ВЕЖБЕ 6

РЕКУРЕНТНЕ РЕЛАЦИЈЕ

РЕКУРЕНТНА РЕЛАЦИЈА реда k

$$a_{n+k} = F(n, a_n, a_{n+1}, a_{n+2}, \ldots, a_{n+k-1})$$

$n \in \mathbb{N}$

$a_n, a_{n+1}, \ldots, a_{n+k}$

узастопни чланови неког низа

ЛИНЕАРНЕ ХОМОГЕНЕ Р.Р. СА КОНСТАНТНИМ КОЕФИЦИЈЕНТИМА

$$c_k a_{n+k} + c_{k-1} a_{n+k-1} + \ldots + c_1 a_{n+1} + c_0 a_n = 0$$

$c_i = \text{const}$

1. Наћи опште решење рекурентне релације

а) $f_{n+2} - 7f_{n+1} + 12f_n = 0$

$f_n \to t^n$

$t^{n+2} - 7t^{n+1} + 12t^n = 0 \quad /:t^n$

$$\boxed{t^2 - 7t + 12 = 0}$$

КАРАКТЕРИСТИЧНА Ј-НА

$t_{1,2} = \dfrac{7 \pm \sqrt{49 - 4 \cdot 12}}{2} = \dfrac{7 \pm 1}{2}$

$t_1 = 4 \qquad t_2 = 3$

$f_n = A \cdot t_1^n + B \cdot t_2^n$

$f_n = A \cdot 3^n + B \cdot 4^n$

б) $f_n + 3f_{n-1} - 10f_{n-2} = 0$

$f_n \to t^n$

$t^n + 3t^{n-1} - 10t^{n-2} = 0 \quad /:t^{n-2}$

$t^2 + 3t - 10 = 0$

$t_{1,2} = \ldots$

$(t+5)(t-2) = 0$

$t_1 = -5 \quad t_2 = 2$

$f_n = A \cdot (-5)^n + B \cdot 2^n$

c) $f_{n+2} - 4f_{n+1} + 13f_n = 0$

$f_n \to t^n$

$t^{n+2} - 4t^{n+1} + 13t^n = 0 \quad / : t^n$

$t^2 - 4t + 13 = 0$

$$t_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2}$$

$t_1 = 2+3i \qquad t_2 = 2-3i$

$$f_n = A \cdot (2+3i)^n + B \cdot (2-3i)^n$$

d) $f_{n+2} + 6f_{n+1} + 9f_n = 0$

$t^2 + 6t + 9 = 0$

$(t+3)^2 = 0$

$t_1 = t_2 = -3$

$\cancel{f_n = A \cdot (-3)^n + B \cdot (-3)^n = (A+B)(-3)^n = C \cdot (-3)^n}$

$$\boxed{f_n = A \cdot (-3)^n + B \cdot n \cdot (-3)^n}$$

e) $f_{n+3} + 3f_{n+2} + 3f_{n+1} + f_n = 0$

$t^3 + 3t^2 + 3t + 1 = 0$

$(t+1)^3 = 0$

$t_1 = t_2 = t_3 = -1$

$f_n = A(-1)^n + Bn(-1)^n + Cn^2(-1)^n$
$= (A + nB + n^2C)(-1)^n$

f) $f_{n+4} + 4f_n = 0$

$t^{n+4} + 4t^n = 0 \ /:t^n$

$t^4 + 4 = 0$

смена: $x = t^2$

$x^2 + 4 = 0$

$x^2 = -4$

$x = \pm 2i$

$x = 2i$

$t = \pm\sqrt{2i}$

$x = -2i$

$t = \pm\sqrt{-2i}$

$f_n = A(\sqrt{2i})^n + B(-\sqrt{2i})^n + C(\sqrt{-2i})^n + D(-\sqrt{-2i})^n$

2. Решити рекурентну релацију

a) $f_n = 5f_{n-1} - 6f_{n-2}$, $\boxed{f_0 = f_1 = 1}$

$f_n - 5f_{n-1} + 6f_{n-2} = 0$

$t^2 - 5t + 6 = 0$

$(t-2)(t-3) = 0$

$t_1 = 2 \quad t_2 = 3$

$f_n = A \cdot 2^n + B \cdot 3^n$

$1 = f_0 = A \cdot 2^0 + B \cdot 3^0 = A + B$

$1 = f_1 = A \cdot 2^1 + B \cdot 3^1 = 2A + 3B$

$$\boxed{\begin{array}{l} A + B = 1 \\ 2A + 3B = 1 \end{array}}$$

$2(A+B) + B = 1$

$2 \cdot 1 + B = 1$

$B = -1$

$A = 1 - B$

$= 2$

$f_n = 2 \cdot 2^n + (-1) \cdot 3^n = \boxed{2^{n+1} - 3^n}$

b) $f_n = 6f_{n-1} - 9f_{n-2}$, $f_0 = f_1 = 2$

$f_n - 6f_{n-1} + 9f_{n-2} = 0$

$t^2 - 6t + 9 = 0$

$(t-3)^2 = 0$

$t_1 = t_2 = 3$

$f_n = (A + Bn)3^n$

$2 = f_0 = (A + B\cdot 0)\cdot 3^0 = A$

$2 = f_1 = (A + B\cdot 1)\cdot 3^1 = 3A + 3B$

$\Rightarrow B = -\frac{4}{3}$

$f_n = \left(2 + \left(-\frac{4}{3}\right)\cdot n\right)3^n = 2\cdot 3^n - 4\cdot n\cdot 3^{n-1}$

c) $f_n = 5f_{n-1} - 6f_{n-2} - 4f_{n-3} + 8f_{n-4}$

$f_0 = 1,\ f_1 = 8,\ f_2 = 12,\ f_3 = 38$

$$f_n - 5f_{n-1} + 6f_{n-2} + 4f_{n-3} - 8f_{n-4} = 0$$

$$t^4 - 5t^3 + 6t^2 + 4t - 8 = 0$$

$\pm 1, \pm 2, \pm 4, \pm 8$

ХОРНЕРОВА ШЕМА:

	1	-5	6	4	-8
~~1~~	~~1~~	~~-4~~	~~2~~	~~6~~	~~-2~~
2	1	-3	0	4	0
2	1	-1	-2	0	

$$t^4 - 5t^3 + 6t^2 + 4t - 8 = (t-2)^2(t^2 - t - 2) = (t-2)^2(t-2)(t+1) = (t-2)^3(t+1)$$

$t_1 = t_2 = t_3 = 2 \quad t_4 = -1$

$$f_n = (A + nB + n^2C)2^n + D(-1)^n$$

$$\boxed{\begin{aligned} 1 = f_0 &= A + D \\ 8 = f_1 &= 2A + 2B + 2C - D \\ 12 = f_2 &= 4A + 8B + 16C + D \\ 38 = f_3 &= 8A + 24B + 72C - D \end{aligned}}$$

$\vdots$

$A = 3 \quad B = -\frac{1}{4} \quad C = \frac{1}{4} \quad D = -2$

$$f_n = \left(3 - \frac{n}{4} + \frac{n^2}{4}\right)2^n - 2(-1)^n$$

3. Решити систем
$$\begin{aligned} f_{n+1} &= 2f_n - g_n \quad (1)\\ g_{n+1} &= f_n + 4g_n \quad (2)\end{aligned}$$
уз почетне услове $f_0=2, g_0=1$.

$(1): \; g_n = 2f_n - f_{n+1} \; (*)$

$(2): \; 2f_{n+1} - f_{n+2} = f_n + 8f_n - 4f_{n+1}$

$f_{n+2} - 6f_{n+1} + 9f_n = 0$

$t^2 - 6t + 9 = 0$

$t_1 = t_2 = 3$

$f_n = (A + B\cdot n)3^n$

$f_0 = 2$

$(1): \; f_1 = 2f_0 - g_0 = 3$

$2 = f_0 = A$

$3 = f_1 = 3A + 3B$

$\Rightarrow B = -1$

$f_n = 2\cdot 3^n - n\cdot 3^n = (2-n)\cdot 3^n$

$(*) \; g_n = 2f_n - f_{n+1} = 2\cdot(2-n)3^n - (\underbrace{2-(n+1)}_{1-n})3^{n+1} =$

$3^n(4-2n-3+3n) = (1+n)3^n$

4. Нађи опште решење једначине $a_{n+2}^2 = 5a_{n+1}^2 - 4a_n^2$. НЕЛИНЕАРНА

Уводимо смену $b_n = a_n^2$

$$b_{n+2} = 5b_{n+1} - 4b_n$$

$$b_{n+2} - 5b_{n+1} + 4b_n = 0$$

$$t^2 - 5t + 4 = 0$$

$$(t-4)(t-1) = 0$$

$$b_n = A \cdot 1^n + B \cdot 4^n$$

$$= A + B \cdot 4^n \xrightarrow[\text{смену}]{\text{враћамо}} \boxed{a_n = \pm\sqrt{A + B4^n}}$$

5. Ако се зна да су сви чланови низа a_n почев од a_2 различити решити

a) $a_{n+2} = \dfrac{a_{n+1}^3}{a_n^2}$, $\boxed{a_0 = 1, a_1 = 2}$

$\log\frac{a}{b} = \log a - \log b$

$\log ab = \log a + \log b$

$\log a^b = b \cdot \log a$

$$a_{n+2} = \frac{a_{n+1}^3}{a_n^2} \quad \Big/ \log_2$$

решење: $\boxed{a_n = 2^{2^n - 1}}$

$b_n = \log_2 a_n$

$\Rightarrow a_n = 2^{b_n}$

$$\log_2 a_{n+2} = \log_2 a_{n+1}^3 - \log_2 a_n^2$$

$$\log_2 a_{n+2} = 3\log_2 a_{n+1} - 2\log_2 a_n$$

смена: $b_n = \log_2 a_n$

$$\boxed{\begin{aligned} b_0 &= \log_2 a_0 = \log_2 1 = 0 \\ b_1 &= \log_2 a_1 = \log_2 2 = 1 \end{aligned}}$$

$$b_{n+2} = 3b_{n+1} - 2b_n$$

$t^2 - 3t + 2 = 0 \qquad (t-2)(t-1) = 0$

$b_n = A + B \cdot 2^n$

$\left.\begin{aligned} A + B &= 0 \\ A + 2B &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} B &= 1 \\ A &= -1 \end{aligned}$

$\boxed{b_n = -1 + 2^n}$

Напомена: Услов да су сви чланови низа почев од a_2 различити обезбеђује да уведена смена буде добро дефинисана.

НЕХОМОГЕНА

6. Наћи општу формулу за следећи низ $a_{n+2}-4a_{n+1}+4a_n=2^n$, ако је $a_0=a_1=0$.

хомогени део:

$$h_{n+2}-4h_{n+1}+4h_n=0$$

$$t^2-4t+4=0$$

$$\boxed{h_n=(A+Bn)2^n}$$

$$\cancel{0=h_0=A}$$

$$\cancel{0=h_1=2A+2B}$$

Тражимо једно партикуларно решење нехомогеног дела

$$\cancel{p_n=C\cdot 2^n}$$

$$2^n=\cancel{p_{n+2}-4p_{n+1}}+4p_n=C\cdot 2^{n+2}-4\cdot C\cdot 2^{n+1}+4\cdot C\cdot 2^n=$$

$$\cancel{2^n(4C-8C+4C)}=2^n\cdot 0=0$$

$$\boxed{p_n=C\cdot n^2\cdot 2^n}$$

Напомена: $=3^n$

$p_n=C\cdot 3^n$

$=3^n+4^n$

$p_n^1=C\cdot 3^n\quad p_n^2=D\cdot 4^n$

$$2^n=p_{n+2}-4p_{n+1}+4p_n=$$

$$=C(n+2)^2\cdot 2^{n+2}-4\cdot C\cdot(n+1)^2\cdot 2^{n+1}+4\cdot C\cdot n^2\cdot 2^n$$

$$=C\,2^n\left(4(n^2+4n+4)-8(n+1)^2+4n^2\right)$$

$$=C\cdot 2^n\left(\cancel{4n^2}+\cancel{16n}+16-\cancel{8n^2}-\cancel{16n}-8+\cancel{4n^2}\right)$$

$$=8\cdot C\cdot 2^n\implies 8C=1\implies \boxed{C=\frac{1}{8}}$$

$$a_n=h_n+p_n$$

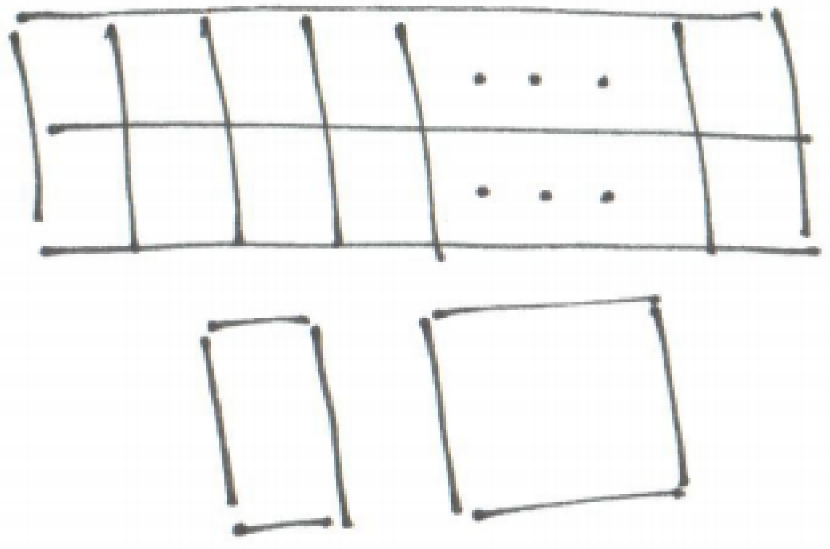
$$0=a_0=h_0+p_0=A+\frac{1}{8}\cdot 0\cdot 2^0\Rightarrow A=0$$

$$0=a_1=h_1+p_1=2A+2B+\frac{1}{8}\cdot 2$$

$$\implies B=-\frac{1}{8}$$

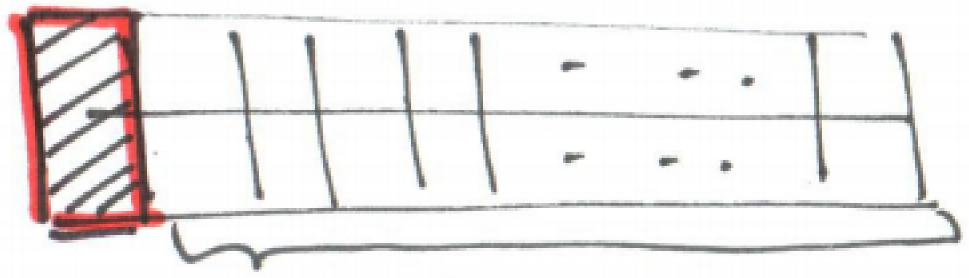
$$a_n=-\frac{1}{8}\cdot n\cdot 2^n+\frac{1}{8}\cdot n^2\cdot 2^n=2^{n-3}n(n-1)$$

7. Правоугаоник величине 2×n издељен је на 2n једнаких квадрата. На располагању имамо домине правоугаоног облика 2×1 и 2×2. На колико начина се цео правоугаоник 2×n може прекрити са овим доминама?



f_n - број начина да прекријемо правоугаоник димензије 2×n датим доминама

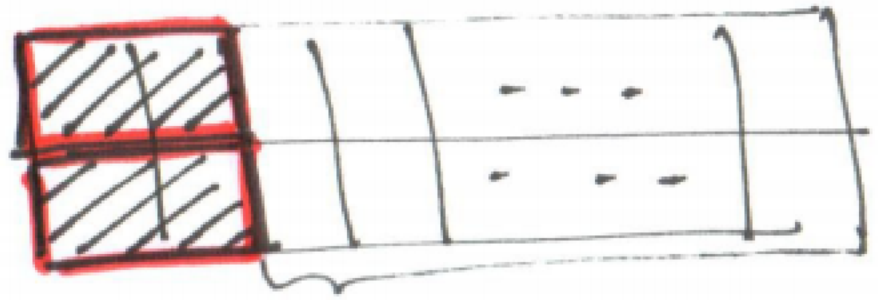
1°



f_{n-1}

2×(n-1)

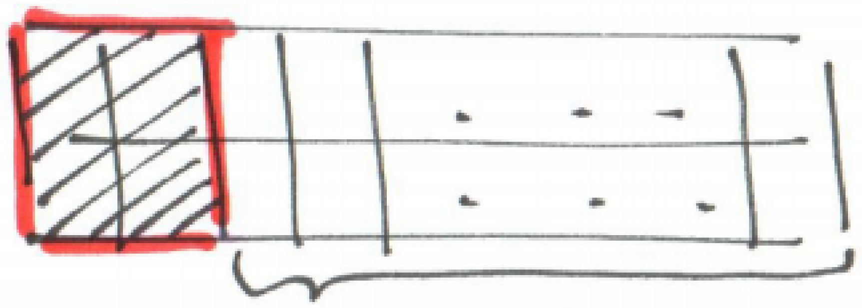
3°



f_{n-2}

2×(n-2)

2°

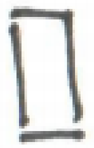
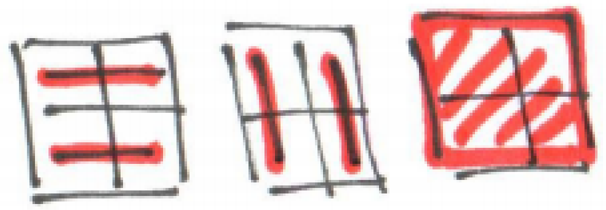


f_{n-2}

2×(n-2)

$$f_n = f_{n-1} + f_{n-2} + f_{n-2} \Rightarrow \boxed{f_n = f_{n-1} + 2f_{n-2}}$$

1° 2° 3°

$f_0 = 1\,!$ $\quad f_1 = 1$  $\quad f_2 = 3$ 

$$f_2 = f_1 + 2f_0 = 1 + 2\cdot 1 = 3$$

8. Колико има речи дужине n над азбуком $A=\{1,2,3\}$ у којима се не појављује подреч 11?

f_n – број тражених речи дужине n

1° $1\ {2 \atop 3}\ \boxed{\quad n-2 \quad}$ $\quad 2\cdot f_{n-2}$

2° $2\ \boxed{\quad n-1 \quad}$ $\quad f_{n-1}$

3° $3\ \boxed{\quad n-1 \quad}$ $\quad f_{n-1}$

$$f_n = 2f_{n-2} + 2f_{n-1}$$

$f_0 = 1$ (празна реч)

$f_1 = 3$

$(f_2 = 3^2 - 1 = 8)$