright_{P}, \Theta \longrightarrow \Omega_{1}
                                                                     By induction on \Theta
          \theta([\Omega]A_0) = \theta([\Gamma]A_0)
                                                                     By above equality EQa
                           = [\Gamma, \triangleright_{P}, \Theta] A_0
                                                                     By Lemma 95 (Substitution Upgrade) (i) (with EQ0)
                           = [\Omega_1]A_0
                                                                     By Lemma 39 (Principal Agreement) (i)
                           = [\Omega_1][\Gamma, \triangleright_P, \Theta]A_0
                                                                     By Lemma 29 (Substitution Monotonicity) (iii)
                                                                     By Lemma 95 (Substitution Upgrade) (iii)
             \theta([\Omega]\Gamma) = [\Omega_1](\Gamma, \triangleright_P, \Theta)
             \theta([\Omega]e) = [\Omega_1]e
                                                                     By Lemma 95 (Substitution Upgrade) (iv)
          [\Omega_1](\Gamma, \blacktriangleright_P, \Theta) \vdash [\Omega_1]e \leftarrow [\Omega_1][\Gamma, \blacktriangleright_P, \Theta]A_0! By above equalities
         dom(\Gamma, \blacktriangleright_P, \Theta) = dom(\Omega_1)
                                                                                        dom(\Gamma) = dom(\Omega)
```

```
\Gamma, \blacktriangleright_{P}, \Theta \vdash e \Leftarrow [\Gamma, \blacktriangleright_{P}, \Theta] A_{0} ! \dashv \Delta'
                                                                                       By i.h.
                         \Delta' \longrightarrow \Omega_2'
                                                                                       "
                        \Omega_1 \longrightarrow \Omega_2'
                   dom(\Delta') = dom(\Omega'_2)
                             \Delta' = (\Delta, \triangleright_P, \Delta'')
                                                                                       By Lemma 22 (Extension Inversion) (ii)
                            \Omega_2' = (\Omega', \blacktriangleright_P, \Omega_Z)
                                                                                       By Lemma 22 (Extension Inversion) (ii)
                           \Delta \longrightarrow \Omega'
                        \Omega_0 \longrightarrow \Omega_2'
                                                                                       By Lemma 33 (Extension Transitivity)
                   \Omega, \blacktriangleright_P \longrightarrow \Omega^7, \blacktriangleright_P, \Omega_Z
                                                                                       By above equalities
                          \Omega \longrightarrow \Omega'
                                                                                       By Lemma 22 (Extension Inversion) (ii)
                    dom(\Delta) = dom(\Omega')
                  \Gamma, \blacktriangleright_{P}, \Theta \vdash e \Leftarrow [\Gamma, \blacktriangleright_{P}, \Theta] A_{0} ! \dashv \Delta, \blacktriangleright_{P}, \Delta''
                                                                                                    By above equality
                             \Gamma \vdash e \Leftarrow ([\Gamma]\sigma = [\Gamma]\tau) \supset [\Gamma]A_0! \dashv \Delta \quad \text{By } \supset I
                             \Gamma \vdash e \Leftarrow [\Gamma](P \supset A_0) ! \dashv \Delta
                                                                                                    By def. of subst.
                                               [\underline{\Omega]\Gamma \vdash [\Omega](e\ s_0) : [\Omega]A_0\ p \gg B\ q}_{\text{Decl}\supset \text{Spine}}
        [\Omega]\Gamma \vdash [\Omega]P true
                   [\Omega]\Gamma \vdash [\Omega](e s_0) : ([\Omega]P) \supset [\Omega]A_0 \mathfrak{p} \gg B \mathfrak{q}
       [\Omega]\Gamma \vdash [\Omega]P true
                                                Subderivation
       [\Omega]\Gamma \vdash [\Omega][\Gamma]P true
                                                By Lemma 29 (Substitution Monotonicity) (ii)
             \Gamma \vdash [\Gamma] P true \dashv \Theta
                                                By Lemma 97 (Completeness of Checkprop)
       \Theta \longrightarrow \Omega_1
                                                 "
      \Omega \longrightarrow \Omega_1
                                                 "
\mathsf{dom}(\Theta) = \mathsf{dom}(\Omega_1)
        \Gamma \longrightarrow \Omega
                                                 Given
     [\Omega]\Gamma = [\Omega_1]\Theta
                                                 By Lemma 57 (Multiple Confluence)
   [\Omega]A_0 = [\Omega_1]A_0
                                                 By Lemma 55 (Completing Completeness) (ii)
               [\Omega]\Gamma \vdash [\Omega](e\ s_0) : [\Omega]A_0\ p \gg B\ q
                                                                                    Subderivation
            [\Omega_1]\Theta \vdash [\Omega](e s_0) : [\Omega_1]A_0 \mathfrak{p} \gg B \mathfrak{q}
                                                                                    By above equalities
                    \Theta \vdash e \ s_0 : [\Theta] A_0 \ p \gg B' \ q \dashv \Delta
                                                                                    By i.h.
                   B' = [\Delta]B'
13
                                                                                     "
         dom(\Delta) = dom(\Omega')
13
                    B = [\Omega']B'
3
                \Delta \longrightarrow \Omega'
             \Omega_1 \longrightarrow \Omega'
               \Omega \longrightarrow \Omega'
                                                                                     By Lemma 33 (Extension Transitivity)
            [\Theta]A_0 = [\Theta][\Gamma]A_0
                                                                                    By Lemma 29 (Substitution Monotonicity) (iii)
         \Theta \vdash e \ s_0 : [\Theta][\Gamma] A_0 \ p \gg B' \ q \dashv \Delta
                                                                                      By above equality
         \Gamma \vdash e \ s_0 : ([\Gamma]P) \supset [\Gamma]A_0 \ p \gg B' \ q \dashv \Delta \quad By \supset Spine
                                                                                      By def. of subst.
         \Gamma \vdash e \ s_0 : [\Gamma](P \supset A_0) \ p \gg B' \ q \dashv \Delta
```

$$\bullet \ \, \textbf{Case} \ \, \underbrace{ \begin{array}{c} [\Omega]\Gamma \vdash [\Omega]e_0 \Leftarrow A_k' \ p \\ \hline [\Omega]\Gamma \vdash \mathsf{inj}_k \ [\Omega]e_0 \Leftarrow \underbrace{A_1' + A_2'}_{[\Omega]A} \ p \end{array} }_{\text{Decl} + l_k}$$

Either $[\Gamma]A = A_1 + A_2$ (where $[\Omega]A_k = A'_k$) or $[\Gamma]A = \hat{\alpha} \in \mathsf{unsolved}(\Gamma)$.

In the former case:

$$\begin{split} [\Omega]\Gamma \vdash [\Omega]e_0 & \Leftarrow A_k' \ p & \text{Subderivation} \\ [\Omega]\Gamma \vdash [\Omega]e_0 & \Leftarrow [\Omega]A_k \ p & [\Omega]A_k = A_k' \\ \Gamma \vdash e_0 & \Leftarrow [\Gamma]A_k \ p \dashv \Delta & \text{By i.h.} \\ & \Delta \longrightarrow \Omega & \text{"} \\ & \text{dom}(\Delta) = \text{dom}(\Omega') & \text{"} \\ & \Gamma \vdash \text{inj}_k \ e_0 & \Leftarrow ([\Gamma]A_1) + ([\Gamma]A_2) \ p \dashv \Delta & \text{By } + I_k \\ & \Gamma \vdash \text{inj}_k \ e_0 & \Leftarrow [\Gamma](A_1 + A_2) \ p \dashv \Delta & \text{By def. of subst.} \end{split}$$

In the latter case, $A = \hat{\alpha}$ and $[\Omega]A = [\Omega]\hat{\alpha} = A_1' + A_2' = \tau_1' + \tau_2'$. By inversion on $\Gamma \vdash \hat{\alpha}$ p *type*, it must be the case that p = y.

$$\begin{array}{ll} \Gamma \longrightarrow \Omega & \text{Given} \\ \Gamma = \Gamma_0[\hat{\alpha}:\star] & \hat{\alpha} \in \mathsf{unsolved}(\Gamma) \\ \Omega = \Omega_0[\hat{\alpha}:\star\!=\!\tau_0] & \text{By Lemma 22 (Extension Inversion) (vi)} \end{array}$$

Let
$$\Omega_2 = \Omega_0[\hat{\alpha}_1 : \star = \tau_1', \hat{\alpha}_1 : \star = \tau_2', \hat{\alpha} : \star = \hat{\alpha}_1 + \hat{\alpha}_2].$$

Let $\Gamma_2 = \Gamma_0[\hat{\alpha}_1 : \star, \hat{\alpha}_2 : \star, \hat{\alpha} : \star = \hat{\alpha}_1 + \hat{\alpha}_2].$

$$\Gamma \longrightarrow \Gamma_2$$
 By Lemma 23 (Deep Evar Introduction) (iii) twice and Lemma 26 (Parallel Admissibility) (ii)

$$\Omega \longrightarrow \Omega_2$$
 By Lemma 23 (Deep Evar Introduction) (iii) twice and Lemma 26 (Parallel Admissibility) (iii)

 $\Gamma_2 \longrightarrow \Omega_2$ By Lemma 26 (Parallel Admissibility) (ii), (ii), (iii)

• Case
$$\frac{[\Omega]\Gamma, x : A_1' p \vdash [\Omega]e_0 \Leftarrow A_2' p}{[\Omega]\Gamma \vdash \lambda x, [\Omega]e_0 \Leftarrow A_1' \to A_2' p} \text{ Decl} \to I$$

We have $[\Omega]A=A_1'\to A_2'$. Either $[\Gamma]A=A_1\to A_2$ where $A_1'=[\Omega]A_1$ and $A_2'=[\Omega]A_2$ —or $[\Gamma]A=\hat{\alpha}$ and $[\Omega]\hat{\alpha}=A_1'\to A_2'$.

In the former case:

$$[\Omega]\Gamma,x:A_1' p \vdash [\Omega]e_0 \in A_2' p \qquad Subderivation \\ A_1' = [\Omega]A_1 \qquad \qquad Known in this subcase \\ = [\Omega][\Gamma]A_1 \qquad By Lemma 30 (Substitution Invariance) \\ Applying Ω on both sides \\ = [\Omega][\Gamma]A_1 \qquad By idempotence of substitution \\ [\Omega]\Gamma,x:A_1' p = [\Omega,x:A_1' p](\Gamma,x:[\Gamma]A_1 p) \qquad By definition of context application \\ [\Omega,x:A_1' p](\Gamma,x:[\Gamma]A_1 p) \vdash [\Omega]e_0 \in A_2' p \qquad By above equality \\ & \Gamma \longrightarrow \Omega \qquad Given \\ & \Gamma,x:[\Gamma]A_1 p \mapsto \Omega,x:A_1' p \qquad By def. of dom(-) \\ & Gom(\Gamma) = dom(\Omega) \qquad Given \\ & dom(\Gamma,x:[\Gamma]A_1 p) = \Omega,x:A_1' p \qquad By def. of dom(-) \\ & \Gamma,x:[\Gamma]A_1 p \vdash e_0 \in A_2 p \vdash \Delta' \qquad By l.h. \\ & \Delta' \longrightarrow \Omega_0' \qquad " \qquad \\ & \Omega_0' = (\Omega',x:A_1' p,\Omega_2) \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_1 \times (A_1' p) = \Omega_1' \times (A_2' p) \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_2 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_2 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_2 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_2 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_2 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_3 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_4 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_4 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_4 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_4 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_4 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_4 \times (A_1' p) = \Omega_1' \qquad By Lemma 22 (Extension Inversion) (v) \\ & R_4 \times (A_1' p) = \Omega_1' \qquad By Lemma 23 (Deep Evar Introduction) (iii) twice \\ & and Lemma 26 (Parallel Admissibility) (ii) \\ & R_4 \times (A_1' p) = \Omega_1' \Rightarrow \alpha_1' \Rightarrow \alpha_2' \Rightarrow \alpha_1' \Rightarrow \alpha_1' \Rightarrow \alpha_2' \Rightarrow \alpha_1' \Rightarrow \alpha_$$

```
\Gamma \longrightarrow \Gamma_2
                      By Lemma 23 (Deep Evar Introduction) (iii) twice
                        and Lemma 26 (Parallel Admissibility) (ii)
\Omega \longrightarrow \Omega_2
                     By Lemma 23 (Deep Evar Introduction) (iii) twice
                        and Lemma 26 (Parallel Admissibility) (iii)
                      By Lemma 26 (Parallel Admissibility) (ii), (ii), (iii)
[\Omega]\Gamma, \chi: \tau_1' \not \vdash [\Omega]e_0 \leftarrow \tau_2' \not \downarrow
                                                    Subderivation
          [\Omega]\Gamma = [\Omega_2]\Gamma_2
                                                    By Lemma 57 (Multiple Confluence)
              \tau_2' = [\Omega] \hat{\alpha}_2
                                                    From above equality
                   = [\Omega_2] \hat{\alpha}_2
                                                    By Lemma 55 (Completing Completeness) (i)
              \tau_1' = [\Omega_2] \widehat{\alpha}_1
                                                    Similar
                [\Omega_2]\Gamma_2, x : \tau_1' \not I = [\Omega_2, x : \tau_1' \not I](\Gamma_2, x : \hat{\alpha}_1 \not I)
                                                                                                   By def. of context application
[\Omega_2, x: \tau_1' \ f](\Gamma_2, x: \hat{\alpha}_1 \ f) \vdash [\Omega]e_0 \Leftarrow [\Omega_2]\hat{\alpha}_2 \ f
                                                                                                   By above equalities
                           dom(\Gamma) = dom(\Omega)
                                                                                                   Given
             \mathsf{dom}(\Gamma_2, x : \hat{\alpha}_1 \not I) = \mathsf{dom}(\Omega_2, x : \tau_1' \not I)
                                                                                                   By def. of \Gamma_2 and \Omega_2
                        \Gamma_2, x : \hat{\alpha}_1 \not I \vdash e_0 \leftarrow [\Gamma_2, x : \hat{\alpha}_1 \not I] \hat{\alpha}_2 \not I \dashv \Delta^+
                                                                                                   By i.h.
                               \Delta^+ \longrightarrow \Omega^+
                                                                                                   "
                        \mathsf{dom}(\Delta^+) = \mathsf{dom}(\Omega^+)
                               \Omega_2 \longrightarrow \Omega^+
        \Gamma_2, x: \hat{\alpha}_1 \not ! \longrightarrow \Delta^+
                                                                         By Lemma 51 (Typing Extension)
                       \Delta^+ = (\Delta, x : \hat{\alpha}_1 \not I, \Delta_Z)
                                                                         By Lemma 22 (Extension Inversion) (v)
                       \Omega^+ = (\Omega', x : \dots !, \Omega_Z)
                                                                         By Lemma 22 (Extension Inversion) (v)
                      \Delta \longrightarrow \Omega'
               \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
13
                      \Omega \longrightarrow \Omega_2
                                                                         Above
                      \Omega \longrightarrow \Omega^+
                                                                         By Lemma 33 (Extension Transitivity)
                      \Omega \longrightarrow \Omega'
                                                                         By Lemma 22 (Extension Inversion) (v)
B
                                                                         By \to \!\! l \hat{\alpha}
                           \Gamma \vdash \lambda x. e_0 \Leftarrow \hat{\alpha} \not I \dashv \Delta
                          \hat{\alpha} = [\Gamma] \hat{\alpha}
                                                                         \hat{\alpha} \in \mathsf{unsolved}(\Gamma)
                           \Gamma \vdash \lambda x. e_0 \leftarrow [\Gamma] \hat{\alpha} \not I \dashv \Delta By above equality
```

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• Case
$$\frac{[\Omega]\Gamma, x : [\Omega]Ap \vdash [\Omega]v \Leftarrow [\Omega]Ap}{[\Omega]\Gamma \vdash rec \ x : [\Omega]v \Leftrightarrow [\Omega]Ap} \ DeclRec$$

$$[\Omega]\Gamma, x : [\Omega]Ap \vdash [\Omega]v \Leftrightarrow [\Omega]Ap \ Subderivation$$

$$[\Omega]\Gamma, x : [\Omega]Ap \vdash [\Omega]v \Leftrightarrow [\Omega]Ap \ Subderivation$$

$$[\Omega]\Gamma, x : [\Omega]Ap = [\Omega, x : [\Omega]Ap](\Gamma, x : [\Gamma]Ap) \ By definition of context application$$

$$[\Omega, x : [\Omega]Ap](\Gamma, x : [\Gamma]Ap) \vdash [\Omega]v \Leftrightarrow [\Omega]Ap \ By above equality$$

$$\Gamma \longrightarrow \Omega \ Given \ By \longrightarrow \forall xr$$

$$dom(\Gamma) = dom(\Omega) \ Given \ By \longrightarrow \forall xr$$

$$dom(\Gamma, x : [\Gamma]Ap) = \Omega, x : [\Omega]Ap \ By def. of dom(-)$$

$$\Gamma, x : [\Gamma]Ap \vdash v \Leftrightarrow [\Gamma]Ap \vdash \Delta' \ By i.h.$$

$$\Delta' \longrightarrow \Omega'_0 \ "$$

$$dom(\Delta') = dom(\Omega'_0) \ "$$

$$\Omega, x : [\Omega]Ap \longrightarrow \Omega'_0 \ "$$

$$\Omega'_0 = (\Omega', x : [\Omega]Ap, \Theta) \ By Lemma 22 (Extension Inversion) (v)$$

$$\Gamma, x : [\Gamma]Ap \longrightarrow \Delta' \ By Lemma 22 (Extension Inversion) (v)$$

$$\Gamma, x : [\Gamma]Ap \longrightarrow \Delta' \ By Lemma 22 (Extension Inversion) (v)$$

$$\Delta, x : \cdots, \Theta \longrightarrow \Omega', x : [\Omega]Ap, \Theta \ By above equalities$$

$$\Delta \longrightarrow \Omega' \ dom(\Delta) = dom(\Omega') \ "$$

$$\Gamma, x : [\Gamma]Ap \vdash v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta \ By above equality$$

$$\Gamma \vdash rec x. v \Leftrightarrow [\Gamma]Ap \vdash A, x : [\Gamma]Ap, \Theta$$

 $A' = [\Theta]A'$

```
\Gamma \longrightarrow \Omega
                                                                                     Given
               [\Omega]\Gamma = [\Omega_{\Theta}]\Theta
                                                                                     By Lemma 57 (Multiple Confluence)
               [\Omega]\Gamma \vdash [\Omega]s_0 : A \ q \gg C \ \lceil p \rceil
                                                                                     Subderivation
           [\Omega_{\Theta}]\Theta \vdash [\Omega]s_0 : [\Omega_{\Theta}]A' \; q \gg C \; \lceil p \rceil
                                                                                     By above equalities
                    \Theta \vdash s_0 : [\Theta]A' \neq C' \lceil p \rceil \dashv \Delta
                                                                                     By i.h.
                   C^{\,\prime} = [\Delta]\,C^{\,\prime}
3
                \Delta \longrightarrow \Omega'
                                                                                     "
13
                                                                                     "
         \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega^{\,\prime})
3
            \Omega_{\Theta} \longrightarrow \Omega'
                    C = [\Omega']C'
13
                    \Theta \vdash s_0 : A' \neq C' \lceil \mathfrak{p} \rceil \dashv \Delta
                                                                                     By above equality
                                                                                     By Lemma 33 (Extension Transitivity)
               \Omega \longrightarrow \Omega'
₽
                      \Gamma \vdash e_0 \ s_0 \Rightarrow C' \ p \dashv \Delta
                                                                                     By \to \!\! E
13
```

```
Case
                                                                       for all C_2.
                                                                        if [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A ! \gg C_2 \text{ // then } C_2 = C DeclSpineRecover
               [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A ! \gg C !
                                                          [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A ! \gg C [!]
                      \Gamma \longrightarrow \Omega
                                                                           Given
                    [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A ! \gg C !
                                                                           Subderivation
                          \Gamma \vdash s : [\Gamma]A ! \gg C' ! \vdash \Delta By i.h.
                     \Delta \longrightarrow \Omega'
      EF
                     \Omega \longrightarrow \Omega'
      ₽
                                                                           "
              \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
      S.
                                                                           "
                        C = [\Omega']C'
      13
                                                                           "
                       C' = [\Delta]C'
      EF
```

Suppose, for a contradiction, that $\mathsf{FEV}([\Delta]C') \neq \emptyset$. That is, there exists some $\hat{\alpha} \in \mathsf{FEV}([\Delta]C')$.

Choose $\hat{\alpha}_R$ such that $\hat{\alpha}_R \in \mathsf{FEV}(C')$ and either $\hat{\alpha}_R = \hat{\alpha}$ or $\hat{\alpha} \in \mathsf{FEV}([\Delta]\hat{\alpha}_R)$. Then either $\hat{\alpha}_R = \hat{\alpha}$, or $\hat{\alpha}_R$ is declared to the right of $\hat{\alpha}$ in Δ .

```
From (NEQ) and (EQ)
     [\Omega_2]C' \neq [\Omega']C'
                \Gamma \vdash s : [\Gamma]A ! \gg C' ! \vdash \Delta
                                                                                  Above
        [\Omega_2]\Gamma \vdash [\Omega_2]s : [\Omega_2][\Gamma]A ! \gg [\Omega_2]C' !
                                                                                  By Theorem 9
                \Gamma \vdash s : [\Gamma]A ! \gg C' ! \vdash \Delta
                                                                                  Above
                 \Gamma \vdash A ! type
                 \Gamma \vdash [\Gamma]A ! type
                                                                                  By Lemma 13 (Right-Hand Substitution for Typing)
FEV([\Gamma]A) = \emptyset
                                                                                   By inversion
\mathsf{FEV}([\Gamma]A) \subseteq \mathsf{dom}(\cdot)
                                                                                  Property of \subseteq
                                                                                   By Lemma 72 (Separation—Main) (Spines)
               \Delta = (\Delta_{L} * \Delta_{R})
    (\Gamma * \cdot) \xrightarrow{*} (\Delta_{L} * \Delta_{R})
  \mathsf{FEV}(\mathsf{C}') \subseteq \mathsf{dom}(\Delta_\mathsf{R})
              \hat{\alpha}_{R} \in FEV(C')
                                                                                  Above
              \hat{\alpha}_R \in \mathsf{dom}(\Delta_R)
                                                                                  Property of \subseteq
  dom(\Delta_{I}) \cap dom(\Delta_{R}) = \emptyset
                                                                                   \Delta well-formed
              \hat{\alpha}_R \notin \mathsf{dom}(\Delta_L)
     \mathsf{dom}(\Gamma)\subseteq \mathsf{dom}(\Delta_L)
                                                                                   By Definition 5
              \hat{\alpha}_R \notin \mathsf{dom}(\Gamma)
```

```
[\Omega_2]\Gamma \vdash [\Omega_2]s : [\Omega_2][\Gamma]A ! \gg [\Omega_2]C' !
                                                                  Above
                   \Omega_2 and \Omega_1 differ only at \hat{\alpha}
                                                                  Above
FEV([\Gamma]A) = \emptyset
                                                                  Above
  [\Omega_2][\Gamma]A = [\Omega_1][\Gamma]A
                                                                  By preceding two lines
             \Gamma \vdash [\Gamma] A type
                                                                  Above
          \Gamma \longrightarrow \Omega_2
                                                                  By Lemma 33 (Extension Transitivity)
          \Omega_2 \vdash [\Gamma] A \text{ type}
                                                                  By Lemma 38 (Extension Weakening (Types))
 \mathsf{dom}(\Omega_2) = \mathsf{dom}(\Omega_1)
                                                                  \Omega_1 and \Omega_2 differ only at \hat{\alpha}
           \Omega_1 \vdash [\Gamma] A  type
                                                                  By Lemma 18 (Equal Domains)
            \Gamma \vdash [\Gamma] A  type
                                                 Above
           \Omega \vdash [\Gamma] A  type
                                                 By Lemma 38 (Extension Weakening (Types))
[\Omega_1][\Gamma]A = [\Omega'][\Gamma]A = [\Omega][\Gamma]A By Lemma 55 (Completing Completeness) (ii) twice
                                                 By Lemma 29 (Substitution Monotonicity) (iii)
              = [\Omega]A
      [\Omega]\Gamma = [\Omega']\Gamma
                                                 By Lemma 57 (Multiple Confluence)
              = [\Omega_1] \Gamma
                                                 By Lemma 57 (Multiple Confluence)
              = [\Omega_2]\Gamma
                                                 Follows from \hat{\alpha}_R \notin dom(\Gamma)
[\Omega_2]s = [\Omega]s \Omega_2 and \Omega differ only in \hat{\alpha}
                    [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A! \gg [\Omega_2]C'!
                                                                       By above equalities
                        C = [\Omega']C'
                                                                       Above
                [\Omega']C' \neq [\Omega_2]C'
                                                                       By def. of subst.
                        C \neq [\Omega_2]C'
                                                                       By above equality
                        C = [\Omega_2]C'
                                                                       Instantiating "for all C_2" with C_2 = [\Omega_2]C'
                       \Rightarrow \Leftarrow
       FEV([\Delta]C') = \emptyset
                                                                       By contradiction
                        \Gamma \vdash s : [\Gamma]A ! \gg C' [!] \dashv \Delta
                                                                       By SpineRecover
B
```

$$\begin{array}{ll} \bullet \ \, \textbf{Case} & \underline{ [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A\ p \gg C\ q} \\ \hline \underline{ [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A\ p \gg C\ [q] } \ \, \textbf{DeclSpinePass} \\ \\ & \underline{ [\Omega]\Gamma \vdash [\Omega]s : [\Omega]A\ p \gg C\ q} \quad \textbf{Subderivation} \\ & \Gamma \vdash s : [\Gamma]A\ p \gg C'\ q \dashv \Delta \quad \textbf{By i.h.} \\ \hline \textbf{Mom} \quad \Delta \longrightarrow \Omega' \qquad \qquad " \\ \hline \textbf{Mom} \quad \text{dom}(\Delta) = \text{dom}(\Omega') \qquad \qquad " \\ \hline \textbf{Mom} \quad \Omega \longrightarrow \Omega' \qquad \qquad " \\ \hline \textbf{Mom} \quad C' = [\Delta]C' \qquad \qquad " \\ \hline \textbf{C}' = [\Delta]C' \qquad \qquad " \\ \hline \textbf{C} = [\Omega']C' \qquad \qquad " \end{array}$$

We distinguish cases as follows:

- If p = 1 or q = 1, then we can just apply SpinePass:

- Otherwise, p = ! and q = \mathcal{Y} . If FEV(C) $\neq \emptyset$, we can apply SpinePass, as above. If FEV(C) = \emptyset , then we instead apply SpineRecover:

$$\qquad \qquad \Gamma \vdash s : [\Gamma] A \ \mathfrak{p} \gg C^{\, \prime} \ \lceil ! \rceil \dashv \Delta \quad \text{By SpineRecover}$$

Here,
$$q' = !$$
 and $q = !$, so $q' \sqsubseteq q$.

Case

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 $\frac{[\Omega]\Gamma \vdash [\Omega]e_0 \Leftarrow [\Omega]A_1 \ q \qquad [\Omega]\Gamma \vdash [\Omega]s_0 : [\Omega]A_2 \ q \gg B \ p}{[\Omega]\Gamma \vdash [\Omega](e_0 \ s_0) : ([\Omega]A_1) \to ([\Omega]A_2) \ q \gg B \ p} \ \mathsf{Decl} \to \mathsf{Spine}$

$$\begin{split} [\Omega]\Gamma \vdash [\Omega]e_0 & \Leftarrow [\Omega]A_1 \ q & \text{Subderivation} \\ \Gamma \vdash e_0 & \Leftarrow A' \ q \dashv \Theta & \text{By i.h.} \\ \Theta & \longrightarrow \Omega_\Theta & '' \\ \Omega & \longrightarrow \Omega_\Theta & '' \\ A & = [\Omega_\Theta]A' & '' \\ A' & = [\Theta]A' & '' \end{split}$$

$$[\Omega]\Gamma \vdash [\Omega]s_0 : [\Omega]A_2 \neq B p$$
 Subderivation

$$\Gamma \vdash e_0 \ s_0 : A_1 \rightarrow A_2 \ q \gg B \ p \dashv \Delta \quad By \rightarrow Spine$$

$$\bullet \ \, \textbf{Case} \ \, \frac{[\Omega]\Gamma \vdash [\Omega]P \ \textit{true} \qquad [\Omega]\Gamma \vdash [\Omega]e \Leftarrow [\Omega]A_0 \ p}{[\Omega]\Gamma \vdash [\Omega]e \Leftarrow ([\Omega]A_0) \land [\Omega]P \ p} \ \, \text{Decl} \land \textbf{I}$$

If e not a case, then:

```
[\Omega]\Gamma \vdash [\Omega]P true
                                                          Subderivation
                  \Gamma \vdash P \ true \dashv \Theta
                                                          By Lemma 97 (Completeness of Checkprop)
             \Theta \longrightarrow \Omega'_0
                                                          "
             \Omega \longrightarrow \Omega'_0
              \Gamma \longrightarrow \Omega
                                                          Given
              \Gamma \longrightarrow \Omega'_0
                                                          By Lemma 33 (Extension Transitivity)
            [\Omega]\Gamma = [\Omega]\Omega
                                                          By Lemma 54 (Completing Stability)
                    = [\Omega'_0]\Omega'_0
                                                          By Lemma 55 (Completing Completeness) (iii)
                    = [\Omega'_0]\Theta
                                                          By Lemma 56 (Confluence of Completeness)
                  \Gamma \vdash A_0 \land P p type
                                                          Given
                  \Gamma \vdash A_0 p type
                                                          By inversion
         [\Omega]A_0 = [\Omega'_0]A_0
                                                          By Lemma 55 (Completing Completeness) (ii)
             [\Omega]\Gamma \vdash [\Omega]e \Leftarrow [\Omega]A_0 p
                                                          Subderivation
          [\Omega'_0]\Theta \vdash [\Omega]e \Leftarrow [\Omega'_0]A_0 p
                                                          By above equalities
                 \Theta \vdash e \Leftarrow [\Theta] A_0 p \dashv \Delta
                                                          By i.h.
             \Delta \longrightarrow \Omega'
37
                                                          "
       \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
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           \Omega_0' \longrightarrow \Omega'
                                                          By Lemma 33 (Extension Transitivity)
             \Omega \longrightarrow \Omega'
13
                  \Gamma \vdash e \Leftarrow A_0 \land P p \dashv \Delta \quad Bv \land I
13
```

Otherwise, we have $e = case(e_0, \Pi)$. Let n be the height of the given derivation.

```
n-1 [\Omega]\Gamma \vdash [\Omega](case(e_0,\Pi)) \Leftarrow [\Omega]A_0 p
                                                                          Subderivation
n-2 [\Omega]\Gamma \vdash [\Omega]e_0 \Rightarrow B!
                                                                          By Lemma 62 (Case Invertibility)
n-2 [\Omega]\Gamma \vdash [\Omega]\Pi :: B \Leftarrow [\Omega]A_0 p
n-2 [\Omega]\Gamma \vdash [\Omega]\Pi \text{ covers } B
n-1 [\Omega]\Gamma \vdash [\Omega]P true
                                                                          Subderivation
                                                                         By Lemma 61 (Interpolating With and Exists) (1)
n-1 [\Omega]\Gamma \vdash [\Omega]\Pi :: B \Leftarrow ([\Omega]A_0) \land ([\Omega]P) p
n-1 \ [\Omega]\Gamma \vdash [\Omega]\Pi :: B \Leftarrow [\Omega](A_0 \land P) \ p
                                                                          By def. of subst.
                   \Gamma \vdash e_0 \Rightarrow B' ! \dashv \Theta By i.h.
              \Theta \longrightarrow \Omega_0'
              \Omega \longrightarrow \Omega'_0
                 B = [\Omega_0']B'
                     = [\Omega'_0][\Theta]B'
                                                  By Lemma 30 (Substitution Invariance)
              [\Omega]\Gamma = [\Omega'_0]\Theta
                                                  By Lemma 57 (Multiple Confluence)
 [\Omega](A_0 \wedge P) = [\Omega'_0](A_0 \wedge P)
                                                  By Lemma 55 (Completing Completeness) (ii)
```

• Case DeclNil: Similar to the first part of the Decl\lambda case.

 $\Gamma \vdash e \Leftarrow \exists \alpha : \kappa. A_0 p \dashv \Delta$ By $\exists I$

 $\Delta \longrightarrow \Omega'$

 $\Omega_0 \longrightarrow \Omega'$

 $\Omega \longrightarrow \Omega_0$ $\Omega \longrightarrow \Omega'$

 $\mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')$

3

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3

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"

By Lemma 33 (Extension Transitivity)

```
\Gamma, \triangleright_{\hat{\alpha}}, \hat{\alpha} : \mathbb{N} \longrightarrow \Gamma'
                                                                                By Lemma 47 (Checkprop Extension)
         \Gamma, \triangleright_{\hat{\alpha}}, \hat{\alpha} : \mathbb{N} \longrightarrow \Omega'_0
                                                                                By Lemma 33 (Extension Transitivity)
                        [\Omega]\Gamma = [\Omega]\Omega
                                                                                By Lemma 54 (Completing Stability)
                                   = [\Omega^+]\Omega^+
                                                                                By def. of context application
                                   = [\Omega'_0]\Omega'_0
                                                                                By Lemma 55 (Completing Completeness) (iii)
                                   = [\Omega'_0]\Gamma'
                                                                                By Lemma 56 (Confluence of Completeness)
                      [\Omega]A_0 = [\Omega^+]A_0
                                                                                By def. of context application
                                   = [\Omega'_0]A_0
                                                                                By Lemma 55 (Completing Completeness) (ii)
                          [\Omega]\Gamma \vdash [\Omega]e_1 \leftarrow [\Omega]A_0 p
                                                                                Subderivation
                      [\Omega'_0]\Gamma' \vdash [\Omega]e_1 \Leftarrow [\Omega'_0]A_0 p
                                                                                By above equalities
   2
                              \Gamma' \vdash e_1 \Leftarrow [\Gamma'] A_0 p \dashv \Theta
                                                                                By i.h.
                          \Theta \longrightarrow \Omega_0''
                        \Omega_0' \longrightarrow \Omega_0''
                               [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow (\text{Vec t}_2 \ [\Omega]A_0) \ \text{/}
                                                                                                                  Subderivation
                               [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow (\text{Vec}([\Omega^+]\hat{\alpha})[\Omega]A_0) \text{ } I
                                                                                                                  By def. of substitution
                           [\Omega_0'']\Theta \vdash [\Omega]e_2 \Leftarrow (\text{Vec}([\Omega_0'']\hat{\alpha})[\Omega_0'']A_0) \not
                                                                                                                 By lemmas
                           By def. of subst.
   3
                                    \Theta \vdash e_2 \Leftarrow [\Theta] A_0 p \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Delta'
                                                                                                                  By i.h.
                 \Delta, \triangleright_{\hat{\alpha}}, \Delta' \longrightarrow \Omega''
                                                                                                                  "
         dom(\Delta, \blacktriangleright_{\hat{\alpha}}, \Delta') = dom(\Omega'')
                                                                                                                  "
                           \Omega_0'' \longrightarrow \Omega''
                                \Omega'' = (\Omega, \blacktriangleright_{\hat{\alpha}}, \dots)
                                                                                                      By Lemma 22 (Extension Inversion) (ii)
                                \Delta \longrightarrow \Omega'
   137
                        \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
   13
            (\Gamma', \blacktriangleright_{\hat{\alpha}}, \dots) \longrightarrow \Omega'
                                                                                                      By Lemma 33 (Extension Transitivity)
                             \Omega \longrightarrow \Omega'
                                                                                                      By Lemma 22 (Extension Inversion) (ii)
   3
                                     \Gamma \vdash e_1 :: e_2 \Leftarrow (\text{Vec t } A_0) p \dashv \Delta By Cons
   34
          \frac{[\Omega]\Gamma \vdash [\Omega]e_1 \Leftarrow A_1' \ p \qquad [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow A_2' \ p}{[\Omega]\Gamma \vdash \langle [\Omega]e_1, [\Omega]e_2 \rangle \Leftarrow A_1' \times A_2' \ p} \ \mathsf{Decl} \times \mathsf{I}
Either [\Gamma]A = A_1 \times A_2 or [\Gamma]A = \hat{\alpha} \in \mathsf{unsolved}(\Gamma).
```

- In the first case ($[\Gamma]A = A_1 \times A_2$), we have $A'_1 = [\Omega]A_1$ and $A'_2 = [\Omega]A_2$.

```
[\Omega]\Gamma \vdash [\Omega]e_1 \Leftarrow A'_1 p
                                                                                                          Subderivation
                  [\Omega]\Gamma \vdash [\Omega]e_1 \Leftarrow [\Omega]A_1 p
                                                                                                          [\Omega]A_1 = A_1'
                       \Gamma \vdash e_1 \leftarrow [\Gamma] A_1 \mathfrak{p} \dashv \Theta
                                                                                                          By i.h.
                 \Theta \longrightarrow \Omega_{\Theta}
                                                                                                           "
         \mathsf{dom}(\Theta) = \mathsf{dom}(\Omega_\Theta)
                 \Omega \longrightarrow \Omega_{\Theta}
                 [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow A_2' p
                                                                                        Subderivation
                 [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow [\Omega]A_2 p
                                                                                       [\Omega]A_2 = A_2'
                  \Gamma \longrightarrow \Theta
                                                                                        By Lemma 51 (Typing Extension)
                [\Omega]\Gamma = [\Omega_{\Theta}]\Theta
                                                                                        By Lemma 57 (Multiple Confluence)
            [\Omega]A_2 = [\Omega_{\Theta}]A_2
                                                                                        By Lemma 55 (Completing Completeness) (ii)
             [\Omega_{\Theta}]\Theta \vdash [\Omega]e_2 \Leftarrow [\Omega_{\Theta}]A_2 p
                                                                                        By above equalities
                       \Theta \vdash e_2 \Leftarrow [\Gamma] A_2 \mathfrak{p} \dashv \Delta
                                                                                        By i.h.
                  \Delta \longrightarrow \Omega'
                                                                                        "
         \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
137
             \Omega_{\Theta} \longrightarrow \Omega'
                 \Omega \longrightarrow \Omega'
                                                                                        By Lemma 33 (Extension Transitivity)
137
                        \Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow ([\Gamma] A_1) \times ([\Gamma] A_2) p \dashv \Delta \quad \text{By } \times I
                        \Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow [\Gamma](A_1 \times A_2) \mathfrak{p} \dashv \Delta
                                                                                                          By def. of subst.
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```

– In the second case, where $[\Gamma]A = \hat{\alpha}$, combine the corresponding subcase for Decl+I_k with some straightforward additional reasoning about contexts (because here we have two subderivations, rather than one).

```
Case
                                                                             [\Omega]\Gamma \vdash [\Omega]e_0 \Rightarrow C q
                [\Omega]\Gamma \vdash [\Omega]\Pi :: C ! \Leftarrow [\Omega]A \ p \qquad \forall D. [\underline{\Omega]\Gamma \vdash [\Omega]e_0} \Rightarrow D \ q \supset [\Omega]\Gamma \vdash [\Omega]\Pi \ \textit{covers} \ D ! DeclCase
                                                              [\Omega]\Gamma \vdash \mathsf{case}([\Omega]e_0, [\Omega]\Pi) \Leftarrow [\Omega]A v
                [\Omega]\Gamma \vdash [\Omega]e_0 \Rightarrow C q
                                                                Subderivation
                      \Gamma \vdash e_0 \Rightarrow C' \neq q \vdash \Theta
                                                                By i.h.
                 \Theta \longrightarrow \Omega_{\Theta}
                                                                "
                                                                 "
         \mathsf{dom}(\Theta) = \mathsf{dom}(\Omega_{\Theta})
                                                                 "
                \Omega \longrightarrow \Omega_{\Theta}
                     C = [\Omega_{\Theta}]C'
                     \Theta \vdash C' \neq type
                                                                By Lemma 63 (Well-Formed Outputs of Typing)
        FEV(C') = \emptyset
                                                                By inversion
         [\Omega_{\Theta}]C' = C'
                                                                By a property of substitution
```

```
\Gamma \longrightarrow \Omega
                                                                                  Given
                  \Delta \longrightarrow \Omega
                                                                                  Given
                  \Theta \longrightarrow \Omega
                                                                                  By Lemma 33 (Extension Transitivity)
                 [\Omega]\Gamma = [\Omega]\Theta = [\Omega]\Delta
                                                                                  By Lemma 56 (Confluence of Completeness)
                   \Gamma \longrightarrow \Theta
                                                                                  By Lemma 51 (Typing Extension)
                   \Gamma \longrightarrow \Omega_{\Theta}
                                                                                  By Lemma 33 (Extension Transitivity)
                [\Omega]\Gamma = [\Omega_{\Theta}]\Theta
                                                                                  By Lemma 57 (Multiple Confluence)
                       \Gamma \vdash A \ type
                                                                                  Given + inversion
                                                                                  By Lemma 38 (Extension Weakening (Types))
                      \Omega \vdash A \ type
                [\Omega]A = [\Omega_{\Theta}]A
                                                                                  By Lemma 55 (Completing Completeness) (ii)
                  [\Omega]\Gamma \vdash [\Omega]\Pi :: C \Leftarrow [\Omega]A p
                                                                                  Subderivation
               [\Omega_{\Theta}]\Theta \vdash [\Omega]\Pi :: [\Omega_{\Theta}]C' \Leftarrow [\Omega_{\Theta}]A \mathfrak{p}
                                                                                  By above equalities
                      \Theta \vdash \Pi :: C' \Leftarrow [\Theta] A \mathfrak{p} \dashv \Delta
                                                                                  By i.h. (v)
                  \Delta \longrightarrow \Omega'
                                                                                  "
           dom(\Delta) = dom(\Omega')
              \Omega_\Theta \longrightarrow \Omega
                 \Omega \longrightarrow \Omega'
                                                                                  By Lemma 33 (Extension Transitivity)
             [\Omega]\Gamma \vdash [\Omega]\Pi covers C
                                                                         Instantiation of quantifier
            [\Omega]\Gamma = [\Omega]\Delta
                    = [\Omega']\Delta
                                                                          By Lemma 57 (Multiple Confluence)
           [\Omega']\Delta \vdash [\Omega]\Pi covers C'
                                                                          By above equalities
             \Delta \longrightarrow \Omega'
                                                                          By Lemma 33 (Extension Transitivity)
                   \Gamma \vdash C'! type
                                                                          Given
              \Gamma \longrightarrow \Delta
                                                                          By Lemma 51 (Typing Extension) & 33
                  \Delta \vdash C'! type
                                                              By Lemma 41 (Extension Weakening for Principal Typing)
          [\Delta]C' = C'
                                                                          By FEV(C') = \emptyset and a property of subst.
                  \Delta \vdash \Pi covers C'
                                                                          By Theorem 11
                   \Gamma \vdash \mathsf{case}(e_0, \Pi) \Leftarrow [\Gamma] \land \mathfrak{p} \dashv \Delta By Case
  13
          \frac{[\Omega]\Gamma \vdash [\Omega]e_1 \Leftarrow A_1 \ \mathfrak{p} \qquad [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow A_2 \ \mathfrak{p}}{[\Omega]\Gamma \vdash \langle [\Omega]e_1, [\Omega]e_2 \rangle \Leftarrow \underbrace{A_1 \times A_2}_{[\Omega]A} \mathfrak{p}} \ \mathsf{Decl} \times \mathsf{I}
Either A = \hat{\alpha} where [\Omega]\hat{\alpha} = A_1 \times A_2, or A = A_1' \times A_2' where A_1 = [\Omega]A_1' and A_2 = [\Omega]A_2'.
In the former case (A = \hat{\alpha}):
We have [\Omega] \hat{\alpha} = A_1 \times A_2. Therefore A_1 = [\Omega] A_1' and A_2 = [\Omega] A_2'. Moreover, \Gamma = \Gamma_0[\hat{\alpha} : \kappa].
      [\Omega]\Gamma \vdash [\Omega]e_1 \Leftarrow [\Omega]A_1' p
                                                                              Subderivation
  Let \Gamma' = \Gamma_0[\hat{\alpha}_1 : \kappa, \hat{\alpha}_2 : \kappa, \hat{\alpha} : \kappa = \hat{\alpha}_1 + \hat{\alpha}_2].
```

```
[\Omega]\Gamma = [\Omega]\Gamma'
                                                                 By def. of context substitution
          [\Omega]\Gamma' \vdash [\Omega]e_1 \Leftarrow [\Omega]A_1' p
                                                                 By above equality
                                                                By i.h.
                \Gamma' \vdash e_1 \Leftarrow [\Gamma']A_1' p' \dashv \Theta
           \Theta \longrightarrow \Omega_1
           \Omega \longrightarrow \Omega_1
    \mathsf{dom}(\Theta) = \mathsf{dom}(\Omega_1)
           [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow [\Omega]A_2' p
                                                                Subderivation
                  [\Omega]\Gamma = [\Omega_1]\Theta
                                                                        By Lemma 57 (Multiple Confluence)
               [\Omega]A_2' = [\Omega_1]A_2'
                                                                        By Lemma 55 (Completing Completeness) (ii)
                [\Omega_1]\Theta \vdash [\Omega]e_2 \Leftarrow [\Omega_1]A_2' p
                                                                        By above equalities
                        \Theta \vdash e_2 \Leftarrow [\Theta] A_2' \mathfrak{p}' \dashv \Delta
                                                                        By i.h.
            dom(\Delta) = dom(\Omega')
                                                                         "
                   \Delta \longrightarrow \Omega'
    13
                \Omega_1 \longrightarrow \Omega'
                   \Omega \longrightarrow \Omega'
                                                                        By Lemma 33 (Extension Transitivity)
    13
                         \Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow \hat{\alpha} p' \dashv \Delta
                                                                        By \times I\hat{\alpha}
In the latter case (A = A_1' \times A_2'):
                                                                           Subderivation
                  [\Omega]\Gamma \vdash [\Omega]e_1 \Leftarrow A_1 p
                  [\Omega]\Gamma \vdash [\Omega]e_1 \Leftarrow [\Omega]A_1' p
                                                                           A_1 = [\Omega] A_1'
                        \Gamma \vdash e_1 \leftarrow [\Gamma] A_1' p \dashv \Theta
                                                                           By i.h.
                  \Theta \longrightarrow \Omega_0
                                                                           "
           \mathsf{dom}(\Theta) = \mathsf{dom}(\Omega_0)
                  \Omega \longrightarrow \Omega_0
                  [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow A_2 p
                                                                           Subderivation
                  [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow [\Omega]A_2' p
                                                                           A_2 = [\Omega]A_2'
                        \Gamma \vdash A_1' \times A_2' p type
                                                                           Given (A = A_1' \times A_2')
                        \Gamma \vdash A'_2 type
                                                                           By inversion
                   \Gamma \longrightarrow \Omega
                                                                           Given
                   \Gamma \longrightarrow \Omega_0
                                                                           By Lemma 33 (Extension Transitivity)
                    \Omega_0 \vdash A_2' type
                                                                           By Lemma 38 (Extension Weakening (Types))
                                                                           By Lemma 55 (Completing Completeness)
                  [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow [\Omega_0]A_2' p
                  [\Omega]\Gamma \vdash [\Omega]e_2 \Leftarrow [\Omega_0][\Theta]A_2' \ \mathfrak{p}
                                                                           By Lemma 29 (Substitution Monotonicity) (iii)
                 [\Omega]\Theta \vdash [\Omega]e_2 \Leftarrow [\Omega_0][\Theta]A_2' p
                                                                           By Lemma 57 (Multiple Confluence)
                       \Theta \vdash e_2 \Leftarrow [\Theta]A_2' p \dashv \Delta
                                                                           By i.h.
                  \Delta \longrightarrow \Omega'
  137
           \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
  ₽
                \Omega_0 \longrightarrow \Omega'
                  \Omega \longrightarrow \Omega'
                                                                           By Lemma 33 (Extension Transitivity)
  ESF
            \Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow ([\Omega]A_1) \times ([\Omega]A_2) \mathfrak{p} \dashv \Delta
                                                                                         By \times I
    \Gamma \vdash \langle e_1, e_2 \rangle \Leftarrow [\Omega](A_1 \times A_2) \mathfrak{p} \dashv \Delta
                                                                                          By def. of substitution
```

Now we turn to parts (v) and (vi), completeness of matching.

- Case DeclMatchEmpty: Apply rule MatchEmpty.
- Case DeclMatchSeq: Apply the i.h. twice, along with standard lemmas.
- Case DeclMatchBase: Apply the i.h. (i) and rule MatchBase.
- Case DeclMatchUnit: Apply the i.h. and rule MatchUnit.
- Case DeclMatch∃: By i.h. and rule Match∃.
- **Case** DeclMatch×: By i.h. and rule Match×.
- Case DeclMatch $+_k$: By i.h. and rule Match $+_k$.
- Case $\frac{[\Omega]\Gamma \: / \: P \vdash \vec{\rho} \Rightarrow e :: [\Omega]A, [\Omega]\vec{A} \: ! \Leftarrow [\Omega]C \: p}{[\Omega]\Gamma \vdash \vec{\rho} \Rightarrow e :: ([\Omega]A \land [\Omega]P), [\Omega]\vec{A} \: ! \Leftarrow [\Omega]C \: p} \: \mathsf{DeclMatch} \land$

To apply the i.h. (vi), we will show (1) $\Gamma \vdash (A, \vec{A})$! *types*, (2) $\Gamma \vdash P$ *prop*, (3) $FEV(P) = \emptyset$, (4) $\Gamma \vdash C p$ *type*, (5) $[\Omega]\Gamma / [\Omega]P \vdash \vec{\rho} \Rightarrow [\Omega]e :: [\Omega]\vec{A}! \Leftarrow [\Omega]C p$, and (6) $p' \sqsubseteq p$.

$$\Gamma \vdash (A \land P, \vec{A})$$
! types Given

 $\Gamma \vdash (A \land P)$! *type* By inversion on PrincipalTypevecWF

 $\Gamma \vdash A ! type$ By Lemma 42 (Inversion of Principal Typing) (3)

(2) $\Gamma \vdash P \ prop$ "

(3) $FEV(P) = \emptyset$ By inversion

(1) $\Gamma \vdash (A, \vec{A})$! types By inversion and PrincipalTypevecWF

(4)
$$\Gamma \vdash C p type$$
 Given

(5)
$$[\Omega]\Gamma / P \vdash \vec{\rho} \Rightarrow [\Omega]e :: [\Omega]A, [\Omega]\vec{A} \Leftarrow [\Omega]C p$$
 Subderivation

(6)
$$p' \sqsubseteq p$$
 Given

$$\Gamma \vdash \vec{\rho} \Rightarrow e :: ([\Gamma]A \land [\Gamma]P), [\Gamma]\vec{A}) \Leftarrow [\Gamma]C \ p' \dashv \Delta \quad \text{By Match} \land \\ \Gamma \vdash \vec{\rho} \Rightarrow e :: [\Gamma] ((A \land P), \vec{A}) \Leftarrow [\Gamma]C \ p' \dashv \Delta \quad \text{By def. of subst.}$$

- Case DeclMatchNeg: By i.h. and rule MatchNeg.
- Case DeclMatchWild: By i.h. and rule MatchWild.
- Case DeclMatchNil: Similar to the DeclMatch∧ case.
- Case DeclMatchCons: Similar to the DeclMatch∃ and DeclMatch∧ cases.

$$\begin{array}{c} \bullet \ \ \text{Case} \\ \frac{\mathsf{mgu}([\Omega]\sigma, [\Omega]\tau) = \bot}{[\Omega]\Gamma \: / \: [\Omega]\sigma = [\Omega]\tau \vdash [\Omega](\vec{\rho} \Rightarrow e) :: [\Omega]\vec{A} \: ! \Leftarrow [\Omega]C \: p \end{array} \text{DeclMatch} \bot$$

```
\Gamma \longrightarrow \Omega
                                                               Given
       EFF
                 FEV(\sigma = \tau) = \emptyset
                                                               Given
                                [\Omega]\sigma = [\Gamma]\sigma By Lemma 39 (Principal Agreement) (i)
                                 [\Omega]\tau = [\Gamma]\tau
                                                               Similar
                   mgu([\Omega]\sigma, [\Omega]\tau) = \bot
                                                                                                                          Given
                      \mathsf{mgu}([\Gamma]\sigma, [\Gamma]\tau) = \bot
                                                                                                                          By above equalities
                                                  \Gamma \ / \ \sigma \stackrel{\circ}{=} \tau : \kappa \dashv \bot
                                                                                                                          By Lemma 94 (Completeness of Elimeq) (2)
                       \Gamma / [\Gamma] \sigma = [\Gamma] \tau \vdash \vec{\rho} \Rightarrow e :: [\Gamma] \vec{A} \Leftarrow [\Gamma] C p \dashv \Gamma
                                                                                                                         By Match⊥
                                            \Omega \longrightarrow \Omega
                                                                                                                          By Lemma 32 (Extension Reflexivity)
         13
                                      dom(\Gamma) = dom(\Omega)
                                                                                                                          Given
Case
                 \frac{\mathsf{mgu}([\Omega]\sigma, [\Omega]\tau) = \theta \qquad \theta([\Omega]\Gamma) \vdash \theta(\vec{\rho} \Rightarrow [\Omega]e) :: \theta([\Omega]\vec{A}) \; ! \Leftarrow \theta([\Omega]C) \; p}{[\Omega]\Gamma \; / \; [\Omega]\sigma = [\Omega]\tau \vdash \vec{\rho} \Rightarrow [\Omega]e :: [\Omega]\vec{A} \; ! \Leftarrow [\Omega]C \; p} \; \mathsf{DeclMatchUnify}
                               ([\Omega]\sigma = [\Gamma]\sigma) and ([\Omega]\tau = [\Gamma]\tau) As in DeclMatch\bot case
        mgu([\Omega]\sigma, [\Omega]\tau) = \theta
                                                                                                        Given
            mgu([\Gamma]\sigma, [\Gamma]\tau) = \theta
                                                                                                        By above equalities
                                       \Gamma / \sigma \stackrel{\circ}{=} \tau : \kappa \dashv (\Gamma, \Theta)
                                                                                                        By Lemma 94 (Completeness of Elimeq) (1)
                                       \Theta = (\alpha_1 = t_1, \dots, \alpha_n = t_n)
                                                                                                         " for all \Gamma \vdash \mathfrak{u} : \kappa
                             [\Gamma, \Theta]u = \theta([\Gamma]u)
                      \theta([\Omega]\Gamma) \vdash \theta(\vec{\rho} \Rightarrow [\Omega]e) :: \theta([\Omega]\vec{A}) \Leftarrow \theta([\Omega]C) p Subderivation
                     \theta([\Omega]\Gamma) = [\Omega, \blacktriangleright_P, \Theta](\Gamma, \blacktriangleright_P, \Theta)
                                                                                               By Lemma 95 (Substitution Upgrade) (iii)
                    \theta([\Omega]\vec{A}) = [\Omega, \blacktriangleright_P, \Theta]\vec{A}
                                                                                               By Lemma 95 (Substitution Upgrade) (i) (over \vec{A})
                    \theta([\Omega]C) = [\Omega, \blacktriangleright_P, \Theta]C
                                                                                               By Lemma 95 (Substitution Upgrade) (i)
           \theta(\vec{\rho} \Rightarrow [\Omega]e) = [\Omega, \blacktriangleright_P, \Theta](\vec{\rho} \Rightarrow e)
                                                                                               By Lemma 95 (Substitution Upgrade) (iv)
         [\Omega, \triangleright_P, \Theta](\Gamma, \triangleright_P, \Theta) \vdash [\Omega, \triangleright_P, \Theta](\vec{\rho} \Rightarrow e) :: [\Omega, \triangleright_P, \Theta]\vec{A} \Leftarrow [\Omega, \triangleright_P, \Theta]C By above equalities
                          \Gamma, \blacktriangleright_{P}, \Theta \vdash (\vec{\rho} \Rightarrow e) :: [\Gamma, \blacktriangleright_{P}, \Theta] \vec{A} \Leftarrow [\Gamma, \blacktriangleright_{P}, \Theta] C p \dashv \Delta, \blacktriangleright_{P}, \Delta'
                 \Delta, \blacktriangleright_{P}, \Delta' \longrightarrow \Omega', \blacktriangleright_{P}, \Omega''
                  \Omega, \blacktriangleright_{P}, \Theta \longrightarrow \Omega', \blacktriangleright_{P}, \Omega''
         dom(\Delta, \blacktriangleright_P, \Delta') = dom(\Omega', \blacktriangleright_P, \Omega'')
                                        \Delta \longrightarrow \Omega'
                                                                                                                      By Lemma 22 (Extension Inversion) (ii)
                                \mathsf{dom}(\Delta) = \mathsf{dom}(\Omega')
         13
                                        \Omega \longrightarrow \Omega'
                                                                                                                      By Lemma 22 (Extension Inversion) (ii)
                \Gamma / [\Gamma] \sigma = [\Gamma] \tau \vdash \vec{\rho} \Rightarrow e :: [\Gamma] \vec{A} \Leftarrow [\Gamma] C p \dashv \Delta By MatchUnify
```