d-cell Ising model Note

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Contents

1 Residual entropy and counting ground state degeneracy

Residual entropy is defined as logrithm of ground state degeneracy. Thus if the ground state has a large degeneracy that is extensive, then we will have an extensive residual entropy. The hamiltonian of our concern is defined as following

$$H_p = \sum_{c_{p+1}} \prod_{i \in c_{p+1}} \sigma_i$$

in our case, this Hamiltonian is defined on a d-dimensional hypercubic lattice, where $d \geq p$, where c_p is a p-*cell*, we will discuss the formal definition of a cell later (it is not needed to understand the physics setup). Here for example 0-cell is a vertice, 1-cell is an edge, and 2-cell is a plaquette, and so on. The element of a p-cell is all $(p-1)$-cell that constructs the p-cell, thus for H_0, we have c_1, thus we have sites on each vertices and the Hamiltonian writes as a standard Ising model with nearest-neighbor interaction,

$$H_0 = \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

and for $p = 1$, we have spins on the edges

$$H_1 = \sum_{\square} \sigma_1 \sigma_2 \sigma_3 \sigma_4$$

for $p = 2$, we have spins on the faces

$$H_2 = \sum_{\square \in \text{faces of the hypercube}} \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6$$

The ground state degeneracy is mainly due to the gauge invariance, for example, for $p = 1, d = 2$, in a plaquette one can exchange the spin on two edges attached to the same vertex without changing the energy of our plaquette. So we can let each vertices have a spin $-1, 1$ to represent the spin on two edges attached to it as showed in Figure 1. No matter how one changes the spin on the edges, the product of edge spins that attached to the same vertex on the vertices is the same. This also means there is an effective spin at the vertex (denotes whether all edges attached to the vertex) is completely free. Thus it creates a freedom of N_0 spins where N_0 is the number of 0-cell (the vertices) , thus the degeneracy is at scale 2^{N_0}.

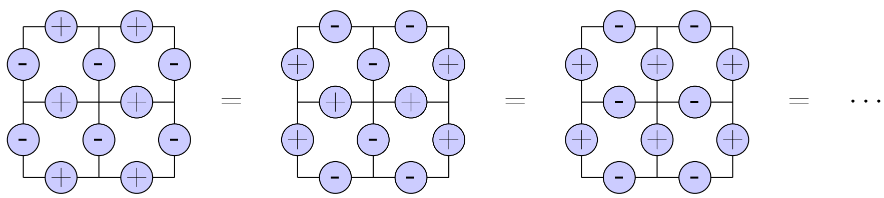


Figure 1: gauge freedom in the 2-cell case (plaquette), H_1 is the same for these configurations. 2nd configuration is obtained by flipping all spins attached to center vertex, 3rd configuration is obtained by flipping all spins attached all vertices.

Similarly, if $p = 2, d = 3$, on the faces of a cube, the 4 faces attached to the same edge can flip their spins. Thus we have a free spin on each edge of number N_1. However, this over-counts because when flipping all the effective spin on the edges that attached to the same vertex corresponding to the same face spins. Thus the total number of free spins should be $N_1 - N_0$.

Thus following this derivation, and generalize the above conclusion. We know, the free spins at p-cell is flipping all spins of p-cell that attached to $p-1$-cell, and the free spins for general p can be obtained by counting the free spins of $p-1$ case. Thus we have

$$\begin{aligned} N_{free} &= N_{p-1} - (N_{p-2} - (N_{p-3} - \cdots (N_1 - N_0))) \\ &= N_{p-1} - N_{p-2} + N_{p-3} - \cdots N_1 \pm N_0 \end{aligned}$$

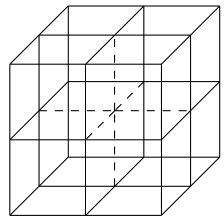
thus the ground state degeneracy is $2^{N_{p-1} - N_{p-2} + N_{p-3} - \cdots N_1 \pm N_0}$ for hypercubic lattice with no defects (the defects will create an $O(1)$ correction to the degeneracy).

This gives the residual entropy of the ground state,

$$S/N = \frac{N_{p-1} - N_{p-2} + N_{p-3} - \cdots N_1 \pm N_0}{N_p} log(2)$$

for $p = 2, d = 3$ case, assume the cubic lattice has length L, we have $N_0 = L^3$, $N_1 = 3L(L-1)^2$, and total number of spins $N = N_2 = 3(L-1)^3$, thus

$$\begin{aligned} S/N &= (N_1 - N_0)/N log(2) \\ &= (3(K+1)K^2 - (K+1)^3)/3K^3 log(2) \quad K = L - 1 \\ &= (K+1)(2K^2 - 2K - 1)/3K^3 log(2) \\ &= (2K^3 - 3K - 1)/3K^3 log(2) \\ &\approx \frac{2}{3} log(2) \quad \text{when } K \to \infty \end{aligned}$$



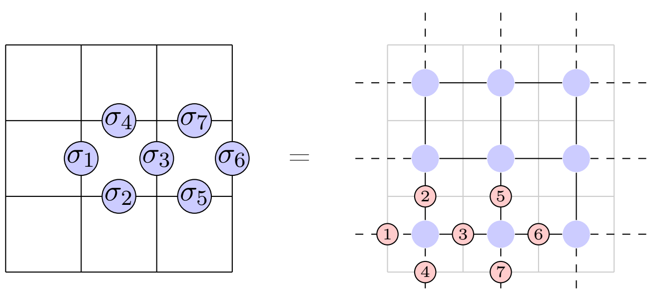
However, the above derivation can be much simpler, which is two sentence in the original self-correcting memory paper[2]. It is actually the Euler characteristic of the cell chain complex. Or the Betti number as the rank of the n-th singular homology group[3, 1].

2 Tensor network calculation of partition function

Let's first start with 2D case, we can write down the partition function for 2D, $2-cell$ case.

$$
\begin{aligned}
H_1 &= -\sum_{\square} \sigma_1\sigma_2\sigma_3\sigma_4 \\
Z &= \sum_{\sigma} \exp[-\beta H] = \sum_{\sigma} \exp\left[\beta \sum_{\square} \sigma_1\sigma_2\sigma_3\sigma_4\right] \\
&= \sum_{\sigma} \prod_{\square} \exp[\beta\sigma_1\sigma_2\sigma_3\sigma_4]
\end{aligned}
$$

the term $\exp[\beta\sigma_1\sigma_2\sigma_3\sigma_4]$ over all σ_i forms rank-4 tensor $T_{\sigma_1,\sigma_2,\sigma_3,\sigma_4}$ of size $2\times2\times2\times2$ on each plaquette, on neighboring plaquette, this requires the spin on neighboring edge to be the same, diagramatically, this is



More generally, because c_p attached to c_{p+1} only shared by two c_{p+1}, thus without using copy tensor, we can write down the general partition function as a tensor network where every c_{p+1} has a $2(p+1)$ leg tensor connects to its neighboring c_{p+1}.

3 Cluster update for the p-cell model

3.1 Fortuin-Kasteleyn cluster decomposition

The Fortuin-Kasteleyn cluster decomposition is straightforward to generalize on our p-cell model by writing

$$
\begin{aligned}
E &= -\sum_{c_{p+1}} \prod_{c_p \in c_{p+1}} \sigma_{c_p} \\
E_{c'_{p+1}} &= -\sum_{c_{p+1} \neq c'_{p+1}} \prod_{c_p \in c_{p+1}} \sigma_{c_p}
\end{aligned}
\tag{1}
$$

now we define partition function on the $(p+1)$-cell that has the same spins and $(p+1)$-cell that has different spins.

$$
Z^{\text{same}}_{c'_{p+1}} = \sum_s \delta_{c'_{p+1}} e^{-\beta E_{c'_{p+1}}} \quad Z^{\text{diff}}_{c'_{p+1}} = \sum_s (1-\delta_{c'_{p+1}}) e^{-\beta E_{c'_{p+1}}} \tag{2}
$$

and the total partition function becomes

$$
Z = e^{\beta} Z^{\text{same}}_{c'_{p+1}} + e^{-\beta} Z^{\text{diff}}_{c'_{p+1}} \tag{3}
$$

furthermore, define

$$
Z^{ind}_{c'_{p+1}} = \sum_s e^{-E_{c'_{p+1}}} = Z^{\text{same}}_{c'_{p+1}} + Z^{\text{diff}}_{c'_{p+1}} \tag{4}
$$

we have the same partition function as standard Ising model

$$
Z = (e^{\beta} - e^{-\beta}) Z^{\text{same}}_{c'_{p+1}} + e^{-\beta} Z^{ind}_{c'_{p+1}} \tag{5}
$$

this means the SW updates and Wolf update still holds as a cluster on $(p+1)$-cell chains.

4 The critial field

General interaction of a system of N_s Ising spins $S(r) = \pm 1$ located at sites r of a lattice can be written as

$$
H = -\sum_b I(b) R(b) \tag{6}
$$

where $I(b) \in \mathcal{R}$ is the interaction strength, and

$$
R(b) = \prod_r S(r)^{\theta(r,b)} \quad \theta(r,b) \in \{0,1\} \tag{7}
$$

then assuming $S(r) = (-1)^{\sigma(r)}, \sigma(r) \in \{0,1\}$, $R(b) = (-1)^{\rho(b)}$, we have $\rho(b) = \bigoplus_r \theta(r,b)\sigma(r)$. Let N_θ be the rank of the matrix $\theta(r,b)$. Then there are 2^{N_θ} different configurations $\{\rho(b)\}$. Assume interaction constants $I(b)$ are positive, the ground states of the system are defined by $R(b) = 1$ for all b. Thus the ground states are determined by

$$
\bigoplus_r \theta(r,b)\sigma_0(r) = 0 \quad \forall b \tag{8}
$$

this has 2^{N_g} solutions with $N_g = N_s - N_\theta$. We associate the unitary operators $U\{\sigma_0\} = \prod_r S_x(r)^{\sigma_0(r)}$

$$
S_x^2(r) = 1, \quad S_x(r) S(r) S_x(r)^{-1} = -S(r) \tag{9}
$$

The operators U commute with all operators R

$$
\begin{aligned}
U\{\sigma_0\}R(b)U\{\sigma_0\}^{-1} &= \prod_r S_x(r)^{\sigma_0(r)} \prod_r (S(r))^{\theta(r,b)} \prod_r S_x^{-1}(r)^{\sigma_0(r)} \\
&= \prod_r (S_x S(r) S_x^{-1})^{\bigoplus_r \theta(r,b)\sigma_0(r)} \\
&= \prod_r (-1)^{\bigoplus_r \theta(r,b)\sigma_0(r)} \\
&= R(b)
\end{aligned} \tag{10}
$$

Therefore, all the operators U commute with the Hamiltonian:

$$
U\{\sigma_0\}HU\{\sigma_0\}^{-1} = H \tag{11}
$$

A product of spins $\prod_r S(r)^{\psi(r)}$, $\psi(r) \in \{0,1\}$, is transformed by $U\{\sigma_0\}$ into

$$
U\{\sigma_0\}\prod_r S(r)^{\psi(r)} U\{\sigma_0\}^{-1} = \prod_r S(r)^{\psi(r)}(-1)^{\oplus\psi(r)\sigma_0(r)} \tag{12}
$$

4.1 The dual relation

Given partition function

$$
Z\{K\} = \sum_{\{S(r)\}} e^{-\beta H\{\mathbf{S}\}} = \sum_{\{S(r)\}} \prod_b e^{K(b)R(b)} \tag{13}
$$

$$
\begin{aligned}
e^{K(b)R(b)} &= \frac{e^{-K(b)} + e^{K(b)}}{2}(1 + R(b)\frac{e^{-K(b)} - e^{K(b)}}{e^{-K(b)} + e^{K(b)}}) \\
&= \frac{1}{2}(e^{-K(b)} + e^{K(b)} + R(b)(e^{-K(b)} - e^{K(b)})) \\
&= e^{-K(b)} \quad \text{or} \quad e^{K(b)} \\
&= \cosh K(b)[1 + R(b)\tanh K(b)] \\
&= \cosh K(b) \sum_{\phi(b)} R(b)^{\phi(b)} \tanh K(b)^{\phi(b)}
\end{aligned} \tag{14}
$$

thus we have

$$
\begin{aligned}
Z\{K\} &= \sum_{\{S(r)\}} \prod_b \cosh K(b) \sum_{\phi(b)} R(b)^{\phi(b)} \tanh K(b)^{\phi(b)} \\
&= \prod_b \cosh K(b) \sum_{\phi(b)} \tanh K(b)^{\phi(b)} \sum_{\{S(r)\}} R(b)^{\phi(b)} \quad \text{(reorder the summation)}
\end{aligned} \tag{15}
$$

If $R(b)^{\phi(b)} = 1$ then we have $\sum_{\{S(r)\}} R(b)^{\phi(b)} = 2^{N_s}$ otherwise the sum vanishes, then we always have the dual relation that $Z\{K\} = Z^*\{K^*\} = 0$. Denote the solution of product of operators $R(b)^{\phi(b)} = 1$ to be $\phi_0(b)$, and we have

$$
\oplus_b \theta(r,b)\phi_0(b) = 0 \quad \forall r \tag{16}
$$

then we have

$$Z\{K\} = 2^{N_s} \prod_b \cosh K(b) \sum_{\{\phi_0(b)\}} \prod_b \tanh(K(b))^{\phi_0(b)} \tag{17}$$

On the other hand, the partition function $Z^*\{K^*\}$ can be written as

$$\begin{aligned} Z^*\{K^*\} &= \sum_{\{S(r^*)\}} e^{-\beta H^* S} = \sum_{\{S(r^*)\}} \prod_b e^{K^*(b) R^*(b)} \\ &= \prod_b e^{K^*(b)} \sum_{\{S(r^*)\}} \prod_b e^{-2K^*(b)\rho^*(b)} \end{aligned} \tag{18}$$

where $R^*(b) = (-1)^{\rho^*(b)} = 1 - 2\rho^*(b)$, if we have the closure condition

$$\oplus \theta(r, b)\theta^*(r^*, b) = 0 \tag{19}$$

we have

$$\oplus_b \theta(r, b)\rho^*(b) = \oplus_b \oplus_r \theta(r, b)\theta^*(r^*, b)\sigma(r^*) = 0 \tag{20}$$

thus we have $\rho^*(b) = \phi_0(b)$, it follows that

$$Z^*\{K^*\} = \prod_b e^{K^*(b)} \sum_{\{\phi_0(b)\}} N\{\phi_0\} \prod_b e^{-2K^*(b)\phi_0(b)} \tag{21}$$

Here $N\{\phi_0\}$ denotes the number of configurations $\{S(r^*)\}$ which obey

$$\phi_0(b) = \oplus_{r^*} \theta(r^*, b)\sigma(r^*) \quad \forall b \tag{22}$$

If for a given set $\{\phi_0(b)\}$ there is no solution then $N\{\phi_0\} = 0$, otherwise $N\{\phi_0\} = 2^{N_s^* - N_\theta^*}$, and for $\beta^* = 0$ it follows that

$$Z^* = 2^{N_s^*} = \sum_{\{\phi_0\}} N\{\phi_0\} \tag{23}$$

There are $2^{N_b - N_\theta} = 2^{N_\theta^*}$ sets $\{\phi_0(b)\}$, where $N_\theta + N_\theta^* = N_b$, and N_b is the number of bonds. Therefore all N obey $N\{\phi_0\} = 2^{N_s^* - N_\theta^*}$. From one obtains

$$Z^*\{K^*\} = 2^{N_g^*} \prod_b e^{K^*(b)} \sum_{\{\phi_0(b)\}} \prod_b e^{-2K^*(b)\phi_0(b)} \tag{24}$$

Thus for the symmetric partition function $Y\{K\}$ where

$$\begin{aligned} Y\{K\} &= Z\{K\} 2^{-(N_s + N_g)/2} \prod_b [\cosh 2K(b)]^{-1/2} \\ Z\{K\} &= \sum_{\{S(r)\}} e^{-\beta H\{S\}} \end{aligned} \tag{25}$$

Now we can check

$$
\begin{aligned}
Y\{K\} &= 2^{(N_\theta)/2}(\prod_b \cosh K(b)\cosh 2K(b)^{-1/2}) \sum_{\{\phi_0(b)\}} (\prod_b \tanh(K(b))^{\phi_0(b)}) \\
&= 2^{(N_\theta)/2}(\prod_b \cosh(K(b))\sinh(2K(b))^{-1/2}(\tanh(2K(b)))^{1/2}) \sum_{\{\phi_0(b)\}} \prod_b e^{-2K^*(b)\phi_0(b)} \\
&= 2^{(N_\theta)/2}(\prod_b \tanh(K(b))^{-1}\sinh(K(b))(2\sinh(K(b))\cosh(K(b)))^{-1/2}) \sum_{\{\phi_0(b)\}} \prod_b e^{-2K^*(b)\phi_0(b)} \\
&= 2^{(N_\theta)/2}(\prod_b 2^{-1/2}\tanh(K(b))^{-1}(\tanh(K(b)))^{1/2}) \sum_{\{\phi_0(b)\}} \prod_b e^{-2K^*(b)\phi_0(b)} \\
&= 2^{(N_\theta)/2}(\prod_b 2^{-1/2}\tanh(K(b))^{1/2}) \sum_{\{\phi_0(b)\}} \prod_b e^{-2K^*(b)\phi_0(b)} \\
&= 2^{(N_\theta - N_b)/2}(\prod_b e^{-K^*(b)}) \sum_{\{\phi_0(b)\}} \prod_b e^{-2K^*(b)\phi_0(b)} \\
&= 2^{-N^*_\theta/2}(\prod_b e^{-K^*(b)}) \sum_{\{\phi_0(b)\}} \prod_b e^{-2K^*(b)\phi_0(b)} \\
&= Y^*\{K^*\}
\end{aligned}
\tag{26}
$$

In general we can define Ising model M_{dn} on the lattice L with n dimentional bonds consists of Ising spins $S(r) = \pm 1$ at all sites $r = r^{(n-1)}$ interacting via

$$
-\beta H_{dn} = K \sum_{r^{(n)}} \prod_r S(r)^{\theta(r, r^{(n)})} + h \sum_r S(r) \tag{27}
$$

where n denotes the $n-$cell, and d denotes the dimention. In general M_{dn} has $N_s = \binom{d}{n} N$ Ising spins located at the centers of $(n-1)-d$ hypercubes.

Definition 1. *(Boundary function). The boundary function $\Theta(r^m, r^{(m-1)})$ is defined as*

$$
\Theta(r^m, r^{(m-1)}) = \begin{cases} 0 & r^{(m-1)} \text{ lies on the boundary of } B(r^{(m)}) \\ 1 & \text{otherwise} \end{cases} \tag{28}
$$

and we have its dual $\Theta^*(r^m, r^{(m-1)})$ defined as

$$
\Theta^*(r^m, r^{(m-1)}) = \begin{cases} 1 & \text{if } r^{(m)} \text{ lies on the boundary of } B^*(r^{(m-1)}) \\ \Theta(r^m, r^{(m-1)}) & \text{otherwise} \end{cases} \tag{29}
$$

that is, if $r^{(m-1)}$ lies on the boundary of $B(r^{(m)})$, then $r^{(m)}$ lies on the boundary of $B^*(r^{(m-1)})$.

Corollary 1. *The m-dimensional boundaries of $B^{(m+1)}$ form a closed m-dimensional hypersurface, two m-dimensional boundaries $B^{(m)}$ of $B^{(m+1)}$ meet in each $(m-1)$-dimensional hypercell at the boundary of $B^{(m+1)}$. Therefore*

$$
\oplus_{r^{(m)}} \Theta(r^{(m-1)}, r^{(m)}) \Theta(r^{(m)}, r^{(m+1)}) = 0 \tag{30}
$$

Theorem 1. *model M_{dn} and model $M^*_{d,d-n+1}$ are related by the duality relation*

$$
Y_{dn}(K, h) = Y^*_{d,d-n+1}(K^*, h^*) \tag{31}
$$

with

$$
\tanh K = e^{-2h^*} \quad \tanh h = e^{-2K^*} \tag{32}
$$

$$
\begin{aligned}
K_{c,dn}(h) &= K_{c,dn}(0) - h^{2n} + \cdots \\
h_{c,dn}(K) &= K_{c,dd-n}(0) - \sinh 2K_{c,dd-n}(0) e^{-4nK} \\
h_{c,43}(K) &= K_{c,41}(0) - \sinh 2K_{c,41}(0) e^{-12K} \\
h_{c,43}(K)/K_{c,41}(0) &= 1 - \frac{\sinh 2K_{c,41}(0)}{K_{c,41}} e^{-12K}
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
K_{c,32}(h) &= K_{c,32}(0) - h^{4} \\
K_{c,31}(h) &= K_{c,31}(0) - h^{2} \\
h_{c,32}(K) &= K_{c,31}(0) - \sinh 2K_{c,3,1}(0) e^{-8K} \\
K_{c,32}^{-1} &= \frac{1}{K_{c,32}(0) - K_c^4 (h/K_c)^4}
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
Y_{dn}(K,h) &= Y^*_{d,d-n+1}(K^*,h^*) \\
\tanh K = e^{-2h^*} &\quad \tanh h = e^{-2K^*}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\tanh K_{dn} &= e^{-2h_{d(d-n)}} \\
\tanh h_{dn} &= e^{-2K_{d(d-n)}}
\end{aligned}
$$

$$
\tanh K = \frac{e^K - e^{-K}}{e^K + e^{-K}} = \frac{1 - e^{-2K}}{1 + e^{-2K}} \approx (1 - e^{-2K})
$$

$$
\begin{aligned}
h_{d(d-n)} &= -\frac{1}{2} \log \tanh K_{dn} \\
h^{2n}_{d(d-n)} &= \frac{1}{2^{2n}} \log \tanh K_{dn}{}^{2n}
\end{aligned}
$$

$$
\begin{aligned}
K_{c,d(d-n)}(h) &= K_{c,d(d-n)}(0) - h^{2n}_{d(d-n)} \\
h_{c,dn} \approx \tanh h_{c,dn} &= e^{-2K_{c,d(d-n)}} \\
&= e^{-2(K_{c,d(d-n)}(0) - h^{2n}_{d(d-n)})} \\
&= e^{-2K_{c,d(d-n)}(0)} e^{-2h^{2n}_{d(d-n)}}
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\tanh h_{c,dn} &= e^{-2K_{c,d(d-n+1)}} \\
\tanh h_{c,dn}{}^{-1} &= e^{2K_{c,d(d-n+1)}} \\
\frac{e^h - e^{-h}}{e^h + e^{-h}} &+ \frac{e^h + e^{-h}}{e^h - e^{-h}} \\
&\frac{e^{2h} + e^{-2h}}{e^{2h} - e^{-2h}}
\end{aligned}
$$

$$
h_{c,dn} = e^{-2K_{c,d(d-n+1)}} = e^{-2K_{c,d(d-n+1)}(0) - 2h^{2n}_{c,d(d-n+1)}} \tag{37}
$$

References

[1] Benjamin Young Denis Sjerve. Algebra toplogy, page 39.

[2] Matthew B Hastings, Grant H Watson, and Roger G Melko. Self-correcting quantum memories beyond the percolation threshold. *Physical review letters*, 112(7):070501, 2014.

[3] Wiki. Euler characteristic.