DISKRETNA MATEMATIKA

- PREDAVANJE -

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1. Binomni koeficijenti
2. Binomna formula
3. Polinomni koeficijenti
4. Polinomna formula

Tema 1

Binomni koeficijenti

Binomni koeficijenti

Neka je $0 \leq m \leq n$.

Definicija

Binomni koeficijent:

$$\binom{n}{0} = 1 \qquad \binom{n}{m} = \frac{n(n-1)\ldots(n-m+1)}{m(m-1)\ldots 2 \cdot 1}, m > 0$$

Primer:

$$\binom{10}{2} = \frac{10 \cdot 9}{2} = 5 \cdot 9 = 45.$$

Binomni koeficijenti - osobina1

Lema

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

Za $m \in \{0, n\}$ imamo

$$\binom{n}{n} = \frac{n!}{n! \cdot 0!} = 1 = \binom{n}{0}.$$

Ako je $1 \leq m \leq n-1$,

$$\binom{n}{m} = \frac{n(n-1)\dots(n-m+1)}{m(m-1)\dots 2 \cdot 1} \cdot \frac{(n-m)!}{(n-m)!} = \frac{n!}{m!(n-m)!}.$$

Binomni koeficijenti - osobina2

Lema

$$\binom{n}{m} = \binom{n}{n-m}$$

Na osnovu osobine1, možemo izvesti sledeće:

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n!}{(n-(n-m))!(n-m)!} = \binom{n}{n-m}.$$

Paskalov identitet (osobina 3)

Lemma

1. $\binom{n}{n} = \binom{n}{0} = 1$
2. $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, 1 \leq m \leq n-1$

1. $\binom{n}{n} = \binom{n}{0} = \frac{n!}{n! \cdot 0!} = 1$
2. $$\begin{aligned} \binom{n-1}{m-1} + \binom{n-1}{m} &= \frac{(n-1)!}{(m-1)!(n-m)!} + \frac{(n-1)!}{m!(n-m-1)!} \\ &= \frac{m \cdot (n-1)! + (n-m) \cdot (n-1)!}{m \cdot (m-1)! \cdot (n-m) \cdot (n-m-1)!} \\ &= \frac{(m+n-m) \cdot (n-1)!}{m!(n-m)!} = \binom{n}{m} \end{aligned}$$

Paskalov identitet (osobina 3)

$$
\begin{array}{lccccccccccc}
n=0 & & & & & & \binom{0}{0} & & & & & \\
n=1 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
n=2 & & & & \binom{2}{0} & & \binom{2}{1} & + & \binom{2}{2} & & & \\
n=3 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
n=4 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
n=5 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
 & & & & & & \cdots & & & & &
\end{array}
$$

Paskalov identitet (osobina 3)

$$
\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & + & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & & \dots & & & & &
\end{array}
$$

Tema 2

Binomna formula

Binomna formula

Teorema (Binomna formula)

Neka je $n \geq 1$. *Tada je*

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{k} x^{n-k} y^k + \ldots + \binom{n}{n} x^0 y^n$$

Kombinatorna interpretacija:

$$\begin{aligned}(x+y)^n =& \underbrace{(x+y)(x+y)\ldots(x+y)}_{n \text{ puta}} \\ =& (xx+xy+yx+yy)(x+y)\ldots(x+y) \\ =& (xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy)(x+y)\ldots(x+y) \\ =& x^n + x^{n-1}y + \ldots + x^{n-1}y + x^{n-2}y^2 + \ldots + x^{n-2}y^2 + \ldots + xy^{n-1} + \ldots + xy^{n-1} + y\end{aligned}$$

Ako iz m zagrada izaberemo y, a iz $n-m$ zagrada izaberemo x: $x^{n-m}y^m$.

Broj načina da izaberemo m zagrada iz kojih ćemo izabrati y jednak je $\binom{n}{m}$

Binomna formula

Teorema (Binomna formula)

Neka je $n \geq 0$. *Tada je*

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + \binom{n}{n}x^0 y^n$$

Dokaz: indukcijom po n
$n = 1 : (x+y)^1 = x+y$
$T_n \Rightarrow T_{n+1}$:

$$\begin{aligned}(x+y)^n(x+y) =& (x^n + nx^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + nxy^{n-1} + y^n)(x+y) \\ =& \begin{array}{ccccccc} x^{n+1} & +nx^n y & +\binom{n}{2}x^{n-1}y^2 & + \ldots & +\binom{n}{n-1}x^2y^{n-1} + & xy^n & \\ & + \ x^n y & +\binom{n}{1}x^{n-1}y^2 & + \ldots & +\binom{n}{n-2}x^2y^{n-1} + & nxy^n & +y^{n+1}. \end{array}\end{aligned}$$

Binomna formula

Teorema (Binomna formula)

Neka je $n \geq 0$. *Tada je*

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{k} x^{n-k} y^k + \ldots + \binom{n}{n} x^0 y^n$$

$$\begin{aligned}
(x+y)^{n+1} &= x^{n+1} + (n+1)x^n y + \left(\binom{n}{1} + \binom{n}{2}\right) x^{n-1} y^2 + \ldots \\
&\quad + \left(\binom{n}{n-1} + \binom{n}{n-2}\right) x^2 y^{n-1} + (n+1)xy^n + y^{n+1} \\
&= x^{n+1} + (n+1)x^n y + \binom{n+1}{2} x^{n-1} y^2 + \ldots \\
&\quad + \binom{n+1}{n-1} x^2 y^{n-1} + (n+1)xy^n + y^{n+1} \\
&= \sum_{m=0}^{n+1} \binom{n+1}{m} x^{n+1-m} y^m .
\end{aligned}$$

Zadaci

1. $$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$
2. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \text{ ako je } n > 0$$
3. $$\sum_{k=0}^{0} (-1)^k \binom{n}{k} = 1$$
4. $$\sum_{k=0}^{n} 2^k \binom{n}{k} = (1+2)^n = 3^n$$

Tema 3

Polinomni koeficijenti

Polinomni koeficijenti

Neka je $l \geq 1, m_1, \ldots, m_l \geq 0$ i $n = m_1 + \ldots + m_l$.

Definicija

Polinomni koeficijent:

$$\binom{n}{m_1, m_2, \ldots, m_l} = \frac{n!}{m_1! \cdot \ldots \cdot m_l!}$$

Primer

$$\binom{5}{1, 3, 1} = \frac{5!}{1!3!1!} = 20$$

Polinomni koeficijenti-osobina1

Lema

$$\binom{n}{m_1, m_2, \ldots, m_l} = \binom{n}{m_1}\binom{n-m_1}{m_2}\binom{n-(m_1+m_2)}{m_3}\cdots\binom{n-(m_1+\ldots+m_{l-1})}{m_l}$$

$$\begin{aligned}
&\binom{n}{m_1}\binom{n-m_1}{m_2}\binom{n-(m_1+m_2)}{m_3}\cdots\binom{m_l}{m_l} \\
&\quad = \frac{n!}{m_1!(n-m_1)!}\frac{(n-m_1)!}{m_2!(n-m_1-m_2)!}\frac{(n-m_1-m_2)!}{m_3!(n-m_1-m_2-m_3)!}\cdots\frac{m_l!}{m_l!0!} \\
&\quad = \frac{n!}{m_1!m_2!\ldots m_l!} = \binom{n}{m_1, m_2, \ldots, m_l}
\end{aligned}$$

Primer

$$\binom{5}{1,3,1} = \binom{5}{1}\cdot\binom{4}{3}\cdot\binom{1}{1} = 5\cdot 4 = 20$$

Polinomni koeficijenti-osobina2

Lema

$$\binom{n}{m_1, m_2, \dots, m_l} = \binom{n}{k_1, k_2, \dots, k_l}, \qquad \{\{m_1, \dots, m_l\}\} = \{\{k_1, \dots, k_l\}\}$$

Primer

$$\binom{4}{0,1,3} = \binom{4}{0,3,1} = \binom{4}{1,0,3} = \binom{4}{1,3,0} = \binom{4}{3,1,0} = \binom{4}{3,0,1} = \frac{4!}{0!1!3!} = 4$$

Polinomni koeficijenti-osobina3

Neka je $1 \leq m_1, \ldots, m_l \leq n-1$.

Lema

$$\binom{n}{m_1, m_2, \ldots, m_l} = \binom{n-1}{m_1-1, m_2, \ldots, m_l} + \binom{n-1}{m_1, m_2-1, \ldots, m_l} + \ldots + \binom{n-1}{m_1, m_2, \ldots, m_l-1}$$

$$P(m_1, m_2, \ldots, m_l) = P(m_1-1, m_2, \ldots, m_l) + P(m_1, m_2-1, \ldots, m_l) + \ldots + P(m_1, m_2, \ldots, m_l-1)$$

Kako je $m_1 \geq 1$, svaka permutacija kao prvu koordinatu ima a_1 ili a_2 ili...ili a_l:

$$\begin{aligned} |P(M)| &= |P(M_{a_1}) \cup P(M_{a_2}) \ldots \cup P(M_{a_l})| \\ &= |P(M \setminus \{a_1\})| + |P(M \setminus \{a_2\})| + \ldots + |P(M \setminus \{a_l\})| \end{aligned}$$

$P(M) =$ skup permutacija multiskupa M

$P(M_{a_i}) =$ skup permutacija multiskupa M sa prvom koordinatom a_i

Polinomni koeficijenti-osobina4

Lema

$$\binom{n}{m_1, m_2, \ldots, m_{l-1}, 0} = \binom{n}{m_1, m_2, \ldots, m_{l-1}}$$

Tema 4

Polinomna formula

Polinomna formula

Teorema (Polinomna formula)

Neka je $l \geq 2$ *i* $n \geq 0$.

$$(x_1 + \ldots + x_l)^n = \sum_{\substack{m_1 + \ldots + m_l = n \\ m_1 \geq 0 \ldots m_l \geq 0}} \binom{n}{m_1, \ldots, m_l} x_1^{m_1} x_2^{m_2} \ldots x_l^{m_l}$$

Polinomna formula

Zadatak

Napisati u razvijenom obliku $(x+y+z)^3$

$$\begin{aligned}
(x+y+z)^3 &= \binom{3}{3,0,0}x^3y^0z^0 + \binom{3}{0,3,0}x^0y^3z^0 + \binom{3}{0,0,3}x^0y^0z^3 \\
&+ \binom{3}{0,1,2}x^0y^1z^2 + \binom{3}{0,2,1}x^0y^2z^1 + \binom{3}{1,0,2}x^1y^0z^2 \\
&+ \binom{3}{1,2,0}x^1y^2z^0 + \binom{3}{2,0,1}x^2y^0z^1 + \binom{3}{2,1,0}x^2y^1z^0 \\
&+ \binom{3}{1,1,1}x^1y^1z^1 \\
&= x^3+y^3+z^3+3yz^2+3y^2z+3xz^2+3xy^2+3x^2z+3x^2y+6xyz
\end{aligned}$$

Polinomna formula

Zadatak

Odrediti koeficijent uz $x^2y^3z^5$ *u razvoju stepena trinoma* $(x+2y-z)^{10}$

Koeficijent uz $x^2y^3z^5$ je sadržan u sabirku

$$\binom{10}{2,3,5}x^2(2y)^3(-z)^5 = \frac{10!}{2!3!5!}x^2 2^3 y^3 (-1)^5 z^5 = -20160x^2y^3z^5$$

Polinomna formula

Zadatak

Odrediti koeficijent uz x u razvoju stepena trinoma $(2x^3 - x + 1)^4$.

$$T_{i,j,k} = \binom{4}{i,j,k}(2x^3)^i(-x)^j = \binom{4}{i,j,k}2^i(-1)^j x^{3i+j}$$

$$\begin{array}{ccccccc} i & + & j & + & k & = & 4 \\ 3i & + & j & & & = & 1 \end{array}$$

odakle je $(i,j,k) \in \{(0,1,3)\}$ i traženi koeficijent je $\binom{4}{0,1,3}2^0(-1)^1 = -4$.

1. Koliko sabiraka ima u razvijenom obliku $(x_1 + \ldots + x_l)^n$?

$$\binom{n+l-1}{l-1}$$

2. $$\sum_{\substack{m_1 + \ldots + m_l = n \\ m_1 \geq 0 \ldots m_l \geq 0}} \binom{n}{m_1, \ldots, m_l} = (1 + \ldots + 1)^n = l^n$$