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Functional modelling of volatility in the Swedish limit order book

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Abstract

The publicly available electronic limit order book at the Stockholm Stock Exchange consists of five levels of prices and quantities of a given stock with a bid and ask side. All changes in the book during one day can be recorded with a time quote. Studying the variation of the quoted price returns as a function of quantity is discussed. In particular, discovering and modelling dynamic behaviours in the volatility of prices and liquidity measures are considered. Applying a functional approach, estimation of the volatility dynamics of the spreads, created as differences between the ask and bid prices, is presented through a case study. For that purpose two-step estimation of functional linear models is used, extending this method to a time series context.

1. Introduction

Measuring the volatility of asset prices is one of the major concerns in financial risk management. In the available literature, volatility has usually been treated as an unobserved component of variation of the prices, normally estimated by the sample standard deviation. The volatility estimation procedures are quite straightforward when constant volatility is assumed and when the data are sampled at daily or lower frequencies. Nowadays, though, it is widely acknowledged that volatility of a financial asset price exhibits some kind of time-dependent variation and predictability. Also, there are empirical evidences that individual financial volatilities show remarkable temporal interdependencies across assets and markets, as argued in Audrino (2006). Furthermore, the growing availability of intra-day high-frequency financial data imposes a need for some new and more efficient procedures for volatility modelling.

In this paper we are interested in modelling daily volatility observed in the Swedish limit order book (LOB) high-frequency data. The nature of this kind of data gives an incentive for the analysis of the prices' (returns') volatility via the study of the price–volume relationships. Although the theoretical works concerned with the information contained in this kind of relationship are rare, there are many empirical studies that find evidence of strong effect of trading volumes on asset prices (see, e.g. Suominen (2001)).

The main objective of this study is to use all available information at any time point in the LOB by creating a measure of the average prices for a given quantity of shares. It is reasonable to believe that, when the prices are calculated as functions of quantities of shares by using the so-called *bid and ask curves*, more precise measures of

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daily volatility are likely to be produced. In this context, a functional volatility measure, called the *functional realized quadratic variation* (FRQV) is used. This measure is a version of the *realized quadratic variation* (RQV) given in Barndorff-Nielsen and Shephard (2002a), modified to suit the Swedish LOB data.

Following the recommendations of Barndorff-Nielsen and Shephard (2005), an appropriate sampling time period is chosen in order to reduce the *market microstructure frictions* (see, e.g. Bandi and Russel (2003) and Campbell et al. (1997)). Then, for each given quantity of shares, the sums of squared functional returns are computed over the chosen time periods. In this way the volatility estimates of daily price returns for a given quantity can be obtained. These estimates may be used as proxies for daily volatilities under, e.g., the stochastic volatility (SV) models or some other models that incorporate time-varying volatility.

In order to study the dynamics in the prices' volatility in the LOB, a functional time series model and its estimation by a *two-step estimation procedure* (Fan and Zhang, 2000) is proposed. The main idea is to fit an autoregressive model to each time series of FRQV estimates in order to obtain the *raw estimates* of the coefficient functions. Then, an existing smoothing technique is applied to each set of the raw estimates producing the smooth estimates of the coefficient functions.

The paper is organized as follows. A brief description of the Swedish LOB market is given in Section 2. Then, in Section 3, the bid and ask curves are discussed and the functional mid-quotes and spreads, based on these curves, are introduced. Section 4 describes measures of volatility by summarizing the theory on RQV. Furthermore, in this section a functional approach to volatility estimation is briefly explained. In Section 5, a model for functional time series is introduced as well as an estimation procedure related to this model. Section 6 presents the empirical results from a case study. Finally, a short conclusion with some recommendations for a future study are given in Section 7.

2. The Swedish limit order book

Generally, the LOB data consists of unexecuted limit orders maintained by the specialists. Two kinds of orders are commonly present in the limit order market, the limit orders and the market orders. Limit orders are orders to buy or sell a fixed quantity of shares of a financial asset at a given limit price or better. A market order is an order to buy or sell a quantity of shares immediately and at the best available price. The submitted limit orders are stored into an electronic limit order book until execution or cancellation. Thus, at any point in time, it is possible to extract the quoted bid (resp. ask) prices of a stock together with the corresponding quantities available at these prices. These quantities give us information about the total demand and supply at given price levels. Hence, the limit orders provide information about liquidity and immediacy in the market (Iori et al., 2003, p. 147).

The Swedish computerized limit order market has an order-driven trading system, similar to the systems on the e.g. Paris Bourse, Toronto Stock Exchange and the Stock Exchange of Hong Kong. The limit order market system is operated by the Stockholm Stock Exchange (SSE). On the SSE limit order market, a submitted limit buy (resp. sell) order is automatically matched against the arriving market or limit orders. Whenever the terms of the orders match, the SSE limit order trading system automatically generates deals. The priority of execution is given first according to prices and then according to times of submission (Sandås, 2001).

The best (highest) bid price is always placed at the top of the bid side of the order book. Correspondingly, the best (lowest) ask price is placed at the top of the opposite (offer) side of the order book. For a very large trading lots it is also possible to close a deal via telephone which is called the *off-change registration*. This kind of deal has to be reported to the SSE limit order market system manually within a very short period of time. Trading in SSE limit order market is only allowed to the authorized exchange members who have to meet certain requirements (Stockholm Stock Exchange, 2007).

Each change in the Swedish LOB is registered in real time. Hence, at any recorded time point, the Swedish LOB is represented by ten observations of prices and ten corresponding quantities. From time series of these observations it is possible to create the mid-quotes (the averages of the best bid and ask quotes) and the spreads (the differences between the best ask and bid quotes). In the related literature, the mid-quotes are usually used as proxies for efficient prices while the spreads are treated as measures of liquidity.

As an illustration, consider a simple scenario based on the made-up limit order book data at a time point t, given in panel A of Table 1.

Assume that a buy limit order arrives, for 20,000 shares specifying the limit price 27.85. The system matches this buy order by working down the sell side of the book. The orders in the first two lines on the sell side are completely

Table 1
Hypothetical limit order book

Panel A: Initial situation					Panel B: Situation after a bid limit order				
Level	Bid price	Bid vol.	Ask price	Ask vol.	Level	Bid price	Bid vol.	Ask price	Ask vol.
1	27.70	8000	27.80	9149	1	27.85	3071	27.90	7795
2	27.65	1190	27.85	7780	2	27.70	8000	27.95	1558
3	27.60	1914	27.90	7795	3	27.65	1190	28.00	7819
4	27.55	1460	27.95	1558	4	27.60	1914	28.05	7242
5	27.50	1809	28.00	7819	5	27.55	1460	28.10	2439

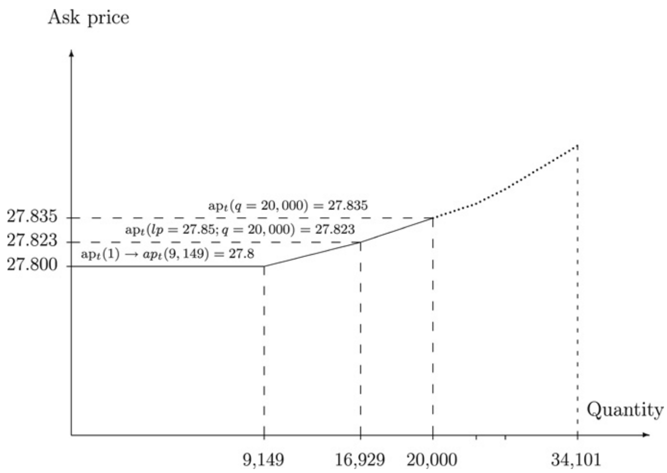

Fig. 1. Hypothetical limit order book at time t (see Table 1), bid order situation: average ask price per share (considering bid limit order for 20,000 shares and limit price $lp = 27.85$ resp. bid market order for 20,000 shares).

filled since there are 9149 shares available at the price 27.80 and 7780 shares available at the limit price 27.85. However, the limit order has not been fully filled since there are outstanding $20{,}000 - 9149 - 7780 = 3071$ shares that cannot be matched since the limit price is achieved. After the transaction, the best bid price will be 27.85 with the remaining quantity 3071 shares, as shown in panel B of Table 1. The execution of the bid order has also moved the best ask price from 27.80 up to 27.90. Then, the average price per share that the investor, who put the bid limit order, has to pay is

$$27.80\frac{9149}{9149+7780} + 27.85\frac{7780}{9149+7780} \approx 27.823.$$

Hence, the additional cost to the investor due to the lack of liquidity in the top level in the order book is $27.823 - 27.80 = 0.023$ per share.

Consider now a situation when a a market buy order, for the same quantity 20,000 shares, arrives. The average price per share for this order will be

$$27.80\frac{9149}{20{,}000} + 27.85\frac{7780}{20{,}000} + 27.90\frac{3071}{20{,}000} \approx 27.835.$$

As the market order worked down to the third level of the order book, the additional cost for the investor caused by the lack of liquidity is now $27.835 - 27.80 = 0.035$ per share. Fig. 1 depicts these two situations showing how the average ask price increases as the demanded bid quantity becomes larger.

3. The bid and ask curves

Many of the available studies about the limit order trading are concerned with some aspects of market liquidity (Ahn et al., 2001; Biais et al., 1995). Some authors (e.g. Iori et al. (2003) and Sandås (2001)) focus on models for double-auction markets by using the price–volume relationship captured by a *price impact function*. These functions are commonly used as measures of liquidity for executing market orders. Ghysels et al. (1998) summarize so-called "stylized facts" concerned with the price–volume relationship in connection to the market microstructure theory, giving arguments for further studies about some unexplored issues regarding this topic.

In Gourieroux and Jasiak (2001, p. 360), the demand and supply functions, created by cumulating volumes on the bid and ask side of the order book, are introduced. The inverses of the demand (resp. supply) functions are called the ask (resp. bid) curves, interpreted as the prices per share to be sold (resp. paid) for a given quantity offered (resp. demanded). As commented in Gourieroux et al. (1999), the size of discrepancy between the bid and ask curves, as well as their slopes and shapes summarize information about the immediate liquidity. Also, the bid and ask curves may be incorporated into the measures of the prices volatility since their slopes represent the marginal effects of demand and supply on the prices, as noted in Gourieroux and Jasiak (2001, p. 386).

In the financial literature, variation of stock prices is commonly studied through the quoted bid and ask prices of a single share. We suspect that using only such prices would not reveal enough information about the stock prices variation in the LOB market. Henceforth, the bid and ask curves, as measures of liquidity that incorporate both prices and quantities, will be used in the creation of some alternative measures of prices volatility. As a formal definition of the bid (resp. ask) curves is not yet available in the academic literature, the bid (resp. ask) curve definition from Olsson (2005) is adopted here, as follows

$$\overline{p}_t(q) = \frac{\sum_{i=1}^{k-1}(P_{it}Q_{it}) + P_{kt}\left(q - \sum_{i=1}^{k-1} Q_{it}\right)}{q} \quad \text{for } k = 1, \ldots, K, \tag{1}$$

meaning that the bid (resp. ask) curve $\overline{p}_t(q)$ is the average price per share for a given quantity q at time t; P_{it} is the quoted (bid or ask) price of an asset at level i and time t; Q_{it} is the volume available (bid or ask) at level i and time t; K is the number of the available price levels in the LOB (in the Swedish market $K = 5$), while k represents the level of the LOB that corresponds to the quantity demanded (supplied), i.e

$$\sum_{i=1}^{k-1} Q_{it} < q \leq \sum_{i=1}^{k} Q_{it},$$

In some other works there are definitions that define approximately the same functions as in (1) (see, e.g., Bowsher (2004)). From (1), the mid-quote and the spread time series may be computed, again as functions of quantities, as follows

$$\text{Mid-quotes} \qquad P_t^*(q) = \log\left[\frac{1}{2}\left(\bar{p}_t^b(q) + \bar{p}_t^a(q)\right)\right], \tag{2a}$$

$$\text{Spreads} \qquad S_t^*(q) = \log\left[\bar{p}_t^a(q)/\bar{p}_t^b(q)\right], \tag{2b}$$

where $\bar{p}_t^b(q)$ and $\bar{p}_t^a(q)$ are the bid respective ask curves.

Thus, using a large number of quantities q at any time record in the LOB it is possible to obtain a collection of mid-quote and spread curves that partly summarizes the content of the order book. As noted in Gourieroux and Jasiak (2001, p. 388), the mid-quotes and the spreads are likely to extract more information when computed for appropriate quantities instead of using quantity equal to one share.

An example of the bid, mid-quote and ask curves is given in Fig. 2. The curves are computed using Ericsson B LOB data on January 2, 2007, at two randomly chosen time points. Notice that the curves have different shapes at different times while for some (very large) quantities the curves are not defined at all.

4. Measures of volatility

4.1. Short introduction to the theory on realized quadratic variation

As an alternative to the existing parametric models for conditional volatility estimation, Barndorff-Nielsen and Shephard (2001, 2002a,b, 2004a,b,c, 2005, 2006) have developed a nonparametric approach for the estimation of

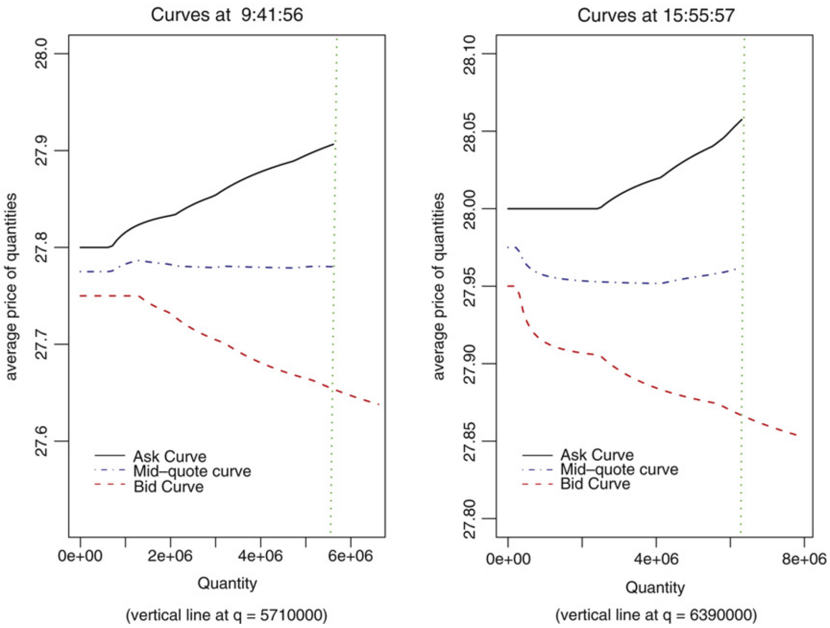

Fig. 2. Bid, mid-quote and ask curves: Jan 2, 2007, Ericsson B LOB; curves computed using quantities from 1 to 8 million shares; vertical lines indicate that information about liquidity beyond the certain quantities is missing.

volatility from high-frequency financial data. The same approach may also be found in concurrent and independent work by Andersen and Bollerslev (1998) and Andersen et al. (2003). This approach is based on estimating the unknown quadratic variation (QV) of the return process $y(t)$ of the log prices of a financial asset at time t, defined as

$$[Y]\,(t) = p - \lim_{M\to\infty} \sum_{j=0}^{M-1} \left(y(t_{j+1}) - y(t_j)\right)^2, \tag{3}$$

where the limit in probability is taken over any sequence of partitions $t_0 = 0 < t_1 < t_2 < \cdots < t_M = t$ and $\sup_{1\le j\le M}{}_j \left\{t_j - t_{j-1}\right\} \to 0$.

Then, over any ith fixed time interval having length $\hbar > 0$, the RQV estimator is computed as the sum of M intra-$\hbar$ squared returns

$$[Y_M]_i = \sum_{j=1}^{M} r_{j,i}^2, \tag{4}$$

where $i = 1, 2, \ldots T$ and T is the total number of such intervals while the jth intra-$\hbar$ return for the ith period is defined as

$$r_{j,i} = y\left((i-1)\hbar + \frac{\hbar j}{M}\right) - y\left((i-1)\hbar + \frac{\hbar(j-1)}{M}\right), \quad j = 1, 2, \ldots, M. \tag{5}$$

Under the general framework of *stochastic volatility* (SV) models and assuming certain regularity conditions, the RQV is shown to be a consistent estimator of the QV (Barndorff-Nielsen and Shephard, 2002b). In other words, the ith period RQV converges in probability to the ith increment of QV as $M \to \infty$:

$$[Y_M]_i \to^P [Y]\,(i\hbar) - [Y]\,((i-1)\,\hbar) = [Y]_i.$$

In practice, the RQV is typically computed using interval of length one day. Consequently, the RQV is commonly used in measuring daily variability of asset prices (returns) as a consistent estimator of the daily increments of QV. The major problem with this estimator in practical applications is the presence of the microstructure effects, as noted in Barndorff-Nielsen and Shephard (2002a). These effects cause some serious biases when the sampling frequency is high. As a countermeasure, a choice of the appropriate sampling frequency should be made by means of the *volatility–signature plots*, firstly introduced in Andersen et al. (2000).

In order to reduce effects of the microstructure noise some alternatives to the RQV estimator are proposed, such as Fourier estimator and wavelet estimator (Nielsen and Frederiksen, 2007) and the *price range* estimator (Alizadeh et al., 2002). See also Kalnina and Linton (2006) for introducing an econometric model that involves a modified *Two Scales Realized Volatility* as an estimator of the QV of the latent price process in the presence of microstructure noise.

4.2. Functional approach to volatility estimation

Statistical analysis for functional data has increased in popularity since Ramsay and Silverman (1997) offered a collection of techniques and methods for analyzing the curves instead of the ordinary observations. At present, most of the applied works concerned with the functional data are oriented to the three major areas: factorial analysis for functional data, regression with functional variables and curves classifications (Mantiega and Vieu, 2007). However, the development of the inference tools for the functional data analysis is still at the very beginning. Some inferential methods have recently been proposed by López-Pintado and Romo (2007) focusing on the idea of statistical *depth*. Another important area related to the functional data analysis is the approach based on nonparametric statistics (see, e.g. Ferraty and Vieu (2006)). Some topics related to the functional approach to time series analysis are also discussed in this book.

As far as we know, none of the available works is concerned with the functional approach in creating volatility measures for the prices from the LOB data. Particularly, the main concern of this paper is volatility estimation of the functional mid-quotes and the functional spreads in the Swedish LOB. This is essentially a nonparametric approach. By reformulating (5), the functional returns of mid-quotes and spreads may be computed for each ith interval using the M intra-$\hbar$ returns, as follows

$$\text{Mid-quotes} \qquad r^{p^*}_{j,i}(q) = P^*_{[(i-1)\hbar+\hbar j/M]}(q) - P^*_{[(i-1)\hbar+\hbar(j-1)/M]}(q), \tag{6a}$$

$$\text{Spreads} \qquad r^{s^*}_{j,i}(q) = S^*_{[(i-1)\hbar+\hbar j/M]}(q) - S^*_{[(i-1)\hbar+\hbar(j-1)/M]}(q). \tag{6b}$$

Then, using the sum of squared returns over each ith interval, a daily functional RQV (FRQV) may be obtained according to

$$[Y_M]_i\,(q) = \sum_{j=1}^{M} r^2_{j,i}(q). \tag{7}$$

However, the bid (resp. ask) curves usually have different "*pre-images*" since the total quantity available on the bid (resp. ask) side of the book may vary with time. Consequently, calculating the values of functions for some (especially very large) quantities q would result in a large number of missing values which would, in turn, entail heavy biases on our estimator. In order to solve this problem, we propose to use relative quantities for computations of the bid and ask curves, as follows

$$\tilde{\bar{p}}\,(w) = \bar{p}\,(wQ_t)\,, \tag{8}$$

for

$$0 < w \le 1 \quad \text{and} \quad Q_t = \sum_{i=1}^{5} Q_{i,t}.$$

Hence the weights w are simple percentages of Q_t, which is the total available (bid or ask) quantity in the LOB at time t. In this way the bid and ask curves will be defined at any time record in the LOB. This method is similar to the idea of *registering* curves, introduced by Ramsay and Silverman (1997, Ch. 5). Registering curves is a transformation of the curves motivated by the fact that the internal dynamics of each curve may be different at the same arguments. The authors discuss two types of variations that can cause the values of two or more functions to differ when compared at the same arguments, the *range* variation and the *domain* variation. The range variation is simple vertical variation caused by the fact that the two or more curves genuinely differ when compared at the same arguments. The domain variation, on the other hand, is a kind of variation that is induced by the fact that two or more functions should not be compared at the same arguments. When the domain variation is suspected, the arguments of each curve have to be transformed in order to be able to compare the curves.

Concerning the LOB data in this study, the total available bid and ask quantities usually differ from each other when observed at the same time point. Also the internal dynamics over time of each bid (resp. ask) curve computed for the fixed quantity q is different since the quantity varies from one time point to another. As a consequence, a kind of domain variation is likely to be present implying that the bid and ask curves should not be computed for the same quantities. Since the aim of this procedure is to create a measure (FRQV in (7)) that incorporates both the bid and the ask curves that are compatible, some kind of transformation of the curves is needed.

5. Functional time series model

So far, we have dealt with the computation of the FRQV in order to establish a method that might be useful in measuring daily volatility of the prices in the LOB. The next step is to study dynamics in these measures. The previous studies about dynamics of the daily realized volatilities include, among others, Andersen and Bollerslev (1998), where a multivariate Gaussian vector autoregressive model is suggested, and Andersen et al. (2003), where the use of the standard volatility models, such as generalized autoregressive heteroscedasticity model, is motivated. Concerning the LOB data, studies of the volatility dynamics are quite limited in the available academic literature. Closely related to this work is a study by Bowsher (2004), where a functional signal plus noise time series model for the analysis of the dynamics of the bid and ask curves is introduced. See, also, Gourieroux and Jasiak (2001, Ch. 14) for a discussion about a linear dynamic factor model for the bid and ask curves.

The idea here is to start from a simple dynamic model, such as the autoregressive model (AR) of a specified order. The motivation for the use of this model is two-fold. An AR model is easy to understand and interpret in economic terms since the observed realized volatility today is likely to be influenced by the past volatilities. Another reason is more subtle as it is related to the condition that the convergence of the RQV is valid under the general stochastic volatility modelling framework. For instance, a process called *Ornstein–Uhlenbeck* (OU) process is commonly used for modelling volatility in continuous time under this framework. The discrete analogue to the continuous time Gaussian OU process is a Gaussian AR process of order 1 (Gourieroux and Jasiak, 2001, p. 251).

Accordingly, we define an autoregressive model of order p for a function $Y_i^*(w)$, as follows

$$Y_i^*(w) = \beta_0\,(w) + \sum_{l=1}^{p} \beta_l\,(w)\,Y_{i-l}^*\,(w) + \varepsilon_i(w), \tag{9}$$

where $Y_i^*(w)$ denote FRQV of the spread (mid-quote) for a percentage w at day i and $\varepsilon_i(w)$ is a zero-mean stochastic process, independent through days i, $(i = 1, 2, \ldots, T)$, but correlated for different w. In other words,

$$\begin{aligned} &E(\varepsilon_i(w)) = 0,\\ &Cov(\varepsilon_i(w), \varepsilon_{i+h}(w)) = 0, \quad \text{for } h \neq 0 \end{aligned} \tag{10}$$

and

$$\begin{aligned} Cov(\varepsilon_i(w_j), \varepsilon_i(w_k)) &= \gamma\left(w_j, w_k\right), \quad \text{for } j \neq k,\\ &= \sigma_\varepsilon^2, \quad \text{for } j = k. \end{aligned} \tag{11}$$

The model fitting procedure is divided into two steps, obtaining the raw estimates of the coefficient functions and smoothing the obtained raw estimates. This procedure, called the *two-step estimation procedure*, is introduced in Fan and Zhang (2000) for the purpose of studying functional linear models with applications to longitudinal data.

5.1. Obtaining the raw estimates

In the first step of the procedure, the parameters of model (9) are estimated separately for each fixed percentage w. Using the standard least squares theory, an estimator of

$$\boldsymbol{\beta}(w) = \left(\beta_0(w), \beta_1(w), \ldots, \beta_p(w)\right)^{\mathrm{T}}$$

is

$$\mathbf{b}(w) = \left(b_0(w), b_1(w), \ldots, b_p(w)\right)^{\mathrm{T}},$$

such that

$$\mathbf{b}(w) = \left(\mathbf{X}_w^{\mathrm{T}}\mathbf{X}_w\right)^{-1}\mathbf{X}_w^{\mathrm{T}}\mathbf{Y}_w^*,$$

where

$$\mathbf{Y}_w^* = \left(Y_T^*(w), Y_{T-1}^*(w), \ldots, Y_{p+1}^*(w)\right)^{\mathrm{T}}$$

and

$$\mathbf{X}_w = \begin{pmatrix} 1 & Y_{T-1}^*(w) & \ldots & Y_{T-p}^*(w) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & Y_p^*(w) & \ldots & Y_1^*(w) \end{pmatrix}.$$

As a result, a matrix $\mathbf{B}$ (of dimension $V \times (p+1)$, where V is the total number of weight-percentages w) consisting of the raw estimates may be obtained, as shown here

$$\mathbf{B} = \begin{pmatrix} b_0(w_1) & b_1(w_1) & \ldots & b_p(w_1) \\ b_0(w_2) & b_1(w_2) & \ldots & b_p(w_2) \\ \vdots & \vdots & \ddots & \vdots \\ b_0(w_V) & b_1(w_V) & \ldots & b_p(w_V) \end{pmatrix},$$

where each column represent a vector of the raw estimates

$$\mathbf{b}_l(w) = \left(b_l(w_1), \ldots, b_l(w_V)\right)^{\mathrm{T}},$$

for the corresponding parameter vector

$$\boldsymbol{\beta}_l(w) = \left(\beta_l(w_1), \ldots, \beta_l(w_V)\right)^{\mathrm{T}}.$$

These raw estimate vectors will be used in the second step in order to compute the final estimates of the coefficient functions.

5.2. Refining the coefficient functions by smoothing

The raw estimates obtained in the previous section are normally inefficient since they do not incorporate the information from the non-design points. Therefore, it is necessary to smooth the raw estimates, $\mathbf{b}_l(w)$, by using some existing nonparametric technique. After this smoothing step, the final estimates $\hat{\boldsymbol{\beta}}_l(w)$ for the coefficient functions are obtained. The smoothing step of this procedure has an extra advantage as it is one-dimensional. Thus, different amounts of smoothing may be applied to different components of the coefficient functions (Fan and Zhang, 2000). We will assume that the chosen smoothing technique is linear in the responses (e.g., splines, kernels, local polynomial regression, etc.). Furthermore, assuming that $\beta_l(w)$ is $(d+1)$ times continuously differentiable, a nonparametric linear estimator of the gth derivative of $\beta_l^{(g)}(w)$, for some $0 \leq g < d+1$, is given as

$$\widehat{\beta_l^{(g)}}(w) = \sum_{v=1}^{V} k_l\left(w_v, w\right) b_l(w_v), \tag{12}$$

and the weights $k_l(w_v, w)$ are constructed according to the relevant smoothing technique.

6. Empirical results

In this section an application to the LOB data for the Ericsson B stock from the SSE is presented. The access to the data is provided by the *Ecovision* information system that offers such data for all stock shares traded at the

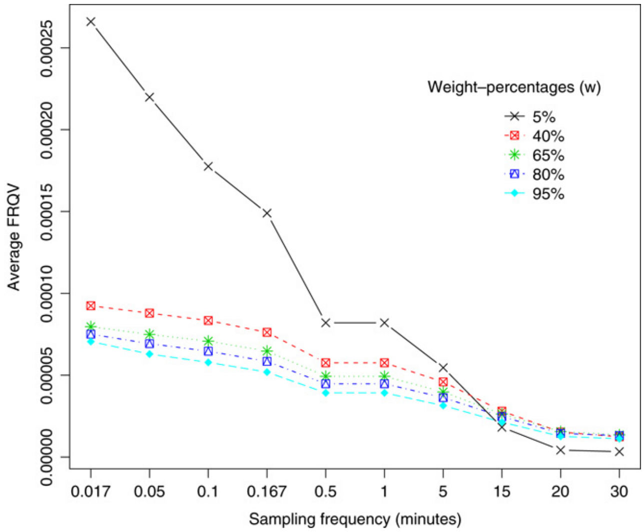

Fig. 3. Volatility–Signature Plots for FRQV of the registered bid and ask curves, 164 days, from May 2, 2005 to December 31, 2005, Ericsson B LOB.

SSE's trading system. Originally, these intra-day LOB data consist of 324 fully working days, from May 2, 2005 to August 25, 2006. Weekends and holidays are not included because of the lack of activity in the market. Due to technical problems there are 7 days missing in this period, which we treat as non-available observations. All registered observations between 9:15:00 and 17:15:00 are collected during each day.

The analysis is performed by fitting model (9) and smoothing the obtained raw estimates of the coefficient functions, as explained in Section 5. In the first step of this analysis, the daily FRQV estimates are obtained by computing the functional spread returns using (6b). The squared returns are computed over consecutive 20 min periods resulting in 24 observations per day. The 20 min sampling frequency is chosen by looking at the volatility–signature plots. These plots are created by computing the average RQV of weight-percentages $w = 0.05, 0.40, 0.65, 0.80, 0.95$, over a sub-sample consisting of 164 days and for different frequencies, as shown in Fig. 3.

Obviously, the average RQVs of each quantity stabilize for frequencies lower than 20 minutes which is an indicator that the microstructure noise would have the smallest impact on estimates at this frequency. Notice that this result is somewhat different from Barndorff-Nielsen and Shephard (2005), where the recommended frequencies were 5 or 10 minutes. As their study is based on the data from the London Stock Exchange's order book, our guess is that the data from the Swedish Stock Exchange is more influenced by the microstructure noise even at the frequencies lower than 10 minutes.

The bid and ask curves, used for computing the $FRQV(w)$, are obtained according to (8). We have chosen 99 weight-percentages (w) for this study which leads to 99 FRQV estimates each day. The weights are given as $w = 0.01, 0.02, \ldots, 0.99$. As a result, we obtain 99 RQV time series, each having length of 324 days.

Then, an AR model of a proper order is fitted separately to each of 99 RQV time series, according to (9). After examining the sample autocorrelation plots of each time series of $Y_t^*(w)$ and comparing several AR models of suitable orders, an AR(3) model is chosen as the most appropriate one according to Akaike Information Criterion. Thus, a global AR(3) model may be written as

$$Y_i^*(w) = \beta_0(w) + \sum_{l=1}^{3} \beta_l(w)\, Y_{i-l}^*(w) + \varepsilon_i(w). \tag{13}$$

As a result of the fitting procedure in (13), the 99 raw estimates for each of the four coefficient functions are obtained. A few weakly significant partial autocorrelations are found in the residuals from the fitted autoregressions that include very small and very large weight-percentages but in general the assumptions for the error terms, given in (10) and (11), seems to hold.

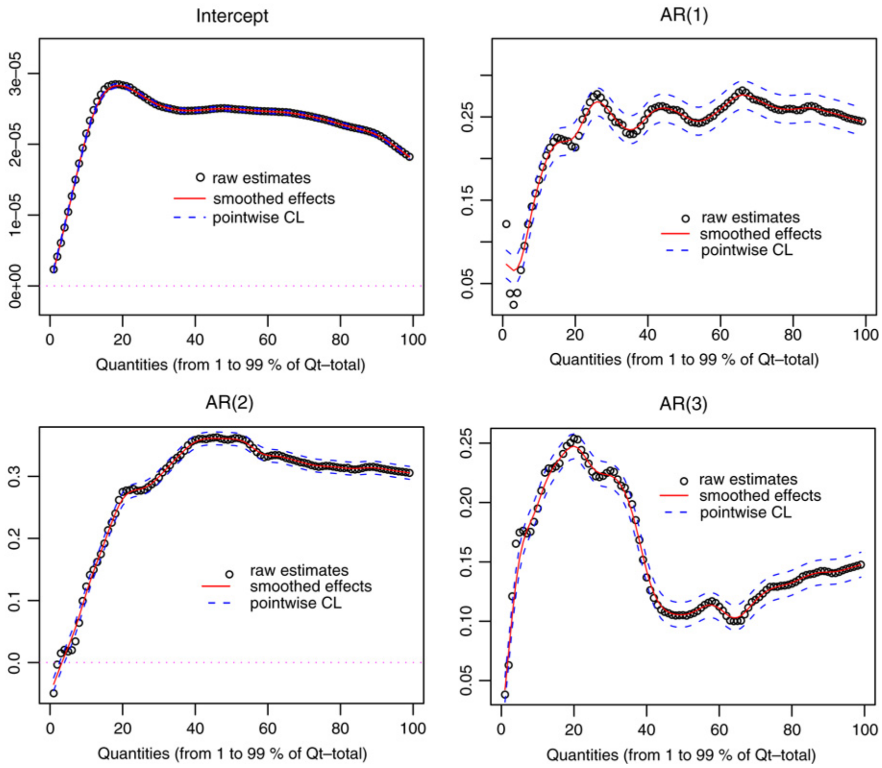

Fig. 4. Final estimates of the coefficient curves with 95% (pointwise) confidence bands for the daily Ericsson B LOB FRQV from an AR(3) model: May 2, 2005 to Aug 25, 2006.

Then, the resulting raw estimates may be used in the smoothing step of this analysis. Accordingly, the raw estimates are refined using a version of the locally weighted regression smoothing, called *the super smoother*. This is performed by using the function *supsmu* in the package *stats*, available in the statistical software *R* (R Development Core Team, 2007). This smoothing method chooses the best *span* by a local cross-validation, implying that the span is not constant over the entire range of predictor values. This particular package is chosen arbitrary as a convenient and relatively fast method. Observe that the choice of smoothing method is not of great importance for the purpose of this study since any other smoothing method would probably produce similar results as well. See Fan and Zhang (2000) for applications using different smoothing techniques.

Fig. 4 depicts the raw estimates for this data set, together with the final estimates of the coefficient functions and with their estimated pointwise ± 2 standard error limits. For convenience, the pointwise confidence bands are computed by using default values of the standard error estimates from the mentioned *R*-function. Notice that the statistical quality of these estimates might be improved further by taking into account the covariance structure among the raw estimates. See also the discussion in Fan and Zhang (2000) about estimation of the standard error bands for a typical linear estimator based on a chosen smoothing technique.

The raw estimates are indicated by circles and the smoothed effects by the full curves, while the ± 2 standard errors pointwise confidence bands for the smoothed fit are shown as the broken lines. Apparently, the pointwise confidence intervals do not include zeroes, in almost all of the cases, indicating a high level of significance of the autoregressive functional coefficients.

It is worth noting that the final estimates of the functional intercepts show strong increasing trend when the relative quantity increases up to a certain level (w around 20%) and then stabilize with a slow decreasing trend. Similarly, the final estimates of the functional AR(1) and AR(2) coefficients have increasing tendency up to a moderate relative quantity level (for AR(1) when w around 30% and for AR(2) when w around 40%) and then they become stable. The curve of the final estimates of the AR(3) functional coefficients has a strong increasing tendency when w increases up to 20% and then fall sharply until w reaches a moderate level of around 50%. Afterwards, this curve has a weak increasing trend as the relative quantity level increases. Obviously, the variations of the functional coefficients depend on the relative quantities used in this analysis.

7. Conclusions

Due to the nature of the Swedish LOB data, an alternative approach to the volatility estimation is presented here. The main idea is to make use of the full available information about the prices and the corresponding quantities at each level of the LOB by creating the bid and ask curves. These functions are then used, instead of the bid and ask prices, in the creation of the FRQV estimator of daily volatility of the stock prices. The dynamics of the obtained FRQV estimates, which are functions of quantities, may be further explored by fitting an appropriate autoregressive time series model by using a two-step estimation procedure. Although this procedure is mainly aimed at studying the volatility in the Swedish LOB data, it may probably be used, with slight modifications, for the LOB data from any other market. Also, the procedure is likely to be helpful in studying some other kinds of functional time series. The main advantage is simplicity and possibility to use the existing software with relatively little programming effort.

In economic terms, the proposed procedure might be viewed as a tool to improve volatility estimation and forecasting in, e.g., derivative pricing, asset allocation or in risk management. The precise and bias-free volatility estimation is of crucial interest for the investors to create optimal strategies. In this respect the method discussed here might contribute since it takes into account more information than it is common to the standard volatility models.

As a suggestion for a future study, we may recommend improving the estimation of the standard errors of the raw estimates, which is also suggested in Fan and Zhang (2000, p. 310). Another recommendation might be to focus on the improvement of the statistical tests that affect the choice of the particular global dynamic model for the collection of functional time series of realized volatilities. Also, the performance of the FRQV estimator and the related two step procedure might be studied by assuming a specific stochastic volatility model, such as the Ornstein–Uhlenbeck (Barndorff-Nielsen and Shephard, 2001) model or the constant elasticity of variance model (Barndorff-Nielsen and Shephard, 2002a). We are currently working on a simulation study under this framework.

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